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Split-Plot Central Composite Designs Robust to a Pair of Missing Observations

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ABSTRACT: This study constructs robust split-plot central composite designs against missing pairs of observations. Split-plot central composite designs (CCD) consist of factorial (f), whole-plot axial (α), subplot axial (β), and center (c) points. A loss function in terms of determinant (D) criterion was formulated based on two different configurations of the factorial and axial parts; losses due to missing pairs of observations of these different categories of points were investigated. Robust split-plot central composite designs against missing pairs of observations were then developed under each of the two configurations. It was observed that the losses, L_{ff}, $L_{\beta\beta}$, and $L_{f\beta}$, due respectively, to missing pairs of observations of the factorial, subplot axial, and, factorial and subplot axial points, were higher than the losses due to missing pairs of observations of the whole-plot axial and center points given by $L_{\alpha\alpha}$ and L_{cc} respectively. Thus the factorial (f) and the subplot axial (β) points were found to be the most influential points in these designs while the whole-plot axial (α) and the center (c) points were less influential. This work has therefore identified and properly classified the losses due to missing design points in the split-plot CCD portions. In this way, the practitioner can avoid the experimental points having less influence from the full CCD experiments and this could lead to a possible increase in design efficiency.

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Response Surface Methodology (RSM) is an area of experimental design which consists of a group of mathematical and statistical techniques used in the development of an adequate functional relationship between a response of interest, y, and a number of associated control (or input) variables denoted by $x_1, x_2, \ldots, x_k$ (Myers *et al*, 2009). RSM has continued to play vital roles in developing, optimizing and improving processes, particularly where several input variables,$x_1, x_2, \ldots, x_k$, potentially influence some performance measure or quality characteristic, y, of the process under study. However, one difficulty in applying classical response surface designs is that they inherently assume that all factors are equally easy to manipulate, thereby allowing for complete randomization of experimental run order. In practice, most industrial experiments cannot be completely randomized due to the presence of some factors with hard-to-change (HTC) levels. Therefore, these experiments are often conducted in a manner that restricts the randomization, which leads to a split-plot structure. In a split-plot design, experimental runs are performed in groups, where in a group, the levels of the HTC factors are not reset from run to run. This creates dependence among the runs in one group, and leads to clusters of correlated errors and responses. The first paper to exclusively focus on conducting response surface experiments with a split-plot structure was by Myers and Lentner (Letsinger *et al,* 1996). The authors investigate efficiency of various second-order response surface designs when conducted with a split-plot structure. This was then followed by other authors like Vining *et al* (2005), Kowalski *et al* (2006) etc., who modified completely-randomized central composite and Box-Behnken designs (CCD and BBD) to accommodate a split-plot structure. Split-plot central composite designs (CCD) consist of factorial (f), whole-plot axial (α), subplot axial (β), and center (c) points.

Even in a carefully-planned experiments, situations of missing observations cannot be avoided due to reasons that are beyond an experimenter's control. When some observations are lost, desirable design properties such as independence, optimality and orthogonality etc., are destroyed. In this situation, parameters of the fitted model cannot be separately estimated and the design may break down. Thus there is strong need for designs that are insensitive or robust to missing observations. A number of researchers have studied the loss due to a set of missing observations over a range of α value associated with completely randomized central composite designs (CCD) using the loss function introduced by Akhtar and Prescott (1986). Since then, minimaxloss designs that are robust to sets of missing observations have effectively emerged. Much of these studies consider the second-order central composite designs differing in number of control variables and configurations of the factorial, axial and center portions. Some of such results include Akhtar and

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Prescott (1986), Akhtar (2001) and Akram (2002). However, little or no work has been done on investigating the loss due to missing observations in response surface designs conducted within a split-plot structure. Therefore, the objective of this study is to assess the loss due to missing pairs of observations in split-plot central composite designs and to construct split-plot central composite designs that are robust against these missing pairs of observations.

MATERIALS AND METHODS

Candidate Split-plot Designs: Two different configurations of split-plot central composite designs (CCDs) were considered. These include:
Split-plot CCDs with single replication of factorial and axial parts (F+A).
Split-plot CCDs with half replicate of factorial part and complete replicate of axial part (1/2 F+A).
For each of these configurations, five (5) different candidate split-plot CCDs with factors between 3 and 6 were selected to establish the validity of the criterion. These designs are given in the table below with *k, w* and *s* denoting, respectively, numbers of design variables, whole-plot variables and subplot variables.

Table.1: Candidate Split-plot CCDs

(F + A)			(1/2F + A)		
k	*w*	*s*	***k***	*w*	*s*
4	1	3	**5**	1	4
4	2	2	**5**	2	3
5	2	3	**6**	2	4
5	3	2	**6**	3	3
			6	1	5

Now, for a k-factor D(*w,s*) split-plot CCD with four categories of design points, there are ten possible pairs of missing observations from factorial (*f*), whole-plot axial (α), subplot axial (β), and center (*c*) points. These include *ff*, *αα, ββ, cc*, *fα, fβ, fc, αβ, αc,* and *βc*. In terms of the derived Minimaxloss criterion, our computations show that the loss in design information due to missing any of the pairs: $\alpha\alpha$, *cc,* $f\alpha$*, fc,* $\alpha\beta$*,* αc and βc, denoted respectively by $L_{\alpha\alpha}$, L_{cc}, $L_{f\alpha}$, L_{fc}, $L_{\alpha\beta}$, $L_{\alpha c}$, and $L_{\beta c}$, did not have as much adverse effect on the design as that due to missing $ff, \beta\beta,$ and $f\beta$. Therefore, we investigate the robustness of these designs in the presence of the missing pairs: *cc,*$\alpha\alpha$,$ff, \beta\beta,$ $f\beta,$ and construct robust designs accordingly.

Loss function with respect to the Determinant Criterion (minimaxloss criterion): For split-plot response surface designs, the generalized least squares (GLS) model is

$$\boldsymbol{y} = X\boldsymbol{\beta} + Z\gamma + \varepsilon \qquad (1)$$

Where $\boldsymbol{y}$ is the *N x 1* vector of responses, *X* is the *N x p* model matrix, β is the *p* x 1 vector of coefficients, *Z* is an *N x b* incidence matrix assigning observations to each of the *b* whole plots; γ is the *N x 1* vector of random whole-plot errors, ε is the *N x 1* vector of random subplot errors. It is assumed that $\gamma_i \sim N(0, \sigma_\gamma^2)$, $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$, $cov(\gamma_i, \varepsilon_{ij}) = 0$.

The variance - covariance matrix for the observation vector **y** is

$$Var(y) = V = \sigma_\varepsilon^2 I_n + \sigma_\gamma^2 ZZ' = \sigma_\varepsilon^2 (I_n + dZZ') \qquad (2)$$

where $d = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$; σ_γ^2 and σ_ε^2 are the whole-plot and subplot error variances respectively.

The matrix $\boldsymbol{ZZ}$' is a block diagonal matrix with diagonal matrices of $_1, J_{n2}, \ldots, J_{nz}$, where J_{ni} is an n_i x n_i matrix of 1's.

The GLS estimates are

$$\hat{\beta}_{GLS} = (X'V^{-1}X)^{-1}X'V^{-1}y$$

$$\hat{y} = X(X'V^{-1}X)^{-1}X'V^{-1}y = Hy \qquad (3)$$

$$\text{Var}\left(\underline{\hat{\beta}}_{GLS}\right) = (X'V^{-1}X)^{-1}X'V^{-1}\text{Var}\left(\underline{y}\right)V^{-1}X(X'V^{-1}X)^{-1} = (X'V^{-1}X)^{-1} \quad \text{, since Var}\left(\underline{y}\right) = V$$

Where $\boldsymbol{X}$ is as stated above and **y** is the vector of responses.
Following Akhtar and Prescott (1986) criterion for completely randomized central composite designs, Angela *et al* (2013) defined the minimaxloss criterion due to a single missing design point *u* in a split-plot central composite design as

$$l_u = \frac{|X'V^{-1}X| - |X'V^{-1}X|_u}{|X'V^{-1}X|} = 1 - \frac{|X'V^{-1}X|_u}{|X'V^{-1}X|}, \qquad u = 1,2,\ldots \quad (4)$$

Where $|X'V^{-1}X|_u$ is the determinant of the reduced information matrix due to the u[th] missing point. l_u is the loss and lies between zero and one. While the authors use this criterion to develop robust split-plot CCDs in the presence of a single missing observation, this present work develops split-plot CCDs that are robust to missing pairs (uv) of observations using

$$l_{uv} = \frac{|X'V^{-1}X| - |X'V^{-1}X|_{uv}}{|X'V^{-1}X|} = 1 - \frac{|X'V^{-1}X|_{uv}}{|X'V^{-1}X|}, \qquad u, v = 1,2,\ldots \quad (5)$$

The missing observations are at the subplot level, and the whole -plot axial point distance equals the subplot axial point distance from the design center (i.e., $\alpha = \beta$).

RESULTS AND DISCUSSIONS

The computed maximum losses (L_{ff}, $L_{\alpha\alpha}$, $L_{\beta\beta}$, L_{cc}, and $L_{f\beta}$) due to missing pairs of observations of factorial(*f*), whole-plot axial(α), subplot axial(β), and center(*c*) points are presented in the tables below for each of the selected split-plot CCDs and under each of

the two configurations. The robust (or minimaxloss) designs are denoted by asterisks with the respective parameters as shown in each of the tables. These are the designs with the smallest maximum loss among all the maximum losses. We first consider the designs with single replication of factorial and axial parts (F+A). Under this configuration, the losses corresponding to missing pairs of observations are given respectively in TABLES 2 –5 below for the split-plot CCDs: D(1,3), D(2,2), D(3,2), and D(2,3). For each of these CCDs, the losses were also plotted against their corresponding α values and the loss curves given respectively as items *a*, *b*, *c*, and *d*, in FIGURE 1.

Comparing the different losses, it can be seen from each of the tables that there is a point (α) where the values of L_{ff} and $L_{\beta\beta}$ coincide, as can also be seen from the corresponding figures. That is, at this point, the maximum loss is minimized. Thus our formulated loss function for split-plot CCDs performs as expected and compared favorably with that of Akhtar and Prescott, (1986) for completely randomized CCDs. From TABLE 2, we see that the four-factor split-plot CCD with $w = 1,\ s = 3,\ n_f = 16,\ n_\alpha = 16,\ n_\beta = 6,\ n_c = 8$ and $\alpha = \beta = 2.059$ is a minimaxloss design robust to a pair of missing observations.

From TABLE 3, we can see that the four-factor split-plot CCD with $w = 2,\ s = 2,\ n_f = 16,\ n_\alpha = 16,\ n_\beta = 4,\ n_c = 4$ and $\boldsymbol{\alpha = \beta = 2.0933}$ is a minimaxloss design robust to a pair of missing observations. From TABLE 4, we observed that the values of L_{ff} and $L_{\beta\beta}$ coincide at the point $\alpha = \beta = 1.6042$ for the D(3,2) design**.** Thus the five-factor split-plot CCD with $w = 3, s = 2,\ n_f = 32,\ n_\alpha = 24, n_\beta = 4,\ n_c = 4,$ and $\alpha = \beta = 1.6042$ is a minimaxloss design robust to a pair of missing observations.

We can observe from TABLE 5 that for the D(2,3) CCD, the values of L_{ff} and $L_{\beta\beta}$ coincide at the point where $\alpha = \beta = 1.339$. Thus, the five-factor split-plot CCD with $w = 2,\ s = 3,\ n_f = 32,\ n_\alpha = 32,\ n_\beta = 6,\ n_c = 8,$ and $\alpha = \beta = 1.339$ is a minimaxloss design robust to a pair of missing observations.

These points of minimum loss can also be seen directly from items *a* – *d* in FIGURE 1, where the loss curves L_{ff} and $L_{\beta\beta}$ intersect at the given value $(\alpha = \beta)$. Similar plots were used in Akhtar (2001) for completely randomized CCD. We can see from items *b* and *c*, that the loss curve, L_{cc}, makes a bell-shaped curve when plotted against α, attaining it's maximum at $\alpha = \sqrt{k}$, while $L_{\alpha\alpha}$ increases gradually with increasing α. Item *a* in FIGURE 1 shows that L_{cc} and $L_{\alpha\alpha}$ overlap and slightly make a bell-shaped curve when plotted against α, while in item *d*, L_{cc} and $L_{\alpha\alpha}$ overlap and continue to increase gradually as α increases.

Then the designs with half replicate of factorial part and complete replicate of axial part (1/2F+A) were considered. Under this configuration, the losses corresponding to missing a pair of observations of each of the different categories of design points are given respectively in TABLES 6 – 10 for the split-plot designs D(1,4), D(2,3), D(2,4), D(3,3), and D(1,5). These losses were also plotted against the various α values and given respectively as items *a*, *b*, *c*, *d*, and *e*, in FIGURE 2. Our computations of losses have shown that, for each of these designs also, the losses corresponding to missing a pair of center points, (L_{cc}), and those corresponding to missing a pair of axial points $(L_{\alpha\alpha})$, were each less than L_{ff} and $L_{\beta\beta}$ for the whole range of α, and also that L_{cc} continues to increase up to the point $\alpha = \sqrt{k}$, and then decreases as α increases beyond this point. It can also be seen from each of the tables that the losses L_{ff} decreases gradually with increasing value of α, whereas $L_{\beta\beta}$ has an increasing trend with increasing α.

Table 2: Losses due to different pairs of missing observations for five-factor D(1,3) split-plot CCD with single replication of factorial and axial parts; $n_f = 16$, $n_\alpha = 16$, $n_\beta = 6$, $n_c = 8$, $N_w = 6$, and $N = 46$.

α	Loses due to missing a pair of center points (L_{cc})	Loses due to missing a pair of factorial points (L_{ff})	Loses due to missing a pair of whole-plot axial points $(L_{\alpha\alpha})$	Loses due to missing a pair of subplot axial points $(L_{\beta\beta})$	Loses due to missing one factorial and one subplot axial points $(L_{f\beta})$	Overall maxloss due to missing pairs of observations
1.0	0.0190	0.8942	0.0263	0.7091	0.8026	0.8942
1.20	0.0222	0.8851	0.0278	0.7335	0.8076	0.8851
1.50	0.0286	0.8699	0.0296	0.7710	0.8168	0.8699
1.75	0.0336	0.8564	0.0307	0.8023	0.8259	0.8564
2.0	0.0357	0.8428	0.0316	0.8328	0.8363	0.8428
2.059	0.0356	0.8397***	0.0318	0.8397***	0.8390	0.8397***
2.2	0.0345	0.8325	0.0322	0.8554	0.8453	0.8554
2.5	0.0304	0.8184	0.0329	0.8846	0.8587	0.8846
3.0	0.0238	0.7981	0.0338	0.9205	0.8791	0.9205

****Minimax loss due to two missing observations.*

Table 3: Losses due to different pairs of missing observations for four-factor D(2,2) split-plot CCD with single replication of factorial and axial parts; $n_f = 16$, $n_\alpha = 16$, $n_\beta = 4$, $n_c = 4$, $N_w = 10$, and $N = 40$.

α	maxloss due to missing a pair of center points (L_{cc})	maxloss due to missing a pair of factorial points (L_{ff})	maxloss due to missing a pair of whole-plot axial points ($L_{\alpha\alpha}$)	maxloss due to missing a pair of subplot axial points ($L_{\beta\beta}$)	maxloss due to missing one factorial and one subplot axial points ($L_{f\beta}$)
1.0	0.0614	0.8663	0.1081	0.6499	0.7651
1.20	0.0761	0.8563	0.1147	0.6773	0.7712
1.50	0.1119	0.8397	0.1229	0.7182	0.7815
1.75	0.1488	0.8244	0.1286	0.7527	0.7870
2.0	0.1666	0.8094	0.1342	0.7901	0.8048
2.0933	0.1642	0.8044***	0.1363	0.8044***	0.8105
2.5	0.1241	0.7873	0.1443	0.8592	0.8359
3.0	0.0837	0.7714	0.1506	0.9046	0.8623

***Minimaxloss due to two missing observations.

Table 4: Losses due to different pairs of missing observations for five-factor D(3,2) split-plot CCD with single replication of factorial and axial parts; $n_f = 32$, $n_\alpha = 24$, $n_\beta = 4$, $n_c = 4$, $N_w = 16$, and $N = 64$.

α	maxloss due to missing a pair of center points (L_{cc})	maxloss due to missing a pair of factorial points (L_{ff})	maxloss due to missing a pair of whole-plot axial points ($L_{\alpha\alpha}$)	maxloss due to missing a pair of subplot axial points ($L_{\beta\beta}$)	maxloss due to missing one factorial and one subplot axial points ($L_{f\beta}$)	Overall maxloss due to missing pairs of observations
1.0	0.0433	0.6846	0.0980	0.6164	0.6395	0.6846
1.20	0.0520	0.6793	0.1031	0.6327	0.6459	0.6793
1.50	0.0746	0.6706	0.1102	0.6584	0.6562	0.6706
1.6042	0.0863	0.6673***	0.1125	0.6673***	0.6599	0.6673***
1.75	0.1066	0.6626	0.1154	0.6796	0.6651	0.6796
2.0	0.1470	0.6538	0.1200	0.7019	0.6750	0.7019
2.236	0.1666	0.6456	0.1247	0.7286	0.6883	0.7286
2.50	0.1448	0.6380	0.1306	0.7644	0.7083	0.7644
3.0	0.0823	0.6272	0.1397	0.8225	0.7450	0.8225

***Minimaxloss due to two missing observations.

Table 5: Losses due to different pairs of missing observations for five-factor D(2,3) split-plot CCD with single replicationof factorial and axial parts; $n_f = 32$, $n_\alpha = 32$, $n_\beta = 6$, $n_c = 8$, $N_w = 10$, and $N = 78$.

α	maxloss due to missing a pair of center points (L_{cc})	maxloss due to missing a pair of factorial points (L_{ff})	maxloss due to missing a pair of whole-plot axial points ($L_{\alpha\alpha}$)	maxloss due to missing a pair of subplot axial points ($L_{\beta\beta}$)	maxloss due to missing one factorial and one subplot axial points ($L_{f\beta}$)	Overall maxloss due to missing pairs of observations
1.0	0.0142	0.7115	0.0234	0.6789	0.6819	0.7115
1.25	0.0171	0.7060	0.0252	0.6970	0.6894	0.7060
1.339	0.0185	0.7039***	0.0258	0.7039***	0.6923	0.7039***
1.50	0.0216	0.6999	0.0268	0.7167	0.6979	0.7167
1.75	0.0275	0.6933	0.0281	0.7375	0.7075	0.7375
2.0	0.0333	0.6862	0.0291	0.7597	0.7185	0.7597
2.236	0.0357	0.6795	0.0300	0.7825	0.7307	0.7825
2.5	0.0331	0.6725	0.0309	0.8086	0.7459	0.8086

***Minimaxloss due to two missing observations.

Among all the possible groups of pairs within every factorial whole plot for the D(1,4) CCD, we observed that ff1, ff3, and ff5 were empty, i.e., they don't exist, and only ff2 and ff4 were non-empty. Our computations of losses have shown that

$$L_{ff4} > L_{ff2};\ \ L_{\alpha\alpha0} > L_{\alpha\alpha1}; \text{and } L_{f\beta1} > L_{f\beta2}.$$

Minimaxloss designs were therefore developed for the groups with maximum losses. From the losses in TABLE 6 for the D (1,4) design, we observed that the maximum loss ($L_{\beta\beta} = 0.9893$) is minimum when $L_{ff} = L_{\beta\beta} = 0.9893$ for $\alpha = \beta = 5.995$ as indicated in the last column of the table. Thus, the five-factor D(1,4) split-plot CCD with half replication of factorial part and complete replication of axial part, and with $n_f = 16,\ n_\alpha = 16,\ n_\beta = 8,\ n_c = 8,$ and $\alpha = \beta =$ 5.995 is a minimaxloss design robust to a pair of missing observations.

Among all the possible groups of pairs within every factorial whole plot for the D(2,3) CCD, we observed that ff1, ff3, and ff5 were empty. The only non-empty pairs were ff2 and ff4. Our computations of losses have shown that

$$L_{ff4} > L_{ff2};\ \ L_{\alpha\alpha0} > L_{\alpha\alpha1} > L_{\alpha\alpha2}; \text{and } L_{f\beta1} > L_{f\beta2}.$$

From TABLE 7, we observed that the maximum loss ($L_{\beta\beta} = 0.9617$) is minimum when $L_{ff} = L_{\beta\beta} = 0.9617$ for $\alpha = \beta = 4.043$. Thus, the five-factor D(2,3) split-plot CCD with half replication of factorial part and complete replication of axial part, and with $n_f = 16,\ n_\alpha = 16,\ n_\beta = 6,\ n_c = 4,$ and $\alpha = \beta =$

4.043 is a minimaxloss design robust to a pair of missing observations.

Among all the possible groups of pairs within every factorial whole plot for the D(2,4) CCD, we observed that ff1, ff3, ff5, and ff6 were empty. The only non-empty pairs were ff2 and ff4. Our computations of losses have shown that

$$L_{ff4} > L_{ff2};\ L_{\alpha\alpha0} > L_{\alpha\alpha1} > L_{\alpha\alpha2}; \text{and } L_{f\beta1} > L_{f\beta2}.$$

We can observe from TABLE 8 that the maximum loss ($L_{\beta\beta} = 0.8394$) is minimum when $L_{ff} = L_{\beta\beta} = 0.8394$ for $\alpha = \beta = 2.665$. Thus, the six-factor D(2,4) split-plot CCD with half replication of factorial part and complete replication of axial part and with $n_f = 32,\ n_\alpha = 32,\ n_\beta = 8,\ n_c = 8,$ and $\alpha = \beta = 2.665$ is a minimaxloss design robust to a pair of missing observations.

Among all the possible groups of pairs within every factorial whole plot for the D(3,3) CCD, we observed that ff1, ff3, ff4, ff5, and ff6 were all empty. The only non-empty pair was the ff2. Our computations of losses have shown that

$$L_{\alpha\alpha0} > L_{\alpha\alpha1};\ L_{f\beta1} > L_{f\beta2}.$$

We can observe from TABLE 9 that the maximum loss ($L_{\beta\beta} = 0.8278$) is minimum when $L_{ff} = L_{\beta\beta} = 0.8278$ for $\alpha = \beta = 2.739$. Thus, the six-factor D(3,3) split-plot CCD with half replication of factorial part and complete replication of axial part and with $n_f = 32,\ n_\alpha = 24,\ n_\beta = 6,\ n_c = 4,$ and $\alpha = \beta = 2.739$ is a minimaxloss design robust to a pair of missing observations. Among all the possible groups of pairs within every factorial whole plot for the D(1,5) CCD, we observed that ff1, ff3, ff5, and ff6, and ff6 were all empty. The only non-empty pairs were ff2 and ff4. Our computations of losses have shown that

$$L_{ff2} > L_{ff4};\ L_{\alpha\alpha0} > L_{\alpha\alpha1} > L_{\alpha\alpha2};\ L_{f\beta1} > L_{f\beta2}.$$

We can observe from TABLE 10 that the maximum loss ($L_{\beta\beta} = 0.8526$) is minimum when $L_{ff} = L_{\beta\beta} = 0.8526$ for $\alpha = \beta = 2.706$. Thus, the six-factor D(1,5) split-plot CCD with half replication of factorial part and complete replication of axial part and with $n_f = 32,\ n_\alpha = 32,\ n_\beta = 10,\ n_c = 16,$ and $\alpha = \beta = 2.706$ is a minimaxloss design robust to a pair of missing observations.

We can see from item *a* in FIGURE 2 that the curves L_{cc} and $L_{\alpha\alpha}$ overlap each other up to the point $(2.5000 < \alpha < 3.000)$, and that the two curves each makes a bell-shape when plotted against α. In items *b* and *d*, L_{cc} makes a bell-shaped curve when plotted against α, while $L_{\alpha\alpha}$ increases sharply as α increases. In item *c*, L_{cc} and $L_{\alpha\alpha}$ overlap each other and make a bell-shaped curve with the horizontal (α) axis. In item *e*, L_{cc} and $L_{\alpha\alpha}$ overlap each other and maintain a parallel position with the horizontal (α) axis.

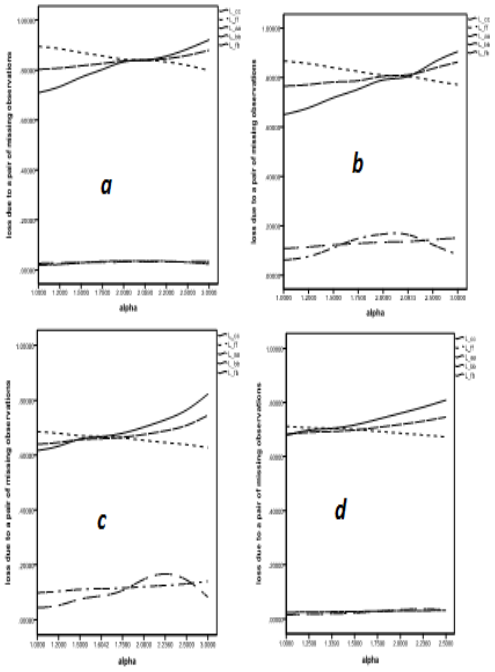


Figure 1: Loss curves due to a pair of missing observations for (a) [D(1,3)] (b) [D(2,2)] (c) [D(3,2)] and (d) [D(2,3)] with single replicate of factorial and axial points (F+A)

Table 6: Losses due to different pairs of missing observations for five-factor D(1,4) split-plot CCD with half replication of factorial and complete replication of axial parts; n_f = 16 from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 16, n_β = 8, n_c = 8, N_w = 6, and N = 48.

α	maxloss due to missing a pair of center points (L_{cc})	maxloss due to missing a pair of factorial points (L_{ff})	maxloss due to missing a pair of whole-plot axial points ($L_{\alpha\alpha}$)	maxloss due to missing a pair of subplot axial points ($L_{\beta\beta}$)	maxloss due to missing one factorial and one subplot axial points ($L_{f\beta}$)	Overall maxloss due to missing pairs of observations
1.0	0.0183	0.9994	0.0265	0.7383	0.9861	0.9994
1.20	0.0207	0.9991	0.0281	0.7611	0.9821	0.9991
1.50	0.0255	0.9984	0.0298	0.7958	0.9767	0.9984
2.0	0.0343	0.9966	0.0317	0.8496	0.9007	0.9966
2.5	0.0342	0.9945	0.0330	0.8944	0.9675	0.9945
3.0	0.0278	0.9931	0.0338	0.9269	0.9676	0.9931
3.5	0.0226	0.9921	0.0343	0.9490	0.9693	0.9921
4.0	0.0194	0.9915	0.0346	0.9639	0.9717	0.9915
5.995	0.0146	0.9893***	0.0352	0.9893***	0.9813	0.9893***

6.3	0.0143	0.9392	0.0352	0.9909	0.9824	0.9909

***Minimaxloss due to two missing observations.

Table 7**:** Losses due to different pairs of missing observations for five-factor D(2,3) split-plot CCD with half replication of factorial and complete replication of axial parts; n_f = 16 from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 16, n_β = 6, n_c = 4, N_w = 10, and N = 42.

α	maxloss due to missing a pair of center points (L_{cc})	maxloss due to missing a pair of factorial points (L_{ff})	maxloss due to missing a pair of whole-plot axial points ($L_{\alpha\alpha}$)	maxloss due to missing a pair of subplot axial points ($L_{\beta\beta}$)	maxloss due to missing one factorial and one subplot axial points ($L_{f\beta}$)
1.0	0.0627	0.9985	0.1094	0.7173	0.9853
1.20	0.0731	0.9974	0.1163	0.7381	0.9813
1.50	0.0975	0.9949	0.1251	0.7748	0.9761
2.0	0.1550	0.9886	0.1359	0.8314	0.9700
2.5	0.1541	0.9809	0.1444	0.8821	0.9678
3.0	0.1082	0.9739	0.1506	0.9196	0.9688
3.5	0.0786	0.9677	0.1545	0.9442	0.9710
4.043	0.0619	0.9617***	0.1573	0.9617***	0.9738
4.5	0.0541	0.9574	0.1590	0.9717	0.9762
5.0	0.0488	0.9533	0.1604	0.9793	0.9787

***Minimaxloss due to two missing observations.

Table 8: Losses due to different pairs of missing observations for six-factor D(2,4) split-plot CCD with half replication of factorial and complete replication of axial parts; n_f = 32 from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 32, n_β = 8, n_c = 8, N_w = 10, and N = 80.

α	maxloss due to missing a pair of center points (L_{cc})	maxloss due to missing a pair of factorial points (L_{ff})	maxloss due to missing a pair of whole-plot axial points ($L_{\alpha\alpha}$)	maxloss due to missing a pair of subplot axial points ($L_{\beta\beta}$)	maxloss due to missing one factorial and one subplot axial points ($L_{f\beta}$)
1.0	0.0145	0.8711	0.0235	0.7089	0.8070
1.20	0.0163	0.8681	0.0250	0.7226	0.8100
1.50	0.0201	0.8629	0.0270	0.7451	0.8153
2.0	0.0300	0.8532	0.0294	0.7851	0.8261
2.5	0.0356	0.8427	0.0310	0.8261	0.8396
2.665	0.0343	0.8394***	0.0315	0.8394***	0.8446
3.0	0.0287	0.8329	0.0322	0.8643	0.8550
3.5	0.0208	0.8240	0.0331	0.8951	0.8699
4.0	0.0159	0.8158	0.0336	0.9188	0.8835

***Minimaxloss due to two missing observations.

Table 9**:** Losses due to different pairs of missing observations for 6 -factor D(3,3) split-plot CCD with half replicationof factorial and complete replication of axial parts; n_f = 32 from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 24, n_β = 6, n_c = 4, N_w = 16, and N = 66.

α	maxloss due to missing a pair of center points (L_{cc})	maxloss due to missing a pair of factorial points (L_{ff})	maxloss due to missing a pair of whole-plot axial points ($L_{\alpha\alpha}$)	maxloss due to missing a pair of subplot axial points ($L_{\beta\beta}$)	maxloss due to missing one factorial and one subplot axial points ($L_{f\beta}$)
1.0	0.0477	0.8653	0.0988	0.6824	0.7951
1.50	0.0713	0.8558	0.1116	0.7210	0.8033
2.0	0.1247	0.8447	0.1223	0.7619	0.8137
2.5	0.1659	0.8328	0.1313	0.8054	0.8274
2.739	0.1470	0.8278***	0.1355	0.8278***	0.8359
3.0	0.1149	0.8230	0.1395	0.8504	0.8455
3.5	0.0705	0.8148	0.1455	0.8853	0.8624
4.0	0.0492	0.8074	0.1497	0.9115	0.8773

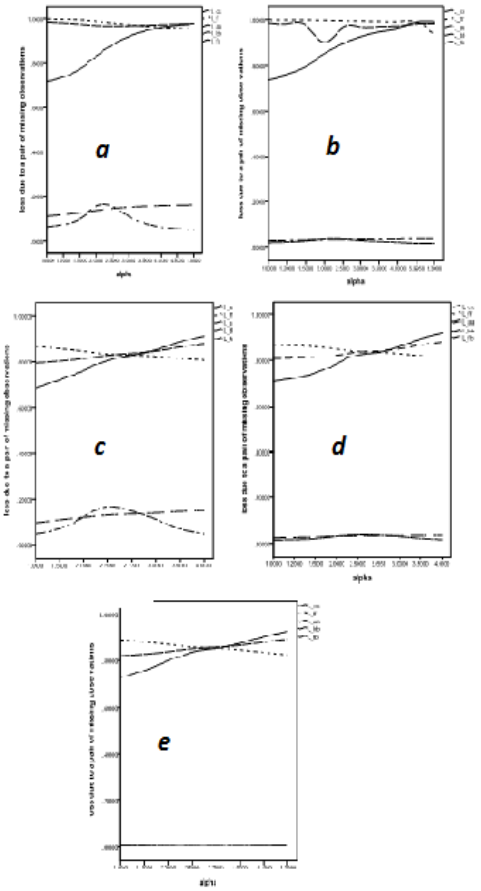
**Minimaxloss due to two missing observations.

Table 10**:** Losses due to different pairs of missing observations for six-factor D(1,5) split-plot CCD with halfreplicationof factorial and complete replication of axial parts; n_f = 32 from half replicate of factorial part with highest-order interaction as defining contrast, n_α = 32, n_β = 10, n_c = 16, N_w = 6, and N = 90.

α	maxloss due to missing a pair of center points (L_{cc})	maxloss due to missing a pair of factorial points (L_{ff})	maxloss due to missing a pair of whole-plot axial points ($L_{\alpha\alpha}$)	maxloss due to missing a pair of subplot axial points ($L_{\beta\beta}$)	maxloss due to missing one factorial and one subplot axial points ($L_{f\beta}$)
1.0	0.0043	0.8860	0.0062	0.7241	0.8218
1.50	0.0056	0.8779	0.0070	0.7592	0.8292
2.0	0.0074	0.8679	0.0074	0.7984	0.8390
2.5	0.0083	0.8571	0.0077	0.8374	0.8510
2.706	0.0080	0.8526***	0.0077	0.8526***	0.8564
3.0	0.0073	0.8463	0.0078	0.8724	0.8642
3.5	0.0060	0.8362	0.0080	0.9008	0.8771
4.0	0.0050	0.8269	0.0080	0.9230	0.8893

***Minimaxloss due to two missing observations.

Loss curves due to a pair of missing observations for split-plot CCDs with half replicate of factorial part and single replicate of axial part



Figures 2**:** Loss curves due to a pair of missing observations for (a) [D(1,4)] (b) [D(2,3)] (c) [D(2,4)] (d) [D(3,3)] and (e) [D(1,5)] with half replicate of factorial part and complete replicate of axial part (1/2F+A).

Conclusions: This study has established the robustness potentials of split-plot central composite designs when different pairs of observations were missing. The study has identified the most influential points of these designs to be the factorial (f) and the subplot axial (β) points while the less influential ones are the whole-plot axial and the center points. The study revealed a gradual decrease in the losses, L_{ff}, due to missing pairs of factorial points and an increase in the losses, $L_{\beta\beta}$, due to missing pairs of subplot axial points as the axial point distance of these designs increases from the design center. Thus design efficiency can be increased considerably by avoiding less influential points from the full split-plot CCD.

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