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HAL Id: halshs-00479819

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Submitted on 2 May 2010

Geometrical and Arithmetical Methods
in Early Medieval Perspective

Dominique Raynaud[*]

ABSTRACT. This paper examines the hypothesis that early perspective paintings were drawn arithmetically, without vanishing points. The best argument for this hypothesis is that the division of two parallel lines by straight lines intersecting each other at the vanishing point (geometrical method) is equivalent to the division of those parallel lines in proportional parts (arithmetical method). If arithmetical method had been used, then the vanishing points exhibited *ex post* should be purely fortuitous. But the lack of multiples and submultiples of measurement units, the absence of proportionality ratios, the length of the operating series, and the correspondence of vanishing points to visible loci of the picture offer sound objections for this hypothesis. The use of optics and geometrical method is more probative—though it does not mean that painters were using concepts of linear perspective, which would be an anachronism.

KEYWORDS. Middle Ages, perspective, painting, arithmetic, geometry.

INTRODUCTION

Perspective is a legitimate object for the history of science when painting layouts obey a mathematical scheme. In spite of the alleged contributions by Brunelleschi and Alberti, the history of perspective *as a science* begins only in Piero della Francesca's *De prospectiva pingendi* (1470)[1]. The present paper will try to apply methods in the history of science to the duecento and trecento, as well as to more distant periods. This is a tentative essay calling for further research, and we deliberately focus on a situation that is poorly documented: the frescoes painted on the lower register of the Upper Church in Assisi. Here are the reasons justifying this choice:

1° Contemporaneous documents concerning these mural paintings are practically nonexistent. The names of the painters that were working on the site are only assumed. As for many ancient large-scale projects, numerous hands have been suggested: Jacopo Torriti, Cimabue, the Master of the Capture, the Master of Isaac, Giotto, etc. Take the particular case of *The Legend of St Francis* scenes. The spectrum of attributions ranges from Giotto di Bondone[2] and the Master

of Isaac, sometimes identified with Arnolfo di Cambio[3], to an unknown Roman painter such as Pietro Cavallini or Filippo Rusuti[4].

2° We know just as little about the dates of the decoration campaign. The church was built between the laying of the foundation stone by Gregory IX (17 July 1228) and its consecration by Innocent IV (11 June 1253). The period of decoration is known less accurately. It ranges from about 1254 to 1338. According to Vasari,[5] Giotto painted the thirty-two scenes for the *Legend of St Francis* on the lower register of the Upper Church circa 1296-1305. The *terminus post quem* (1296) coincides with the election of Giovanni da Morrovalle as the general minister of the Franciscan order. The *terminus ante quem* (1305) can be deduced from the fact that the tower of the Palazzo del Capitano, seen unfinished on the frescoes, was completed at that date. Certain art historians have restricted this interval to 1296-1299, or even 1296-1297, on the basis of stylistic criteria.[6] But the present tendency is to antedate the paintings. The reasons are (1) Vasari's obvious bias in favor of the Medici and Florentine contribution to art; (2) the growing evidence of a Roman influence on the Assisi frescoes; (3) the long vacancy of the Throne between 1292 and 1294, which does not corroborate any key decision relative to the Pontifical Church in this period; (4) the recent discovery of a fourteenth-century Franciscan manuscript that reports the intentions and commission by Nicolas IV. Thus there are good reasons to date the beginning of the fresco campaign ca. 1290-1292.[7]

3° Very little is known concerning the time techniques. The only testimonies on fresco painting are those by Vasari and Cennini,[8] at several centuries of distance. The material examination[9] of the frescoes partly compensates for the gap, but one must infer a complex process from scattered clues. Nevertheless, three main stages of fresco painting are discernible: (1) sketching the drawing *(disegno)*; (2) squaring the pattern *(gratta)* onto the rough plaster underlayer *(arricio)*; (3) transferring the pattern onto the layer of fresh plaster *(intonaco)*, with straight lines being drawn with a ruler and sometimes redrawn with an awl *(puntaruolo)*; circles being

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[3] A.M. ROMANINI, *Arnolfo pittore: pittura e spazio virtuale nel cantiere gotico*, «Arte medievale», 11, 1997, pp. 3-33.

[4] H.B.J. MAGINNIS, A. LADIS, *Assisi today: the upper church*, «Source», 18, 1998, pp. 1-6.

[5] "Finite queste cose [in Arezzo, Giotto] si condusse in Ascesi, città dell'Umbria, essendovi chiamato da fra' Giovanni di Muro della Marca, allora Generale de' frati di san Francesco, dove nella chiesa di sopra dipinse a fresco, sotto il corridoio che attraversa le finestre, dai due lati della chiesa, trentadue storie della vita e fatti di San Francesco", G. VASARI, *Le Vite de' più eccellenti Pittori Scultori e Architettori nelle redazioni del 1550 e 1568*, a cura di R. Bertanini e P. Barocchi, Firenze, Sansoni editore, 1967, II, p. 100.

[6] E. LUNGHI, *La Basilica di San Francesco di Assisi*, Antella, Scala, 1996, pp. 66-67.

[7] L. BELLOSI, G. RAGIONIERI, *Giotto e le storie di San Francesco, op. cit.*; D. COOPER, J. ROBSON, *Pope Nicholas IV and the upper church at Assisi*, «Apollo», 157, 2003, pp. 31-35.

[8] G. VASARI, *Le Vite, op. cit.*, p. 199; C. CENNINI, *Il Libro dell'arte o Trattato della pittura*, a cura di C. e G. Milanesi, Firenze, Le Monnier, 1859, p. 60.

drawn with a compass; curves being directly drawn on the coating; and faces being transferred by means of drawings used as a kind of template *(patroni)*. In general, preliminary drawings were not kept.[10] The frescoes of the Upper Church at Assisi are known only scarcely: the crumbling coating enables us to discover the reddish brown pigment underdrawing *(sinopia)* on the left side of *St Francis Preaching before Pope Honorius III*;[11] the examination in low-angled light makes it possible to see the sign of the line in the fresh coating, the perforations of the nails maintaining the line and, at times, the painter's fingerprints, as in the *Extasis*.[12] The double transfer of the *disegno* to the *sinopia*, and then onto the *intonaco*, poses a difficult problem to the history of perspective. The layout, required from the earliest stage, was partly disappearing in the double transfer. Seen in low-angled light, the vanishing lines in perspective are generally limited to the edges of the buildings.[13]

4° Documentary gaps preclude a direct apprehension of the knowledge that was underlying the perspective layouts. That is why these mural paintings have given rise to multifarious interpretations, encompassing the whole spectrum from intellectualism to empiricism. There is no best way. Intellectualism runs the risk of anachronism by accrediting concepts of horizon, vanishing point, etc., that were undefined at that time, whereas empiricism encounters an underevaluation of the overt or tacit knowledge that is necessary to draw in perspective. And yet the frescoes of the Upper Church at Assisi are unquestionably pioneering trials in the rationalization of the visual space. These frescoes thus await an interpretation able to reconcile empirical evidence with conceptual minimalism.[14]

1. THE ARITHMETICAL CONSTRUCTION HYPOTHESIS

In order to explain the duecento and trecento improvement in the expression of depth, historians first supposed that artists already had at their disposal a basic knowledge of convergence,

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[9] *Pittura a fresco. Tecniche esecutive, cause di degrado, restauro,* a cura di G. Basile, Firenze, Le Monnier, 1989; B. ZANARDI, C. FRUGONI, F. ZERI, *Il Cantiere di Giotto. Le storie di San Francesco ad Assisi*, Milano, Skira, 1996.

[10] B. DEGENHART, A. SCHMITT, *Einleitung a Corpus der Italienischen Zeichnungen 1300-1450*, I-1. *Sud- und Mittelitalien*, Berlin, Gebr. Mann Verlag, 1968, p. XIX.

[11] "[…] Subito sopra l'aureola du Francesco è visibile una piccolissima traccia, del tutto indecifrabile, d'un disegno in rosso sinopia. Ma, al di là di questa minima prova è l'enorme complessità iconografica del ciclo a rendere impossibile l'ipotesi di una esecuzione a fresco delle scene in assenza d'un detagliatissimo progetto su carta (o pergamena) da riportare al vere sull'arriccio in forma di sinopia […] Questi dati materiali impongono di nuovo di dar per certo un assai dettagliato lavoro di progettazione del ciclo francescano, prima con disegni su carta (o pergamena) e poi sull'arriccio", B. ZANARDI, *Il cantiere di Giotto, op. cit.*, pp. 24, 32.

[12] *Ibid.*, pp. 24-32.

[13] *Ibid.*, p. 29 (fig. 18, 19, 20), p. 31 (fig. 22, 23, 24).

[14] J. WHITE, *The Birth and Rebirth of Pictorial Space*, London, Faber and Faber, 1957, French transl. *Naissance et renaissance de l'espace pictural*, Paris, Adam Biro, 1992, p. 32.

infinity, etc. This interpretation[15] has given rise to well-known objections: (1) Mathematical knowledge that was at the disposal of craftsmen is poorly known, and mostly hypothetical; (2) The convergence on a defined vanishing point is only occasional: we observe more often a convergence on a "vanishing axis" or on a less accurately defined "vanishing region"; (3) Trials in the rationalization of pictorial space have led to many competing systems, hence reducing the importance of linear perspective as such. Such difficulties invalidate much of the credibility granted to Panofsky's opinion that painters such as Lorenzetti had a mathematical consciousness of the vanishing point, thought of as a "symbol of infinity."[16]

Empiricism is the opposite of intellectualism. Among the empiricist stances, that of Andrés de Mesa Gisbert demandes attention because it combines rigor with minimalism. The author proposed a hypothesis to explain the convergence of vanishing lines in fourteenth-century painting.[17] This is in fact a method for building a perspective arithmetically. "Si disponemos dos rectas paralelas con cualquier distancia entre sí, y luego de dividir una de ellas en un número cualquiera de partes lo hacemos en forma similar sobre la segunda paralela, guardando exactamente las mismas proporciones con las que se lo ha hecho inicialmente, al unir los puntos correspondientes con líneas rectas, en su prolongación *obtendremos la convergencia de todas ellas sobre un solo y único punto sin necesidad de haber operado con él*" (*Ibid*., p. 33, italics mine).

The author's basic insight is that convergence of segments *AD*, *BE*, *CF…* may be obtained either by drawing the lines *OA*, *OB*, *OC…* whose segments *AD*, *BE*, *CF…* are the visible parts *(geometrical method)*, or by proportioning segments *DE*, *EF…* to segments *AB*, *BC…*, the proportionality ratio between *AB* and *DE*, *BC* and *EF* being sufficient to ensure the existence of the virtual vanishing point *O (arithmetical method)*. Among the relations on similar triangles, craftsmen would have extracted relation (1), leaving aside relations (2), (3), etc., which make the concurrent point *O* intervene (*Ibid*., pp. 33-34, Figure 1).

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[15] G.J. KERN, *Die Anfänge der zentralperspektivischen Konstruktion in der italienischen Malerei des 14. Jahrhunderts* , «Mitteilungen des Kunsthistorischen Instituts in Florenz», 2, 1913, pp. 39-65; E. PANOFSKY, *Die Perspective als symbolische Form*, «Vorträge der Bibliothek Warburg», 1924, pp. 258-331, French transl. *La Perspective comme 'forme symbolique'*, Paris, Éditions de Minuit, 1975; D. GIOSEFFI, *Perspectiva artificialis. Per la storia della prospettiva, spigolature e appunti*, Trieste, Tipografia Smolars, 1957. The thesis has been amended by R. SINISGALLI, *Per la storia della prospettiva, 1405-1605*, Roma, L'Erma, 1978.

[16] E. PANOFSKY, *Die Perspective als symbolische Form, op. cit.*, p. 125.

[17] A. DE MESA GISBERT, *El 'fantasma' del punto de fuga en los estudios sobre la sistematización geometrica de la pintura del siglo XIV*, «D'Art», 15, 1989, pp. 29-50. The author has henceforth specialized in architectural measured survey.

$$(1) \ \frac{DE}{EF} = \frac{AB}{BC} \qquad (2) \ \frac{OE}{ED} = \frac{OB}{BA} \qquad (3) \ \frac{OE}{EF} = \frac{OB}{BC}$$
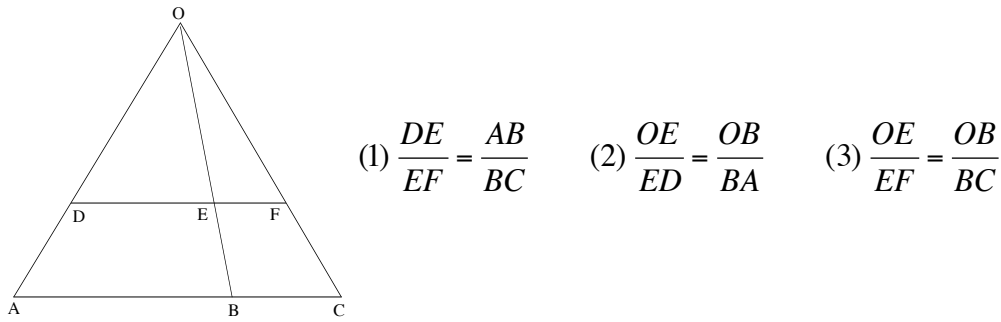
Figure 1: The arithmetical hypothesis

The arithmetical hypothesis explains quite simply why *ex post* layouts show a point at infinity: it would be a by-product of the analysis that presupposes the use of geometrical devices. But, in fact, painters could have used a proportionality rule to divide the frontal-horizontal lines and distribute them in depth.

In addition, de Mesa's hypothesis solves three problems affecting pre-renaissance paintings: (1) it accounts for the arbitrary behavior of certain lines, invoking that proportional ratios were not defined on all lines (observe parallel edges of the abaci of the capitals in *St Francis Preaching before Pope Honorius III)*; (2) it explains by way of the same reason the additional presence of a vanishing axis as in *The Pentecost*; and (3) the convergence on a vanishing region is seen as a secondary effect of the errors made during the transfer of proportional segments (any inaccurate positioning of points *A, B, C* will induce a deviation of vanishing lines *AD, BE, CF)*; this arithmetical scheme is often cited to emphasize anachronism in the attempts to recognize the beginnings of linear perspective in medieval times.[18]

Despite its obvious ingenuity and usefulness, the arithmetical method gives rise to difficulties that were never systematically explored. This is understandable: when a conclusion seems correct, rarely do we thoroughly question its premises. The arithmetical hypothesis is highly liable: nothing proves that medieval craftsmen made use of vanishing points. Until now the arithmetical method has been hypothetical, not factual. There is a need for further scrutiny.

2. LACK OF MULTIPLES OR SUBMULTIPLES OF MEASUREMENT UNITS

In a perspective painting composed according to the arithmetical method, frontal lines ought to be divided into multiples or submultiples of common measurement units. The choice of a set of measures is never arbitrary, whether having a symbolic or practical value. The history of ar-

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[18] Consequently, arithmetical method supports the idea that perspective was a Renaissance invention: the author speaks of Brunelleschi's contribution, *Ibid*, p. 35. For criticism, see D. RAYNAUD, *L'Hypothèse d'Oxford, Essai sur les origines de la perspective*, Paris, PUF, 1998, pp. 4-9, 132-150.

chitecture provides many illustrations of this. Take for example the alleged geometric layout of the cappella Pazzi by Brunelleschi.[19] Konrad Hecht[20] inaugurated the critical approach by paying attention to the discrepancy of seventeen regulator layouts published between 1867 and 1957. In the same vein, Jean Guillaume[21] has shown that the regulator layouts imagined to explain the architectural composition do not match well with the Cappella dei Pazzi measurements, to which Brunelleschi always gave integer or simple values—e.g. pilasters are $1\frac{1}{2}$ *braccia* in width.

Transposed to the case of mural painting, the discovery of units multiples would be a good clue that the arithmetical method had been used: (1) it is easier to calculate proportional ratios on simple measurements; and (2) the means is favorable with regard to memory and communication. Do early perspective paintings exhibit multiples or submultiples of measurement units?

In Umbria, the two competing systems used were the *braccio* and the *piede*. As in many other regions, there were two values for the *braccio:* the *braccio corto* (0,599 m) and the *braccio lungo* (0,668 m) documented in Perugia, Foligno, Orvieto, Spoleto, etc. Choose the second one, inasmuch as the *braccio corto* was called "*da legname*" and "*da muratori*", whereas the *braccio lungo* was specified as a unit "*da lana*", "*da panno*" or "*da seta*". This choice established the system of measures: *braccio* (599 mm), *oncia* (49,92 mm), *soldo* (29,95 mm). The Umbrian value for the *piede* "*da legname e da fabbrica*" was 0,363 m, from which we can deduce a second system of measures: *piede* (363 mm), *palmo* (90,75 mm), *pollice* (30,25 mm), and *dito* (22,69 mm)[22].

	br.	o.	s.	q.
br.	1	12	20	60
o.	$\frac{1}{12}$	1	$\frac{5}{3}$	5
s.	$\frac{1}{20}$	$\frac{3}{5}$	1	3
q.	$\frac{1}{60}$	$\frac{1}{5}$	$\frac{1}{3}$	1

braccio umbro (0,599 m)

	p.	pa.	po.	d.
p.	1	4	12	16
pa.	$\frac{1}{4}$	1	3	4
po.	$\frac{1}{12}$	$\frac{1}{3}$	1	$\frac{4}{3}$
d.	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{4}$	1

piede umbro (0,363 m)

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[19] A similar case occurs in Lescot's façade for the Louvre, whose measurements are multiple of the pied du Roi (326,6 mm) used in Paris around 1546, J.-P. SAINT AUBIN, *Photogrammétrie et étude des ordres: le Louvre de Lescot,* in *L'Emploi des ordres à la Renaissance*, ed. by J. Guillaume, Actes du colloque de Tours (9-14 juin 1986), Paris, Picard, 1992, pp. 219-226.

[20] K. HECHT, *Maßverhältnisse und Maße der Cappella Pazzi*, «Architectura», 6, 1976, pp. 148-174.

[21] J. GUILLAUME, *Désaccord parfait: ordres et mesures dans la chapelle des Pazzi*, «Annali di Architettura», 2, 1991, pp. 9-23.

[22] "Brachium continet 12 vntias", "Pes palmorum quattuor, pollicum seu vnciarum duodecim, digitorum vero sexdecim". We leave aside the *quattrino*, whose narrow step (9,98 mm) is not differential enough. R.E. ZUPKO, *Italian Weights and Measures from the Middle Ages to the Nineteenth Century*, Philadelphia, The American Philosophical Society, 1981, pp. 47-48 (braccio), 197 (piede).

Look now at *The Confirmation of the Franciscan Rule*, which de Mesa *(op. cit.*, fig. 11-14) presumes to be an exemplary application of arithmetical method. Draw the layout of this fresco,[23] and report the main measurements on it. Mark on the layout: above the line on the left side, the coffer radius; below the line, the coffer diameter; above the line on the right side (underlined), the total length of the vaulted ceiling (Figure 2).
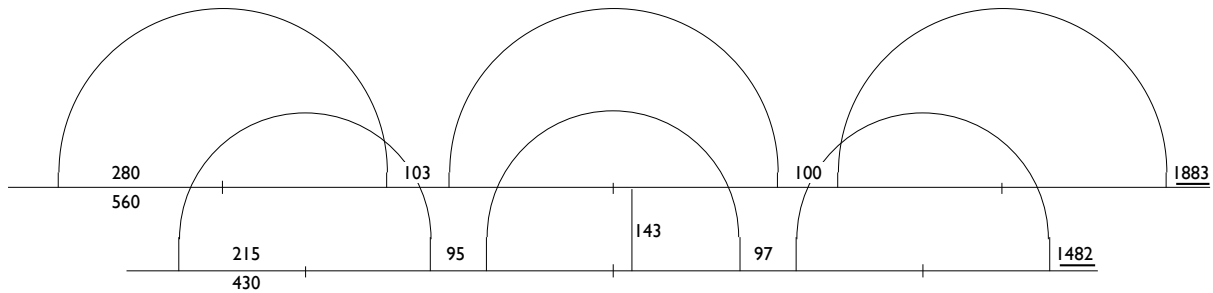

Figure 2: The Confirmation of the Franciscan Rule

Braccio-soldi system

	Horizontal dim. (mm)		Vertical dim. (mm)
549,08 < 560 < 569,05	419,30 < 430 < 449,25	1856,90 < 1883 < 1886,85	119,80 < 143 < 149,75
269,55 < 280 < 299,50	209,65 < 215 < 239,60	1467,55 < 1482 < 1497,50	
99,83 < 103 < 119,80	89,85 < 95 < 99,83		
99,83 < 100 < 119,80	89,85 < 97 < 99,83		

Number of concordances plus or minus the error: 2.
List of concordances: 99,83 mm = 2 o.

Piede-palmi system

	Horizontal dim. (mm)		Vertical dim. (mm)
544,50 < 560 < 567,19	423,50 < 430 < 431,06	1875,50 < 1883 < 1883,06	136,13 < 143 < 151,25
272,25 < 280 < 294,94	211,75 < 215 < 226,88	1474,69 < 1482 < 1482,25	
90,75 < 103 < 113,44	90,75 < 95 < 113,44		
90,75 < 100 < 113,44	90,75 < 97 < 113,44		

Number of concordances plus or minus the error: 3.
List of concordances: 431,06 mm = 1 br. 3 d., 1482,25 mm = 4 br. 1 po., 1883,06 mm = 5 br. 3 d.

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[23] We have relied upon the photographic survey by B. ZANARDI, C. FRUGONI, F. ZERI, *Il Cantiere di Giotto*, p. 128. The survey scale can be deduced from the dimensions of the fresco (363 × 357 cm), the height of the standing St Francis (122 cm), the height of the figures (122, 111, 123 cm), and the diameter of the halo (35,5 cm). We have systematically checked the parallelism and lack of distorsion on the fresco, following D. RAYNAUD, *La théorie des erreurs et son application à la reconstruction des tracés perspectifs*, in *L'Artiste et l'Oeuvre à l'épreuve de la perspective*, ed. by M. Dalai Emiliani, M. Le Blanc, P. Dubourg Glatigny, Rome, École Française de Rome, 2006, pp. 411-430.

The dimensions of the coffers can hardly be expressed in one or another system of units. The measurements corresponding to integer values ± 3 mm are 2 out of 11 in the *braccio-soldi* system, 3 out of 11 in the *piede-palmi* system. This fact raises a doubt on the existence of an underlying system of measurements.

Apply now the same analytic method to the fresco of *St Francis Preaching before Pope Honorius III*, which represents a groined-vault roofed space.[24]
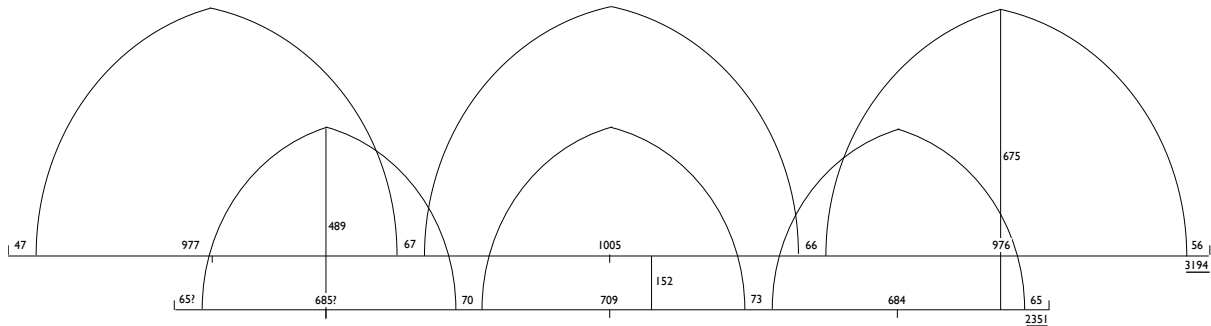

Figure 3: St Francis Preaching before Pope Honorius III

Braccio-soldi system

	Horizontal dim. (mm)		Vertical dim. (mm)
998,33 < 1005 < 1018,30	698,83 < 709 < 718,80	3174,70 < 3194 < 3194,67	658,90 < 675 < 688,85
958,40 < 977 < 988,35	658,90 < 685 < 688,85	2346,08 < 2351 < 2366,05	479,20 < 489 < 499,17
958,40 < 976 < 988,35	658,90 < 684 < 688,85		149,75 < 152 < 179,70
59,90 < 67 < 89,85	59,90 < 73 < 89,85		
59,90 < 66 < 89,85	59,90 < 70 < 89,85		
49,92 < 56 < 59,90	59,90 < 65 < 89,85		
29,95 < 47 < 49,92	59,90 < 65 < 89,85		

Number of concordances plus or minus the error: 3.
List of concordances: 49,92 mm = 1 o., 149,75 mm = 5 s., 3194,67 mm = 5 br. 4 o.

Piede-palmi system

	Horizontal dim. (mm)		Vertical dim. (mm)
998,25 < 1005 < 1020,94	703,31 < 709 < 726,00	3176,25 < 3194 < 3198,97	665,50 < 675 < 680,63
975,56 < 977 < 998,25	680,63 < 685 < 695,75	2336,83 < 2351 < 2359,52	484,00 < 489 < 499,13
975,56 < 976 < 998,25	680,63 < 684 < 695,75		151,25 < 152 < 158,81
60,50 < 67 < 68,06	68,06 < 73 < 90,75		

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[24] B. ZANARDI, *Il Cantiere di Giotto, op. cit.*, p. 242.

60,50 < <u>66</u> < <u>68,06</u> | <u>68,06</u> < <u>70</u> < 90,75

45,38 < 56 < 60,50 | 60,50 < 65 < 68,06

<u>45,38</u> < <u>47</u> < 60,50 | 60,50 < 65 < 68,06

Number of concordances plus or minus the error: 7.
List of concordances: 45,38 mm = 2 d., 68,06 mm = 3 d., 151,25 mm = 1 pa. 2 po., 975,56 mm = 2 br. 2 pa. 3 d.

The main measurements of this space can hardly be expressed in one or another systems of units. The dimensions close to an integer value ± 3 mm are 3 out of 19 in the *braccio-soldi* system, 7 out of 19 in the *piede-palmi* system.

Consider now *The Recovery of the Wounded Man of Lerida*, whose flat-coffered ceiling is a checkered pattern in perspective.[25] On the perspective scheme (Figure 4): the first number on the left side represents the width of the narrowest coffer of the row; the second one, the coffer mean width; close to the axis, the height of the central coffers; on the right side, the total width of the row; below the axis, the total height of the ceiling.
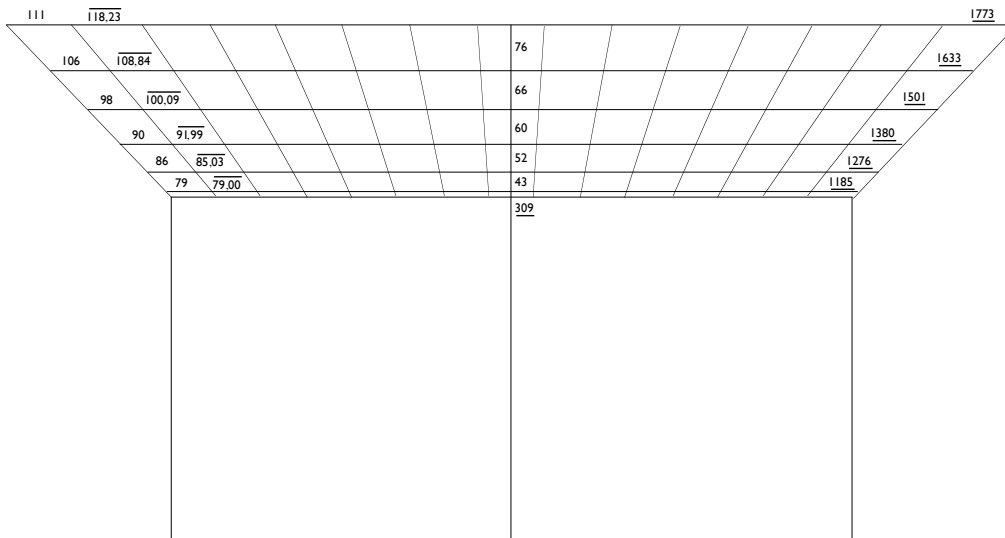

Figure 4: The Recovery of the Wounded Man of Lerida

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[25] *Ibid.*, p. 332. *The Recovery of the Wounded Man* is one of the earliest works that present a correct foreshortening of the intervals, but it is not a case of linear perspective, because the correctness of the perspective is limited to the coffered ceiling. (1) Side ceilings are put in oblique perspective while the main ceiling is in central perspective. (2) The horizon is situated 722 mm above the eye line, with which it should coincide. (3) There is a lack of regularity in the foreshortening. The remotest horizontal line of the ceiling produces an interval as high as the previous one, probably because of the mix-up between this line and the one that marks the boundary of the coffered space. In perspective, however, two equal intervals ought to be of different heights. (4) The fresco shows

Braccio-soldi system

Horizontal dim. (mm)				*Vertical dim.* (mm)
99,83 < 111 < 119,80	99,83 < <u>118,23</u> < <u>119,80</u>	1767,05 < 1773 < 1797,00	59,90 < 76 < 89,85
99,83 < 106 < 119,80	99,83 < 108,84 < 119,80	1617,30 < 1633 < 1647,25	59,90 < 66 < 89,85
89,85 < <u>98</u> < <u>99,83</u>	<u>99,83</u> < <u>100,09</u> < 119,80	1497,50 < 1501 < 1527,45	<u>59,90</u> < <u>60</u> < 89,85
<u>89,85</u> < <u>90</u> < 99,83	89,85 < <u>91,99</u> < 99,83	<u>1377,70</u> < <u>1380</u> < 1397,67	<u>49,92</u> < <u>52</u> < 59,90
59,90 < 86 < 89,85	59,90 < 85,03 < 89,85	1257,90 < 1275 < 1287,85	29,95 < 43 < 49,92
59,90 < 79 < 89,85	59,90 < 79,00 < 89,85	1168,05 < 1185 < 1198,00	299,50 < 309 < 329,45

Number of concordances plus or minus the error: 8.
List of concordances: 49,92 mm = 1 o., 59,90 mm = 2 s., 89,85 mm = 3 s., 99,83 mm = 2 o., 119,80 mm = 4 s., 1377,70 mm = 2 br. 5 s.

Piede-palmi system

Horizontal dim. (mm)				*Vertical dim.* (mm)
90,75 < <u>111</u> < <u>113,44</u>	113,44 < <u>118,23</u> < <u>121,00</u>	1769,63 < 1773 < 1784,75	68,06 < 76 < 90,75
90,75 < 106 < 113,44	90,75 < 108,84 < 113,44	1610,81 < <u>1633</u> < <u>1633,50</u>	60,50 < <u>66</u> < <u>68,06</u>
90,75 < 98 < 113,44	90,75 < 100,09 < 113,44	1497,38 < 1501 < 1512,50	45,38 < <u>60</u> < <u>60,50</u>
68,06 < <u>90</u> < <u>90,75</u>	<u>90,75</u> < <u>91,99</u> < 113,44	1361,25 < 1380 < 1383,94	45,38 < 52 < 60,50
68,06 < 86 < 90,75	68,06 < 85,03 < 90,75	1270,50 < 1275 < 1293,19	30,25 < <u>43</u> < <u>45,38</u>
68,06 < 79 < 90,75	68,06 < 79,00 < 90,75	1179,75 < 1185 < 1202,44	302,50 < 309 < 317,63

Number of concordances plus or minus the error: 8.
List of concordances: 45,38 mm = 2 d., 60,50 mm = 2 po., 68,06 mm = 3 d., 90,75 mm = 1 pa., 113,44 mm = 1 pa. 1 d., 121,00 mm = 1 pa. 1 po., 1633,50 mm = 4 br. 2 pa.

No measurement taken onto the three frescoes come down to a simple combination of multiples or submultiples of time units.

1. In the *Recovery of the Wounded Man of Lerida*, the coffers' mean dimensions (column 2), which are by nature more reliable than the single ones, are not better adjusted to integer values than the most erroneous values taken onto the fresco (column 1).

2. Each series shows many dimensions removed from integer values (59,90 < 79 < 89,85; 299,50 < 309 < 329,45; 1257,90 < 1275 < 1287,85, etc.), hence being inexpressible in one or another system of units.

3. There are very few measurements close to integer values. In the *Approval of the Franciscan Rule*, there are 2 out of 11 in the *braccio-soldi* system, and 3 out of 11 in the *piede-palmi*

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small errors of drawing. For example, the axis of the ceiling is shifted 12 mm to the right compared to the one of the composition.

system. In *St Francis Preaching before Pope Honorius III*, there are 3 out of 19 in the *braccio-soldi* system, and 7 out of 19 in the *piede-palmi* system. In the *Recovery of the Wounded Man of Lerida*, there are 8 out of 24 in the *braccio-soldi* system, and 8 out of 24 in the *piede-palmi* system. If we combine all the measurements, there are 13 out of 54 (24%) that are close to an integer value in the first system; and 18 out of 54 (33%) in the second one. The number of values being superior to 50, it is comparable to chance level. In the *braccio-soldi* system, integer values fall on every 29,95 mm and 49,92 mm ± 3 mm. One *braccio* (LCM of *oncia* and *soldo)* contains twelve *oncie* error intervals and twenty *soldi* error intervals, four of which are overlapping. The probability of obtaining an integer value ± 3 mm in the LCM interval is therefore equal to $\frac{6(20+12-4)}{599,00} = \frac{168}{599} = 28\%$. In the *piede-palmi* system, integer values fall on every 22,69 mm and 30,25 mm ± 3 mm. One *palmo* (LCM of *pollice* and *dito)* contains three *pollici* error intervals, four *dita minuti* error intervals, among which one in common. The probability of getting an integer value in the LCM interval is $\frac{6(3+4-1)}{90,75} = \frac{36}{90,75} = 39\%$. Theoretical and empirical numbers of integer values are of the same order of magnitude: 24% ≈ 28% and 33% ≈ 39%. Thereby, empirical concordances do not exceed random level. The hardly higher result in the *piede-palmi* system is due to the *pollici* and *dita minuti* narrower step.

4. Finally, suppose a variation of the units used by craftsmen around the ones historically documented. It is necessary to introduce such a variation, because the usual medieval units were not as accurate as present standards. This variation results from several factors: (1) Historical fluctuations. For example, fifteenth-century architectural measured surveys set the *braccio fiorentino* at 0,5875, 0,5860 or 0,5836 m[26]; (2) Professional fluctuations. *Braccio* and *piede* values were specific to different crafts, as proved by their names: *"agrimensorio"*, *"da legname"*, *"da muratori"*, *"da panno"*, *"da mercatori"*, etc.; (3) Regional fluctuations. In the sole case of Tuscany, the *braccio* took on the common value of 0,584 m (in Arezzo, Florence, S. Miniato, Pistoia, Siena, Montepulciano, Lucca, Pisa, Volterra, etc.), but shorter and longer values of the *braccio* were nevertheless used in Fivizanno (0,486 m), Massa (0,495 m), Montecarlo (0,593 m), and Pontremoli (0,692 m)[27]; The actual or putative presence of foreign masters (Cavallini, Rusuti, Giotto, etc.) on the site of the Upper Church precludes the exclusion of one or another of the units used in Assisi (*br*. 0,599 m; *p*. 0,363 m), Rome (*br*. 0,636 m; *p*. 0,298 m) or Florence (*br*. 0,584 m).

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[26] These *braccio* values are given by K. HECHT, *Maßverhältnisse und Maße der Cappella Pazzi*, *op. cit*.; L. BENEVOLO, S. CHIEFFI, G. MEZETTI, *Indagine sul S. Spirito di Brunelleschi*, «Istituto di Storia dell'Architettura, Quaderni», 85/90, 1968, pp. 1-52; C.L. FROMMEL, *Der Römische Palastbau der Hochrenaissance*, 3 Bde, Tübingen, Verlag E. Wasmuth, 1973.
[27] R.E. ZUPKO, *Italian Weights and Measures, op. cit*., p. 46.

In order to take into account historical, professional and regional fluctuations, let us produce a continuous variation of *braccio* from 525,6 mm (*br. fl.* – 10%) to 699,6 mm (*br. rom.* + 10%) and then study the fluctuation of the number of integer values *y* as a function of this extensible *braccio x*. If function $y = f(x)$ admits a maximum almost equal to the total number of the frescoes' measurements, this maximum will be the value of the unit searched for. An optimization algorithm enables us to detect the maxima. The function admits 10 values (550,01 ≤ *br.* ≤ 550,24) as a *minimum minimorum*, and 27 values (652,40 ≤ *br.* ≤ 652,44) as a *maximum maximorum*. The range of integer values plus or minus the margin of error is from $\frac{10}{54} = 18\%$ to $\frac{27}{54} = 50\%$, i.e. less than a half. Consider now a continuous variation of the *piede* from 268,2 mm (*p. rom.* – 10%) to 399,3 mm (*p. umbro* + 10%). The number of integer values fluctuates from 12 (375,34 ≤ *p.* ≤ 375,54) to 31 (273,30 ≤ *p.* ≤ 273,35). Thus, the range of integer values plus or minus the margin of error is from $\frac{12}{54} = 22\%$ to $\frac{31}{54} = 57\%$. It follows that none of the unit systems can convert the dimensions measured on the frescoes into integer or simple measures. The frescoes' dimensions reveal no numerical consistency and, in nearly half of the cases, they come to a standstill after the division by the smallest subunit. This conclusion exhibit a significant discrepancy with respect to the arithmetical hypothesis that requires discontinuity, not continuity.

3. ABSENCE OF SIMPLE PROPORTIONAL RATIOS

There is another possibility to save the arithmetical hypothesis. Just imagine that painters used arithmetical formulas with non-metric instruments as lines that can be folded in equal parts to determine a given ratio. In this case, arithmetic still works, though without standard dimensions. Suppose that, in the classical manner, duecento and trecento painters used modules without unit. Due to the use of arithmetical method, homologous parts should nevertheless be in simple proportional ratio a_n/a_{n+1}. This is exactly what Andrés de Mesa supposes in taking a_1/a_2 = 2 (*op. cit.*, p. 34, fig. 8). But, in fact, such ratios are at random in the frescoes examined. In the *Confirmation of the Franciscan Rule*, the ratios $\frac{430}{560} = 0,7678\ldots$, $\frac{1482}{1883} = 0,7870\ldots$ do not coincide with elementary fractions $\frac{3}{4} = 0,75$ or $\frac{4}{5} = 0,8$. In *St Francis Preaching before Pope Honorius III*, $\frac{685}{977} = 0,7011\ldots$, $\frac{709}{1005} = 0,7054\ldots$, $\frac{684}{976} = 0,7008\ldots$, $\frac{2351}{3194} = 0,7360\ldots$ likewise differ from $\frac{2}{3} = 0,6666\ldots$ or $\frac{3}{4} = 0,75$. The result is even clearer in the third fresco. *The Recovery of the Wounded Man of Lerida* allows us to calculate similar ratios from more accurate mean values. But ratios $\frac{79,00}{85,03} = 0,9290\ldots$, $\frac{85,03}{91,99} = 0,9243\ldots$, $\frac{91,99}{100,09} = 0,9190\ldots$, $\frac{100,09}{108,84} = 0,9196\ldots$, and

$\frac{108,84}{118,23} = 0,9205\ldots$ fluctuate around $\frac{12}{13} = 0,9230\ldots$, which is an unlikely fraction, owing to its denominator. Therefore, craftsmen seem to have left aside elementary fractions.

It is also necessary to submit to analysis de Mesa's argument *(op. cit.*, pp. 43-45) that painters used the *superbipartiens* rule in order to plan the foreshortening. This proportionality rule consists in making an interval two-thirds as high as the previous one. Alberti already criticized this method:

> Hic essent nonnulli qui unam ab divisa aequedistantem lineam intra quadrangulum ducerent, spatiumque, quod inter utrasque lineas adsit, in tres partes dividerent. Tum huic secundae aequedistanti lineae aliam item aequedistantem hac lege adderent, ut spatium quod inter primam divisam et secundam aequedistantem lineam est, in tres partes divisum una parte sui excedat spatium id quod sit inter secundam et tertiam lineam, ac deinceps reliquas lineas adderent ut semper sequens inter lineas esset spatium ad antecedens, ut verbo mathematicorum loquar, superbipartiens [...][28].

The passage can be interpreted in several ways. The narrowest interpretation consists in attaching more importance to the two-thirds rule than to the name of *superbipartiens* proportion that Alberti gave to it. The broader interpretation acts on the contrary: it considers the two-thirds rule as only an instance of the general case of *superbipartiens* proportion.

Works studied in this paper do not match the narrow interpretation. Take again *The Recovery of the Wounded Man of Lerida* that presents a 15 × 5 checkered ceiling. We can compare its intervals to two *superbipartiens* series, resulting from the foreshortening of the largest interval: $76\frac{2}{3} = 50,7$, $76\frac{4}{9} = 33,8$, $76\frac{8}{27} = 22,5$, $76\frac{16}{81} = 15,0$ (series 1) or the enlargement of the smallest interval: $43\frac{3}{2} = 64,5$, $43\frac{9}{4} = 96,7$, $43\frac{27}{8} = 145,1$, $43\frac{81}{16} = 217,7$ (series 2). Neither of the two series corresponds to the intervals of *The Recovery* (series 0):

Series 1	<u>76</u>	50,7	33,8	22,5	15,0
Series 0	<u>76</u>	66	60	52	<u>43</u>
Series 2	217,7	145,1	96,7	64,5	<u>43</u>

The broad interpretation is based on the fact that the *superbipartiens* proportion concept regards a class of ratios. This concept comes from the medieval theory of proportions.[29] *Super-*

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[28] L.B. ALBERTI, *De la peinture / De pictura (1435)*, pref., transl., and notes by J.-L. Schefer, introd. by S. Deswarte-Rosa, Paris, Macula, 1992, pp. 116-117, commentary p. 242.

[29] The most influential texts were those by BOETHIUS, *De Institutione arithmeticae*, ed. By J.-I. Guillaumin, Paris, Les Belles Lettres, 1995; Jordanus de NEMORE, *De Elementis arithmetice artis. A medieval treatise on number theory*, ed. by H.L.L. Busard, Stuttgart, Franz Steiner, 1991; H.L.L. BUSARD, *Die Traktate 'De proportionibus' von Jordanus Nemorarius und Campanus*, «Centaurus», 15, 1971, pp. 193-227. Correct definitions can

partiens genus characterizes every proportion $(n + \frac{p}{q} : 1)$, *n*, *p*, *q* natural integers, $n = 1$, $p < q$, $p \geq 2$. Among the species of this genus, the *superbipartiens*[30] proportion meets the additional condition that $p = 2$. The two-thirds rule $(1 + \frac{2}{3} : 1)$ is then given for $q = 3$. It is a subspecies of the *superbipartiens* species. It is thus possible to read in the *De Pictura* a foreshortening rule extended to any proportional ratio $(1 + \frac{2}{q} : 1)$, $q \in N$, $q > 2$. Unfortunately, none of such ratios correspond to the intervals of the *Recovery of the Wounded Man*. The series are diverging as *q* is growing, so the best matching occurs for $q = 3$, which is an unsatisfactory case.[31] Consequently there is no proof that *superbipartiens* rule has been used.

We can nevertheless imagine a *latissimo sensu* interpretation of the rule, by extending it to all the cases in which a given term of the series is a constant ratio of the previous one. The rule is thus widened to *multiplex, superparticularis, superpartiens, multiplex superparticularis* and *multiplex superpartiens* proportions. This interpretation does not match any better *The Recovery's* intervals, because they do not follow a constant ratio. The series 0,827 (43/52); 0,867 (52/60); 0,909 (60/66); 0,868 (66/76) can be compared to the corresponding ratios of a checkered pattern in linear perspective. Establish first that constant ratios are antagonistic with perspective.
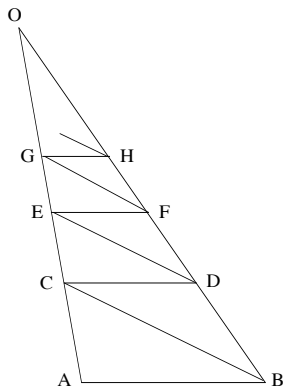

Figure 5

1) Lines *AB, CD, EF, GH…* are horizontal, and points *A, C, E, G…* collinear (Figure 5). Therefore, $\angle BAC = \angle DCE = \angle FEG…$

2) By hypothesis, intervals are in constant ratio, hence *AC / AB = CE / CD = EG / EF…*

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also be found in less-known treatises, such as this anonym *Tractatus proportionum*: "Prima species superpartienti generis est proportio superbipartiens, que fit quando maior numerus continet totum minorem et insuper eius duas partes, ut 5 ad 3, 7 ad 5. Secunda species supradicti generis est proportio supertripartiens, que fit quando maior numerus continet in se totum minorem in se et insuper eius tres partes, ut sunt 7 ad 4, 11 ad 8, etc.", Saint-Dié, B. M., ms. 42, fol. 119r.

[30] *Supertripartiens*, *superquadripartiens*, and *superquinquepartiens* proportions were formed on the same pattern.

3) After *Elements*, VI, 7, it follows that $\angle ABC = \angle CDE = \angle EFG$…

4) Since *AB, CD, EF, GH*… are parallels, the diagonals *BC, DE, FG*… are parallel to each other and cannot intersect whereas linear perspective requires a concurrent point (distance point). Consequently, successive equal intervals in linear perspective cannot be in constant ratio. Reciprocally, a painting in which successive equal intervals are in constant ratio is not a linear perspective.

Knowing that the *Recovery's* intervals are not in constant ratio, its ceiling foreshortening ought to be compared to linear perspective. Empirical values are already known. Theoretical values can be found by means of analytic geometry. Begin with a simplified scheme of the fresco, on which the coordinates (x, y) of the points useful for obtaining theoretical values are marked (Figure 6).
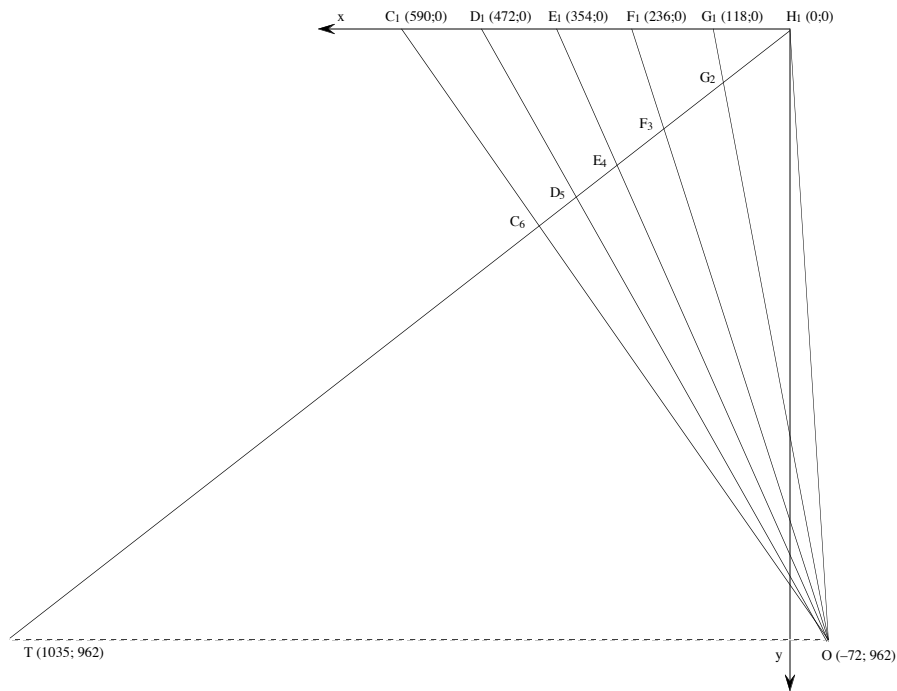

Figure 6. Scheme for the Recovery of the Wounded Man of Lerida

Let us mark in system H_1xy, the coordinates of points C_1, D_1, E_1, F_1, G_1, and the coordinates of central point O and point of distance T that should be used in linear perspective. Find the ordinates of points C_6, D_5, E_4, F_3, G_2 setting the height of intervals. Each point belongs to two lines: $C_6 = OC_1 \cap H_1T$; $D_5 = OD_1 \cap H_1T$… Therefore each point solves a system of equations that describe the lines the point belongs to. For example, point G_2 solves the system:

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[31] To match the observed series to the *superbipartiens* series, a non integer *q* must be taken (optimal matching is for $q = 2,3$ that provides the terms 43,45; 49,97; 57,47; 66,09; 76,00). But this is impossible by definition.

$$\begin{cases} y = \dfrac{962}{1305}x & (1: \text{éq. } H_1T) \\[2mm] y = -\dfrac{962}{118+72}x + 597{,}45 & (2: \text{éq. } G_1O) \end{cases}$$

By introducing the value x of (1) in (2), we obtain:

$$-\frac{962}{118+72}\left(\frac{1305}{962}\,y\right) + 597{,}45 - y = 0$$

And after simplification and factorization:

$$-y\left(\frac{1305}{118+72} + 1\right) + 597{,}45 = 0$$

Whence:

$$y = \frac{597{,}45}{1 + \dfrac{1305}{118+72}} = \frac{597{,}45}{7{,}869} = 75{,}93$$

The difference between ordinates of points G_1 (x; 0) and G_2 (x; 75,93) determines the height of the first interval, i.e. 75,93 mm. The ordinates of points C_6, D_5, E_4, F_3 ought to be calculated in the same way. Deduce now the height of all intervals: 75,93; 64,44; 56,36; 48,84; 42,98 (series 3), and compare these theoretical values with the empirical values derived from the fresco. The values are fitted with an slight error e_{max} = 3,64 mm:

Series 0	76	66	60	52	43
Series 3	75,93	64,44	56,36	48,84	42,98

The ceiling of the *Recovery of the Wounded Man of Lerida* uses a foreshortening method that is indiscernible from the one that is required by linear perspective. Consequently no *super-bipartiens* method—understood *stricto sensu*, *lato sensu* or *latissimo sensu*—has been used in the laying out of the fresco. The fact opposes the arithmetical hypothesis. Why is it that painters of the Upper Church did not use proportionality ratios to solve the problem of the foreshortening? Perhaps because they took the representational problem as geometers, whose frame of mind always favors a continuum.

4. LENGTH OF THE OPERATING SERIES[32]

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[32] For details, see D. RAYNAUD, *Las primeras perspectivas de los siglos XIII y XIV según el enfoque del modus operandi*, in *Perspectiva: fundamentação teórica e cultural*, ed. by Magno Mello, Belo Horizonte, Argumentum, 2009, pp. 41-62.

The apparent simplicity of arithmetical method is partly due to the fact that it is deliberately vague about certain operations. For instance, the proportionality of intervals is established on parallels lines, but the existence of those parallels is taken for granted, whereas they need to be built. Many pre-perspective paintings such as *The Approval of the Franciscan Rule, St Francis Preaching before Pope Honorius III,* or *Christ among the Doctors* present an axis of symmetry. But to draw an axis of symmetry by means of proportional ratios is not an easy matter. It is thus necessary to describe all the constructive operations in order to reasonably compare geometrical and arithmetical methods in perspective.

Consider again *The Recovery of the Wounded Man of Lerida*, and, in order to neutralize all previous objections, suppose that intervals obey simple dimensions and proportions. Assume the layout to be a ruler-and-compass construction, according to the usual devices of geometry. The operating series are describable at different scales. One should distinguish between *m.o. macro-operations* (draw a perpendicular, divide a line into n equal parts…) and *e.o. elementary operations* (take a given aperture of a compass, join two points with a ruler…)

For example, the *m.o.* "draw a perpendicular to a given line" contains five *e.o.*: "fix the metal point of the compass on a point of the given line", "draw a circle of any aperture", "fix the metal point on another point of the given line", "draw a circle of same aperture", "join the circles' intersections with a ruler" *(Elements,* I, 11, Figure 7).
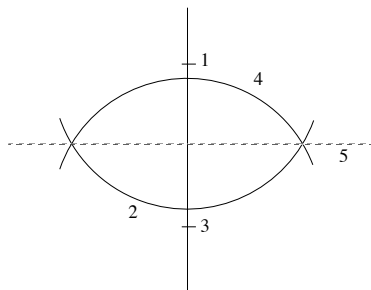

Figure 7

We can compare the length of the different operating series by means of the minimum number of operations required *(m.m.o.* and *m.e.o.* respectively).

Arithmetical method operating series (Figure 9)

1° Draw axis OS (take two marks, draw a vertical), 7 e.o.

2° Draw first horizontal $A_1P_1 \perp OS$ (draw a perpendicular), 5 e.o.

3° Draw horizontal A_2P_2 at the distance A_1P_1 from H_1H_2 (draw a parallel), 6 e.o.

4° Calculate $H_2H_3 = k.\ H_1H_2$ (apply a proportional ratio), 2 e.o.

5° Draw horizontal A_3P_3 at the distance A_2P_2 from H_2H_3, 6 e.o.

Repeat 3 times operations 4 and 5 to get lines $A_4P_4 \ldots A_6P_6$, 24 e.o.

12° Calculate $\frac{1}{2} H_1 I_1$ (apply a ratio), 3 e.o.

13° Draw $H_1 I_1$ on $A_1 P_1$ on both sides of axis OS (draw a circle of radius $\frac{1}{2} H_1 I_1$), 2 e.o.

14° Transfer interval $H_1 I_1$ little by little onto $A_1 P_1$ with a compass, 15 e.o.

15° Calculate $K = k^5$, 5 e.o.

16° Calculate $\frac{1}{2} H_6 I_6 = K \cdot \frac{1}{2} H_1 I_1$ (apply a ratio), 3 e.o.

17° Draw $H_6 I_6$ on $A_6 P_6$ on both sides of axis OS, 2 e.o.

18° Transfer interval $H_6 I_6$ little by little onto $A_6 P_6$ with a compass, 15 e.o.

19° Join $A_1 A_6$ with a ruler, 1 e.o.

Repeat 15 times operation 19 to get all segments $B_1 B_6 \dots P_1 P_6$, 15 e.o.

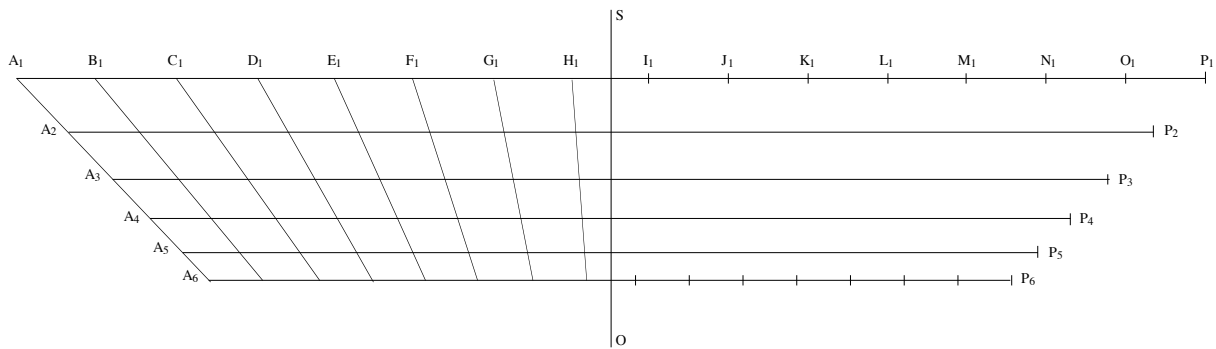*m.m.o.* = 34   *m.e.o.* = 111

Figure 9. Arithmetical layout for the Recovery of the Wounded Man

Geometrical method operating series (Figure 10)

1° Draw axis OS (draw a vertical), 3 e.o.

2° Draw first horizontal $A_1 P_1 \perp OS$ (draw a perpendicular), 5 e.o.

3° Calculate $\frac{1}{2} H_1 I_1$ (apply a ratio), 3 e.o.

4° Draw $H_1 I_1$ on $A_1 P_1$ on both sides of axis OS (draw a circle of radius $\frac{1}{2} H_1 I_1$), 2 e.o.

5° Transfer interval $H_1 I_1$ little by little onto $A_1 P_1$ with a compass, 15 e.o.

6° Draw pencil lines $A_1 O \dots P_1 O$ with a ruler, 16 e.o.

7° Draw horizontal $OT \perp OS$ (draw a perpendicular), 5 e.o.

8° Draw diagonal $H_1 T$, 3 e.o.

9° Draw horizontal $A_2 P_2$ through point $H_1 T \cap G_1 O \dots$, 6 e.o.

Repeat 4 times operation 9 to get all lines $A_3 P_3 \dots A_6 P_6$, 24 e.o.
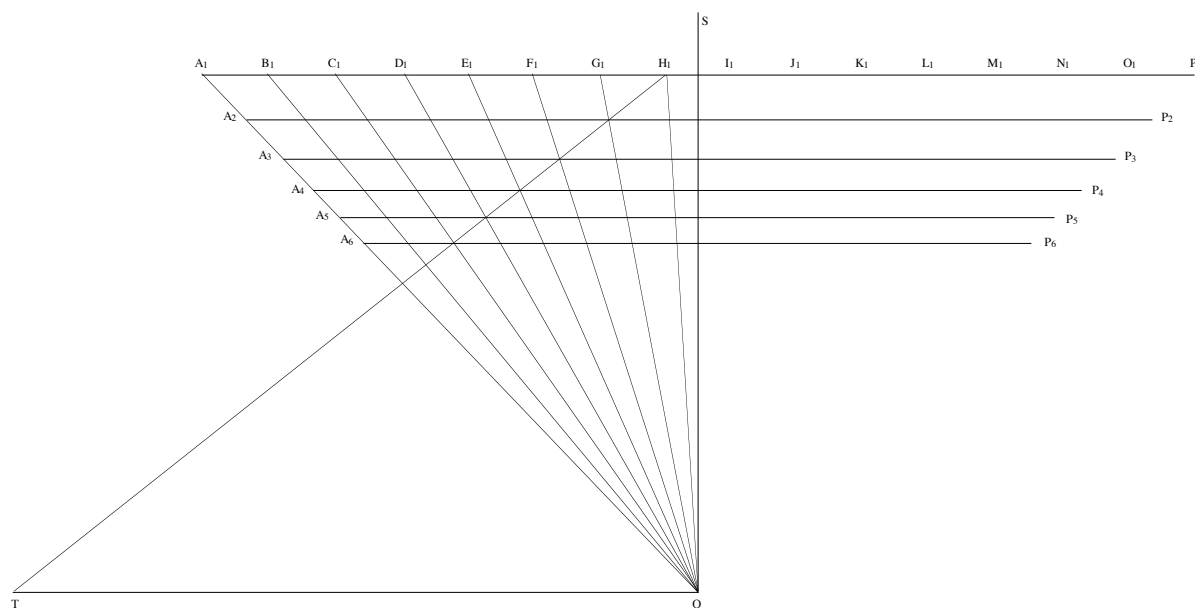
m.m.o. = 13   *m.e.o.* = 83

Figure 10. Geometric layout for the Recovery of the Wounded Man

The comparison of the operating series is definitely unfavorable to arithmetical method because it requires more operations (34 *m.m.o.* and 111 *m.e.o.*) than does geometrical method (13 *m.m.o.* and 83 *m.e.o.*). Furthermore, one should note that the present comparison has been limited to the devices of learned geometry,[33] supposing all constructions to be drawn with ruler and compass. But practical geometry[34] freed oneself from such constraints, by having recourse to many geometrical instruments. The fact has been known since earliest antiquity. Introduce only the main instruments documented in the thirteenth century: ruler, compass, square, level, and *libella*. Thus, arithmetical method appears to be twice as long (34 *m.m.o.* and 111 *m.e.o.*) as the geometrical-instrumental one (13 *m.m.o.* and 63 *m.e.o.*). This difference in length is not due to

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[33] The most popular text was Adelard of Bath's version of the *Elements*: EUCLID OF ALEXANDRIA, *The First Latin Translation of Euclid's* Elements *commonly ascribed to Adelard of Bath: Books I-VIII and Books X.36-XV.2* ed. by H.L.L. Busard. Toronto, The Pontifical Institute of Medieval Studies, 1983 (for a mathematical commentary, see EUCLID, *Les Éléments*, ed. by B. Vitrac, 4 vols, Paris, PUF, 1990-2001), and its commentary by AL-NAYRÎZÎ, *Anaritii in decem libros priores Elementorum Euclidis commentarii ex interpretatione Gherardi Cremonensis*, edidit M. Curze, Leipzig, B. G. Teubner, 1899.

[34] See the treatises by BOETHIUS (M. FOLKERTS, *«Boetius» Geometrie II: Ein mathematisches Lehrbuch des Mittelalters*, Wiesbaden, Steiner Verlag, 1970); ABRAHAM BAR H̲IYYA (M. CURZE, *Der Liber Embadorum des Abraham bar Chijja Savasorda in der Übersetzung des Plato von Tivoli*, Leipzig, B. G. Teubner, 1902); *Artis cuiuslibet consummatio* (S.K. VICTOR, *Practical Geometry in the High Middle Ages*, Philadelphia, The American Philosophical Society, 1979); Leonardo FIBONACCI (*Scritti di Leonardo Pisani, matematico del secolo decimoterzo*, publicati da B. Boncompagni, vol. II. *Leonardi Pisani Practica geometriae ed opuscoli*, Roma, Tipografia delle scienze matematiche e fisiche, 1862); ABÛ AL-WAFÂ' AL-BÛZJÂNÎ, *Kitâb fî mâ yah̲tâju ilayhi al-ṣâni' min a'mâl al-handasa / L'œuvre scientifique d'Abû al-Wafâ' al-Bûzjânî*, II. *La géométrie pratique d'Abû al-Wafâ'*, ed. J. Aghayani Chavosi, thèse de doctorat, Université de Paris VII, 1997; N. CHUQUET, *La Géométrie. Première géométrie algébrique en langue française (1484)*, introduction, texte et notes par Hervé L'Huillier, Paris, Vrin, 1979). For further references, see the works quoted at note 36.

arithmetical method complexity, but to the fact it presupposes a great number of background operations. Once the implicit operations have been rendered, the method is much more tedious.

5. COINCIDENCE OF POINTS AT INFINITY WITH VISIBLE LOCI

Since points at infinity (central and lateral vanishing points) are useless according to arithmetical method, painters should not have used them. Consequently, they should not coincide with visible loci of the pictorial composition. In contrast, if a systematic and accurate correspondence is found, it will support the idea that duecento and trecento painters actually used geometrical devices instead of arithmetical ones.

The systematic correspondence with visible loci of the composition is already a feature of certain paintings of the Giotto epoch. For instance, in *The Recovery of the Wounded Man*, central point O and lateral points $T\,T'$ are aligned, thus forming a line: the so-called horizon of classical perspective. But with the side ceiling being divided in six by four coffers, it appears that this line coincides with a perfectly visible line that divides lateral ceilings in two equal parts. In addition, the checkered ceiling is marked by diagonals $A_1B_2C_3\ldots$ $B_1C_2D_3\ldots$ $C_1D_2E_3\ldots$ on the left side, and by diagonals $P_1O_2N_3\ldots$ $O_1N_2M_3\ldots$ $N_1M_2L_3\ldots$ on the right side. These diagonals converging on lateral points $T\,T'$ are exactly those that should be used to obtain the perspective foreshortening. If medieval painters had ever followed the arithmetical method, why are the horizon and diagonals lines so neatly visible? This correspondence is not occasional. A squared coffer divided by the diagonal is usual in Assisiate frescoes. It works as a spatial roofing pattern in several scenes: *St Francis Honored by a Simple Man, St Francis before the Sultan (Trial by Fire), Death of the Knight of Celano*, and *Christ among the Doctors*. Thus we cannot exclude that such diagonals could have served as an empirical rule to place receding lines in depth.

6. REINTERPRETING PERSPECTIVE ANOMALIES

The comparison of arithmetical and geometrical methods in early perspective shows the prominence of geometrical devices in duecento and trecento paintings. This is a somewhat counter-intuitive finding. One should then clarify the gap between the present result and the usual opinion on the topic, interpreting anew the main errors affecting medieval paintings.

1. The arbitrary behavior of small elements, such as the abaci of capitals, is not an exclusive clue of arithmetical method. Such anomalies are understandable considering the material constraints exerted upon the laying out. Fast-drying coating required a quick transfer of the pictorial scheme onto the *intonaco*. Consequently only the main lines were transferred, not the small elements.

2. The convergence of edges on a vanishing region is hardly a better proof in favor of an arithmetical formula. Methodology shows that all perspective painting—whether derived from an

arithmetical or geometrical formula—is subject to errors of construction. In a geometrical layout, an error of parallelism is sufficient to make vanishing lines deviate.

3. The presence of a vanishing axis is not so easily attributable to errors of construction. This pictorial scheme indeed presents a systematization that departs from random. The use of axial composition could have resulted from the theory of binocular vision expounded by Ibn al-Haytham, latinized as Alhazen *(Kitâb al-manâẕir / De aspectibus*, III, 2, 12 sq.), and by his Latin commentators: Roger Bacon, *Perspectiva*, II, II, 2; John Pecham, *Perspectiva communis*, I, 80; and Witelo, *Perspectiva*, III, 27 sq. This idea has been tested onto works whose edges converge on two central vanishing points.[35] They correspond to the case of homonym diplopia discerned by Ibn al-Haytham.

7. EARLY PERSPECTIVE IN THE EDUCATIONAL CONTEXT

Inserting arithmetical and geometrical methods in the educational context is suitable for testing craftsmen's possibilities of applying mathematics in workshop practices. Perspective as a science took its departure with Piero della Francesca's first geometrical proofs. Before this date, there was—strictly speaking—no more than affinity between perspective and the medieval sciences.

1. The common opinion is that painters and craftsmen, not being admitted to the university, benefited only from abacus school training. This could be a sound argument for the arithmetical hypothesis—as abacus means properly arithmetic—but any confusion between *abaco* and *scuola d'abaco* should be avoided. Arithmetic was not the one and only teaching of abacus schools. *Arismetricha, geometria*,[36] *edifichare e prospettiva*[37] were also on the program, as at Santa Trinità in Florence, founded by Paolo Dagomari dell'Abaco. Thus, there are some reasons to think that it was there that medieval painters acquired basic notions of geometry and optics they afterwards put into practice.

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[35] D. RAYNAUD, *Ibn al-Haytham sur la vision binoculaire: un précurseur de l'optique physiologique*, «Arabic Sciences and Philosophy», 13, 2003, pp. 79-99; ID., *Une application méconnue des principes de la vision binoculaire: Ibn al-Haytham et les peintres du Trecento (1295-1450)*, «Oriens/Occidens», 5, 2004, pp. 93-131.

[36] On practical geometry, see A. SIMI and L. TOTI-RIGATELLI, *Some 14th and 15th century texts on practical geometry*, in M. FOLKERTS and J. P. HOGENDIJK, eds., *Vestigia Mathematica. Studies in Medieval and Early Modern Mathematics in Honour of H.L.L. Busard*, Amsterdam/Atlanta, Rodopi, 1993, pp. 453-470; A. SIMI, Problemi caratteristici della geometria pratica nei secoli XIV-XVI, *Scienze mathematiche e insegnamento in epoca medioevale*, a cura di P. Freguglia, L. Pellegrini e R. Paciocco, Napoli, ESI, 2000, pp. 153-200.

[37] On practical perspective, see G. ARRIGHI, *Un estratto dal "De visu" di M° Grazia de'Castellani (dal Codice Ottoboniano latino 3307 della Biblioteca Apostolica Vaticana)*, «Atti della Fondazione Giorgio Ronchi», 22, 1967, pp. 44-58; F. CAMEROTA, *Misurare "per perspectiva" : geometria practica e prospectiva pingendi*, in *La prospettiva*, a cura di R. Sinisgalli, Firenze, Cadmo, 1998, pp. 340-378; F. CECCHINI, *Ambiti di diffusione del sapere ottico nel Duecento. Tracce per uno studio sulle conoscenze scientifiche degli artisti italiani del XIII secolo*, in *L'Artiste et l'Oeuvre à l'épreuve de la perspective, op. cit.*, pp. 19-42.

The chances for architecture *(edifichare)* and perspective to cross-fertilize each other benefited from the high popularity of Vitruvius' *De architectura* in the Middle Ages.[38] According to Vitruvius, the architect ought to be "*peritus graphidos, eruditus geometria [et optices non ignarus]*"[39]. He explicitly deals with perspective *(scaenographia)*, referring to Agatharcus, Democritus, and Anaxagorus[40]. Except for Agatharcus who was working as a painter, his sources are in the field of optics and geometry. Vitruvius attributes a minor position to arithmetic: "*Per arithmeticen uero sumptus aedificiorum consummantur, mensurarum rationes explicantur difficilesque symmetriarum quaestiones geometricis rationibus et methodis inueniuntur*"[41]. Medieval craftsmen seem to have appreciated in the same way the affinities between perspective, optics, and geometry. For instance, a Pistoia pulpit epigraph reports that Giovanni Pisano was learned in optics: "*Sculpsit Johannes… doctum super omnia visa*". Similarly, Villard de Honnecourt relies on geometry to introduce the art of drawing: "*Ci comence li force des trais de portraiture si con li ars de iometrie les ensaigne*"[42]. At folios 20r-21r, Villard subsequently presents some devices to measure an inaccessible height or distance, a usual problem of practical geometry and perspective.

8. EARLY PERSPECTIVE IN THE CLASSIFICATION OF SCIENCES

The medieval classification of sciences[43] can help us to appreciate the links between perspective, geometry and arithmetic. According to the classification by al-Fârâbî, transmitted to the Latin World by Gerard of Cremona's and Dominicus Gundissalinus' translations[44], *perspectiva* is threefold: *optica* (direct rays), *catoptrica* (reflected rays), *dioptrica* (refracted rays). Picto-

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[38] "Le texte de Vitruve n'a cessé d'être connu (et donc recopié) de l'Antiquité à la Renaissance. Aussi ne faut-il pas s'étonner du nombre relativement important (près d'une centaine) de manuscrits aujourd'hui recensés qui contiennent des extraits, des parties ou l'ensemble du *De architectura*", VITRUVE, *De l'architecture*, Livre I, ed. by Ph. Fleury, Paris, Les Belles Lettres, 1990, p. LIII. The *editio princeps* goes back to the fifteenth century, *L. Vitruuii Polionis ad Cesarem Augustum de architectura libri decem*, Rome, Johannes Sulpicius, 1487.

[39] "He must be well-read, expert in drawing, learned in geometry [and not ignorant in optics]", VITRUVIUS, *De l'architecture, Livre I, op. cit.*, p. 5 (I, 3). The portion enclosed in brackets is provided by a few manuscripts.

[40] VITRUVIUS, *De l'architecture, Livre VII, op. cit.*, p. 5 (praef. 12).

[41] VITRUVIUS, *De l'architecture, Livre I, op. cit.*, p. 5-6 (I, 4). This passage makes arithmetic thoroughly useless for perspective purposes, if we take care to translate *symmetria* by "modularity" or "common scale of measures".

[42] H. R. HAHNLOSER, *Villard de Honnecourt, Kritische Gesamtausgabe des Bauhüttenbuches*, Wien, Akademische Druck- und Verlagsanstalt, 1935, taf. 36 (fol. 18v).

[43] J.A. WEISHEIPL, *Classification of the sciences in medieval thought*, «Mediaeval Studies», 27, 1965, pp. 54-90; ID., *The nature, scope, and classification of the sciences*, in D.C. LINDBERG, ed., *Science in the Middle Ages*, Chicago, University of Chicago Press, 1978, pp. 461-482; G. FEDERICI VESCOVINI, *L'inserimento della 'perspectiva' tra le arti del quadrivio*, Actes du IVe Congrès international de Philosophie médiévale, Montréal, 1967, Paris, Librairie J. Vrin, 1969, pp. 969-974.

[44] H. HUGONNARD ROCHE, *La classification des sciences de Gundissalinus et l'influence d'Avicenne*, in J. JOLIVET et R. RASHED, eds., *Études sur Avicenne*, Paris, Les Belles Lettres, 1984, pp. 41-63; J. JOLIVET, *Classification des sciences*, in R. RASHED, ed., *Histoire des sciences arabes*, 3, Paris, Le Seuil, 1997, pp. 255-270.

rial perspective is tied to *optica* only. The relations between sciences are described by the Aristotelian concept of subalternation.[45] There is subalternation when a superior science *(scientia subalternans)* provides the *propter quid* of a fact presented by an inferior science *(scientia subalternata)*. Ever since Aristotle's *Posterior Analytics*, optics has been subordinated to geometry. This has given rise either to its outright absorption by geometry, as in Boethius' *De Trinitate*, or to its insertion in the geometrical sciences, as in Dominicus de Clivassio's *Quaestiones perspectivae*.[46] Many classifications made a marked distinction between theoretical and practical sciences, as in the *Etymologiae* by Isidorus of Sevilla or the *Didascalion* by Hugh of St Victor. In the Arabic tradition, on the contrary, scholars instead imagined a continuous gradation from the speculative to the practical sciences.[47] Along the lines of Fârâbî, Dominicus Gundissalinus names seven mathematical sciences, having both theoretical and practical aspects, including optics *(de aspectibus)*, statics *(de ponderibus)*, and engineering *(de ingeniis)*. Relying on the same tradition, Roger Bacon devoted an entire chapter of his *Communia mathematica* to *Geometria speculativa et practica*,[48] and fra' Luca Pacioli expounded the "parte principale de tutta l'opera de Geometria, in tutti li modi theorica e pratica."[49] Such connections explain why early perspective depended mainly from geometrical sciences and why, though a practical art, it was able to benefit from the contribution of speculative geometry and *perspectiva naturalis*.

Last, but not least, it must be mentionedthat mathematical sciences were dichotomized according to their subject: arithmetic as a science of discrete quantities (πλῆθος), geometry as a science of continuous quantities (μέγεθος). This dichotomy was in place from the time of the works by Aristotle, Proclus, and Geminus,[50] right up to the Italian Renaissance treatises that still

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[45] Aristotle rejects the genres mixing in the course of a demonstration but admits, under certain conditions, the subordination of sciences, ARISTOTLE, *Posterior Analytics*, ed. by H. Tredennick, Cambridge, Harvard University Press, 1966, I, IX, 66-69 et I, XIII, 88-90. Aristotle recognizes that optics is subordinated to geometry, I, XIII, 88-90. Metaphysical considerations have sometimes contributed to emancipate optics. On the *subalternatio scientiae* in the Middle Ages, see S.J. LIVESEY, *Science and Theology in the Fourteenth Century: the Subalternate Sciences in Oxford Commentaries on the Sentences*, «Synthese», 83, 1990, pp. 273-292.

[46] "Est sciendum quod quinque sunt scientiae mathematicae, scilicet arismetica, geometria, musica, astrologia et perspectiva quae differunt secundum quod visum in prima conclusione; secundo sciendum quod ordo istius ad alias mathematicas est iste: quoniam arismetica et geometria ordine doctrinae ista praecedunt, sed astrologia et musica ipsa secuntur", Dominicus de Clivassio, *Quaestiones perspectivae*, Firenze, BNCF, San Marco, Conv. Soppr. J X 19, quaest. 1, ff. 44r-44v, G. FEDERICI VESCOVINI, *Studi sulla prospettiva medievale*, Torino, Giappichelli, 1964, p. 210.

[47] The insertion of practical sciences in the classification of sciences seems to be of a Greek origin. Pappus reports that Heron's disciples divided mechanics into theoretical parts: geometry, arithmetics, astronomy, physics, and manual parts: architecture (ὀικοδομική), ironworks (χαλκευτική), carpentry (τεκτονική), and painting (ζωγραφική), PAPPI ALEXANDRINI *Collectionis quae supersunt*, ed. F. Hultsch, Berlin, Weidman, 1876-1878, pp. 1022.3-1028.3 (VIII, praef. 1-3).

[48] R. BACON, *Communia mathematica Fratris Rogeri*, ed. by R. Steele, Oxford, Clarendon, 1940 (I, 3, 2).

[49] L. PACIOLI, *Summa de aritmetica, geometria, proportione et proportionalita*, Venice, Paganino de Paganini, 1494, fol. 75r.

[50] B. VITRAC in EUCLIDE, *Éléments, op. cit.*, 2, pp. 19, 22.

identified devices *per numero* and *per linea*.[51] Having recourse to these categories, we understand better why the drawing of the frescoes in the Upper Church at Assisi presents such poor affinities with arithmetical method. Though further research is needed on medieval paintings, it appears that continuous geometrical insight could have served as a guide to the earliest perspective experiments.

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[51] L. PACIOLI, *Divina proportione*, Venice, Paganino de Paganini, 1509; P. CATANEO, *L'Architettura*, Venice, Aldo, 1567; A. PALLADIO, *I Quattro Libri de architettura*, Venice, Domenico dei Franceschi, 1570. About the devices *per numero* and *per linea*, see S. GESSNER, *Les Mathématiques dans les écrits d'architecture italiens, 1545-1570*, thèse, Paris, Université de Paris VII, 2006, pp. 109-144.