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A Asymptotics of Stationary and Nonstationary AR(1) Processes with Multiple Structural Breaks in Mean

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The Asymptotics of Stationary and Nonstationary AR(1) Processes with Multiple Structural Breaks in Mean

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Abstract

This study provides the asymptotic distributions of the least squares estimator (LSE) of an AR(1) process in the presence of structural breaks in the mean when the break sizes and the break durations are assumed to be generated by ergodic stationary processes. We further examine the special case where the process has a unit root. When there is a finite number of structural breaks, (1) the rate of convergence to the limiting distribution becomes much slower than when there is no structural break, (2) the persistence level tends to increase depending on the break sizes and the break durations, and (3) there exist multiple asymptotic results for the case of unit root. We propose a feasible unit root test with a new limiting distribution that is more robust than the DF test in detecting a unit root. The analysis is also extended to the case of an infinite number of structural breaks.

Keywords: AR(1), Unit Root, Structural Breaks, Asymptotic Properties, The Dicky-Fuller (DF) test, The Least Squares Estimator (LSE)

JEL Classification: C12, C13

1 Introduction

The seminal study of Perron (1989) shows that unit roots are less likely to be rejected when the data generating process is stationary about a broken linear trend. Subsequently a huge literature has evolved addressing the issue of unit root tests in the presence of structural breaks (see, *inter alia*, Perron, 1990; Zivot and Andrews, 1992; Bai, Lumsdaine, and Stock, 1998; Bai and Perron, 1998). Most of these studies investigate the effects of structural breaks in nonstationary processes such as trend stationary or unit root processes. They show that structural breaks could cause an I(1) process to be even more persistent, and that stationarity is easily confused with unit root when there are breaks.[1]

Despite the huge literature the asymptotic properties of the estimated coefficients of stationary processes and unit root processes have not yet been proposed in the presence of multiple structural breaks. Most analytical results in structural breaks assume one or two breaks in a series. The effects of multiple breaks on a process can be investigated using simulations, which becomes relatively easier with the recent computing development in software and hardware. This could help us understand the effects of structural breaks, but may not enhance our knowledge in this area.

Our study provides the asymptotic distributions of the LSE of the AR parameter of stationary processes and unit root processes in the presence of multiple structural breaks in mean, highlighting an important topic in econometrics - unit root tests in the presence of structural breaks. The following two cases are also investigated; the number of structural breaks is (1) fixed regardless of the sample size or (2) proportional to the sample size.

For this purpose, we use a simple zero mean AR(1) process as the data generating process (DGP). Although this DGP is simple and restrictive, we will show in this paper that the asymptotic property of the LSE of the AR parameter of the zero mean AR process are indeed complicated in the presence of multiple structural breaks. This could be a reason why asymptotic properties of a process in the presence of multiple breaks have not yet been proposed in the literature. The asymptotic results with more generalised processes could be less intuitive and appealing, if they become too complicated and thus simulations may be required to investigate the asymptotic properties.

The contribution of this paper is twofold. The first is to examine the asymptotic properties of the LSE of an AR(1) process when there are structural breaks generated by ergodic stationary processes. We show that the limiting distributions we derive for the LSE of the AR parameter are neither the usual Gaussian distribution nor the asymptotic distribution we obtain when the DGP has a unit root. The LSE without considering the structural breaks is not asymptotically consistent with the true AR parameter. The convergence rate becomes much slower than when there is no structural break and the bias depends on the ratio of break size. Multiple asymptotic results are obtained when the DGP has a unit root. Second, we propose a feasible structural break unit root test which is independent of any nuisance parameter with a new limiting distribution. This test is shown to be robust against the alternatives of stationarity with structural breaks in the mean or a unit root without structural breaks for DGPs in our framework. Thus, this test can reject the null of unit root in the presence of structural breaks when the alternative of stationarity or unit root without structural breaks is true.

This paper is organized as follows. In the next section, we investigate the effects of structural breaks in the mean. We first propose our DGP and underlying assumptions. Then, given our DGP, we derive the asymptotic properties of the LSE of the AR parameter in the presence of structural breaks in the mean. Section 3 contains brief concluding remarks.

2 The Asymptotics of the LSE of the AR parameter in the Presence of Multiple Breaks

2.1 Data Generating Processes and Assumptions

The DGP reflects what we are interested in, i.e., the type of structural breaks and processes. Following the work of Perron (1989) many studies investigate the effects of structural breaks on various types of nonstationary processes. A simple DGP used by Engle and Smith (1999), Diebold and Inoue (2001), and Granger and Hyung (2004) is the occasional breaks in mean plus noise model:

$$z_t = c_t^* + \xi_t, \tag{1}$$
$$c_t^* = c_{t-1}^* + I_t \epsilon_t,$$

where $\xi_t \sim IID(0, \sigma^2)$, $\epsilon_t \sim IID(0, \sigma_\epsilon^2)$, and I_t is an indicator variable which equals 1 with probability (p) and is independent of ϵ_t. It follows that Tp is the expected number of structural breaks in the sample.

Note that shocks in c_t^* are accumulated over time whenever $I_t = 1$, and thus by definition, the process in (1) is non-stationary. To keep the process stationary p should decrease with the sample size (T). Diebold and Inoue (2001) and Granger and Hyung (2004) show how long memory can be obtained by making p a hyperbolically decaying function of T. This type of DGP may be appropriate for level data, in particular when ϵ_t is allowed to have a positive mean so that z_t tends to increase over time.

In this study, we choose a simple $AR(1)$ process as the DGP and assume that the mean has multiple structural breaks. By allowing the AR parameter to be one, we also investigate the effects of structural breaks on unit root processes. More general DGPs such as ARMA processes with seasonal dummies are desirable. However it is likely that the asymptotic distribution of the LSE of the AR parameters becomes too complicated and thus less intuitive. We shall focus on a simple AR(1) process in this study, and leave the case of general DGPs for future studies.

The AR(1) process with structural breaks in mean (SBMs), is similar to (1) except that c_t^* does not follow a unit root process but moves around zero. Using the same notation as in (1), we propose the following AR process with SBMs

$$\begin{aligned} z_t &= c_t + \phi z_{t-1} + \xi_t, \\ c_t &= (1 - I_t)c_{t-1} + I_t \epsilon_t, \end{aligned} \tag{2}$$

where ξ_t is an ergodic martingale difference sequence (MDS) which satisfy $E[\xi_t|\mathcal{F}_{t-1}] = 0$, $E[\xi_t^2|\mathcal{F}_{t-1}] = \sigma^2$, where $\mathcal{F}_{t-1}$ is an information filtration generated by $\{c_0,\ z_0,\ \epsilon_0,\ \xi_0,$ $...,\ c_{t-1},\ z_{t-1},\ \epsilon_{t-1},\ \xi_{t-1}\}$, and

$$E[|\xi_t|\mathbf{1}_{(|\xi_t|>a|\alpha_n|^{-1})}|\mathcal{F}_{t-1}] \xrightarrow{P} 0, \tag{3}$$

where the sequence $|\alpha_n| \longrightarrow |\alpha^*| < 1$ as $n \to \infty$ for some $a \in (0, 1]$. If all the moments of ξ_t are finite and $a = 1$, then by the Markov and Holder inequalities,

$$E[|\xi_t|\mathbf{1}_{(|\xi_t|>|\alpha_n|^{-1})}|\mathcal{F}_{t-1}] \leq |\alpha_n|^r E[|\xi_t|^r] = o(1)$$

as $n,\ r \longrightarrow \infty$. Thus, (3) is obviously satisfied; ϵ_t in (2) is an ergodic stationary process with the stationary distribution $F_\epsilon(\bullet)$ which has zero mean and variance σ_ϵ^2.

The sequence of ergodic martingale differences includes the sequence of independently and identically distributed random variables as a special case. However, the results in our paper can be shown to hold under a fairly general assumption as used in Phillips (1988), namely that $\{\epsilon_t, \xi_t\}$ is strong mixing, though we do not use the strong mixing assumption for the innovation processes because our analytical results may not be necessarily improved given the complicated nature of this assumption.

For a given sample size T, we assume that there are K different means in the sample, i.e., $Tp = K - 1$. The process c_t follows a renewal - reward process where renewal and reward are assumed to be independent. The break size c_t is a reward at renewal. The break point series, $T(1)+1,\ T(1)+T(2)+1,\ ...,\ \sum_{k=1}^{K-1} T(k)+1$ are realizations of a renewal process where $T(k)$ is the number of observations in the k-th sub-AR process, and $\sum_{k=1}^{K} T(k) = T$. Let τ_k denote the fraction of the sample such that $T(k) = [T\tau_k]$, where $[\bullet]$ is the largest integer that is less or equal to $\bullet$. It is easy to show that $\sum_{i=1}^{K}[T\tau_i] = [T\sum_{i=1}^{K}\tau_i] = T$ since $\sum_{i=1}^{K}\tau_i = 1$. Therefore, the true trajectory of the mean process as defined in (2) is given as follows

$$c_k = c_{t_k}\left(= \epsilon_{\sum_{i=1}^{k-1}[T\tau_i]+1}\right),\ \forall\ t_k \in \left\{\sum_{i=1}^{k-1}[T\tau_i]+1, \sum_{i=1}^{k}[T\tau_i]\right\}. \tag{4}$$

Note that as in Bai and Perron (1998) τ_k needs to be asymptotically distinct. In order to investigate the asymptotic behaviour of the LSE of the AR parameter in the presence of structural breaks, we need the following assumptions.

Assumption 1 *For a given $K \in (0,\infty]$, the duration process τ_k is an ergodic process independent of ϵ_t and ξ_t. The sample paths of τ_k satisfy $\tau_K = 1 - \sum_{k=1}^{K-1}\tau_k$, $E(\tau_k^2) = \sigma_\tau^2(K)$ and $E(\tau_k^3) = \lambda_\tau(K) > 0$. τ_k has the stationary distribution $F_\tau(\bullet)$.*

Note that $\tau_k > 0\ \forall k$, and thus the non-central skewness of τ_k is always positive. Since τ_k generally decreases as K increases, $\sigma_\tau^2(K)$ and $\lambda_\tau(K)$ are decreasing functions of K.

Assumption 2 *$p = O(T^{-1})$ such that $\lim_{T\to\infty} Tp = K-1$.*

Assumption 2 is useful for investigating the asymptotic properties of a process with a small number of structural breaks in large samples. Many studies such as Diebold and Inoue (2001), Leipus and Surgailis (2003), and Granger and Hyung (2004) allow $p \to 0$ to explain long memory with structural breaks in similar processes to equation (1). The assumption says that $T(k)$ tends to increase with T so that the number of structural breaks remains finite. However, as explained in Diebold and Inoue (2001) and Granger and Hyung (2004), this sample size-dependent probability may not reflect reality. In general as T increases K is also expected to increase as in the following assumption.

Assumption 3 *$K = O(T)$ such that* $\lim_{T\to\infty} \frac{K-1}{T} = p$.

We note however that the probability of breaks is usually small and thus K may be still small even if T increases to a large number, and thus the asymptotic results with Assumption 3 in this study should be interpreted with care. Nevertheless the assumption is useful for the investigation of the effects of structural breaks since it provides much simplified asymptotic results.

Assumption 4 *The changing points $z_{\sum_{i=1}^{k}[T\tau_i]+1}$ have finite first and second order moments, i.e., $E\left(z_{\sum_{i=1}^{k}[T\tau_i]+1}\right) < \infty$ and $E\left(z^2_{\sum_{i=1}^{k}[T\tau_i]+1}\right) < \infty$.*

Assumption 4 is equivalent to $\left|z_{\sum_{i=1}^{k}[T\tau_i]+1}\right| < \infty$ and $z^2_{\sum_{i=1}^{k}[T\tau_i]+1} < \infty$ almost sure. This follows from the Borel-Cantelli lemma. Since

$$\sum_{K=1}^{\infty} P\left\{\omega : \left(\left|z_{\sum_{i=1}^{k}[T\tau_i]+1}\right| \geq K\right)\right\} < \sum_{K=1}^{\infty} \frac{E\left|z_{\sum_{i=1}^{k}[T\tau_i]+1}\right|^2}{K^2} < \infty,$$

then

$$\begin{aligned} P\left\{\bigcap_{n=1}^{\infty}\bigcup_{K=n}^{\infty}\left[\omega : \left(\left|z_{\sum_{i=1}^{k}[T\tau_i]+1}\right| \geq K\right)\right]\right\} &= P\left\{\omega : \left(\left|z_{\sum_{i=1}^{k}[T\tau_i]+1}\right| \geq \infty\right)\right\} \\ &= 0. \end{aligned}$$

Thus, $\left|z_{\sum_{i=1}^{k}[T\tau_i]+1}\right| < \infty$ is almost sure (a.s.), and similarly we can prove that $z^2_{\sum_{i=1}^{k}[T\tau_i]+1} < \infty$ a.s.

The effect of structural breaks in the mean of an AR(1) process on the persistence level can be investigated when the structural breaks are disregarded. The misspecified zero mean AR(1) process is given by

$$z_t = \varphi z_{t-1} + \eta_t, \tag{5}$$

where $\eta_t \sim IID(0, \sigma_\eta^2)$. Our concern here is the effects of changes in c_t around zero on the LSE of φ ($\hat{\varphi}$). In our study, these effects are investigated for both a stationary autoregressive process ($|\phi| < 1$) and a unit root process ($\phi = 1$).

2.2 Stationary Case

Here the AR parameter of the true DGP is assumed to be less than one, i.e., $|\phi| < 1$.

Theorem 1 *Under Assumptions 2 and 4, the asymptotic conditional distribution of the LSE of φ of the misspecified model in (5) for the true DGP in (2) is given by*

$$\hat{\varphi} - \phi\Big|_{\{\tau_k, c_k\}_{k=1}^K} \overset{W}{\Longrightarrow} \frac{A}{B} = \mathcal{A}, \tag{6}$$

where

$$\begin{aligned} A &= \sum_{k=1}^{K} c_k \tau_k \left(\frac{c_k}{1-\phi} + \frac{\sigma}{\sqrt{1-\phi^2}} W_k(1) \right), \\ B &= \sum_{k=1}^{K} \tau_k \left(\frac{c_k}{1-\phi} + \frac{\sigma}{\sqrt{1-\phi^2}} W_k(1) \right)^2, \end{aligned}$$

and $W_k(1)$ is standard Brownian motion and $\overset{W}{\Longrightarrow}$ denotes convergence in distribution (or weak convergence). The LHS of (6) *is the bias conditional on a sample path of* (τ_k, ϵ_t).

Proof. See the Appendix. ■

The rate of convergence decreases due to SBMs in a surprising way. When there are SBMs in a stationary AR(1) process, $\hat{\varphi} - \phi = O_p(1)$ and $\hat{\varphi} - \phi = o_p(T^{-\alpha})\ \forall\ \alpha > 0$ whereas $\hat{\varphi} - \phi = o_p(T^{-1/2})$ when there is no SBM in the AR(1) process. This property is also different from that of the unit root case where the limiting distribution of $(\hat{\varphi} - 1)$ is asymmetric but converges at rate of T^{-1}. The support of the limiting distribution of $\hat{\varphi} - \phi$ is the entire real line since B is always positive whilst A can take any value of the entire real line. Therefore, in the presence of SBMs $\hat{\varphi} - \phi$ does not converge to zero whereas it is not the case when there is no SBM.

The LSE is not consistent with the true AR parameter and is positively biased in mean. As opposed to the usual case where the limiting distribution of $\sqrt{T}(\hat{\varphi} - \phi)$ is symmetric around zero, the limiting distribution of $\hat{\varphi} - \phi$ is asymmetric around a positive value since the denominator in (6) contains a $\chi^2(1)$ random variable and the numerator includes a positive component $\frac{c_k^2}{1-\phi}$. By taking expectations the unconditional distribution of the bias can be obtained;

$$\mathcal{L}(\hat{\varphi} - \phi | P) \Longrightarrow \int_{\substack{(0,1)^K \\ \sum_{k=1}^K \tau_k = 1}} \int_{\mathbb{R}^K} \mathbb{P}(\mathcal{A} | \{\tau_k, c_k\}_{k=1}^K) \prod_{k=1}^K dF_\epsilon(c_k) dF_\tau(\tau_k),$$

where $\mathcal{L}(X|P)$ denotes the distribution of X when X is defined on the probability space with the probability distribution P; $\mathbb{P}$ is the conditional distribution of $\mathcal{A}$.

The limiting conditional distribution depends on the parameters c_k, τ_k and σ, which are break size, break duration and the volatility of the error term in the AR process, respectively. When ϕ is large and negative the value of $\mathcal{A}$ is influenced by Brownian motion relatively more than the break size c_k because of the different weights on these components, e.g., $\frac{1}{1-\phi}$ and $\frac{1}{1-\phi^2}$ respectively. Therefore break sizes are weighted more for a large positive ϕ and the asymmetric patterns in $\mathcal{A}$ between positive and negative values of ϕ are expected.

The result in Theorem 1 is obtained with the assumption that the number of structural breaks (K) does not increase in proportion to the sample size (T). We now adopt Assumption 3 to obtain the limits of $\hat{\varphi} - \phi$ when $T \to \infty$ and $K \to \infty$.

Proposition 1 *Let us suppose that*

$$\sum_{k=1}^K \frac{c_k^\alpha}{k^\alpha} < +\infty \text{ for some } \alpha \in (0, 2]. \tag{7}$$

Then under Assumptions 1, 3, and 4,

$$\hat{\varphi} - \phi \overset{P}{\Longrightarrow} \frac{V}{\frac{V}{1-\phi} + \frac{1}{1+\phi}} \geq 0, \tag{8}$$

where $V = \sigma_\epsilon^2/\sigma^2$. *When* $V > \frac{(1-\phi)^2}{(1+\phi)^2}$, *the bias decreases.*

Proof. See the Appendix. ■

Proposition 1 confirms what has been observed in many previous studies mainly by simulations; the LSE tends to be more persistent than ϕ suggests. For example Granger and Hyung (2004) show that persistence level increases with the magnitude of breaks. Our results show that it is the relative break size to error term that increases the bias. When the volatilities of break size and error term increase simultaneously with the same ratio, the bias does not change. These results can explain why it is hard to find persistence in asset returns even if there exist structural breaks. This is because asset returns are highly volatile relative to the magnitude of structural breaks (see Getmansky, Lo, and Makarov, 2003 for example).

The bias decreases when $V > \frac{(1-\phi)^2}{(1+\phi)^2}$. The right hand side of this inequality shows that as $\phi \to 1$, $\frac{(1-\phi)^2}{(1+\phi)^2} \to 0$ and thus, *ceteris paribus*, we have $V > \frac{(1-\phi)^2}{(1+\phi)^2}$. Although $\hat{\varphi}$ is positively biased in the presence of SBMs, the bias decreases as $\phi \to 1$ and eventually becomes zero. On the other hand as $\phi \to -1$, $\frac{(1-\phi)^2}{(1+\phi)^2} \to \infty$ and thus the bias increases. However, the magnitude of the bias depends on the value of V. For example for small V, $\hat{\varphi} - \phi \overset{P}{\Longrightarrow} 0$ as $\phi \to -1$, while $\hat{\varphi} - \phi \overset{P}{\Longrightarrow} 1.8$ when $\phi = -0.8$ for large V. In general the bias in $\hat{\varphi}$ lies between 0 and 2.

When $\phi = 0$, the DGP is a mean plus noise process whose mean occasionally changes.

Theorem 2 *With Assumptions 2 and 4, the asymptotic conditional distribution of the LSE of* φ *in the misspecified model in (5) for the true DGP in (2) when* $\phi = 0$ *is given by*

$$\hat{\varphi}\Big|_{\{\tau_k, c_k\}_{k=1}^K} \overset{a.s.}{\Longrightarrow} \frac{\sum_{k=1}^K \tau_k c_k^2}{\sum_{k=1}^K \tau_k c_k^2 + \sigma^2} \tag{9}$$

Again under Assumptions 3 and 4 we have

$$\hat{\varphi} \overset{a.s.}{\Longrightarrow} \frac{V}{V+1}. \tag{10}$$

Proof. See the Appendix. ■

Although there are some similarities between Theorems 1 and 2, the asymptotic result in (9) cannot be obtained from Theorem 1 by simply putting $\phi = 0$. The convergence in (9) and (10) is "almost sure" which is stronger than the convergence in probability in (8). Again $\hat{\varphi}$ is asymmetric because of c_k^2. Equation (10) shows that the asymptotic positive bias increases with V. When break sizes are relatively large so that $V(=\sigma_\epsilon^2/\sigma^2)$ increases, $\hat{\varphi}$ increases even if the DGP is mean plus noise process. However the process does not become a nonstationary process since $0 \leq \frac{\sum_{k=1}^{K}\tau_k c_k^2}{\sum_{k=1}^{K}\tau_k c_k^2+\sigma^2} < 1$ almost sure. Therefore contrary to the arguments of Campos et al. (1996), a stationary process with structural breaks in the mean is not necessarily a unit root though the persistence level of this process increases, and the DF test is biased toward accepting a stationary process with breaks in mean.

2.3 Non-stationary Case

In this section the true DGP is assumed to follow a unit root process (i.e., $\phi = 1$) with structural breaks in mean;

$$z_t = (1 - I_t)c_{t-1} + I_t\epsilon_t + z_{t-1} + \xi_t. \tag{11}$$

The DGP is similar to 'model (B)' of Perron (1989) and those used by Leybourne et al. (1998) and Kim et al (2000). Leybourne et al. (1998) and Kim et al. (2000) show that the unit root null hypothesis is spuriously rejected when the true break in the series is relatively early or soon after the assumed break. This is partially because the DF test always converges under the null of a unit root whether or not the break occurs; and this makes it difficult to distinguish between a unit root with SBMs and a unit root without SBMs (see Leybourne et al., 1998). Hence, when the durations of breaks are considerable so that the influence of breaks in the limiting distribution is significant, the DF test can reject the null of unit root more often than in the conventional case. Our analysis extends their research in the presence of multiple breaks.

Again, the misspecified AR model in (5) is estimated and the limiting distributions of the LSE of the AR parameter for z_t are derived.

Theorem 3 *When the DGP follows (11), under Assumptions 2 and 4 the LSE of φ for the misspecified model in (5) has the following limiting conditional distributions:*

$$\text{Case 1:} \qquad \hat{\varphi} - 1 = o_p(1), \ \ or \tag{12}$$

$$\text{Case 2:} \qquad T^{3/2}(\hat{\varphi} - \gamma(T))\Big|_{\{\tau_k, c_k\}_{k=1}^K} \overset{W}{\Longrightarrow} \frac{\sigma \sum_{k=1}^K c_k \int_0^{\tau_k} s dW_{2k}(s)}{\sum_{k=1}^K \frac{c_k^2(\tau_k)^3}{3}} = \mathcal{B}, \ or \tag{13}$$

$$\text{Case 3:} \qquad T(\hat{\varphi} - 1)\Big|_{\{\tau_k, c_k\}_{k=1}^K} \overset{W}{\Longrightarrow} \frac{\sum_{k=1}^K \left[\frac{c_k^2(\tau_k)^2}{2} - c_{k-1}^2(\tau_{k-1})^2\right]}{\sum_1^K \frac{c_k^2(\tau_k)^3}{3}} = \mathcal{C}, \tag{14}$$

where $\gamma(T) = \frac{\sum_{k=1}^K \left\{\sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2 + c_k \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}\right\}}{\sum_{k=1}^K \left\{z_{T(k-1)}^2 + \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2\right\}}$ *and* $\lim_{T \longrightarrow \infty} \gamma(T) = 1$.

Proof. See the Appendix for proof. ∎

By taking expectations, the limiting unconditional distributions of the biases as defined in (13) and (14) are given by

$$\mathcal{L}(T^{3/2}(\hat{\varphi} - \gamma(T))|P) \Longrightarrow \int_{\substack{(0,1)^K \\ \sum_{k=1}^K \tau_k = 1}} \int_{\mathbb{R}^K} \mathbb{P}(\mathcal{B}|\{\tau_k, c_k\}_{k=1}^K) \prod_{k=1}^K dF_\epsilon(c_k) dF_\tau(\tau_k), \tag{15}$$

$$\mathcal{L}(T(\hat{\varphi} - 1)|P) \Longrightarrow \int_{\substack{(0,1)^K \\ \sum_{k=1}^K \tau_k = 1}} \int_{\mathbb{R}^K} \mathbb{P}(\mathcal{C}|\{\tau_k, c_k\}_{k=1}^K) \prod_{k=1}^K dF_\epsilon(c_k) dF_\tau(\tau_k)$$

respectively.

When the DGP follows a unit root process with SBMs, we have three different limiting distributions. As in Kim et al. (2004), the limiting distributions display a wide range of different characteristics, and the asymptotic properties of $\hat{\varphi} - 1$ are achieved by using various normalizing terms. When $\hat{\varphi} - 1$ is normalized by 1 (the first case), $\hat{\varphi}$ is the consistent estimate of the unit root. However, $\hat{\varphi}$ converges to unit much slower than that in the standard unit root process. The second case shows that the LSE is biased; the convergence rate is very fast, i.e., $T^{-3/2}$. The third case shows that the convergence rate is T^{-1}, the rate of the conventional DF test. Case 3 shows that the order of structural breaks matters when there are multiple breaks in the series. This is consistent with Leybourne et al. (1998) and Kim et al. (2000) who show that the rejection of the unit root null hypothesis depends on where the true break in the series lies.

It is well-known that without structural breaks in the DGP, the conventional DF unit root test based on the statistics $T(\widehat{\phi} - 1)$ is consistent for the null hypothesis $H_o : \phi = 1$ against the alternative $H_1 : \phi < 1$ since the test is convergent under the null whilst divergent under the alternative at a rate proportional to $T^{1/2}$. Thus, the DF test has power against the alternative hypothesis. However, when there are SBMs, the DF test could diverge under the null hypothesis at a rate proportional to $T^{1/2}$ in view of Case 2. This implies that under the null and the alternative the DF test could diverge. As a result, the DF test is not robust to SBMs as conjectured in Leybourne et al. (1998).

An immediate question is whether we can distinguish unit root or stationary process without SBMs from unit root with SBMs. This hypothesis can effectively be written as $H_0 : \{\phi = 1 \text{ and } c_k \neq c_j \ \forall \ k \neq j\}$ and $H_1 : \{\phi = 1 \text{ and } c_k = 0 \ \forall \ k\} \bigcup \{\phi < 1\}^2$, and the test for the null against the alternative can be constructed based on $T^{3/2}(\widehat{\varphi} - \gamma(T))$ in which τ_k, c_k and K can be estimated consistently by the method of Bai and Perron (1998). Since it is straightforward to verify that the test is convergent under the null and divergent under the alternative, the test is consistent, and consequently it has power against the alternative H_1.

Therefore using the consistency of the test based on $T^{3/2}(\widehat{\varphi} - \gamma(T))$, we could resolve the spurious rejection of the DF test in the presence of SBMs by using the following two stage procedure: First, use the test based on $T^{3/2}(\widehat{\varphi}-\gamma(T))$ to test the unit root. Second, if the test based on $T^{3/2}(\widehat{\varphi}-\gamma(T))$ rejects the null then the set of remaining alternatives is $\{\phi < 1\}\bigcup\{\phi = 1 \text{ and } c_k = 0 \ \forall \ k\}$ which are then tested by the conventional DF test. Furthermore, as the limiting unconditional distribution in (15) does not depend on any nuisance parameter except for the number of regimes (K), thus the null is an intersection hypothesis, i.e., $H_0 : \ \bigcap_{\substack{K\in\Gamma \\ \Gamma\neq\varnothing}}\{\phi = 1 \text{ for a given number of breaks } K\}$. The test rejects the null if $T^{3/2}(\hat{\varphi}-\gamma(T))$ is too small, thus the rejection region of the test is

$$\bigcup_{\substack{K\in\Gamma \\ \Gamma\neq\varnothing}}\{T^{3/2}(\hat{\varphi}-\gamma(T))\Big|_{\{\tau_k,c_k\}_{k=1}^K} < R\} = \inf_{\substack{K\in\Gamma \\ \Gamma\neq\varnothing}}\{T^{3/2}(\hat{\varphi}-\gamma(T))\Big|_{\{\tau_k,c_k\}_{k=1}^K}\} < R,$$

where R is a critical value. Therefore, the test is efficient with the limiting distribution

$$\int_{(0,1)^{||\Gamma||}}\int_{\mathbb{R}^{||\Gamma||}}\mathbb{P}\left(\inf_{\substack{K\in\Gamma \\ \Gamma\neq\varnothing}}\left(\mathcal{B}|\{\tau_k,c_k\}_{k=1}^K, \ \sum_{k=1}^{K}\tau_k = 1\right)\right)\prod_{k=1}^{||\Gamma||}dF_\epsilon(c_k)dF_\tau(\tau_k),$$

where $||\Gamma||$ is the size of the set Γ. The critical values of the test can be tabulated by rather complicated simulations. For this reason, we leave further details for future study.

Proposition 2 *With Assumptions 1, 3, and 4, as $K \to \infty$ the second case of Theorem 3 is*

$$T^{3/2}(\hat{\varphi}-\gamma(T)) \overset{P}{\Longrightarrow} 0.$$

For the third case we have

$$T(\hat{\varphi}-1) \overset{P}{\Longrightarrow} -\frac{3\sigma_\tau^2(\infty)}{2\lambda_\tau(\infty)} \leq 0, \tag{16}$$

where $\sigma_\tau^2(\infty)$ and $\lambda_\tau(\infty)$ are defined in Assumption 1.

Proof. See the Appendix for proof. ■

The first result is straightforward. Equation (16) confirms the results of Leybourne at el. (1998) in the presence of multiple breaks. If there are structural breaks in mean in unit root process, rejections of the unit root do not necessarily imply that the true process is stationary. The negative bias is a function of noncentral variance and skewness of the duration variable τ_k.

Equation (16) suggests that break size becomes less important as K increases. This point can be seen clearly when the break durations are presumably the same.

Remark 1 *If $\tau_k = \frac{1}{K}$ $\forall k$ such that $\sigma_\tau^2(K) = \frac{1}{K^2}$ and $\lambda_\tau(K) = \frac{1}{K^3}$, then*

$$\hat{\varphi} - 1 \overset{P}{\Longrightarrow} -\frac{3}{2}p,$$

where p is the probability of breaks.

The result indicates that the asymptotic behaviour of $\hat{\varphi}$ does not depend on break sizes but on break probability. As the number of breaks increases with the number of observations, break size and the order of breaks do not appear in the asymptotic properties.

3 Simulations

The asymptotic results in theorems and propositions are not easy to interpret even with this simple AR(1) process. We need to better understand how fast the asymptotic conditional distributions could be achieved, and how the sample least squares (LS) estimates behave for different values of ϕ, different break sizes, and different numbers of breaks. To answer these questions we simulate the asymptotic conditional distributions and the compare the results with the sample LS estimates we obtain by estimating (5).

AR(1) series are generated for the sample sizes of T=100, 200, 500, 1000, and 3000. For the numbers of breaks, we set $K-1$=4, 9, 49, and 99 where K is the number of sub-processes. The error term in the AR process follows standard normal, $\xi_t \sim N(0,1)$, while the sizes of structural breaks are drawn from three different normal distributions, i.e.,

$\epsilon_t \sim N(0, 0.1^2)$, $N(0,1)$, and $N(0, 10^2)$, which have average break sizes ($=\sigma_\epsilon/\sigma = \sqrt{V}$) of 0.1, 1, and 10, relative to the AR error term. For the values of ϕ, we take 1, 0.9, 0.5, 0, and -0.5. Our main concern is AR processes with positive values of ϕ, but we also include -0.5 to see the effects of SBMs for negative values of ϕ. For the asymptotic conditional distributions we generate Brownian motions ($W_k(1)$) with 10000 i.i.d. standard normal variates. We repeat the procedure 10000 times to obtain the sample LS estimates and asymptotic conditional distributions.

Table 1 reports the results of $\widehat{\varphi} - \phi$ at various percentage points for various numbers of breaks, and break sizes, when the AR parameter is 0.5. When the numbers of breaks are small and break sizes are small, i.e., $K = 4$ and $\sigma_\epsilon/\sigma = 1$ or 0.1, our asymptotic conditional distribution does not explain the sample distributions well. However except these few cases the asymptotic conditional distributions in Theorems 1 and 2 explain the sample distributions well. As expected as break size increases, the positive bias increases. Sample distributions do not seem to approach the limiting distributions quickly. In some cases 500 to 1000 observations are required to approximate the limiting distributions.

Panel A also shows the distributions of $\widehat{\varphi} - \phi$ as K increases. As mentioned in Assumption 3, we do not expect K is infinite in reality. The panel shows that $\widehat{\varphi} - \phi$ approaches $\frac{V}{\frac{V}{1-\phi}+\frac{1}{1+\phi}}$ slowly as K increases from 4 to 99. In Panel B the effects of SBMs for different values of ϕ are reported; $\widehat{\varphi} - \phi$ is negatively related with the values of ϕ. As ϕ increases $\widehat{\varphi} - \phi$ decreases, which confirms Proposition 1. The case of $\phi = -0.5$ at the bottom of Panel B shows that our analytical results also work well for negative ϕ.

Figure 1 reports the Gaussian kernel densities of $\widehat{\varphi} - \phi$ for various numbers of samples and break sizes in the presence of SBMs when ϕ=0.5 and K=9. For a large value of σ_ϵ/σ the mass of the densities move towards 0.5 with a larger negative skewness. As T increases the negative skewness decreases and thus the sample densities approach the limiting distributions at the bottom of Figure 1.

We repeat a similar procedure for the unit root case in Theorem 3. Figure 2 shows that the sample LS estimates of $\widehat{\varphi}$ are always negatively biased and thus the null unit root hypothesis will be rejected, which is consistent with the limiting conditional distribution of case 3 of Theorem 3. We find that $T^{3/2}(\widehat{\varphi}-1)$ in case 2 is symmetric around zero and is not different from zero.[3] Although the limiting conditional distribution may show non-trivial non-zero values of $T^{3/2}(\widehat{\varphi}-\gamma(T))$ for smaller values of σ_ϵ/σ, it approaches zero as K increase as in Proposition 2. In addition, the limiting conditional distribution in case 3 shows that $T(\widehat{\varphi}-1)$ is not trivial but $\widehat{\varphi}-1$ approaches a small negative number as T increases. Thus though we have three different limiting distributions in the presence of SBMs when the true process is unit root, the simulation results suggest that it is case 3 that dominates the other two.

4 Conclusion

These results suggest several conclusions. First, by some asymptotic arguments and the proposed conditional DGP approach we show that the LSE of the AR parameter has an asymmetric limiting distribution when there are structural breaks in the mean of a zero mean stationary $AR(1)$ process. Structural breaks in mean lead to an increase in persistence level that further depends on break sizes and break durations. On the other hand, when the DGP follows a unit root with structural breaks in the mean, three different limiting cases emerge. Monte Carlo simulations support our theoretical findings.

Second, our results suggest that when there are SBMs, the conventional statistics we use for inferences may not be appropriate. Because of slow convergence rates and biases in the persistence level, the conventional Gaussian t test and the DF test do not have power for these processes. The analysis of time series such as forward discounts, equity premium, or volatility becomes more difficult; and the differentiation between stationarity and nonstationarity is ambiguous in the presence of structural breaks. Therefore, we propose a new unit root test as a remedy to the problem of spurious rejection of unit root by the DF test in the presence of SBMs as found in Leybourne et al. (1998).

Notes

[1]Besides the relationship between structural breaks and persistence, a significant number of studies focus on cointegration and long memory in the presence of structural breaks. Some recent studies on these topics are Campos et al. (1996), Lobato and Savin (1998), Diebold and Inoue (2001) and Granger and Hyung (2004). A vast literature has also developed around forecasting in the presence of structural breaks. See the prominent work of Clements and Hendry (1996), and Pesaran and Timmermann (2005) for a recent development in this area.

[2]Since we have assumed that the sample paths of the mean process with zero expectation vary around zero, $c_k = 0 \; \forall \; k$ when there is no structural breaks in the mean.

[3]In the simulations we set $\gamma(T)$ equal to 1.

Appendix

Liptser and Shiryayev (1989, page 582) show that if $\xi^{(T)} = (\xi_{T,t}, \mathcal{F}_t^{(T)})$ is a martingale difference array, and the following conditions

$$\sum_{t=1}^{[T\tau]} E[\xi_{T,t}^2 | \mathcal{F}_{t-1}^{(T)}] \stackrel{P}{\Longrightarrow} C_\tau,$$

$$\sum_{t=1}^{[T\tau]} E\left[|\xi_{T,t}| \mathbf{1}_{(|\xi_{T,t}|>a)} | \mathcal{F}_{t-1}^{(T)}\right] \stackrel{P}{\Longrightarrow} 0 \textit{ for } a \in (0,1] \textit{ and } \tau \in (0,1]$$

are satisfied, then

$$\sum_{k=1}^{[T\tau]} \xi_{T,t} \stackrel{W}{\Longrightarrow} W(C_\tau). \tag{17}$$

Using the above result, we derive the following lemma and corollary to prove theorems and propositions in our study.

Lemma 1 *Let ξ_t be a sequence of martingale differences with $E[\xi_t|\mathcal{F}_{t-1}] = 0$ and $E[\xi_t^2|\mathcal{F}_{t-1}] = \sigma^2$. Let $S_{[T\tau]} = \sum_{t=1}^{[T\tau]} \phi_t \xi_t$, $\forall\ \tau \in (0,1]$. If*

$$\lim_{T\longrightarrow\infty} \sum_{t=1}^{[T\tau]} \phi_t^2 = C.g(\tau), \tag{18}$$

where C is a positive constant and $g(\tau)$ is a positive finite function of τ, then

$$\lim_{T\longrightarrow\infty} \frac{S_{[T\tau]}}{\sigma\sqrt{C}} \stackrel{W}{\Longrightarrow} W(g(\tau)), \tag{19}$$

where $W(g(\tau))$ is a Brownian motion with variance $g(\tau)$.

Proof. The proof follows by verifying the conditions of Liptser and Shiryayev's result above:

$$\lim_{T\longrightarrow\infty}\sum_{t=1}^{[T\tau]}\phi_t^2 E[\xi_t^2|\mathcal{F}_{t-1}] = \sigma^2 \lim_{T\longrightarrow\infty}\sum_{t=1}^{[T\tau]}\phi_t^2 = C.g(\tau)$$

$$\lim_{T\longrightarrow\infty}\sum_{t=1}^{[T\tau]}\phi_t^2 E\left[|\xi_t|\mathbf{1}_{(|\xi_t|>\phi_t^{-1}a)}|\mathcal{F}_{t-1}\right] \leq \lim_{T\longrightarrow\infty}\max_{0<t\leq T}\phi_t^2\sum_{t=1}^{[T\tau]}E\left[|\xi_t|\mathbf{1}_{(|\xi_t|>\frac{a}{\max_{0<t\leq T}\phi_t})}|\mathcal{F}_{t-1}\right] = 0$$

in view of equation (3). ■

The following corollary is a special case of Lemma 1.

Corollary 1 *If $\phi_t = \phi^t$ and $|\phi| < 1$ then*

$$\lim_{T\longrightarrow\infty}\frac{S_{[T\tau]}}{\sigma\sqrt{\frac{\phi^2}{1-\phi^2}}} \stackrel{W}{\Longrightarrow} W(1) \tag{20}$$

Proof. The result follows since $\lim_{T\longrightarrow\infty}\sum_{t=1}^{[T\tau]}\phi^{2t} = \frac{\phi^2}{1-\phi^2}$. ■

Proof of Theorem 1

Since

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} z_t &= \frac{1}{T}\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} z_{t_k} \\
&= \sum_{k=1}^{K}\frac{T(k)-T(k-1)}{T}\frac{1}{T(k)-T(k-1)}\sum_{t_k=T(k-1)+1}^{T(k)} z_{t_k} \\
&\stackrel{m.s.}{\Longrightarrow}\sum_{k=1}^{K}\tau_k\frac{c_k}{1-\phi}\stackrel{a.s.}{\Longrightarrow} 0
\end{aligned}$$

by the weak law of large number for covariance-stationary processes, the sample mean of z_t is asymptotically zero. Therefore the LSE of φ is

$$\hat{\varphi} = \frac{\sum_{t=1}^{T} z_t z_{t-1}}{\sum_{t=1}^{T} z_{t-1}^2}. \tag{21}$$

Note

$$\sum_{t=1}^{T} z_t z_{t-1} = \sum_{t=1}^{T(1)} z_t z_{t-1} + ... + \sum_{t=T(k)+1}^{T(k)} z_t z_{t-1} + ... + \sum_{t=T(K)+1}^{T(K)} z_t z_{t-1},$$

and

$$\sum_{t=1}^{T} z_{t-1}^2 = \sum_{t=1}^{T(1)} z_{t-1}^2 + ... + \sum_{t=T(k)+1}^{T(k)} z_{t-1}^2 + ... + \sum_{t=T(K)+1}^{T(K)} z_{t-1}^2,$$

where $T(k) = \sum_{i=1}^{k}[T\tau_i]$ with $T(0) = 0$. We then obtain

$$\begin{aligned}
\hat{\varphi} &= \frac{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} z_{t_k} z_{t_k-1}}{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} z_{t_k-1}^2} \qquad (22)\\
&= \frac{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} (c_k + \phi z_{t_k-1} + \xi_{t_k}) z_{t_k-1}}{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} z_{t_k-1}^2}\\
&= \frac{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} c_k z_{t_k-1} + \sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} \phi z_{t_k-1}^2 + \sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} \xi_{t_k} z_{t_k-1}}{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} z_{t_k-1}^2}\\
&= \phi + \frac{\begin{array}{c}\sum_{k=1}^{K} c_k \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1} + \sum_{k=1}^{K} c_k z_{T(k-1)}\\ + \sum_{k=1}^{K}\sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} z_{t_k-1} + \sum_{k=1}^{K} \xi_{T(k-1)+1} z_{T(k-1)}\end{array}}{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1}^2 + \sum_{k=1}^{K} z_{T(k-1)}^2}
\end{aligned}$$

with $\xi_0 = 0$ and $z_0 = 0$. Note that z_{t_k-1} can be rewritten as

$$z_{t_k-1} = \frac{c_k}{1-\phi} + \phi^{-T(k-1)-1}\left(\phi^{t_k-1}\tilde{z}_{T(k-1)+1} + \sum_{j=T(k-1)+1}^{t_k-2} \phi^j \xi_{t_k-j+T(k-1)}\right), \tag{23}$$

where $\tilde{z}_{T(k-1)+1} = z_{T(k-1)+1} - \frac{c_k}{1-\phi}$ and $\sum_{j=T(k-1)-1}^{T(k-1)} \phi^j \xi_{t_k-j+T(k-1)} = 0$. We now derive asymptotic properties of each individual elements in equation (22).

1. Let A_1^* defined as follows;

$$\begin{aligned} A_1^* &\equiv \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} z_{t_k-1} \\ &= \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} \left\{ \frac{c_k}{1-\phi} + \phi^{-T(k-1)-1} \left(\phi^{t_k-1} \tilde{z}_{T(k-1)+1} + \sum_{j=T(k-1)+1}^{t_k-2} \phi^j \xi_{t_k-j+T(k-1)} \right) \right\}. \end{aligned}$$

- The first component of A_1^*: Note that since ξ_{t_k} is an ergodic MDS we have

$$\begin{aligned} T^{-1/2} \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} &= T^{-1/2} \sum_{t_k=1}^{T(k)-T(k-1)-1} \xi_{t_k+T(k-1)+1} \\ &\stackrel{d}{=} T^{-1/2} \sum_{t_k=1}^{T(k)-T(k-1)-1} \xi_{t_k} \\ &= T^{-1/2} \sum_{t_k=1}^{[T\tau_k]-1} \Delta S_{2t_k}, \end{aligned} \tag{24}$$

where $S_{2t_k} = \sum_{j=1}^{t_k} \xi_j$ and Δ is the difference operator. Since

$$T^{-1/2} S_{2[T\tau_k]} \stackrel{W}{\Longrightarrow} \sigma W_{2k}(\tau_k) \tag{25}$$

by Donsker's IP (see Theorem 14.1 of Billingsley (1999)), we have the following limit

$$T^{-1/2} \frac{c_k}{1-\phi} \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} \stackrel{W}{\Longrightarrow} \frac{c_k \sigma}{1-\phi} W_{2k}(\tau_k), \tag{26}$$

where $\tau_k = \lim_{T \longrightarrow \infty} \frac{[T\tau_k]-1}{T}$ and W_{2k} is a Brownian motion.

- The second component of A_1^*: With a similar argument above we have

$$\begin{aligned}
\phi^{-T(k-1)} \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} \phi^{t_k-1} &= \sum_{t_k=1}^{T(k)-T(k-1)-1} \phi^{t_k} \xi_{t_k+T(k-1)+1} \\
&\stackrel{d}{=} \sum_{t_k=1}^{T(k)-T(k-1)-1} \phi^{t_k} \xi_{t_k} \\
&= \sum_{t_k=1}^{[T\tau_k]-1} \phi^{t_k} \xi_{t_k} \\
&= S_{1[T\tau_k]-1},
\end{aligned}$$

where $S_{1[T\tau_k]-1} = \sum_{t_k=1}^{[T\tau_k]-1} \phi^{t_k} \xi_{t_k}$. Using Corollary 1 we have

$$S_{1[Ts]-1} \stackrel{W}{\Longrightarrow} \sigma\sqrt{C} W_{1k}(1), \tag{27}$$

where $C = \frac{\phi^2}{1-\phi^2}$ and W_{1k} is the Brownian motion. Therefore we obtain

$$z_{T(k-1)+1} \phi^{-T(k-1)} \sum_{T(k-1)+2}^{T(k)} \xi_{t_k} \phi^{t_k-1} \stackrel{W}{\Longrightarrow} z_\infty \sigma\sqrt{C} W_{1k}(1), \tag{28}$$

where z_∞ denote the initial changing point of the process z_t by Assumption 4.

- The third component of A_1^*: We have

$$
\begin{aligned}
T^{-1/2}\phi^{-T(k-1)} & \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} \sum_{j=T(k-1)+1}^{t_k-2} \phi^j \xi_{t_k-j+T(k-1)} \\
& = T^{-1/2} \sum_{t_k=1}^{T(k)-T(k-1)-1} \xi_{t_k+T(k-1)+1} \sum_{j=1}^{t_k-1} \phi^j \xi_{t_k-j} \\
& \overset{d}{=} T^{-1/2} \sum_{t_k=1}^{T(k)-T(k-1)-1} \xi_{t_k} \sum_{j=1}^{t_k-1} \phi^j \xi_{t_k-j} \\
& = T^{-1/2} \sum_{t_k=1}^{T(k)-T(k-1)-1} \Delta S_{2t_k} S_{1t_k-1} \\
& = \sum_{t_k=1}^{[T\tau_k]-1} \int_{\frac{t_k}{T}}^{\frac{t_k+1}{T}} \frac{\Delta S_{2[Ts]}}{\sqrt{T}} S_{1[Ts]-1}.
\end{aligned} \tag{29}
$$

Since S_{1t_k} and S_{2t_k} are independent, an application of equation (27) and the continuous mapping theorem yields the following limit:

$$
T^{-1/2}\phi^{-T(k-1)-1} \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} \sum_{j=T(k-1)+1}^{t_k-2} \phi^j \xi_{t_k-j+T(k-1)} \overset{W}{\Longrightarrow} \frac{\sigma^2\sqrt{C}}{\phi} W_{1k}(1) W_{2k}(\tau_k).
$$

- Therefore the asymptotic property of A_1 is

$$
\begin{aligned}
T^{-1/2}A_1 &= T^{-1/2} \sum_{k=1}^{K} \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} z_{t_k-1} \\
&\overset{W}{\Longrightarrow} \sum_{k=1}^{K} \frac{c_k \sigma}{1-\phi} W_{2k}(\tau_k) + \sum_{k=1}^{K} \frac{\sigma^2\sqrt{C}}{\phi} W_{1k}(1) W_{2k}(\tau_k) + o_p(1)
\end{aligned} \tag{30}
$$

2. Using Equation (23), we have

$$\sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1} = \frac{c_k}{1-\phi}([T\tau_k]-1) + \phi^{-T(k-1)-1}\tilde{z}_{T(k-1)+1} \sum_{T(k-1)+2}^{T(k)} \phi^{t_k-1} + \phi^{-T(k-1)-1} \sum_{t_k=T(k-1)+2}^{T(k)} \sum_{j=T(k-1)+1}^{t_k-2} \phi^j \xi_{t_k-j+T(k-1)}.$$

Note that

$$T^{-1}\phi^{-T(k-1)-1}\tilde{z}_{T(k-1)+1} \sum_{T(k-1)+2}^{T(k)} \phi^{t_k-1} \Longrightarrow T^{-1} \sum_{t_k=1}^{[T\tau_k]-1} \phi^{t_k} = o_p(1).$$

In addition

$$\begin{aligned} T^{-1}\phi^{-T(k-1)-1} \sum_{t_k=T(k-1)+2}^{T(k)} \sum_{j=T(k-1)+1}^{t_k-2} \phi^j \xi_{t_k-j+T(k-1)} &= T^{-1}\phi^{-1} \sum_{t_k=1}^{T(k)-T(k-1)-1} \sum_{j=1}^{t_k-1} \phi^j \xi_{t_k-j} \\ &\stackrel{d}{=} T^{-1}\phi^{-1} \sum_{t_k=1}^{[T\tau_k]-1} S_{1t_k-1} \\ &= \phi^{-1} \sum_{t_k=1}^{[T\tau_k]-1} \int_{\frac{t_k}{T}}^{\frac{t_k+1}{T}} S_{1[Ts]-1} ds \\ &\stackrel{W}{\Longrightarrow} \frac{\sigma\sqrt{C}\tau_k W_{1k}(1)}{\phi} \end{aligned}$$

in view of Corollary 1. Therefore we obtain

$$T^{-1} \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1} \stackrel{W}{\Longrightarrow} \tau_k \frac{c_k}{1-\phi} + \frac{\tau_k\sigma\sqrt{C}}{\phi} W_{1k}(1) + o_p(1), \tag{31}$$

and

$$\begin{aligned} T^{-1}A_2 &= T^{-1}\sum_{k=1}^{K} c_k \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1} \\ &\overset{W}{\Longrightarrow} \sum_{k=1}^{K} \frac{\tau_k c_k^2}{1-\phi} + \sum_{k=1}^{K} \frac{\tau_k c_k \sigma\sqrt{C}}{\phi} W_{1k}(1) + o_p(1). \end{aligned}$$

3. Since

$$z_{T(k-1)} = \frac{c_{k-1}}{1-\phi} + \phi^{-T(k-2)-1}\left(\phi^{T(k-1)}\tilde{z}_{T(k-2)+1} + \sum_{j=T(k-2)+1}^{T(k-1)-1} \phi^j \xi_{T(k-1)+T(k-2)-j+1}\right), \tag{32}$$

we have

$$\begin{aligned} T^{-1/2}A_3 &\equiv T^{-1/2}\sum_{k=1}^{K} \xi_{T(k-1)+1} z_{T(k-1)} \\ &= T^{-1/2}\sum_{k=1}^{K} \xi_{T(k-1)+1}\left(\frac{c_{k-1}}{1-\phi} + \phi^{T(k-1)-T(k-2)-1}\tilde{z}_{T(k-2)+1}\right) \\ &\quad + T^{-1/2}\phi^{-T(k-2)-1}\sum_{k=1}^{K} \xi_{T(k-1)+1} \sum_{j=T(k-2)+1}^{T(k-1)-1} \phi^j \xi_{T(k-1)+T(k-2)-j+1} \\ &\Longrightarrow o_p(1), \end{aligned} \tag{33}$$

using equation (28) and similar arguments above.

4. From equation (32), we have

$$\begin{aligned} \phi^{-T(k-2)-1}\sum_{j=T(k-2)+1}^{T(k-1)-1} \phi^j \xi_{T(k-1)+T(k-2)-j+1} &= \phi^{-1}\sum_{j=1}^{T(k-1)-T(k-2)-1} \phi^j \xi_{T(k-1)-j+1} \\ &\overset{d}{=} \phi^{-1}\sum_{j=1}^{T(k-1)-T(k-2)-1} \phi^j \xi_j \\ &\overset{W}{\Longrightarrow} \frac{\sigma\sqrt{C}W_{1k-1}(1)}{\phi} \end{aligned}$$

in view of Corollary 1. In addition the limit of the second component in (32) is

$$\frac{c_{k-1}}{1-\phi} + \phi^{T(k-1)-T(k-2)-1} z_{T(k-2)+1} \Longrightarrow \frac{c_{k-1}}{1-\phi} + o_p(1).$$

Therefore we have

$$\begin{aligned} A_4 &\equiv \sum_{k=1}^{K} c_k z_{T(k-1)} \\ &\overset{W}{\Longrightarrow} \frac{c_{k-1}}{1-\phi} + \sum_{k=1}^{K} \frac{c_k \sigma \sqrt{C} W_{1k-1}(1)}{\phi} + o_p(1). \end{aligned}$$

5. Let $A_5 = \sum_{k=1}^{K} \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1}^2$. Using similar asymptotic arguments, we also obtain

$$\begin{aligned} T^{-1}A_5 &= T^{-1} \sum_{k=1}^{K} \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2 \\ &\overset{W}{\Longrightarrow} \sum_{k=1}^{K} \tau_k \left(\frac{c_k}{1-\phi}\right)^2 + 2\sum_{k=1}^{K} \frac{c_k \sigma \tau_k \sqrt{C}}{\phi(1-\phi)} W_{1k}(1) + \sum_{k=1}^{K} \frac{\sigma^2 \tau_k C}{\phi^2} W_{1k}^2(1) + o_p(1) \end{aligned} \quad (34)$$

6. Finally we have

$$\begin{aligned} A_6 &\equiv \sum_{k=1}^{K} z_{T(k-1)}^2 && (35) \\ &\overset{W}{\Longrightarrow} \sum_{1}^{K} \frac{c_{k-1}^2}{(1-\phi)^2} + 2\frac{c_{k-1}\sigma\sqrt{C}}{\phi(1-\phi)} \sum_{1}^{K} W_k(1) + \frac{\sigma^2 C}{\phi^2} \sum_{1}^{K} W_k^2(1) + o_p(1). && (36) \end{aligned}$$

Therefore, we arrive at the following expression:

$$\varphi - \phi \overset{W}{\Longrightarrow} \frac{O_p(T^{-1/2}) + \sum_{k=1}^{K} \frac{\tau_k c_k^2}{1-\phi} + \sum_{k=1}^{K} \frac{c_k \tau_k \sigma \sqrt{C}}{\phi} W_k(1)}{O_p(T^{-1}) + \sum_{k=1}^{K} \frac{\tau_k c_k^2}{(1-\phi)^2} + 2\sum_{k=1}^{K} \frac{c_k \tau_k \sigma}{(1-\phi)\sqrt{1-\phi^2}} W_k(1) + \sum_{k=1}^{K} \frac{\tau_k \sigma^2}{1-\phi^2} W_k^2(1)}. \quad (37)$$

Proof of Proposition 1

From equation (7) and Assumption 3, an application of Loeve's SLLN yields

$$\sum_{k=1}^{K} \tau_k c_k W_k(1) = \sum_{k=1}^{K} \tau_k W(\sqrt{c_k}) \overset{a.s}{\Longrightarrow} 0, \tag{38}$$

since $\sum_{k=1}^{K} \tau_k = 1$ where $\tau_k \in (0,1)$ and τ_k and c_k are independent. Therefore

$$\sum_{k=1}^{K} \frac{c_k \tau_k \sigma \sqrt{C}}{\phi} W_k(1) \overset{a.s}{\Longrightarrow} 0, \text{ and}$$
$$\frac{2}{(1-\phi)\sqrt{1-\phi^2}} \sum_{k=1}^{K} c_k \tau_k \sigma W_k(1) \overset{a.s}{\Longrightarrow} 0.$$

In addition, applying Etemadi's SLLN for non-negative random variables we have

$$\begin{aligned} \sum_{k=1}^{K} \frac{\tau_k \sigma^2}{1-\phi^2} W_k^2(1) &= \frac{\sigma^2}{1-\phi^2} \sum_{k=1}^{K} W_k^2(\sqrt{\tau_k}) \\ &\overset{a.s}{\Longrightarrow} \frac{\sigma^2}{1-\phi^2}, \end{aligned} \tag{39}$$

since

$$\sum_{k=1}^{K} W_k^2(\sqrt{\tau_k}) \overset{a.s}{\Longrightarrow} \sum_{k=1}^{K} \tau_k = 1.$$

Furthermore,

$$\lim_{K \longrightarrow \infty} \sum_{k=1}^{K} \tau_k c_k^2 = \lim_{K \longrightarrow \infty} \sum_{k=1}^{K} \tau_k \epsilon_k^2 \overset{a.s}{\Longrightarrow} \sigma_\epsilon^2, \tag{40}$$

since the random component ϵ_t moves around zero with volatility σ_ϵ^2. Therefore equations (38), (39), (40), and (7) together with Assumptions 3 give

$$\begin{aligned}
\lim_{T\longrightarrow\infty,K\longrightarrow\infty}\frac{A}{B} &= \frac{\lim_{T\longrightarrow\infty,K\longrightarrow\infty}\sum_{k=1}^{K}\left\{\frac{\tau_k c_k^2}{1-\phi}+\frac{c_k\tau_k\sigma\sqrt{C}}{\phi}W_k(1)\right\}}{\lim_{T\longrightarrow\infty,K\longrightarrow\infty}\sum_{k=1}^{K}\left\{\frac{\tau_k c_k^2}{(1-\phi)^2}+2\frac{c_k\tau_k\sigma}{(1-\phi)\sqrt{1-\phi^2}}W_k(1)+\frac{\tau_k\sigma^2}{1-\phi^2}W_k^2(1)\right\}}\\
&= \frac{V}{\frac{V}{1-\phi}+\frac{1}{1+\phi}} \qquad (41)\\
&\equiv \frac{A_\infty}{B_\infty}\geq 0,
\end{aligned}$$

where $V=\sigma_\epsilon^2/\sigma^2$, and A_∞ and B_∞ are obviously defined. It is straight forward to show that $\frac{\partial}{\partial V}\left(\frac{A_\infty}{B_\infty}\right)=\frac{1}{1+\phi}\left(\frac{V}{1-\phi}+\frac{1}{1+\phi}\right)^{-2}>0$ and $\frac{\partial}{\partial\phi}\left(\frac{A_\infty}{B_\infty}\right)=-V\left(\frac{V}{1-\phi}+\frac{1}{1+\phi}\right)^{-2}\left(\frac{V}{(1-\phi)^2}-\frac{1}{(1+\phi)^2}\right)$. The second case shows that we have $\frac{\partial}{\partial\phi}\left(\frac{A_\infty}{B_\infty}\right)<0$ when $V>\frac{(1-\phi)^2}{(1+\phi)^2}$.

Proof of Theorem 2

Note that

$$\begin{aligned}
\widehat{\varphi} &= \frac{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} z_{t_k}z_{t_k-1}}{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)} z_{t_k-1}^2}\\
&= \frac{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)}(c_{t_k}+\xi_{t_k})(c_{t_k-1}+\xi_{t_k-1})}{\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)}(c_{t_k-1}+\xi_{t_k-1})^2}\\
&= \frac{\begin{array}{c}\sum_{k=1}^{K}(c_kc_{k-1}+c_k^2([T\tau_k]-1))+\sum_{k=1}^{K}c_k\sum_{t_k=T(k-1)+1}^{T(k)}\xi_{t_k-1}\\ +\sum_{k=1}^{K}(c_{k-1}\xi_{T(k-1)}+c_k\sum_{t_k=T(k-1)+2}^{T(k)}\xi_{t_k})+\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)}\xi_{t_k}\xi_{t_k-1}\end{array}}{\begin{array}{c}\sum_{k=1}^{K}(c_{k-1}^2+c_k^2([T\tau_k]-1))+\sum_{k=1}^{K}\sum_{t_k=T(k-1)+1}^{T(k)}\xi_{t_k-1}^2\\ +2\sum_{k=1}^{K}(c_{k-1}\xi_{T(k-1)}+c_k\sum_{t_k=T(k-1)+2}^{T(k)}\xi_{t_k-1})\end{array}} \qquad (42)
\end{aligned}$$

We have

$$\sum_{T(k-1)+1}^{T(k)} \xi_{t_k}\xi_{t_k-1} = \sum_{t_k=1}^{T(k)-T(k-1)} \xi_{t_k+T(k-1)}\xi_{t_k+T(k-1)-1} \stackrel{d}{=} \sum_{t_k=1}^{T(k)-T(k-1)} \xi_{t_k}\xi_{t_k-1}.$$

Since $E(\xi_t\xi_{t-1}|\mathcal{F}_{t-1}) = \xi_{t-1}E(\xi_t|\mathcal{F}_{t-1}) = 0$ thus $\xi_t\xi_{t-1}$ is a sequence of martingale differences. An application of SLLN for martingale difference sequences (White, 1984) yields

$$\frac{1}{[T\tau_k]}\sum_{t=1}^{[T\tau_k]} \xi_t\xi_{t-1} \xrightarrow{a.s} 0, \text{ thus} \tag{43}$$

$$\frac{1}{\max[\{[T\tau_k]\}_{k=1}^K]}\sum_{t=1}^{[T\tau_k]} \xi_t\xi_{t-1} \xrightarrow{a.s} 0. \tag{44}$$

- Since ξ_t is an ergodic MDS, we have $\sum_{t_k=T(k-1)+1}^{T(k)} \xi_{t_k} \stackrel{d}{=} \sum_{t_k=1}^{[T\tau_k]} \xi_{t_k}$. We apply SLLN for MDS to obtain

$$\frac{1}{[T\tau_k]}\sum_{t_k=1}^{[T\tau_k]} \xi_{t_k} \xrightarrow{a.s} 0, \text{thus},$$

$$\frac{1}{\max[\{[T\tau_k]\}_{k=1}^K]}\sum_{t_k=1}^{[T\tau_k]} \xi_{t_k} \xrightarrow{a.s} 0.$$

- Since ξ_t is an ergodic MDS, we obtain $\sum_{T(k-1)+1}^{T(k)} \xi_{t_k-1}^2 \stackrel{d}{=} \sum_{t_k=1}^{[T\tau_k]} \xi_{t_k}^2$. By applying Etemadi's SLLN for non-negative $\mathcal{L}_2$ random variables we get

$$\frac{1}{\max[\{[T\tau_k]\}_{k=1}^K]}\sum_{t_k=1}^{[T\tau_k]} \xi_{t_k}^2 \xrightarrow{a.s} \frac{\sigma^2\tau_k}{\max[\{\tau_k\}_{k=1}^K]}. \tag{45}$$

By dividing both the numerator and denominator of equation (42) by $\max[\{\tau_k\}_{k=1}^K]$

together with equations (43) to (45) we have

$$\widehat{\varphi} \xrightarrow{a.s} \frac{\sum_{k=1}^{K} \tau_k c_k^2}{\sum_{k=1}^{K} \tau_k c_k^2 + \sigma^2} \tag{46}$$

Since c_k is a sample path of the occasionally changing process c_t generated by ϵ_t, an application of SLLN for MDS yields

$$\sum_{k=1}^{K} \tau_k c_k^2 \xrightarrow{a.s} \sigma_\epsilon^2, \text{ as } K \to \infty. \tag{47}$$

Therefore, we obtain

$$\widehat{\varphi} \xrightarrow{a.s} \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma^2} = \frac{V}{V+1}. \tag{48}$$

Proof of Theorem 3

The LSE of φ is given by

$$\hat{\varphi} = \frac{\sum_{k=1}^{K} z_{T(k-1)+1} z_{T(k-1)} + \sum_{k=1}^{K} \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k} z_{t_k-1}}{\sum_{k=1}^{K} z_{T(k-1)}^2 + \sum_{k=1}^{K} \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1}^2}. \tag{49}$$

Since

$$\sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k} z_{t_k-1} = c_k \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1} + \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1}^2 + \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} z_{t_k-1},$$

we obtain

$$\hat{\varphi} - 1 = \frac{\begin{array}{c}\sum_{k=1}^{K} z_{T(k-1)+1} z_{T(k-1)} + \sum_{k=1}^{K} c_k \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1} \\ + \sum_{k=1}^{K} \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} z_{t_k-1} - \sum_{k=1}^{K} z_{T(k-1)}^2\end{array}}{\sum_{k=1}^{K} z_{T(k-1)}^2 + \sum_{1}^{K} \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2}, \tag{50}$$

or

$$\hat{\varphi} - \frac{\sum_{k=1}^{K}\left\{\sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2 + c_k \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}\right\}}{\sum_{k=1}^{K}\left\{z_{T(k-1)}^2 + \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2\right\}} = \frac{\sum_{k=1}^{K}\left\{z_{T(k-1)+1} z_{T(k-1)} + \sum_{T(k-1)+2}^{T(k)} \xi_{t_k} z_{t_k-1}\right\}}{\sum_{k=1}^{K}\left\{z_{T(k-1)}^2 + \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2\right\}} \quad (51)$$

Let's derive the limiting distributions of the terms in equation (50).

1. Using

$$z_{t_k-1} = z_{T(k-1)+1} + (t_k - T(k-1) - 2)c_k + \sum_{j=T(k-1)+2}^{t_k-1} \xi_j,$$

we define B_1 as follows

$$\begin{aligned} B_1 &\equiv \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2 \\ &= \sum_{t_k=T(k-1)+2}^{T(k)} \left\{ \begin{array}{c} z_{T(k-1)+1}^2 + c_k^2 (t_k - T(k-1) - 2)^2 + \left(\sum_{j=T(k-1)+2}^{t_k-1} \xi_j\right)^2 \\ +2c_k z_{T(k-1)+1}(t_k - T(k-1) - 2) + 2z_{T(k-1)+1} \sum_{j=T(k-1)+2}^{t_k-1} \xi_j \\ +2c_k (t_k - T(k-1) - 2) \sum_{j=T(k-1)+2}^{t_k-1} \xi_j \end{array} \right\} \end{aligned} \quad (52)$$

- First we have

$$T^{-1} \sum_{t_k=T(k-1)+2}^{T(k)} z_{T(k-1)+1}^2 = z_{T(k-1)+1}^2 T^{-1}(T(k) - T(k-1) - 1) \Longrightarrow z_\infty \tau_k, \quad (53)$$

- and

$$\begin{aligned} T^{-3} \sum_{t_k=T(k-1)+2}^{T(k)} (t_k - T(k-1) - 2)^2 &= T^{-3} \sum_{t_k=0}^{T(k)-T(k-1)-2} t_k^2 \\ &\Longrightarrow \frac{(\tau_k)^3}{3}. \end{aligned} \quad (54)$$

- The third component becomes

$$
\begin{aligned}
T^{-2}\sum_{t_k=T(k-1)+2}^{T(k)}\left(\sum_{j=T(k-1)+2}^{t_k-1}\xi_j\right)^2 &= T^{-2}\sum_{t_k=1}^{T(k)-T(k-1)-1}\left(\sum_{j=1}^{t_k-1}\xi_{j+T(k-1)+1}\right)^2 \\
&= \sum_{t_k=1}^{[T\tau_k]-1}\int_{\frac{t_k}{T}}^{\frac{t_k+1}{T}}\frac{\left(S_{2[Ts]-1}\right)^2}{T}\frac{1}{T} \\
&\stackrel{W}{\Longrightarrow} \sigma^2\int_0^{\tau_k} W_k^2(s)ds, \qquad (55)
\end{aligned}
$$

using Donsker's IP and the fact that ξ_t is an ergodic MDS.

- Using a similar method as in equation (54), we also have

$$
\begin{aligned}
c_k z_{T(k-1)+1}T^{-2}\sum_{t_k=T(k-1)+2}^{T(k)}(t_k - T(k-1) - 2) &= c_k z_{T(k-1)+1}T^{-2}\sum_{t_k=0}^{T(k)-T(k-1)-2} t_k \\
&\Longrightarrow c_k z_\infty \frac{(\tau_k)^2}{2}, \qquad (56)
\end{aligned}
$$

by the same argument as (54) and

$$
\begin{aligned}
T^{-3/2}z_{T(k-1)+1}\sum_{t_k=T(k-1)+2}^{T(k)}\sum_{j=T(k-1)+2}^{t_k-1}\xi_j &\stackrel{d}{=} z_{T(k-1)+1}T^{-3/2}\sum_{t_k=1}^{T(k)-T(k-1)-1}\sum_{j=1}^{t_k-1}\xi_j \\
&= z_{T(k-1)+1}\sum_{t_k=1}^{[T\tau_k]-1}\int_{\frac{t_k}{T}}^{\frac{t_k+1}{T}}\frac{S_{2t_k-1}}{\sqrt{T}}\frac{1}{T} \\
&\stackrel{W}{\Longrightarrow} z_\infty\sigma\int_0^{\tau_k} W_k(s)ds, \qquad (57)
\end{aligned}
$$

by the same argument as (55).

- Finally

$$
\begin{aligned}
T^{-5/2} \sum_{t_k=T(k-1)+2}^{T(k)} (t_k - T(k-1) - 2) \sum_{j=T(k-1)+2}^{t_k-1} \xi_j &\overset{d}{=} T^{-5/2} \sum_{t_k=1}^{T(k)-T(k-1)-1} t_k \sum_{j=1}^{t_k-1} \xi_j \\
&= \sum_{t_k=1}^{[T\tau_k]-1} \int_{\frac{t_k}{T}}^{\frac{t_k+1}{T}} \frac{[Ts]}{T} \frac{S_{2[Ts]-1}}{\sqrt{T}} \frac{1}{T} \\
&\overset{W}{\Longrightarrow} \sigma \int_0^{\tau_k} sW_{2k}(s)ds,
\end{aligned}
$$

by the same argument as (55)

- Therefore the asymptotic property of B_1 is

$$
T^{-3} B_1 \overset{W}{\Longrightarrow} \frac{c_k^2 (\tau_k)^3}{3} + O_p(T^{-1/2}) \tag{58}
$$

2. We next define B_2 and investigate the asymptotic property of B_2.

$$
\begin{aligned}
B_2 &\equiv \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} z_{t_k-1} \\
&= \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} \left[z_{T(k-1)+1} + c_k(t_k - T(k-1) - 2) + \sum_{j=T(k-1)+2}^{t_k-1} \xi_j \right], \qquad (59)
\end{aligned}
$$

where

$$
\begin{aligned}
T^{-1} \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} \sum_{j=T(k-1)+2}^{t_k-1} \xi_j &= T^{-1} \sum_{t_k=1}^{T(k)-T(k-1)-1} \xi_{t_k+T(k-1)+2} \sum_{j=1}^{t_k-1} \xi_{j+T(k-1)+1} \\
&\overset{d}{=} T^{-1} \sum_{t_k=1}^{T(k)-T(k-1)-1} \xi_{t_k} \sum_{j=1}^{t_k-1} \xi_j \\
&= T^{-1} \sum_{t_k=1}^{T(k)-T(k-1)-1} \Delta S_{2t_k} S_{2t_k-1} \\
&= \sum_{t_k=1}^{T(k)-T(k-1)-1} \int_{\frac{t_k}{T}}^{\frac{t_k+1}{T}} \frac{S_{2[Ts]}}{\sqrt{T}} \Delta \frac{S_{2[Ts]}}{\sqrt{T}} \\
&\overset{W}{\Longrightarrow} \sigma^2 \int_0^{\tau_k} W_{2k}(s) dW_{2k}(s). \qquad (60)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
T^{-3/2} c_k \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k}(t_k - T(k-1) - 2) &\overset{d}{=} T^{-3/2} c_k \sum_{t_k=1}^{T(k)-T(k-1)-1} t_k \xi_{t_k} \\
&= c_k \sum_{t_k=1}^{T(k)-T(k-1)-1} \int_{\frac{t_k}{T}}^{\frac{t_k+1}{T}} \frac{[Ts]}{T} \frac{\Delta S_{2[Ts]+1}}{\sqrt{T}} \\
&\overset{W}{\Longrightarrow} \sigma c_k \int_0^{\tau_k} s dW_{2k}(s). \qquad (61)
\end{aligned}
$$

Finally

$$
\begin{aligned}
T^{-1/2} \sum_{t_k=T(k-1)+2}^{T(k)} \xi_{t_k} z_{T(k-1)+1} &= z_{T(k-1)+1} T^{-1/2} \sum_{t_k=1}^{[T\tau_k]-1} \xi_{t_k} \\
&\overset{W}{\Longrightarrow} z_\infty \sigma W_{2k}(\tau_k). \qquad (62)
\end{aligned}
$$

Thus,

$$
T^{-3/2} B_2 \overset{W}{\Longrightarrow} \sigma c_k \int_0^{\tau_k} s dW_{2k}(s) + O_p(T^{-1/2}). \qquad (63)
$$

3. Let us define

$$
\begin{aligned}
B_3 &\equiv z^2_{T(k-1)} \\
&= z^2_{T(k-2)+1} + ([T\tau_{k-1}]-1)^2 c^2_{k-1} + \left(\sum_{j=T(k-2)+2}^{T(k-1)} \xi_j\right)^2 + 2z_{T(k-2)+1}c_{k-1}([T\tau_{k-1}]-1) \\
&\quad +2z_{T(k-2)+1}\sum_{j=T(k-2)+2}^{T(k-1)} \xi_j + 2c_{k-1}([T\tau_{k-1}]-1)\sum_{j=T(k-2)+2}^{T(k-1)} \xi_j,
\end{aligned}
$$

where

$$
\begin{aligned}
&T^{-2}c^2_{k-1}([T\tau_{k-1}]-1)^2 \Longrightarrow c^2_{k-1}(\tau_{k-1})^2, \\
&T^{-1}\left(\sum_{j=T(k-2)+2}^{T(k-1)} \xi_j\right)^2 \overset{W}{\Longrightarrow} \sigma^2 W^2_{2k-1}(\tau_{k-1}), \\
&T^{-1}z_{T(k-2)+1}c_{k-1}([T\tau_{k-1}]-1) \Longrightarrow z_\infty c_{k-1}\tau_{k-1}, \\
&T^{-1/2}z_{T(k-2)+1}\sum_{j=T(k-2)+2}^{T(k-1)} \xi_j \overset{W}{\Longrightarrow} z_\infty \sigma W_{k-1}(\tau_{k-1}), \\
&T^{-3/2}c_{k-1}([T\tau_{k-1}]-1)\sum_{j=T(k-2)+2}^{T(k-1)} \xi_j \overset{W}{\Longrightarrow} \tau_{k-1}c_{k-1}\sigma W_{k-1}(\tau_{k-1}).
\end{aligned}
$$

Thus

$$
T^{-2}B_3 = T^{-2}z^2_{T(k-1)} \overset{W}{\Longrightarrow} c^2_{k-1}(\tau_{k-1})^2 + O_p(T^{-1/2}). \tag{64}
$$

4. Let

$$
\begin{aligned}
B_4 = \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1} &= \sum_{t_k=T(k-1)+2}^{T(k)} z_{T(k-1)+1} + \sum_{t_k=T(k-1)+2}^{T(k)} (t_k - T(k-1)-2)c_k \\
&\quad + \sum_{t_k=T(k-1)+2}^{T(k)} \sum_{j=T(k-1)+2}^{t_k-1} \xi_j \\
&\overset{d}{=} z_{T(k-1)+1}([T\tau_{k-1}]-1) + c_k \sum_{t_k=0}^{T(k)-T(k-1)-2} t_k + \sum_{t_k=1}^{T(k)-T(k-1)-1} \sum_{j=1}^{t_k-1} \xi_j,
\end{aligned}
$$

since ξ_t is an ergodic MDS. We have

$$
\begin{aligned}
&T^{-1} z_{T(k-1)+1}([T\tau_{k-1}] - 1) \Longrightarrow z_\infty \tau_k, \\
&T^{-2} c_k \sum_{t_k=0}^{T(k)-T(k-1)-2} t_k \Longrightarrow \frac{c_k(\tau_k)^2}{2} \text{ in view of equation (56),} \\
&T^{-3/2} \sum_{t_k=1}^{T(k)-T(k-1)-1} \sum_{j=1}^{t_k-1} \xi_j \overset{W}{\Longrightarrow} \sigma_k \int_0^{\tau_k} W_k(s)ds \text{ in view of equation (57).}
\end{aligned}
$$

Therefore, we obtain

$$
T^{-2} c_k B_4 = T^{-2} c_k \sum_{t_k=T(k-1)+2}^{T(k)} z_{t_k-1} \Longrightarrow \frac{c_k^2 \tau_k^2}{2} + O_p(T^{-1/2}).
$$

5. Finally since $z_{T(k-1)} = z_{T(k-2)+1} + (T(k-1) - T(k-2) - 1)c_{k-1} + \sum_{j=T(k-2)+2}^{T(k-1)} \xi_j$, we have

$$
\begin{aligned}
B_5 &\equiv z_{T(k-1)+1} z_{T(k-1)} \\
&= z_{T(k-1)+1} \left[z_{T(k-2)+1} + c_{k-1}(T(k-1) - T(k-2) - 1) + \sum_{j=T(k-2)+2}^{T(k-1)} \xi_j \right],
\end{aligned}
$$

where

$$
\begin{aligned}
&z_{T(k-1)+1} z_{T(k-2)+1} \Longrightarrow (z_\infty)^2, \\
&T^{-1} z_{T(k-1)+1} c_{k-1}(T(k-1) - T(k-2) - 1) \Longrightarrow z_\infty c_{k-1} \tau_{k-1}, \\
&T^{-1/2} z_{T(k-1)+1} \sum_{j=T(k-2)+2}^{T(k-1)} \xi_j \overset{d}{=} T^{-1/2} z_{T(k-1)+1} \sum_{j=1}^{T(k-1)-T(k-2)-1} \xi_j \overset{W}{\Longrightarrow} z_\infty \sigma W_k(\tau_{k-1}).
\end{aligned}
$$

Therefore

$$
T^{-1} B_5 = T^{-1} z_{T(k-1)+1} z_{T(k-1)} \Longrightarrow z_\infty c_{k-1} \tau_{k-1} + O_p(T^{-1/2}). \qquad (65)
$$

Summarising, we have

$$T^{-3}B_1 \Longrightarrow \frac{c_k^2(\tau_k)^3}{3} + O_p(T^{-1/2}) \tag{66}$$

$$T^{-3/2}B_2 \overset{W}{\Longrightarrow} \sigma c_k \int_0^{\tau_k} s dW_{2k}(s) + O_p(T^{-1/2}). \tag{67}$$

$$T^{-2}B_3 \overset{W}{\Longrightarrow} c_{k-1}^2(\tau_{k-1})^2 + O_p(T^{-1/2}). \tag{68}$$

$$T^{-2}c_k B_4 \Longrightarrow \frac{c_k^2(\tau_k)^2}{2} + O_p(T^{-1/2}). \tag{69}$$

$$T^{-1}B_5 \overset{W}{\Longrightarrow} c_{k-1}z_\infty^* \tau_{k-1} + O_p(T^{-1/2}) \tag{70}$$

Equations (66) to (70) yield the following asymptotic properties of $\hat{\varphi}$:

- From Equation 49 we have:

$$\hat{\varphi} - 1 = o_p(1)$$

- On the other hand from equation (51) we have:

$$T^{3/2}(\hat{\varphi} - 1) \overset{W}{\Longrightarrow} \frac{\sigma \sum_{k=1}^K c_k \int_0^{\tau_k} s dW_{2k}(s) + O_p(T^{-1})}{\frac{\sum_{k=1}^K c_k^2(\tau_k)^3}{3} + O_p(T^{-1/2})},$$

  in view of

$$\lim_{T \longrightarrow \infty} \frac{\sum_{k=1}^K \left\{ \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2 + c_k \sum_{T(k-1)+2}^{T(k)} z_{t_k-1} \right\}}{\sum_{k=1}^K \left\{ z_{T(k-1)}^2 + \sum_{T(k-1)+2}^{T(k)} z_{t_k-1}^2 \right\}} = \lim_{T \to \infty} \gamma(T) \left(= \frac{\frac{c_k^2(\tau_k)^3}{3}}{\frac{c_k^2(\tau_k)^3}{3}} = 1 \right).$$

- Finally from equation (50) we have :

$$T(\hat{\varphi} - 1) \overset{W}{\Longrightarrow} \frac{\sum_1^K \frac{c_k^2(\tau_k)^2}{2} - \sum_1^K c_{k-1}^2(\tau_{k-1})^2 + O_p(T^{-1/2})}{\sum_1^K \frac{c_k^2(\tau_k)^3}{3} + O_p(T^{-1})}.$$

Proof of Proposition 2

Assumptions 1 and 3 and an application of SLLN for ergodic MDS give:

$$\lim_{K\longrightarrow\infty}\frac{1}{K}\sum_{k=1}^{K}c_k^2(\tau_k)^2=\lim_{K\longrightarrow\infty}\frac{1}{K}\sum_{k=1}^{K}c_{k-1}^2(\tau_{k-1})^2=E[\epsilon_k^2\tilde{\tau}_k^2]=\sigma_\epsilon^2\sigma_\tau^2(\infty),$$

$$\lim_{K\longrightarrow\infty}\frac{1}{K}\sum_{k=1}^{K}c_k^2(\tau_k)^3=E[\epsilon_k^2\tilde{\tau}_k^3]=\sigma_\epsilon^2\lambda_\tau(\infty).$$

Proposition 2 follows.

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Table 1 Percentage Points of the Distributions of $\varphi-\phi$ for Various Number of Breaks, Break Sizes, and AR Parameters In the Presence of Structural Breaks in Mean

AR(1) series are generated for the sample sizes of T=100, 200, 500, 1000, and 3000. For the numbers of breaks, we use 4, 9, 49, and 99. The error term in the AR process follows standard normal, N(0,1), while the sizes of structural breaks are drawn from three different normal distributions, i.e., $N(0,0.1^2)$, N(0,1), and $N(0,10^2)$, which have average break sizes of 0.1, 1, and 10, relative to the AR error term. For the values of ϕ, we take 1, 0.9, 0.5, 0, and -0.5. For the asymptotic distributions we generate Brownian motions with 10000 i.i.d. standard normal variates. We repeat the procedure 10000 times to obtain the sample LS estimates and asymptotic distributions.

A. The Effects of Structural Breaks in Mean for Various Number of Breaks, Break Sizes, and Number of Observations

AR Parameter	Number of Breaks	Break Size	Total Number of Observations	Percentage Points of the Distributions								
				1%	2.50%	5%	10%	50%	90%	95%	97.50%	99%
0.5	4	10	100	0.321	0.374	0.407	0.434	0.473	0.488	0.491	0.492	0.494
			200	0.403	0.432	0.449	0.463	0.485	0.493	0.495	0.496	0.496
			500	0.434	0.456	0.468	0.479	0.492	0.496	0.497	0.498	0.498
			1000	0.443	0.464	0.475	0.484	0.495	0.498	0.498	0.498	0.499
			Limit	0.439	0.455	0.465	0.474	0.498	0.522	0.532	0.542	0.556
		1	100	-0.055	-0.002	0.043	0.099	0.292	0.404	0.422	0.435	0.447
			200	-0.014	0.028	0.075	0.128	0.309	0.413	0.429	0.441	0.452
			500	0.028	0.061	0.095	0.140	0.313	0.416	0.433	0.445	0.456
			1000	0.033	0.064	0.099	0.147	0.314	0.416	0.434	0.445	0.456
			Limit	-0.057	0.063	0.131	0.195	0.374	0.556	0.625	0.702	0.819
		0.1	100	-0.224	-0.184	-0.155	-0.121	-0.001	0.104	0.131	0.152	0.177
			200	-0.151	-0.123	-0.104	-0.079	0.004	0.081	0.101	0.118	0.135
			500	-0.090	-0.073	-0.059	-0.043	0.009	0.059	0.073	0.085	0.098
			1000	-0.058	-0.048	-0.038	-0.028	0.010	0.048	0.059	0.068	0.077
			Limit	-0.153	-0.111	-0.082	-0.055	0.015	0.085	0.111	0.142	0.184
0.5	49	10	100	0.135	0.164	0.186	0.211	0.290	0.351	0.366	0.377	0.389
			200	0.333	0.347	0.357	0.368	0.404	0.430	0.437	0.442	0.448
			500	0.435	0.441	0.445	0.449	0.462	0.472	0.475	0.477	0.479
			1000	0.467	0.470	0.472	0.474	0.481	0.486	0.487	0.488	0.489
			Limit	0.485	0.487	0.489	0.491	0.498	0.506	0.508	0.509	0.511
		1	100	-0.025	0.011	0.041	0.073	0.175	0.257	0.277	0.294	0.311
			200	0.137	0.164	0.183	0.206	0.273	0.327	0.340	0.351	0.363
			500	0.233	0.252	0.266	0.281	0.331	0.371	0.380	0.388	0.396
			1000	0.268	0.282	0.294	0.307	0.351	0.386	0.395	0.402	0.409
			Limit	0.268	0.288	0.304	0.320	0.375	0.429	0.446	0.459	0.477
		0.1	100	-0.229	-0.190	-0.158	-0.121	-0.005	0.099	0.124	0.145	0.169
			200	-0.151	-0.125	-0.103	-0.078	0.002	0.078	0.099	0.115	0.132
			500	-0.084	-0.069	-0.056	-0.042	0.010	0.058	0.070	0.082	0.096
			1000	-0.055	-0.045	-0.035	-0.024	0.012	0.047	0.056	0.064	0.074
			Limit	-0.026	-0.019	-0.013	-0.006	0.015	0.036	0.042	0.048	0.055
0.5	99	10	200	0.191	0.210	0.225	0.241	0.294	0.339	0.350	0.359	0.370
			500	0.389	0.396	0.401	0.407	0.425	0.441	0.445	0.448	0.452
			1000	0.445	0.448	0.451	0.454	0.463	0.471	0.472	0.474	0.475
			Limit	0.489	0.490	0.492	0.493	0.498	0.504	0.505	0.506	0.508
		1	200	0.049	0.072	0.091	0.113	0.182	0.241	0.256	0.270	0.283
			500	0.215	0.230	0.242	0.255	0.296	0.332	0.341	0.349	0.358
			1000	0.269	0.280	0.290	0.301	0.335	0.363	0.370	0.376	0.383
			Limit	0.304	0.316	0.324	0.336	0.375	0.414	0.425	0.434	0.446
		0.1	200	-0.155	-0.127	-0.107	-0.081	0.001	0.073	0.091	0.109	0.130
			500	-0.090	-0.074	-0.058	-0.043	0.008	0.055	0.067	0.079	0.091
			1000	-0.055	-0.044	-0.034	-0.025	0.011	0.044	0.054	0.062	0.071
			Limit	-0.013	-0.009	-0.005	0.000	0.014	0.030	0.034	0.039	0.043

B. The Effects of Structural Breaks in Mean for Different AR Parameters

AR Parameter	Number of Breaks	Break Size	Total Number of Observations	Percentage Points of the Distributions								
				1%	2.50%	5%	10%	50%	90%	95%	97.50%	99%
0.9	9	10	100	0.035	0.050	0.061	0.069	0.088	0.095	0.096	0.097	0.098
			200	0.070	0.076	0.082	0.086	0.094	0.098	0.098	0.099	0.099
			1000	0.094	0.096	0.097	0.097	0.099	0.100	0.100	0.100	0.100
			Limit	0.097	0.098	0.098	0.099	0.100	0.101	0.102	0.102	0.103
		1	100	-0.013	0.008	0.024	0.039	0.073	0.088	0.091	0.092	0.094
			200	0.033	0.047	0.056	0.065	0.083	0.092	0.094	0.095	0.096
			1000	0.065	0.073	0.078	0.082	0.092	0.096	0.097	0.097	0.098
			Limit	0.069	0.074	0.078	0.082	0.095	0.107	0.111	0.115	0.121
		0.1	100	-0.175	-0.138	-0.112	-0.085	-0.011	0.035	0.045	0.052	0.059
			200	-0.095	-0.077	-0.062	-0.046	0.001	0.034	0.042	0.047	0.053
			1000	-0.028	-0.022	-0.016	-0.009	0.011	0.029	0.034	0.038	0.042
			Limit	-0.028	-0.020	-0.012	-0.005	0.016	0.037	0.043	0.051	0.061
0.5	9	10	100	0.355	0.383	0.400	0.418	0.457	0.478	0.482	0.485	0.488
			200	0.425	0.439	0.449	0.458	0.478	0.488	0.490	0.492	0.493
			1000	0.478	0.483	0.486	0.488	0.494	0.497	0.497	0.497	0.498
			Limit	0.464	0.471	0.476	0.482	0.498	0.514	0.520	0.525	0.533
		1	100	0.023	0.080	0.122	0.169	0.306	0.389	0.405	0.417	0.429
			200	0.090	0.129	0.165	0.206	0.327	0.402	0.417	0.428	0.440
			1000	0.134	0.171	0.203	0.238	0.343	0.412	0.427	0.438	0.448
			Limit	0.119	0.174	0.211	0.252	0.375	0.497	0.543	0.584	0.638
		0.1	100	-0.223	-0.186	-0.155	-0.119	0.002	0.106	0.132	0.154	0.177
			200	-0.147	-0.123	-0.100	-0.076	0.006	0.082	0.102	0.119	0.139
			1000	-0.056	-0.045	-0.036	-0.025	0.012	0.048	0.058	0.066	0.076
			Limit	-0.081	-0.064	-0.048	-0.032	0.015	0.063	0.079	0.096	0.115
0	9	10	100	0.641	0.702	0.751	0.791	0.884	0.937	0.948	0.955	0.965
			200	0.786	0.828	0.854	0.879	0.935	0.964	0.970	0.974	0.978
			1000	0.910	0.928	0.942	0.953	0.976	0.987	0.989	0.990	0.992
			Limit	0.876	0.900	0.916	0.935	0.990	1.045	1.062	1.079	1.102
		1	100	0.004	0.059	0.111	0.174	0.397	0.592	0.639	0.682	0.721
			200	0.057	0.108	0.156	0.209	0.412	0.607	0.657	0.695	0.735
			1000	0.104	0.143	0.181	0.230	0.429	0.619	0.662	0.702	0.743
			Limit	-0.098	0.013	0.118	0.214	0.499	0.782	0.878	0.979	1.100
		0.1	100	-0.218	-0.185	-0.156	-0.122	0.009	0.136	0.172	0.205	0.237
			200	-0.156	-0.130	-0.107	-0.082	0.008	0.100	0.125	0.148	0.179
			1000	-0.065	-0.054	-0.043	-0.031	0.009	0.050	0.061	0.072	0.085
			Limit	-0.111	-0.088	-0.068	-0.047	0.010	0.066	0.085	0.102	0.128
-0.5	9	10	100	0.685	0.798	0.885	0.976	1.189	1.323	1.350	1.372	1.394
			200	0.936	1.035	1.101	1.160	1.308	1.392	1.408	1.420	1.432
			1000	1.188	1.246	1.290	1.328	1.413	1.453	1.460	1.465	1.471
			Limit	1.156	1.214	1.264	1.312	1.458	1.597	1.643	1.691	1.753
		1	100	-0.036	0.004	0.042	0.091	0.297	0.548	0.626	0.692	0.759
			200	0.013	0.044	0.077	0.117	0.304	0.536	0.613	0.668	0.752
			1000	0.055	0.083	0.107	0.140	0.303	0.528	0.602	0.661	0.748
			Limit	-0.390	-0.248	-0.128	0.003	0.371	0.735	0.858	0.971	1.114
		0.1	100	-0.163	-0.136	-0.116	-0.089	0.015	0.135	0.171	0.203	0.241
			200	-0.123	-0.104	-0.086	-0.066	0.009	0.093	0.117	0.141	0.169
			1000	-0.055	-0.047	-0.039	-0.029	0.005	0.042	0.053	0.064	0.074
			Limit	-0.098	-0.076	-0.060	-0.043	0.006	0.055	0.072	0.087	0.108

Figure 1 Kernel Densities of $\widehat{\varphi} - \phi$ for Various Number of Samples and Break Sizes In the Presence of Structural Breaks in Mean When $\phi = 0.5$ and $K - 1 = 9$

T	Relative Break Size(σ_ϵ/σ)		
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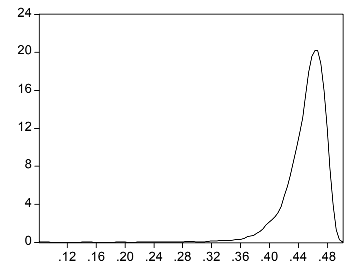
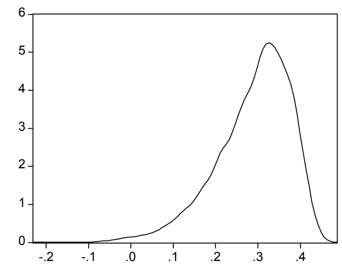
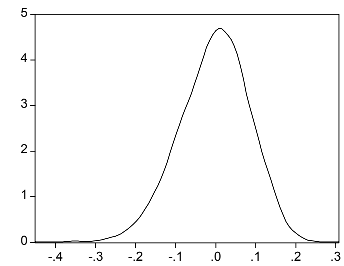
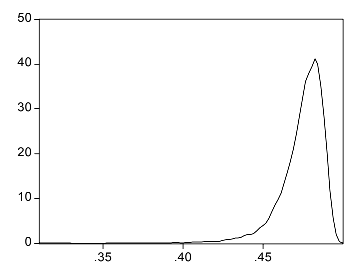
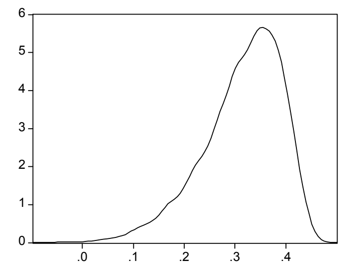
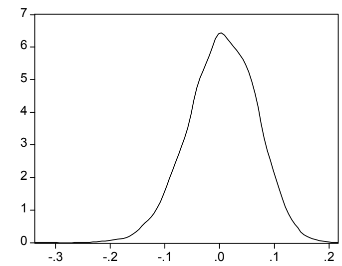
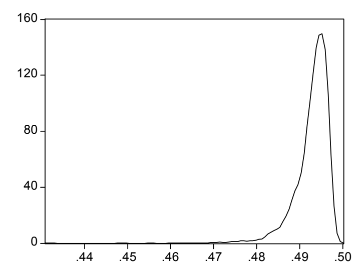
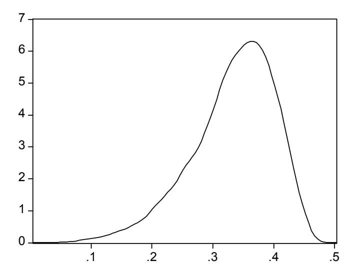
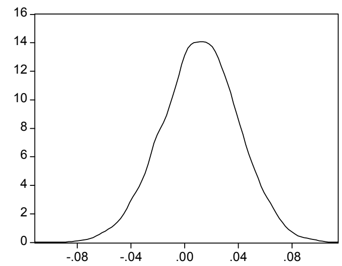
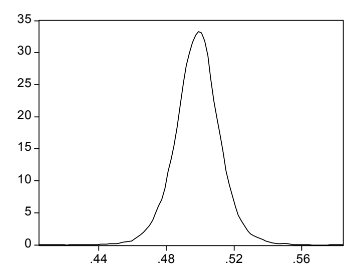
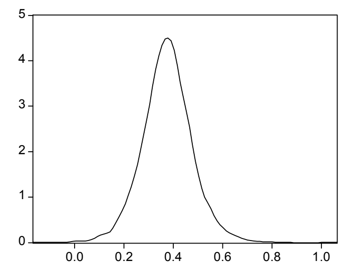
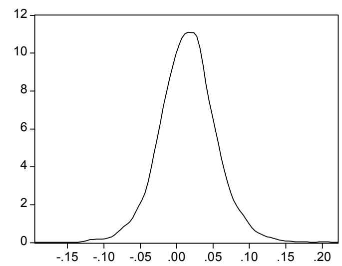
100			
200			
1000			
Limit			

Figure 2 Kernel Densities of $\widehat{\varphi}-1$ for Various Number of Samples and Break Sizes In the Presence of Structural Breaks in Mean When $K-1=9$

<table>
<tr><td>T</td><td colspan="3">Relative Break Size(σ_ϵ/σ)</td></tr>
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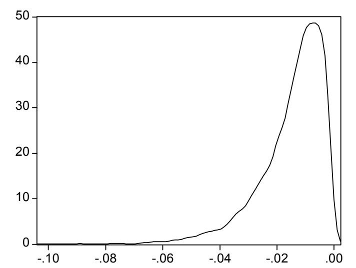
<td>
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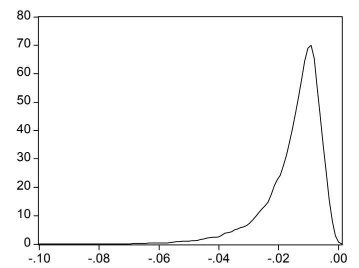
<td>
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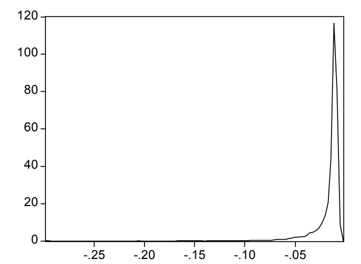
<td>
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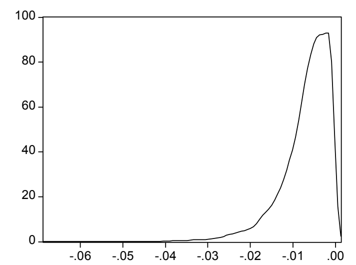
<td>
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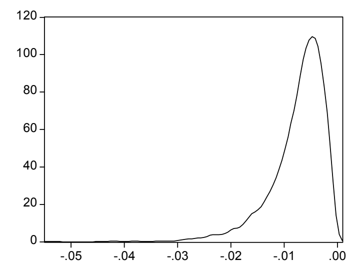
<td>
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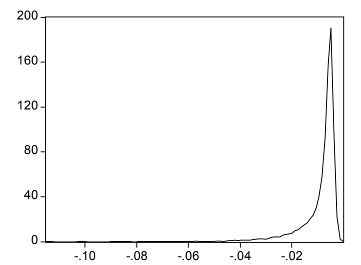
<td>
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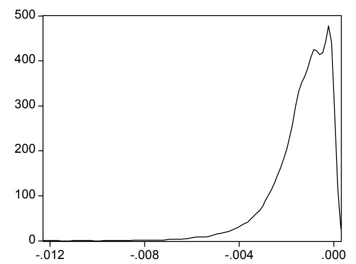
<td>
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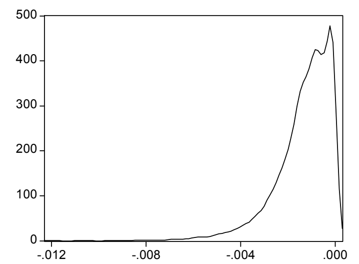
<td>
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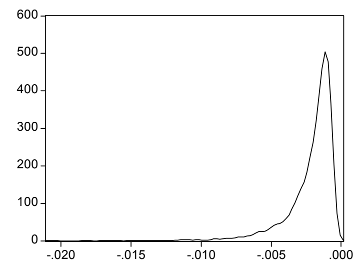
<td>
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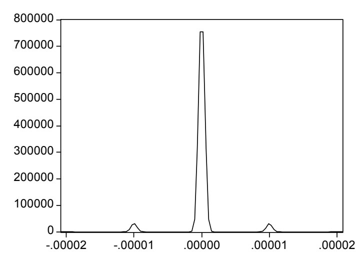
<td>
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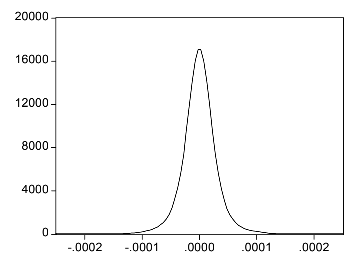
<td>
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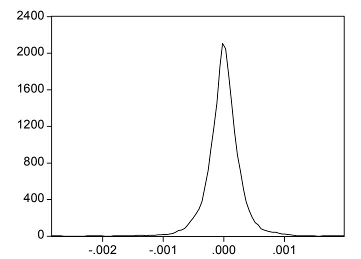
<td>
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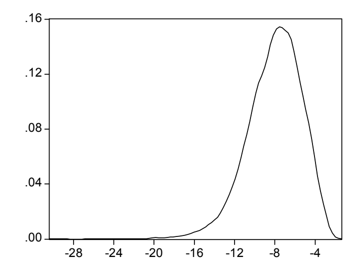
<td>
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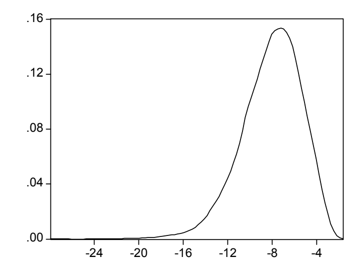
<td>
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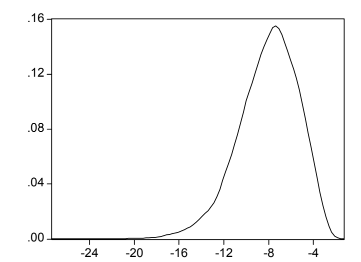
<td>
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