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Résumé: Les jeux irréversibles sont des jeux stochastiques où une fois un état est quitté, il n'est plus jamais revisité. Cette classe contient les jeux absorbants. Cet article démontre l'existence et une caractérisation de la valeur asymptotique pour tout jeu irréversible fini à information incomplète des deux côtés. Cela généralise Mertens et Zamir 1971 pour les jeux répétés à information incomplète des deux côtés et Rosenberg 2000 pour les jeux absorbants à information incomplète d'un côté.

Abstract: Irreversible games are stochastic games in which once the play leaves a state it never revisits that state. This class includes absorbing games. This paper proves the existence and a characterization of the asymptotic value for any finite irreversible game with incomplete information on both sides. This result extends Mertens and Zamir 1971 for repeated games with incomplete information on both sides, and Rosenberg 2000 for absorbing games with incomplete information on one side.

Mots clés : Jeux stochastiques, jeux répétés, information incomplète, valeur asymptotique, principe de comparaison, inégalités variationelles.

Key Words : Stochastic games, repeated games, incomplete information, asymptotic value, comparison principle, variational inequalities.

Classification JEL: C73, D82.

Classification AMS: 91A15, 91A20, 93C41, 49J40, 58E35, 45B40, 35B51.

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Irreversible Games with Incomplete Information: The Asymptotic Value

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Abstract

Irreversible games are stochastic games in which once the play leaves a state it never revisits that state. This class includes absorbing games [4]. This paper proves the existence and a characterization of the asymptotic value for any finite irreversible game with incomplete information on both sides. This result extends Mertens and Zamir [10] for repeated games with incomplete information on both sides, and Rosenberg [13] for absorbing games with incomplete information on one side.

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1 Introduction

A stochastic game [14] is a repeated game where the current state is observed by the players and the state changes from stage to stage according to a transition probability distribution depending on the current state and on the moves of the players. We consider two player zero sum games played by players I (the maximizer) and J (the minimizer). The evolution of the game is specified by a state space Ω, move sets I and J, and a transition probability function Q from $I \times J \times \Omega$ to Ω. The stage payoff function g is defined from $I \times J \times K \times L \times \Omega$ to $\mathbf{R}$. All sets under consideration are finite.

The stochastic game is *absorbing* if only one state $\omega_0 \in \Omega$ is non-absorbing, that is $Q(i, j, \omega)(\omega) = 1$ for all states $\omega \neq \omega_0$ and all i and j. More generally, a stochastic game is *irreversible* if once the play leaves a state, it never re-visits it in the future. Formally, for every sequence $(i_t, j_t, \omega_t)_{t=1,...,T-1}$, if $\omega_1 \neq \omega_0$ and $\omega_T = \omega_0$ then $\prod_{t=0}^{T-1} q(i_t, j_t, \omega_t)(\omega_{t+1}) = 0$.

A stochastic game with incomplete information is played in discrete time as follows. At stage 0, nature chooses $k \in K$ according to some probability[1] distribution $p \in \Delta(K)$ and chooses $l \in L$ according to some probability distribution $q \in \Delta(L)$. Player I privately learns his type k, player J learns l. An initial state ω_1 is given and known to the players. Inductively, at stage $t = 1, 2, ...$, knowing the past history of moves and states $h_t = (\omega_1, i_1, j_1, ..., i_{t-1}, j_{t-1}, \omega_t)$, simultaneously, player I chooses $i_t \in I$ and player J chooses $j_t \in J$. The new state $\omega_{t+1} \in \Omega$ is drawn according

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[1]For a finite set F, $\Delta(F)$ is the set of probability distribution on F.

to the probability distribution $Q(i_t, j_t, \omega_t)(\cdot)$. The triplet (i_t, j_t, ω_{t+1}) is publicly announced and the situation is repeated at stage $t+1$. The stage payoff $r(t) = g(i_t, j_t, k, l, \omega_t)$ is not known to the players before the end of the game. This description is public knowledge.

Let $H_t = \Omega \times (I \times J \times \Omega)^{t-1}$ be the set of public histories up to stage t and $H = \cup_t H_t$. A behavioral strategy σ for player I is a mapping from $K \times H$ to $\Delta(I)$ and a behavioral strategy τ for player J is a mapping from $L \times H$ to $\Delta(J)$.

We study the discounted game $\Gamma_\lambda(p, q, \omega_1)$ where the payoff is $\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} r(t)$. It is well known [17] that this game has a value $v_\lambda(p, q, \omega_1)$. The objective of this paper is to study the problems of existence and characterization of the asymptotic value $v(p, q, \omega_1) = \lim_{\lambda \to 0} v_\lambda(p, q, \omega_1)$.

In the deterministic case (Ω is a singleton), the game is reduced to a repeated game with incomplete information à la Aumann-Maschler [1]. When information is incomplete on one side (L is a singleton), Aumann and Maschler proved the existence of the asymptotic value and provided an explicit characterization, the famous $v(p) = Cav_{\Delta(K)}(u)(p)$ formula[2]. Mertens and Zamir [10] extended this result to repeated games with incomplete information on both sides and provided an elegant system of functional equations that characterizes the asymptotic value, the famous $v(p, q) = Cav_{\Delta(K)} \min(u, v)(p, q) = Vex_{\Delta(L)} \max(u, v)(p, q)$ system[3].

In the complete information case (K and L are singletons), the game is reduced to a stochastic game à la Shapley. Bewley and Kohlberg [3] proved the existence of the asymptotic value using semi-algebraic tools.

Few results are known when the repeated game is a stochastic game with incomplete information. Sorin [15] and [16] was the first to prove the existence of the asymptotic value v for big match games with incomplete information on one side, and provided an explicit characterization for the limit (see the next section for some examples). An operator approach allows Rosenberg [13] to prove the existence of asymptotic value v for absorbing games with incomplete information on one side (without characterization). Rosenberg and Vieille [12] proved the existence of asymptotic value v for recursive games[4] with incomplete information on one side[5] (without characterization).

In the present paper, we follow the variational approach developed in Laraki [6], [7] and [9], and show that the asymptotic value v exists for any irreversible game with incomplete information on both sides *and* is uniquely characterized as the solution of a system of variational inequalities that extends Mertens-Zamir's [10] system. In fact, the variational inequalities are necessarily satisfied by any accumulation point of v_λ in any stochastic game with incomplete information.

The sketch of the proof in the variational approach as well as in the operator approach developed in [11] and [13] is similar to the one used in differential games with fixed duration to prove the existence and the characterization of the value [2]. One starts by establishing a dynamic programming principle for some discretization w_n of the differential game, shows that the $\{w_n\}_{n \in \mathbf{N}}$ are equi-continuous, and proves that any accumulation point w of w_n satisfies two variational inequalities (i.e. w is an upper and a lower viscosity uniformly continuous solution of some Hamilton-Jacobi-Belman equation with boundary). A maximum principle then shows that there is at most one uniformly continuous viscosity solution of the HJB equation.

The main difficulty in this paper is the identification of the "right" variational inequalities in the class of absorbing games with incomplete information. Once the result is established for this class, it is easily extended, by induction, to all irreversible games with incomplete information.

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[2]Where $u(p)$ is the value of the non revealing game and $Cav_{\Delta(K)}(u)(\cdot)$ is the smallest concave function $g(\cdot)$ on $\Delta(K)$ greater than $u(\cdot)$.

[3]Where $u(p, q)$ is the value of the non revealing game and, for each p, $Vex_{\Delta(L)}(u)(\cdot, q)$ is the greatest convex function f on $\Delta(L)$ smaller than $u(\cdot, p)$ (with respect to the q variable, for each fixed p).

[4]A stochastic game is *recursive* if the stage payoff is 0 in all non-absorbing states.

[5]Sorin [15] and [16] and Rosenberg-Vieille [12] proved more. They show the existence of the uniform maxmin and minmax and prove that they may differ. Rosenberg and Vieille observed, in their conclusion, that their proof extends to recursive games with incomplete information on both sides.

2 Examples

In the following examples, the game is of incomplete information on one side with two possible types for player I (the row player, the maximizer) and two actions for each player.

The first is a classical example of a repeated game taken from Aumann and Maschler [1] (chapter 1, section 2).

	L	R
T	1	0
B	0	0

p

and

	L	R
T	0	0
B	0	1

$\widehat{p}$

Each matrix represents a type of player I (the row player). Nature selects the left matrix with probability p and the right matrix with probability $\widehat{p} = 1 - p$. Player I (but not player J) is informed. The players then play repeatedly the matrix game that has been selected. The non-revealing game (in which the players do not use their information or equivalently do not observe their types) is the matrix game:

	L	R
T	p	0
B	0	$\widehat{p}$

Its value is $u(p) = p\widehat{p}$. Aumann and Maschler's general result implies that $v(p) = Cav(u)(p) = p\widehat{p}$. Since in general u is an algebraic function, so is the asymptotic value $v = Cav(u)$.

Sorin [16] slightly changes this game to obtain the following absorbing game with incomplete information on one side:

	L	R
T	1^*	0
B	0^*	0

p

and

	L	R
T	0^*	0
B	0^*	1

$\widehat{p}$

Here, entries/payoffs with a $*$ are absorbing: they lead to an absorbing state where the payoff is the one that appears in the entry, regardless of the moves of the players in subsequent stages. Hence if the payoff is given by the left matrix and at some stage player J plays L and player I plays T, the game stops and the stage payoffs from that stage on are 1 for player I and -1 for player J. Such an absorbing game, in which one player controls the transition, is called a *big match game*. If the informed player controls the transition, the game is called a big match game of type 1, whereas if the uninformed player controls the transition, it is called a big match game of type 2. The non-revealing game (in which the informed player does not use his information) is the following absorbing game with complete information:

	L	R
T	p^*	0
B	0^*	$\widehat{p}$

It can be shown that its asymptotic value is again $u(p) = p\widehat{p}$ (the proof is not so trivial but not difficult either). Sorin proved that for every big match of type 2, $v(p) = Cav(u)(p) = p\widehat{p}$.

This result is surprising when compared with the last example of Sorin [15]. Change now the structure of absorptions to obtain a big match of type 1:

	L	R
T	1^*	0^*
B	0	0

p

and

	L	R
T	0^*	0^*
B	0	1

$\widehat{p}$

The asymptotic value of the non-revealing game is again $u(p) = p\widehat{p}$ but the asymptotic value $v(p) = \widehat{p}(1 - exp(-\frac{p}{\widehat{p}}))$ which is not an algebraic function of p.

Sorin provides an explicit characterization of $v(p)$ in all big match games of type 1 as the value of an auxiliary game in continuous time. However, it is not clear how to extend his characterization to all absorbing games (or to all irreversible games), it is not known how this characterization is related to the Mertens-Zamir system or how the asymptotic value $v(\cdot)$ may be deduced directly from the value of the non-revealing game $u(\cdot)$. This paper answers the last questions for irreversible games with incomplete information on both sides.

3 The Shapley operator

Let $\mathcal{F}$ denote the set of real valued functions f on $\Delta(K)\times\Delta(L)\times\Omega$ bounded by $C = \max_{i,j,k,l,\omega} |g(i,j,k,l,\omega)|$, concave in p, convex in q, and Lipschitz on (p,q) with constant $2C$ for the L_1 norm. That is, for every $(p_1, q_1, p_2, q_2, \omega)$:

$$|f(p_1, q_1, \omega) - f(p_1, q_2, \omega)| \leq 2C\|p_1 - p_2\|_1 + 2C\|q_1 - q_2\|_1,$$

where $\|p_1 - p_2\|_1 = \sum_{k\in K} |p_1^k - p_2^k|$ and $\|q_1 - q_2\|_1 = \sum_{l\in L} |q_1^l - q_2^l|$.

The Shapley operator [17] $T(\lambda, \cdot)$ associates to a function f in $\mathcal{F}$ the function:

$$\begin{aligned} T(\lambda, f)(p,q,\omega) &= \max_{x\in\Delta(I)^K} \min_{y\in\Delta(J)^L} \left[\begin{array}{c} \lambda g(x,y,p,q,\omega) \\ +(1-\lambda)\sum_{i,j,\tilde{\omega}} \overline{x}(i)\overline{y}(j)Q(i,j,\omega)(\tilde{\omega})f(p(i),q(j),\tilde{\omega}) \end{array} \right] \\ &= \min_{y\in\Delta(J)^L} \max_{x\in\Delta(I)^K} \left[\begin{array}{c} \lambda g(x,y,p,q,\omega) \\ +(1-\lambda)\sum_{i,j,\tilde{\omega}} \overline{x}(i)\overline{y}(j)Q(i,j,\omega)(\tilde{\omega})f(p(i),q(j),\tilde{\omega}) \end{array} \right], \end{aligned}$$

where $g(x,y,p,q,\omega) = \sum_{i,j,k,l} p^k q^l x^k(i) y^l(j) g(k,l,i,j,\omega)$ is the expected stage payoff, $\overline{x}(i) = \sum_{k\in K} p^k x^k(i)$ is the total probability that I plays i, $\overline{y}(j) = \sum_{l\in L} q^l y^l(j)$ is the total probability that J plays j, $p^k(i) = \frac{p^k x^k(i)}{\overline{x}(i)}$ is the conditional probability that the type of player I is k given the move i and $q^l(j) = \frac{q^l y^l(j)}{\overline{y}(j)}$ is the conditional probability that the type of player J is l given the move j.

Lemma 1 *The Shapley operator $T(\lambda, \cdot)$ is defined from $\mathcal{F}$ to itself. Its unique fixed point is v_λ.*

Proof. It is well known that $T(\lambda, f)$ is concave in p and convex in l, bounded by C and that v_λ is its unique fixed point (see [17] and [13]).

The fact that the image of a $2C$-Lipschitz function is also $2C$-Lipschitz was an open question in Rosenberg [13]. To prove it, recall that the famous splitting procedure (see for example [17] proposition 2.3) says that for each $\alpha \in \Delta(I)$ and each $(\pi(i))_{i\in I} \in \Delta(K)^I$ if $\sum_{i\in I} \alpha(i)\pi(i) = p \in \Delta(K)$ then there exists $x \in \Delta(I)^K$ such that $\alpha(i) = \overline{x}(i) = \sum_{k\in K} p^k x^k(i)$ and $\pi^k(i) = p^k(i) = \frac{p^k x^k(i)}{\overline{x}(i)}$. The inverse is, of course, always true. Similarly, for each $\beta \in \Delta(J)$ and each $(\rho(j))_{j\in J} \in \Delta(L)^J$ if $\sum_{j\in J} \beta(j)\rho(j) = q \in \Delta(L)$ then there exists $y \in \Delta(J)^L$ such that $\beta(j) = \overline{y}(j) = \sum_{l\in L} q^l y^l(j)$ and $\rho^l(j) = q^l(j) = \frac{q^l y^l(j)}{\overline{y}(j)}$.

Consequently, if $\Gamma(p) = \{(\alpha,\pi) \in \Delta(I) \times \Delta(K)^I : \sum_{i\in I} \alpha(i)\pi(i) = p\}$ and $\Lambda(q) = \{(\beta,\rho) \in \Delta(J) \times \Delta(L)^J : \sum_{j\in J} \beta(j)\rho(j) = q\}$ then

$$\begin{aligned} T(\lambda, f)(p,q,\omega) &= \max_{(\alpha,\pi)\in\Gamma(p)} \min_{(\beta,\rho)\in\Lambda(q)} \left(\begin{array}{c} \lambda \sum_{i,j} \alpha(i)\beta(j)g(i,j,\pi(i),\rho(j),\omega) \\ +(1-\lambda)\sum_{i,j,\tilde{\omega}} \alpha(i)\beta(j)Q(i,j,\omega)(\tilde{\omega})f(\pi(i),\rho(j),\tilde{\omega}) \end{array} \right) \\ &= \min_{(\beta,\rho)\in\Lambda(q)} \max_{(\alpha,\pi)\in\Gamma(p)} \left(\begin{array}{c} \lambda \sum_{i,j} \alpha(i)\beta(j)g(i,j,\pi(i),\rho(j),\omega) \\ +(1-\lambda)\sum_{i,j,\tilde{\omega}} \alpha(i)\beta(j)Q(i,j,\omega)(\tilde{\omega})f(\pi(i),\rho(j),\tilde{\omega}) \end{array} \right) \end{aligned}$$

Let $(\alpha_1,\pi_1)\in\Gamma(p_1)$ be optimal for the maximizer above at (p_1,q_1,ω) and let $(\beta_2,q_2)\in\Lambda(q_2)$ be optimal for the minimizer at (p_2,q_2,ω). From the optimal splitting of probabilities lemma 8.4 in Laraki [8], there exists $\pi_2\in\Delta(K)^I$ such that $(\alpha_1,\pi_2)\in\Gamma(p_2)$ and

$$\sum_{i\in I}\alpha_1(i)\|\pi_2(i)-\pi_1(i)\|_1=\|p_2-p_1\|_1.$$

Similarly there is $\rho_1\in\Delta(L)^J$ such that $(\beta_2,\rho_1)\in\Lambda(q_1)$ and

$$\sum_{j\in J}\beta_2(j)\|\rho_2(j)-\rho_1(j)\|_1=\|q_2-q_1\|_1.$$

Consequently:

$$\begin{array}{rcl} T(\lambda,f)(p_1,q_1,\omega) & = & \min\limits_{(\beta,\rho)\in\Lambda(q_1)}\left(\begin{array}{c}\lambda\sum_{i,j}\alpha_1(i)\beta(j)g(i,j,\pi_1(i),\rho(j),\omega)\\ +(1-\lambda)\sum_{i,j,\tilde\omega}\alpha_1(i)\beta(j)Q(i,j,\omega)(\tilde\omega)f(\pi_1(i),\rho(j),\tilde\omega)\end{array}\right)\\ & \leq & \begin{array}{c}\lambda\sum_{i,j}\alpha_1(i)\beta_2(j)g(i,j,\pi_1(i),\rho_1(j),\omega)\\ +(1-\lambda)\sum_{i,j,\tilde\omega}\alpha_1(i)\beta_2(j)Q(i,j,\omega)(\tilde\omega)f(\pi_1(i),\rho_1(j),\tilde\omega)\end{array}\end{array}$$

and

$$\begin{array}{rcl} T(\lambda,f)(p_2,q_2,\omega) & = & \max\limits_{(\alpha,\pi)\in\Gamma(p_2)}\left(\begin{array}{c}\lambda\sum_{i,j}\alpha(i)\beta_2(j)g(i,j,\pi(i),\rho_2(j),\omega)\\ +(1-\lambda)\sum_{i,j,\tilde\omega}\alpha(i)\beta_2(j)Q(i,j,\omega)(\tilde\omega)f(\pi(i),\rho_2(j),\tilde\omega)\end{array}\right)\\ & \geq & \begin{array}{c}\lambda\sum_{i,j}\alpha_1(i)\beta_2(j)g(i,j,\pi_2(i),\rho_2(j),\omega)\\ +(1-\lambda)\sum_{i,j,\tilde\omega}\alpha_1(i)\beta_2(j)Q(i,j,\omega)(\tilde\omega)f(\pi_2(i),\rho_2(j),\tilde\omega)\end{array}\end{array}$$

Consequently:

$$\begin{array}{cl} & T(\lambda,f)(p_1,q_1,\omega)-T(\lambda,f)(p_2,q_2,\omega)\\ \leq & \begin{array}{c}\lambda\sum_{i,j}\alpha_1(i)\beta_2(j)\left(g(i,j,\pi_1(i),\rho_1(j),\omega)-g(i,j,\pi_2(i),\rho_2(j),\omega)\right)\\ +(1-\lambda)\sum_{i,j,\tilde\omega}\alpha_1(i)\beta_2(j)Q(i,j,\omega)(\tilde\omega)\left(f(\pi_1(i),\rho_1(j),\tilde\omega)-f(\pi_2(i),\rho_2(j),\tilde\omega)\right)\end{array}\\ \leq & \begin{array}{c}\lambda 2C\sum_{i,j}\alpha_1(i)\beta_2(j)\left(\|\pi_2(i)-\pi_1(i)\|_1+\|\rho_2(j)-\rho_1(j)\|_1\right)\\ +(1-\lambda)2C\sum_{i,j,\tilde\omega}\alpha_1(i)\beta_2(j)Q(i,j,\omega)(\tilde\omega)\left(\|\pi_2(i)-\pi_1(i)\|_1+\|\rho_2(j)-\rho_1(j)\|_1\right)\end{array}\\ \leq & 2C\|p_1-p_2\|_1+2C\|q_1-q_2\|_1.\end{array}$$

■

4 The auxiliary absorbing game

From know on (unless specified), the stochastic game with incomplete information is supposed to be irreversible. We will prove, by induction on the cardinality of Ω that v_λ converges uniformly as λ goes to zero.

An irreversible game is of cardinality M at an initial state ω if the number of stages (including ω) that may be visited with a positive probability during a play is at most M. When the cardinality of Ω is 1, the result[6] follows from Mertens and Zamir [10].

[6] Laraki [7] gives a short proof and a new characterization of the Mertens-Zamir's result [10]. The link between the two characterizations is established in section 7. The proof in sections 5 and 6 establishes the uniform converges of the discounted values in all absorbing games with incomplete information, in which the payoff after absorption is fixed forever. Consequently, our paper does not really need to assume the Mertens-Zamir's result as given. This assumption is made only to simplify the induction argument.

Suppose that uniform convergence of the asymptotic values holds for all irreversible games of cardinality smaller than M and prove it for some fixed irreversible game with cardinality $M+1$. Let ω_0 denotes the initial state and that $\Omega = \{\omega_0, ..., \omega_M\}$. Because the game is irreversible, all states ω_m, $m = 1, ..., M$ are of cardinality at most M. By the induction hypothesis, $v_\lambda(\cdot,\cdot,\omega_m)$ converges uniformly to some function $v(\cdot,\cdot,\omega_m)$ for all $m = 1, ..., M$. So, we are reduced to prove the uniform convergence of $v_\lambda(\cdot,\cdot,\omega_0)$.

The family $(v_\lambda(\cdot,\cdot,\omega_0))_{0<\lambda<1}$ being equi-continuous (by lemma 1), Ascoli's theorem implies the existence of $\lambda_n \to 0$ such that $v_{\lambda_n}(\cdot,\cdot,\omega_0)$ uniformly converges to some function $w(\cdot,\cdot,\omega_0) \in \mathcal{F}$.

The rest of the paper establishes two variational inequalities that w should satisfy. Then a maximum principle allows to deduce that at most one function $w \in \mathcal{F}$ satisfies the two properties. Consequently, $v_\lambda(\cdot,\cdot,\omega_0)$ has exactly one possible accumulation point as $\lambda \to 0$, and so it converges uniformly.

Towards this aim, introduce an auxiliary finite zero-sum absorbing game with complete information (that depends on g, Q and $v(\cdot,\cdot,\omega_m)$, $m = 1, ..., M$). The auxiliary game is played in discrete time and starts at state ω_0. At stage $t = 1, 2, ...$, simultaneously, player I chooses $i_t \in I$ and player J chooses $j_t \in J$:

- the payoff at stage t is $g(i_t, j_t, p, q, \omega_0)$;
- with probability $Q^*(i_t, j_t, \omega_0) = 1 - Q(i_t, j_t, \omega_0)(\omega_0) = \sum_{m=1}^{M} Q(i_t, j_t, \omega_0)(\omega_m)$ the game is absorbed and the payoff in all future stages $s > t$ is

$$f(i_t, j_t, p, q, \omega_0) := \frac{1}{Q^*(i_t, j_t, \omega_0)} \sum_{m=1}^{M} Q(i_t, j_t, \omega_0)(\omega_m) v(p, q, \omega_m);$$

- with probability $Q(i_t, j_t, \omega_0) = Q(i_t, j_t, \omega_0)(\omega_0)$ the situation is repeated at step $t+1$.

From [4], this absorbing game admits an asymptotic value, denoted $u(p, q, \omega_0)$. The function u will play in our characterization a role as the value of the non-revealing game in Aumann-Maschler [1], Mertens-Zamir [10] and Sorin [15], [16].

Using a variational approach, Laraki [9] proved the existence of the asymptotic value for any absorbing game and provided an explicit formula for it, which we present now.

Denote by $M_+(I) = \{\alpha = (\alpha^i)_{i \in I} : \alpha^i \geq 0\}$ the set of non-negative measures on I. Observe that $\Delta(I) \subset M_+(I)$. For any $i \in I$ and $j \in J$, let $g^*(i, j, p, q, \omega_0) = Q^*(i, j, \omega_0) f(i, j, p, q, \omega_0)$. This is the absorbing payoff, not conditioned on absorption. For any $(\alpha, j) \in M_+(I) \times J$ and $\varphi : I \times J \to [-C, C]$, φ is extended linearly as follows $\varphi(\alpha, j) = \sum_{i \in I} \alpha^i \varphi(i, j)$.

Lemma 2 *The asymptotic value of the auxiliary absorbing game satisfies:*

$$u(p, q, \omega_0) = \sup_{x \in \Delta(I), \alpha \in M_+(I)} \min_{j \in J} \left(\begin{array}{c} \frac{g^*(x,j,p,q,\omega_0)}{Q^*(x,j,\omega_0)} \mathbf{1}_{\{Q^*(x,j,\omega_0)>0\}} \\ + \frac{g(x,j,p,q,\omega_0) + g^*(\alpha,j,p,q,\omega_0)}{Q(x,j,\omega_0) + Q^*(\alpha,j,\omega_0)} \mathbf{1}_{\{Q^*(x,j,\omega_0)=0\}} \end{array} \right),$$

and it is a $2C$-Lipschitz function.

Proof. In the formula as stated in theorem 3 in Laraki [9], it is required in the $\sup_{x \in \Delta(I), \alpha \in M_+(I)}$ above that for every $i \in I$, $x^i > 0 \Rightarrow \alpha^i = 0$. As Guillaume Vigeral noted[7], this is not necessary since the α_i is active in the formula above only when $x_i = 0$.

That u is $2C$-Lipschitz could easily be proved directly from the formula or by noting that u is the uniform limit of the $2C$-Lipschitz functions u_λ , where u_λ is the value of the λ-discounted

[7]A personal communication.

auxiliary absorbing game. That u_λ is $2C$-Lipschitz can be proved as in the lemma 1 (but it is not needed to use the lemma of optimal splitting of probabilities). ■

Implicitly, the auxiliary game defines an operator U that associates to each stochastic game with incomplete information and state space $\Omega = \{\omega_0, ..., \omega_M\}$, each family of $2C$-Lipschitz functions $v(p, q, \omega_m)$, $m = 0, ..., M$ on $\Delta(K) \times \Delta(L)$, and each state $\omega_m \in \Omega$ a $2C$-Lipschitz function $u(p, q, \omega_m) = U\left[w(\cdot, \cdot, \omega_k)_{k \neq m}\right](p, q, \omega_m)$ on $\Delta(K)\times\Delta(L)$. This operator will be used to extend the variational inequalities in the next section to any stochastic game with incomplete information.

5 The variational characterization

Since $v_{\lambda_n}(\cdot, \cdot, \omega_0)$, converges uniformly to $w(\cdot, \cdot, \omega_0)$ and since by the induction hypothesis $v_\lambda(\cdot, \cdot, \omega_m)$ converges uniformly to $v(\cdot, \cdot, \omega_m)$ for all $m = 1, ..., M$, taking λ converging to zero in the functional equation $v_\lambda = T(\lambda, v_\lambda)$ implies the following.

Lemma 3 *For any accumulation point $w(\cdot, \cdot, \omega_0)$ of $v_\lambda(\cdot, \cdot, \omega_0)$ and all p, q:*

$$\begin{aligned} w(p, q, \omega_0) &= \max_{x \in \Delta(I)^K} \min_{y \in \Delta(J)^L} \sum_{i,j} \overline{x}(i) \overline{y}(j) \left[\begin{array}{c} Q(i, j, \omega_0)(\omega_0) w(p(i), q(j), \omega_0) \\ + \sum_{m=1}^M Q(i, j, \omega_0)(\omega_m) v(p(i), q(j), \omega_m) \end{array} \right] \\ &= \min_{y \in \Delta(J)^L} \max_{x \in \Delta(I)^K} \sum_{i,j} \overline{x}(i) \overline{y}(j) \left[\begin{array}{c} Q(i, j, \omega_0)(\omega_0) w(p(i), q(j), \omega_0) \\ + \sum_{m=1}^M Q(i, j, \omega_0)(\omega_m) v(p(i), q(j), \omega_m) \end{array} \right]. \end{aligned}$$

This defines an operator Φ from $\mathcal{F}$ to itself called the reduced Shapley operator.

Let $X(0, p, q, \omega_0, w) \subseteq \Delta(I)^K$ be the set of strategies for player I that achieves the maximum in the first equation and let $Y(0, p, q, \omega_0, w) \subseteq \Delta(J)^L$ be the set of strategies for player J that achieves the minimum in the second equation (in short, the set of optimal strategies in the reduced zero-sum Shapley game, which has a value since the maxmin and minmax commute).

A strategy $x \in \Delta(I)^K$ of player I is called non-revealing if $p(i) = p$ for all $i \in I$ or, equivalently, if x^k does not depend on k. Similarly, a strategy $y \in \Delta(J)^L$ for player J is non-revealing if $q(j) = q$ for all $j \in J$ or, equivalently, if y^l does not depend on l. A subset of strategies is non-revealing if *all* its elements are non-revealing.

Lemma 4 *For any accumulation point w of $v_\lambda(\cdot, \cdot, \omega_0)$ and all p, q:*

- ***P1:*** *If $X(0, p, q, \omega_0, w)$ is non-revealing then $w(p, q, \omega_0) \leq u(p, q, \omega_0)$.*
- ***P2:*** *If $Y(0, p, q, \omega_0, w)$ is non-revealing then $w(p, q, \omega_0) \geq u(p, q, \omega_0)$.*

The idea of the proof is as follows. If all elements of $X(0, p, q, \omega_0, w)$ are non-revealing then asymptotically, player I should not use his information as long as the state is ω_0. Since player J has always the option to ignore his information, the asymptotic payoff of player I should not exceed the asymptotic payoff of the "non-revealing game".

The next section shows there is at most one fixed point of the reduced Shapley operator that satisfies the necessary optimality conditions (implying uniqueness). When Ω is reduced to a singleton (the game is deterministic), the conditions are shown to be equivalent to the Mertens-Zamir [10] system of functional equations.

Proof. Let p and q be such that all elements of $X(0, p, q, \omega_0, w)$ are non-revealing.

Let x_n be optimal in $T(\lambda_n, v_{\lambda_n})(p, q, \omega_0)$ for player I (the maximizer) and let j be any non-revealing pure action of player J. Thus:

$$\begin{aligned} v_{\lambda_n}(p, q, \omega_0) &\leq \lambda_n g(x_n, j, p, q, \omega_0) \\ &+ (1 - \lambda_n) \sum_{i \in I} \overline{x}_n(i) \left[\begin{array}{c} Q(i, j, \omega_0)(\omega_0) v_{\lambda_n}(p_n(i), q, \omega_0) \\ + \sum_{m=1}^M Q(i, j, \omega_0)(\omega_m) v_{\lambda_n}(p_n(i), q, \omega_m) \end{array} \right]. \end{aligned}$$

Since $\Delta(I)^K$ is compact, without loss of generality one can assume that x_n converges to some x. Since all elements of $X(0,p,q,\omega_0,w)$ are non-revealing, x should be non-revealing: actually by the Berge maximum theorem, the set of optimal strategies is upper-semi-continuous as $\lambda \to 0$. Consequently, $p_n(i) \to p$ for all $i \in I$.

Case 1: $Q^*(x,j,\omega_0) > 0$.

Recall that $Q^*(x,j,\omega_0) = 1 - Q(x,j,\omega_0)(\omega_0) = \sum_{m=1}^{M} Q(x,j,\omega_0)(\omega_m) > 0$. Thus, letting $\lambda_n \to 0$ in the formula above implies:

$$\begin{aligned} w(p,q,\omega_0) &\leq \frac{\sum_{m=1}^{M} Q(x,j,\omega_0)(\omega_m) v(p,q,\omega_m)}{1 - Q(x,j,\omega_0)(\omega_0)} \\ &= \frac{g^*(x,j,p,q,\omega_0)}{Q^*(x,j,\omega_0)}. \end{aligned}$$

Case 2: $Q^*(x,j,\omega_0) = 0$.

Jensen's inequality implies that $\sum_{i\in I} \overline{x}_n(i) v_{\lambda_n}(p_n(i),q,\omega_0) \leq v_{\lambda_n}(p,q,\omega_0)$. Since $Q(i,j,\omega_0)(\omega_0) = 1 - \sum_{m=1}^{M} Q(i,j,\omega_0)(\omega_m)$, one deduces that:

$$\begin{aligned} &\lambda_n v_{\lambda_n}(p,q,\omega_0) + (1-\lambda_n) \sum_i \overline{x}_n(i) \sum_{m=1}^{M} Q(i,j,\omega_m)(\omega_m) v_{\lambda_n}(p_n(i),q,\omega_0) \\ \leq\ &\lambda_n g(x_n,j,p,q,\omega_0) + (1-\lambda_n) \sum_{i\in I} \overline{x}_n(i) \sum_{m=1}^{M} Q(i,j,\omega_0)(\omega_m) v_{\lambda_n}(p_n(i),q,\omega_m). \end{aligned}$$

Let $\alpha_n = \left(\frac{(1-\lambda_n)\overline{x}_n(i)}{\lambda_n}\right)_{i\in I} \in M_+(I)$. Consequently:

$$\begin{aligned} &v_{\lambda_n}(p,q,\omega_0) + \sum_i \alpha_n(i) \sum_{m=1}^{M} Q(i,j,\omega_m)(\omega_m) v_{\lambda_n}(p_n(i),q,\omega_0) \\ \leq\ &g(x_n,j,p,q,\omega_0) + \sum_{i\in I} \alpha_n(i) \sum_{m=1}^{M} Q(i,j,\omega_0)(\omega_m) v_{\lambda_n}(p_n(i),q,\omega_m). \end{aligned}$$

thus, after normalizing one gets:

$$\begin{aligned} &\frac{v_{\lambda_n}(p,q,\omega_0) + \sum_i \alpha_n(i) \sum_{m=1}^{M} Q(i,j,\omega_m)(\omega_m) v_{\lambda_n}(p_n(i),q,\omega_0)}{1 + \sum_i \alpha_n(i) \sum_{m=1}^{M} Q(i,j,\omega_m)(\omega_m)} \\ \leq\ &\frac{g(x_n,j,p,q,\omega_0) + \sum_{i\in I} \alpha_n(i) \sum_{m=1}^{M} Q(i,j,\omega_0)(\omega_m) v_{\lambda_n}(p_n(i),q,\omega_m)}{1 + \sum_i \alpha_n(i) \sum_{m=1}^{M} Q(i,j,\omega_m)(\omega_m)}. \end{aligned}$$

Take a subsequence such that for all $j \in J$ such that $Q^*(x,j,\omega_0) = 0$, the left and right hand terms converge (both are bounded since normalized). The left hand term is a convex combination of values that converge to $w(p,q,\omega_0)$. Since $g(x_n,j,p,q)$ converges to $g(x,j,p,q)$ and $v_{\lambda_n}(p_n(i),q,\omega_m)$ to $v(p,q,\omega_m)$, one deduces that for each $\epsilon > 0$, there is α such that for all j such that $Q^*(x,j,\omega_0) = 0$ one has:

$$w(p,q,\omega_0) \leq \frac{g(x,j,p,q,\omega_0) + \sum_{i\in I} \alpha(i) \sum_{m=1}^{M} Q(i,j,\omega_0)(\omega_m) v(p,q,\omega_m)}{1 + \sum_i \alpha(i) \sum_{m=1}^{M} Q(i,j,\omega_m)(\omega_m)} + \epsilon,$$

since $Q(x,j,\omega_0) = 1 - Q^*(x,j,\omega_0) = 1$, one deduces:

$$w(p,q,\omega_0) \leq \frac{g(x,j,p,q,\omega_0) + g^*(\alpha,j,p,q,\omega_0)}{Q(x,j,\omega_0) + Q^*(\alpha,j,\omega_0)} + \epsilon.$$

Consequently, $w(p,q,\omega_0) \leq u(p,q,\omega_0)$. By symmetry[8], one deduces that if $Y(0,p,q,\omega_0,w)$ is non-revealing then $w(p,q,\omega_0) \geq u(p,q,\omega_0)$. ∎

Observe that to each stochastic game with incomplete information and state space $\Omega = \{\omega_0, ..., \omega_M\}$ and each family of $2C$-Lipschitz functions $w(\cdot,\cdot,\omega_m)$, $m = 0, ..., M$ on $\Delta(K) \times \Delta(L)$ and each state $\omega_m \in \Omega$, a reduced Shapley operator $\Phi\left[w(\cdot,\cdot,\omega_k)_{k\neq m}\right]$ could be defined as follows:

$$\begin{aligned} &\Phi\left[w(\cdot,\cdot,\omega_k)_{k\neq m}\right](f)(p,q) \\ = &\max_{x\in\Delta(I)^K} \min_{y\in\Delta(J)^L} \sum_{i,j} \overline{x}(i)\overline{y}(j) \left[Q(i,j,\omega_m)(\omega_m) f(p(i),q(j)) + \sum_{k\neq m} Q(i,j,\omega_m)(\omega_k) w(p(i),q(j),\omega_k)\right] \end{aligned}$$

in which the max and the min permute (the game has a value). This will be used to extend the variational characterization to any stochastic game with incomplete information.

6 The maximum principle

The following lemma extends the one in Laraki [7], first established in a different form by Mertens and Zamir [10].

Lemma 5 *Let w_1 and w_2 be two fixed points of Φ in $\mathcal{F}$ and suppose that:*

- *If $X(0,p,q,\omega_0,w_1)$ is non-revealing then $w_1(p,q) \leq u(p,q,\omega_0)$.*
- *If $Y(0,p,q,\omega_0,w_2)$ is non-revealing then $w_2(p,q) \geq u(p,q,\omega_0)$.*

Then $w_1 \leq w_2$.

Proof. By contradiction, suppose $\max_{p\in\Delta(K),q\in\Delta(L)} w_1(p,q) - w_2(p,q) = \delta > 0$.
Let $C = \arg\max_{p\in\Delta(K),q\in\Delta(L)} w_1(p,q) - w_2(p,q)$. This is a compact set. Let (p_0, q_0) be an extreme point of $co(C)$, the convex hull of C. By Caratheodory's theorem, this is also an element of C. Let $x \in X(0,p_0,q_0,\omega_0,w_1)$ and $y \in Y(0,p_0,q_0,\omega_0,w_2)$. Thus:

$$\begin{aligned} \delta &= w_1(p_0,q_0) - w_2(p_0,q_0) \\ &\leq \sum_{i,j} \overline{x}(i)\overline{y}(j) \left[Q(i,j,\omega_0)(\omega_0)(w_1(p(i),q(j)) - w_2(p(i),q(j)))\right] \\ &\leq \delta \sum_{i,j} \overline{x}(i)\overline{y}(j) \left[Q(i,j,\omega_0)(\omega_0)\right] \end{aligned}$$

Consequently, $\sum_{i,j} \overline{x}(i)\overline{y}(j) Q(i,j,\omega_0)(\omega_0) = 1$ so that for all i and j such that $\overline{x}(i)\overline{y}(j) > 0$, one should have $Q(i,j,\omega_0)(\omega_0) = 1$, that is, when the players play x and y respectively, then the game stays in the state ω_0 with probability 1. Hence,

$$\begin{aligned} \delta &\leq \sum_{i,j} \overline{x}(i)\overline{y}(j)(w_1(p(i),q(j)) - w_2(p(i),q(j))) \\ &\leq \sum_{i,j} \overline{x}(i)\overline{y}(j)\delta \\ &= \delta \end{aligned}$$

[8]Observe that in that case, one should use a different formula for u in which player J minimizes over y and β and player I maximizes over i. Since there is a unique asymptotic value, both formulas for u must coincide (which is not obvious from their definition).

Consequently, for all i and j such that $\overline{x}(i)\overline{y}(j) > 0$, $(p(i), q(j)) \in C$. Since $\sum_{i,j} \overline{x}(i)\overline{y}(j)(p(i), p(j)) = (p_0, q_0)$ and since (p_0, q_0) be an extreme point of $co(C)$, $(p(i), p(j)) = (p_0, q_0)$ for all i, j so that x and y must be non-revealing. Thus, $w_1(p_0, q_0) \leq u(p_0, q_0, \omega_0)$ and $w_2(p_0, q_0) \geq u(p_0, q_0, \omega_0)$, implying that $w_1(p_0, q_0) - w_2(p_0, q_0) \leq 0$, a contradiction. ■

Consequently:

Theorem 6 $v_\lambda(\cdot, \cdot, \omega_0)$ *converges uniformly to the unique fixed point* v *of* Φ *that satisfies the variational inequalities P1 and P2.*

To compute v_λ for all ω, one should proceed recursively. Draw a tree from the initial state ω_0. The successors of ω_0 are all the $\omega_m \in \Omega$ that may be reached with a positive probability from ω_0 and so one. One then uses the characterization to compute the functions $v(\cdot, \cdot, \omega)$ by backward induction from ω at the end of the tree to the top.

Recall that when the game is of incomplete information on one side, the existence of the asymptotic value is already known from Rosenberg [13] for absorbing games (and so, by induction, for irreversible games).

7 The Mertens-Zamir system

The game is controllable by player I at ω_0 if there is $i \in I$ such that for every $j \in J$, the game remains at ω_0 with probability 1 ($Q(i, j, \omega_0)(\omega_0) = 1$). A similar definition holds for player J.

For a bounded real valued function f on $\Delta(K)$, $Cav_{\Delta(K)}(f)$ is the smallest concave function greater than f. Similarly, for a bounded real valued function h on $\Delta(L)$, $Vex_{\Delta(L)}(h)$ is the greatest convex function on $\Delta(L)$ smaller than h.

Theorem 7 *If the game is controllable by J then* $v(p, q, \omega_0) = Cav_{p\in\Delta(K)} [\min(u, v)] (p, q, \omega_0)$ *and if it is controllable by player I then* $v(p, q, \omega_0) = Vex_{q\in\Delta(L)} [\max(u, v)] (p, q, , \omega_0)$.

Consequently, if the game is controllable by both players (as in the deterministic case studied in [10]), v is the solution of the Mertens-Zamir system with respect to u.

Proof. Suppose player J controls the game. Let us show that:

- *P1'*: for all (p_0, q_0), if p_0 is an extreme point of the epigraph of $p \to v(p, q_0, \omega_0)$ then $v(p_0, q_0, \omega_0) \leq u(p_0, q_0, \omega_0)$.

Since v satisfies $P1$, it is sufficient to show that $X(0, p, q, \omega_0, v)$ contains only non-revealing strategies. Let $x \in X(0, p, q, \omega_0, v)$ let $j \in J$ such that for all i, $Q(i, j, \omega_0)(\omega_0) = 1$. Then:

$$\begin{aligned} v(p_0, q_0, \omega_0)) &= \min_{y\in\Delta(J)^L} \sum_{i,j} \overline{x}(i)\overline{y}(j) \left[Q(i, j, \omega_0)(\omega_0)v(p(i), q(j), \omega_0) + \sum_{m=1}^{M} Q(i, j, \omega_0)(\omega_m)v(p(i), q(j), \omega_m) \right] \\ &\leq \overline{x}(i)(\omega_0)v(p(i), q, \omega_0). \end{aligned}$$

Since p_0 is an extreme point of the epigraph of $p \to v(p, q_0, \omega_0)$, one should have $p(i) = p$ for every i, that is x should be non-revealing. Consequently, $v(p_0, q_0, \omega_0) \leq u(p_0, q_0, \omega_0)$ and so v satisfies *P1'*. From Laraki's [7] characterization of the Mertens-Zamir system, a function $v(\cdot, \cdot, \omega_0)$ satisfies *P1'* if and only if $v(p, q, \omega_0) = Cav_{\Delta(K)} [\min(u, v)] (p, q, \omega_0)$. This characterization has also been established independently by Rosenberg and Sorin [11]. ■

This extends a similar result in Sorin [16] when the game is a big match of type 2 (as in example 2). Example 3 shows that the *Cav* formula does not always hold if player J does not control the game. The variational characterization shows that big match games and more generally in absorbing games in which the game stops after absorption, the asymptotic value v is a function of the value of the non-revealing game u in a subtle way where the reduced shapley operator plays an important role. This is not obvious from Sorin's characterization in big match games of type 1.

8 Extensions

We are working to prove that the values of the finitely repeated games v_n uniformly converges, as the length of the game n goes to infinity, to the same limit. The proof needs the introduction of other somehow complex technics.

An interesting open question is to establish the existence and the characterization of the uniform maxmin and minmax. Both exist and may differ when the game is deterministic with incomplete information on both sides [1], when information is one side incomplete in (1) big match games (Sorin [15], [16]) and (2) recursive games (Rosenberg and Vieille [12]).

Conclude the paper by remarking that the variational characterization is a necessary condition that any accumulations points of any stochastic game with (or without) incomplete information must satisfy.

Corollary 8 *In any stochastic game with incomplete information, if* $(w(\cdot,\cdot,\omega_m))_{m=0,\dots,M}$ *is an accumulation point of the equicontinuous family* $(v_\lambda(\cdot,\cdot,\omega_m))_{m=0,\dots,M}$ *then for each* $\omega_m \in \Omega$, $w(\cdot,\cdot,\omega_m)$ *is the unique function that satisfies P1 and P2 with with respect to* $u(p,q,\omega_m) = U\left[w(\cdot,\cdot,\omega_k)_{k\neq m}\right](p,q,\omega_m)$ and *the reduced Shapley operator* $\Phi\left[w(\cdot,\cdot,\omega_k)_{k\neq m}\right]$.

One can easily construct a one player stochastic game of complete information in which this necessary condition is not sufficient to characterize the asymptotic value (consider one action at each state and two states where the transition goes deterministically for one state to the other). The condition may however be sufficient in many games (as irreversible games) and has the merit to explain how information is optimally used. It also shows that absorbing games may play an important role in general stochastic games with incomplete information (an idea already known [5]). The characterization implies that if v exists in a stochastic game (which is conjectured in all finite stochastic games with incomplete information) then, asymptotically, players must play at sequence of stages where the state is fixed as if they were in an auxiliary absorbing game with incomplete information. Moreover, if at some stage t they are at state ω_t and if the set $X(0,p_t,q_t,v,\omega_t)$ is non-revealing then player I must not use asymptotically his information at all and should play as if he was in the auxiliary absorbing game, until a new state is reached. This intuition may perhaps be helpful in proving the conjecture.

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