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Uniquely List Colorability of Complete Split Graphs

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Abstract
The join of null graph O_m and complete graph K_n, denoted by $S(m,n)$, is called a complete split graph. In this paper, we characterize unique list colorability of the graph $G = S(m,n)$. We shall prove that G is uniquely 3-list colorable graph if and only if $m \geq 4$, $n \geq 4$ and $m + n \geq 10$, $m(G) \leq 4$ for every $1 \leq m \leq 5$ and $n \geq 6$.

Keywords . Chromatic number, list- chromatic number, uniquely list colorable graph, complete split graph.

1. Introduction. All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$, $E(G)$ (or V, E in short) and $\overline{G}$ will denote its vertex-set, its edge-set and its complementary graph, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$, the degree of v (resp., the degree of v with respect to W), denoted by deg(v) (resp., deg$_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. The null graphs and complete graphs of order n are denoted by O_n and K_n, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* defined is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.

A graph $G = (V, E)$ is called a *split graph* if there exists a partition $V = I \cup K$ such that $G[I]$ and $G[K]$ are null and complete graphs, respectively. We will denote such a graph by $S(I \cup K, E)$. The join of null graph O_m and complete graph K_n, $O_m + K_n = S(m,n)$, is called a *complete split graph*.The notion of split graphs was introduced in 1977 by Földes and Hammer [7]. A role that split graphs play in graph theory is clarified in [7] and in [3], [4], [15], [17], [20], [21], [22]. These graphs have been paid attention also because they have connection with packing and knapsack problems [5], with the matroid theory [8], with Boolean functions [18], with the analysis of parallel processes in computer programming [11] and with the task allocation in distributed systems [12]. Many generalizations of split graphs have been made. The newest one is the notion of bisplit graphs introduced by Brandstädt et al. [2].

Let $G = (V, E)$ be a graph and λ is a positive integer.

A λ-*coloring* of G is a bijection $f : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer λ such that G has a λ-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. We say that a graph G is n-*chromatic* if $n = \chi(G)$.

Let $(L_v)_{v \in V}$ be a family of sets. We call a coloring f of G with $f(v) \in L_v$ for all $v \in V$ is a *list coloring from the lists* L_v. We will refer to such a coloring as an L-coloring. The graph G is called λ-*list-colorable*, or λ-*choosable*, if for every family $(L_v)_{v \in V}$ with $|L_v| = \lambda$ for all v, there is a coloring of G from the lists L_v. The smallest positive integer λ such that G has a λ-choosable is called the *list-chromatic number*, or *choice number* of G and is denoted by $ch(G)$.

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Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors L_v, such that there exists a unique L-coloring for G, then G is called a *uniquely k-list colorable graph* or a UkLC graph for short. The idea of uniquely colorable graph was introduced independently by Dinitz and Martin [6] and by Mahmoodian and Mahdian [16] (Mahmoodian and Mahdian have obtained some results on the uniquely k-list colorable complete multipartite graphs).

The list coloring model can be used in the channel assignment. The fixed channel allocation scheme leads to low channel utilization across the whole channel. It requires a more effective channel assignment and management policy, which allows unused parts of channel to become available temporarily for other usages so that the scarcity of the channel can be largely mitigated [24]. It is a discrete optimization problem. A model for channel availability observed by the secondary users is introduced in [24].

There have been many interesting and insightful research results on these issues for different graph classes (see [9], [13], [14], [16]). However, these are still issues that have not been resolved thoroughly, so much more attention is needed. In this paper, we shall characterize unique list colorability of the graph $G = S(m, n)$. Namely, we shall prove that G is uniquely 3-list colorable graph if and only if $m \geq 4$, $n \geq 4$ and $m + n \geq 10$, $m(G) \leq 4$ for every $1 \leq m \leq 5$ and $n \geq 6$.

2. Preliminaries. If a graph G is not uniquely k-list colorable, we also say that G has property $M(k)$. So G has the property $M(k)$ if and only if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist at least two list colorings. The least integer k such that G has the property $M(k)$ is called the *m-number* of G, denoted by $m(G)$. This conception was originally introduced by Mahmoodian and Mahdian in [16].

Lemma 2.1 ([16]). *Each UkLC graph is also a $U(k-1)LC$ graph.*

Lemma 2.2 ([16]). *The graph G is UkLC if and only if $k < m(G)$.*

Lemma 2.3 ([16]). *A connected graph G has the property $M(2)$ if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.*

Lemma 2.4 ([16]). *For every graph G we have $m(G) \leq |E(\overline{G})| + 2$.*

Lemma 2.5 ([16]). *Every UkLC graph has at least $3k - 2$ vertices.*

For example, one can easily see that the graph $S(2, 2)$ has the property M(3) and it is U2LC, so $m(S(2, 2)) = 3$.

Proposition 2.1. *Let $G = S(m, n)$ be a UkLC graph with $k \geq 2$. Then*

(i) $m \geq 2$;

(ii) $k < \frac{m^2-m+4}{2}$;

(iii) $k \leq \left\lfloor \frac{m+n+2}{3} \right\rfloor$. *Proof:* (i) If $m = 1$ then G is a complete graph K_{n+1}. Lemma 2.3, G has the property $M(2)$, a contradiction.

(ii) It is not difficult to see that $|E(\overline{G})| = \frac{m(m-1)}{2}$. By Lemma 2.4, we have

$$m(G) \leq |E(\overline{G})| + 2 = \frac{m^2 - m + 4}{2}.$$

By Lemma 2.2, we have $k < \frac{m^2-m+4}{2}$.

(iii) Assertion (iii) follows immediately from Lemma 2.5.

Let $G = S(m, n)$ be a UkLC graph with $V(G) = I \cup K, G[I] = O_m, G[K] = K_n, m \geq 2, n \geq 1, k \geq 3$. Set

$$I = \{u_1, u_2, \ldots, u_m\}, K = \{v_1, v_2, \ldots, v_n\}.$$

Suppose that, for the given k-list assignment L:

$L_{u_i} = \{a_{i,1}, a_{i,2}, \ldots, a_{i,k}\}$ for every $i = 1, \ldots, m$,

$L_{v_i} = \{b_{i,1}, b_{i,2}, \ldots, b_{i,k}\}$ for every $i = 1, \ldots, n$,

there is a unique k-list color f:

$f(u_i) = a_{i,1}$ for every $i = 1, \ldots, m$,

$f(v_i) = b_{i,1}$ for every $i = 1, \ldots, n$.

Proposition 2.2.

(i) $|f(I)| \geq 2$;

(ii) $|f(I)| \leq m - 2$, where $m \geq 4$.

Proof: (i) For suppose on the contrary that $|f(I)| = 1$, then $a_{1,1} = a_{2,1} = \ldots = a_{m,1} = a$. Set $H = G - I$, it is not difficult to see that H is a complete graph K_n. We assign the following lists L'_v for the vertices v of H:

If $a \in L_v$ then $L'_v = L_v \setminus \{a\}$,

If $a \notin L_v$ then $L'_v = L_v \setminus \{b\}$, where $b \in L_v$ and $b \neq f(v)$.

It is clear that $|L'_v| = k - 1 \geq 2$ for every $v \in V(H)$. By Lemma 2.3, H has the property $M(2)$. So by Lemma 2.1, H has the property $M(k-1)$. It follows that with lists L'_v, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists L_v, there exist at least two list colorings for the vertices v of G, a contradiction.

(ii) For suppose on the contrary that $|f(I)| \geq m-1$. We consider separately two cases.

Case 1: $|f(I)| = m-1$.

Without loss of generality, we may assume that $a_{1,1} = a_{2,1}$ and $a_{i,1} \neq a_{j,1}$ for every $i, j \in \{2, \ldots, m\}, i \neq j$. Set graph $G' = (V', E')$, with

$$V' = I \cup K, E' = (E(G) \cup \{u_i u_j | i, j = 1, 2, \ldots, m; i \neq j\}) \setminus \{u_1 u_2\}.$$

It is clear that G' is complete split graph $S(2, m+n-2)$ with $V(G') = I' \cup K'$, where

$$I' = \{u_1, u_2\}, K' = \{u_3, u_4, \ldots, u_m, v_1, v_2, \ldots, v_n\}.$$

Since $a_{1,1} = a_{2,1}$, it is not difficult we have got a contradiction.

Case 2: $|f(I)| = m$.

In this case, $a_{i,1} \neq a_{j,1}$ for every $i, j \in \{1, 2, \ldots, m\}, i \neq j$. Set graph $G'' = (V'', E'')$, with

$$V'' = I \cup K, E'' = E(G) \cup \{u_i u_j | i, j = 1, 2, \ldots, m; i \neq j\}.$$

It is clear that G'' is a complete graph K_{m+n}. By Lemma 2.3, G'' has the property $M(2)$, so with lists L_v, there exist at least two list colorings for the vertices v of G''. Since $V(G) = V(G'')$, it is not difficult to see that with lists L_v, there exist at least two list colorings for the vertices v of G, a contradiction.

3. Main Results. We need the following Lemmas 3.1–3.9 to prove our results.

Lemma 3.1. *(i)* $m(S(1,n)) = 2$ *for every* $n \geq 1$*;*

(ii) $m(S(r,1)) = 2$ *for every* $r \geq 1$*;*

(iii) $m(S(2,n)) = 3$ *for every* $n \geq 2$*.* *Proof:* (i) It is clear that $S(1,n)$ is a complete graph for every $n \geq 1$, by Lemma 2.3, $m(S(1,n)) = 2$ for every $n \geq 1$.

(ii) It is clear that $S(r,1)$ is a complete bipartite graph for every $r \geq 1$, by Lemma 2.3, $m(S(r,1)) = 2$ for every $r \geq 1$.

(iii) By Lemma 2.3, $G = S(2,n)$ is U2LC for every $n \geq 2$.

It is not difficult to see that $|E(\overline{G})| = 1$. By Lemma 2.4, $m(S(2,n)) \leq 3$ for every $n \geq 2$.

Thus, $m(S(2,n)) = 3$ for every $n \geq 2$.

Lemma 3.2 ([9]). $m(S(3,n)) = 3$ *for every* $n \geq 2$*;*

Lemma 3.3 ([9]). *For every* $r \geq 2$*,* $m(S(r,3)) = 3$.

Lemma 3.4 ([10]). *Graphs* $S(5,4)$ *and* $S(4,4)$ *have property* $M(3)$*.*

Lemma 3.5 ([19]). *The graph* $S(4,5)$ *has property* $M(3)$*.*

Lemma 3.6. $G = S(4,n)$ *has the property* $M(4)$ *for every* $n \geq 2$*;* *Proof:* Let $G = S(4,n)$ is a complete split graph with $V(G) = I \cup K, G[I] = O_4, G[K] = K_n, n \geq 2$. Set

$$I = \{u_1, u_2, u_3, u_4\}, K = \{v_1, v_2, \ldots, v_n\}.$$

For suppose on the contrary that graph $G = S(4,n)$ is U4LC. So there exists a list of 4 colors L_v for each vertex $v \in V(G)$, such that there exists a unique L-coloring f for G. By (i) and (ii) of Proposition 2.2, $|f(I)| = 2$.

Let $f(I) = \{a, b\}$. Set graph $H = G - I$, it is not difficult to see that H is a complete graph K_n. We assign the following lists L'_v for the vertices v of H:

(a) If $a, b \in L_v$ then $L'_v = L_v \setminus \{a, b\}$,

(b) If $a \in L_v, b \notin L_v$ then $L'_v = L_v \setminus \{a, c\}$, where $c \in L_v$ and $c \neq f(v)$,

(c) If $a \notin L_v, b \in L_v$ then $L'_v = L_v \setminus \{b, c\}$, where $c \in L_v$ and $c \neq f(v)$,

(d) If $a, b \notin L_v$ then $L'_v = L_v \setminus \{c, d\}$, where $c, d \in L_v, c \neq d$ and $c, d \neq f(v)$.

It is clear that $|L'_v| = 2$ for every $v \in V(H)$. By Lemma 2.3, H has the property $M(2)$. It follows that with lists L'_v, there exist at least two list colorings for the vertices v of H. So it is not difficult to see that with lists L_v, there exist at least two list colorings for the vertices v of G, a contradiction.

Lemma 3.7 ([25]). *(i) For every* $n \geq 2$*,* $S(5,n)$ *has the property* $M(4)$*;*

(ii) If $n \geq 5$ *then* $m(S(5,n)) = 4$*.*

Lemma 3.8 ([23]). *For every* $m \geq 1, k \geq 2$*,* $S(m, 2k-3)$ *has the property* $M(k)$*.*

Lemma 3.9 ([23]). *For every* $n \geq 1, k \geq 2$*,* $S(2k-3, n)$ *has the property* $M(k)$*.*

Now we prove our results.

Theorem 3.1. *The graph* $G = S(m,n)$ *is uniquely* 3*-list colorable graph if and only if* $m \geq 4$*,* $n \geq 4$ *and* $m+n \geq 10$*.*

Proof: Firrst we prove the necessity. Suppose that $G = S(m,n)$ is U3LC. If $m < 4$ or $n < 4$ then by Lemma 3.8 and Lemma 3.9, it is not difficult to see that G has the property $M(3)$, a contradiction.

Therefore, $m \geq 4$ and $n \geq 4$. It follows that $m + n \geq 8$. If $m + n = 8$ then $m = 4$ and $n = 4$, by Lemma 3.4, G has property $M(3)$, a contradiction. If $m + n = 9$ then $(m, n) \in \{(4, 5), (5, 4)\}$, by Lemma 3.4 and Lemma 3.5, G has property $M(3)$, a contradiction. Thus, $m + n \geq 10$.

Now we prove the sufficiency. Suppose that $m \geq 4$, $n \geq 4$ and $m+n \geq 10$. Let $V(G) = I \cup K, G[I] = O_m, G[K] = K_n, I = \{u_1, u_2, \ldots, u_m\}, K = \{v_1, v_2, \ldots, v_n\}$. We prove G is U3LC by induction on $m + n$. If $m + n = 10$, then we consider separately three cases.

(i) $m = 4$ *and* $n = 6$.

We assign the following lists for the vertices of G:

$L_{u_1} = \{1, 3, 4\}, L_{u_2} = \{1, 7, 8\}, L_{u_3} = \{2, 5, 6\}, L_{u_4} = \{2, 7, 8\}$;

$L_{v_1} = \{1, 2, 3\}, L_{v_2} = \{1, 2, 4\}, L_{v_3} = \{1, 2, 5\}, L_{v_4} = \{1, 2, 6\}, L_{v_5} = \{1, 2, 7\}, L_{v_6} = \{1, 2, 8\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_1) = 1, f(u_2) = 1, f(u_3) = 2, f(u_4) = 2$;

$f(v_1) = 3, f(v_2) = 4, f(v_3) = 5, f(v_4) = 6, f(v_5) = 7, f(v_6) = 8$.

(ii) $m = 5$ *and* $n = 5$.

We assign the following lists for the vertices of G:

$L_{u_1} = \{1, 4, 5\}, L_{u_2} = \{1, 3, 6\}, L_{u_3} = \{2, 3, 7\}, L_{u_4} = \{2, 4, 5\}, L_{u_5} = \{2, 6, 7\}$;

$L_{v_1} = \{1, 2, 3\}, L_{v_2} = \{1, 2, 4\}, L_{v_3} = \{1, 2, 5\}, L_{v_4} = \{1, 2, 6\}, L_{v_5} = \{1, 2, 7\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_1) = 1, f(u_2) = 1, f(u_3) = 2, f(u_4) = 2, f(u_5) = 2$;

$f(v_1) = 3, f(v_2) = 4, f(v_3) = 5, f(v_4) = 6, f(v_5) = 7$.

(iii) $m = 6$ *and* $n = 4$.

We assign the following lists for the vertices of G:

$L_{u_1} = \{1, 3, 5\}, L_{u_2} = \{1, 4, 5\}, L_{u_3} = \{2, 3, 6\}, L_{u_4} = \{2, 3, 4\}, L_{u_5} = \{2, 4, 6\}, L_{u_6} = \{2, 5, 6\}$;

$L_{v_1} = \{1, 2, 3\}, L_{v_2} = \{1, 2, 4\}, L_{v_3} = \{1, 2, 5\}, L_{v_4} = \{1, 2, 6\}$.

A unique coloring f of G exists from the assigned lists:

$f(u_1) = 1, f(u_2) = 1, f(u_3) = 1, f(u_4) = 2, f(u_5) = 2, f(u_6) = 2$;

$f(v_1) = 3, f(v_2) = 4, f(v_3) = 5, f(v_4) = 6$.

Now let $m + n > 10$ and assume the assertion for smaller values of $m + n$. We consider separately two cases.

Case 1: $m \geq 5$.

Set $G' = G - u_m = S(m - 1, n)$. By the induction hypothesis, for each vertex v in G', there exists a list of 3 colors L'_v, such that there exists a unique f' for G'. We assign the following lists for the vertices of G:

$L_{u_m} = L'_{u_{m-1}}, L_v = L'_v$ if $v \in V(G')$.

A unique coloring f of G exists from the assigned lists:

$f(u_m) = f'(u_{m-1}), f(v) = f'(v)$ if $v \in V(G')$.

Case 2: $n \geq 5$.

Set $G' = G - v_n = S(m, n - 1)$. By the induction hypothesis, for each vertex v in G', there exists a list of 3 colors L'_v, such that there exists a unique f' for G'. We assign the following lists for the vertices of G:

$L_{v_n} = \{f'(v_{n-1}), f'(v_{n-2}), t\}$ with $t \notin f'(G'), L_v = L'_v$ if $v \in V(G')$.

A unique coloring f of G exists from the assigned lists:

$f(v_n) = t, f(v) = f'(v)$ if $v \in V(G')$.

Corollary 3.1. $m(S(4, n)) = 4$ *for every* $n \geq 6$. *Proof:* It follows from Theorem 3.1 and Lemma 3.6.

Theorem 3.2. $m(S(r, n)) \leq 4$ *for every* $1 \leq r \leq 5$ *and* $n \geq 6$. *Proof:* It follows from Lemma 3.1 to Lemma 3.7.

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