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Price and Wealth Dynamics in a Speculative Market with Generic Procedurally Rational Traders *

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Abstract

An agent-based model of a simple financial market with arbitrary number of traders having relatively general behavioral specifications is analyzed. In a pure exchange economy with two assets, riskless and risky, trading takes place in discrete time under endogenous price formation setting. Traders' demands for the risky asset are expressed as fractions of their individual wealths, so that the dynamical system in terms of wealth and return is obtained. Agents' choices, i.e. investment fractions, are described by means of the generic smooth functions of an infinite information set. The choices can be consistent with (but not limited to) the solutions of the expected utility maximization problems.

A complete characterization of equilibria is given. It is shown that irrespectively of the number of agents and of their behavior, all possible equilibria belong to a one-dimensional "Equilibrium Market Line". This geometric tool helps to illustrate possibility of different phenomena, like multiple equilibria, and also can be used for comparative static analysis. The stability conditions of equilibria are derived for general model specification and allow to discuss the relative performances of different strategies and the selection principle governing market dynamics.

JEL codes: G12, D83.

Keywords: Asset Pricing Model, Procedural Rationality, Heterogeneous Agents, CRRA Framework, Equilibrium Market Line, Stability Analysis, Multiple Equilibria.

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1 Introduction

This paper is devoted to the analytic investigation of an asset pricing model where an arbitrary number of heterogeneous generic traders participate in a speculative activity. We consider a simple, pure exchange economy where one asset is a riskless security, yielding a constant return on investment, and another asset is a risky equity, paying a stochastic dividend. Trading takes place in discrete time and in each trading period the relative price of the risky asset is fixed through a market clearing condition. Agents participation to the market is described in terms of their individual demand for the risky asset. We impose only one restriction on the way in which the individual demands of traders are formed. Namely, the amount of the risky security demanded by any trader is assumed to be proportional to his current wealth. Corresponding *investment shares* of the agents' wealth are chosen at each period on the basis of the commonly available information.

This behavioral assumption is consistent with a number of strategies based on optimization, and in particular on the maximization of expected utility function with constant relative risk aversion (CRRA). However, the framework is not limited to such rational behaviors. Works of Herbert Simon (see e.g. Simon (1976)) emphasize that agents operating in the markets may not be optimizers, but still avoid to behave in completely random or irrational manner. That is even if they are not "rational" in the sense as this word is widely used in economics, the traders can follow some deliberately chosen or invented procedures. Such agents can be called *procedurally* rational to stress the difference with respect to the smaller class of substantively rational optimizers. We model procedural rationality by means of smooth *investment functions* which map the information set to the present investment share.

The presence of procedural rationality in the market naturally leads to the idea of heterogeneity of agents. There are not doubts that even rational agents differ in terms of preferences and implied actions. In the last years, many contributions emphasize an importance of the heterogeneity in expectations for explanation of observed "anomalies" of financial markets (i.e. those facts that cannot be explained by classic financial models) like huge trading volume or excess volatility, see e.g. Brock (1997). In this paper investment functions are agent-specific and, thus, describe the outcome of an idiosyncratic procedures which can be defined as the collective description of the preferences, beliefs and implied actions.

Using assumption of CRRA-type of behavior but avoiding the precise specification of investment functions we derive the dynamical system governing asset price and agents' wealths. The natural rest points of this system turn out to correspond to the constant levels of price *return* and *relative* wealths. We provide a complete characterization of such equilibria in terms of few parameters related with investment functions and derive their stability conditions. We find that, irrespectively of the number of agents operating in the market and of the shape of investment functions, there exist a simple relation between equilibrium price return and investment fractions. A simple function, the "Equilibrium Market Line", previously introduced in Anufriev, Bottazzi, and Pancotto (2006), can be used to obtain a geometric characterization of both the location of all possible equilibria and the conditions of their stability.

Our model can be confronted with the last contributions in the field of the Heterogeneous Agent Models (HAMs) extensively reviewed in Hommes (2006). First, the majority of HAMs are built assuming independence of the agents' demands from their wealths. In terms of expected utility theory, it amounts to consider constant absolute risk averse (CARA) traders. CARA-type of behavior is assumed for the sake of simplicity, in order to decouple wealth evolution from the system and concentrate on the analysis of price dynamics. Our choice of the CRRA framework is motivated, instead, by the empirical and experimental evidence in its favor (see the discussions in Levy, Levy, and Solomon (2000) and Campbell and Viceira (2002)) and also by a relative rarity of corresponding HAMs (exceptions are Chiarella and He (2001), Chiarella, Dieci, and Gardini (2006), Anufriev, Bottazzi, and Pancotto (2006)). Second, since HAMs are concentrated on the heterogeneity in expectations, it is typical to work with common demand functions (i.e. preferences, attitude towards risk, etc.). Furthermore, the expectations are modeled in the simplest possible way sufficient to reflect different stylized behaviors, like "fundamental", "trend chaser" or "contrarian" attitude. Keeping investment functions generic, we intend to avoid unrealistic simplicity of the agents' expectations and assumption of fixed preferences. Consequently, the role of such parameters as the length of memory or coefficient of trend extrapolation can be analyzed *without* changing the model set-up. Third, HAMs usually deal either few types of investors (e.g. two types in DeLong, Shleifer, Summers, and Waldmann (1991) and Chiarella and He (2001), three in Day and Huang (1990) and up to four types in Brock and Hommes (1998)) or with the limiting properties of the market when the number of types is large enough (Brock, Hommes, and Wagener, 2005) to apply some variation of the Central Limit Theorem. Instead, our results are valid for any finite number of investors.

At the same time, the current model can also be compared with so-called, evolutionary finance literature (see, e.g. Blume and Easley (1992), Sandroni (2000) and Hens and Schenk-Hoppé (2005)). These are analytic investigations of the market with many assets populated by the agents of CRRA-type behavior. One important drawback of these contributions is the assumption of short life of the assets leading to the ignorance of the capital gain on the agents' wealths. Our work can be seen, thus, as an extension of the evolutionary finance analysis in this direction, even if we consider simpler market setting with only one risky asset.

Finally, one can consider this model as an analytic counterpart of the numerous simulations of the artificial financial markets with CRRA agents (see, e.g. Levy, Levy, and Solomon (1994), Levy, Levy, and Solomon (2000) and Zschischang and Lux (2001) and recent review in LeBaron (2006)). The need of such analytic investigation seems apparent because of the inherent difficulty to interpret the results of simulations in a systematic way. Model with *generic* agents' behaviors is especially useful given the tendency to simulate markets with many different types of behavior.

There is one common question which unify all three streams of the literature mentioned above. Is the Milton Friedman's hypothesis about impossibility for the non-rational agents to survive in the market valid? Our general results provide a simple and clear answer to this question. Indeed, we show that the survivors in the market are determined not only by their strategies, but instead by the total behavioral ecology. Consequently, the Friedman's hypothesis is not valid in our framework, even if it can hold for some particular cases. E.g. applying our results to the set of expected utility maximizing behaviors one can re-obtain findings of Blume and Easley (1992) that the survivor is rational agent but not any rational agent will survive.

The rest of the paper is organized as follows. In Section 2 we describe our economy, presenting assumptions and briefly discussing them. First, we explicitly write the traders' inter-temporal budget constraints. Second, we derive the resulting dynamics in terms of returns and wealth shares. Finally, we introduce agent specific investment functions. In Section 3 we start the equilibrium and stability analysis of the system from the simplest case of a single agent in the market. The Equilibrium Market Line is derived, and its use is discussed. The stability conditions are obtained in general and, then, specified for some economically important special cases. Section 4 is devoted to the analysis of the general case with arbitrarily large number of traders in the market. Our findings and implications are summarized in Section 5.

2 Model Definition

Consider a simple pure exchange economy where trading activities take place in discrete time. The economy is composed by a riskless asset (bond) giving in each period a constant interest rate $r_f > 0$ and a risky asset (equity) paying a random dividend D_t at the beginning of each period t. The riskless asset is considered the numéraire of economy and its price is fixed to 1. The ex-dividend price P_t of the risky asset is determined at each period, on the basis of the aggregate demand, through market-clearing condition. The resulting intertemporal budget constraint is derived below and the main hypotheses, on the nature of the investment choices and of the fundamental process, are discussed. These hypotheses will allow us to derive the explicit dynamical system governing the evolution of the economy.

2.1 Intertemporal budget constraint

We consider general situation when the economy is populated by a fixed number N of traders[1]. Let $W_{t,n}$ stand for the wealth of trader n at time t and let $x_{t,n}$ stand for the fraction of this wealth invested into the risky asset. We consider the model without consumption where total agent's wealth has to be reinvested. Thus, after the trading session at time t, agent n possesses $x_{t,n}\,W_{t,n}/P_t$ shares of the risky asset and $(1-x_{t,n})\,W_{t,n}$ shares of the riskless security. In the beginning of time $t+1$ the agent gets (in terms of the numéraire) random dividends D_{t+1} per each share of the risky asset and constant interest rate r_f for all shares of the riskless asset. Therefore, at time $t+1$ the wealth of agent n reads

$$W_{t+1,n}(P_{t+1}) = (1-x_{t,n})\,W_{t,n}\,(1+r_f) + \frac{x_{t,n}\,W_{t,n}}{P_t}\,(P_{t+1}+D_{t+1}) \quad . \tag{2.1}$$

Through the capital gain, the new wealth depends on the price P_{t+1} of the risky asset, which is fixed so that aggregate demand equals aggregate supply. Assuming a constant supply of risky asset, whose quantity can then be normalized to 1, price P_{t+1} is defined as the solution of the equation

$$\sum_{n=1}^{N} x_{t+1,n}\,W_{t+1,n}(P_{t+1}) = P_{t+1} \quad . \tag{2.2}$$

Simultaneous solution of (2.1) and (2.2) provides new price P_{t+1}. Once the price is fixed, the new portfolios and wealths are determined and economy is ready for the next round.

The dynamics defined by (2.1) and (2.2) describe an exogenously growing economy due to the continuous injections of new riskless assets, whose price remains, under the assumption of totally elastic supply, unchanged. It is convenient to remove this exogenous economic expansion from the dynamics of the model. To this purpose we introduce rescaled variables

$$w_{t,n} = W_{t,n}/(1+r_f)^t \ , \quad p_t = P_t/(1+r_f)^t \ , \quad e_t = D_t/(P_{t-1}\,(1+r_f)) \quad , \tag{2.3}$$

[1] Using the terminology of the literature about heterogeneous agent modeling, we consider N *types* of agents (cf. Brock, Hommes, and Wagener (2005)). However, the relative wealth of all traders with the same investment behavior (type) is constant inside our framework. Furthermore, only the total wealth belonging to all traders of the same type matters for the aggregate market dynamics. Consequently, we associate each type with a sole trader. In the terminology of the evolutionary finance literature we deal with N different *strategies* (cf. Hens and Schenk-Hoppé (2005)).

denoted with lower case names. The last quantity, e_t, represents (to within the factor) the dividend yield. Rewriting (2.1) and (2.2) using new variables one obtains

$$\begin{cases} p_{t+1} = \sum_{n=1}^{N} x_{t+1,n}\, w_{t+1,n} \\ w_{t+1,n} = w_{t,n} + w_{t,n}\, x_{t,n} \left(\dfrac{p_{t+1}}{p_t} - 1 + e_{t+1} \right) \quad \forall n \in \{1, \ldots, N\} \quad . \end{cases} \tag{2.4}$$

These equations represent an evolution of state variables $w_{t,n}$ and p_t over time, provided that stochastic process $\{e_t\}$ is given and the set of investment shares $\{x_{t,n}\}$ is specified.

In this paper the agents' investment shares are assumed to be independent of the contemporaneous price and wealth, the assumption which will be formalized in Section 2.3. Under such assumption, the dynamics imply a simultaneous determination of the equilibrium price p_{t+1} and of the agents' wealths $w_{t+1,n}$, so that $N+1$ equations in (2.4) define the state of the system at time $t+1$ only implicitly. Indeed, the N variables $w_{t+1,n}$, defined in the second equation, appear on the right-hand side of the first, and, at the same time, the variable p_{t+1}, defined in the first equation, appears in the right-hand side of the second. For analytical purposes, one has to derive the explicit equations that govern the system dynamics.

2.2 The dynamical system for wealth shares and price return

Let a_n be an agent specific variable, dependent or independent from time t. We denote with $\langle a \rangle_t$ the *wealth weighted average* of this variable at time t on the population of agents, i.e.

$$\langle a \rangle_t = \sum_{n=1}^{N} a_n\, \varphi_{t,n} \ , \quad \text{where} \quad \varphi_{t,n} = \frac{w_{t,n}}{w_t} \quad \text{and} \quad w_t = \sum_{n=1}^{N} w_{t,n}. \tag{2.5}$$

The transformation of the implicit dynamics (2.4) into an explicit one is not, in general, possible also because the market price should remain positive over time. On the other hand, the agents are allowed to have negative wealth, which is interpreted as debt in that case. Therefore, $\varphi_{t,n}$ are arbitrary numbers whose sum over all agents is equal to 1 for any period t.

The next result gives the condition for which the dynamical system implicitly defined in (2.4) can be made explicit without violating the requirement of positiveness of prices.

Proposition 2.1. *Let us assume that initial price p_0 is positive. From equations (2.4) it is possible to derive a map $\mathbb{R}^N \to \mathbb{R}^N$ that describes the evolution of traders' wealth $w_{t,n}$ with positive prices $p_t \in \mathbb{R}^+$ $\forall t$ provided that*

$$\Big(\langle x_t \rangle_t - \langle x_t\, x_{t+1} \rangle_t \Big) \Big(\langle x_{t+1} \rangle_t - (1 - e_{t+1}) \langle x_t\, x_{t+1} \rangle_t \Big) > 0 \qquad \forall t \quad . \tag{2.6}$$

If previous condition is met, the growth rate of (rescaled) price $r_{t+1} = p_{t+1}/p_t - 1$ reads

$$r_{t+1} = \frac{\langle x_{t+1} - x_t \rangle_t + e_{t+1} \langle x_t\, x_{t+1} \rangle_t}{\langle x_t\, (1 - x_{t+1}) \rangle_t} \quad , \tag{2.7}$$

the individual growth rates of (rescaled) wealth $\rho_{t+1,n} = w_{t+1,n}/w_{t,n} - 1$ are given by

$$\rho_{t+1,n} = x_{t,n} \left(r_{t+1} + e_{t+1} \right) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{2.8}$$

and the agents' (rescaled) wealth shares $\varphi_{t,n}$ evolve according to

$$\varphi_{t+1,n} = \varphi_{t,n} \frac{1 + (r_{t+1} + e_{t+1})\, x_{t,n}}{1 + (r_{t+1} + e_{t+1}) \left\langle x_t \right\rangle_t} \qquad \forall n \in \{1, \ldots, N\} \quad . \tag{2.9}$$

Proof. See appendix A. □

The market evolution is *explicitly* described by the system of $N + 1$ equations in (2.7) and (2.8), or, equivalently, in (2.7) and (2.9). The dynamics of rescaled price p_t can be derived from (2.7) in a trivial way, but price will remain positive only if condition (2.6) is satisfied[2]. Finally, using (2.4), one can easily obtain the evolution of unscaled price P_t.

Expression (2.7) for the return determination stresses the role of the relative agents' wealths in our model. Agents who are more rich have a higher influence on the price determination. However, our model differs from other contributions with the same feature (Blume and Easley, 1992; Hens and Schenk-Hoppé, 2005) in that the capital gain is included into the price return so that the latter depends on the investment decisions from *two* consequent periods. Wealth dynamics (2.8) reveal that individual returns are proportional to the gross return (capital gain or loss plus the dividend yield), which is typical for the market with the CRRA agents. Finally, equation (2.9) describes the evolution of the relative wealth. One can interpret this relation as a *replicator dynamics*, initially used in mathematical biology and then in evolutionary economics, since it shows that an market influence of any agent changes according to his performance *relative* to the average performance. Furthermore, one has to take the (rescaled) wealth return as a measure of performance.

Having obtained the explicit dynamics for the evolution of price and wealth one is interested in the asymptotic behavior of the system. It turns out that the dynamics defined by (2.7) and (2.8) does not possess any interesting fixed point in terms of the levels of price and wealth. Indeed, if the price and the wealth were constant, one would have $r_{t+1} = \rho_{t+1,n} = 0$ for all t and n. This implies that for all agents $x_{t,n} = 0$ in those periods when a positive dividend is paid, i.e. there is no demand for the risky asset (and (2.6) is violated). The reason is that the rescaling of variables in (2.3) does not remove an expansion due to the wealth growth of the CRRA agents. The presence of this expansion suggests to look for possible asymptotic states of *steady growth*. In order to guarantee that the dynamics given in Proposition 2.1 is defined in terms of the price return and wealth shares[3] we make the following

Assumption 1. The dividend yields e_t are i.i.d. random variables obtained from a common distribution with positive support.

This assumption implies that price and dividends grow at the same rate. Even if this property characterizes the fundamental price in an economy with geometrically growing dividend, Assumption 1 is restrictive in our framework. The price in our model is determined through the market clearing condition and is not necessary fixed on the fundamental level. On the other hand, the annual historical data for the Standard&Poor 500 index suggest that yield can be reasonably described as a bounded positive random variable whose behavior is roughly stationary. Moreover, Assumption 1 is also common to several works in literature (Chiarella and He, 2001; Anufriev, Bottazzi, and Pancotto, 2006).

[2]In general, it may be quite difficult to check the validity of this condition at each time step. However, if agents are diversifying and do not go short, then inequality (2.6) is satisfied (Anufriev, Bottazzi, and Pancotto, 2006).

[3]Recall that the dividend yield term, appearing in (2.7) and (2.9), contains asset price in the denominator.

2.3 Agents' investment functions

In definition of the agents' behavior we intend do not rely on any particular specification of the functional form for the demand. Instead, we define it as general as possible inside our framework. One of the principle leading to our definition of the investment decisions is that these decisions, being idiosyncratic and endogenous, have to be independent of the contemporaneous price and wealth levels. Moreover, since in this paper we are mainly concerned with the effect of *speculative* behaviors on the market aggregate performance, we let aside those issues which may occur under asymmetric knowledge of the underlying fundamental process. Thus, we assume that the structure of the yield process defined in Assumption 1 is known to everybody. Along the same line, we assume that all agents base their investment decisions at time t exclusively on the public and commonly available information set $\mathcal{I}_{t-1}$ formed by past realized prices. This set can alternatively be defined through the past return realizations as $\mathcal{I}_{t-1} = \{r_{t-1}, r_{t-2}, \dots\}$ and we make the following

Assumption 2. For each agent n there exists a differentiable *investment function* f_n which maps the present information set into his investment share:

$$x_{t,n} = f_n(\mathcal{I}_{t-1}) \quad . \tag{2.10}$$

Function f_n in the right-hand side of (2.10) gives a complete description of the investment decision of the n-th agent. The knowledge about the fundamental process is not explicitly inserted in the information set but is embedded in the functional form of f_n. Past realizations of the fundamental process do not affect agents' decisions, which, rather, tend to adapt to observed price fluctuations. One can refer to this investment behavior, common in the agent-based literature (e.g. Brock and Hommes (1998)), as *"technical trading"*, stressing the similarity with trading practices observed in real markets. At the same time, Assumption 2 rules out other possible dependencies in the investment function f_n, like an explicit relation of the present investment choice with past investment choices or with investment choices of other traders. It is also clearly violated for those agents whose demand is independent of the wealth, like in case with CARA expected utility maximizers.

The investment choice described by (2.10) can be obtained as the result of two distinct steps. In the first step agent n, using a set of estimators $\{g_{n,1}, g_{n,2}, \dots\}$, forms his expectation at time t about the behavior of future prices, $\theta_{n,j} = g_{n,j}(\mathcal{I}_{t-1})$ where $\theta_{\cdot,j}$ stands for some statistics of the returns distribution at time $t+1$, e.g. the average return, the variance or the probability that a given return threshold be crossed. With these expectations, using a choice function h_n possibly derived from some optimization procedure, he computes the fraction of the wealth invested in the risky asset $x_{t+1,n} = h_n(\theta_{n,1}, \theta_{n,2}, \dots)$. For such interpretations, the investment function f_n from Assumption 2 would be a composition of estimators $\{g_{n,\cdot}\}$ and choice function h_n. This intuitive and common in the economic literature interpretation is not required by our framework, even if it is perfectly compatible with (2.10). In our model agents are not forced to use some specific predictors, rather they are allowed to map the past return history into the future investment choice, using whatever smooth function they like. Using the terminology coined by Herbert Simon, the traders modeled here are *procedurally* rational. Investment functions describe the outcome of an idiosyncratic procedures which can be defined as the collective description of the preferences, beliefs and implied actions. As a consequence, many behavioral specifications both from the classical framework with rational optimizers and also from the agent-based models with boundedly rational agents can be represented by suitable investment function. It opens a great space for applicability of our framework. Let us briefly outline two examples. (See Anufriev (2005) for a thorough discussion.)

Consider an agent who maximizes expected utility of his wealth and has power utility $U(W, \gamma) = W^{1-\gamma}/(1-\gamma)$, where $\gamma > 0$ is the relative risk aversion coefficient. Solution of this problem has a desirable property that investment share x^* is wealth independent. This property holds for any distribution of the next period return which agent is assumed to perceive now. However, x^* is unavailable in explicit form for continuous distributions. This technical issue had two consequences for the agent-based modeling. On one hand, the majority of elaborated analytic models were built in the CARA framework, which is less realistic than the CRRA one. On the other hand, the models with CRRA expected utility maximizers were based on the computer simulations, i.e. they lacked rigorous treatment. Now notice that one can define a certain class of investment functions, describing solutions of the expected utility maximization for different γ_n and the agent's perceptions of the return distribution. In other words, all such rational behaviors are covered by our framework. Furthermore, developed below geometric representation of equilibria allows to perform comparative static exercises within this class even without explicit knowledge of the investment functions from it.

The second example is inspired by the first analytical attempt to investigate a model with heterogeneous agents in the CRRA framework. Chiarella and He (2001) consider the market where all agents have mean-variance demand function, while their expectations about the future return and variance are heterogeneous. Apparently, our framework incorporates such agents' behaviors for *any* type of the expectation formation, not only for those considered in the original paper.

Under Assumption 1, the dynamics in terms of price return, wealth shares and investment shares are described by (2.7), (2.9) and (2.10). In order to analyse a finite-dimensional system we restrict each agent n to base his decision on the past L_n price returns. Without loss of generality we can assume that the "memory span" is the same for all traders and denote it as L. For the following discussion L must be finite, but can be arbitrarily large.

3 Single Agent Case

We start with the analysis of the very special situation in which a single agent operates on the market. The main reason to perform this analysis rests in its relevance for the multi-agent case, as we will see in the next Section. In particular, some type of the generic equilibrium in the setting with N heterogeneous traders requires, as necessary condition for stability, the stability of a suitably defined single agent equilibrium. Another application of the single agent analysis is to provide a succinct description of the aggregate properties of a system with many relatively homogeneous agents (Anufriev and Bottazzi, 2006).

This Section starts laying down the dynamics of the single agent economy as a multidimensional dynamical system of difference equations of the first order. All possible equilibria of the system are identified and the associated characteristic polynomial, which can be used to analyze their stability, derived.

3.1 Dynamical system

In the case of one single agent the dynamical system describing the market evolution can be considerably simplified since the explicit evolution of wealth shares in (2.8) is not needed. As a consequence, the whole system can be described with only $L+1$ variables representing one investment choice and L pas returns. We denote the price return at time $t-l$ as $r_{t,l}$, so that the system reads

$$\begin{cases} x_{t+1} &= f\big(r_{t,0}, r_{t,1}, \ldots, r_{t,L-1}\big) \\ r_{t+1,0} &= R\Big(f\big(r_{t,0}, r_{t,1}, \ldots, r_{t,L-1}\big),\ x_t,\ e_t\Big) \\ r_{t+1,1} &= r_{t,0} \\ \quad\vdots & \\ r_{t+1,L-1} &= r_{t,L-2} \end{cases} \quad . \tag{3.1}$$

Notice that we introduce function representing the right hand-side of (2.7):

$$R(x', x, e) = \frac{x' - x + e\,x'\,x}{(1 - x')\,x} \quad , \tag{3.2}$$

with variables x' and e denoting the current (contemporaneous with return) investment choice and dividend yield, and variable x denoting the *previous period* investment choice.

Dynamical system (3.1) contains the noise component e_t which is i.i.d. random variable according to Assumption 1. For the rigorous mathematical analysis we substitute the yield by its mean value $\bar{e}$ and consider the *deterministic skeleton* of (3.1). Resulting system gives, in a sense, the "average" representation of the stochastic dynamics. It turns out that independent of the agent's behavior, all possible equilibria of the skeleton belong to the one-dimensional geometric locus.

3.2 Equilibrium market line

Let us introduce the following

Definition 3.1. The *Equilibrium Market Line* (EML) is the function $l(r)$ defined as

$$l(r) = \frac{r}{\bar{e} + r} \quad . \tag{3.3}$$

Let x^* denote the agent's wealth share invested in the risky asset at equilibrium and let r^* be the equilibrium return. In any fixed point the realized returns are constant, so that $r_0 = r_1 = \cdots = r_{L-1} = r^*$. One has the following

Proposition 3.1. *Let $\boldsymbol{x^*} = (x^*; r^*, \ldots, r^*)$ be a fixed point of the deterministic skeleton of* (3.1)*. Then it is*

(i) Equilibrium return r^* satisfies

$$l(r^*) = f(r^*, \ldots, r^*) \quad , \tag{3.4}$$

while the equilibrium investment share $x^* = f(r^*, \ldots, r^*)$.

(ii) Equilibrium $\boldsymbol{x^*}$ generates positive prices, if either $x^* < 1$ or $x^* \geq 1/(1 - \bar{e})$.

(iii) The equilibrium growth rate of the agent's wealth is equal to price return r^*.

Proof. See appendix B. □

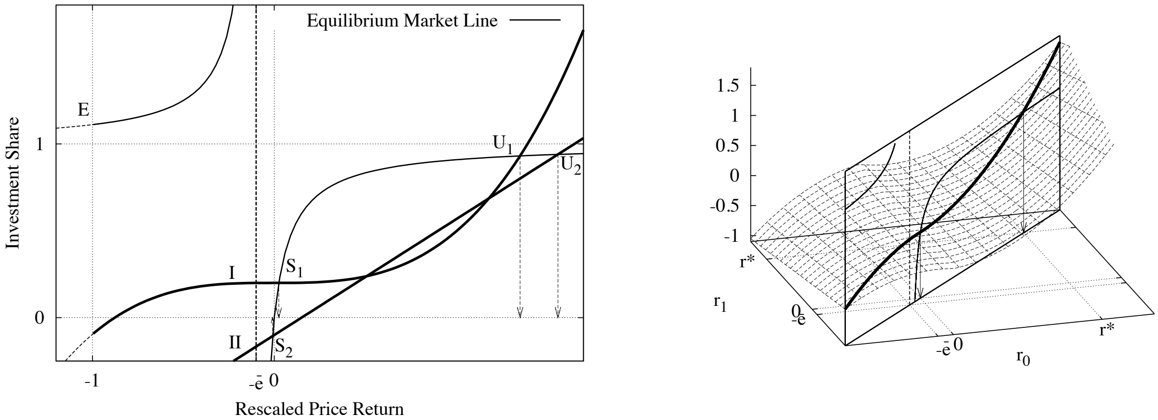

Figure 1: **Left panel:** Equilibria of system (3.1) are intersections of the EML with symmetrizations of the agents' investment functions (shown as thick lines and labeled as I and II). There are two equilibria in both cases: S_1 and U_1 in the market with agent I, and S_2 and U_2 in the market with agent II. **Right panel:** Investment function based on the last two realized returns $f(r_{t-1}, r_{t-2})$ and its intersection (thick line) with the plane $r_{t-1} = r_{t-2}$. Equilibria are found in this plane as intersections with the EML.

This statement justifies the introduction of the Equilibrium Market Line in Definition 3.1. Indeed, according to (3.4) all fixed points of the dynamics can be found as the intersections of the EML with the symmetrization of function f, i.e. with the restriction of this function to the one-dimensional subspace defined as $r_0 = r_1 = \cdots = r_{L-1}$. The main reason for such simple characterization of equilibria is the underlying requirement of consistency between fixed agent's action and resulting dynamics. Thus, the EML is the locus of points where this consistency condition is satisfied.

We illustrate condition (3.4) in Fig. 1. In the left panel the hyperbolic curve shown as a thin line represents the EML (3.3), while two thick lines depict two investment functions. More precisely, these lines are symmetrizations of some investment functions, so that the diagram shows the section of many-dimensional graph by the hyper-plane $r_0 = \cdots = r_{L-1}$. E.g. the right panel shows two-dimensional investment function depending on the two last realized returns, but only diagonal section, analogous to the left panel, is relevant for the question of equilibria location. The intersections of the investment function with the EML are all possible equilibria of the system. The ordinate of the intersection gives the value of equilibrium investment share x^*, while the abscissa gives the equilibrium return r^*.

According to Proposition 3.1*(ii)*, economically meaningful equilibria are characterized by values of the investment share inside the intervals $(-\infty, 1)$ or $[1/(1-\bar{e}), +\infty)$. It amounts to require $r^* > -1$, which implies that part of the EML on the left from point E is meaningless. On the remaining part of the Line one can distinguish between three qualitatively different scenarios.

In equilibria with $r^* \in [-1, -\bar{e})$ the return is negative and, hence, rescaled price p_t of the risky asset decreases to 0. Wealth of the agent is positive at any moment of time and also vanishes[4]. The agent possesses total supply of the risky asset, while his amount of the numéraire $B_t < 0$ as one can check writing the market clearing condition as $x^*(B_t + P_t) = P_t$.

[4]In general, to guarantee the positiveness of the price at the initial period one has to choose initial wealth appropriately. Since $p_0 = x^* w_0$, for positive x^* the initial (and consequent) agent's wealth is positive, while for negative x^* the wealth is negative.

Thus, the agent has to borrow money in order to keep his relatively high demand for the risky asset. Due to the decrease in the agent's wealth, this demand is unsufficient to provide high (even positive) return.

If $r^* \in (-\bar{e}, 0)$ the capital gain on the risky asset is negative and price of the asset falls. However, the contribution from the dividend makes the gross return $r^* + \bar{e}$ positive. Furthermore, agent has to be in debt and have negative wealth in order to guarantee the positiveness of price. From Proposition 3.1*(iii)* it follows that his wealth increases to 0. The agent possesses total supply of the risky asset and negative amount of the numéraire B_t. In this case agent has to borrow money in order to keep positive demand of the asset. Ultimately, the dividend payment allows the agent's wealth to increase. Equilibrium S_2 for the agent II in the left panel of Fig. 1 is of such kind.

Finally, if the rescaled return is positive, the price p_t of the asset increases. Agent has a positive amount of the numéraire and his total wealth is positive and increases. Such situation will be observed in equilibria S_1, U_1 and U_2.

What can be said about the dynamics of price P_t in all these three scenarios? To answer this question it is important to bear in mind the following relation between the scaled return r_t and return R_t in terms of unscaled price:

$$1 + R_t = (1 + r_t)\,(1 + r_f) \quad .$$

Thus, in the third scenario where the rescaled price increases, the unscaled price also increases with the higher rate. However, in the first and second scenarios where the rescaled price is falling down, the price before scaling may increase for high enough risk-free interest rate.

To conclude our discussion about equilibrium properties notice that in all possible equilibria there exists a non-zero equity premium, i.e. difference in the total return of the riskless and risky asset. The equity premium is observed in the real markets (Mehra and Prescott, 1985). It can be explained within the classical paradigm as a monetary incentive existing in the equilibrium to encourage an optimizing risk-averse representative agent to hold a risky asset. In our framework an equity premium in equilibrium can be easily computed as

$$\frac{P_{t+1} - P_t + D_{t+1}}{P_t} - r_f = \frac{\bar{e}\,(1 + r_f)}{1 - x^*} \quad .$$

In our framework the risk premium is endogenously generated due to the interplay of the total wealth reinvesting and dynamics feedback of the return to the wealth level. Consequently, equity premium increases with the dividend yield, risk-free interest rate and agent's investment share.

3.3 Stability of single-agent equilibria

As the next natural step we move to discuss the stability conditions of the equilibria. The stability conditions are derived from the analysis of the roots of the characteristic polynomial associated with the Jacobian of system (3.1) computed at equilibrium. The characteristic polynomial does, in general, depend on the behavior of the individual investment function f in an infinitesimal neighborhood of the equilibrium $\boldsymbol{x}^*$. This dependence can be summarized with the help of the following

Definition 3.2. The *stability polynomial* $P(\mu)$ of the investment function f in $\boldsymbol{x}^*$ is

$$P_f(\mu) = \frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}}\ , \tag{3.5}$$

where all the derivatives are computed in point $(r^*, \ldots, r^*)$.

Using the previous definition the equilibrium stability conditions can be formulated in terms of the equilibrium return r^*, and of the slope of the EML in equilibrium

$$l'(r^*) = \frac{\bar{e}}{(\bar{e}+r^*)^2} \, .$$

The following applies

Proposition 3.2. *The fixed point* $\boldsymbol{x^*} = (x^*; r^*, \ldots, r^*)$ *of system* (3.1) *is (locally) asymptotically stable if all the roots of the polynomial*

$$Q(\mu) = \mu^{L+1} - \frac{P_f(\mu)}{r^* \, l'(r^*)} \Big(\big(1+r^*\big)\, \mu - 1 \Big) \quad , \tag{3.6}$$

are inside the unit circle.

The equilibrium $\boldsymbol{x^*}$ *is unstable if at least one of the roots of* $Q(\mu)$ *lies outside the unit circle.*

Proof. The condition above is a direct consequence of the characteristic polynomial of the Jacobian matrix at equilibrium. See appendix C for derivation. □

Once investment function f is known, polynomial $P_f(\mu)$ and, in turn, polynomial $Q(\mu)$ are explicitly derived. The analysis of $L+1$ roots of $Q(\mu)$, which are usually called *multipliers*, can be performed in order to reveal the role of the different parameters in stabilization of a given equilibrium. Such rigorous analysis is often unfeasible even for simple investment functions, so one should rely on the computational approach, mostly. For illustrative purposes we present below three special cases, where analytical results are available to a certain extent. The reader is referred to Appendix D for justification of the results and further discussion.

Example 1. Agent with short memory, $L = 1$.

Consider an agent with single memory lag, i.e. with investment share depending only on the past return, $x_{t+1} = f(r_t)$. This is satisfied, e.g. for any agent with naïve forecast of the next period return. The stability polynomial is simple in this case, $P_f = f'(r^*)$, and multipliers are roots of the second-degree polynomial

$$Q(\mu) = \mu^2 - \frac{f'(r^*)}{l'(r^*)\, r^*} \Big(\big(1+r^*\big)\, \mu - 1 \Big) \quad . \tag{3.7}$$

Using general results reproduced in Propositions D.1 and D.2, one gets

Proposition 3.3. *The fixed point* $\boldsymbol{x^*} = (x^*; r^*)$ *of system* (3.1) *with* $L = 1$ *is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)} \frac{1}{r^*} < 1 \, , \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \textit{and} \qquad \frac{f'(r^*)}{l'(r^*)} \frac{2+r^*}{r^*} > -1 \quad . \tag{3.8}$$

Fixed point exhibits Neimark-Sacker*,* fold *or* flip *bifurcation if the first, second or third inequality in* (3.8) *turns to equality, respectively.*

The stability region defined by inequalities in (3.8) is shown in the upper left panel of Fig. 2 in coordinates r^* and $f'(r^*)/l'(r^*)$. The second coordinate is the relative slope of the investment function at equilibrium with respect to the slope of the Equilibrium Market Line. Notice that if the slope of f at the equilibrium increases, the system tends to lose its stability. In particular, the second inequality in (3.8) requires the slope of investment function to be smaller than the slope of function $l(r)$.

Let us come back to the example in the left panel of Fig. 1 and assume that $L = 1$ for each agent. Both equilibrium U_1 for agent I and U_2 for agent II are unstable, since the second inequality in (3.8) is violated. On the contrary, equilibrium S_1 is (presumably) stable, since the slope of the investment function in S_1 is very small. From Fig. 2 it follows that if the slope of function I in S_1 increases, this equilibrium loses stability through a Neimark-Sacker bifurcation, implying smooth cyclical return trajectory for parameters close to bifurcation value. The increase of the slope of the second function in S_2 would, instead, lead to flip bifurcation and return time series close to 2-cycle.

Example 2. EWMA forecasting rule

In agreement with the two-step procedure mentioned in Section 2.3, we now assume that agent formulates at time $t+1$ forecast y_t for the next period return and then invests fraction $x_{t+1} = f(y_t)$ of his wealth. Furthermore, we assume the following forecasting rule

$$y_t = C_L(\lambda)\,\big(r_t + \lambda\, r_{t-1} + \cdots + \lambda^{L-1}\, r_{t-L+1}\big) \quad , \tag{3.9}$$

where the normalization coefficient $C_L(\lambda) = (1-\lambda)/(1-\lambda^L)$. Thus, agent uses an Exponentially Weighted Moving Average (EWMA) of the past returns as forecast. The decay factor $\lambda \in [0, 1)$ determines how the relative weights in the average y_t are distributed across more recent and older observations. The last available observation r_t has the highest weight.

To explore stability of the system with a single EWMA-forecaster, notice that at equilibrium the forecast coincides with return, $y_t = r^*$, and the stability polynomial reads

$$P_f(\mu) = f'(r^*)\,\frac{1-\lambda}{1-\lambda^L}\,\frac{\mu^L - \lambda^L}{\mu - \lambda} \quad . \tag{3.10}$$

Substitution of the last polynomial into (3.6) gives characteristic polynomial $Q(\mu)$. Let us denote as $S_{\lambda,L}$ the stability region drawn in the coordinates of Fig. 2, i.e. area where all $L+1$ multipliers lie inside the unit circle. We are interested in the dependence of the shape of this region on the decay factor λ and memory span L.

Let us, first, fix λ. When $L = 1$ we find ourselves in the limits of the previous example (independently of λ). On the other extreme, when $L = \infty$, one can prove the following

Proposition 3.4. *The fixed point* $\boldsymbol{x^*} = (x^*; r^*)$ *of system* (3.1) *with* $L = \infty$ *is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)}\,\frac{1}{r^*} < \frac{1}{1-\lambda}\,, \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \text{and} \qquad \frac{f'(r^*)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -\,\frac{1+\lambda}{1-\lambda} \quad . \tag{3.11}$$

This fixed point exhibits Neimark-Sacker, fold *or* flip *bifurcation if the first, second or third inequality in* (3.11) *turns to equality, respectively.*

Proof. The result can be obtained rigorously through reduction of infinite-dimensional system (3.1) to the two-dimensional one, using the recursive relation available for the EWMA estimator in case $L = \infty$. See Anufriev, Bottazzi, and Pancotto (2006). In Appendix D we sketch an alternative proof. □

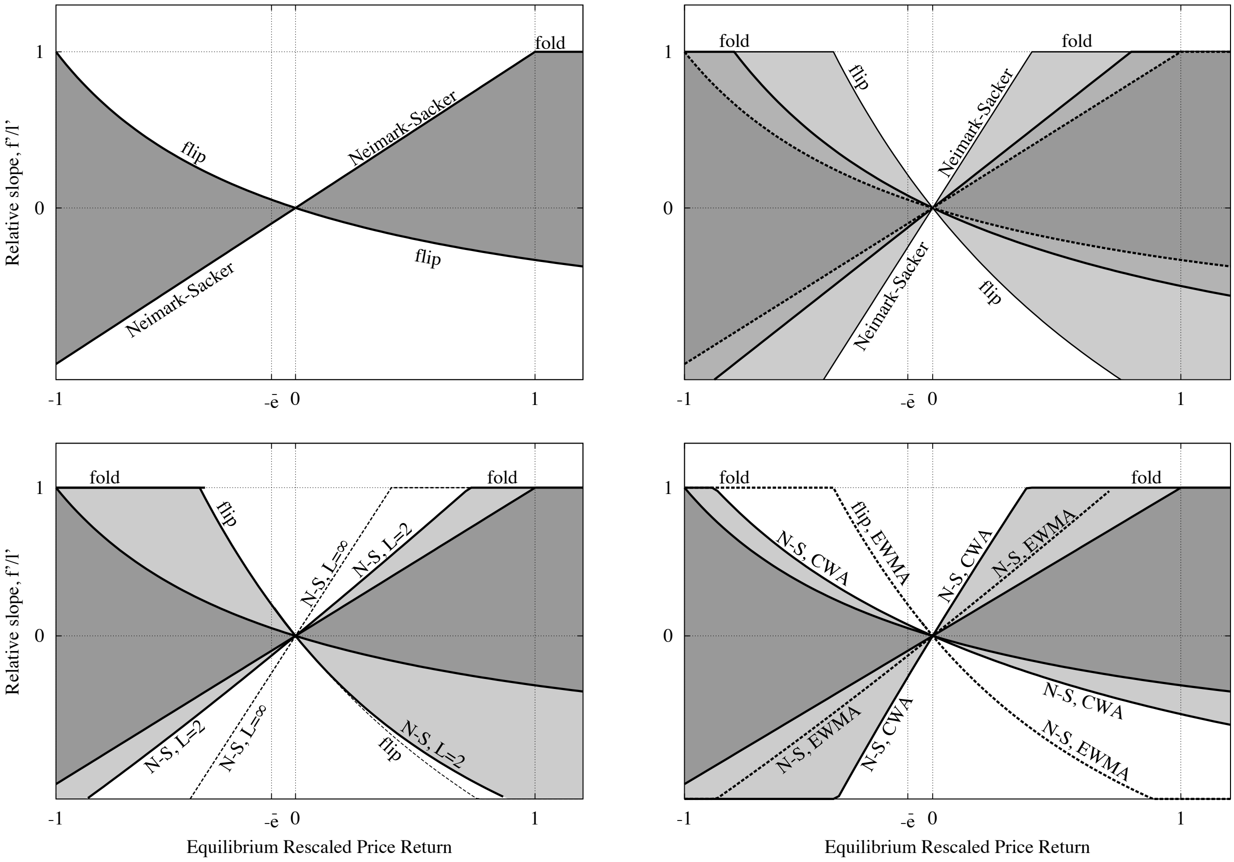

Figure 2: Stability regions and types of bifurcations for system (3.1) in special cases. **Upper Left Panel:** Example 1, $L = 1$. Fixed point is stable if $(r^*, f'/l')$ belongs to the dark gray area. **Upper Right Panel:** Example 2, EWMA estimator for different λ and $L = \infty$. For $\lambda = 0$ the stability region is the same as in example 1, shown as the dark gray area. If $\lambda = 0.2$ it expands and becomes the union of the dark and semi-dark gray areas. When $\lambda = 0.6$ the region expands further and contains the light gray areas in addition. **Lower Left Panel:** Example 2, EWMA estimator with $\lambda = 0.6$ for different L. When $L = 1$ the stability region is the same as in example 1, shown as the dark gray area. If $L = 2$ the region expands and contains the light gray areas in addition. The boundaries for $L = \infty$ are shown as dotted lines. **Lower Right Panel:** Example 3, CWA estimator for different L. When $L = 1$ the stability region is the same as in example 1, shown as the dark gray area. For $L = 2$ the region expands and becomes the union of the dark and light gray areas. The bifurcation loci for the EWMA case with $L = 2$ and $\lambda = 0.6$ are shown as dotted lines for comparison.

The regions $S_{0,\infty}$, $S_{0.2,\infty}$ and $S_{0.6,\infty}$ defined by (3.11) for different values of λ are depicted on the upper right panel of Fig. 2.

For the intermediate case, when $L > 1$ but finite, analytic results are limited. In Appendix D we derive the relations between parameters when one of the multipliers crosses the unit circle for the case $L = 2$. The corresponding curves are shown, when $\lambda = 0.6$, in the lower left panel of Fig. 2. They are labeled as “N-S”, “flip” and “fold” depending on where this crossing happens exactly (e.g. ”N-S” curve corresponds to those points where two complex conjugated multipliers cross the circle). Since all equilibria generated by the horizontal investment functions are stable, the points on the horizontal axes should lie in the stability region. The construction of this region can now be finalized using the argument of continuous dependence of the roots of polynomial on its coefficients. Notice that investment functions with negative slope and $r^* > 0$ generate now the Neimark-Sacker (and not flip) bifurcations.

For larger L one can show that the set of those parameters where one multiplier is equal to 1, is invariant and given by line $f'/l' = 1$. The locus of parameters where a multiplier is equal to -1 depends on whether L even or odd. For even L the locus is invariant and coincides with the corresponding boundary of region $S_{\lambda,\infty}$. When L is odd, the locus changes and point-wise converges to the boundary of $S_{\lambda,\infty}$. We can make a conjecture, therefore, that the expansion of stability region is not monotone with L. However, as Proposition 3.4 guarantees, the increase in the memory span *ultimately* brings the stability to all fixed points belonging to region $S_{\lambda,\infty}$. Since for given λ this region does not cover the whole parameter space $(r^*, f'/l')$, not all fixed points can be stabilized by change of L.

Can still unstable points become stable by appropriate choice of the decay factor λ? The general answer to this question is no, since condition $f'/l' < 1$ has to be satisfied. However, all fixed points for which this condition holds can be stabilized through the increase of λ. Indeed, from (3.11) it follows that region $S_{\lambda,\infty}$ enlarges with λ and for $\lambda \to 1$ contains all equilibria with $f'(r^*)/l'(r^*) < 1$. (See also the upper right panel of Fig. 2.)

Let us summarize our findings, referring again on the EML plot in Fig. 1. For behavior based on the EWMA estimator, equilibria U_1 and U_2 cannot be stabilized neither by increase of the memory span L, nor by increase of the decay factor λ. Instead, both equilibria S_1 and S_2 will become stable if, first, one chooses large enough λ and, second, appropriate L.

Example 3. Sample average forecasting rule

In our final example we modify the forecasting rule (3.9) in order to use the simple sample average as a forecast, as e.g. in Levy, Levy, and Solomon (2000) and Chiarella and He (2001). Such Constant Weighted Average (CWA) is given as follows

$$y_t = \frac{1}{L}\left(r_t + r_{t-1} + \cdots + r_{t-L+1}\right) \quad . \tag{3.12}$$

When $L = 1$ we are again in the situation of the first example. What happens if L increases? The line where a multiplier crosses the unit circle in point $\mu = 1$ is still given as $f'/l' = 1$. The locus of parameters such that a multiplier is equal to -1 does not exist for even L. Thus, the system can lose stability only through Neimark-Sacker or fold bifurcation. Example with $L = 2$ is depicted in the lower right panel of Fig. 2. Thus, the stability region enlarges with increase of L, analogously to the previous case. There is one important difference, however. Namely, the increase of memory span L leads now to the stabilization of *any* fixed point of the system with $f'/l' < 1$.

4 Economy with Many Agents

This Section extends the previous results to the case of a finite, but arbitrarily large, number of heterogenous agents. Each agent n possesses his own investment function f_n based on a finite number L of past market realizations. We start this section with the derivation of the $2N + L - 1$ dimensional stochastic dynamical system which describes the evolution of the economy and, then, identify all possible equilibria of the associated deterministic skeleton and analyze their stability.

4.1 Dynamical system

The evolution of agents' wealths is not any longer decoupled from the system and, consequently, all equations in (2.9) are relevant for the dynamics. The first-order dynamical system will be defined in terms of the following $2N+L-1$ independent variables

$$x_{t,n} \quad \forall n \in \{1,\dots,N\}\,; \qquad \varphi_{t,n} \quad \forall n \in \{1,\dots,N-1\}\,; \qquad r_{t,l} \quad \forall l \in \{0,\dots,L-1\}\,, \tag{4.1}$$

where $r_{t,l}$ denotes the price return at time $t-l$. Notice that only $N-1$ wealth shares are needed. Indeed, since wealth shares are summed up to 1 at any time step t, $\varphi_{t,N} = 1 - \sum_{n=1}^{N-1}\varphi_{t,n}$. The dynamics of the system is provided by the following

Lemma 4.1. *The* $2N+L-1$ *dynamical system defined by* (2.7) *and* (2.9) *under Assumptions 1 and 2 reads*

$$\begin{aligned}
\mathcal{X}: &\left[\begin{array}{lcl}
x_{t+1,1} &=& f_1\big(r_{t,0},\dots,r_{t,L-1}\big)\\
\quad\vdots & \vdots & \vdots\\
x_{t+1,N} &=& f_N\big(r_{t,0},\dots,r_{t,L-1}\big)
\end{array}\right.\\
\mathcal{W}: &\left[\begin{array}{lcl}
\varphi_{t+1,1} &=& \varPhi_1\Big(x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1};\\
&& \qquad R\big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1});\\
&& \qquad\quad x_{t,1},\dots,x_{t,1};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}\big)\Big)\\
\quad\vdots & \vdots & \vdots\\
\varphi_{t+1,N-1} &=& \varPhi_{t,N-1}\Big(x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1};\\
&& \qquad R\big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1});\\
&& \qquad x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}\big)\Big)
\end{array}\right.\\
\mathcal{R}: &\left[\begin{array}{lcl}
r_{t+1,0} &=& R\Big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1});\\
&& \qquad x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}\Big)\\
r_{t+1,1} &=& r_{t,0}\\
\quad\vdots & \vdots & \vdots\\
r_{t+1,L-1} &=& r_{t,L-2}
\end{array}\right.
\end{aligned}\tag{4.2}$$

where

$$\begin{aligned}
&R\Big(y_1,y_2,\dots,y_N;x_1,x_2,\dots,x_N;\varphi_1,\varphi_2,\dots,\varphi_{N-1};e\Big) =\\
&= \frac{\sum_{n=1}^{N-1}\varphi_n\big(y_n\,(1+e\,x_n)-x_n\big) + \Big(1-\sum_{n=1}^{N-1}\varphi_n\Big)\,\big(y_N\,(1+e\,x_N)-x_N\big)}{\sum_{n=1}^{N-1}\varphi_n\,x_n\,(1-y_n) + (1-\sum_{n=1}^{N-1}\varphi_n)\,x_N\,(1-y_N)}
\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}
&\varPhi_n\big(x_1,x_2,\dots,x_N;\varphi_1,\varphi_2,\dots,\varphi_{N-1};e;R\big) =\\
&= \varphi_n\,\frac{1+x_n\,(R+e)}{1+(R+e)\Big(\sum_{m=1}^{N-1}\varphi_m\,x_m + \big(1-\sum_{m=1}^{N-1}\varphi_m\big)\,x_N\Big)} \qquad \forall n \in \{1,\dots,N-1\}\,.
\end{aligned}\tag{4.4}$$

Proof. We ordered the equations to obtain three separated blocks: $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{R}$. In block $\mathcal{X}$ there are N equations defining the investment choices of agents. Block $\mathcal{W}$ contains $N-1$ equations describing the evolution of the wealth shares. Finally, block $\mathcal{R}$ is composed by L equations which describe the evolution of the return. In the last block equations are in ascending order with respect to the time lag.

The set $\mathcal{X}$ is immediately obtained from the definition of the investment functions. The first equation of block $\mathcal{R}$ is (2.7) rewritten in terms of variables (4.1) using (4.3) and (2.5), while the remaining equations are just the result of a "lag" operation. Notice that (4.3) reduces to (3.2) in the case of a single agent. Finally, the evolution of wealth shares described in block $\mathcal{W}$ is obtained from (2.9) once the notation introduced in (2.5) is explicitly expanded. Notice that, due to the presence of function R in the last expression, all functions Φ_n depend on the same set of variables as R. □

The rest of this Section is devoted to the analysis of the *deterministic skeleton* of (4.2): we replace the yield realizations $\{e_t\}$ by their mean value $\bar{e}$ and analyze the equilibria of the resulting deterministic system.

4.2 Determination of equilibria

The characterization of fixed points of system (4.2) is in many respects similar to the single agent case discussed above. Let $\boldsymbol{x^*} = (x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_{N-1}^*; r^*, \ldots, r^*)$ denotes a fixed point where r^* is the equilibrium return, and x_n^* and φ_n^* stay for the equilibrium value of the investment function and the equilibrium wealth share of agent n, respectively. Let us introduce the following

Definition 4.1. Agent n is said to *survive* in $\boldsymbol{x^*}$ if his equilibrium wealth share is different from zero, $\varphi_n^* \neq 0$. Agent n is said to *dominate* the economy if he is the only survivor, i.e. $\varphi_n^* = 1$.

One can recognize the parallel between our definition above and the frameworks in DeLong, Shleifer, Summers, and Waldmann (1991) and Blume and Easley (1992). We adopt here the deterministic version of the concepts of survival and dominance used in that papers. The following statement characterizes all possible equilibria of system (4.2).

Proposition 4.1. *Let $\boldsymbol{x^*}$ be a fixed point of the deterministic skeleton of system* (4.2)*. Then equilibrium investment shares are defined according to*

$$x_n^* = f_n(r^*, \ldots, r^*) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{4.5}$$

and three mutually exclusive cases are possible:

(i) ***Single agent survival.*** *In $\boldsymbol{x^*}$ only one agent survives and, therefore, dominates the economy. Without loss of generality we can assume this agent to be agent* 1 *so that $\varphi_1^* = 1$ and all other equilibrium wealth shares are equal to zero.*

The equilibrium return r^* is determined as the solution of

$$l(r^*) = f_1(r^*, \ldots, r^*) \quad , \tag{4.6}$$

and equal to the wealth growth rate of the survivor.

(ii) ***Many agents survival.*** *In $\boldsymbol{x}^*$ more than one agent survives. Without loss of generality one can assume that the agents with non-zero wealth shares are the first k agents (with $k > 1$) so that the equilibrium wealth shares satisfy*

$$\varphi_n^* = 0 \quad \textit{if } n > k \qquad \textit{and} \qquad \sum_{n=1}^{k} \varphi_n^* = 1 \quad . \tag{4.7}$$

The equilibrium return r^* must simultaneously satisfy the following set of k equations

$$l(r^*) = f_n(r^*, \ldots, r^*) \qquad \forall n \in \{1, \ldots, k\} \quad , \tag{4.8}$$

implying that the first k agents have the same investment share $x^*_{1\diamond k}$ at equilibrium. The wealth growth rates of all survivors are equal to r^*.

(iii) ***Absence of Equity Premium.*** *In $\boldsymbol{x}^*$ the equilibrium return $r^* = -\bar{e}$. The wealth shares of agents satisfy to*

$$\sum_{n=1}^{N} x_n^* \varphi_n^* = 0 \qquad \textit{and} \qquad \sum_{n=1}^{N} \varphi_n^* = 1 \quad . \tag{4.9}$$

The wealth growth rates of all agents are equal to 0.

Proof. See appendix E. □

Strictly speaking, we could distinguish between only two situations in this Proposition. The first situation is described by item *(ii)* with arbitrary k, so that when $k = 1$ item *(i)* becomes a particular case. We show below that, asymptotically, such situation is equivalent to the single-agent scenario. The second situation is described in item *(iii)* and differs from the previous one, because of the absence of the risk premium which can not happen in the single agent market.

Notice the difference between items *(i)* and *(ii)*. In the first case, when a single agent survives, Proposition 4.1 defines a precise value for each component (x^*, φ^* and r^*) of the equilibrium $\boldsymbol{x}^*$, so that a single point is uniquely determined. In the second case, on the contrary, there is a residual degree of freedom in the definition of the equilibrium: while r^* and investment shares x^*'s are uniquely defined, the only requirement on the equilibrium wealth shares of the surviving agents is the fulfillment of the second equality in (4.7). Consequently we have

Corollary 4.1. *Consider the deterministic skeleton of system* (4.2)*. If it possesses one equilibrium $\boldsymbol{x}^*$ with k survivors, it possesses a $k-1$-simplex of equilibria with k-survivors constituted by all the points obtained from $\boldsymbol{x}^*$ through a change in the relative wealths of the survivors. If the first k agents survive as in* (4.7)*, this set can be written as*

$$\left\{ \left(x_1^*, \ldots, x_N^*; \varphi_1, \ldots, \varphi_k, \underbrace{0, \ldots, 0}_{N-1-k}; \underbrace{r^*, \ldots, r^*}_{L}; \right) \,\middle|\, \sum_{j=1}^{k} \varphi_j = 1 \right\} .$$

The differences among the first two cases of Proposition 4.1 does not only regard the geometrical nature of the *locus* of equilibria. Indeed, while in the first case no requirements are imposed on the behavior of the investment function of the different agents, in the second

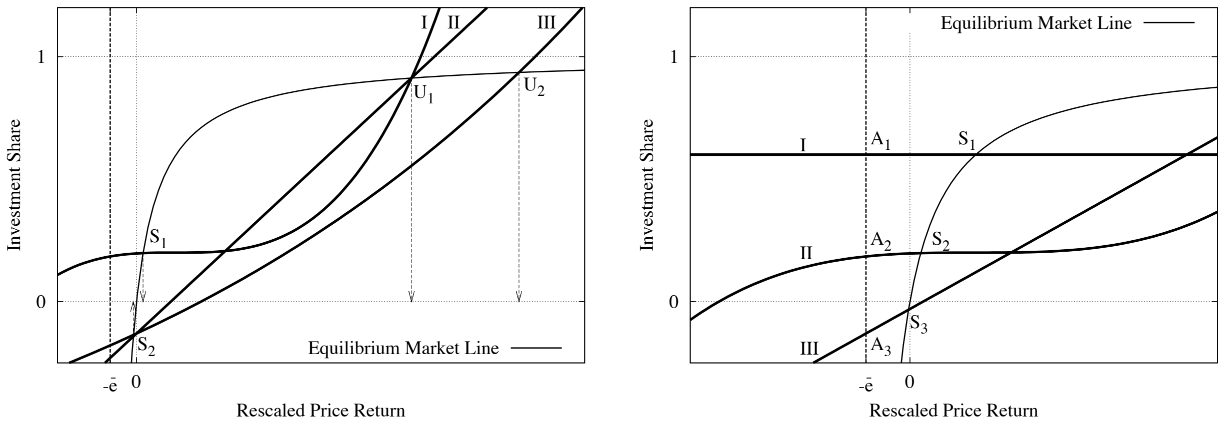

Figure 3: **Left panel**: Non-generic situations in which in S_2 agents II and II survive, while in U_1 agents I and II survive. **Right panel**: No-equity-premium equilibria with three survivors, who invest in points A_1, A_2 and A_3.

type of solutions all the investment shares $x_1^*, \dots, x_k^*$ must *at the same time* be equal to a single value $x_{1\diamond k}^*$. The equilibrium with $k > 1$ survivors exists only in the particular case in which the k investment functions $f_1, \dots, f_k$ satisfy this restriction. Consequently, an economy composed by N agents having generic, so to speak "randomly defined", investment functions, has probability zero of displaying any equilibrium with multiple survivors. In other terms, such many survivors equilibria are non-generic.

Both types of multi-agent equilibria derived in Proposition 4.1*(i)* and *(ii)* are strictly related to "special" single-agent equilibria. As in the single agent case, the growth rate of the total wealth is equal to the equilibrium price return and is determined by the growth rate of those agents who survive in the equilibrium. Moreover, the determination of the equilibrium return level r^* for the multi-agent case in (4.6) or (4.8) is identical to the case where the agent, or one of the agents, who would survive in the multi-agent equilibrium, is present alone in the market. An useful consequence of this fact is that the geometrical interpretation of market equilibria presented in Section 3.2 can be extended to illustrate how equilibria with many agents are determined.

As an example let us consider the left panel of Fig. 1 and suppose that agents with two investment functions shown in the left panel *simultaneously* operate in the market. According to Proposition 4.1*(i)* all possible equilibria can be found as intersections of one of the functions with the EML (c.f. (4.6)). In this case there are four possible equilibria. In two of them (S_1 and U_1) the agent I survives such that $\varphi_1^* = 1$. In the other two equilibria (S_2 and U_2) the agent II survives and $\varphi_1^* = 0$. In each equilibrium, the intersection of the investment function of the surviving agent with the EML gives both the equilibrium return r^* and the equilibrium investment share of the survivor. The equilibrium investment share of the other agent can be found, accordingly to (4.5), as the intersection of his own investment function with the vertical line passing through the equilibrium return. Since two investment functions shown in the left panel of Fig. 1 do not possess common intersections with the EML, the equilibria with more than one survivors are impossible. An example of investment functions which allow for multiple survivors equilibria is reported in the left panel of Fig. 3. The common intersection of different investment functions with the EML define the manifold of the multiple survivors equilibria.

Let us now turn to the "no-equity-premium" equilibria identified in Proposition 4.1*(iii)*. In these equilibria many agents survive, and their investment and wealth shares are balanced in such a way that the capital gain and the dividend yield offset each other so that the riskless and the risky assets have the same expected return. As opposite to the situation described in Proposition 4.1*(ii)*, these are generic equilibria with many survivors. Furthermore, if $N > 2$ then definition of any no-arbitrage equilibrium has additional degrees of freedom corresponding to a change in the relative wealths of the survivors. Namely, there exist the following $N-2$-dimensional manifold of equilibria

$$\left\{ \left(x_1^*, \ldots, x_N^*; \varphi_1, \ldots, \varphi_{N-1-k}; \underbrace{-\bar{e}, \ldots, -\bar{e}}_{L} \right) \Bigg| \sum_{j=1}^{N} \varphi_j = 1 \, , \sum_{j=1}^{N} \varphi_j x_j^* = 0 \right\} .$$

Geometrically, the "no-equity-premium" equilibria can be represented by the vertical asymptote of the EML. Points A_1, A_2 and A_3 in the right panel of Fig. 3 represents corresponding investment shares of the agents, while the wealth shares are defined from (4.9).

4.3 Stability of multi-agents equilibria

This Section presents the stability analysis of the equilibria defined in Proposition 4.1. The three Propositions below provide the stability region in the parameter space for the cases enumerated in Proposition 4.1, i.e. for generic case of a single survivor, for non-generic case of many survivors and for generic case with many survivors and without the equity premium. The derivation of these Propositions requires quite cumbersome algebraic manipulations and we refer the reader to Appendix F for the intermediate lemmas and final proofs.

For the generic case of a single survivor equilibrium we have the following

Proposition 4.2. *Let $\boldsymbol{x}^*$ be a fixed point of* (4.2) *associated with a single survivor equilibrium. Without loss of generality we can assume that the survivor is the first agent. Let $P_{f_1}(\mu)$ denote the $(L-1)$-dimensional stability polynomial associated with the investment function of the survivor.*

Equilibrium $\boldsymbol{x}^*$ is (locally) asymptotically stable if the two following conditions are met:

***1)** all the roots of polynomial*

$$Q_1(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\, l'(r^*)}\, P_{f_1}(\mu) \quad , \tag{4.10}$$

are inside the unit circle.

***2)** the equilibrium investment shares of the non-surviving agents satisfy to*

$$-2 - r^* < x_n^* \left(r^* + \bar{e} \right) < r^* \, , \quad 1 < n \leq N \, . \tag{4.11}$$

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of polynomial in (4.10) *is outside the unit circle or if at least one of the inequalities in* (4.11) *holds with the opposite (strict) sign.*

In particular, the system exhibits a fold bifurcation if one of the $N-1$ right-hand inequalities in (4.11) *becomes an equality and a flip bifurcation if one of the $N-1$ left-hand inequalities becomes an equality.*

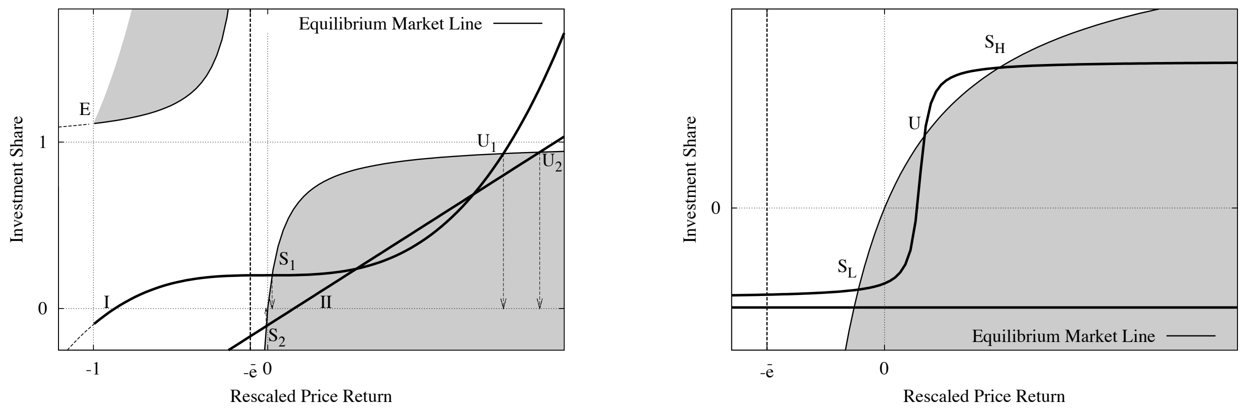

Figure 4: **Left Panel:** Four equilibria in the market with two agents. The region where condition (4.11) is satisfied is shown gray. **Right Panel:** Non-linear investment function leading to multiple equilibria. S_H and S_L are stable while U is unstable. A second linear investment function always lays below the first.

Thus, the stability condition for a generic fixed point in the multi-agent economies is twofold. First, equilibrium should be "self-consistent", i.e. remain stable even if any non-surviving agent would be removed from the economy. This very intuitive result strictly follows from the comparison between $Q_1(\mu)$ and polynomial $Q(\mu)$ in (3.6). This is however not enough. A further requirement comes from the inequalities in (4.11). In particular, according to the left-hand inequality, the wealth growth rate of those agents who do not survive in the stable equilibrium should be strictly less than the wealth growth rate of the survivors r^*. Thus, in those equilibria where $r^* > -\bar{e}$ the surviving agent must be the most aggressive and invest a higher wealth share in the risky asset. On the other hand, in those equilibria where $r^* < -\bar{e}$ the survivor has to be the least aggressive.

The EML plot can be used to obtain a geometric illustration of the previous Proposition. In Fig. 4 we draw again the two investment functions discussed in Section 3. Let us now suppose that they are both present on the market at the same time. The region where the additional condition (4.11) is satisfied is reported in gray. In Section 4.2 we found four possible equilibria: S_1, S_2, U_1 and U_2. Proposition 4.2 states that, first, the dynamics cannot be attracted by an equilibrium which was unstable in the respective single-agent cases. And, second, it cannot be attracted by an equilibrium in which non-surviving agent invests in the point belonging to a white region. Points U_1 and U_2 will be unstable if an agent uses EWMA or CWA forecast (cf. examples 2 and 3 in Section 3.3). Therefore, they are unstable also in the multi-agents market. From item **2)** of Proposition 4.2 it follows that S_1 is the only stable equilibrium of the system with two agents. Notice, indeed, that in the abscissa of S_1, i.e. for the equilibrium return, the linear investment function of the non-surviving agent II passes below the investment function of the surviving agent and belongs to the gray area. On the contrary, in the abscissa of S_2, the investment function of the non-surviving agent I has greater value and does not belong to the gray area. Consequently, this equilibrium is unstable.

Let us move now to consider the non-generic case, when k different agents survive in the equilibrium. The following applies

Proposition 4.3. *Let $\boldsymbol{x}^*$ be a fixed point of* (4.2) *with k survivors defined by* (4.5), (4.7) *and* (4.8).

The fixed point $\boldsymbol{x}^*$ is never hyperbolic and, consequently, never (locally) asymptotically stable. Its non-hyperbolic submanifold is the $k-1$-dimensional manifold defined in Corollary 4.1.

Let $P_{f_n}(\mu)$ be the stability polynomial of investment function f_n. The equilibrium $\boldsymbol{x}^*$ is (locally) stable if the two following conditions are met:

***1)** all the roots of polynomial*

$$Q_{1\diamond k}(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\, l'(r^*)} \sum_{n=1}^{k} \varphi_n^*\, P_{f_n}(\mu) \quad , \tag{4.12}$$

are inside the unit circle.

***2)** the equilibrium investment shares of the non-surviving agents satisfy to*

$$-2 - r^* < x_n^*\,(r^* + \bar{e}) < r^* \,, \quad k < n \leq N \,. \tag{4.13}$$

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of polynomial in (4.12) *is outside the unit circle or if at least one of the inequalities in* (4.13) *holds with the opposite (strict) sign.*

The non-hyperbolic nature of the equilibria with many survivors turns out to be a direct consequence of their non-unique specifications. The motion of the system along the $k-1$ dimensional subspace consisting of the continuum of equilibria leaves the aggregate properties of the system invariant so that all these equilibria can be considered equivalent. Proposition 4.3 also provides the stability conditions for perturbations in the hyperplane orthogonal to the non-hyperbolic manifold formed by equivalent equilibria. The polynomial $Q_{1\diamond k}(\mu)$ is quite similar to the corresponding polynomial in Proposition 4.2, except that one has to weight the stability polynomial of the different investment functions $P_{f_k}(\mu)$ with the weights corresponding to the relative wealth of survivors in the equilibrium. At the same time, the constraint on the investment shares in (4.13) is identical to the one obtained in (4.11). In particular, similar to the case with a single survivor, in those equilibria where $r^* > -\bar{e}$ all surviving agents must be more aggressive than those who do not survive, and *vice versa*.

Finally, let us analyse those equilibria where $r^* = -\bar{e}$ and, therefore, there is no equity premium. We consider general situation and allow some agents to have zero wealth shares. Without loss of generality, we assume that first $k \leq N$ agents survive in the equilibrium. The following result characterizes the stability of such equilibria

Proposition 4.4. *Let $\boldsymbol{x}^*$ be a fixed point of system* (4.2) *belonging to a $N-2$-dimensional manifold of k-survivors equilibria defined by* (4.5) *and* (4.9).

If $N \geq 3$, the fixed point $\boldsymbol{x}^*$ is non-hyperbolic and, consequently, is not (locally) asymptotically stable. The equilibrium $\boldsymbol{x}^*$ is (locally) stable if all the roots of the following polynomial are inside the unit circle

$$\mu^{L+1} + \frac{\mu - 1}{\langle x^2 \rangle} \sum_{j=1}^{k} \varphi_j^*\, P_{f_j}(\mu) \quad , \tag{4.14}$$

where $P_{f_j}(\mu)$ is the stability polynomial of investment function f_j computed in point $(-\bar{e}, \ldots, -\bar{e})$, and $\langle x^2 \rangle = \sum_{n=1}^{k} \varphi_n^*\, {x_n^*}^2$.

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of polynomial in (4.14) *is outside the unit circle.*

As in the case of Proposition 4.3, the "no-equity-premium" equilibria can be non-hyperbolic, due to possibility to change wealth between agents without changing the aggregate properties of the dynamics. For the complete stability analysis, the roots of polynomial (4.14) should be analyzed for specific investment functions, analogously to our analysis in Section 3.3. In particular, when all investment functions are horizontal in point $-\bar{e}$, the equilibrium (if it exists) is always stable.

4.4 Optimal selection and multiple equilibria

In this Section, using the geometric interpretation based on the EML we discuss some relevant implications of Proposition 4.2 about the asymptotic behavior of the model and its global properties. We confine the discussion to the generic case of equilibria with a single survivor.

The first implication concerns the aggregate dynamics of the economy. Let us consider a stable many-agent equilibrium $\boldsymbol{x^*}$. Let us suppose that r^* is the equilibrium return in $\boldsymbol{x^*}$ and that the first agent survives. Then his wealth return is equal to r^* and this is also the asymptotic growth rate of the *total* wealth. Then, we can interpret the second requirement of Proposition 4.2 as saying that, in the dynamic competition, those agent survives who allows the economy to have the highest possible rate of growth. Indeed, if any other agent $n \neq 1$ survived, the economy would have grown with rate $x_n^* (r^* + \bar{e})$, which is less than r^* according to (4.11). This result can be called an *optimal selection principle* since it clearly states the market endogenous selection toward the best aggregate outcome.

To be a bit more specific, notice that in equilibria with $r^* > -\bar{e}$ the overall wealth of the economy grows (in particular for $r^* < 0$ the negative wealth grows to 0), while in equilibria with $r^* < -\bar{e}$ the wealth of the economy falls. Thus, according to the optimal selection principle the surviving agent must be the most aggressive in equilibria where the economy grows and must be the least aggressive investor in equilibria where the economy shrinks.

Notice, however, that this selection does not apply to the whole set of equilibria, but only to the subset formed by equilibria associated with stable fixed points in the single agent case (c.f. (4.12)). For instance, with the investment functions shown in the left panel of Fig. 4, the dynamics will never end up in U_2, even if this is the equilibrium with the highest possible return. Furthermore, the variety of possible investment functions implies that the optimal selection principle has a local character. Indeed, even if we exclude all unstable single-agent equilibria, the market will not choose the equilibrium with the highest growth rate. Sometimes it can be the case like in the left panel of Fig. 4. However, it is often not the case and a simple counter-example is provided by a single investment function possessing *multiple stable equilibria* as shown in the right panel of Fig. 4. For this investment function both S_L and S_H are stable equilibria. Now suppose that an agent possessing this function competes on the market with other agents which are more risk averse than him and always invest smaller shares of wealth in the risky asset. An example of more risk averse behavior is provided by the linear investment function in the same plot. In this situation, these two equilibria of the nonlinear function remain stable and the riskier agent will ultimately dominate the market. The resulting market equilibrium will only depend on the initial conditions.

Possible presence of the equilibria without equity premium, identified for the multi-agent market in Proposition 4.1*(iii)*, is another source of multiple equilibria. For example, in the situation depicted in the right panel of Fig. 3 there is a stable equilibrium where agent I survives alone (point S_1) and many no-equity-premium stable equilibria where all three agents survive and agent I possesses high enough wealth share φ_1^* (cf. polynomial (4.14) in Proposition 4.4).

5 Conclusion

This paper introduces novel results concerning the characterization and stability of equilibria in speculative pure exchange economies with heterogeneous CRRA traders. The framework is relatively general in terms of agents' behaviors and differs from most of the models with heterogeneous agents in two important respects.

First, we analyze the aggregate dynamics and asymptotic behavior of the market when *an arbitrary large number* of traders participate to the trading activity. Second, we do not restrict in any way the procedure used by agents in order to build their forecast about future prices, nor the way in which agents can use this forecast to obtain their present asset demand. In our terms, agent with any smooth investment function mapping the information set to the present investment choice can present in the model.

Even if consideration of an arbitrary number of generic agents' behaviors leads us to study dynamical systems of an arbitrarily large dimension, we are able to provide a complete characterization of market equilibria and a description of their stability conditions in terms of few parameters characterizing the traders investment strategies. In particular, we find that, irrespectively of the number of agents operating in the market and of the structure of their demand functions, only three types of equilibria are possible:

- generic equilibria, associated with isolated fixed points, where a single agent asymptotically possesses the entire wealth of the economy,
- non-generic equilibria, associated with continuous manifolds of fixed points, where many agents possess a finite shares of the total wealth,
- generic equilibria associated with many survivors, where the economy does not possess the equity premium.

Furthermore, we show in total generality that a simple function, the "Equilibrium Market Line", can be used to obtain a geometric characterization of the location of all these types of equilibria. Furthermore, some results about stability conditions can also be inferred from the same EML.

Our general results provide, we believe, a simple and clear description of the principles governing the asymptotic market dynamics resulting from the competition of different trading strategies. The optimizing agents may dominate non-optimizing agents but may also be dominated by them. In general, the ultimate result of competition between agents depends on the whole market ecology. The EML is a handy and useful tool for demonstration such phenomena as absence of equilibrium, presence of multiple equilibria, and also for comparative statics exercises. From this plot (and results of stability analysis of Section 4.3) the following two "impossibility theorems" follow in an obvious way. First, there exist no "best" strategy, independently of what "best" means exactly, since any possible market equilibrium can be destabilized by some investment function. Second, it is impossible to build a dominance order relation inside the space of trading strategies, since two strategies may generate multiple stable equilibria with *different* survivors in them, so that the outcome will depend on the initial conditions or noise.

The present analysis can be extended in many directions. First of all, one may raise the question of the robustness of the results with respect to Assumption 1 about constant dividend yield. Our preliminary results of the analytic investigation in this direction show that some results (like presence of equity premium in equilibria and possibility to represent the equilibria by the EML) are actually robust and do not depend on the exact dividend specification. Second, in the limits of our framework, one can wonder about other possible dynamics. For instance, we have shown that there is a theoretical possibility do not have any equilibrium at all. The dynamics in this case remain unknown. Also the dynamics after bifurcations, which is the key question in many heterogeneous agent models, were not investigated. Probably numerical methods can be effectively applied to study these questions and also clarify the role of initial conditions and the determinants of the relative size of the basins of attraction for multiple equilibria scenarios. Third, our general CRRA-framework led us in Proposition 2.1 to the system in terms of returns and wealth shares. There are numerous behavioral specifications which were not analyzed here and still consistent with such framework. These specifications range from the evaluation of the "fundamental" value of the asset, possibly obtained from a private source of information, to a strategic behavior that try to keep in consideration the reaction of other market participants to the revealed individual choices. Furthermore, one may ask what are the consequences of the optimal selection principle for a market in which the set of strategies is not "frozen", but instead is evolving in time, plausibly following some adaptive process. For instance, one can assume that agents imitate the behavior of other traders (see e.g. Kirman (1991)) or that they update strategies according to recent relative performances (see e.g. Brock and Hommes (1998)). The analysis of the consequences of the introduction of such strategies on the optimal selection principle may, ultimately, refute the statement about the impossibility of defining a dominance relation among strategies.

APPENDIX

A Proof of Proposition 2.1

Plugging the expression for $w_{t+1,n}$ from the second equation in system (2.4) into the right-hand side of the first equation of the same system, and assuming that $p_t > 0$ and, consistently with (2.6), $p_t \neq \sum x_{t+1,n}\, x_{t,n}\, w_{t,n}$ one gets

$$p_{t+1} = \left(1 - \frac{1}{p_t}\sum_{n=1}^{N} x_{t+1,n}\, x_{t,n}\, w_{t,n}\right)^{-1} \left(\sum_{n=1}^{N} x_{t+1,n}\, w_{t,n} + \big(e_{t+1} - 1\big)\sum_{n=1}^{N} x_{t+1,n}\, w_{t,n}\, x_{t,n}\right) =$$

$$= p_t\, \frac{\sum_n x_{t+1,n}\, w_{t,n} + (e_{t+1} - 1)\sum_n x_{t+1,n}\, w_{t,n}\, x_{t,n}}{\sum_n x_{t,n}\, w_{t,n} - \sum_n x_{t+1,n}\, x_{t,n}\, w_{t,n}} =$$

$$= p_t\, \frac{\big\langle x_{t+1}\big\rangle_t - \big\langle x_t\, x_{t+1}\big\rangle_t + e_{t+1}\big\langle x_t\, x_{t+1}\big\rangle_t}{\big\langle x_t\big\rangle_t - \big\langle x_t\, x_{t+1}\big\rangle_t} \quad ,$$

where we used the first equation of (2.4) rewritten for time t to get the second equality. Condition (2.6) is obtained imposing $p_{t+1} > 0$, and the dynamics of price return in (2.7) are immediately derived. From the second equation of (2.4) it follows that

$$w_{t+1,n} = w_{t,n}\,\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{A.1}$$

leading to (2.8). To obtain the wealth share dynamics, divide both sides of (A.1) by w_{t+1} to have

$$\begin{aligned}\varphi_{t+1,n} &= \frac{w_{t,n}}{\sum_m w_{t+1,m}}\big(1+x_{t,n}\,(r_{t+1}+e_{t+1})\big) = \\ &= \frac{w_{t,n}}{\sum_m w_{t,m} + (r_{t+1}+e_{t+1})\sum_m x_{t,m}w_{t,m}}\big(1+x_{t,n}\,(r_{t+1}+e_{t+1})\big) = \\ &= \frac{\varphi_{t,n}}{1+(r_{t+1}+e_{t+1})\sum_m x_{t,m}\varphi_{t,m}}\big(1+x_{t,n}\,(r_{t+1}+e_{t+1})\big) \quad ,\end{aligned}$$

where (A.1) has been used to get the second line and we divided both numerator and denominator by the total wealth at time t to get the third.

B Proof of Proposition 3.1

Plugging the equilibrium values of the variables in the first equation of (3.1), one gets $x^* = f(r^*, \dots, r^*)$. Now using $R(x^*, x^*, e) = e\,x^*/(1-x^*)$ one can invert the second equation to get (3.4). Item *(ii)* follows directly from condition (2.6) written at equilibrium. Finally, from (2.8) using the previous relations one has $\rho^* = x^*(r^*+\bar{e}) = l(r^*)(r^*+\bar{e}) = r^*$.

C Proof of Proposition 3.2

The $(L+1)\times(L+1)$ Jacobian matrix $\boldsymbol{J}$ of system (3.1) reads

$$\left\|\begin{array}{ccccccc} 0 & \frac{\partial f}{\partial r_0} & \frac{\partial f}{\partial r_1} & \frac{\partial f}{\partial r_2} & \dots & \frac{\partial f}{\partial r_{L-2}} & \frac{\partial f}{\partial r_{L-1}} \\ R^x & R^f\frac{\partial f}{\partial r_0} & R^f\frac{\partial f}{\partial r_1} & R^f\frac{\partial f}{\partial r_2} & \dots & R^f\frac{\partial f}{\partial r_{L-2}} & R^f\frac{\partial f}{\partial r_{L-1}} \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{array}\right\| , \tag{C.1}$$

where the derivatives of function R defined in (3.2) can be computed as follows

$$R^x = \frac{\partial R(x^*, x^*)}{\partial x} = -\frac{1}{x^*\,(1-x^*)}\,, \qquad R^f = \frac{\partial R(x^*, x^*)}{\partial x'} = \frac{1+r^*}{x^*\,(1-x^*)}\,. \tag{C.2}$$

The stability condition of equilibrium are provided by the following

Lemma C.1. *The characteristic polynomial $P_J(\mu)$ of system* (3.1) *in the equilibrium $\boldsymbol{x^*}$ is*

$$P_J(\mu) = (-1)^{L-1}\left(\mu^{L+1} - \frac{(1+r^*)\mu - 1}{x^*(1-x^*)}P_f(\mu)\right) \tag{C.3}$$

where $P_f(\mu)$ denotes the stability polynomial of function f introduced in (3.5).

Proof. Consider (C.1) and introduce $(L+1)\times(L+1)$ identity matrix $\boldsymbol{I}$. Expanding the determinant of $\boldsymbol{J}-\mu\,\boldsymbol{I}$ by the elements of the first column and using Lemma F.1 one has

$$\begin{aligned}\det\big(\boldsymbol{J}-\mu\,\boldsymbol{I}\big) &= (-\mu)\,(-1)^{L-1}\left(\left(R^f\frac{\partial f}{\partial r_0}-\mu\right)\mu^{L-1} + R^f\frac{\partial f}{\partial r_1}\,\mu^{L-2} + \dots + R^f\frac{\partial f}{\partial r_{L-1}}\right) - \\ &\quad - R^x\,(-1)^{L-1}\left(\frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \dots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}}\right) = \\ &= (-1)^{L-1}\left(\mu^{L+1} - \left(\mu R^f + R^x\right)\sum_{k=0}^{L-1}\frac{\partial f}{\partial r_k}\,\mu^{L-1-k}\right)\end{aligned} \quad ,$$

which, using relations in (C.2) and definition of stability polynomial in (3.5) reduces to (C.3). □

Using the relationship $l'(r^*) = x^*(1-x^*)/r^*$ it is immediate to see that, apart from irrelevant sign, (C.3) is identical to (3.6).

D Analysis of stability of single agent equilibrium

For the extensive theory of the stability and bifurcation analysis of the autonomous dynamical systems see e.g. Guckenheimer and Holmes (1983) or Kuznetsov (1995). We start below with two general results for two-dimensional non-linear dynamical systems.

Stability of 2-dimensional system and proof of Proposition 3.3

Consider a general 2-dimensional non-linear dynamical system

$$\begin{cases} x_{1,t+1} = f_1(x_{1,t}, x_{2,t}) \\ x_{2,t+1} = f_2(x_{1,t}, x_{2,t}) \end{cases} , \tag{D.1}$$

and suppose that (x_1^*, x_2^*) is a fixed point of the system. Moreover, let $J(x_1^*, x_2^*)$ denote the Jacobian matrix of the system (D.1) computed in this fixed point:

$$J(x_1^*, x_2^*) = \begin{Vmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{Vmatrix} . \tag{D.2}$$

Let $t = J_{1,1} + J_{2,2}$ and $d = J_{1,1}\, J_{2,2} - J_{1,2}\, J_{2,1}$ be, respectively, the trace and the determinant of matrix (D.2). Then the following result takes place:

Proposition D.1 (Sufficient conditions for the local stability). *The fixed point* (x_1^*, x_2^*) *of system* (D.1) *is locally asymptotically stable if the following conditions are satisfied*

$$d < 1 \quad , \qquad t < 1 + d \quad , \qquad t > -1 - d \quad . \tag{D.3}$$

Proof. The characteristic polynomial computed in a fixed point in our notations reads: $\mu^2 - t\,\mu + d = 0$. Thus, stability analysis reduces to the analysis of region in (t, d) space, where the modulus of both roots

$$\mu_\pm = \frac{t \pm \sqrt{t^2 - 4\,d}}{2}$$

are less than 1. There are two cases.

If $t^2 \geq 4\,d$ there are two real roots $\mu_- \leq \mu_+$ (coinciding when $t^2 = 4\,d$) and for the local stability we need that both $\mu_+ < 1$ and also $\mu_- > -1$. It is straightforward to see that it leads to the following conditions on the trace and determinant:

$$t < 2 \quad , \qquad 1 - t + d > 0 \qquad \text{and} \qquad t > -2 \quad , \qquad 1 + t + d > 0 \quad .$$

Area S_2 in Fig. 5 contains those points where these conditions are satisfied and both roots are real.

Instead, if $t^2 < 4\,d$ there are two complex conjugate roots and the stability condition reads

$$|\lambda| = \sqrt{\lambda_+ \lambda_-} = \sqrt{d} < 1$$

Corresponding region is labeled S_1 in Fig. 5. All points corresponding to stable fixed point lie, therefore, in the triangle shaped by the union of S_1 and S_2. This triangle is described by conditions (D.3). □

According to Proposition D.1, if all 3 inequalities in (E.2) hold, the fixed point is stable. The fixed point will be unstable if at least one of the roots of characteristic polynomial lies outside the unit circle, i.e. when at least one of the inequalities in (E.2) has an opposite sign. The situation in which the change of one or more parameters of a system leads to the cross of the unit circle by a root of characteristic polynomial is called a *bifurcation.* Three types of generic bifurcations are usually considered, depending on where exactly the root of the polynomial crosses the unit circle. Bifurcation is called *Neimark-Sacker* (*fold*, *flip*) if a root crosses the unit circle being non-real (equal to 1, equal to −1), respectively. The following Proposition summarizes the information about types of bifurcations.

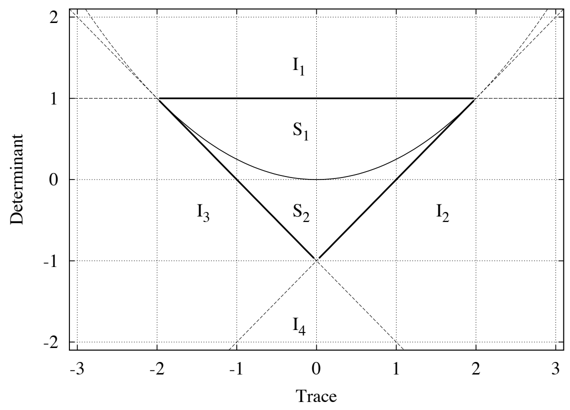

Figure 5: Stability region for a fixed point of a two-dimensional dynamical system in the Trace-Determinant coordinates. The system is stable in the triangle defined as union of the areas S_1 and S_2.

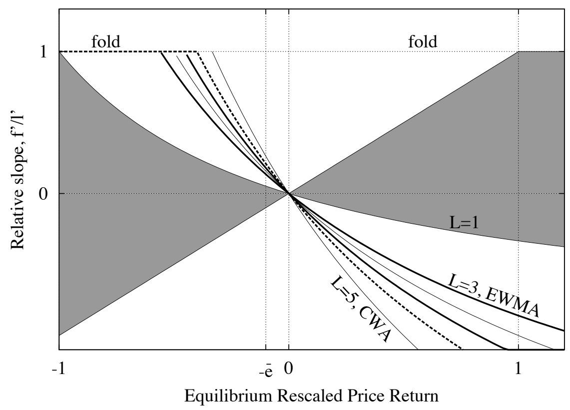

Figure 6: Loci where flip bifurcations can occur in the system with EWMA forecast and $\lambda = 0.6$ (thick lines) and with CWA forecast (thin lines). If $L = 1$ in both cases the loci define the stability region shown as dark. For the EWMA behavior: the loci for odd L rotate clock-wise and approach the locus corresponding to even L (thick dotted curve). For the CWA behavior: for even L loci do not exist, while for odd L they rotate faster w.r. to EWMA case and approach the ordinate axes.

Proposition D.2. *The fixed point of the system* (D.1) *looses its stability when one of the inequalities in conditions* (D.3) *is changing its sign. Moreover, the system exhibits*

- *Neimark-Sacker bifurcation, if* $d = 1$,
- *fold bifurcation, if* $t = 1 + d$,
- *flip bifurcation, if* $t = -1 - d$.

Proof. From the proof of the previous proposition it is clear that when the parameters cross the line separating region S_1 from I_1 at Fig. 5 (when $d = 1$), the modulus of the *complex* eigenvalues become larger than 1. Thus the system exhibits a Neimark-Sacker bifurcation. When the line between S_2 and I_2 is crossed (and so $t = 1+d$) the largest real eigenvalue becomes greater than 1, so that a fold bifurcation is observed. Finally, if the line between S_2 and I_3 is crossed (i.e. $t = -1 - d$) the smallest real eigenvalue becomes less than -1, and the flip bifurcation is observed. $\square$

Proposition 3.3 follows straight-forwardly from the last two Propositions applied to polynomial (3.7).

Stability for the system with EWMA forecast

Using stability polynomial (3.10), the characteristic polynomial for this case can be written as follows:

$$Q(\mu) = \mu^{L-1} \left(\mu^2 - \frac{1-\lambda}{1-\lambda^L} \frac{1-(\lambda/\mu)^L}{1-\lambda/\mu} \frac{Y}{X} \Big((1+X)\mu - 1 \Big) \right) \quad , \tag{D.4}$$

where variables $X = r^*$ and $Y = l'(r^*)/f'(r^*)$ correspond to the abscissa and ordinate of the stability region diagrams in Fig. 2.

Bifurcations Loci

Unfortunately, for the polynomials of degree higher than 2, conditions on the roots analogous to (D.3) are unavailable. One can, however, characterize the *loci* of different types of bifurcations, where by locus of, say, flip bifurcation we mean the set of those pairs (X, Y) under which one of the roots of (D.4) is equal to -1. Notice that, in general, not any point on such locus corresponds to flip bifurcation, since bifurcation happens only if all other roots are inside the unit circle. With this remark in mind, we formulate

Lemma D.1. *Consider polynomial* (D.4) *for arbitrary* λ *and* L. *The locus of* fold *bifurcations is given by* $Y = 1$, *while the locus of* flip *bifurcations is provided as follows:*

$$1 + Y\,\frac{2+X}{X}\,\frac{1-\lambda}{1+\lambda}\,\frac{1+\lambda^L}{1-\lambda^L} = 0 \qquad \textit{for } L \textit{ odd} \qquad \textit{and} \qquad 1 + Y\frac{2+X}{X}\,\frac{1-\lambda}{1+\lambda} = 0 \qquad \textit{for } L \textit{ even}.$$

Proof. The computations are pretty straight-forward: one has to let $\mu = 1$ and $\mu = -1$ in (D.4). □

We illustrate the last result in Fig. 6, where we show the locus of fold bifurcations, and loci of flip bifurcations when $\lambda = 0.6$ for $L = 1, 3, 5$ and for even L (coinciding with the case $L = \infty$). Notice that the central interval in the locus of fold bifurcations does not correspond, in reality, to any bifurcation.

The last possible boundary of the stability region is a locus of *Neimark-Sacker* bifurcation. Plugging $\mu = e^{i\psi}$, where ψ is arbitrary angle and i is the imaginary unit, into equation $Q(\mu) = 0$ one can derive corresponding locus. For high L the corresponding curve may have high dimension, so we confine ourselves on the example with $L = 2$.

Lemma D.2. *Consider polynomial* (D.4) *for* $L = 2$ *and arbitrary* λ. *The locus of Neimark-Sacker bifurcations is given by the following curve of the second order*

$$Y^2\,\lambda\,\big(X + 1 + \lambda\big) + Y\,X\,(1+\lambda)\,\big(1 - \lambda - \lambda X\big) - X^2\,(1+\lambda)^2 = 0 \quad , \tag{D.5}$$

subject to constraint $4(1+\lambda^2)X^2 - (1+\lambda+X)^2Y^2 > 0$.

Proof. One has to solve the following equation

$$(1+\lambda)Xe^{3i\psi} - (1+X)Ye^{2i\psi} + \big(1 - \lambda(1+X)\big)Ye^{i\psi} + \lambda Y = 0 \quad . \tag{D.6}$$

Using the Euler formula and equalizing the real and imaginary parts of the left- and right-hand sides, one gets the system of two equations. From the equation for the imaginary parts it follows that

$$\sin\psi\Big((1+\lambda)X(3 - 4\sin^2\psi) - 2(1+X)Y\cos\psi + Y - \lambda(1+X)Y\Big) = 0 \quad .$$

Since, we are interested in the pure complex roots of the characteristic polynomial, let us assume that $\sin\psi \neq 0$. Thus, the expression in the big parenthesis is equal to 0. Then, we substitute the resulting expression for $3 - 4\sin^2\psi$ in the equation of the real parts of (D.6) and, using relation $\cos 3\psi = \cos\psi(1 - 4\sin^2\psi)$, obtain

$$\cos\psi = \frac{Y\,(1+X) + Y\,\lambda}{2\,(1+\lambda)\,X} \quad .$$

Plugging this expression into both the equation for the real and for imaginary parts of (D.6), we get the following system

$$\begin{aligned}(1+\lambda+X)\,Y\,\Big(-\lambda(1+\lambda)Y^2 + (1+\lambda)(1+\lambda+\lambda Y)X^2 - (1-\lambda^2+\lambda Y)XY\Big) &= 0\\ \sqrt{4(1+\lambda)^2X^2 - (1+\lambda+X)^2Y^2}\,\Big(-\lambda(1+\lambda)Y^2 + (1+\lambda)(1+\lambda+\lambda Y)X^2 - (1-\lambda^2+\lambda Y)XY\Big) &= 0\,.\end{aligned}$$

In any solution of this system, the common expression in the parenthesis must be equal to zero, which gives (D.5). Additional constraint guarantees that the squared root in the second equation is defined. □

We depict curve (D.5) in the lower left panel of Fig. 2 and complete the construction of the stability regions applying the property of the continuity of the roots of polynomial on its coefficients.

Limiting case, $L \to \infty$

Finally, we sketch here heuristic proof of Proposition 3.4, concerning the stability region of the system when $L \to \infty$. Consider the region outside of the unit circle of complex plane (including the circle itself) and fix $\mu = \mu_0$. Since $|\mu_0| \geq 1$, the first term in (D.4) cannot be equal to zero. Therefore, in order μ_0 would be a root of the characteristic polynomial $Q(\mu)$, it should be the case that the expression in the parenthesis of (D.4) cancels out. But when $L \to \infty$, since $\lambda < 1$, it is $(\lambda/\mu_0)^L \to 0$. Therefore, the zeroes of the $Q(\mu)$ lie in the neighborhoods of solutions of the following *quadratic* equation

$$\mu^2 - (1-\lambda)\frac{1}{1-\lambda/\mu}\,\frac{Y}{X}\Big((1+X)\mu - 1\Big) = 0 \quad . \tag{D.7}$$

Applying conditions derived in Proposition D.1 to the last equation we find out that its roots are inside of the unit circle if and only if conditions (3.11) are satisfied. Now, if the values of the parameters are such that the roots $\mu_{1,2}$ of this equation are inside of the unit circle, it means that for large enough L the polynomial (D.4) cannot have roots outside of the unit circle or on it.

Stability for the system with Constant Weighted forecast

We proceed through the same steps as in the previous Section. The characteristic polynomial for this case can be written as follows:

$$Q(\mu) = \mu^{L-1} \left(\mu^2 - \frac{1}{L} \frac{1-(1/\mu)^L}{1-1/\mu} \frac{Y}{X} \Big((1+X)\mu - 1 \Big) \right) \quad , \tag{D.8}$$

where, like before, variables $X = r^*$ and $Y = l'(r^*)/f'(r^*)$ correspond to the abscissa and ordinate of the stability region diagrams in Fig. 2.

Bifurcations Loci

We characterize the *loci* of fold and flip bifurcations. Recall, that by locus of some bifurcation we mean those pairs (X, Y) under which a root of (D.8) crosses the unit circle in the corresponding points (in 1 for fold and in -1 for flip).

Lemma D.3. *Consider polynomial* (D.8) *for arbitrary L. The locus of* fold *bifurcations is given by $Y = 1$, while the locus of* flip *bifurcations does not exist for even L and is provided by following expression for odd L:*

$$Y = -\frac{L\,X}{2+X} \quad .$$

Proof. The direct substitution of $\mu = 1$ and $\mu = -1$ to (D.8) leads to the conclusion. □

The flip bifurcations loci are shown as thin curves in Fig. 6 for $L = 1$ (boundary of the dark region), $L = 3$ (unlabeled thin curve) and $L = 5$ (labeled thin curve). For higher L the curves rotate clock-wise and approach vertical axes with $L \to \infty$.

For the locus of *Neimark-Sacker* bifurcations we consider the case $L = 2$ and prove the following

Lemma D.4. *Consider polynomial* (D.8) *for $L = 2$. The locus of Neimark-Sacker bifurcations is given by the following curve of the second order*

$$Y^2\,(2+X) - 2YX\,(2+X) + 4\,X^2 = 0 \quad , \tag{D.9}$$

subject to constraint $16\,X^2 - (2+X)^2 Y^2 > 0$.

Proof. It is completely analogous to proof of Lemma D.4. □

The stability region implied by the last two lemmas for the case $L = 2$ is depicted in the lower right panel of Fig. 2.

Limiting case, $L \to \infty$

The behavior of the locus of flip bifurcations with increasing L may suggest that any point of the system (apart from those where $f'/l' > 1$) can be stabilized. Indeed, let us consider the region outside of the unit circle (including the circle itself), fix $\mu = \mu_0$ and let $L \to \infty$. Since $|\mu_0| \geq 1$, the first term in (D.8) cannot be equal to zero. Therefore, in order μ_0 would be a root of the characteristic polynomial $Q(\mu)$, it should be the case that the expression in the parenthesis of (D.4) cancels out. However, when $L \to \infty$ it leads to $\mu_0 = 0$ which contradicts to our choice of μ_0. Therefore with high L all zeroes of $Q(\mu)$ are inside the unit circle independent of X and Y, and any point is stable.

E Proof of Proposition 4.1

From block $\mathcal{X}$ one immediately has (4.5). From the first row of block $\mathcal{R}$ it is

$$r^* = \bar{e}\,\frac{\sum_{n=1}^{N-1}\varphi_n^* {x_n^*}^2 + \big(1-\sum_{n=1}^{N-1}\varphi_n^*\big){x_N^*}^2}{\sum_{n=1}^{N-1}\varphi_n^*\,x_n^*\,(1-x_n^*) + \big(1-\sum_{n=1}^{N-1}\varphi_n^*\big)\,x_N^*\,(1-x_N^*)}\,. \tag{E.1}$$

Let us, first, assume that $r^* + \bar{e} \neq 0$. Then, from block $\mathcal{W}$ using (4.4) one obtains

$$\varphi_n^* = 0 \quad \text{or} \quad \sum_{m=1}^{N-1}\varphi_m^*\,x_m^* + \left(1-\sum_{m=1}^{N-1}\varphi_m^*\right)x_N^* = x_n^* \qquad \forall n \in \{1,\dots,N-1\}\,. \tag{E.2}$$

Together with (E.1) this equation admits two types of solutions, depending on whether one or many equilibrium wealth shares are different from zero.

If only one of the wealth shares is zero, we assume that $\varphi_1^* = 1$. In this case (E.2) is satisfied for all agents. From (E.1) one has $x_1^* = r^*/(\bar{e}+r^*)$ which together with (4.5) leads to (4.6).

If, instead, many agents survive we assume (4.7). In this case, the second equality of (E.2) must be satisfied for any $n \leq k$. Since its left-hand side does not depend on n, a $x_{1\diamond k}^*$ must exist such that $x_1^* = \cdots = x_k^* = x_{1\diamond k}^*$. Substituting $\varphi_n^* = 0$ for $n > k$ and $x_n^* = x_{1\diamond k}^*$ for $n \leq k$ in (E.1) one gets $x_{1\diamond k}^* = r^*/(\bar{e}+r^*)$. The equilibrium return r^* is implicitly defined combining this last relation with (4.5) for $n \leq k$.

Consider now the case when $r^* + \bar{e} = 0$. Then all equations in block $\mathcal{W}$ are satisfied, while (E.2) straightforwardly leads to (4.9).

The equilibrium wealth growth rates of the survivors are immediately obtained from (2.8) and (4.5).

F Proofs of Propositions of Section 4.3

We start with the simple result which will be useful in what follows

Lemma F.1.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ 1 & -\mu & 0 & \dots & 0 & 0 \\ 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\mu & 0 \\ 0 & 0 & 0 & \dots & 1 & -\mu \end{vmatrix} = (-1)^{n+1}\sum_{k=1}^{n} x_k\,\mu^{n-k} \quad , \tag{F.1}$$

Proof. Consider this determinant as a sum of elements from the first row multiplied on the corresponding minor. The minor of element x_k, whose corresponding sign is $(-1)^{k+1}$, is a block-diagonal matrix consisting of two blocks. The upper-left block is an upper-diagonal matrix with 1's on the diagonal. The lower-right block is a lower-diagonal matrix with $-\mu$'s on the diagonal. The determinant of this minor is equal to $(-\mu)^{n-k}$ and the relation above immediately follows. □

Before proving Propositions 4.2, 4.3 and 4.4 we need some preliminary results. The Jacobian matrix of the deterministic skeleton of system (4.2) is a $\big(2N+L-1\big)\times\big(2N+L-1\big)$ matrix. Using the block structure introduced in Section 4.1 it is separated in 9 blocks

$$\boldsymbol{J} = \begin{Vmatrix} \frac{\partial\mathcal{X}}{\partial\mathcal{X}} & \frac{\partial\mathcal{X}}{\partial\mathcal{W}} & \frac{\partial\mathcal{X}}{\partial\mathcal{R}} \\ \frac{\partial\mathcal{W}}{\partial\mathcal{X}} & \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{R}} \\ \frac{\partial\mathcal{R}}{\partial\mathcal{X}} & \frac{\partial\mathcal{R}}{\partial\mathcal{W}} & \frac{\partial\mathcal{R}}{\partial\mathcal{R}} \end{Vmatrix} , \tag{F.2}$$

The block $\partial\mathcal{X}/\partial\mathcal{X}$ is a $N \times N$ matrix containing the partial derivatives of the agents' present investment choices with respect to the agents' past investment choices. According to (2.10) the investment choice of any agent does not explicitly depend on the investment choices in the previous period, therefore

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial f_n}{\partial x_m} = 0\,, \qquad 1 \leq n,m \leq N \quad ,$$

and this block is a zero matrix.

The block $\partial\mathcal{X}/\partial\mathcal{W}$ is a $N \times (N-1)$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' wealth shares. According to (2.10) this is a zero matrix and

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{W}}\right]_{n,m} = \frac{\partial f_n}{\partial\varphi_m} = 0\,, \qquad 1 \le n \le N\,, \quad 1 \le m \le N-1 \quad .$$

The block $\partial\mathcal{X}/\partial\mathcal{R}$ is a $N \times L$ matrix containing the partial derivatives of the agents' investment choices with respect to the past returns. We use the following notation

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{R}}\right]_{n,l} = \frac{\partial f_n}{\partial r_{l-1}} = f_n^{r_{l-1}}\,, \qquad 1 \le n \le N\,, \quad 1 \le l \le L \quad .$$

The definitions of the next blocks will make use of the functions R and Φ_n which have been defined in (4.3) and (4.4), respectively. Function R depends on the agents' contemporaneous investment choices given by the investment functions f_m, the agents' previous investment choices $x_{t,m}$ and the agents' wealth shares $\varphi_{t,m}$. We denote the corresponding derivatives as R^{f_m}, R^{x_m} and R^{φ_m}. Function Φ_n depends on the agents' previous investment choices $x_{t,m}$, the agents' wealth shares $\varphi_{t,m}$ and the value of return given by function R. The corresponding derivatives are denoted as $\Phi_n^{x_m}$, $\Phi_n^{\varphi_m}$ and Φ_n^{R}.

The block $\partial\mathcal{W}/\partial\mathcal{X}$ is $(N-1) \times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' investment choices. It is

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial x_m} = \Phi_n^{x_m} + \Phi_n^{R} \cdot R^{x_m}\,, \qquad 1 \le n \le N-1\,, \quad 1 \le m \le N \quad . \tag{F.3}$$

The block $\partial\mathcal{W}/\partial\mathcal{W}$ is a $(N-1) \times (N-1)$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' wealth shares. It is

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial\varphi_m} = \Phi_n^{\varphi_m} + \Phi_n^{R} \cdot R^{\varphi_m}\,, \qquad 1 \le n,m \le N-1 \quad . \tag{F.4}$$

The block $\partial\mathcal{W}/\partial\mathcal{R}$ is a $(N-1) \times L$ matrix containing the partial derivatives of the agents' wealth share with respect to lagged returns. It is

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{R}}\right]_{n,l} = \frac{\partial\varphi_n}{\partial r_{l-1}} = \Phi_n^{R} \cdot \sum_{m=1}^{N} R^{f_m} f_m^{r_{l-1}}\,, \quad 1 \le n \le N-1\,, \quad 1 \le l \le L \quad . \tag{F.5}$$

The block $\partial\mathcal{R}/\partial\mathcal{X}$ is the $L \times N$ matrix containing the partial derivatives of the lagged returns with respect to the agents' investment choices. Its structure is simple, since only the first line can contain non-zero elements. It reads

$$\left[\frac{\partial\mathcal{R}}{\partial\mathcal{X}}\right]_{l,n} = \begin{cases} R^{x_n} & l=1 \\ 0 & \text{otherwise} \end{cases}\,, \qquad 1 \le l \le L\,, \quad 1 \le n \le N \quad .$$

The block $\partial\mathcal{R}/\partial\mathcal{W}$ is the $L \times (N-1)$ matrix containing the partial derivatives of the lagged returns with respect to the agents' wealth shares. It also has $L-1$ zero rows and reads

$$\left[\frac{\partial\mathcal{R}}{\partial\mathcal{W}}\right]_{l,n} = \begin{cases} R^{\varphi_n} & l=1 \\ 0 & \text{otherwise} \end{cases}\,, \qquad 1 \le l \le L\,, \quad 1 \le n \le N-1 \quad .$$

The block $\partial\mathcal{R}/\partial\mathcal{R}$ is the $L \times L$ matrix containing the partial derivatives of the lagged returns with respect to themselves.

$$\left[\frac{\partial\mathcal{R}}{\partial\mathcal{R}}\right] = \begin{Vmatrix} R^{r_0} & R^{r_1} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{Vmatrix}\,,$$

where we introduce a notation for $R^{r_l} = \sum_{m=1}^{N} R^{f_m} f_m^{r_l}$.

With the previous definitions and differentiating the correspondent functions, one obtains

Lemma F.2. *Let $\boldsymbol{x}^*$ be an equilibrium of system (4.2). The corresponding Jacobian matrix, $\boldsymbol{J}(\boldsymbol{x}^*)$, has the following structure, where the actual values of non-zero elements vary dependently on whether there exist an equity premium in $\boldsymbol{x}^*$.*

$$
\left\|\begin{array}{cccccc|cccccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & f_N^{r_0} & \dots & f_N^{r_{L-2}} & f_N^{r_{L-1}} \\
\hline
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star & \star & \dots & \star & \blacksquare \\
[& \Phi^x &] & \vdots & \ddots & \vdots & [& \Phi^\varphi_{1,k} &] & [& \Phi^\varphi_{k+1,N} &] & [& \Phi^r &] & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star & \star & \dots & \star & \blacksquare \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \bigstar & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \bigstar & 0 & \dots & 0 & 0 \\
\hline
R^{x_1} & \dots & R^{x_k} & 0 & \dots & 0 & R^{\varphi_1} & \dots & R^{\varphi_k} & R^{\varphi_{k+1}} & \dots & R^{\varphi_{N-1}} & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0
\end{array}\right\|
$$

Namely, the "varying" elements belong to 4 blocks filled by $\star$'s and denoted as $[\Phi^x]$, $[\Phi^\varphi_{1,k}]$, $[\Phi^\varphi_{k+1,N}]$ and $[\Phi^r]$. In particular, all the elements of the latter block are zeros in the equilibrium with equity premium. Furthermore, in such equilibrium the elements labeled as $\blacksquare$'s are also 0's. Finally, the elements labeled as $\bigstar$'s on the diagonal of the central block are all equal to 1 in the no-equity-premium equilibria.

Proof. For the computation of the elements in the second big row-block, it will be useful to establish the values of relevant derivatives in equilibrium. In general, for corresponding m and n, it is

$$
\Phi_n^{x_m} = \varphi_n^* \frac{(\delta_{n,m} - \varphi_m^*)(r^* + \bar{e})}{1 + (r^* + \bar{e})\left\langle x^* \right\rangle} \quad , \qquad R^{x_m} = \varphi_m^* \frac{x_m^*(r^* + \bar{e}) - 1 - r^*}{\left\langle x^*(1 - x^*) \right\rangle} \quad ,
$$

$$
\Phi_n^{R} = \varphi_n^* \frac{x_n^* - \left\langle x^* \right\rangle}{1 + (r^* + \bar{e})\left\langle x^* \right\rangle} \quad , \qquad R^{f_m} = \varphi_m^* \frac{1 + x_m^*(r^* + \bar{e})}{\left\langle x^*(1 - x^*) \right\rangle} \quad ,
$$

$$
\Phi_n^{\varphi_m} = \frac{\delta_{n,m}\big(1 + x_n^*(r^* + \bar{e})\big) - \varphi_n^*(r^* + \bar{e})(x_m^* - x_N^*)}{1 + (r^* + \bar{e})\left\langle x^* \right\rangle} \quad ,
$$

$$
R^{\varphi_m} = \frac{(\bar{e} + r^*)(x_m^{*^2} - x_N^{*^2}) - r^*(x_m^* - x_N^*)}{\left\langle x^*(1 - x^*) \right\rangle} \quad .
$$

Consider equilibrium with $r^* \neq -\bar{e}$, i.e. one described in Proposition 4.1*(i)* and *(ii)*. All survivors invest the same in such an equilibrium. It immediately implies that $\Phi_n^R = 0$ for any agent n, so that

$$
\begin{aligned}
\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}}\right]_{n,m} &= \begin{cases} /m, n \leq k/ & = \varphi_n^* \left(\delta_{n,m} - \varphi_m^*\right)(\bar{e} + r^*)/(1 + r^*) \\ /\text{otherwise}/ & = 0 \end{cases} \\
\left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} &= \begin{cases} /n > k\,,\ \ n \neq m/ & = 0 \\ /\text{otherwise}/ & = \Big(\delta_{n,m}\big(1 + x_n^*(r^* + \bar{e})\big) - \varphi_n^*(r^* + \bar{e})(x_m^* - x_N^*)\Big)/(1 + r^*) \end{cases} \\
\left[\frac{\partial \mathcal{W}}{\partial \mathcal{R}}\right]_{n,m} &= 0\,, \quad \forall n, m \quad .
\end{aligned}
\tag{F.6}
$$

On the other hand, in equilibrium with $r^* = -\bar{e}$, i.e. one described in Proposition 4.1*(iii)*, it is $\Phi_n^{x_m} = 0$,

$\Phi_n^{\varphi_m} = \delta_{n,m}$ and $\Phi_n^R = \varphi_n^* x_n^*$ for all possible n and m. Thus, one has

$$\begin{aligned}
\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}}\right]_{n,m} &= \begin{cases} /\text{for } m,n \le k/ & = (1-\bar{e})\,\varphi_n^*\,\varphi_m^*\,x_n^* / \langle x^2 \rangle \\ /\text{otherwise}/ & = 0 \end{cases} \quad ; \\
\left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} &= \begin{cases} /\text{for } n \le k/ & = \delta_{n,m} - \bar{e}\,\varphi_n^*\,x_n^*(x_m^* - x_N^*)/\langle x^2 \rangle \\ /\text{otherwise}/ & = \delta_{n,m} \end{cases} \quad ; \\
\left[\frac{\partial \mathcal{W}}{\partial \mathcal{R}}\right]_{n,m} &= \begin{cases} /\text{for } n \le k/ & = -\varphi_n^* x_n^* \sum_{m=1}^N \varphi_m^* f_m^{r_{l-1}} / \langle x^2 \rangle \\ /\text{otherwise}/ & = 0 \end{cases} \quad .
\end{aligned} \tag{F.7}$$

□

Lemma F.3. *Consider equilibrium $\boldsymbol{x}^*$ with $r^* \neq -\bar{e}$. The characteristic polynomial P_J of the matrix $\boldsymbol{J}(\boldsymbol{x}^*)$ reads*

$$P_J(\mu) = (-1)^{N+L}\,\mu^{N-1}\,(1-\mu)^{k-1} \prod_{j=k+1}^{N} \left(\frac{1 + x_j^*\,(r^* + \bar{e})}{1 + r^*} - \mu\right)$$
$$\left(\mu^{L+1} - \frac{(1+r^*)\mu - 1}{x_{1\diamond k}^*(1 - x_{1\diamond k}^*)} \sum_{j=1}^{k} \varphi_j^*\, P_{f_j}(\mu)\right) \tag{F.8}$$

where P_{f_n} is the stability polynomial associated to the n-th investment function as defined in (3.5).

Proof. The following proof is constructive: we will identify in succession the factors appearing in (F.8). At each step, a set of eigenvalues is found and the problem is reduced to the analysis of the residual matrix obtained removing the rows and columns associated with the relative eigenspace.

Consider the Jacobian matrix in Lemma F.2. The last $N-k$ columns of the left blocks contain only zero entries so that the matrix possesses eigenvalue 0 with (at least) multiplicity $N-k$. Moreover, in each of the last $N-1-k$ rows in the central blocks the only non-zero entries belong to the main diagonal. Consequently, $\Phi_j^{\varphi_j}$ for $k+1 \le j \le N-1$ are eigenvalues of the matrix, with multiplicity (at least) one. A first contribution to the characteristic polynomial is then determined as

$$(-\mu)^{N-k} \prod_{j=k+1}^{N-1} (\Phi_j^{\varphi_j} - \mu) = (-\mu)^{N-k} \prod_{j=k+1}^{N-1} \left(\frac{1 + x_j^*(r^* + \bar{e})}{1 + r^*} - \mu\right) \tag{F.9}$$

where we used (F.6) to compute $\Phi_j^{\varphi_j}$ at equilibrium.

In order to find the remaining part of the characteristic polynomial we eliminate the rows and columns associated to the previous eigenvalues to obtain

$$\boldsymbol{L} = \left\|\begin{array}{ccc|ccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\
\hline
\Phi_1^{x_1} & \dots & \Phi_1^{x_k} & \Phi_1^{\varphi_1} & \dots & \Phi_1^{\varphi_k} & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_k^{x_1} & \dots & \Phi_k^{x_k} & \Phi_k^{\varphi_1} & \dots & \Phi_k^{\varphi_k} & 0 & \dots & 0 & 0 \\
\hline
R^{x_1} & \dots & R^{x_k} & R^{\varphi_1} & \dots & R^{\varphi_k} & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0
\end{array}\right\| \tag{F.10}$$

This quadratic matrix has $2k + L$ rows when $k < N$. If $k = N$, representation (F.10) is, strictly speaking, not correct. Indeed, there exist only $N-1$ wealth shares φ's in the original system, therefore the central block of the matrix has maximal dimension $(N-1)\times(N-1)$. Therefore, in this case, the correct matrix has dimension $(2N + L - 1) \times (2N + L - 1)$ and can be obtained from (F.10) through the elimination of the last row and the last column in the central blocks. We will compute now the characteristic polynomial, i.e. determinant $\det(\boldsymbol{L}-\mu\boldsymbol{I})$, where $\boldsymbol{I}$ denotes an identity matrix of the corresponding dimension. We consider separately the following two cases: when $k<N$ and when $k=N$.

If $k<N$, then from (F.6) it follows that for $n,m\leq k$ it is

$$\Phi_n^{\varphi_m}=\begin{cases}1-\varphi_n^*\,v & \text{if } n=m\\ -\varphi_n^*\,v & \text{otherwise}\end{cases}\,,\qquad\text{where}\qquad v=\left(x^*_{1\diamond k}-x^*_N\right)\frac{\bar{e}+r^*}{1+r^*}\,. \tag{F.11}$$

Moreover, since all survivors invest share $x_{1\diamond k}$, it follows that for $m\leq k$

$$R^{\varphi_m}=v\,b\,,\qquad\text{where}\qquad b=x^*_N\,\frac{1+r^*}{x^*_{1\diamond k}\,(1-x^*_{1\diamond k})}\quad . \tag{F.12}$$

The central column block in the determinant $\det(\boldsymbol{L}-\mu\boldsymbol{I})$ can be represented as $\|\; v\,\boldsymbol{b}+\boldsymbol{b}_1 \mid \ldots \mid v\,\boldsymbol{b}+\boldsymbol{b}_k\;\|$, where the following column vectors have been introduced

$$\begin{array}{rcl}
\boldsymbol{b} &=& \left\|\begin{array}{ccc|ccc|cccc} 0 & \ldots & 0 & -\varphi_1^* & \ldots & -\varphi_k^* & b & 0 & \ldots & 0\end{array}\right\|\;,\\
\boldsymbol{b}_1 &=& \left\|\begin{array}{ccc|ccc|cccc} 0 & \ldots & 0 & 1-\mu & \ldots & 0 & 0 & 0 & \ldots & 0\end{array}\right\|\;,\\
&\vdots&\\
\boldsymbol{b}_k &=& \left\|\begin{array}{ccc|ccc|cccc} 0 & \ldots & 0 & 0 & \ldots & 1-\mu & 0 & 0 & \ldots & 0\end{array}\right\|\;.
\end{array}$$

We consider each of the columns in the central block as a sum of two terms and, applying the multilinear property of the discriminant, end up with a sum of 2^k determinants. Many of them are zeros, since they contain two or more columns proportional to vector $\boldsymbol{b}$. There are only $k+1$ non-zero elements in the expansion. One of them has the following structure of the central column block: $\|\;\boldsymbol{b}_1 \mid \ldots\quad\ldots\quad\ldots \mid \boldsymbol{b}_k\;\|$, while k others possess similar structure in the central column block, with column $v\,\boldsymbol{b}$ on the ν'th place instead of $\boldsymbol{b}_\nu$, i.e. for all $\nu\in\{1,\ldots,k\}$ the blocks look like $\|\;\boldsymbol{b}_1 \mid \ldots \mid v\,\boldsymbol{b} \mid \ldots \mid \boldsymbol{b}_k\;\|$.

The central matrix in the one obtained from the former block is diagonal and, therefore, its determinant is equal to $(1-\mu)^k\det\boldsymbol{M}(k)$, where

$$\boldsymbol{M}(k)=\left\|\begin{array}{ccc|cccc}
-\mu & \ldots & 0 & f_1^{r_0} & \ldots & f_1^{r_{L-2}} & f_1^{r_{L-1}}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \ldots & -\mu & f_k^{r_0} & \ldots & f_k^{r_{L-2}} & f_k^{r_{L-1}}\\
\hline
R^{x_1} & \ldots & R^{x_k} & R^{r_0}-\mu & \ldots & R^{r_{L-2}} & R^{r_{L-1}}\\
0 & \ldots & 0 & 1 & \ldots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \ldots & 0 & 0 & \ldots & 1 & -\mu
\end{array}\right\| \tag{F.13}$$

Other k determinants can be simplified in analogous way, so that

$$\det(\boldsymbol{L}-\mu\boldsymbol{I})=(1-\mu)^k\det\boldsymbol{M}(k)+(1-\mu)^{k-1}\sum_{\nu=1}^{k}\det\boldsymbol{M}_\nu(k)\quad , \tag{F.14}$$

where for all $\nu\in\{1,\ldots,k\}$ we define the following matrix

$$\boldsymbol{M}_\nu(k)=\left\|\begin{array}{ccc|c|cccc}
-\mu & \ldots & 0 & 0 & f_1^{r_0} & \ldots & f_1^{r_{L-2}} & f_1^{r_{L-1}}\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \ldots & -\mu & 0 & f_k^{r_0} & \ldots & f_k^{r_{L-2}} & f_k^{r_{L-1}}\\
\hline
\Phi_\nu^{x_1} & \ldots & \Phi_\nu^{x_k} & -v\,\varphi_\nu^* & 0 & \ldots & 0 & 0\\
\hline
R^{x_1} & \ldots & R^{x_k} & v\,b & R^{r_0}-\mu & \ldots & R^{r_{L-2}} & R^{r_{L-1}}\\
0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0
\end{array}\right\|\;. \tag{F.15}$$

We expand the last matrix on the minors of the elements of the central column. For this purpose we for each $\nu \in \{1,\dots,k\}$ introduce yet another matrix

$$\boldsymbol{N}_\nu(k) = \left\| \begin{array}{ccc|ccccc} -\mu & \dots & 0 & f_1^{r_0} & f_1^{r_1} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_k^{r_0} & f_k^{r_1} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\ \hline \Phi_\nu^{x_1} & \dots & \Phi_\nu^{x_k} & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{array} \right\| \quad .$$

Then it is:

$$\det \boldsymbol{M}_\nu(k) = v\big(-\varphi_\nu^* \det \boldsymbol{M}(k) - b \det \boldsymbol{N}_\nu(k)\big) \quad . \tag{F.16}$$

Let us compute the determinant of matrix $\boldsymbol{M}(k)$ in a recursive way. Consider the expansion by the minors of the elements from the first column. The minor of the first element $-\mu$ is a matrix with a structure similar to $\boldsymbol{M}(k)$, whose determinant we denote as $\boldsymbol{M}(k-1)$. The minor associated with R^{x_1} has a left upper block with $k-1$ entries equal to $-\mu$ below the main diagonal. This block generates a contribution μ^{k-1} to the determinant and once its columns and rows are eliminated, one remains with a matrix of the type in (F.1). Applying Lemma F.1 one then has

$$\det \boldsymbol{M}(k) = (-\mu)\, \det \boldsymbol{M}(k-1) + (-1)^k R^{x_1} \mu^{k-1} \,(-1)^{L-1}\, P_{f_1}(\mu) \ ,$$

where P_{f_1} is the stability polynomial associated with the first investment function. Applying recursively the relation above, the dimension of the determinant is progressively reduced. At the end the lower right block of the original matrix remains, which is again a matrix similar to (F.1). Applying once more Lemma F.1 one has for $\boldsymbol{M}(k)$ the following

$$\det \boldsymbol{M}(k) = (-1)^{L-1+k}\, \mu^{k-1} \sum_{j=1}^{k} R^{x_j}\, P_{f_j}(\mu) + (-1)^{L-1+k}\, \mu^k \left(\sum_{j=0}^{L-1} R^{r_j} \mu^{L-1-j} - \mu^L \right) \ . \tag{F.17}$$

The determinant of matrix $\boldsymbol{N}_\nu(k)$ can be computed using the similar strategy. The only difference is that in the last recursive step one of the matrix has zero determinant. Therefore, we have:

$$\det \boldsymbol{N}_\nu(k) = (-\mu)\, \det \boldsymbol{N}_\nu(k-1) + (-1)^k \Phi_\nu^{x_1} \mu^{k-1}\,(-1)^{L-1}\, P_{f_1}(\mu) = (-1)^{L-1+k}\, \mu^{k-1} \sum_{j=1}^{k} \Phi_\nu^{x_j}\, P_{f_j}(\mu) \quad ,$$

which, taking into account (F.16), implies

$$\begin{aligned} \sum_{\nu=1}^{k} \det \boldsymbol{M}_\nu(k) &= v \left(-\det \boldsymbol{M}(k) - b \sum_{\nu=1}^{k} \det \boldsymbol{N}_\nu(k) \right) = \\ &= -v\, \det \boldsymbol{M}(k) + v\, b\, (-1)^{L+k}\, \mu^{k-1} \sum_{\nu=1}^{k} \sum_{j=1}^{k} \Phi_\nu^{x_j}\, P_{f_j}(\mu) = -v\, \det \boldsymbol{M}(k) \quad . \end{aligned}$$

(The last equality above follows directly from expression for $\Phi_\nu^{x_j}$.) Let us now substitute the last relation into (F.14). Then using the expression for $\det \boldsymbol{M}(k)$ from (F.17) where the corresponding values of the derivatives of function R are computed in accordance with results of Lemma F.2, the last contribution into characteristic polynomial follows:

$$\begin{aligned} \det(\boldsymbol{L} - \mu \boldsymbol{I}) &= (1-\mu)^{k-1}\,(1-\mu-v)\, \det \boldsymbol{M}(k) = \\ &= (-1)^{L-1+k}\, \mu^{k-1}\,(1-\mu)^{k-1} \left(\frac{1 + x_N^*(r^* + \bar{e})}{1+r^*} - \mu \right) \left(\frac{(1+r^*)\mu - 1}{x_{1\diamond k}^*(1 - x_{1\diamond k}^*)} \sum_{j=1}^{k} \varphi_j^*\, P_{f_j}(\mu) - \mu^{L+1} \right) . \end{aligned}$$

If $k = N$, i.e. all agents survive, then all investment shares are the same. In this case, from (F.6), all elements in the central column block of matrix (F.10) are zeros apart from the 1's on the diagonal in the central matrix. It contributes to the characteristic polynomial by the factor $(1-\mu)^{N-1}$. The remaining part is the determinant of matrix $\boldsymbol{M}$ in this case. This is consistent with the expression above.

The product of the $\det(\boldsymbol{L} - \mu\boldsymbol{I})$ and (F.9) gives (F.8), what completes the proof. □

Lemma F.4. *Consider no-equity-premium equilibrium $\boldsymbol{x}^*$ with $r^* = -\bar{e}$. The characteristic polynomial P_J of the matrix $\boldsymbol{J}(\boldsymbol{x}^*)$ reads*

$$P_J(\mu) = (-1)^{L+N}\,(1-\mu)^{N-2}\,\mu^{N-1}\,\big(\mu+\bar{e}-1\big)\left(\mu^{L+1} + \frac{\mu-1}{\langle x^2\rangle}\sum_{j=1}^{k}\varphi_j^*\,P_{f_j}(\mu)\right) \tag{F.18}$$

Proof. Since the proof is analogous to the proof of Lemma F.3. some details are omitted. In particular, we confine analysis on the case $k < N$. From the Jacobian matrix in Lemma F.2 we can immediately identify that in each of the $N-1-k$ last rows belonging to the central row clock of the matrix the only non-zero entries belong to the main diagonal of $[\partial\mathcal{W}/\partial\mathcal{W}]$ and equal to 1. In addition, the last $N-k$ columns of the leftmost blocks contain only zero entries. Together, it gives the first entry in the characteristic polynomial:

$$(-\mu)^{N-k}\,(1-\mu)^{N-1-k} \quad , \tag{F.19}$$

while the rows and columns associated to the previous eigenvalues can be eliminated. We obtain

$$\boldsymbol{L} = \left\Vert\begin{array}{ccc|ccc|ccc}
0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \quad \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & f_N^{r_0} & \dots f_N^{r_{L-2}} & f_N^{r_{L-1}} \\
\hline
\Phi_1^R R^{x_1} & \dots & \Phi_1^R R^{x_k} & 1+\Phi_1^R R^{\varphi_1} & \dots & \Phi_1^R R^{\varphi_k} & \Phi_1^{r_0} & \dots \Phi_1^{r_{L-2}} & \Phi_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \quad \vdots & \vdots \\
\Phi_k^R R^{x_1} & \dots & \Phi_k^R R^{x_k} & \Phi_k^R R^{\varphi_1} & \dots & 1+\Phi_k^R R^{\varphi_k} & \Phi_k^{r_0} & \dots \Phi_k^{r_{L-2}} & \Phi_k^{r_{L-1}} \\
\hline
R^{x_1} & \dots & R^{x_k} & R^{\varphi_1} & \dots & R^{\varphi_k} & R^{r_0} & \dots R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots \quad 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \quad \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots \quad 1 & 0
\end{array}\right\Vert \tag{F.20}$$

To compute $\det(\boldsymbol{L}-\mu\boldsymbol{I})$, where $\boldsymbol{I}$ is an identity matrix of the corresponding dimension, we apply the multilinear property of the determinant to the central block of columns in matrix $\boldsymbol{L}$. In order to implement this idea, we introduce the following column vectors of the length $2k + L$:

$$\begin{aligned}
\boldsymbol{d} &= \left\Vert\ 0\ \dots\ 0 \ \middle|\ \Phi_1^R\ \dots\ \Phi_k^R \ \middle|\ 1\ 0\ \dots\ 0\ \right\Vert \quad , \\
\boldsymbol{d}_1 &= \left\Vert\ 0\ \dots\ 0 \ \middle|\ 1-\mu\ \dots\ 0 \ \middle|\ 0\ 0\ \dots\ 0\ \right\Vert \quad , \\
&\ \ \vdots \\
\boldsymbol{d}_k &= \left\Vert\ 0\ \dots\ 0 \ \middle|\ 0\ \dots\ 1-\mu \ \middle|\ 0\ 0\ \dots\ 0\ \right\Vert \quad .
\end{aligned}$$

The central column block in the determinant $\det(\boldsymbol{L}-\mu\boldsymbol{I})$ can be represented as $\Vert\ R^{\varphi_1}\,\boldsymbol{d}+\boldsymbol{d}_1 \mid \dots \mid R^{\varphi_k}\,\boldsymbol{d}+\boldsymbol{d}_k\ \Vert$. We consider each of the columns in the central block as a sum of two terms and end up with a sum of 2^k determinants. Notice, however, that many of them are zeros, since they contain two or more columns proportional to vector $\boldsymbol{d}$. There are only $k+1$ non-zero elements in the expansion. The determinant of the matrix with the structure $\Vert\ \boldsymbol{d}_1 \mid \dots \quad \dots \quad \dots \mid \boldsymbol{d}_k\ \Vert$ in the central part is equal to $(1-\mu)^k \det \boldsymbol{N}(k)$, where matrix $\boldsymbol{N}(k)$ is identical to the matrix $\boldsymbol{M}(k)$ defined in (F.13). (We use here another notation in order to stress that the partial derivatives R^{x_j} and R^{f_j} used in these two matrices have different values in different equilibria.) Using (F.17) together with (F.7) it is immediate to see that

$$\det \boldsymbol{N}(k) = (-1)^{L+k}\,\mu^{k-1}\left(\mu^{L+1} + \frac{\mu+\bar{e}-1}{\langle x^2\rangle}\sum_{j=1}^{k}\varphi_j^*\,P_{f_j}(\mu)\right) \quad . \tag{F.21}$$

Other non-zero elements possess similar structure in the central column block, with column $R^{\varphi_\nu}\,\boldsymbol{d}$ on the ν'th place instead of $\boldsymbol{d}_\nu$ for all $\nu \in \{1,\dots,k\}$. Their determinants can be represented as $(1-\mu)^{k-1}\det \boldsymbol{N}_\nu(k)$, where for all $\nu \in \{1,\dots,k\}$ we define matrix

$$\boldsymbol{N}_\nu(k) = \left\| \begin{array}{ccc|c|cccc} -\mu & \dots & 0 & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\ \hline \Phi_\nu^R R^{x_1} & \dots & \Phi_\nu^R R^{x_k} & R^{\varphi_\nu}\Phi_v^R & \Phi_\nu^{r_0} & \dots & \Phi_\nu^{r_{L-2}} & \Phi_\nu^{r_{L-1}} \\ \hline R^{x_1} & \dots & R^{x_k} & R^{\varphi_\nu} & R^{r_0}-\mu & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{array} \right\| .$$

This matrix can be simplified, since its central row is (almost) proportional to the next row (the first row in the bottom block). Applying multilinear property of the determinant, and computing the determinant of the resulting matrix we get

$$\det \boldsymbol{N}_\nu(k) = (-\mu)^{L+k} R^{\varphi_\nu}\Phi_v^R \quad .$$

Using the corresponding expressions from Lemma F.2, one can check that $\sum_{\nu=1}^k \Phi_\nu^R R^{\varphi_\nu} = -\bar{e}$. Therefore,

$$\begin{aligned} \det(\boldsymbol{L}-\mu\boldsymbol{I}) &= (1-\mu)^k \det \boldsymbol{N}(k) + (1-\mu)^{k-1}\sum_{\nu=1}^k \det \boldsymbol{N}_\nu(k) = \\ &= (1-\mu)^k \det \boldsymbol{N}(k) - (1-\mu)^{k-1}(-\mu)^{L+k}\bar{e} = \\ &= (1-\mu)^{k-1}(-1)^{L+k}\,\mu^{k-1}\left((1-\mu)\mu^{L+1} + (1-\mu)\frac{\mu+\bar{e}-1}{\langle x^2\rangle}\sum_{j=1}^k \varphi_j^*\,P_{f_j}(\mu) - \mu^{L+1}\bar{e} \right) = \\ &= (1-\mu)^{k-1}(-1)^{L+k}\,\mu^{k-1}\,(\mu+\bar{e}-1)\left(\mu^{L+1} + \frac{\mu-1}{\langle x^2\rangle}\sum_{j=1}^k \varphi_j^*\,P_{f_j}(\mu) \right) \end{aligned}$$

Combining now the last expression with (F.19) we get polynomial (F.18). □

Using the characteristic polynomial of the Jacobian matrix in the corresponding equilibrium, it is straightforward to derive the equilibrium stability conditions of Section 4.3.

The case of one survivor: Proof of Proposition 4.2

If $k=1$ the characteristic polynomial (F.8) reduces to

$$P_J(\mu) = (-1)^{N+L}\,\mu^{N-1}\prod_{j=2}^N\left(\frac{1+x_j^*\,(r^*+\bar{e})}{1+r^*}-\mu\right)\left(\mu^{L+1} - \frac{(1+r^*)\mu-1}{x_1^*(1-x_1^*)}\,P_{f_1}(\mu)\right) .$$

From the expression of the derivative of the EML at equilibrium $l'(r^*)$ one can see that last factor corresponds to the polynomial Q_1 in (4.10). The conditions in (4.11) are derived from the requirement

$$\left|\frac{1+x_j^*\,(r^*+\bar{e})}{1+r^*}\right| < 1 \qquad j \geq 2\,,$$

and the Proposition is proved.

The case of many survivors: Proof of Proposition 4.3

In the case of $k > 1$ survivors the characteristic polynomial in (F.8) possesses a unit root with multiplicity $k-1$. Consequently, the fixed point is non-hyperbolic.

To find the eigenspace associated with the eigenvalue 1 we subtract from the initial Jacobian matrix (F.2) computed at equilibrium the identity matrix of the corresponding dimension and analyze the kernel of the resulting $\boldsymbol{J} - \boldsymbol{I}$ matrix. This can be done through the analysis of the kernel of the matrix obtained by the substitution of the identity matrix from matrix $\boldsymbol{L}$ given in (F.10). Let us consider the $k < N$ and the $k = N$ cases separately.

When $k < N$, as we showed in the proof of Lemma F.3, in the matrix obtained as a result of subtraction of an identity matrix from (F.10), the central $k-1$ columns are identical, see (F.11) and (F.12). Therefore, the kernel of the matrix $\boldsymbol{J} - \boldsymbol{I}$ can be generated by a basis containing the following $k-1$ vectors

$$\boldsymbol{u}_n = \Big(\underbrace{0,\dots,0}_{N};\underbrace{0,\dots,0}_{n-1},1,\underbrace{0,\dots,0}_{k-n-1},-1;\underbrace{0,\dots,0}_{N-1-k};\underbrace{0,\dots,0}_{L}\Big)\ , \quad 1 \le n \le k-1 \quad . \tag{F.22}$$

Notice that the direction of vector $\boldsymbol{u}_n$ corresponds to a change in the relative wealths of the n-th and k-th survivor.

If, instead, $k = N$, then the last $k-1$ columns in the resulting (from (F.10)) matrix are zero vectors, and then the kernel of the matrix $\boldsymbol{J} - \boldsymbol{I}$ can be generated with the $N-1$ vectors of the canonical basis

$$\boldsymbol{v}_n = \Big(\underbrace{0,\dots,0}_{N};\underbrace{0,\dots,0}_{n-1},1,\underbrace{0,\dots,0}_{N-n-1};\underbrace{0,\dots,0}_{L}\Big)\ , \quad 1 \le n \le N-1 \quad . \tag{F.23}$$

whose direction corresponds to a change in the relative wealths of the n-th and N-th survivors.

If the system is perturbed away from equilibrium $\boldsymbol{x}^*$ along the directions defined in (F.22) or (F.23), a new fixed point is reached. Then, the system is stable, but not asymptotically stable, with respect to these perturbations.

Moreover, since the eigenspaces identified above do not depend on the system parameters, it is immediate to realize that they do constitute not only the tangent spaces to the corresponding non-hyperbolic manifolds, but the manifolds themselves.

The polynomial (4.12) is the last factor in (F.8), while conditions (4.13) are obtained by imposing

$$\left|\frac{1 + x_j^*\,(r^* + \bar{e})}{1 + r^*}\right| < 1 \qquad j > k+1\ ,$$

which completes the proof.

The case of "no-equity-premium" equilibria. Proof of Proposition 4.4

Independently of the number of survivors, the characteristic polynomial in (F.18) possesses a unit root with multiplicity $N-2$. Consequently, the fixed point $\boldsymbol{x}^*$ is never hyperbolic, when $N \geq 3$. It is easy to see that in this case all equilibria belong to the manifold of dimension $N-2$ and that this is exactly a non-hyperbolic manifold of $\boldsymbol{x}^*$. For the stability of equilibrium $\boldsymbol{x}^*$ with respect to the perturbations in the directions *orthogonal* to this manifold, it is sufficient to have all other eigenvalues inside the unit circle. If this condition is satisfied, then equilibrium $\boldsymbol{x}^*$ of the system is stable, but not asymptotically stable. Since $\bar{e} > 0$, this sufficient condition can be expressed through the roots of the last term in (F.18). This term is exactly polynomial (4.14).

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