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Non-Manipulable Agenda Setting for Voting with Single-Peaked Preferences*

Yuta Nakamura †

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Abstract

Even though there are many potential alternatives, in reality, not all of them are given to voters. This is because voters' set of choices, the *agenda*, is created by agenda setters. However, agenda setters may have a vested interest to set an agenda such that an outcome they prefer is chosen by voters. The purpose of this study is to find conditions regarding agenda setters' preferences under which voting rules cannot be manipulated by agenda setters. Assuming that voters' preferences are single-peaked, we show that, whenever monetary rewards to agenda setters are not allowed, any unanimous voting rule is manipulable via agenda setting. We also show that in a mechanism that gives monetary rewards to agenda setters, the median rule is non-manipulable via agenda setting if and only if no agenda setter has a special interest in alternatives. A policy implication of these results is that agenda setters should be totally irrelevant to political issues.

Keywords: Agenda setting; Single-peaked preferences; Median rule; Condorcet winner; Mechanism design.

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†School of Economics and Business Administration, Yokohama City University, Yokohama 236-0027, Japan; y_naka@yokohama-cu.ac.jp

1 Introduction

Even though there are many potential alternatives, in actuality, not all of them are given to voters. This is because voters' set of choices, the *agenda*, is created by agenda setters, e.g., political parties that propose some policies in referendums or committee members who decide which films to nominate. However, since any alternative outside of the agenda cannot be chosen by voters, agenda setters may have an incentive to set an agenda such that an outcome they prefer is chosen.[1] This paper finds conditions regarding agenda setters' preferences under which such agenda manipulation does not occur.

Since the seminal works by Black (1948a, b) and Moulin (1980), it has been well-known that when voters' preferences are single-peaked, the median rule is quite an appealing voting rule that satisfies *efficiency*, *anonymity*, and *strategy-proofness*. Moreover, whenever the entire set of alternatives is provided to the voters, the median rule selects the Condorcet winner, an alternative that beats any other alternatives in pairwise-majority comparisons. However, in many real-life political situations, such a rich set is not given because the agenda is set by agenda setters. In this case, the median rule outcome may differ from the Condorcet winner. For example, let us suppose that there are two agenda setters and five voters whose preferences are as illustrated in Figure 1. Each agenda setter and voter has a single-peaked preference regarding four alternatives: x_1, x_2, x_3, and x_4, with $x_1 < x_2 < x_3 < x_4$. For instance, Voter 1 prefers x_3 to x_2, x_2 to x_1, and so on. Here, x_3 is unanimously supported by voters, and hence, it is the Condorcet winner. However, let us suppose that agenda setter a_1 proposes x_1 and agenda setter a_2 proposes x_2 and that the agenda is composed of these proposed alternatives, $\{x_1, x_2\}$. Then, since all voters prefer x_2 to x_1, the median rule, in this case, selects

[1] For example, in the 2015 Osaka Metropolitan Plan referendum in Osaka city, only the special wards (*tokubetsu-ku*) plan favored by the Osaka Restoration Association was put to a vote, even though the LDP and Komeito had proposed another plan.

Agenda setter	Voter
$a_1 : x_1 x_2 x_3 x_4$	$1 : x_3 x_2 x_1 x_4$
$a_2 : x_2 x_1 x_3 x_4$	$2 : x_3 x_4 x_2 x_1$
	$3 : x_3 x_2 x_1 x_4$
	$4 : x_3 x_2 x_4 x_1$
	$5 : x_3 x_2 x_1 x_4$

Figure 1: Single-Peaked Preferences of Agenda Setters and Voters

x_2 from this agenda. However, the outcome x_2 is not the Condorcet winner x_3. Moreover, this pair of proposed alternatives (x_1, x_2) constitutes a dominant strategy equilibrium of a game played by agenda setters, i.e., neither of them can benefit by proposing another alternative.[2]

We say that a voting rule is *manipulable via agenda setting* if an alternative that is unanimously supported by voters is not a unique dominant strategy outcome in the game played by agenda setters. Theorem 1 shows that whenever monetary rewards to agenda setters are not allowed, any unanimous voting rule is manipulable via agenda setting. In particular, this result implies that the median rule is manipulable via agenda setting. To remove agenda setters' incentives that may affect the outcome, we next introduce a mechanism that provides monetary rewards to agenda setters. In this mechanism, each agenda setter receives a monetary reward r/ℓ if their proposed alternative is selected by voters and the number of agenda setters who propose the same selected alternative is ℓ. Conversely, the agenda setter receives no monetary reward if their proposed alternative is

[2] To see that x_1 is agenda setter a_1's dominant strategy, for example, consider a case in which agenda setter a_1 proposes x_3 and agenda setter a_2 proposes x_2. Then, the median rule selects x_3 from an agenda $\{x_2, x_3\}$; however, this is unprofitable to agenda setter a_1.

not selected by voters. Theorem 2 shows that in this mechanism, the median rule is non-manipulable via agenda setting if and only if no agenda setter has a special interest in alternatives, regardless of the size of r.

This paper is related to the seminal work by Downs (1957), which shows a policy convergence in a two-party competition.[3] Although Downs's purpose is to analyze political parties' behavior in a two-party system, our purpose is a bit different. We are interested in finding conditions regarding agenda setters' preferences under which agenda manipulation does not occur. The importance of considering the impacts of agenda manipulation has been widely recognized since the time of McKelvey (1976, 1979).[4] In a multidimensional spatial model, he shows that if only one agenda setter has complete control over agendas and no Condorcet winner exists, then this agenda setter can always construct a sequence of binary agendas from which his ideal alternative is finally selected by voters. Dutta, Jackson, and Le Breton (2001, 2002) analyze candidates' incentives to enter or exit an election such that an alternative they prefer is chosen by voters. The authors of that study show that the outcome of any unanimous voting rule is affected by the incentives of candidates to enter or exit the election. In contrast to their studies, we analyze agenda setters' incentives to propose an alternative such that an alternative that they prefer is chosen by voters.

The rest of this paper is organized as follows: section 2 provides our model, section 3 presents our main results, and section 4 makes concluding comments. All proofs are relegated to the Appendix.

2 The Model

Let $[0,1]$ be the set of all potential *alternatives*, $A = \{a_1, a_2, \ldots, a_m\}$ the finite set of *agenda setters* and $I = \{1, 2, \ldots, n\}$ the finite set of *voters*.

[3]As is well-known, Downs's (1957) discussion dates back to Hotelling (1927).

[4]For a survey of the agenda manipulation literature, we refer to Ordeshook (1986).

We assume that $|A| = m \geq 2$ and that n is odd.

A *single-peaked preference* is a complete, transitive, and anti-symmetric binary relation[5] $\succsim_i$ on $[0,1]$ such that there exists a best alternative $b(\succsim_i) \in [0,1]$ for which

$$[x < y < b(\succsim_i) \text{ or } b(\succsim_i) < y < x] \Longrightarrow b(\succsim_i) \succ_i y \succ_i x \quad \forall x, y \in [0,1].$$

Let $\mathscr{S}$ be the set of single-peaked preferences. A *preference profile* of n voters is as follows:

$$\succsim = (\succsim_1, \succsim_2, \ldots, \succsim_n) \in \mathscr{S}^n.$$

Each agenda setter $a_k \in A$ proposes an alternative $s_k \in [0,1]$, and each profile of proposed alternatives $s \equiv (s_1, s_2, \ldots, s_m) \in [0,1]^m$ generates an agenda $\{s_1, s_2, \ldots, s_m\} \subset [0,1]$. For each $s \in [0,1]^m$ and each $a_k \in A$, let

$$s_{-k} = (s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_m) \in [0,1]^{m-1},$$

and for each $s'_k \in [0,1]$, let

$$(s'_k, s_{-k}) = (s_1, \ldots, s_{k-1}, s'_k, s_{k+1}, \ldots, s_m).$$

For each $\succsim \in \mathscr{S}^n$ and each $s \in [0,1]^m$, we denote the restriction of $\succsim$ to $\{s_1, s_2, \ldots, s_m\}$ by $\succsim|_{\{s_1, s_2, \ldots, s_m\}}$.

Definition 1. A *voting rule* is a function $f : \mathscr{S}^n \times [0,1]^m \to [0,1]$ such that

(i) for each $\succsim \in \mathscr{S}^n$ and each $s \in [0,1]^m$, $f(\succsim, s) \in \{s_1, s_2, \ldots, s_m\}$,

(ii) for each $\succsim \in \mathscr{S}^n$ and each $s, s' \in [0,1]^m$, if $\{s_1, s_2, \ldots, s_m\} = \{s'_1, s'_2, \ldots, s'_m\}$, then $f(\succsim, s) = f(\succsim, s')$.

[5] Completeness: for each $x, y \in [0,1]$, either $x \succsim_i y$ or $y \succsim_i x$, Transitivity: for each $x, y, z \in [0,1]$, $x \succsim_i y$ and $y \succsim_i z$ together imply $x \succsim_i z$, Anti-symmetry: for each $x, y \in [0,1]$, $x \succsim_i y$ and $y \succsim_i x$ imply $x = y$. In addition, for each $x, y \in [0,1]$, we write $x \succ_i y$ if and only if $x \succsim_i y$ and $y \not\succsim_i x$.

(iii) for each $\succsim, \succsim' \in \mathscr{S}^n$ and each $s \in [0,1]^m$,

$$\big[\forall i \in I,\ \succsim_i |_{\{s_1, s_2, \ldots, s_m\}} = \succsim'_i |_{\{s_1, s_2, \ldots, s_m\}}\big] \Longrightarrow f(\succsim, s) = f(\succsim', s).$$

Condition (i) requires that a voting rule select an alternative from an agenda. Condition (ii) requires that the voting rule be independent of which ever agenda setter proposes an alternative. Condition (iii) requires that the voting rule be independent of voters' preferences on unproposed alternatives. Let $\mathscr{F}$ be the set of voting rules.

For each $\succsim_i \in \mathscr{S}$ and each $s \in [0,1]^m$, let $b(\succsim_i, s) \in \{s_1, \ldots, s_m\}$ be i's best alternative in $\{s_1, \ldots, s_m\}$, that is,

$$b(\succsim_i, s) \succsim_i x \quad \forall x \in \{s_1, \ldots, s_m\}.$$

Definition 2. A voting rule $f \in \mathscr{F}$ is *unanimous* if for each $\succsim \in \mathscr{S}^n$, each $s \in [0,1]^m$ and each $x \in \{s_1, \ldots, s_m\}$,

$$[\exists x \in \{s_1, \ldots, s_m\}, \forall i \in I,\ b(\succsim_i, s) = x] \Longrightarrow f(\succsim, s) = x.$$

The median rule is a voting rule that selects the median alternative in any given agenda.

Definition 3. The *median rule* is a voting rule $f^m \in \mathscr{F}$ such that for each $(\succsim, s) \in \mathscr{S}^n \times [0,1]^m$,

$$|\{i \in I : f^m(\succsim, s) \geq b(\succsim_i, s)\}| \geq \frac{n}{2} \text{ and } |\{i \in I : f^m(\succsim, s) \leq b(\succsim_i, s)\}| \geq \frac{n}{2}$$

We now discuss the incentives of agenda setters. Each agenda setter $a_k \in A$ is concerned with a voting outcome and the monetary reward he receives. We assume that each agenda setter $a_k \in A$ has a quasi-linear utility function $u_k : [0,1] \times \mathbb{R} \to \mathbb{R}$, i.e., there exists a function $v_k : [0,1] \to \mathbb{R}$ such that for each $x \in [0,1]$ and each $r_k \in \mathbb{R}$, $u_k(x, r_k) = v_k(x) + r_k$. We also assume that for each $a_k \in A$, v_k is weakly single-peaked, that is, there exists $x \in [0,1]$ such that

$$[z \leq y \leq x \text{ or } x \leq y \leq z] \Longrightarrow v_k(x) \geq v_k(y) \geq v_k(z) \quad \forall y, z \in [0,1].$$

We next introduce a mechanism that provides monetary rewards to agenda setters. For each voting outcome $x \in [0,1]$ and each $s \in [0,1]^m$, each agenda setter $a_k \in A$ receives a monetary reward $r_k(x,s) \in \mathbb{R}_+$ such that

$$r_k(x,s) = \begin{cases} \dfrac{r}{|\{a_{k'} \in A : s_{k'} = x\}|} & \text{if } s_k = x, \\ \\ 0 & \text{if } s_k \neq x, \end{cases}$$

where $r \in \mathbb{R}_+$ is the total monetary reward, given by the society. On the one hand, we can interpret r as the total cost of avoiding agenda manipulation. On the other hand, we can alternatively interpret r as the factor of determining the agenda setters' payoffs of winning measured in money, i.e., $r_k(x,s)$. In the next section, we first consider the case in which no monetary rewards are provided to agenda setters ($r = 0$), and then we consider the case in which some monetary rewards are provided to agenda setters ($r > 0$).

For each $(f, \succsim) \in \mathscr{F} \times \mathscr{S}^n$, and each $a_k \in A$, $s_k \in [0,1]$ is agenda setter a_k's *dominant strategy* at $(f, \succsim)$ if for each $s_{-k} \in [0,1]^{m-1}$ and each $s'_k \in [0,1]$,

$$\begin{aligned} &u_k\big(f(\succsim, s_k, s_{-k}), r_k\big(f(\succsim, s_k, s_{-k}), s_k, s_{-k}\big)\big) \\ &\quad \geq u_k\big(f(\succsim, s'_k, s_{-k}), r_k\big(f(\succsim, s'_k, s_{-k}), s'_k, s_{-k}\big)\big). \end{aligned}$$

Let $\mathrm{DS}_k(f, \succsim)$ be the set of agenda setter a_k's dominant strategies at $(f, \succsim) \in \mathscr{F} \times \mathscr{S}^n$. Then, the set of dominant strategy equilibria at $(f, \succsim) \in \mathscr{F} \times \mathscr{S}^n$ is given by $\mathrm{DE}(f, \succsim) \equiv \prod_{k=1}^m \mathrm{DS}_k(f, \succsim)$.

3 Main Results

In this section, we search for conditions regarding agenda setters' preferences under which a voting rule is non-manipulable via agenda setting.

For each $f \in \mathscr{F}$ and each $\succsim \in \mathscr{S}^n$, let

$$f\big(\succsim, \mathrm{DE}(f, \succsim)\big) = \big\{x \in [0,1] : x = f(\succsim, s) \text{ for some } s \in \mathrm{DE}(f, \succsim)\big\}.$$

A voting rule is said to be manipulable via agenda setting if an alternative that is unanimously supported by voters is not a unique dominant strategy outcome.

Definition 4. A voting rule $f \in \mathscr{F}$ is *manipulable via agenda setting* if there exists $\succsim \in \mathscr{S}^n$ and $x \in [0,1]$ such that

$$[\ x \succsim_i y \quad \forall i \in I,\ \forall y \in [0,1]\ ] \quad \text{and} \quad \big[\ f\big(\succsim, \mathrm{DE}(f, \succsim)\big) \neq \{x\}\ \big]\,.$$

Our first main result states that whenever monetary rewards to agenda setters are not allowed, any unanimous voting rule is manipulable via agenda setting.

Theorem 1. *If $r = 0$, then any unanimous voting rule is manipulable via agenda setting.*

Proof. See the Appendix. □

Corollary 1. *If $r = 0$, then the median rule is manipulable via agenda setting.*

Proof. Immediately follows from Theorem 1. □

We next consider a mechanism that provides positive monetary rewards to agenda setters. For each $\succsim \in \mathscr{S}^n$, $x \in [0,1]$ is the *Condorcet winner* at $\succsim$ if

$$|\{i \in I : x \succ_i y\}| > |\{i \in I : y \succ_i x\}| \quad \forall y \in [0,1] \setminus \{x\}.$$

Since the seminal works by Black (1948a, b), it has been well-known that for each $\succsim \in \mathscr{S}^n$, the Condorcet winner exists at $\succsim$. For each $\succsim \in \mathscr{S}^n$, let $c(\succsim) \in [0,1]$ be the Condorcet winner at $\succsim$.

We say that a voting rule is strongly non-manipulable via agenda setting if the Condorcet winner is the unique dominant strategy outcome.

Definition 5. A voting rule $f \in \mathscr{F}$ is *strongly non-manipulable via agenda setting* if for any $\succsim \in \mathscr{S}^n$,

$$f\big(\succsim, \mathrm{DE}(f, \succsim)\big) = \{c(\succsim)\}.$$

Note that the notion of strong non-manipulability is even stronger than the negation of manipulability in Definition 4. Theorem 2 states that the median rule is strongly non-manipulable via agenda setting if and only if no agenda setter has a special interest in alternatives. In particular, since Theorem 2 only requires $r > 0$, the cost of preventing agenda manipulation can be arbitrarily small.

Theorem 2. *Whenever $r > 0$, the median rule f^m is strongly non-manipulable via agenda setting if and only if for any $a_k \in A$ and any $x, y \in [0, 1]$, $v_k(x) = v_k(y)$.*

Proof. See Appendix. □

The intuition of the proof for Theorem 2 is as follows: First, let us consider the "if" part. Suppose that no agenda setter has a special interest in alternatives. Consider any $\succsim \in \mathscr{S}^n$. Then, for any agenda setter $a_k \in A$, proposing the Condorcet winner $c(\succsim)$ is the unique dominant strategy because each agenda setter is only concerned with their monetary reward and the median rule selects the Condorcet winner whenever it is included in an agenda. Therefore, the Condorcet winner is the unique dominant strategy outcome, i.e., $f^m\big(\succsim, \mathrm{DE}(f^m, \succsim)\big) = \{c(\succsim)\}$.

Next, let us consider the "only if" part. Suppose that for any $\succsim \in \mathscr{S}^n$, $f^m\big(\succsim, \mathrm{DE}(f^m, \succsim)\big) = \{c(\succsim)\}$. Suppose, by contradiction, that there exists an agenda setter $a_k \in A$ and an alternative $x \in (0, 1)$ such that $v_k(x) > v_k(0)$. Consider a profile $\succsim \in \mathscr{S}^n$ such that 0 is the best alternative for any voter. Then, 0 is not agenda setter a_k's dominant strategy because if any agenda setter other than a_k proposes 1, then agenda setter a_k is strictly better off by proposing x rather than proposing 0. Similarly, we can also show that any $z \in [0,1] \setminus \{0\}$ is not agenda setter a_k's dominant strategy, i.e., agenda setter a_k has no dominant strategy. However, it contradicts the assumption $f^m\big(\succsim, \mathrm{DE}(f^m, \succsim)\big) = \{c(\succsim)\}$. Therefore, for any $a_k \in A$ and any $x \in (0,1)$, $v_k(x) \leq v_k(0)$. Similar arguments show that for any $a_k \in A$ and any $x \in (0,1)$, $v_k(x) \geq v_k(0)$, $v_k(x) \leq v_k(1)$ and $v_k(x) \geq v_k(1)$. This implies that for any $a_k \in A$ and any $x, y \in [0,1]$, $v_k(x) = v_k(y)$.

4 Conclusion

This paper found conditions regarding agenda setters' preferences under which a voting rule is (strongly) non-manipulable via agenda setting. First, we showed that whenever monetary rewards to agenda setters are not allowed, any unanimous voting rule is manipulable via agenda setting. Second, we showed that in a mechanism that provides monetary rewards to agenda setters, the median rule is strongly non-manipulable via agenda setting if and only if no agenda setter has a special interest in alternatives. A policy implication of these results is that agenda setters should be totally irrelevant to political issues. An interesting future topic would be to study the case in which agenda setters are also elected by voters.

Appendix

Proof of Theorem 1:

Take any unanimous voting rule $f : \mathscr{S}^n \times [0,1]^m \to [0,1]$. Suppose, by contradiction, that for any $\succsim \in \mathscr{S}^n$ and any $x \in [0,1]$, if $x \succsim_i y$ for all $i \in I$ and all $y \in [0,1]$, then $f\big(\succsim, \mathrm{DE}(f, \succsim)\big) = \{x\}$.

Step 1. For any $a_k \in A$ and any $\succsim \in \mathscr{S}^n$, if $x \in \mathrm{DS}_k(f, \succsim)$, then

$$[\ x \succsim_i y \quad \forall i \in I,\ \forall y \in [0,1]\ ] \Longrightarrow [v_k(x) \geq v_k(y) \quad \forall y \in [0,1]]\,.$$

Take any $a_k \in A$ and any $\succsim \in \mathscr{S}^n$. Suppose that $x \in \mathrm{DS}_k(f, \succsim)$ and $x \succsim_i y$ for all $i \in I$ and all $y \in [0,1]$. Suppose, by contradiction, that there exists $y \in [0,1]$ with $v_k(y) > v_k(x)$. Let $s_{-k} \in [0,1]^{m-1}$ be such that $s_{-k} = (\underbrace{y, \ldots, y}_{m-1})$. Then, $f(\succsim, y, s_{-k}) = y$. By unanimity of f, $f(\succsim, x, s_{-k}) = x$. Thus,

$$v_k(f(\succsim, y, s_{-k})) = v_k(y) > v_k(x) = v_k(f(\succsim, x, s_{-k})),$$

a contradiction to $x \in \mathrm{DS}_k(f, \succsim)$.

Step 2. For any $\succsim \in \mathscr{S}^n$, if $x \succsim_i y$ for all $i \in I$ and all $y \in [0,1]$, then there exists $a_k \in A$ such that $\mathrm{DS}_k(f, \succsim) = \{x\}$.

Take any $\succsim \in \mathscr{S}^n$. Suppose that $x \succsim_i y$ for all $i \in I$ and all $y \in [0,1]$. If there exists $a_k \in A$ with $\mathrm{DS}_k(f, \succsim) = \emptyset$, then $\mathrm{DE}(f, \succsim) = \emptyset$ and

$$\{x\} = f(\succsim, \mathrm{DE}(f, \succsim)) = \emptyset,$$

a contradiction. Therefore, for any $a_k \in A$, $\mathrm{DS}_k(f, \succsim) \neq \emptyset$.

Now, suppose, by contradiction, that for any $a_k \in A$, $\mathrm{DS}_k(f, \succsim) \neq \{x\}$. Let $s \in [0,1]^m$ be such that for any $a_k \in A$, $s_k \in \mathrm{DS}_k(f, \succsim) \setminus \{x\}$. Then, $s \in \mathrm{DE}(f, \succsim)$, and hence

$$x \neq f(\succsim, s) \in f(\succsim, \mathrm{DE}(f, \succsim)),$$

a contradiction to $f(\succsim, \mathrm{DE}(f, \succsim)) = \{x\}$.

Step 3. For any $x \in [0,1]$, there exists $a_{k(x)} \in A$ that satisfies the following two conditions:

(i) for any $y \in [0,1]$, $v_{k(x)}(x) \geq v_{k(x)}(y)$,

(ii) for any $y \in [0,1]$ with $x > y$, $v_{k(x)}(x) > v_{k(x)}(y)$.

Take any $x \in [0,1]$. Let $\succsim \in \mathscr{S}^n$ be such that for any $i \in I$, $b(\succsim_i) = x$ and $y \succ_i z$ for any $y \in [0,x]$ and any $z \in (x,1]$. By Step 2, there exists

$a_{k(x)} \in A$ such that $\mathrm{DS}_{k(x)}(f, \succsim) = \{x\}$. Therefore, by Step 1, for any $y \in [0,1]$, $v_{k(x)}(x) \geq v_{k(x)}(y)$.

Let us show Condition (ii). Take any $y \in [0,1]$ with $x > y$. Suppose, by contradiction, that $v_{k(x)}(x) = v_{k(x)}(y)$. Then, for any $z \in [y, x]$,

$$v_{k(x)}(y) = v_{k(x)}(z) = v_{k(x)}(x).$$

Let us show that $y \in \mathrm{DS}_{k(x)}(f, \succsim)$. Take any $s_{-k(x)} \in [0,1]^{m-1}$ and $s'_{k(x)} \in [0,1]$. Then, by unanimity of f and definition of $\succsim$, $f(\succsim, y, s_{-k(x)}) \in [y, x]$. Thus,

$$v_{k(x)}(f(\succsim, y, s_{-k(x)})) = v_{k(x)}(x) \geq v_{k(x)}(f(\succsim, s'_{k(x)}, s_{-k(x)})).$$

Therefore, $y \in \mathrm{DS}_{k(x)}(f, \succsim)$, a contradiction to $\mathrm{DS}_{k(x)}(f, \succsim) = \{x\}$. Hence $v_{k(x)}(x) > v_{k(x)}(y)$.

Step 4. For any $x, y \in [0,1]$ with $x \neq y$, $a_{k(x)} \neq a_{k(y)}$.

Take any $x, y \in [0,1]$ with $x \neq y$. Without loss of generality, suppose that $x > y$. Then, by Step 3, $v_{k(x)}(x) > v_{k(x)}(y)$ and $v_{k(y)}(y) \geq v_{k(y)}(x)$. Therefore, $a_{k(x)} \neq a_{k(y)}$.

Steps 3 and 4 imply that there is an injection from $[0,1]$ to A, a contradiction to $|A| = m \in \mathbb{N}$. □

Proof of Theorem 2:

"If" part

Suppose that for any $a_k \in A$ and any $x, y \in [0,1]$, $v_k(x) = v_k(y)$.

Step 1. For any $\succsim \in \mathscr{S}^n$ and any $s \in [0,1]^m$, $c(\succsim) \in \{s_1, s_2, \ldots, s_m\}$ implies $f^m(\succsim, s) = c(\succsim)$.

Take any $\succsim \in \mathscr{S}^n$ and any $s \in [0,1]^m$. Suppose that $c(\succsim) \in \{s_1, s_2, \ldots, s_m\}$. It suffices to show that for any $y \in \{s_1, s_2, \ldots, s_m\} \setminus \{f^m(\succsim, s)\}$,

$$|\{i \in I : f^m(\succsim, s) \succ_i y\}| > |\{i \in I : y \succ_i f^m(\succsim, s)\}|.$$

We shall show that for any $i \in I$ and any $x, y \in \{s_1, \ldots, s_m\}$,

$$[x < y < b(\succsim_i, s) \text{ or } b(\succsim_i, s) < y < x] \Longrightarrow b(\succsim_i, s) \succ_i y \succ_i x.$$

Take any $i \in I$ and $x, y \in \{s_1, \ldots, s_m\}$. Suppose that $x < y < b(\succsim_i, s)$. Let us show that $y < b(\succsim_i)$. Suppose, by contradiction, that $b(\succsim_i) \leq y$. Then, by definition of single-peaked preferences, $y \succ_i b(\succsim_i, s)$, a contradiction to definition of $b(\succsim_i, s)$. Thus, $x < y < b(\succsim_i)$. Then, by definitions of $b(\succsim_i, s)$ and single-peaked preferences, $b(\succsim_i, s) \succ_i y \succ_i x$. A similar argument shows that $b(\succsim_i, s) < y < x$ implies $b(\succsim_i, s) \succ_i y \succ_i x$. Then, since preferences are single-peaked on $\{s_1, s_2, \ldots, s_m\}$, for any $y \in \{s_1, s_2, \ldots, s_m\} \setminus \{f^m(\succsim, s)\}$,

$$|\{i \in I : f^m(\succsim, s) \succ_i y\}| > |\{i \in I : y \succ_i f^m(\succsim, s)\}|.$$

Step 2. For any $\succsim \in \mathscr{S}^n$ and any $a_k \in A$, $\mathrm{DS}_k(f^m, \succsim) = \{c(\succsim)\}$.

Take any $\succsim \in \mathscr{S}^n$ and $a_k \in A$. We first show that $c(\succsim) \in \mathrm{DS}_k(f^m, \succsim)$. Take any $s_{-k} \in [0,1]^{m-1}$ and any $s'_k \in [0,1]$. Since

$$c(\succsim) \in \{s_1, \ldots, s_{k-1}, c(\succsim), s_{k+1}, \ldots, s_m\},$$

by Step 1,

$$f^m(\succsim, c(\succsim), s_{-k}) = c(\succsim).$$

Suppose that

$$|\{a_\ell \in A \setminus \{a_k\} : s_\ell = c(\succsim)\}| = 0.$$

Since $v_k(x) = v_k(y)$ for any $x, y \in [0,1]$,

$$\begin{aligned}
&v_k\big(f^m(\succsim, c(\succsim), s_{-k})\big) + r_k\big(f^m(\succsim, c(\succsim), s_{-k}), c(\succsim), s_{-k}\big)\\
&= v_k(c(\succsim)) + r\\
&\geq v_k(c(\succsim)) + r_k\big(f^m(\succsim, s'_k, s_{-k}), s'_k, s_{-k}\big)\\
&= v_k\big(f^m(\succsim, s'_k, s_{-k})\big) + r_k\big(f^m(\succsim, s'_k, s_{-k}), s'_k, s_{-k}\big).
\end{aligned}$$

Next, suppose that $|\{a_\ell \in A \setminus \{a_k\} : s_\ell = c(\succsim)\}| > 0$. If $s'_k = c(\succsim)$, then clearly

$$\begin{aligned} & v_k\big(f^m(\succsim, c(\succsim), s_{-k})\big) + r_k\big(f^m(\succsim, c(\succsim), s_{-k}), c(\succsim), s_{-k}\big) \\ & \geq v_k\big(f^m(\succsim, s'_k, s_{-k})\big) + r_k\big(f^m(\succsim, s'_k, s_{-k}), s'_k, s_{-k}\big). \end{aligned}$$

Suppose that $s'_k \neq c(\succsim)$. Since $|\{a_\ell \in A \setminus \{a_k\} : s_\ell = c(\succsim)\}| > 0$, $c(\succsim) \in \{s_1, \ldots, s_{k-1}, s'_k, s_{k+1}, \ldots, s_m\}$. Hence by Step 1, $f^m(\succsim, s'_k, s_{-k}) = c(\succsim)$. Since $c(\succsim) = f^m(\succsim, c(\succsim), s_{-k})$ and $s'_k \neq c(\succsim) = f^m(\succsim, s'_k, s_{-k})$,

$$\begin{aligned} & v_k\big(f^m(\succsim, c(\succsim), s_{-k})\big) + r_k\big(f^m(\succsim, c(\succsim), s_{-k}), c(\succsim), s_{-k}\big) \\ & = v_k(c(\succsim)) + \frac{r}{|\{a_\ell \in A \setminus \{a_k\} : s_\ell = c(\succsim)\}| + 1} \\ & > v_k(c(\succsim)) + 0 \\ & = v_k\big(f^m(\succsim, s'_k, s_{-k})\big) + r_k\big(f^m(\succsim, s'_k, s_{-k}), s'_k, s_{-k}\big). \end{aligned}$$

Thus, $c(\succsim) \in \mathrm{DS}_k(f^m, \succsim)$.

We next show that for any $y \in [0,1] \setminus \{c(\succsim)\}$, $y \notin \mathrm{DS}_k(f^m, \succsim)$. Take any $y \in [0,1] \setminus \{c(\succsim)\}$. Let $s_{-k} \in [0,1]^{m-1}$ be such that $s_{-k} = (\underbrace{c(\succsim), \ldots, c(\succsim)}_{m-1})$.

Then, $c(\succsim) = f^m(\succsim, c(\succsim), s_{-k})$ and $y \neq c(\succsim) = f^m(\succsim, y, s_{-k})$. Hence

$$\begin{aligned} & v_k\big(f^m(\succsim, c(\succsim), s_{-k})\big) + r_k\big(f^m(\succsim, c(\succsim), s_{-k}), c(\succsim), s_{-k}\big) \\ & = v_k(c(\succsim)) + \frac{r}{m} \\ & > v_k(c(\succsim)) + 0 \\ & = v_k\big(f^m(\succsim, y, s_{-k})\big) + r_k\big(f^m(\succsim, y, s_{-k}), y, s_{-k}\big). \end{aligned}$$

Therefore, $y \notin \mathrm{DS}_k(f^m, \succsim)$, and hence $\mathrm{DS}_k(f^m, \succsim) = \{c(\succsim)\}$.

Step 3. For any $\succsim \in \mathscr{S}^n$, $f^m(\succsim, \mathrm{DE}(f^m, \succsim)) = \{c(\succsim)\}$.

Take any $\succsim \in \mathscr{S}^n$. Then, by Step 2, $\mathrm{DE}(f^m, \succsim) = \big\{(\underbrace{c(\succsim), \ldots, c(\succsim)}_{m})\big\}$.

Thus,

$$f^m(\succsim, \mathrm{DE}(f^m, \succsim)) = \{c(\succsim)\}.$$

□

“Only if” part

Suppose that for any $\succsim \in \mathscr{S}^n$,

$$f^m(\succsim, \mathrm{DE}(f^m, \succsim)) = \{c(\succsim)\}.$$

Take any $a_k \in A$. Let us show that for any $x \in (0,1)$, $v_k(x) = v_k(0)$. Suppose, by contradiction, that there exists $x \in (0,1)$ such that $v_k(x) \neq v_k(0)$. First, we consider the case with $v_k(x) > v_k(0)$. Let $\succsim \in \mathscr{S}^n$ be such that for any $i \in I$, $b(\succsim_i) = 0$. Then, $c(\succsim) = 0$. Let $s_{-k} \in [0,1]^{m-1}$ be such that $s_{-k} = (\underbrace{1, \ldots, 1}_{m-1})$. Then, $f^m(\succsim, x, s_{-k}) = x$ and $f^m(\succsim, 0, s_{-k}) = 0$. Thus,

$$\begin{aligned} & v_k\big(f^m(\succsim, x, s_{-k})\big) + r_k\big(f^m(\succsim, x, s_{-k}), x, s_{-k}\big) \\ & = v_k(x) + r \\ & > v_k(0) + r \\ & = v_k\big(f^m(\succsim, 0, s_{-k})\big) + r_k\big(f^m(\succsim, 0, s_{-k}), 0, s_{-k}\big). \end{aligned}$$

Therefore, $0 \notin \mathrm{DS}_k(f^m, \succsim)$.

Let us show that $\mathrm{DS}_k(f^m, \succsim) = \emptyset$. Take any $y \in (0,1]$. Let $s_{-k} \in [0,1]^{m-1}$ be such that $s_{-k} = (\underbrace{0, \ldots, 0}_{m-1})$. Then, $f^m(\succsim, 0, s_{-k}) = 0$ and $f^m(\succsim, y, s_{-k}) = 0$. Thus,

$$\begin{aligned} & v_k(f^m(\succsim, 0, s_{-k})) + r_k\big(f^m(\succsim, 0, s_{-k}), 0, s_{-k}\big) \\ & = v_k(0) + \frac{r}{m} \\ & > v_k(0) \\ & = v_k(f^m(\succsim, y, s_{-k})) + r_k\big(f^m(\succsim, y, s_{-k}), y, s_{-k}\big). \end{aligned}$$

Therefore, $y \notin \mathrm{DS}_k(f^m, \succsim)$, and hence $\mathrm{DS}_k(f^m, \succsim) = \emptyset$. Then $\mathrm{DE}(f^m, \succsim) = \emptyset$. It in turn implies that

$$\{0\} = \{c(\succsim)\} = f^m(\succsim, \mathrm{DE}(f^m, \succsim)) = \emptyset,$$

a contradiction.

Next, we consider the case with $v_k(0) > v_k(x)$. Let $\succsim \in \mathscr{S}^n$ be such that for any $i \in I$, $b(\succsim_i) = x$ and $0 \succ_i 1$. Then, $c(\succsim) = x$. Let $s_{-k} \in [0,1]^{m-1}$ be such that $s_{-k} = (\underbrace{1,\dots,1}_{m-1})$. Then, $f^m(\succsim, 0, s_{-k}) = 0$ and $f^m(\succsim, x, s_{-k}) = x$. Thus,

$$\begin{aligned}
& v_k(f^m(\succsim, 0, s_{-k})) + r_k\big(f^m(\succsim, 0, s_{-k}), 0, s_{-k}\big) \\
& = v_k(0) + r \\
& > v_k(x) + r \\
& = v_k(f^m(\succsim, x, s_{-k})) + r_k\big(f^m(\succsim, x, s_{-k}), x, s_{-k}\big).
\end{aligned}$$

Therefore, $x \notin \mathrm{DS}_k(f^m, \succsim)$.

Let us show that $\mathrm{DS}_k(f^m, \succsim) = \emptyset$. Take any $y \in [0,1] \setminus \{x\}$. Let $s_{-k} \in [0,1]^{m-1}$ be such that $s_{-k} = (\underbrace{x,\dots,x}_{m-1})$. Then,

$$f^m(\succsim, x, s_{-k}) = x \quad \text{and} \quad f^m(\succsim, y, s_{-k}) = x.$$

Thus,

$$\begin{aligned}
& v_k(f^m(\succsim, x, s_{-k})) + r_k\big(f^m(\succsim, x, s_{-k}), x, s_{-k}\big) \\
& = v_k(x) + \frac{r}{m} \\
& > v_k(x) \\
& = v_k(f^m(\succsim, y, s_{-k})) + r_k\big(f^m(\succsim, y, s_{-k}), y, s_{-k}\big).
\end{aligned}$$

Therefore, $y \notin \mathrm{DS}_k(f^m, \succsim)$, and hence $\mathrm{DS}_k(f^m, \succsim) = \emptyset$. Then $\mathrm{DE}(f^m, \succsim) = \emptyset$. It in turn implies that

$$\{x\} = \{c(\succsim)\} = f^m(\succsim, \mathrm{DE}(f^m, \succsim)) = \emptyset,$$

a contradiction. Therefore, for any $x \in (0,1)$, $v_k(x) = v_k(0)$. A similar argument shows that for any $x \in (0,1)$, $v_k(x) = v_k(1)$. Hence for any $x, y \in [0,1]$, $v_k(x) = v_k(y)$, as desired. □

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