GLOBAL JOURNAL OF PURE AND APPLIED SCIENCES VOL 16, NO. 4 2010: 461- 467

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QUANTUM THEORY OF DAMPED HARMONIC OSCILLATOR

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(Received 7 June 2009; Revision Accepted 2, February 2010)

ABSTRACT

The exact solutions of the Schrödinger equation for damped harmonic oscillator with pulsating mass and modified Caldirola-Kanai Hamiltonian are evaluated. We also investigated the case of under-damped for the two models constructed and the results obtained in both cases do not violate Heisenberg uncertainty principle.

KEY WORDS: Damped oscillator, Harmonic oscillator with pulsating mass, modified Caldirola-Kanai Oscillator, uncertainty relation, wave function.

I INTRODUCTION

The aim of this paper is to evaluate the damped harmonic oscillator. The damping is here considered in the frame of Caldirola-Kanai model (Caldirola, 1941 and Kanai, 1948) and the recently developed model (Ikot et al, 2008). However, the problem of quantum oscillator with time-varying frequency had been solved (Um et al, 1987). The Hamiltonian of this model is usually quadratic in co-ordinates and momentum operators (Ikot et al, 2008).

The quantum calculation is applied because it will give the information about the particle at intermediate levels (discrete energy and wave function can be determined at a particular point in time) unlike the mechanic or statistical mechanic that will give a continuum result (say from -∞ to ∞ without telling us what happen to the particle in between the two boundaries). Our main goals will be to construct two Lagrangians for this simple damped system and use the constructed Lagrangians to evaluate the equation of motion for the damped Harmonic oscillators, and also evaluate the minimum uncertainty relation for under damped regime in each case of Lagrangian.

II. REVIEW OF CALDIROLA-KANAI OSCILLATOR

The Caldirola-Kanai oscillator with a variable mass 9 has Hamiltonian of the form (Caldirola, 1941, Kanai, 1948 and Kim et al, 2003).

$$\hat{H}_{ck} = \frac{1}{2me^{\gamma t/m}} p^2 + \frac{m\omega^2 e^{\gamma t/m}}{2} q^2 \qquad (1)$$

or in the form (Caldirola, 1941, Kanai, 1948, and Um and Yeon, 2002).

$$\hat{H}_{ck} = \frac{1}{2} m e^{2\gamma t}\omega^2(t)\hat{q}^2 + \frac{1}{2m} e^{-2\gamma t}\hat{p}^2 \qquad (2)$$

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where m is the mass of the oscillator, γ is the damping coefficient, $\hat{q}$ and $\hat{p}$ are the co-ordinate and momentum operators and w(t) is time-dependent frequency of the oscillator. The Lagrangian associated with equation (1) and equation (2) are given as:

$$L(q,\dot{q},t)=\frac{me^{\gamma t/m}}{2}\left[\dot{q}^2+\frac{\gamma}{m}\dot{q}+\omega^2(t)q\right] \tag{3}$$

$$L(q,\dot{q},t)=\frac{me^{\gamma t}}{2}\left[\dot{q}^2+2\gamma\dot{q}+\omega^2(t)q\right] \tag{4}$$

respectively. The equation of motion for the classical co-ordinate *q* and momentum *p* of Equation (3) and Equation (4) are of the forms

$$\ddot{q}(t)+\frac{\gamma}{m}\dot{q}+\omega^2(t)q=0, \tag{5}$$

$$\ddot{q}(t)+2\gamma\dot{q}+\omega^2(t)q=0, \tag{6}$$

The time-dependent Schrödinger Wave Equation (SWE) describing this system is given as:

$$H\psi_n(q,t)=E_n\psi_n(q,t). \tag{7}$$

The wave function and energy eigen value of equation (1) can be obtained as:

$$\psi_n(y,t)=\left(\frac{\left(m\omega/\hbar\right)^{1/2}}{\pi^{1/2}2^n n!}\right)^{\frac{1}{2}} H_n\left[e^{\frac{\gamma t}{m}}\left(\frac{m\omega}{\hbar}\right)^{1/2}q\right]$$

$$\times Exp\left[-i\omega t\left(n+\frac{1}{2}\right)e^{\frac{\gamma +}{m}}-\frac{1}{2}\left(\frac{m\omega}{\hbar}\right)e^{\frac{2\gamma t}{m}}q^2\right] \tag{8}$$

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right)e^{-\frac{\gamma t}{m}} \tag{9}$$

respectively.

And for equation (2), the corresponding wave function and energy eigen value is given as:

$$\psi_n(q,t)=\left(\frac{\left(m\omega/\hbar\right)^{1/2}}{\pi^{1/2}2^n n!}\right)Exp\left[-i\left(n+\frac{1}{2}\right)\omega te^{-2\gamma t}\right.$$

$$\left.-\frac{1}{2}\left(\frac{m\omega}{\hbar}\right)e^{2\gamma t}q^2\right]\times H_n\left[e^{\gamma t}\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}q\right] \tag{10}$$

and

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right)e^{2\gamma t} \tag{11}$$

respectively.

III. Harmonic Oscillator With Pulsating Mass An Dmodified Caldirola-Kanai Oscillator

In this section we are going to construct two Lagrangians and use in evaluating the equation of motion for the damped Harmonic Oscillators and the uncertainty relation for the two models will be obtained.

A. HARMONIC OSCILLATOR WITH PULSATING MASS

We write the time-dependent Hamiltonian with time varying frequency as:

$$\hat{H} = Cosh^{-2}\left(\frac{\gamma t}{2}\right)\frac{\hat{p}^2}{2m_0} + \left(\frac{1}{2}m_0 Cosh^2\left(\frac{\gamma t}{2}\right)\right)\omega^2(t)q^2 \quad (12)$$

using equations (7) and (12) we have the wave function of this oscillator as

$$\psi_n = \left(\frac{\left(\frac{m_0\omega}{\hbar}\right)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}2^n n!}\right)^{\frac{1}{2}} H_n\left[\left(\frac{m_0\omega}{\hbar}\right)^{\frac{1}{2}} q\cosh^2\frac{\gamma t}{2}\right]$$

$$\exp\left[-i\cosh^{-2}\frac{\gamma t}{2}\left(n+\frac{1}{2}\right)\omega t - \frac{m_0\omega}{\hbar}q\cosh^4\frac{\gamma t}{2}\right] \quad (13)$$

The lagrangian of equation (12) becomes:

$$L = \frac{1}{2}m_0 Cosh^2\left(\frac{\gamma t}{2}\right)\left[\dot{q}^2 - \omega^2(t)q^2\right] \quad (14)$$

and its equation of motion for the classical co-ordinate *q* and momentum *p* takes the form

$$\ddot{q}(t) + \left[\gamma\tanh\left(\frac{\gamma t}{2}\right)\right]\dot{q}(t) + \omega^2(t)q = 0 \quad (15)$$

The solution of Equation (12) is

$$q(t) = e^{-\left(\frac{\gamma t}{2}\tanh\frac{\gamma t}{2}\right)}\left[Ae^{i\Omega t} + Be^{-i\Omega t}\right], \quad (16)$$

where $\Omega(t) = \sqrt{\omega^2(t) - \frac{\gamma^2}{4}\tanh^2\frac{\gamma t}{2}}$

We summarized the general solution of Equation (15) for the over damped (OD), critically damped (CD) and under-damped (UD) as:

$$q(t) = e^{-\left(\frac{\gamma t}{2}\tanh\frac{\gamma t}{2}\right)}\left[ACosh\Omega t + BSinh\Omega t\right], \quad (17)$$

$$q(t) = e^{-\left(\frac{\gamma t}{2}\tanh\frac{\gamma t}{2}\right)}\left[A + Bt\right], \quad (18)$$

and $$q(t) = e^{-\left(\frac{\gamma t}{2}\tanh\frac{\gamma t}{2}\right)}\left[ACosh\Omega t + BSinh\Omega t\right], \quad (19)$$

respectively.

B. Modified Caldirola-Kanai Oscillator

We write the modified Caldirola-Kanai mode as:

$$\hat{H} = \frac{e^{-(Sin\gamma t + \gamma t)}}{2m_0}\hat{p}^2 + \frac{1}{2}e^{(Sin\gamma t + \gamma t)}m_0\omega^2(t)\hat{q}^2 \quad (20)$$

and the Lagrangian of this modified oscillator is:

$$L(q, \dot{q}, t) = e^{(Sin\gamma t + \gamma t)}\frac{m_0}{2}\left(\dot{q}^2 - \omega^2(t)q^2\right) \quad (21)$$

and its equation of motion takes the form:

$$q(t) + (\gamma Cos\gamma t + \gamma)q(t) + \omega^2(t)q = 0 \tag{22}$$

The solution of equation (22) is

$$q(t) = e^{\frac{-(\gamma Cos\gamma t + \gamma t)}{2}}\left[Ae^{i\Omega t} + Be^{-i\Omega t}\right], \tag{23}$$

where $\Omega(t) = \sqrt{\omega^2(t) - \left(\frac{\gamma Cos\gamma t + \gamma}{2}\right)^2}$

and approximating $Cos\gamma t \approx 1$ for small damping yields:

$$\Omega(t) = \sqrt{\omega^2(t) - \gamma^2} \tag{24}$$

Substituting equation (22) into equation (21) results:

$$q(t) = e^{-\left(\frac{\gamma Cos\gamma t + \gamma t}{2}\right)}\left[Ae^{i\omega t\sqrt{1-\frac{\gamma^2}{\omega^2}}} + Be^{-i\omega t\sqrt{1-\frac{\gamma^2}{\omega^2}}}\right]. \tag{25}$$

substituting equation (20) into equation (7) we have the wave function as

$$\psi_n = \left(\frac{\left(\frac{m_0\omega}{\hbar}\right)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}2^n n!}\right)^{\frac{1}{2}} H_n\left[\left(\frac{m_0\omega}{\hbar}\right)^{\frac{1}{2}} qe^{\sin\gamma t + \gamma t}\right]$$

$$\times\exp\left[-\frac{1}{2}\left(\frac{m_0\omega}{\hbar}\right)e^{2(\sin\gamma t + \gamma t)}q^2 - i\left(n + \frac{1}{2}\right)e^{-(\sin\gamma t + \gamma t)}\omega t\right] \tag{26}$$

We can now summarized the general solution of Equation (22) for the over damped (OD), critically damped (CD) and under-damped (UD) as

$$q(t) = e^{-\frac{\gamma t}{2}(1+Cos\gamma t)}\left[ACosh\Omega t + BSinh\Omega t\right] \tag{27}$$

$$q(t) = e^{-\frac{\gamma t}{2}(1+Cos\gamma t)}\left[A + Bt\right] \tag{28}$$

and

$$q(t) = e^{-\frac{\gamma t}{2}(1+Cos\gamma t)}\left[ACos\Omega t + BSin\Omega t\right] \tag{29}$$ respectively.

IV. Investigation Of The Under-Damped (Ud) Oscillator For The Two Models Constructed

A. The Under Damped Oscillation Of Harmonic Oscillator With Pulsating Mass.

We consider the quantum damped oscillator with time-dependent varying frequency given by equation (16). Subjecting Equation (16) to continuity condition (Man'ko, unpublished), *q(0)=1 and* $\dot{q}(0) = i\Omega,$ we obtain the arbitrary constant A and B as:

$$\left.\begin{aligned} A &= 1 \\ B &= i \end{aligned}\right\} \tag{30}$$

and the classical trajectory becomes:

$$q(t) = e^{\frac{-\gamma t}{2}\tanh\frac{\gamma t}{2}}\left[Cos\Omega t + iSin\Omega t\right], \tag{31}$$

where

$$\Omega^2 = \omega^2(t) - \frac{\gamma^2}{4}\tanh^2\frac{\gamma t}{2}$$

An invariant operator for the general time-dependent oscillator whose eigen function is an exact quantum state up to a time-dependent phase factor had been introduced by Lewis and Risendeld (Lewis and Riesendeld, 1969).

Following Lewis and Risendeld, we can now introduce a pair of operators first order in both position and momentum operators (Kim et al , 2003 and Kim and Page, 2001).

$$\hat{a}(t) = i\left[\varepsilon^*(t)\hat{p} - \dot{\varepsilon}^*(t)\hat{q}\right] \qquad (32)$$

$$\hat{a}^+(t) = i\left[\varepsilon(t)\hat{p} - \dot{\varepsilon}(t)\hat{q}\right]$$

And they are require to satisfy the quantum Liouville-Von Neumann equation defined as:

$$i\hbar\frac{\partial}{\partial t}\hat{a}(t) + \left[\hat{a}(t), \hat{H}(t)\right] = 0$$

$$i\hbar\frac{\partial}{\partial t}\hat{a}^+(t) + \left[\hat{a}^+(t), \hat{H}(t)\right] = 0 \qquad (33)$$

where $\varepsilon(t)$ in equation (32) must satisfies the classical damped equation of equation (15).

The operator in equation (32) and its Hermitian conjugate satisfy at any time t the boson commutation relation, and $\varepsilon(t)$ must also satisfy the Wronskian condition (Kim, 2004)

$$e^{\frac{d}{d\beta}\ln Cosh^2\beta\frac{\gamma t}{2}}\left[\varepsilon^*(t)\varepsilon(t) - \dot{\varepsilon}(t)\varepsilon^*\hat{q}\right] = i \qquad (34)$$

The number operator defined by (Ikot et al, 2008)

$$\hat{N}(t) = \hat{a}^+(t)\hat{a}(t) \qquad (35)$$

also satisfies equation (28), such that each number state

$$\hat{N}(t)|n,t\rangle = n|n,t\rangle \qquad (36)$$

is also an exact quantum state of the time dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}(t)\psi(x,t) \qquad (37)$$

The quantum dispersion coordination is obtained as (Ikot et al, unpublished).

$$\langle\hat{q}^2\rangle = \frac{\hbar}{2m_0\Omega}\varepsilon^*(t)\varepsilon(t) = \frac{\hbar}{2m_0\Omega}e^{-\gamma t\tanh\frac{\gamma t}{2}} \qquad (38)$$

And the uncertainty in momentum is given as

$$\langle\hat{p}^2\rangle = \frac{\hbar}{2m_0\Omega}m'(t)^2\dot{\varepsilon}^*(t)\dot{\varepsilon}(t) = \frac{\hbar}{2}m_0\Omega e^{\gamma t\tanh\frac{\gamma t}{2}}$$

$$\text{x}\left[1 + \frac{1}{\Omega^2}\left(-\frac{\gamma}{2}\tanh\frac{\gamma t}{2} - \frac{\gamma^2 t}{2}Sech^2\frac{\gamma t}{2}\right)^2\right] \qquad (39)$$

where $m'(t)$ is the reduced mass of the oscillator, which is defined as (Ikot et al, unpublished).

$$m'(t) = m_0 e^{\frac{d}{d\beta}\ln Cosh^2\beta\frac{\gamma t}{2}} \qquad (40)$$

and β is a variable parameter that takes values 1,2,3,…;n. By setting $\beta = 1$ equation (39) is obtained.

The generalized uncertainty relation has the value:

$$(\Delta q \Delta p)^2 = \frac{\hbar^2}{4}\left[1 + \left(\frac{\gamma}{2\Omega}\right)^2 \left(\tanh\frac{\gamma t}{2} + \frac{\gamma t}{2} Sech^2 \frac{\gamma t}{2}\right)^2\right] \qquad (41)$$

Equation (41) is a generalized uncertainty relation and it satisfies the Heisenberg uncertainty relation. The product of equations (38) and (39) gives a generalized Heisenberg relation which reduces to the exact when the damping coefficient γ is set to zero.

B. Under Damped Oscillation of Modified Caldirol-Kanai Oscillator

Considering the quantum damped oscillator equation of equation (30) and imposing the boundary conditions we obtain the arbitrary constants A and B as:

$$A = \left(1 - \frac{i\gamma}{2\Omega}\right) \text{ and } B = \frac{i\gamma}{2\Omega} \qquad (42)$$

Substituting equation (42) into equation (29) we obtain the classical trajectory of the form:

$$q(t) = e^{-\frac{\gamma t}{2}(1+Cos\gamma t)}\left[\left(1 - \frac{i\gamma}{2\Omega}\right)Cos\Omega t + \frac{i\gamma}{2\Omega} Sin\Omega t\right] \qquad (43)$$

The quantity $\varepsilon(t)$ must satisfy the Wronskian condition

$$e^{\frac{d}{d\beta}(Sin\beta\gamma t + \beta\gamma t)}\left[\dot{\varepsilon}^*(t)\varepsilon(t) - \dot{\varepsilon}(t)\varepsilon^*(t)\right] = i; \qquad (44)$$

By using the same procedure as in part A, we have the uncertainty in the co-ordinate as:

$$\langle \hat{q}^2 \rangle = \frac{\hbar}{2m_0\Omega} e^{-\gamma t(1+Cos\gamma t)}\left[1 + \frac{\gamma^2}{\Omega^2} Sin^2\Omega t + \frac{\gamma}{\Omega} Sin2\Omega t\right] \qquad (45)$$

and the dispersion of momentum by setting the variable parameter β to unity takes the form:

$$\langle \hat{p}^2 \rangle = \frac{\hbar}{2} m_0 e^{\gamma t(1+Cos\gamma t)}\left[1 + \frac{\gamma^2}{\Omega^2} Sin^2\Omega t + \left(\frac{\sigma(t)}{\Omega}\right)^2 \right.$$

$$\left(\frac{\gamma}{2\Omega} Cos2\Omega t - \frac{\gamma}{\Omega} Sin2\Omega t - \frac{1}{2}\left(\frac{\gamma}{\Omega}\right)^2 - 1\right)$$

$$\left. + \frac{\gamma}{\Omega^2}\sigma(t)\left(\frac{\gamma}{\Omega} Sin2\Omega t - 2Cos2\Omega t\right) + \frac{\gamma}{\Omega} Sin2\Omega t \right], \qquad (46)$$

where $\sigma(t) = \frac{\gamma}{2}\left[-\left(1 + Cos\gamma t\right) + \gamma t Sin\gamma t\right]$

and the reduced mass in this case is defined as:

$$m'(t) = m_0 e^{\frac{d}{d\beta}(Sin\beta\gamma t + \beta\gamma t)} = m_0 e^{\gamma t(1+\beta\gamma t)} \qquad (47)$$

The uncertainty relation has the value

$$(\Delta q \Delta p)^2 = \frac{\hbar^2}{4}\left[1 + g(t)\right], \qquad (48)$$

where

$$g(t) = \left(1 + \frac{\gamma^2}{\Omega^2} Sin^2\Omega t + \frac{\gamma}{\Omega} Sin2\Omega t\right)\left\{\ 1 + \frac{\gamma^2}{\Omega^2} Sin^2\Omega t\right.$$

$$+\left(\frac{\sigma(t)}{\Omega}\right)^2\left[\frac{1}{2}\left(\frac{\gamma}{\Omega}\right)^2 Cos2\Omega t-\frac{\gamma}{\Omega}Sin2\Omega t-\frac{1}{2}\left(\frac{\gamma}{\Omega}\right)^2-1\right]$$

$$\left.+\frac{\gamma\sigma(t)}{\Omega}\left[\frac{\gamma}{\Omega}Sin2\Omega t-2Cos2\Omega t\right]+\frac{\gamma}{\Omega}Sin2\Omega t\right\} \qquad (49)$$

Equation (48) obeys basic quantum principle and when the damping term is set to zero it will return to the exact solution of time independent harmonic oscillator which is already known.

V. CONCLUSION

We have evaluated within the frame of Caldirola-Kanai model the damped harmonic oscillator in the under-damped regime. Here, we obtained the uncertainty relation for the under damped oscillator for the two models constructed and the results obtained satisfy the basic quantum principle (Heisenberg uncertainty principle). In this work, we restricted ourselves to the investigation of under damped oscillation. However, there is need for further investigation of the oscillation of over damped and critically damped oscillators.

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