An efficient optimisation method in groundwater resource management

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Abstract

Uncertainty in input parameters to groundwater flow problems has been recognised as an impediment to designing efficient groundwater management strategies. The most popular approach to tackling this problem has been through the Monte Carlo approach. However, this approach is generally too expensive in terms of computer time because of the number of scenarios required to ensure reliable statistics. Furthermore, solutions obtained through this approach are not necessarily robust. In this paper, it is shown how groundwater management problems, where input parameters are uncertain can be reformulated as second-order cone optimisation (SOCO) problems, which are efficiently solved by recently developed interior-point methods. Results for a real-world case application of a groundwater aquifer found in Kenya are presented.

Keywords: Groundwater management, uncertainty, second-order cone, Laikipia, Kenya

Introduction

Theories of groundwater flow are well developed and over the last three decades, several applications have been made to model groundwater flow systems. A large number of these works have been based on deterministic flow simulation, and over the last decade a considerable volume of literature has been written and theories developed in the field of stochastic subsurface hydrology. In reality, many real-world aquifers are characterised by a few measurement points, which are used to derive the aquifer characteristics. Traditionally, the few measurement points available have been used for zonation purposes after which the management problem is solved deterministically. However, recognising the fact that the material forming aquifers varies enormously spatially, it is not immediately clear how optimal management strategies designed deterministically perform in an environment of uncertainty. Having noted that the parameters of the earth material which dictate the water-flow conditions vary a great deal spatially, one may ask the questions: Does it make sense to model groundwater systems deterministically using the sparse data available? How do solutions based on such an approach perform in a real-world scenario?

Geohydrologists, have for some time now, used the Monte Carlo approach in an attempt to desensitise the optimal solutions, thus including some robustness within the optimisation problem. However, the Monte Carlo approach is generally CPU-intensive (because of the large number of scenarios which have to be considered to arrive at a relatively insensitive optimal solution), hence its main drawback. Recently, some researchers have applied the multi-stage optimisation approach (in particular a two-stage optimisation approach) with some promising results (Ndambuki et al., 2000a; Wagner et al., 1992; Mulvey et al., 1995; Mark et al., 1999).

In this paper, we transform our stochastic groundwater quantity management problem into a second-order cone optimisation problem, which is then solved by some powerful interior-point method. We first present a general introduction to second-order cone optimisation (SOCO) problems followed by the formulation of a second-order cone groundwater quantity management problem. Subsequently, we present results for a real world case aquifer found in Laikipia District, Kenya.

Consider a linear optimisation problem (LOP) of the following form:

$$\text{minimize } c^T x \tag{1}$$

subject to:

$$a_i^T x \le b_i, \ i = 1,...,m \tag{2}$$

$$x \ge 0 \tag{3}$$

where:
$c, a_i \in R^n$; $b_i \in R$ are the problem parameters; while
x are the optimisation variables.

Assuming that all the problem parameters except a_i are accurately known and that a_i is uncertain but lying in ellipsoids e_i defined as:

$$a_i \in \varepsilon_i = \left\{ \overline{a_i} + \mathbf{P}_i u_i \mid \|u_i\| \le 1 \right\} \tag{4}$$

where:
$\mathbf{P} = \mathbf{P}^T$ are $n \times n$ perturbation matrices;
a_i overstrike are the nominal values and the norm of u_i ensure convexity.

Then a robust solution of the optimisation problem given by Eqs. (1) to (3) is as follows:

$$\text{minimize } c^T x \tag{5}$$

subject to:

$$a_i^T x \le b_i, \ \forall a_i \in \varepsilon_i, \ i = 1,...,m \tag{6}$$

$$x \ge 0 \tag{7}$$

Figure 1
Study area

Figure 2
Modelled area
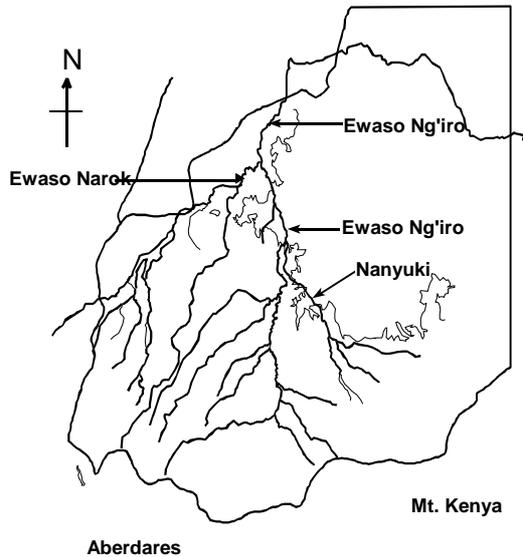
An optimisation problem defined by Eqs. (5) to (7), though now deterministic, has infinitely many constraints and a solution to this robust optimisation problem is feasible if for all $i=1,...m$, the following holds (see e.g. Ben-Tal and Nemirovsky, 1998):

$$\left[\overline{a}_i^T x + \left(P_i u_i\right)^T x - b_i\right] \leq 0, \ \forall u_i : u_i^T u_i \leq 1 \tag{8}$$

which can equivalently be reformulated by a single constraint as:

$$\overline{a}_i^T x + \|P_i x\| \leq b_i \tag{9}$$

Constraints of the form Eq. (9) are referred to as second-order cone constraints (otherwise known as Lorentz cone or ice-cream cone constraints). Thus the optimisation problem defined by Eqs. (1) to (3) can explicitly be written as a second-order cone optimisation problem as follows:

$$\text{minimize } c^T x \tag{10}$$

subject to:

$$\overline{a}_i^T x + \|P_i x\| \leq b_i, \ i = 1,...,m \tag{11}$$

The norm term is the usual Euclidean norm and can be thought of as a penalty term which introduces some robustness within the optimisation problem.

SOCO problems can be solved efficiently through the interior-point methods developed recently (Andersen and Andersen, 1999; Boyd et al., 1994; Sturm, 1999). A few applications of SOCO problems have been reported in the literature. They include antenna array weight design, filter design, grasping force optimisation, portfolio optimisation, truss design, and equilibrium of systems with piecewise-linear springs design. From the literature, the reported applications are basically in the areas of electrical engineering, mechanical engineering, economics, and structural engineering (Ndambuki et al., 2000a,b; Boyd et al., 1994; Lobo et al., 1998; Ben-Tal and Nemirovsky, 1998; Boyd et al., 1998). To the best of our knowledge, this is the first time this novel tool has been applied to a real-world case aquifer.

The study area

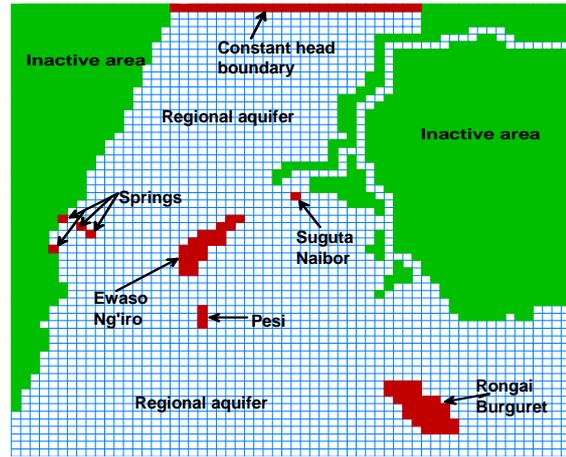The research was conducted within the Ewaso Ng'iro catchment situated in the central part of Kenya. This area extends from about 0°15′ south of the equator to 10°30′ north and between longitudes 36°15′ east and 39°45′ east. It is the domain formed by the area of Upper Ewaso Ng'iro Basin, upstream of the confluence of the Ewaso Ng'iro and Ewaso Narok Rivers and the area south-east of the confluence. This area, shown in Fig. 1 is wholly underlain by Precambrian basement rocks and is bounded to the west by the Laikipia Escarpment and to the south-east and south-west by the volcanic rocks of the Aberdares (3 999 m) and Mount Kenya (5 199 m). The rocks from the higher parts of the study area are mainly volcanic, while in the lower areas, metamorphic rocks of the basement system predominate.

Discretisation of the model

To delineate the area of interest (regional aquifer system) from the local aquifers, the whole study area was discretised into 60 x 60 cells each of 1 850 m x 1 850 m. The modelled area of interest is shown in Fig. 2.

Three main rivers, i.e. Ewaso Ng'iro, Ewaso Narok and Nanyuki, were also modelled within the area of interest (Fig. 1). Further, four swamps (Ewaso Narok, Pesi, Rongai Burguret and Suguta Naibor Swamps) and four perennial springs were included in the modelling exercise as well (Fig. 2).

The overall modelling effort was intended to manage the water resources (especially groundwater) of this area in such a manner so as to maximise benefits derived without substantially imparting undesirable consequences on the ecosystem.

Statement of the management problem

Following the exposition on SOCO problems already presented, the SOCO groundwater management problem was formulated as follows:

$$\text{mimimize } \sum_{j=1}^{N_w} \lambda_j r_j x_j \tag{12}$$

subject to:

$$\overline{a}_i^j x + \|P_i x\| \leq b_i; \ i = 1,...,N_c \tag{13}$$

$$\sum_{j \in Z_e} x_j \geq D_l; \ l = 1,...,3, \ Z_e = 1,...,N_w \tag{14}$$

$$\sum_{j \in Z_e} x_j \geq D_g \tag{15}$$
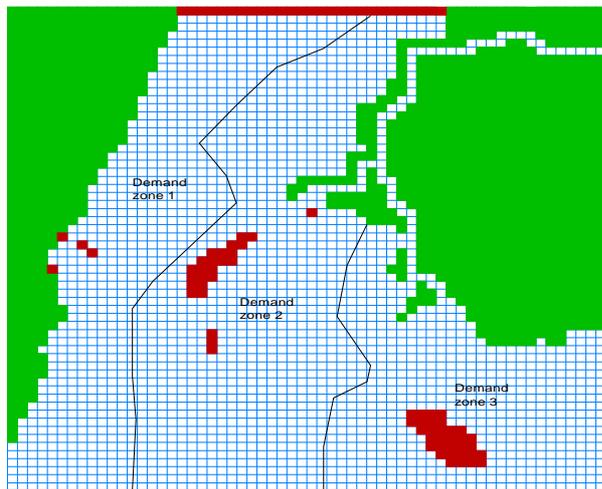
Figure 3
Local demand zones

Figure 4
Variation of optimal value with number of Monte Carlo realisations

$$\sum_{j \in Z_e} x_j \le R_t \tag{16}$$

$$x_j^{min} \le x_j \le x_j^{max} \tag{17}$$

where:

N_w is the number of pumping wells

N_c is the number of control points

Z_e is the number of control points

λ_j is daily operational cost in monetary unit (MU) per unit volume per unit lift in cell j

x_j is pumping rate in cell j

r_j is the pumping lift in cell j given by (H_j-h_j)

H_j is height of ground surface (measured from bottom of aquifer) at cell j

h_j is the head in pumping cell j

a_i_overbar j is the mean response at control point i due to pumping in cell j

b_i is the constraining value at control point i

R_l is recharge in demand zone l

x_j^{min} and x_j^{max} are the minimum and maximum allowable pumping rates;

D_g is total water demand

l refers to different water demand zones as shown in Fig. 3.

Solution methodology

To solve the above formulated robust single objective SOCO groundwater quantity problem, the following steps were followed:

- A number of realisations, w, of the uncertain parameter (in our case, the aquifer transmissivity field) were generated. Each realisation,w, has different spatial values of transmissivity. This means that every realisation,w, gives rise to different responses a_i at locations of interest and hence different values of mean a_i.s and $\mathbf{P}_i$.
- Using the groundwater simulation software, MODFLOW (McDonald and Harbaugh, 1984) and the generated transmissivity fields, the groundwater aquifer responses, a_i, at all points of interest due to unit pumping rates at all candidate well locations were determined.
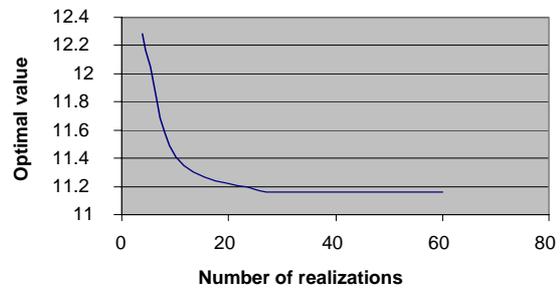- The nominal values of a_i and the perturbations, $\mathbf{P}_i$ were computed.

TABLE 1 Parameters used to generate transmissivity fields		
	Standard deviation (log10)	Correlation length/ field width
Case 1	0.5	0.5
Case 2	1.0	0.5
Case 3	1.5	0.5

- Using the single objective optimisation software, SeDuMi (Sturm, 1999), the single objective SOCO problem was solved.

Input data

The following data were used in this research:

Recharge 24.7x10³ m³·d⁻¹

Mean (harmonic) transmissivity 6.073 m²·d⁻¹

Current amount of water supplied 6.2 x 10³ m³·d⁻¹

Projected global (year 2000) water demand 11.1 x 10³ m³·d⁻¹

Projected Zone 1 water demand 2.07 x 10³ m³·d⁻¹

Projected Zone 2 water demand 5.13 x 10³ m³·d⁻¹

Projected Zone 3 water demand 3.86 x 10³ m³·d⁻¹.

Results

The grid size used is big, hence the results should be viewed within that context. To arrive at a reasonable number of Monte Carlo realisations to be used in the estimation of the uncertainty ellipsoids, we carried out an investigation to determine how the optimal objective value changes as the number of Monte Carlo realisations were increased. Figure 4 shows how the optimal objective value changes with the number,Ω, of Monte Carlo realisations considered. It shows that the optimal objective value changes widely for number of realisations ranging between 4 and 30. However, for more than 30 Monte Carlo realisations, the optimal value tends to stabilise. Thus, it was felt that consideration of 30 Monte Carlo realisations was a good compromise between CPU time and solution accuracy.

The results reported in this paper, therefore, were based on 30 realisations of transmissivity fields. These fields were generated using a mean (harmonic) transmissivity (log10) of 0.7834 and other parameters shown in Table 1. In total, three cases with the parameters shown in Table 1 were analysed.

As is evident from Table 2 and Fig. 4 increasing the value of the

TABLE 2 Optimal objective values for the cases shown in Table 1	
	Objective value (MU)
Case 1	1.23
Case 2	1.62
Case 3	3.52
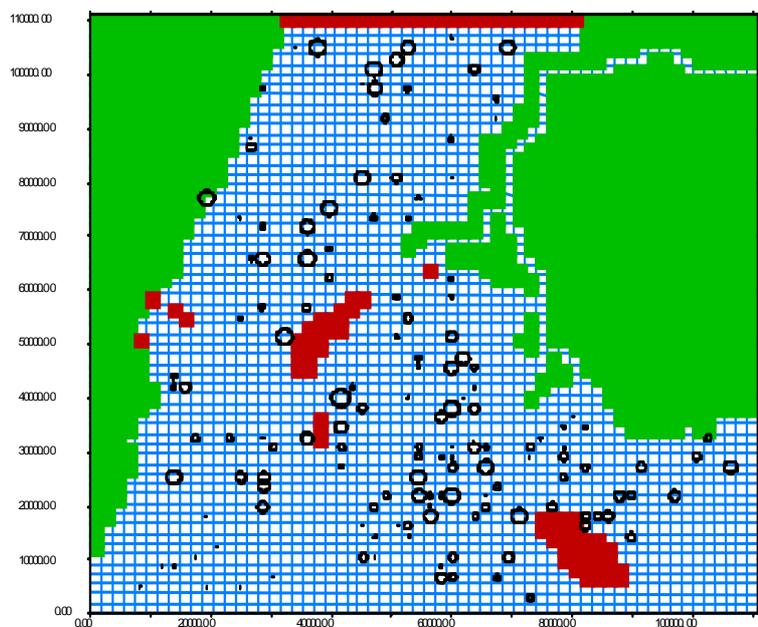
Figure 5
Un-optimised strategy
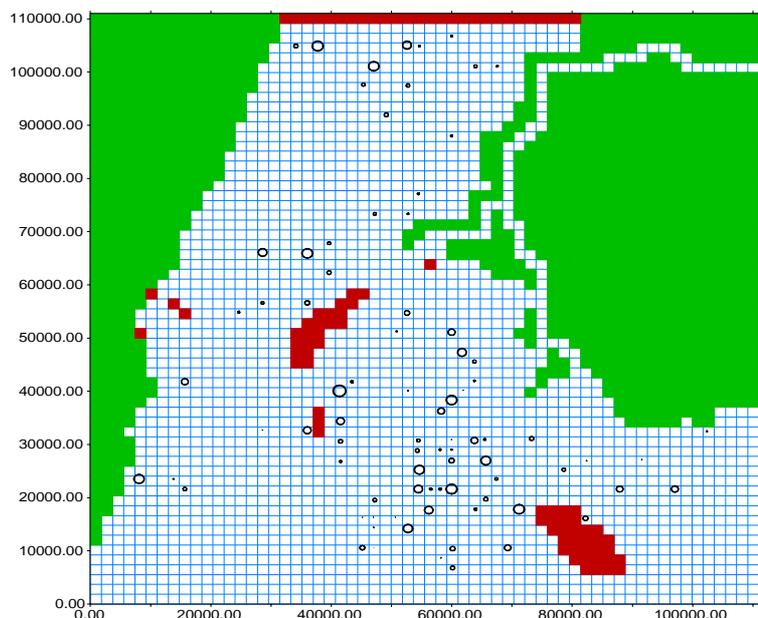
Figure 6
Optimised strategy corresponding to Case 1 of Table 1

standard deviation results in an increase in the cost of the optimal solution as expected. This is because increasing the value of standard deviation is equivalent to increasing the volume of the uncertainty ellipsoids. Thus an optimal solution sought when considering high magnitudes of uncertainty is likely to be more costly since such optimal solutions will have to be robust against higher levels of uncertainty.

Figure 5 shows the present un-optimised pumping strategy, able to supply a total water demand of $6.2 \times 10^3 \, m^3 \cdot d^{-1}$ (which is about 50% of the required supply), while Fig. 6 shows the realised optimal pumping solution. The realised optimal pumping strategy recommends the mobilisation of a total of 140 pumping wells stressing the aquifer at rates higher than the minimum permissible rates. Thus for the groundwater aquifer to satisfy the projected water demand of $11.1 \times 10^3 \, m^3 \cdot d^{-1}$, an additional 56 boreholes will have to be rejuvenated.

Comparison of Fig. 5 with Fig. 6 shows that the optimised strategy has phased out some of the active pumping wells within the present un-optimised strategy, an indication that such pumping wells were not optimally placed during the initial phases of aquifer exploitation.

Conlusions

Because it is virtually impossible to resolve the uncertainty in groundwater aquifer material properties (transmissivity) to such a detail, which would quantify the use of deterministic optimisation approaches, water managers need robust solutions which do not change with small changes in the input transmissivity values. In this paper, we have shown how groundwater quantity optimisation problems whose transmissivity values are uncertain can conveniently be reformulated as second-order cone optimisation problems, which are amenable to solution through interior-point methods. Moreover, we have proposed a solution methodology to such problems and its suitability is clearly shown through a real-world case optimisation problem.

An outstanding feature of this approach is that one is able to circumvent the shortfalls of the popular Monte Carlo approach through consideration of only a few realisations of the uncertain transmissivity field in order to estimate the sizes and shapes of the uncertainty ellipsoids. This leads to significant saving in terms of CPU time, hence allowing optimisation problems of high dimensions to be solved within a relatively shorter time. An extra added advantage (though not discussed in this paper) is the versatility of the approach. This is borne out by the ease with which one can increase or decrease the robustness of the optimal solution by either scaling up or scaling down the uncertainty ellipsoids at very little extra effort in terms of CPU time.

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