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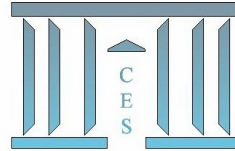
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## The Power of some Standard tests of stationarity against changes in the unconditional variance

Ibrahim AHAMADA, Mohamed BOUTAHAR

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The Power of some Standard tests of stationarity against changes  
in the unconditional variance.

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## Abstract

Abrupt changes in the unconditional variance of returns have been recently revealed in many empirical studies. In this paper, we show that traditional KPSS-based tests have a low power against nonstationarities stemming from changes in the unconditional variance. More precisely we show that even under very strong abrupt changes in the unconditional variance, the asymptotic moments of the statistics of these tests remain unchanged. To overcome this problem, we use some CUSUM-based tests adapted for small samples. These tests do not compete with KPSS-based tests and can be considered as complementary. CUSUM-based tests confirm the presence of strong abrupt changes in the unconditional variance of stock returns, whereas KPSS-based tests do not. Consequently, traditional stationary models are not always appropriate to describe stock returns. Finally we show how a model allowing abrupt changes in the unconditional variance is well appropriate for CAC 40 stock returns.

**Keywords:** KPSS test, Panel stationarity test, Unconditional variance, Abrupt changes, Stock returns, Size-Power curve.

**JEL classification:** C12; C15; C23.

## Résumé

Dans ce papier nous montrons dans un premier temps que les moments asymptotiques des statistiques du test KPSS et ses extensions en panel restent inchangés aux variations brusques de la variance inconditionnelle même si l'ampleur du saut reste élevé. Dans un deuxième temps nous étudions des tests complémentaires ainsi que leurs propriétés asymptotiques. Les tests complémentaires s'adaptent bien aux échantillons réduits puisque nous donnons aussi des valeurs simulées des moments des statistiques en fonction de  $T$ , nombre des observations. Enfin une illustration concrète est proposée à partir de la série SP500.

**Mots-clés :** Test KPSS, Tests de stationnarité en panel, Variance inconditionnelle, Changements brusques, Rendements financiers, Courbe Taille-Puissance

**Classification-JEL:** C12; C15; C23

# 1 Introduction

Many popular econometric models assume that observations come from covariance-stationary processes. The **KPSS test** (Kwiatkowski, Phillips, Schmidt and Shin (1992)) is often used to test the null hypothesis of stationarity against its alternative of unit root. This test has become popular and it is systematically implemented by many commercial software programs (thus increasing its usage). Extensions of this test for heterogeneous panel data are developed by Hadri (2000). The null hypothesis of Hadri's approach is the stationarity in all units (i.e., all individual time series) against the alternative of unit root in all units. Hadri and Larsson (2005) extended the Hadri test by determining exactly the first and the second moment of the test statistics when the time dimension of the panel is fixed. Asymptotically, the moments of Hadri test statistics and those of Hadri and Larsson (**HL test**) coincide. The HL test is concerned by the null of stationarity in all units against the alternative of unit root for some units (i.e. the HL test allows some of the individual series to be stationary under the alternative). All these tests are residual-based LM tests. These tests are among the so called *first generation tests* that are designed for cross-sectionally independent panels. Many authors have investigated the performance of these tests (see for example Hlouskova and Wagner (2006)).

But this paper focuses essentially on the behavior of the HL test and the KPSS test against a particular form of nonstationarity which is the one explained by abrupt changes in the unconditional variance of the processes. Because the null hypothesis of these tests is stationarity, many practitioners conclude unambiguously that the data come from a covariance-stationary process when the null is not rejected. The first aim of this paper is to show that this conclusion is hasty. We show that even under very strong abrupt changes in the unconditional variance, the asymptotic moments of the statistics of these tests remain unchanged. The null is not rejected while the process is not really covariance stationary (since the variance is not constant). Hence a stationary model can be wrongly applied to the data if the null is not rejected. Among many authors, Starica and Granger (2005) noted that some stylized facts of financial returns data can be explained by jumps in the unconditional variance. So, traditional stationary models (stationary GARCH model, stationary long-memory model, etc..) are not always appropriate. Starica and Granger (2005) proposed a nonstationary model that takes into account the jumps in the unconditional variance of financial returns data. The second

aim of this paper is to examine some possible complementary tests of stationarity that would be able to detect abrupt changes in the unconditional variance. These complementary tests are based on the cumulative sums of squares of residuals (CUSUM) and have a Kolmogorov-Smirnov limiting distribution under the null. A Monte Carlo study shows that the power of the CUSUM-based tests increases with the size of the jumps in the unconditional variance. Moreover, as in the HL test case, the CUSUM-based test for panel data is also appropriate for small samples. These CUSUM-based tests do not compete with the KPSS test and the HL test but they can be used as complementary tests that are able to detect other forms of nonstationarity while other tests cannot. An illustration based on French stock returns (CAC40 index) consolidates the nonstationary model proposed by Starica and Granger (2005). This paper is organized as follows: the second section investigates the behaviour of the KPSS test and HL test against changes in the unconditional variance. In Section Three we describe the CUSUM-based tests (complementary tests) and their asymptotic properties. In Section Four an empirical illustration is proposed before the conclusion of the paper in the last section. Proofs are given in the Appendix.

## 2 The Power of the KPSS test and the HL test against changes in unconditional variance.

### 2.1 Presentation of the KPSS test and the HL test.

In this section we briefly describe the KPSS test and the HL test before analysing their abilities to detect nonstationarity explained by changes in unconditional variance.

Let us consider the following process:

$$y_t = r_t + \varepsilon_t \quad t = 1, \dots, T \quad (1)$$

where  $r_t = r_{t-1} + u_t$  is a random walk and  $\varepsilon_t$  is a zero mean stationary process with  $E(u_t \varepsilon_{t'}) = 0$ ;  $E(u_t^2) = \sigma_u^2 \geq 0$  and  $E(\varepsilon_t^2) = \sigma_\varepsilon^2 > 0$ . The null hypothesis is  $H_0 : E(u_t^2) = \sigma_u^2 = 0$ , which means that the component  $r_t = r_0$  is a constant instead of unit root. Under the null hypothesis,  $y_t$  is stationary around constant  $r_0$ . The alternative hypothesis is given by  $H_1 : \sigma_u^2 > 0$ . The statistic of the KPSS test for

stationarity around  $r_0$  is given by:

$$\hat{\eta}_\mu = \frac{T^{-2} \sum_{t=1}^T S_t^2}{\hat{\sigma}^2} \quad (2)$$

where  $S_t = \sum_{j=1}^t \hat{e}_j$ ,  $\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^T \hat{e}_t^2$  and  $\hat{e}_t$ 's are the residuals from regression  $y_t = r_0 + \varepsilon_t$ . If  $\varepsilon_t$  is not a white noise, a long run variance can be estimated by using non parametric approach. Under the null the statistic  $\hat{\eta}_\mu$  is asymptotically distributed as  $\int_0^1 V_1(r)^2 dr$  where  $V_1(r) = W(r) - rW(1)$  is a standard Brownian bridge where critical values are simulated. According to Hadri (2000) and Hadri and Larsson (2005), the asymptotic moments of  $\hat{\eta}_\mu$  under the null are known, i.e.,

$$\lim_{T \rightarrow \infty} E(\hat{\eta}_\mu) = \frac{1}{6} \quad \text{and} \quad \lim_{T \rightarrow \infty} \text{var}(\hat{\eta}_\mu) = \frac{1}{45}.$$

The HL stationarity test is an extension of the KPSS test. Let us consider the following model:

$$y_{it} = r_{it} + \varepsilon_{it} \quad i = 1, \dots, N \quad \text{and} \quad t = 1, \dots, T \quad (3)$$

where  $r_{it} = r_{it-1} + u_{it}$  is a random walk and  $\varepsilon_{it}$  is a stationary process with the following conditions: *The  $\varepsilon_{it}$ 's and  $u_{it}$ 's are gaussian and i.i.d across  $i$  and over  $t$  with  $E(\varepsilon_{it}) = 0$ ,  $\text{var}(\varepsilon_{it}) = \sigma_{\varepsilon i}^2$ ,  $E(u_{it}) = 0$  and  $\text{var}(u_{it}) = \sigma_{ui}^2 \geq 0$ . The null hypothesis is  $H_0 : E(u_{it}^2) = \sigma_{ui}^2 = 0$ ,  $\forall i = 1, \dots, N$  which means that each component  $r_{it}$  is a constant instead of a unit root. Under the null hypothesis, each individual times series is stationary around constant  $r_{i0}$ . The alternative hypothesis is given by,  $H_1 : \sigma_{ui}^2 > 0$  for  $i = 1, \dots, N_1$  where  $N_1 \leq N$ . So, the alternative of the HL test allows some individual units to be stationary while for the Hadri test (2000) all individual times series are nonstationary under the alternative hypothesis. The statistic of the HL test for stationarity around  $r_{i0}$  is given by:*

$$Z_{\mu NT} = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \frac{\hat{\eta}_{\mu i T} - \xi_{\mu T}}{\zeta_{\mu T}} \right) \quad (4)$$

where for each fixed value of  $i$ , the statistic  $\hat{\eta}_{\mu i T}$  is computed as  $\hat{\eta}_\mu$  in equation (2). Under the null, the first and second moments of  $\hat{\eta}_{\mu i T}$  are precisely given by  $\xi_{\mu T} = E(\hat{\eta}_{\mu i T}) = \frac{T+1}{6T}$  (i.e., the asymptotic mean is  $1/6$ ) and  $\zeta_{\mu T}^2 = \text{var}(\hat{\eta}_{\mu i T}) = \frac{2T^2 - 5T + 2}{90T^2}$  (i.e., the asymptotic variance is  $1/45$ ). Hence the HL test is particularly attractive since it uses exact moments for any fixed value of  $T$  while the Hadri test (2000) uses asymptotic moments. From the Lindeberg-Levy central limit theorem, the limiting distribution of  $Z_{\mu NT}$  when  $N \rightarrow \infty$  is the standard  $N(0, 1)$ .

## 2.2 The Power of the HL test and the KPSS test against abrupt changes in the unconditional variance.

In this section we investigate the respective powers of the HL test and the KPSS test against the alternative of nonstationarity explained by changes in unconditional variance. Let us consider the following nonstationary process:

$$Y_{it} = r_{i0} + h_{it}\varepsilon_{it}, \quad i = 1, \dots, N; \quad t = 1, \dots, T. \quad (5)$$

where for each fixed value of  $i$ , the sequence  $(h_{it})$  is a bounded deterministic sequence and  $\varepsilon_{it} \sim i.i.d(0, \sigma_{\varepsilon_i}^2)$ . The variance of process  $\{Y_{it}\}$  is given by  $var(Y_{it}) = \sigma_{\varepsilon_i}^2 h_{it}^2$ . Hence if  $h_{it}$  is a deterministic step function then there are abrupt changes in the unconditional variance of the process  $Y_{it}$ .

Let  $m_T = E(\hat{\eta}_{\mu iT})$ ,  $\sigma_T^2 = var(\hat{\eta}_{\mu iT})$  the moments of  $\hat{\eta}_{\mu iT}$  if it is computed by using process (5). Then we have the following:

**Theorem.1.** *Assume in model (5) that for each fixed value of  $i$  the sequence  $(h_{it})$  is a bounded deterministic sequence with the following condition:*

$$\frac{1}{T} \sum_{t=1}^T h_{it}^2 \rightarrow \bar{h}_{i2}^2 \text{ as } T \rightarrow \infty \quad (i)$$

Then

$$Z_{\mu NT} = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \frac{\hat{\eta}_{\mu iT} - m_T}{\sigma_T} \right) \xrightarrow{\mathcal{L}} N(0, 1), \quad (j)$$

$$\lim_{T \rightarrow \infty} m_T = \frac{1}{6} \quad (jj)$$

and

$$\lim_{T \rightarrow \infty} \sigma_T^2 = \frac{1}{45}. \quad (jjj)$$

Proof: See Appendix.

Theorem 1 shows that the asymptotic mean and variance of the statistic  $\hat{\eta}_{\mu iT}$  remain unchanged while process (5) is not really covariance stationary. The condition (i) is obviously satisfied if the  $h_{it}$ 's are step functions. Consequently for large values of  $T$ , the behaviors of the KPSS-test and the HL-test under



unconditional variance abrupt changes are the same as under the null of covariance-stationarity. Let us now consider the Monte Carlo experiments based on a Data Generating Process according to equation(5), which we denote by  $DGP_{H_1}$ . We suppose that the  $h_{it}$ 's are step functions for  $i = 1, \dots, N_0$ . More precisely for  $i = 1, \dots, N_0$ ,  $h_{it} = \sigma_{i,j+1} > 0$  if  $t = T_{i,j-1} + 1, \dots, T_{i,j}$ ,  $T_{i,j} = [\lambda_{i,j}T]$ ,  $0 < \lambda_{i,1} < \dots < \lambda_{i,m} < 1$  where  $\sigma_{i,j} \neq \sigma_{i,k}$ . There are abrupt changes in the unconditional variances of some individual times series and the dates of changes are given by the  $T_{i,j}$ 's. According to Starica and Granger (2005), the stylized facts of the SP500 returns data can be explained by this form of nonstationarity instead of the traditional stationary models. For simplicity we take  $m = 1$ ,  $\sigma_{i1} = \sigma_1$  and  $\sigma_{i2} = \sigma_2$  for  $i = 1, \dots, N_0$ . The ratio  $\boldsymbol{\lambda} = \frac{\sigma_2}{\sigma_1}$  gives the size of the jumps in the unconditional variance while  $N_0$  denotes the number of the individual nonstationary processes ( $\mathbf{n}_0 = N_0/N$  is the proportion of the nonstationary processes). The  $\varepsilon_{it}$ 's are generated from the standard  $N(0, 1)$ . Fixed effects  $r_{i0}$  are coming from the uniform distribution  $U[0, 1]$ . **Table 1** gives the rejection frequencies of the null hypothesis by using many values of  $(\mathbf{n}_0, \boldsymbol{\lambda})$  at 5% level of significance. The values of  $N$  and  $T$  are fixed to  $N = 100$  and  $T = 100$ . We can see that, when  $\mathbf{n}_0$  increases, the rejection frequencies of the null hypothesis always remain near the nominal size  $\alpha = 0.05$ . This also remains true when increasing the size of the jumps in the unconditional variances (i.e. when increasing  $\lambda = \frac{\sigma_2}{\sigma_1}$ ). Hence, we can conclude that under nonstationarity explained by jumps in the unconditional variances (even for very high jumps, i.e.,  $\lambda = 20$ ), the statistic  $Z_{\mu NT}$  seems to have the same behavior as under the null hypothesis of stationarity although the process is not covariance stationary. The HL test seriously fails to detect abrupt changes in the unconditional variance. So, rejection of the null hypothesis by the HL test must be accompanied by a complementary test to take into account possible abrupt changes in the unconditional variance. In table 1, results about KPSS statistic are obtained by taking  $N = N_0 = 1$ .

### 3 Some complementary tests.

#### 3.1 Definition

Let us consider model (1) and suppose that the null is true, i.e.,  $r_t = r_0$  for  $t = 1, \dots, N$ . According to Section 2.2 process  $y_t$  is not necessarily covariance stationary. Indeed, changes in the unconditional variance

of process can be hidden as follows:

$$y_t = r_0 + h_t \varepsilon_t, \quad t = 1, \dots, T, \quad (6)$$

where the  $h_t$  is supposed to be a step deterministic function and  $\varepsilon_t$  a covariance stationary process. Without loss of generality we can always consider in (6) that  $\text{var}(\varepsilon_t) = \sigma^2 = 1$  and thus  $\text{var}(y_t) = h_t^2$ . It is thus necessary to use a complementary test to make sure that  $h_t$  is constant. For this, we extend the test suggested by Inclan and Tiao (1994) to model (6) and to panel data. We focus on the following null hypothesis

$$H_0 : h_t = h \text{ in (6)}.$$

Let us consider the statistic

$$\tau_T = \max_{t=1, \dots, T} \sqrt{\frac{T}{2}} |D_t|, \quad (7)$$

where  $D_t = \frac{C_t}{C_T} - \frac{t}{T}$ ,  $C_k = \sum_{j=1}^k \widehat{e}_j^2$  and  $\widehat{e}_t$  are the residuals from (6).

**Assumption.1:** In (6),  $(\varepsilon_t)$  is assumed to be *i.i.d.*  $N(0, 1)$ .

**Theorem.2.** *Under the null hypothesis and Assumption 1, we have the following results:*

*i) The limiting distribution of  $\tau_T$  is given by the one of  $W^0 = \sup_r (|W^*(r)|)$  where  $W^*(r)$  is a Brownian Bridge,*

*ii)  $Pr(\tau_\infty < a) = F(a) = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} \exp(-2k^2 a^2)$ ,*

*iii) The first moment of  $\tau_\infty$  is given by:  $E(\tau_\infty) = \ln(2) \sqrt{\frac{\pi}{2}} = \mu$ ,*

*iv) The centered second moment of  $\tau_\infty$  is given by:  $\text{var}(\tau_\infty) = \frac{\pi}{2} [\frac{\pi}{6} - (\log(2))^2] = \nu^2$ .*

The statistic  $\tau_T$  given by (7) is defined with the  $\max(\cdot)$  function, i.e. the Kolmogorov-Smirnov distance. It allows us to evaluate if the maximal size of the jumps in the unconditional variance is significant. The theorem allows us to know exactly the asymptotic critical values of  $\tau_\infty$ . From (ii) of Theorem.2, we have  $F(1.36) \approx 0.95$ . So, the critical value at level  $\alpha = 0.05$  is  $C_{0.05} \approx 1.36$ .

The statistic  $\tau_T$  allows us to investigate covariance stationarity for a single times series. It can easily be

extended to the panel model

$$y_{it} = r_{i0} + h_{it}\varepsilon_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T, \quad (8)$$

where the  $h_{it}$ 's are supposed to be deterministic functions and  $h_{it} > 0$ .

**Assumption 2:**  $(\varepsilon_{it})$  and  $(u_{it})$  are gaussian and *i.i.d* across  $i$  and over  $t$  with  $E(\varepsilon_{it}) = 0$ ,  $var(\varepsilon_{it}) = \sigma_{\varepsilon_i}^2$ ,  $E(u_{it}) = 0$  and  $var(u_{it}) = \sigma_{u_i}^2 \geq 0$ .

We consider the same assumptions as in the HL test. The initial values  $r_{i0}$ 's are treated as fixed unknown values playing the role of heterogeneous intercepts under the null hypothesis of stationarity.

We are concerned by the following null hypothesis:

$$\mathbf{H}_0 : h_{it} = h_i, \forall i = 1, \dots, N \quad \text{in (8)}$$

against the alternative that the  $h_{it}$ 's are not constants for some  $i = i_2, \dots, N_2$  where  $N_2 \leq N$ . Now we define the statistic for cross-sectional data as follows:

$$K_{\mu\nu} = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \frac{\tau_{iT} - \mu}{\nu} \right) \quad (9)$$

where  $\tau_{iT}$  is constructed as in (7) for each fixed  $i = 1, \dots, N$ .

**Corollary** *Under the null hypothesis and Assumption 2, the limiting distribution of  $K_{\mu\nu}$  is a standard normal,  $N(0, 1)$ , as  $T \rightarrow \infty$  and  $N \rightarrow \infty$ .*

The statistic  $K_{\mu\nu}$  given by (9) is more appropriate for the asymptotic case i.e. high values of  $N$  and  $T$ . For low values of  $T$ , we can see in **Table 2** that the simulated values of  $E(\tau_{iT})$  and  $var(\tau_{iT})$  deviate significantly from the asymptotic values given by Theorem 2 (i.e.  $\mu$  and  $\nu$ ). So, a test based on statistic  $K_{\mu\nu}$  can be seriously biased for low values of  $T$ . As in the case of the HL test, we adapt the statistic  $K_{\mu\nu}$  for finite values of  $T$  to improve the finite sample properties of the test (see also Harris and Tzavalis, 1999). So, the statistic  $K_{\mu\nu}$  can be corrected as follows:

$$K_{\mu_T \nu_T} = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \frac{\tau_{iT} - \mu_T^*}{\nu_T^*} \right) \quad (10)$$

where  $\mu_T^*$  and  $\nu_T^*$  are respectively simulated values of  $E(\tau_{iT})$  and  $[\text{var}(\tau_{iT})]^{1/2}$ . It is easy to see that from the Lindeberg-Levy central limit theorem, the statistic  $K_{\mu_T\nu_T}$  is distributed as  $N(0,1)$  when  $N \rightarrow \infty$ , under the null hypothesis and Assumption 2. Assumption 2 guarantees the independence of the individual statistics  $\tau_{iT}$ . Some values of  $\mu_T^*$  and  $\nu_T^*$  are given in Table 2. The values of  $\mu_T^*$  and  $\nu_T^*$  are respectively obtained by taking the average and the empirical standard deviation of 10.000 replications of the statistic  $\tau_T$  given by (7) for each fixed value of  $T$ . The numerical values of the statistic  $\tau_T$  are obtained by using the following DGP:  $y_t = \alpha + \varepsilon_t$ ,  $t = 1, \dots, T$ ,  $\varepsilon_t \rightsquigarrow N(0,1)$ . For each replication of  $\tau_T$ , the constant value  $\alpha$  is coming from the uniform distribution in  $U(0;5)$ .

## 3.2 The Power and size of the complementary tests.

### 3.2.1 Power and size of the test based on the statistic $\tau_T$ .

To investigate the power of the complementary test based on the statistic  $\tau_T$  given by (7) against abrupt changes in the unconditional variances, we again consider the  $DGP_{H_1}$  given by  $y_t = r_0 + h_t\varepsilon_t$ ,  $t = 1, \dots, T$  where  $h_t = I(1 \leq t \leq t^*) + \lambda I(t^* + 1 \leq t \leq T)$ ,  $I(\cdot)$  equals one if its argument is true and zero otherwise, and  $\varepsilon_t \sim i.i.N(0,1)$ . Parameter  $\lambda$  can be regarded as the amplitude of the jump of  $h_t$  whereas  $t^*$  is the break date. For each replication of  $DGP_{H_1}$  the parameter  $r_0$  is generated from the uniform distribution  $U[0,1]$  and the break date is chosen as  $t^* = [\alpha T]$  where  $\alpha \sim U[0.1, 0.9]$ . Rejection frequencies are based on  $N_R = 10000$  replications generated from the  $DGP_{H_1}$  and the nominal significance level  $\alpha = 0.05$ . **Table 3** shows the results for many fixed values of  $\lambda$ . We can observe that the power of the test increases with the amplitude of the variance change  $\lambda$  and the size  $T$  of the data. The size of the test corresponds to  $\lambda = 1$  since in this case the process is covariance stationary. The empirical size becomes closer to the nominal size as sample size  $T$  increases. It seems that there is no size distortion. Thus this test can be used as a complementary test of the KPSS test against breaks in the unconditional variance.

### 3.2.2 Power of the test based on the statistic $K_{\mu_T\nu_T}$ .

Another way to examine the power of a test can be based on the size-power curve of the test statistic (see Davidson and Mackinnon(1998)). According to the authors, these graphs convey much more information

and in a more easily assimilated form than classical tables do. In this section we investigate the power of the test based on the statistic  $K_{\mu_T \nu_T}$  by using some size-power curves. This method is especially convenient when the theoretical distribution of the studied statistic is not complicated to implement. This is the case for statistic  $K_{\mu_T \nu_T}$ , since its distribution is the standard  $N(0,1)$ . A size-power curve is constructed as follows. Let us consider a statistic  $\tau$  having asymptotic distribution function  $F$  under the null. Let us denote by  $\{\tau_j\}_{j=1}^{N_R}$ ,  $N_R$  realizations of the statistic  $\tau$  that are obtained by using a DGP which satisfies the null hypothesis ( $DGP_{H_0}$ ). The P-value of each  $\tau_j$  is defined as follows :  $p_j = 1 - F(\tau_j) = \Pr(\tau > \tau_j)$ . Let us now consider  $\widehat{F}_0$ , the empirical distribution function of  $\{p_j\}_{j=1}^{N_R}$  defined in (0,1) as follows:

$$\widehat{F}_0(x_i) = \frac{1}{N_R} \sum_{j=1}^{N_R} I(p_j \leq x_i). \quad (11)$$

Davidson and MacKinnon(1998) suggest the following choice of  $\{x_i\}_{i=1}^m$ :

$$x_i = 0.001, 0.002, \dots, 0.010, 0.015, \dots, 0.990, 0.991, \dots, 0.999 \quad (m = 215). \quad (12)$$

Let us now consider  $\widehat{F}_1$  constructed in the same way as  $\widehat{F}_0$  but instead of  $\tau_j$  we now use  $N_R$  realizations,  $\{\tau'_j\}_{j=1}^{N_R}$ , generated by using a DGP which satisfies the alternative hypothesis. The size-power curve of the statistic  $\tau$  is the locus of points  $(\widehat{F}_0(x_i), \widehat{F}_1(x_i))$  when  $x_i$  describes the (0,1) interval. Let us consider two alternative hypotheses  $H_1^{(a)}$  and  $H_1^{(b)}$  for the statistic  $\tau$ . If the test based on  $\tau$  is more powerful against  $H_1^{(a)}$  than against  $H_1^{(b)}$  then the size-power curve constructed using a DGP coming from  $H_1^{(a)}$  ( $DGP_{H_1^{(a)}}$ ) converges more quickly to the horizontal line  $y = 1$  than the size-power curve constructed with a DGP coming from  $H_1^{(b)}$  ( $DGP_{H_1^{(b)}}$ ). For more details about the concept of the size-power curve, see Davidson and Mackinnon (1998).

Now, we use the concept of the size-power curve to study the ability of statistic  $K_{\mu_T \nu_T}$  to detect abrupt changes in the unconditional variances. The  $DGP_{H_1^{(a)}}$  is constructed as follows:  $y_{it} = r_{i0} + h_{it}\varepsilon_{it}$  where the  $h_{it}$ 's are step functions with amplitude  $\lambda = 2$  for  $i = 1, \dots, N_0$  and  $h_{it} = h_i = \text{constant}$  for  $i = N_0 + 1, \dots, N$ . The quantity  $n_0 = N_0/N$  gives the proportion of individual nonstationary time series in the panel. We set  $n_0 = N_0/N$  for the experiment. The  $DGP_{H_1^{(b)}}$  is constructed as  $DGP_{H_1^{(a)}}$  but with  $\lambda = 5$ . Our aim is to show that the power of the test increases with  $\lambda$ . The  $DGP_{H_0}$  is constructed as follows:  $y_t = r_{i0} + \varepsilon_{it}$ . For all DGP's, the  $r_{i0}$  are chosen from the uniform distribution  $U(0, 1)$ ,  $\varepsilon_{it} \sim i.i.d.N(0, 1)$  for each fixed value of

$i$ , the dimensions of the panel are set to  $(N, T) = (100, 100)$  and the number of replications is  $N_R = 10000$ . **Figure 1** shows the size-power curves. We can see that the size-power curve constructed by using the  $DGP_{H_1^{(b)}}$  (i.e. with  $\lambda = 5$ ) converges faster to the horizontal line  $y = 1$  than that of the  $DGP_{H_1^{(a)}}$  (i.e. with  $\lambda = 2$ ). The power of the test increases with the amplitude of the variance break. For comparison purposes, we present in **Figure 2** the size-power curves of the HL test statistic by using the same DGP's. In contrast with the case of statistic  $K_{\mu_T\nu_T}$ , we can now see that the two curves coincide with the  $45^\circ$  line, even for strong abrupt change ( $\lambda = 5$ ). Thus for the HL test, there is no contrast between the  $DGP_{H_0}$  and the two nonstationary DGP's, i.e.,  $DGP_{H_1^{(a)}}$  and  $DGP_{H_1^{(b)}}$ . So the statistic  $K_{\mu_T\nu_T}$  can be used as a complementary test of the HL test. Monte Carlo experiments based on many values of  $(\lambda, n_0, N, T)$  were also realized. The results showed that the power of  $K_{\mu_T\nu_T}$  increases with  $\lambda$  and  $n_0$ . We do not present the results to save space but they are available on request.

### 3.2.3 Size of the test based on the statistic $K_{\mu_T\nu_T}$ .

Another approach to investigate the sizes of the statistics is the one based on the so called "**P-value discrepancy plot**" (for more details see Davidson and Mackinnon (1998)). This is simply the graph of  $\widehat{F}_0(x_i) - x_i$  against  $x_i$  where  $\widehat{F}_0$  is given by (11). We consider again the stationary  $DGP_{H_0}$  presented in section 3.2.2. If the distribution of the studied statistic is correct, then the  $p_j$ 's (defined in Section 3.2.2) should be distributed as uniform in  $(0, 1)$ . Therefore, when  $\widehat{F}_0(x_i) - x_i$  is plotted against  $x_i$ , the graph should be close to the horizontal axis  $y = 0$ . **Figure 3** shows the P-value discrepancy plots for the statistic  $K_{\mu_T\nu_T}$  and the one for the HL test statistic. We can notice the proximity of both curves with the horizontal line  $y = 0$  (values of discrepancies are approximately under 0.01) especially in the case of the statistic  $K_{\mu_T\nu_T}$  (i.e. the complementary test). So, there is no size distortion for the two tests. Let us note that in the case of  $K_{\mu_T\nu_T}$  the curve is nearer to the  $y = 0$  line than in the case of the HL test statistic. Hence, the size of  $K_{\mu_T\nu_T}$  seems to be slightly better than in the HL test. Monte Carlo experiments based on many values of  $(N, T)$  were also performed. The results confirmed comparable behaviors for the two tests especially in the case of low values of  $(N, T)$ . We do not present the results to save space but they are available on request.

## 4 Empirical illustration.

There has been a long debate about the modeling of the stylized facts of daily stock returns. The absolute returns data often show a slow decay of the sample's autocorrelation function. Starica and Granger (2005) addressed the following fundamental questions about the persistence of the sample autocorrelation function of stock returns : "How should we interpret the slow decay of the sample autocorrelation function(ACF) of absolute returns? Should we take it at face value, supposing that events that happened a number of years ago have an effect on the present dynamics of returns? Or are the nonstationarities in the returns responsible for its presence?". This question summarizes the classical debate about the modeling of the stock returns. Indeed, the long memory stationary model is often used by many authors to describe the behavior of the sample correlation function of stock returns. Other authors adopt the nonstationary models explained by abrupt changes in the unconditional variance to explain these stylized facts. For an illustration we consider the data of the French financial index (CAC40) in logarithm,  $\{\ln(\mathbf{indexcac}_t)\}$  and returns  $\{\Delta \ln(\mathbf{indexcac}_t)\}$ (**Figure 4**). We consider daily data from January 1, 2003 to January 1, 2004 (the sample size sample  $T = 260$ ). **Table 4** indicates the results of the KPSS test and the complementary test based on the statistic  $\tau_T$ . We can see that both tests reject the null of stationarity for  $\ln(\mathbf{indexcac}_t)$  ( the critical value for the KPSS test is  $C_{0.05} = 0.463$  and the one for  $\tau_T$  is  $C_{0.05} = 1.36$  ). Then application of the  $\Delta$ -filter is often used by many practitioners hoping to obtain covariance stationary data. We can see that the KPSS test does not allow us to reject the null and one can conclude wrongly that  $\Delta \ln(\mathbf{indexcac}_t)$  is covariance stationary. This conclusion allows us to use some traditional stationary models to describe  $\Delta \ln(\mathbf{indexcac}_t)$  (stationary GARCH model, stationary long-memory model, etc). However, as indicated by the complementary test ( $\tau_T$ ), the covariance stationarity of  $\Delta \ln(\mathbf{indexcac}_t)$  is doubtful. The value taken by  $\tau_T$  for  $\Delta \ln(\mathbf{indexcac}_t)$  (i.e.  $\tau_T = 3.7669207$ ) remains twice as large as the critical value ( $C_{0.05} = 1.36$ ). This result is more compatible with the following model

$$\Delta \ln(\mathbf{indexcac}_t) = r_0 + h_t \varepsilon_t \quad (13)$$

where  $\varepsilon_t$  is the covariance stationary process.  $h_t$  is a step function representing the multiple changes in the unconditional standard deviation of  $\Delta \ln(\mathbf{indexcac}_t)$  . This is exactly the nonstationary model retained by Starica and Granger (2005) for the SP500 stock returns. They found that the forecasts based on this

nonstationary model were superior to those obtained in the framework of stationary *GARCH* models. They found also that this model reproduces the classical stylized facts observed in returns.

We also consider panel data  $\{\ln(\mathbf{cac}_{it}), i = 1, \dots, N; t = 1, \dots, T$  where  $\mathbf{cac}_{it}$  is the quotation of the firms  $i$  which make up the CAC40 index. We consider daily data from January 1, 2003 to January 1, 2004 ( $N = 40$  and  $T = 260$ ). Table 4 shows that the two tests reject the null of stationarity for  $\ln(\mathbf{cac}_{it})$  (the critical value of the two tests is  $C_{0.05} = 1.96$ ) while only the statistic  $K_{\mu_T \nu_T}$  indicates abrupt changes in variance. This result conforms the preceding remarks about the KPSS-test and the statistic  $\tau_T$ .

#### 4.1 Estimating $h_t$

Starica and Granger (2005) developed an approach based on the stability of the spectral density to compute an estimate of  $h_t$  (see also Ahamada et al. 2004). Inclan and Tiao (1994) used an iterative algorithm based on statistic  $\tau_T$  to estimate  $h_t$ . Another method allowing us to compute  $\hat{h}_t$  easily and effectively is to apply the Bai and Perron (2003) approach to the centered data  $(\Delta \ln(\mathbf{indexcac}_t) - \hat{r}_0)$  where  $\hat{r}_0$  the empirical mean of  $\Delta \ln(\mathbf{indexcac}_t)$  (we found  $\hat{r}_0 = -0.000423$ ). According to (13) one can consider the following regression with multiple breaks:

$$y_t = \mu_k + v_t \tag{14}$$

where  $y_t = \ln(|\Delta \ln(\mathbf{indexcac}_t) - \hat{r}_0|)$ ,  $\mu_k = \ln(|h_k|)$  if  $t = t_{k-1}, \dots, t_k$ , the set  $\{t_k, k = 1, \dots, m\}$  gives the dates of breaks in the unconditional variance. More precisely these breaks occur in the logarithm of the unconditional standard deviation, i.e.  $\ln(h_t)$ . Bai and Perron (2003) addressed the problem of estimation of the break dates  $t_k$  and presented an efficient algorithm to obtain global minimizers of the sum of squared residuals. The algorithm is based on the principle of dynamic programming. They addressed the issue of testing for structural changes under very general conditions on the errors. The issue of estimating the number of breaks  $m$  is also considered by the authors. From the Bai and Perron approach applied to (14) we obtain the following results:  $\hat{m} = 1$ , i.e. one break date located at  $\hat{t}_1 = 166$ ,  $\hat{h}_t = \hat{\alpha}_1 I(1 \leq t \leq \hat{t}_1) + \hat{\alpha}_2 I(\hat{t}_1 + 1 \leq t \leq T)$  where  $\hat{\alpha}_1 = 0.0113$  with a 95% confidence interval (0.0096, 0.01273),  $\hat{\alpha}_2 = 0.022$  with a 95% confidence interval (0.0179, 0.0256). This result shows that the standard deviation of returns is twice



as great after date  $\hat{t}_1 = 166$  (i.e.,  $\hat{\alpha}_2 \simeq 2\hat{\alpha}_1$ ), hence the amplitude of change is  $\hat{\lambda} = 2$ . Figure 4 allows us to observe this clustering of unconditional volatility.

## 4.2 Validity of the assumptions in residuals

The results of statistic  $\tau_T$  are valid under the condition  $\varepsilon_t \sim i.i.d.N(0, 1)$ . This assumption was also supposed by Starica and Granger (2005) in model (13). Let us consider  $\hat{\varepsilon}_t = (\Delta \ln(\mathbf{indexcac}_t) - \hat{r}_0)/\hat{h}_t$ . The Portmanteau test for white noise applied to  $\hat{\varepsilon}_t$  gives a p-value  $Prob = 0.0969$ . The Bartlett periodogram-based white noise test gives, a p-value  $Prob = 0.6597$  (see **Figure 6**). One sample bilateral t-test of the mean ( $H_0 : mean(\hat{\varepsilon}_t) = 0$ ) gives a p-value  $Prob = 0.7633$  with an empirical mean,  $\overline{\hat{\varepsilon}_t} = -0.01$ . One sample chi2 test of variance ( $H_0 : var(\hat{\varepsilon}_t) = 1$ ) gives a p-value  $Prob = 0.9135$  with empirical  $std = 1.0035$ . **Figure 5** shows that the empirical distribution of  $\hat{\varepsilon}_t$  coincides almost perfectly with the theoretical distribution. All these adequacy tests seem to confirm that  $\varepsilon_t \sim i.i.d.N(0, 1)$ .

## 5 Conclusion

In this paper, we have shown that the KPSS test and its extension to panel data, suggested by Hadri and Larsson (2005), has a low power against nonstationarity coming from changes in the unconditional variance. CUSUM-based tests allow to fill this gap. These tests do not compete with the KPSS-based tests and can be considered as complementary to them. CUSUM-based test for panel data is well adapted to finite samples because the moments of the statistic test are simulated for the small sample sizes. These complementary tests must be applied as follows: First, apply the KPSS-based test. If the null hypothesis is rejected, then conclude that the data contain a unit root, i.e. there is nonstationarity. If the null hypothesis is not rejected, then there is no unit root but a shift in the variance is possible. Then apply the CUSUM-based test. If the null is not rejected, then there is a complete covariance stationarity. Else, if the null is rejected, then conclude that there is no unit root but the data have variance shift and the process is not covariance stationary. An empirical illustration based on the French financial index supports our findings.

## APPENDIX 1

### Proof of Theorem 1

i. Since  $\widehat{\eta}_{\mu iT}$  are i.i.d with mean  $m_T$  and variance  $\sigma_T^2$  the convergence (j) follows from the Lindeberg-Levy central limit theorem.

ii. We omit index  $i$  in the proof and without loss of generality we assume that the intercept is zero.

The residuals are  $e_t = \varepsilon_t - \bar{\varepsilon}$ ,  $\bar{\varepsilon} = \sum_{t=1}^T \varepsilon_t / T$ . It can be shown that  $\widehat{\eta}_{\mu T}$  can be written as the ratio of quadratic form in  $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_T)'$ , *i.e.*

$$\widehat{\eta}_{\mu T} = T^{-1} \frac{\underline{\varepsilon}' C' A C \underline{\varepsilon}}{\underline{\varepsilon}' C \underline{\varepsilon}}$$

where  $C = I_T - \underline{1}\underline{1}'T^{-1}$ ,  $I_T$  is the identity matrix of dimension  $T$  and  $\underline{1}$  is a  $T$ -dimensional vector of ones, and  $\underline{\varepsilon} \sim N(0, \Gamma)$ ,  $\Gamma = \text{diag}(h_1^2, \dots, h_T^2)$ . Let  $Q$  represent an orthogonal matrix (*i.e.*  $Q'Q = I_T$ ) such that

$$Q\Gamma^{1/2}C\Gamma^{1/2}Q = D = \text{diag}(d_1, \dots, d_T),$$

where  $\Gamma^{1/2} = \text{diag}(h_1, \dots, h_T)$  and let  $\Lambda = Q'\Gamma^{1/2}C\Gamma^{1/2}Q = (\lambda_{i,j})$  then the exact moments of  $\widehat{\eta}_{\mu T}$  (see Jones (1987)) are given by

$$\begin{aligned} E(\widehat{\eta}_{\mu T}) &= m_T = T^{-1} \int_0^{+\infty} \sum_{i=1}^T \frac{\lambda_{i,i}}{1 + 2d_i t} \Phi(t, d) dt, \\ E(\widehat{\eta}_{\mu T}^2) &= T^{-2} \int_0^{+\infty} \sum_{i=1}^T \sum_{j=1}^T \frac{\lambda_{i,i} \lambda_{j,j} + 2\lambda_{i,j}}{(1 + 2d_i t)(1 + 2d_j t)} \Phi(t, d) dt, \end{aligned} \quad (15)$$

where  $\Phi(t, d) = \prod_{i=1}^T (1 + 2d_i t)^{-1/2}$ . For small  $T$ , one can use the numerical methods proposed by Paoletta (2003) to compute the moments. But for large  $T$  such methods are time consuming and the asymptotic values of  $E(\widehat{\eta}_{\mu T})$  and  $E(\widehat{\eta}_{\mu T}^2)$  will be useful. To prove (jj) and (jjj) we need the following

**Lemma.** Let  $\widehat{\sigma}_T^2 = \sum_{t=1}^T e_t^2 / T$  and  $e_t$  are the residuals from regression:  $y_t = r_0 + h_t \varepsilon_t$ ,  $\varepsilon_t \sim i.i.d.N(0, 1)$  where  $(h_t)$  is a bounded deterministic sequence such that

$$\frac{1}{T} \sum_{t=1}^T h_t^2 \rightarrow \bar{h}_2^2 \text{ as } T \rightarrow \infty \quad (16)$$

then

$$\frac{1}{T} \sum_{t=1}^T e_t^2 \xrightarrow{a.s.} \bar{h}_2^2. \quad (17)$$

Proof. Assume that  $r_0 = 0$ . We have  $e_t = h_t \varepsilon_t - \overline{h\varepsilon}$ ,  $\overline{h\varepsilon} = \sum_{t=1}^T h_t \varepsilon_t / T$  hence

$$\frac{1}{T} \sum_{t=1}^T e_t^2 = \frac{1}{T} \sum_{t=1}^T (h_t \varepsilon_t)^2 - (\overline{h\varepsilon})^2 \quad (18)$$

$M_T = \sum_{t=1}^T h_t \varepsilon_t$  is a square integrable martingale adapted to the  $\sigma$ -field  $F_T = \sigma(\varepsilon_1, \dots, \varepsilon_T)$  with the increasing process

$$\begin{aligned} \langle M_T \rangle &= \sum_{t=1}^T E((h_t \varepsilon_t)^2 | F_{t-1}) \\ &= \sum_{t=1}^T h_t^2 \rightarrow \infty \end{aligned}$$

The application of theorem 1.3.15 of Duflo (2003) leads to  $M_T / \langle M_T \rangle \rightarrow 0$  almost surely, this with (16) imply that

$$\overline{h\varepsilon} = \frac{1}{T} \sum_{t=1}^T h_t \varepsilon_t \xrightarrow{a.s.} 0 \quad (19)$$

Likewise  $M_T = \sum_{t=1}^T h_t^2 (\varepsilon_t^2 - 1)$  is a square integrable martingale adapted to  $F$ , with increasing process  $\langle M_T \rangle = 2 \sum_{t=1}^T h_t^2$ , hence theorem 1.3.15. in Duflo (2003) implies that

$$\frac{1}{\langle M_T \rangle} \sum_{t=1}^T h_t^2 (\varepsilon_t^2 - 1) \xrightarrow{a.s.} 0 \text{ almost surely on } \{\langle M_\infty \rangle = \infty\} \quad (20)$$

where  $\langle M_\infty \rangle = \lim_{T \rightarrow \infty} \langle M_T \rangle$ . Since

$$\left( \sum_{t=1}^T h_t^2 \right)^2 \leq T \sum_{t=1}^T h_t^4, \quad (21)$$

The assumption (16) implies that there exist an universal constants  $0 < K_1 < K_2 < \infty$  such that

$$K_1 < \frac{1}{T} \sum_{t=1}^T h_t^2 < K_2,$$

this together with (21) implies that  $\langle M_T \rangle \geq 2TK_1^2$  which implies that  $\{\langle M_\infty \rangle = \infty\} = \Omega$  and hence

$$\frac{1}{\langle M_T \rangle} \sum_{t=1}^T h_t^2 (\varepsilon_t^2 - 1) \xrightarrow{a.s.} 0, \quad (22)$$

Since  $(h_t)$  is a bounded deterministic sequence, then there exists an universal  $K > 0$  such that  $h_t^4 \leq K$  for all  $t \geq 1$ , hence  $\langle M_T \rangle \leq TK$  for all  $T$ , therefore

$$\begin{aligned} \left| \frac{1}{T} \sum_{t=1}^T h_t^2 (\varepsilon_t^2 - 1) \right| &= \frac{\langle M_T \rangle}{T} \left| \frac{1}{\langle M_T \rangle} \sum_{t=1}^T h_t^2 (\varepsilon_t^2 - 1) \right| \\ &\leq K \left| \frac{1}{\langle M_T \rangle} \sum_{t=1}^T h_t^2 (\varepsilon_t^2 - 1) \right|, \end{aligned}$$

using (22), it follows that

$$\frac{1}{T} \sum_{t=1}^T h_t^2 (\varepsilon_t^2 - 1) \xrightarrow{a.s.} 0. \quad (23)$$

Combining (16) and (18), (19) and (23) we obtain (17).

From lemma we deduce that  $\widehat{\eta}_{\mu T}$  has the same limiting distribution as

$$\widehat{\psi}_{\mu T} = \frac{1}{T^2 h_2} \sum_{t=1}^T S_t^2, S_t = \sum_{j=1}^t e_j, \quad (24)$$

and hence

$$\lim_{T \rightarrow \infty} m_T = \lim_{T \rightarrow \infty} E(\widehat{\psi}_{\mu T}) \text{ and } \lim_{T \rightarrow \infty} \sigma_T^2 = \lim_{T \rightarrow \infty} \text{var}(\widehat{\psi}_{\mu T}). \quad (25)$$

Let  $D_t$  the  $(2, T)$  matrix given by

$$D_t = \begin{pmatrix} \frac{t}{T} & \cdots & \cdots & \cdots & \cdots & \frac{t}{T} \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

the ones are repeated  $t$  times. Since  $S_t = e_1 D_t \underline{u}$ , where  $e_1 = (-1, 1)'$ ,  $\underline{u} = (u_1, \dots, u_T)'$ ,  $u_t = h_t \varepsilon_t$ ,  $S_t^2$  can be written as a quadratic form in  $\underline{u}$ ,  $S_t^2 = \underline{u}' D_t' e_1 e_1' D_t \underline{u}$ . Consequently (see Magnus (1986))

$$\begin{aligned} E(S_t^2) &= \text{trace}(D_t' e_1 e_1' D_t \Gamma), \Gamma = \text{diag}(h_1^2, \dots, h_T^2) \\ &= \text{trace}(\Gamma^{1/2} D_t' e_1 e_1' D_t \Gamma^{1/2}) \\ &= \left\| \Gamma^{1/2} D_t' e_1 \right\|^2, \end{aligned}$$

where  $\|x\|^2$  is the Euclidian norm of the vector  $x$ .

Now

$$\begin{aligned} \left\| \Gamma^{1/2} D_t' e_1 \right\|^2 &= \sum_{j=1}^t h_j^2 \left(1 - \frac{t}{T}\right)^2 + \sum_{j=t+1}^T h_j^2 \left(\frac{t}{T}\right)^2 \\ &= \left(\sum_{j=1}^T h_j^2\right) \left(\frac{t}{T}\right)^2 + \sum_{j=1}^t h_j^2 \left(1 - \frac{2t}{T}\right) \end{aligned} \quad (26)$$

Hence

$$\begin{aligned}
E(\widehat{\psi}_{\mu T}) &= \frac{1}{T^2 \bar{h}_2} \sum_{t=1}^T E(S_t^2) \\
&= \frac{1}{T^2 \bar{h}_2} \left\{ \sum_{t=1}^T \left( \sum_{j=1}^T h_j^2 \right) \left( \frac{t}{T} \right)^2 + \sum_{j=1}^t h_j^2 \left( 1 - \frac{2t}{T} \right) \right\} \\
&= \frac{1}{\bar{h}_2} \left\{ \left( \frac{1}{T} \sum_{j=1}^T h_j^2 \right) \frac{1}{T^3} \sum_{t=1}^T t^2 + \frac{1}{T^2} \sum_{t=1}^T \sum_{j=1}^t h_j^2 - \frac{2}{T^3} \sum_{t=1}^T t \sum_{j=1}^t h_j^2 \right\}
\end{aligned}$$

The convergence (16) implies (see lemma 2 of Boutahar and Deniau (1996)) that

$$\begin{aligned}
\frac{1}{T^2} \sum_{t=1}^T \sum_{j=1}^t h_j^2 &\rightarrow \frac{\bar{h}_2}{2} \\
\frac{1}{T^3} \sum_{t=1}^T t \sum_{j=1}^t h_j^2 &\rightarrow \frac{\bar{h}_2}{3}
\end{aligned}$$

Therefore

$$\lim_{T \rightarrow \infty} E(\widehat{\psi}_{\mu T}) = \frac{1}{3} + \frac{1}{2} - \frac{2}{3} = \frac{1}{6} \quad (27)$$

and (jj) follows from (25) and (27).

$$\text{var} \left( \frac{1}{T^2} \sum_{t=1}^T S_t^2 \right) = \frac{1}{T^4} \left\{ \text{var} \left( \sum_{t=1}^T S_t^2 \right) + 2 \sum_{s < t} \text{cov}(S_s, S_t) \right\}. \quad (28)$$

Let  $\Lambda_t = D'_t e_1 e'_1 D'_t$ , and  $\Gamma^{1/2} = \text{diag}(h_1, \dots, h_T)$ , we have

$$\begin{aligned}
\text{var}(S_t) &= 2 \text{trace}(\Lambda_t \Gamma \Lambda_t \Gamma) \\
&= 2 \text{trace}(D'_t e_1 e'_1 D'_t \Gamma D'_t e_1 e'_1 D'_t \Gamma) \\
&= 2 \text{trace} \left( \left( \Gamma^{1/2} D'_t e_1 e'_1 D'_t \Gamma^{1/2} \right) \left( \Gamma^{1/2} D'_t e_1 e'_1 D'_t \Gamma^{1/2} \right) \right) \\
&= 2 \left\| \Gamma^{1/2} D'_t e_1 \right\|^4.
\end{aligned}$$

By using (26) we get

$$\text{var}(S_t) = 2 \left[ \left( \sum_{j=1}^T h_j^2 \right) \left( \frac{t}{T} \right)^2 + \sum_{j=1}^t h_j^2 \left( 1 - \frac{2t}{T} \right) \right]^2$$

By using (i), a straightforward computation leads to

$$\frac{1}{T^4} \text{var} \left( \sum_{t=1}^T S_t^2 \right) \sim \frac{\bar{h}_2^4}{15T}, \quad (29)$$

where  $a_T \sim b_T$  means that  $a_T/b_T \rightarrow 1$  as  $T \rightarrow \infty$ . For  $s < t$

$$\begin{aligned}
\text{cov}(S_s, S_t) &= 2\text{trace}(\Lambda_s \Gamma \Lambda_t \Gamma) \\
&= 2\text{trace}(D'_s e_1 e'_1 D'_s \Gamma D'_t e_1 e'_1 D'_t \Gamma) \\
&= 2\text{trace}\left(\left(\Gamma^{1/2} D'_s e_1 e'_1 D'_s \Gamma^{1/2}\right) \left(\Gamma^{1/2} D'_t e_1 e'_1 D'_t \Gamma^{1/2}\right)\right) \\
&= 2\text{trace}((xx')(yy')) \\
&= 2(x'y)^2,
\end{aligned}$$

where  $x = \Gamma^{1/2} D'_t e_1, y = \Gamma^{1/2} D'_s e_1$ .

Since

$$x' = (h_1(1 - t/T), \dots, h_s(1 - t/T), \dots, h_t(1 - t/T), -h_{t+1}t/T, \dots, -h_T t/T)$$

and

$$y' = (h_1(1 - s/T), \dots, h_s(1 - s/T), -h_{s+1}s/T, \dots, -h_T s/T).$$

Therefore

$$\text{cov}(S_s, S_t) = 2 \left\{ \sum_{j=1}^s h_j^2 \left(1 - \frac{s}{T}\right) \left(1 - \frac{t}{T}\right) - \sum_{j=s+1}^t \frac{s}{T} \left(1 - \frac{t}{T}\right) + \sum_{j=t+1}^T h_j^2 \frac{ts}{T^2} \right\}^2.$$

By using (i), a straightforward computation leads to

$$\begin{aligned}
\frac{1}{T^4} \sum_{s < t} \text{cov}(S_s, S_t) &\sim \frac{2\bar{h}_2^4}{T^4} \sum_{s=1}^T \sum_{t=s+1}^T \left\{ \frac{st}{T} + s \left(1 - \frac{(s+t)}{T}\right) - (t-s) \frac{s}{T} \right\} \\
&\sim \frac{\bar{h}_2^4}{90},
\end{aligned} \tag{30}$$

since

$$\begin{aligned}
\sum_{s=1}^T \sum_{t=s+1}^T (ts)^2 &\sim \frac{T^6}{18}, \\
\sum_{s=1}^T \sum_{t=s+1}^T ts^2 &\sim \frac{T^5}{15}, \\
\sum_{s=1}^T \sum_{t=s+1}^T s^2 &\sim \frac{T^4}{12}.
\end{aligned}$$

From (24), (28), (29) and (30) we deduce that

$$\begin{aligned} \text{var}(\widehat{\psi}_{\mu T}) &= \frac{1}{h_2^4} \text{var} \left( \frac{1}{T^2} \sum_{t=1}^T S_t^2 \right) \\ &\sim \frac{1}{45}, \end{aligned}$$

from this and (25), (jjj) holds.

**Proof of Theorem 2.** Let us consider model (6). Under  $H_0$ , we also have  $h_t = h$ . Hence, it is easy to see that under the null hypothesis the OLS estimator of  $r_0$  is given by  $\widehat{r}_0 = \frac{1}{T} \sum_{t=1}^T y_t$  with the residuals given by  $\widehat{e}_t = y_t - \widehat{r}_0 = (r_0 + h\varepsilon_t) - \frac{1}{T} \sum_{t=1}^T (r_0 + h\varepsilon_t) = h\varepsilon_t - \frac{1}{T} \sum_{t=1}^T h\varepsilon_t$ . To sum up, under the null hypothesis we have the following results: a)  $E(\widehat{e}_t) = 0$ , b)  $\text{var}(\widehat{e}_t) = h^2(1 - \frac{1}{T})$  and c) For  $t \neq t'$ ,  $E(\widehat{e}_t \widehat{e}_{t'}) = (\frac{1-T}{T^2})h^2$ . Hence the  $\widehat{e}_t$ 's are asymptotically uncorrelated with a constant variance  $h^2$ , i.e. white noise. So the  $\widehat{e}_t$ 's are asymptotically independent and identically distributed as  $N(0, h^2)$  since  $\varepsilon_t$  are supposed to be  $\varepsilon_t \sim i.i.d.N(0, 1)$ . So the theorem of Inclan and Tiao (1994) is satisfied for large  $T$ . Hence,  $\tau_T = \max_{t=1, \dots, T} \sqrt{\frac{T}{2}} |D_t|$  is asymptotically distributed as  $W^0 = \sup_r (|W^*(r)|)$  when  $T \rightarrow \infty$ , where  $W^*(r)$  is a Brownian Bridge (weak convergence). Hence conclusion (i) of the theorem holds. From Billingsley (1968),  $Pr(W^0 > a) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \exp(-2k^2 a^2) = 1 - F(a)$ , i.e.,  $F(a) = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} \exp(-2k^2 a^2)$  is the theoretical distribution function of  $W^0$ . Hence conclusion (ii) of the theorem also holds. The conclusions (iii) and (iv) of the theorem follow from  $E(W^0) = \int_0^{+\infty} x \frac{\partial F}{\partial x}(x) dx = \ln(2) \sqrt{\frac{\pi}{2}} = \mu$  and  $\text{var}(W^0) = \int_0^{+\infty} (x - \mu)^2 \frac{\partial F}{\partial x}(x) dx = \frac{\pi}{2} [\frac{\pi}{6} - (\log(2))^2] = \nu^2$ .

**Proof of the corollary.** Let us denote by " $\implies$ " the weak convergence. Under  $H_0$  and Assumption 1, the  $\tau_{iT}$ 's are independent across  $i$  (Assumption.1) and following Theorem 2 they are asymptotically distributed as random variables which we note by  $X_i$ 's having the same distribution as that of  $W^0 = \sup_r (|W^*(r)|)$  with mean  $E(W^0) = \mu$  and variance  $\text{var}(W^0) = \nu^2$ . For a fixed value of  $N$ , if  $T \rightarrow \infty$  we obtain  $\mathbf{K}_{\mu\nu} = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \frac{\tau_{iT} - \mu}{\nu} \right) \implies \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \frac{X_i - \mu}{\nu} \right)$ . Now if we allow  $N \rightarrow \infty$ , the Lindeberg-Levy central limit theorem applied to  $\{X_i\}$  allows us to have  $\frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \frac{X_i - \mu}{\nu} \right) \implies N(0, 1)$ . According to the theory of the sequential limit (see Phillips and Moon (1999)), we have  $\mathbf{K}_{\mu\nu} \implies N(0, 1)$  when  $T \rightarrow \infty$  and  $N \rightarrow \infty$ .

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**TABLES.**

**Table 1.** The Power of the HL-test and the KPSS-test under the alternative of jumps in the unconditional variance.

	$\lambda = \frac{\sigma_2}{\sigma_1}$				
	<b>2</b>	<b>5</b>	<b>10</b>	<b>15</b>	<b>20</b>
$\mathbf{n}_0 = 1\%$	0.049	0.046	0.048	0.044	0.046
$\mathbf{n}_0 = 5\%$	0.048	0.059	0.058	0.057	0.053
$\mathbf{n}_0 = 10\%$	0.045	0.052	0.051	0.053	0.057
<b>KPSS</b>	0.0489	0.052	0.0530	0.0520	0.0527

**Table 2.** Simulated values of  $E(\tau_{iT})$  and  $[var(\tau_{iT})]^{1/2}$  for each fixed value of  $T$ .

=====

<b>T</b>	<b>10</b>	<b>15</b>	<b>25</b>	<b>50</b>	<b>100</b>	<b>200</b>	..... $\infty$
$\mu_T^*$	0.6144068	0.6649516	0.71384303	0.7605456	0.7931306	0.81562	$\ln(2)\sqrt{\frac{\pi}{2}}^{**} \simeq 0.868$
$\nu_T^*$	0.2273081	0.2399637	0.24905655	0.2535578	0.2579918	0.25907	$\sqrt{\frac{\pi}{2}[\frac{\pi}{6} - \ln(2)^2]}^{**} \simeq 0.26$

=====

\*\* :Exact asymptotic values.  $\mu_T^*$  and the  $\nu_T^*$  are respectively the simulated values of  $E(\tau_{iT})$  and  $[var(\tau_{iT})]^{1/2}$  .

**Table 3.** The Power of the  $\tau_T$  against jumps in the unconditional variance.

	$\lambda$					
	1	$\sqrt{1.5}$	$\sqrt{2}$	$\sqrt{2.5}$	$\sqrt{3}$	2
$T=50$	0.030	0.076	0.22	0.363	0.537	0.760
$T=100$	0.042	0.193	0.480	0.769	0.90	0.984
$T=200$	0.048	0.386	0.860	0.982	0.997	1

**Table 4:** Nonstationarity in CAC 40

	Statistic( $\tau_T$ )	Statistic(KPSS)
$\ln(\text{indexcac}_t)$	2.5527116	19.623319
$\Delta \ln(\text{indexcac}_t)$	3.7669207***	0.15595496
	Statistic( $K_{\mu_T \nu_T}$ )	Statistic( HL)
$\ln(\text{cac}_{it})$	48.047794	648.13188
$\Delta \ln(\text{cac}_{it})$	55.006095***	-0.97417716

FIGURES.

Figure 1. Performance of the complementary test against abrupt changes in the variance.

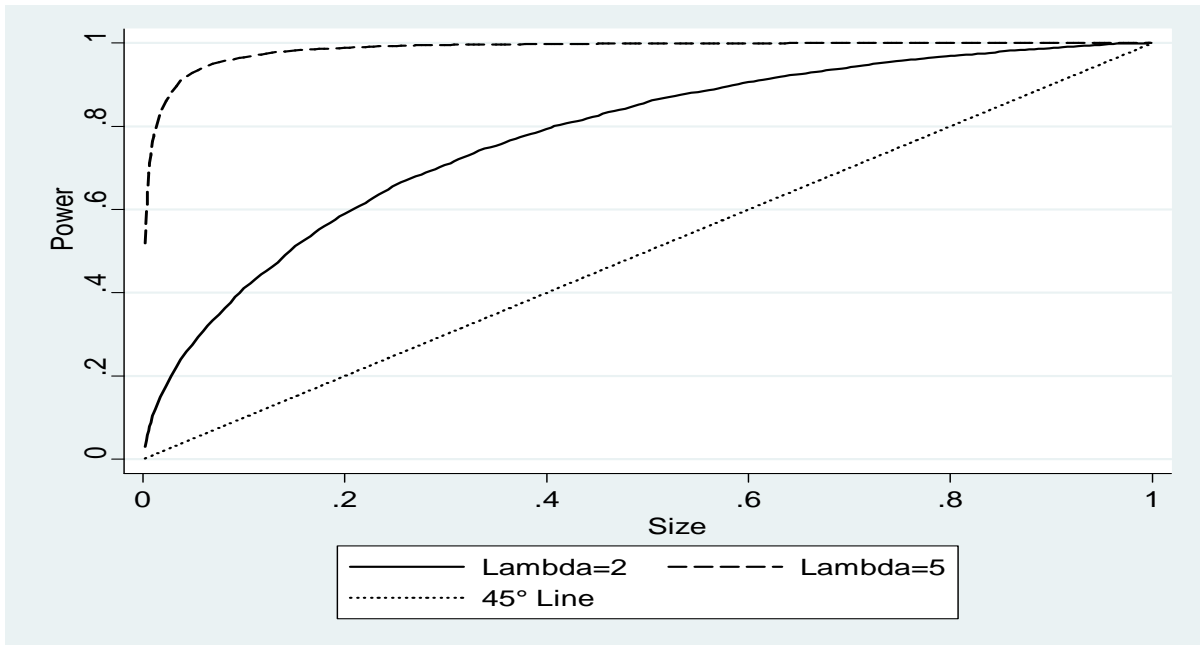
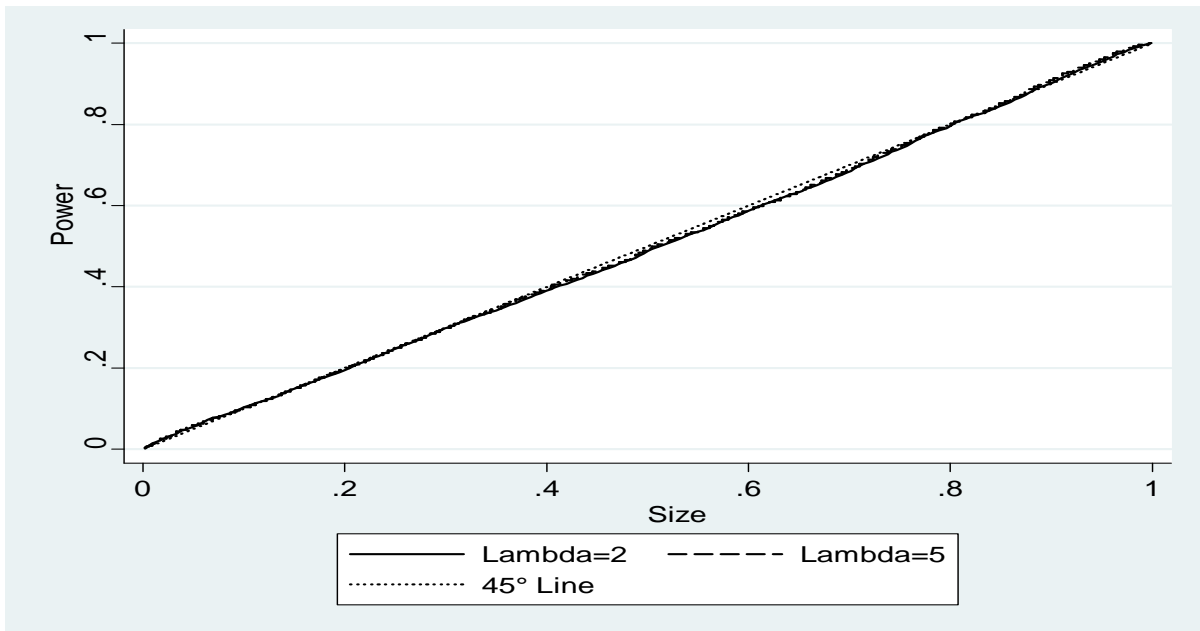
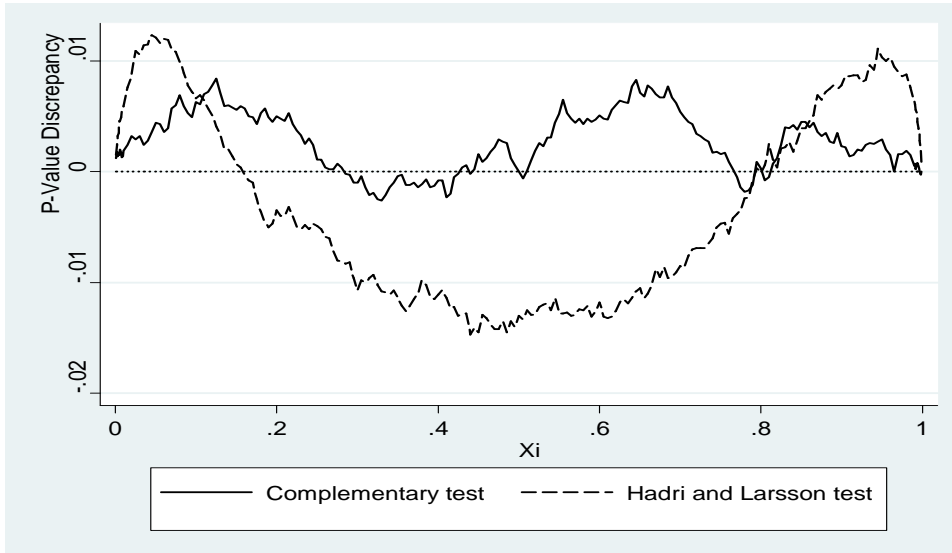


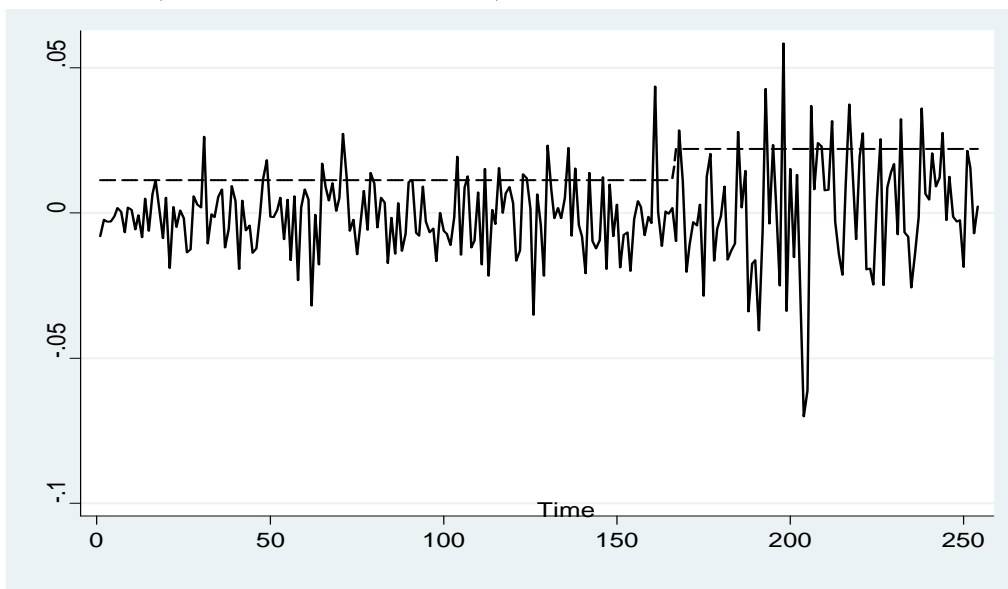
Figure 2. Performance of the HL test against abrupt changes in the variance.



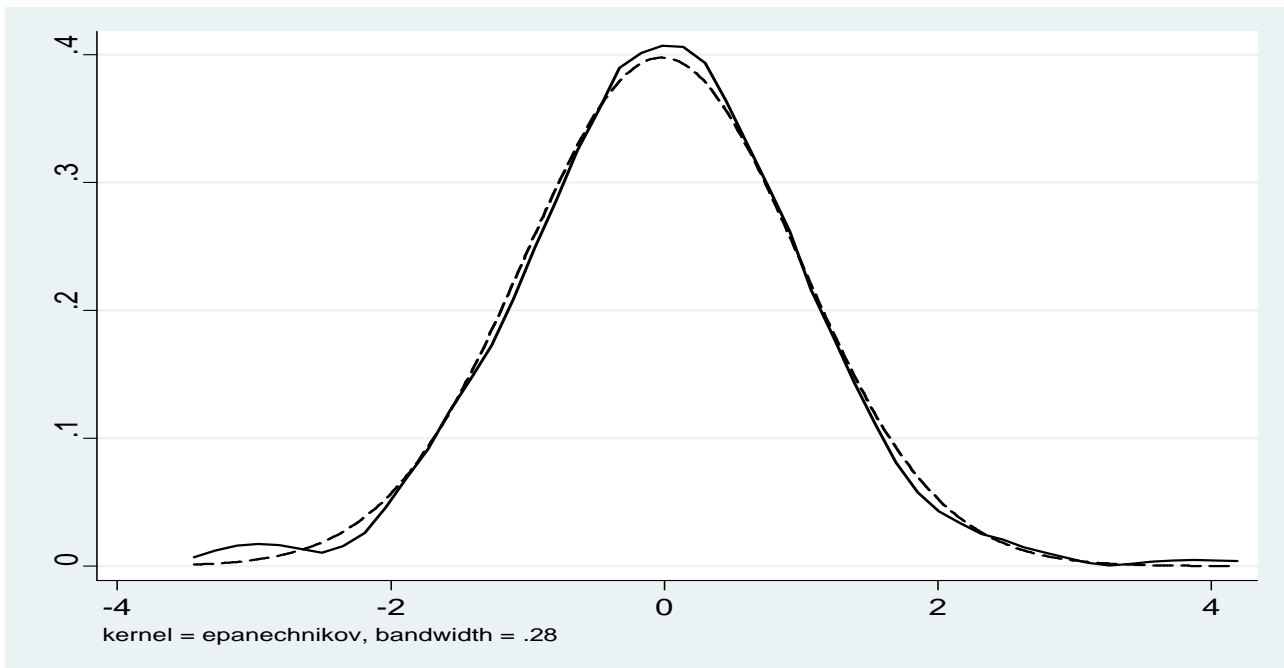
**Figure 3.** P-value Discrepancy for the two tests.



**Figure 4.**  $\hat{h}_t$  (horizontal discontinuous line) and returns.



**Figure 5:** Kernel density estimate of  $\hat{\varepsilon}_t$  (continuous line) and Normal density (discontinuous line)



**Figure.6:** Cumulative Periodogram White-Noise Test for  $\hat{\varepsilon}_t$ .

