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GROWTH OF SUBHARMONIC FUNCTIONS OF ORDER GREATER THAN HALF

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ABSTRACT: In this paper we shall study the growth and asymptotic behaviour of sub-harmonic functions of order greater than half near Pólya peaks. In some way our result is a generalization of Paley's conjecture. The method employed is a non-asymptotic via a normal family of subharmonic functions.

Key words/phrases: Order, Pólya peaks, star function, subharmonic

INTRODUCTION

Let u be a subharmonic function defined on the complex plane C. We set

$$B(r, u) = \sup_{|z|=r} u(z)$$

and define the Nevanlinna characteristic of u by

$$T(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u^+(re^{i\theta})d\theta$$

where $u^+(z) = \max(u(z), 0)$.

The lower order λ of u is given by

$$\lambda = \underline{\lim} \frac{T(r,u)}{\log r}$$

A sequence $\{r_n\}$ of positive numbers is said to be a sequence of Pólya peaks for T(r, u) of order $\lambda > 0$ if there is a sequence $\{\in_n\}$, $\in_n > 0$ such that $\in_n \to 0$ and $\in_n r_n \le t \le \in_n^{-1} r_n$ imply

$$T(t, u) \le (1 + \in_n) \left(\frac{t}{r_n}\right)^{\lambda} T(r_n, u).$$

It is well known that T(r, u) has a sequence of Pólya peaks of lower order $\lambda > 0$ (see Edrei, 1965). It is also easy to see that a subsequence of Pólya peaks for T(r, u) is also a sequence of Pólya peaks for T(r,u).

Let u be a subharmonic function of lower order $\lambda > 1/2$ and $\{r_n\}$ be a sequence of Pólya peaks for T(r, u). We will prove that

$$\overline{\lim} \frac{B(rr_n, u)}{T(r_n, u)} \le \pi\lambda r^{\lambda}, 0 < r < \infty. \qquad (1)$$

For r = 1, (1) is Paley's conjecture (Paley, 1932) and, for its proof see Rossi and Weitsman (1983) in case u = log |f| where f is an entire function.

The subharmonic function

$$u(re^{i\theta}) = \begin{cases} \pi\lambda r^{\lambda} \cos\lambda\theta, |\theta| \le \dfrac{\pi}{2\lambda} \\ 0 \qquad \textit{otherwise} \end{cases} \qquad (2)$$

satisfies $T(r,u) = r^{\lambda}$, $B(r, u) = \pi\lambda r^{\lambda}$ and is extremal for (1), *i.e.*, equality holds in (1). We will show that subharmonic functions of lower order $\lambda > 1/2$, for which equality holds in (1) for some r > 0, behave asymptotically as rotations of the function given in (2). Indeed we have the following theorem.

Theorem 1

Let u be a subharmonic function of lower order $\lambda > 1/2$ and $\{r_n\}$ a sequence of Pólya peaks for T(r, u) of order λ. Then the following statements hold:

a) $\overline{\lim} \dfrac{B(rr_n, u)}{T(r_n, u)} \le \pi\lambda r^{\lambda} \quad (0 < r < \infty)$.

b) If equality holds in (a) for some r > 0 then equality holds for all r > 0.

c) $\lim_{n\to\infty} \dfrac{T(rr_n, u)}{T(r_n, u)} = r^{\lambda}, \quad (0 < r < \infty)$.

d) There is a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $u(r_{n_k} e^{i\theta}) = (o(1) + v(re^{i\theta}))\, T(r_{n_k}, u)$ as $k \to \infty$ for almost all θ, $|\theta - \alpha| \le \pi/2\lambda$, where $v(re^{i\theta}) = \pi\lambda r^{\lambda} \cos\lambda(\theta - \alpha)$, $\alpha \in [-\pi, \pi]$.

We remark that if equality holds in (a) for r = 1 it is proved that (c) holds, (see Edrei and Fuchs, 1976).

DEFINITIONS AND FACTS

In this section we assemble some of the definitions and facts pertinent to prove Theorem 1. Let u be a subharmonic function in the plane. The *- function of u, u* introduced by Baernstein (1974) is defined by

$$u^*(re^{i\theta}) = \frac{1}{2\pi} \sup \int_E u(re^{i\varphi})d\varphi, \ (0 \le \theta \le \pi, r > 0)$$

where the supermum is taken over all sets $E \subseteq [-\pi, \pi]$ with $m(E) = 2\theta$ (m = Lebesgue measure on the real line). Baernstein (1974) proved that u* is subharmonic in the upper half plane, π^+ and continuous in the closure of π^+ except possibly at the origin. Further, u* also satisfy (see also Hyman, 1989, Chap 9)

$$T(r, u) = \max_{0\le\theta\le\pi} u^*(re^{i\theta}), \qquad u^*(r) = 0$$

$$B(r, u) = \pi \frac{\partial}{\partial\theta} u^*(re^{i\theta}) \,|_{\theta=0} \ \ldots\ldots (3)$$

Let $\{r_n\}$ be a sequence of Pólya peaks for T(r, u) of lower order $\lambda > 0$. We set

$$u_n(re^{i\theta}) = \frac{u(rr_n e^{i\theta})}{T(r_n, u)}, \quad (n = 1, 2, \ldots),$$

a sequence of subharmonic functions. A well known result due to Anderson and Baernstein (1978) asserts that there is a subharmonic function v and a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that

$$\lim_{k\to\infty} \int_0^{2\pi} \left| u_{n_k}(re^{i\theta}) - v(re^{i\theta}) \right| d\theta = 0 \quad (0 < r < \infty)$$

and .... (4)

$$\lim_{k\to\infty} \frac{T(r\, r_{n_k}, u)}{T(r_{n_k}, u)} = \lim_{k\to\infty} T(r, u_{n_k}) = T(r, v) \le r^{\lambda} \ (0<r<\infty).$$

In this paper any subharmonic function v which satisfies (4) for some subsequence of $\{r_n\}$ will be referred as a limit function of $\{u_n\}$.

We also need the well-known convolution inequality due to Petrenko (1969). Let u be a subharmonic function in the plane, $0 < \gamma < 1$, we set

$$k(t, \gamma) = \gamma^{-2} \frac{t^{1/\gamma}}{\left(t^{1/\gamma} + 1\right)^2}$$

We have Petrenkós inequality

$$B(r, u) \le \int_0^R u^*(t\, e^{i\pi\gamma})\ k\left(\frac{r}{t}, \gamma\right) \frac{dt}{t}$$

$$+C\left(\frac{r}{R}\right)^{\frac{1}{\gamma}} T(2R, u), \left(0 < r < \frac{R}{2}\right) \ \ldots\ldots (5)$$

for an absolute constant C.

Proofs of the above inequality are also given by Essén (1975) and by Edrei and Fuchs (1976) where it was shown that the Mellin transform of k (t, γ) is

$$\hat{k}(s, \gamma) = \int_0^\infty k(t, \gamma) \frac{dt}{t^{1+s}} = \frac{\pi s}{\sin \pi\gamma s}, \ (0<s<\frac{1}{\gamma}). \ \ldots\ldots (6)$$

Proof of Theorem 1

Lemma

Let u be a subharmonic function of lower order $\lambda > 0$ and $\{r_n\}$ a sequence of Pólya peaks for T(r, u) of order λ. Then there is limit function v of $\{u_n\}$. Such that

$$\overline{\lim}\, B(r, u_n) = B(r, v)$$

Proof.

Let w be any limit function of $\{u_n\}$. Then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ for which (4) holds.

Let r > 0 be fixed and $B(r, u_{n_k}) = u_{n_k}\left(re^{\beta_k}\right)$ (k = 1,2, …) and assume $\beta_k \to \beta$ as $k \to \infty$. Let 0 < t < r and $B(t, w) = w(te^{i\alpha})$. Then we have

$$B(t, w) \le \frac{1}{2\pi} \int_0^{2\pi} w(re^{i\theta}) P_{t/r}(\theta - \alpha)\, d\theta$$

$$= \lim_{k\to\infty} \frac{1}{2\pi} \int_0^{2\pi} u_{n_k}(re^{i\theta}) P_{t/r}(\theta-\alpha)d\theta \le \overline{\lim}\, B(r, u_{n_k})$$

where

$$P_s(\theta) = \frac{1-s^2}{1+s^2-2s\cos\theta}$$ is the Poisson kernel.

Thus $B(t, w) \le \overline{\lim}\, B(r, u_{n_k})$ holds for t<r. Since B(s, v) is a continuous function of s, letting $t \to r$ we have

$$B(r, w) \le \overline{\lim}\ B(r, u_{n_k}) \ \ldots\ldots (7)$$

On the other hand for t > r we have

$$B(r, u_{n_k}) \le \frac{1}{2\pi} \int_0^{2\pi} u_{n_k}(te^{i\theta}) P_{r/t}(\theta - \beta_k)d\theta.$$

Consequently,

$$\overline{\lim}\, B(r, u_{n_k}) \le \overline{\lim}\, \frac{1}{2\pi}\int_0^{2\pi} u_{n_k}(te^{i\theta}) P_{r/t}(\theta-\beta_k)\, d\theta$$

$$= \lim_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_{n_k}(te^{i\theta}) P_{r/t}(\theta-\beta_k)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} w(te^{i\theta}) P_{r/t}(\theta-\beta)\, d\theta$$

$$\le B(t, w)$$

Letting $t \to r$, we have

$$\overline{\lim}\, B(r, u_{r_k}) \le B(r, w). \quad (8)$$

Thus from (7) and (8)

$$B(r, w) = \overline{\lim}\, B(r, u_{n_k}) \le \overline{\lim}\, B(r, u_n). \quad (9)$$

To complete the proof of the Lemma, let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$ such that

$$\lim_{k\to\infty} B(r, u_{n_k}) = \overline{\lim}\, B(r, u_n).$$

Since $\{r_{n_k}\}$ is a subsequence of Pólya peaks, it is a sequence of Pólya peaks for T(r, u). Following the above argument and by (9) there is limit function v of $\{u_{n_k}\}$ such that

$$B(r, v) = \lim_{k\to\infty} B(r, u_{n_k}) = \overline{\lim}\, B(r, u_n).$$

Now if u is a subharmonic function of lower order $\lambda > 1/2$ and v is any limit function of $\{u_n\}$ we have by (3) and (4)

$$v^*(re^{i\theta}) \le T(r, v) \le r^{\lambda}, \quad 0 \le \theta \le \pi/2\lambda.$$

Since $v^*(r) = 0$ and $v^*(re^{i\pi/2\lambda}) \le r^{\lambda}$, by Phragmén Lindelöf Principle we conclude that

$$v^*(re^{i\theta}) \le r^{\lambda} \sin \lambda\theta, \; 0 \le \theta \le \pi/2\lambda.$$

Consequently by (3)

$$B(r, v) = \pi \frac{\partial v^*}{\partial \theta}(re^{i\theta}) \,|\, \theta = 0 \le \pi\lambda r^{\lambda}.$$

Thus by the Lemma,

$$\overline{\lim}\, B(r, u_n) \le \pi\lambda r^{\lambda} \quad (0 < r < \infty). \quad (10)$$

This proves assertion (a) of Theorem 1.

Theorem 2

Let u be a subharmonic function of lower order $\lambda > 1/2$ and $\{r_n\}$ a sequence of Pólya peaks for T(r,u) of order λ. If for some $r_1 > 0$ $\overline{\lim}\, B(r_1, u_n) = \pi\lambda r_1^{\lambda}$, then there is a limit function v of $\{u_n\}$ such that

i) $v^*(re^{i\theta}) = r^{\lambda}\sin \lambda\theta, \; 0 \le \theta \le \pi/2\lambda, \; r > 0$ and consequently $T(r, v) = r^{\lambda}$ and $B(r, v) = \pi\lambda r^{\lambda}$.

ii) $v(re^{i\theta}) = \pi\lambda r^{\lambda} \cos\lambda(\theta-\alpha)$ for $|\theta-\alpha| \le \pi/2\lambda$ and for some $\alpha \in [-\pi, \pi]$.

Proof.

By the Lemma there is a subharmonic function v such that $\overline{\lim}\, B(r, u_n) = B(r, v)$. Since $\overline{\lim}\, B(r_1, u_n) = \pi\lambda r_1^{\lambda}$, we have $B(r_1, v) = \pi\lambda r_1^{\lambda}$, applying Peteronko's inequality (5) with $\gamma = 1/2\lambda$ and using (6) we obtain

$$\pi\lambda r_1^{\lambda} = B(r_1, v) \le \int_0^{\infty} v^*(te^{\frac{i\pi}{2\lambda}})\, k\left(\frac{r_1}{t}, \lambda\right)\frac{dt}{6} \quad (11)$$

$$\le \int_0^{\infty} t^{\lambda} k\left(\frac{r_1}{t}, \gamma\right)\frac{dt}{6}$$

$$= \pi\lambda r_1^{\lambda}$$

Hence equality holds through out in (11), which implies

$$v^*\left(te^{i\pi/2\lambda}\right) = t^{\lambda}, \qquad (0 < t < \infty).$$

Since $v^*(te^{i\theta}) \le T(t, v) \le t^{\lambda}$, it follows that

$$T(t,v) = t^{\lambda}. \quad (12)$$

Let $0 < \alpha < \pi/2\lambda$, $\gamma = \alpha/\pi < 1$ and apply (5) to get $v^*(te^{i\alpha}) = t^{\lambda} \sin \lambda\alpha$.

Since $v^*(te^{i\theta}) \le t^{\lambda} \sin \lambda\theta \;\; 0 \le \theta \le \pi/2\lambda$, it follows by the maximum principle

$$v^*(re^{i\theta}) = r^{\lambda} \sin \lambda\theta, \; 0 \le \theta \le \pi/2\lambda \quad (13)$$

which proves (i) of Theorem 2. It follows from (13) and (3) that

$$B(r, v) = \pi\lambda r^{\lambda} \qquad (0 < r < \infty).$$

Thus,

$$\overline{\lim}\, B(r, u_n) = \pi\lambda r^{\lambda} \qquad (0 < r < \infty).$$

This proves assertion (b) of Theorem 1.

Since $v^*(z)$ is harmonic in the region $0 < \arg z < \pi/2\lambda$, it follows (Essén and Shea, 1978/79) that

$v(re^{i\theta}) = \pi\lambda r^{\lambda} \cos \lambda(\theta - \alpha)$ for $|\theta - \alpha| \le \pi/2\lambda$ and for some $\alpha \in [-\pi, \pi]$.

Assertion (d) of Theorem 1 follows for (4) and an application of results in real analysis. We remark that the above results hold if we replace lower order by order of the subharmonic function.

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