Interpretations of Stability for Twisted Quiver Representations on the Projective Line

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Abstract

The Kobayashi-Hitchin correspondence shows that the moduli space of stable Higgs bundles $\mathcal{M}_X(r,d)$ corresponds directly with solutions to the Hitchin equations, which are self-dual, dimensionally-reduced Yang-Mills equations written on a smooth Hermitian bundle E of rank $r \geq 1$ and degree d on a smooth compact Riemann surface X of genus $g \geq 2$ [5]. We may expand this correspondence to all $g \geq 0$ when we consider twisted versions of the Hitchin equations. As surveyed by Rayan [14], the moduli space $\mathcal{M}_X(r,d)$ can be equipped with a natural $U(1)$ action and the fixed points of this action can be encoded in a "twisted" representation of an A-type quiver,

$$\underset{(r_1,d_1)}{\bullet} \xrightarrow{\phi_1} \underset{(r_2,d_2)}{\bullet} \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} \underset{(r_n,d_n)}{\bullet},$$

where $\sum_{i=1}^n r_i = r$, $\sum_{i=1}^n d_i = d$ and ϕ_i is a bundle map from a rank r_i, degree d_i, bundle to a rank r_{i+1} and degree d_{i+1} bundle tensored by a fixed holomorphic line bundle L. Moreover, in the special case when X is the projective line, the Birkhoff-Grothendieck theorem says that vector bundles in the above quiver decompose into a direct sum of line bundles. Expanding each node accordingly, this allows for many interesting types of quivers, such as argyle quivers as explored by Rayan and Sundbo [15].

This thesis aims to introduce the reader to stable quiver representations in a twisted category of bundles on X. We begin by reviewing the standard theory of linear quiver representations as well as the theory of holomorphic vector bundles on algebraic curves. After this background material, we introduce the notion of a stable vector bundle defined in terms of the Mumford slope condition [9] and then extend this definition more generally to stable twisted quiver representations in the category of bundles on X. From these twisted representations we introduce several associated induced ordinary quiver representations. Finally, we present necessary conditions for stability as linear programming problems when $X = \mathbb{P}^1$ for quiver representations of type (2,1) and type (2,2) and discuss how these necessary stability conditions are manifested in the aforementioned induced ordinary quiver representations.

Acknowledgements

I would like to sincerely thank both of my supervisors Dr. Steven Rayan and Dr. Jenna Rajchgot for all their wisdom in navigating these extraordinary times.

Contents

1 Introduction

The heart of this thesis lies in two mathematical objects, holomorphic vector bundles and linear quiver representations. The first several chapters of this thesis are dedicated to introducing these topics, with Chapter 2 introducing quiver representations and Chapter 3 introducing holomorphic vector bundles. In Chapter 4, we combine these ideas to construct stable quiver representations in a twisted category of bundles on an algebraic curve. This thesis can be seen as directly motivated by work of Gothen and King [4], Schmitt [18], Rayan [13], and Rayan and Sundbo [15]. The notion of stability is present throughout these works, as it is the necessary mathematical condition for the construction of well-formed moduli spaces. The stability condition present in this thesis is a generalization of the one discovered by Mumford [9] involving the so called slope of a vector bundle. This notion of stability was adapted by Hitchin [5] as a tool to study a particularly important moduli space, that of Higgs bundles. The study of this rich moduli space is another direct motivation for our work. The following development of twisted quiver representations, which occupies the rest of this introduction, follows Rayan in the survey paper [14].

Unless otherwise denoted, let X be a smooth compact Riemann surface of genus g. Then X can be equipped with a natural line bundle, the cotangent bundle, which we denote ω_X. If $\mathcal{E}$ is a holomorphic bundle on X and ϕ is a holomorphic section $\phi \in H^0\left(X, \mathrm{End}(\mathcal{E}) \otimes \omega_x\right)$ then the data $(\mathcal{E}, \phi)$ is referred to as a *Higgs bundle*. Higgs bundles have gained prominence as they arise as a source of solutions to the Hitchin equations, which are self-dual, dimensionally-reduced Yang-Mills equations written on a smooth Hermitian bundle E of rank $r \geq 1$ and degree d on X where $g \geq 2$ [5]. We may expand this correspondence to all $g \geq 0$ when we consider twisted versions of the Hitchin equations. However, only some Higgs bundles arise as solutions to the Hitchin equations. Higgs Bundles that arise as solutions to the Hitchin equations are said to be stable and the Kobayashi-Hitchin correspondence establishes the connection between stable Higgs bundles and solutions to the Hitchin equations. Just as with Mumford's [9] earlier work, stable Higgs bundles can be classified in terms of a slope condition. That is, a Higgs bundle $(\mathcal{E}, \phi)$ is stable if and only if for each subbundle $0 \subsetneq \mathcal{U} \subsetneq \mathcal{E}$ such that $\phi\left(\mathcal{U}\right) \subseteq \mathcal{U} \otimes \omega_X$ we have that

$$\frac{\deg\left(\mathcal{U}\right)}{\mathrm{rk}\left(\mathcal{U}\right)} < \frac{\deg\left(\mathcal{E}\right)}{\mathrm{rk}\left(\mathcal{E}\right)}.$$

As we might expect, the space of stable Higgs bundles forms a "nice", meaning Hausdorff, moduli space. As highlighted in [14], the space of all stable pairs $(\mathcal{E}, \phi)$ with underlying smooth bundle E can be considered a moduli space by taking the conjugation action of the group of holomorphic automorphisms of $\mathcal{E}$. This moduli space has the structure of a non-singular quasi-projective variety and is denoted $\mathcal{M}_X(r, d)$.

Until this point, we have only encountered the first mathematical object at the heart of this thesis, namely Higgs bundles which include vector bundles as a special case. Our ultimate goal for this introduction is to show how the moduli space of stable Higgs bundle has a deep fundamental connection with our second mathematical object, twisted quiver representations.

Let $(\mathcal{E}, \phi)$ be a stable Higgs bundle in $\mathcal{M}_X(r,d)$ and let $U(1)$ be the circle group. Then there is a natural group action $U(1) \times \mathcal{M}_X(r,d) \to \mathcal{M}_X(r,d)$ given by $(\lambda, (E, \phi)) \to \left(E, e^{i\lambda}\phi\right)$ where $\lambda \in [0, 2\pi)$ so that ϕ is rotated by λ. We wish to understand the fixed points of this group action. In other words, we wish to determine all $(\mathcal{E}, \phi)$ such that $(\mathcal{E}, \phi) = (\mathcal{E}, e^{i\lambda}\phi)$ for all $\lambda \in [0, 2\pi)$. At first glance, this equation is satisfied if and only if $\phi = 0$. But $\mathcal{M}_X(r,d)$ is a moduli space, so we need to view this equality only up to class equivalence. That is, the equation is satisfied in the moduli space whenever $(\mathcal{E}, e^{i\lambda}\phi) \in [(\mathcal{E}, \phi)]$. Thus, as the equivalence class is defined by taking the conjugation action of the group of holomorphic automorphisms of $\mathcal{E}$ we have that this equation is satisfied if and only if for each λ there exists an automorphism ρ_λ of $\mathcal{E}$ such that $(\mathcal{E}, \rho_\lambda^{-1}\phi\rho_\lambda) = (\mathcal{E}, e^{i\lambda}\phi)$ or simply $\rho_\lambda^{-1}\phi\rho_\lambda = e^{i\lambda}\phi$. Viewing λ as ranging over the real numbers, ρ_λ is really a 1-parameter family of automorphisms of $\mathcal{E}$. Thus, this family has a generator which we denote $\frac{d\rho_\lambda}{d\lambda} \mid_{\lambda=0} = \Lambda$.

Differentiating both sides of the fixed point equation with respect to λ gives

$$\left(-\rho_\lambda^{-1}\left(\frac{d\rho_\lambda}{d\lambda}\right)\rho_\lambda^{-1}\right)\phi\rho_\lambda + \rho_\lambda^{-1}\phi\frac{d\rho_\lambda}{d\lambda} = ie^{i\lambda}\phi.$$

Consider now the special case where $\lambda = 0$. Then the equation becomes

$$\left(-\rho_0^{-1}\left(\frac{d\rho_\lambda}{d\lambda}\mid_{\lambda=0}\right)\rho_0^{-1}\right)\phi\rho_0 + \rho_0^{-1}\phi\left(\frac{d\rho_\lambda}{d\lambda}\mid_{\lambda=0}\right) = i\phi.$$

When $\lambda = 0$, we must then have $\rho_0^{-1}\phi\rho_0 = e^{i(0)}\phi$ or simply $\rho_0^{-1}\phi\rho_0 = \phi$. But our group action is rotation by λ. Therefore when $\lambda = 0$, the action must be trivial. Thus, we must have $\rho_0^{-1} = \rho_0 = \mathrm{Id}_{\mathcal{E}}$. Therefore

$$\begin{aligned}
i\phi &= \left(-\rho_0^{-1}\left(\frac{d\rho_\lambda}{d\lambda}\mid_{\lambda=0}\right)\rho_0^{-1}\right)\phi\rho_0 + \rho_0^{-1}\phi\left(\frac{d\rho_\lambda}{d\lambda}\mid_{\lambda=0}\right) \\
&= -\left(\frac{d\rho_\lambda}{d\lambda}\mid_{\lambda=0}\right)\phi + \phi\left(\frac{d\rho_\lambda}{d\lambda}\mid_{\lambda=0}\right) \\
&= [\Lambda, \phi].
\end{aligned}$$

Thus, $(\mathcal{E}, \phi)$ is fixed if and only if $[\Lambda, \phi] = i\phi$. But Λ is a linear map from $\mathcal{E} \to \mathcal{E}$, so it must have eigenvalues given by sections of $\mathcal{E}$ and eigenspaces given by subbundles of $\mathcal{E}$. Thus, let $B_1, \cdots, B_n$ be the eigenspaces of Λ and $s_1, \cdots, s_n$ be the eigenvalues. Then, as B_k is an eigenspace we must have

$$\begin{aligned} i\phi(B_k) &= [\Lambda, \phi] B_k \\ &= \Lambda\phi(B_k) - \phi(\Lambda B_k) \\ &= \Lambda\phi(B_k) - \phi(s_k B_k) \\ &= \Lambda\phi(B_k) - s_k\phi(B_k) \end{aligned}$$

and therefore $\Lambda\phi(B_k) = (s_k + i)\phi(B_k)$. So s_k+i is an eigenvalue of Λ and $\phi(B_k)$ is the associated eigenspace. Recalling that $\phi \in H^0(X, \mathrm{End}(\mathcal{E}) \otimes \omega_x)$ we must have that $\phi(B_k) \subseteq B_j \otimes \omega_X$ for some j and moreover $s_j = s_k + i$. Thus, we see that ϕ is actually a map from eigenspaces of Λ in to eigenspaces of Λ by increasing the eigenvalue by i. We can re-index the eigenspaces as necessary so that $s_n = s_{n-1} + i = \cdots = s_1 + (n-1)i$ and $\phi : B_{n-1} \to B_n \otimes \omega_X$. Thus, we have a holomorphic chain

$$B_1 \xrightarrow{\phi} B_2 \otimes \omega_X \xrightarrow{\phi} B_3 \otimes \omega_X^2 \xrightarrow{\phi} \cdots \xrightarrow{\phi} B_{n-1} \otimes \omega_X^{n-2} \to B_n \otimes \omega_X^{n-1}.$$

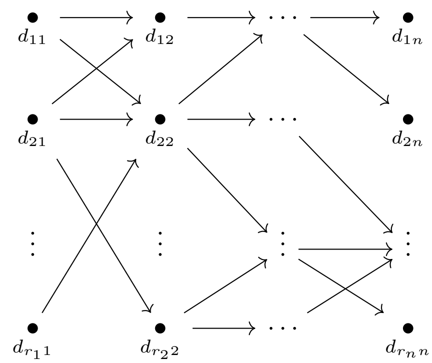
In summary, as the B_j's are eigenspaces of $\mathcal{E}$ we must have the decomposition of $\mathcal{E}$ into subbundles $\mathcal{E} = \oplus_{j=1}^n B_j$ and ϕ acts like a chain with respect to this decomposition. Equivalently, if $\mathrm{rk} B_j = r_j$ and $\deg B_j = d_j$ we see all of the data of the solutions to this fixed point problem can be represented in a A-type quiver

$$\underset{(r_1,d_1)}{\bullet} \xrightarrow{\phi_1} \underset{(r_2,d_2)}{\bullet} \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} \underset{(r_n,d_n)}{\bullet}$$

which lives in the ω_X-twisted category of bundles on X. The Higgs field ϕ may be reconstructed as

$$\phi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \phi_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \phi_{n-1} & 0 \end{bmatrix}.$$

Moreover, in the special case that X is the 2-sphere S^2 with unique complex manifold structure, denoted symbolically as $\mathbb{P}^1$, the Birkhoff-Grothendieck Theorem (see Theorem 3.4.3) says that holomorphic vector bundles decompose into a direct sum of line bundles (each of which are rank 1). So our quiver may be expanded to one of the form



where each node is a line bundle of the specified degree $d_{r_i j}$.

Thus, we see that quivers are connected with stability on the ω_X-twisted category of bundles on X. These mathematical objects are known as "quiver bundles" or equivalently as "twisted" quiver representations along the curve X. In the special case where $X = \mathbb{P}^1$, the Birkhoff-Grothendieck Theorem allows us to consider and study many different types of quivers. This thesis aims to explore the connection between stability on the twisted setting and properties of ordinary quiver representations. To do so, we make attempts to recast stability for twisted quiver representations on $\mathbb{P}^1$ as a solution to a linear programming problem. This study appears in Chapter 4. In particular, we develop linear programs that provide necessary conditions for stable quiver representations of type (2,1) and type (2,2), and then discuss how these necessary conditions are manifested in two induced ordinary quiver representations. Finally, in Chapter 5 we speculate on future directions for further research.

2 Quiver Representations

The goal of this chapter is to develop some of the basic theory of quiver representations that will be needed for the development of twisted quiver representations in future chapters. We begin this chapter by introducing the definition of a quiver followed by several examples. We follow this introduction to quivers with formalizing the notion of a quiver representation and present several examples of these representations. Next, morphisms of quivers are introduced as well as several algebraic properties associated to quiver representations including the notion of indecomposable representations and the Krull-Schmidt Theorem. Lastly, we finish this chapter by introducing the path algebra of a quiver representation and highlight some representations of path algebras through examples. For a more expansive introduction to quiver representations, we refer the reader to the first two chapters of [17], from which much of theory presented here is adapted.

2.1 Introduction to Quiver Representations

We first begin with the definition of a quiver, which informally can be thought of as a directed graph with a finite number of vertices and arrows. We assume the reader is familiar with basic definitions from graph theory.

Definition 2.1.1. A **quiver** Q is a collection of data $Q = (Q_0, Q_1, s, t)$ where

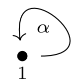
- Q_0 is a set of vertices.
- Q_1 is a set of arrows.
- s is a map $s : Q_1 \to Q_0$ which maps an arrow to its starting point.
- t is a map $t : Q_1 \to Q_0$ which maps an arrow to its terminal (end) point.

Thus, for each element $\alpha \in Q_1$, visually we have an arrow:

$$s(\alpha) \xrightarrow{\quad\alpha\quad} t(\alpha).$$

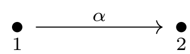
Throughout this thesis, we assume that Q_0 and Q_1 will both be finite sets. We provide some classic examples of quivers to introduce the reader to these fundamental mathematical objects. The following examples are adapted from [7].

Example 2.1.2 (Jordan Quiver)**.** Suppose $Q_0 = \{1\}$, $Q_1 = \{\alpha\}$, and $s(\alpha) = t(\alpha) = 1$. Visually, we represent this quiver as:



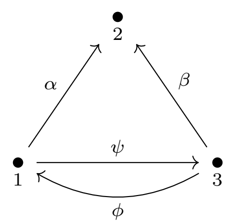
$\diamondsuit$

Example 2.1.3. Suppose $Q_0 = \{1, 2\}$, $Q_1 = \{\alpha\}$, $s(\alpha) = 1$, and $t(\alpha) = 2$. Visually, we represent this quiver as:



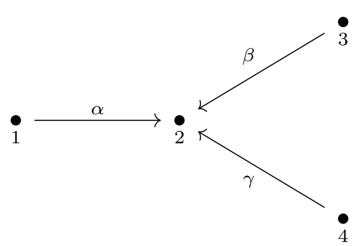
$\diamondsuit$

Example 2.1.4. Suppose $Q_0 = \{1, 2, 3\}$, $Q_1 = \{\alpha, \beta, \psi, \phi\}$, $s(\alpha) = 1 = s(\psi)$, $s(\beta) = 3 = s(\phi)$, $t(\alpha) = 2 = t(\beta)$, $t(\psi) = 3$, and $t(\phi) = 1$. Visually, we represent this quiver as:



$\diamondsuit$

Example 2.1.5 (D_4-type quiver)**.** Suppose $Q_0 = \{1, 2, 3\}$, $Q_1 = \{\alpha, \beta, \gamma\}$, $s(\alpha) = 1$, $s(\beta) = 3, s(\gamma) = 4$, and $t(\alpha) = t(\beta) = t(\gamma) = 2$. Visually, we represent this quiver as:



$\diamondsuit$

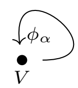
We now introduce the topic at the heart of this chapter: quiver representations. For a more expansive introduction, see [17].

Definition 2.1.6. A **k-linear representation** M of a quiver Q is a collection of data $M = (M_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ where

- $M_i, i \in Q_0$ is a collection of k-vector spaces.
- $\phi_\alpha, \alpha \in Q_1$ is a collection of k-linear maps $\phi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$.

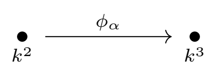
We say that a representation M is finite when each of the vector spaces M_i, where $i \in Q_0$, is finite dimensional. Of course, a representation is always given with respect to a field k. It will often be the case that we assume that k is algebraically closed. Furthermore, for much of the work in this thesis, it will be the case that $k = \mathbb{C}$. One of the primary goals in the study of quiver representations is to attempt to classify all representations of a given quiver up to isomorphism. We illustrate this in the following classical examples adapted from [7]:

Example 2.1.7 (Jordan quiver)**.** Consider again the Jordan quiver where $Q_0 = \{1\}$, $Q_1 = \{\alpha\}$, $s(\alpha) = t(\alpha) = 1$ with representation $M_1 = V$ and linear map ϕ_α. Therefore we have the following representation.

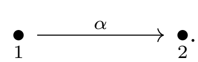


Thus, all representations of a Jordan quiver are pairs (V, ϕ_α). Classifying all representations of the Jordan quiver amounts to finding all endomorphisms of V up to change of basis. From linear algebra, it is well understood that this amounts to finding all conjugacy classes of these endomorphisms. Furthermore, if we assume k is algebraically closed, then every square matrix is similar to a matrix in Jordan canonical form and so we can always choose as a representative from each of these conjugacy classes a matrix that is in Jordan form. This is the inspiration for the name of this fundamental quiver. ♢

Example 2.1.8. We have that

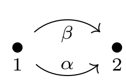


where $\phi_\alpha = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a representation of the quiver

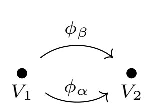


♢

Example 2.1.9 (Kronecker quiver)**.** Consider the quiver Q



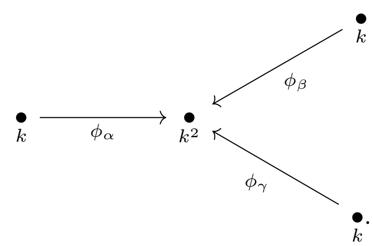
with representation M given by



Classifying all representation of Q is a difficult process. So we consider the sub-case where dim V_1 = dim $V_2 = 1$. Fix a basis B_1 and B_2 for V_1 and V_2 respectively. Then $\phi_\alpha : V_1 \to V_2$ is represented by a matrix $[x]$ and $\phi_\beta : V_1 \to V_2$ is represented by a matrix $[y]$ for $x, y \in k$. A change of basis for either V_1 or V_2 just rescales $[x]$ or $[y]$ by some nonzero $\lambda \in k$ respectively ($x \to \lambda x$ or $y \to \lambda y$). Thus, it is should now be obvious that the representations of Q are in bijection with the set $\frac{k^2}{k^*}$ where k^2 comes from our choice of x and y and we mod out by all none zero scalars k^*. Letting $k = \mathbb{C}$, as we will do in our later work, we see that the one dimensional representations of Q are in bijection with $\frac{\mathbb{C}^2}{\mathbb{C}^*}$ under the above limitations. Of course, as $\frac{(\mathbb{C}^2 \setminus (0,0))}{\mathbb{C}^*} \approxeq \mathbb{P}^1$, we see that removing the zero map and quotienting out by rescaling leads to the representations of Q to be in bijection with $\mathbb{P}^1$. Removing the zero map, which is required for the quotient space to be topologically separated, anticipates the notion of a stability condition which will arise in our future work.

This gives us a very interesting construction of $\mathbb{P}^1$ as the one dimensional representations of the Kronecker quiver over $\mathbb{C}$, excluding the zero map. ♢

Example 2.1.10 (D_4-type quiver)**.** Consider the quiver Q given in Example 2.1.5. A representation M of Q is given by:



Of course, in this case, each of the maps ϕ_i are of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ for $a, b \in k$.

♢

2.2 Morphisms of Quivers

Like most mathematical objects, the study of morphisms or maps between objects allow for rich theory to be developed. The same is true for quiver representations. We now provide a formal definition of a morphism of quiver representations from [17].

Definition 2.2.1. Let Q be a quiver and suppose $M = (M_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $N = (N_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ are two representations of Q. A **morphism** $f : M \to N$ of representations is a tuple of maps $(f_1, f_2, \cdots, f_i, \cdots), \forall i \in Q$ where $f_i : M_i \to N_i$ is a k-linear map such that for each $\alpha \in Q_1$ the diagram

$$\begin{array}{ccc} M_i & \xrightarrow{\phi_\alpha} & M_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ N_i & \xrightarrow{\psi_\alpha} & N_j \end{array}$$

commutes so that $f_j \circ \phi_\alpha(m_i) = \psi_\alpha \circ f_i(m_i), \forall m_i \in M_i$. If each map f_i is a bijection, then we say that f is an **isomorphism** and M and N are isomorphic representations of Q.

Example 2.2.2. Consider again the quiver Q:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

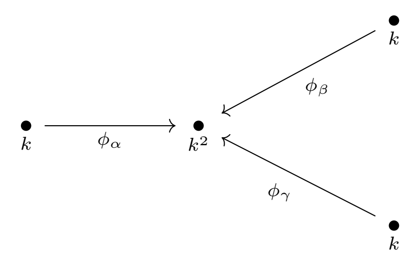
with representations

$$M: \quad \underset{k}{\bullet} \xrightarrow{\phi_\alpha} \underset{k}{\bullet} \qquad\qquad N: \quad \underset{k^2}{\bullet} \xrightarrow{I_2} \underset{k^2}{\bullet}.$$

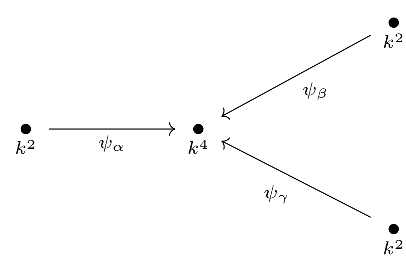
Where $\phi_\alpha = [a]$ for some $a \in k$. Let $f = \left(\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. Then it is clear that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \circ \phi_\alpha(b) = I_2 \circ \begin{bmatrix} a \\ 0 \end{bmatrix}(b)$ so that f is a morphism of M and N. However, it should be clear that f is not an isomorphism since the maps $\begin{bmatrix} a \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are not bijections. $\diamondsuit$

Example 2.2.3. Consider the D_4 quiver Q given in Example 2.1.5 and the following representations:

M :



N :



Suppose that $\phi_i = \begin{bmatrix} x_i \\ x_i \end{bmatrix}$ and $\psi_i = \begin{bmatrix} 0 & x_i \\ y_i & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ for $x_i, y_i \in k$ and $i \in Q_1$. Let $f : N \to M$ be defined by $f = (f_1, f_2, f_3, f_4) = \left(\pi_2, \pi_2, \pi_2, A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)$ where π_2 is the projection map in the second coordinate and $f_4 = A : k^4 \to k^2$. Thus, for any $\begin{bmatrix} a \\ b \end{bmatrix} \in k^2$ we have that:

$$\begin{array}{ccc} \begin{bmatrix} a \\ b \end{bmatrix} & \xrightarrow{\psi_i} & \begin{bmatrix} bx_i \\ ay_i \\ 0 \\ 0 \end{bmatrix} \\ \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle A} \\ [b] & \xrightarrow[\phi_i]{} & \begin{bmatrix} bx_i \\ bx_i \end{bmatrix} \end{array}$$

so that f is a well defined morphism of representations. $\diamondsuit$

With the notion of isomorphic representations, we now have a well defined equivalence relation on the set of all representations of a quiver Q. Before proceeding, we introduce some standard notation. Define Rep Q to be the set of all representations of a quiver Q. Furthermore, we also denote the set of all morphisms between $M, N \in$ Rep Q by Hom (M, N). It can be easily seen that Rep Q forms a category and that Hom (M, N) has the structure of a k-vector space by point-wise addition of morphisms. Our primary goal for the rest of this chapter will be to describe the structure of the category of k-linear representations of a quiver Q.

2.3 Properties of Quiver Representations

Just as one can consider direct sums of vector spaces, one can just as naturally consider direct sums of quiver representations. The following definitions are from [17].

Definition 2.3.1. Let M and N be representations of the quiver Q. Define their **direct product** by $M \oplus N = \left(M_i + N_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \psi_\alpha \end{bmatrix}\right)_{i \in Q_0, \alpha \in Q_1}$.

Of course, applying Definition 2.3.1 recursively we see that we can consider direct sums of any finite number of representations. A natural question that arises from Definition 2.3.1 is which representations M of Q can be represented as a direct sum of non-trivial representations. This leads to the notion of decomposable and indecomposable representations.

Definition 2.3.2. A representation $M \in$ Rep Q is called **indecomposable** if $M \neq 0$ and M cannot be written as the direct sum of two nonzero representations. In other words, M is **indecomposable** if and only if $M \cong N \oplus L$ where $N, L \in$ Rep Q implies $N = 0$ or $L = 0$.

Thus, if a representation is decomposable, we know it can be constructed by considering the direct sum of a finite number of non-trivial representations. A natural next question would be to determine which representations in Rep Q are decomposable, and for these representations that are decomposable, are these decompositions unique? These questions are answered in the following theorem.

Theorem 2.3.3 (Krull-Schmidt Theorem)**.** *Let Q be a quiver and suppose $M \in$ Rep Q. Then there exists a unique decomposition (up to arrangement of the terms) $M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_n$ where $M_i \in$ Rep Q $\forall i$ and each of the M_i are indecomposable.*

Proof. We provide the proof for existence. For a proof of uniqueness, see [1]. The proof is by induction. If M is indecomposable the proof is complete. Suppose then M is decomposable and therefore $M = N_1 \oplus N_2$ for some non-trivial representations $N_1, N_2 \in$ Rep Q of strictly smaller dimension. The fact that N_1 and N_2 are strictly smaller in dimension is due to the fact that M is assumed to be finite dimensional, followed by observing how the linear maps in Definition 2.3.1 are defined. We can then continue recursively in this manner on N_1 or N_2 as needed. This recursion algorithm must terminate due to the assumption that M is finite dimensional and each step in the algorithm strictly reduces dimension. □

In summary, we see that any representation $M \in$ Rep Q can be constructed from indecomposable representations, and that these indecomposable representations form the building blocks of all representations in Rep Q. Thus, our primary goal of classifying all representations of a given quiver up to isomorphism can be simplified greatly by equivalently classifying all indecomposable representations of a given quiver up to isomorphism. We now aim to expand on our theory of morphisms of quivers.

Given a morphism f between two representations $M, N \in$ Rep Q we can use f to construct two important representations of Q. These representations are the kernel and cokernel representations of f. The following constructions are adapted from [17].

- We construct the kernel representation denoted ker f. Suppose that $M = (M_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $N = (N_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ are two representations of a quiver Q and that $f : M \to N$ is a morphism. From elementary linear algebra, we know that ker $f_i = L_i$ is a subspace of the vector space M_i. Define the map $\psi_\alpha : L_i \to L_j$ by $\psi_\alpha = \phi_\alpha \mid L_i$. We show that this map ψ_α is well defined. Suppose $x \in L_i$. Using the fact that f is a morphism, we have $f_j \circ \psi_\alpha(x) = f_j \circ \phi_\alpha(x) = \varphi_\alpha \circ f_i(x) = \varphi_\alpha(f_i(x)) = \varphi_\alpha(0) = 0$ as $x \in$ ker $f_i = L_i$ and therefore $\psi_\alpha(x) \in$ ker $f_j = L_j$ as desired.

- We construct the cokernal representation denoted coker f. Suppose again that $M = (M_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $N = (N_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ are two representations of a quiver Q and that $f : M \to N$ is a morphism. From elementary linear algebra, we know that $\frac{N_i}{f_i(M_i)} = T_i$ is a subspace of the vector space N_i. Define a map $\gamma_\alpha : T_i \to T_j$ by $\gamma_\alpha(n_i + f_i(M_i)) = \varphi_\alpha(n_i) + f_j(M_j)$. We show that this map γ_α is well defined. Suppose that $n_i + f_i(M_i) = n_i' + f_i(M_i)$ for some $n_i, n_i' \in N_i$. Then $n_i - n_i' \in f_i(M_i)$. So using the linear transformation φ_α we have that $\varphi_\alpha(n_i - n_i') = \varphi_\alpha(n_i) - \varphi_\alpha(n_i') \in \varphi_\alpha(f_i(M_i)) = f_j(\phi_\alpha(M_i)) \subseteq f_j(M_j)$ as $\phi_\alpha(M_i) \subseteq M_j$. Thus, $\varphi_\alpha(n_i) + f_j(M_j) = \varphi_\alpha(n_i') + f_j(M_j)$ as desired.

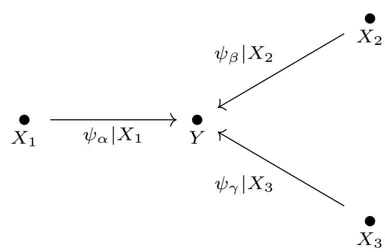
We summarize these two important representations in the following definitions adapted from [17].

Definition 2.3.4. The **kernel representation** denoted ker f is the representation $L = (L_i, \psi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ constructed above.

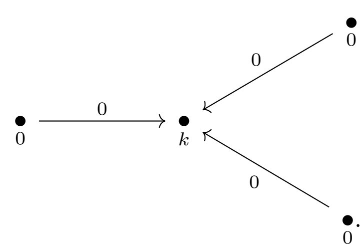
Definition 2.3.5. The **cokernel representation** denoted coker f is the representation $T = (T_i, \gamma_\alpha)_{i\in Q_0, \alpha\in Q_1}$ constructed above.

Example 2.3.6. Consider the morphism $f : N \to M$ given in Example 2.2.3. Let us compute the kernel and cokernal representations of f.

- It is easy to see that $\ker f_1 \cong \ker f_2 \cong \ker f_3 \cong k$ as each of these kernels is of the form $X = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in k \right\}$. On the other hand, simple linear algebra shows that ker A=ker $f_4 \cong k^3$ as the kernel of this map is of the form $Y = \left\{ \begin{bmatrix} 0 \\ a \\ b \\ c \end{bmatrix} : a, b, c \in k \right\}$. Thus, the kernel representation is given by:



- We now compute the cokernal representation. As $f_1 = f_2 = f_3 = \pi_2$ is the surjective projection map from k^2 to k we have that $f_i(N_i) \cong M_i$ and therefore $\frac{M_i}{f_i(N_i)} \cong \{0\}$ for $i = 1, 2, 3$. On the other hand, we see that $\text{im} f_4 = \text{im} A = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} : a \in k \right\} \cong k$. Therefore, $\frac{M_4}{f_4(N_4)} \cong k$. Since all of our outer nodes are 0, we have the cokernal representation:



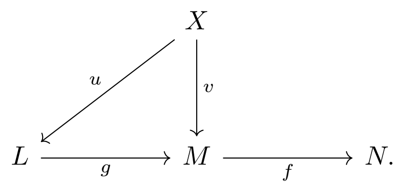
$\diamondsuit$

Of course, given two representations M and N of a quiver Q and a morphism f between them, ker$f$ and coker$f$ give us two additional representations to study. Perhaps more importantly, as readers with a foundation in category theory may be aware, these representations exhibit an extremely important defining algebraic property that is used extensively. We verify that our notion of kernel and cokernel make sense in the context of category theory as done in [17].

Theorem 2.3.7. *Let M, N and L be representations of the quiver Q. Let $f : M \to N$ be a morphisms between them. Then a kernel of f is a morphism $g : L \to M$ such that the following conditions hold:*

1. $f \circ g = 0$.

2. *Given any representation X of Q and any morphism $v : X \to M$ such that $f \circ v = 0$, there is a unique morphism $u : X \to L$ such that $g \circ u = 0$. In symbols:*



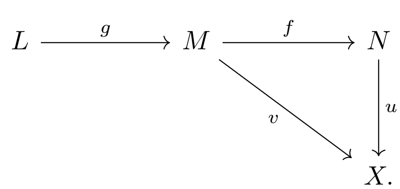
Proof. Our goal is to show that our definition of the kernel representation of f defined in Definition 2.3.4 coincides with this categorical definition. Thus, define L to be the kernel representation of Definition 2.3.4 and let $g : L \to M$ be the identity map. Then clearly for any $l_i \in L_i$, $f_i \circ g_i(l_i) = f_i(l_i) = 0$ by the definition of L_i. To show the second condition, suppose $v : X \to M$ is a map such that $f \circ v = 0$ for some representation $X = (X_i, \chi_\alpha)_{i \in Q_0, \alpha \in Q_1}$. Define the map $u : X \to L$ by $u_i(x_i) = v_i(x_i)$ where $x_i \in X_i$. Thus, as g is simply the identity, we get $g \circ u = v$ as desired. We only need to show that u meets the commutative property of being a morphism. If $x_i \in X_i$ then $\psi_\alpha \circ u_i(x_i) = \varphi_\alpha \circ v_i(x_i) = v_j \circ \chi_\alpha(x_i) = u_j \circ \chi_\alpha(x_i)$ which shows that u is a well defined morphism. Furthermore, since g is the injective identity morphism, this u is unique. □

A similar result can shown to be true for the cokernal representation which is summarized in the following theorem. The proof uses a similar idea to the one given above.

Theorem 2.3.8. *Let M, N and L be representations of the quiver Q. Let $g : L \to M$ be a morphisms. Then a cokernel of g is a morphism $f : M \to N$ such that the following conditions hold:*

1. $f \circ g = 0$.

2. *Given any representation X of Q and any morphism $v : M \to X$ such that $v \circ g = 0$, there is a unique morphism $u : N \to X$ such that $u \circ f = v$. In symbols:*



As we have seen, quiver representations have a number of nice algebraic properties extended directly from vector spaces. One of the most important algebraic tools that we have to study quiver representations come in the form of short exact sequences. For more expansive coverage, see [17].

Definition 2.3.9. A sequence of morphisms $L \xrightarrow{g} M \xrightarrow{f} N$ is **exact** at M if im $g = \ker f$. A sequence of morphisms $\cdots \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots$ is called exact if every representation M_i in the sequence is exact.

Of course, Definition 2.3.9 also extends naturally to the concept of a short exact sequence as we see in the following example.

Example 2.3.10. Consider the quiver Q in Example 2.1.3 and the following representations.

$$S(2): \quad \underset{0}{\bullet} \xrightarrow{\quad 0 \quad} \underset{k}{\bullet}$$

$$S(1): \quad \underset{k}{\bullet} \xrightarrow{\quad 0 \quad} \underset{0}{\bullet}$$

We construct the short exact sequence

$$0 \xrightarrow{\quad 0 \quad} S(2) \xrightarrow{(0,I_1)} S(1) \oplus S(2) \xrightarrow{(I_1,0)} S(1) \xrightarrow{\quad 0 \quad} 0.$$

Which is short exact as $\mathrm{im}(0, I_1) = \{0\} \oplus S(2) = \ker(I_1, 0)$. ♢

We now introduce some important language which arises in both the study of quiver representations and our study of homomorphic vector bundles to come in subsequent chapters.

Definition 2.3.11. A morphism $g : L \to M$ is a called a **section** if there exists a morphism $h : M \to L$ such that $h \circ g = 1_L$. A morphism $f : M \to N$ is called a **retraction** if there exists a morphism $v : N \to M$ such that $f \circ v = 1_N$.

Definition 2.3.12. A short exact sequence

$$0 \xrightarrow{\quad 0 \quad} L \xrightarrow{\quad g \quad} M \xrightarrow{\quad f \quad} N \xrightarrow{\quad 0 \quad} 0$$

splits if g is a section.

It can be easily observed that the short exact sequence given in Example 2.3.10 splits by using the function $\pi_2 : S(1) \oplus S(2) \to S(2)$ given by projection in the second coordinate or $\pi_2(x, y) = y$. We see that $\pi_2 \circ (0, I_1) = I_1 = 1_{S(2)}$. Fortunately, our next theorem gives us a nice relationship between short exact sequences, sections and retractions. Furthermore, the theorem also shows that a short exact sequence induces an important decomposition of the middle representation. Due to the length of its proof, it is omitted. For a detailed proof, see [17].

Theorem 2.3.13. *Let*

$$0 \xrightarrow{\quad 0 \quad} L \xrightarrow{\quad g \quad} M \xrightarrow{\quad f \quad} N \xrightarrow{\quad 0 \quad} 0$$

be a short exact sequence of representations in Rep Q. Then

- *g is a section $\Leftrightarrow$ f is a retraction*
- *If g is a section, so that the sequence is splits, $M \cong L \oplus N$.*

Theorem 2.3.13 shows us the power of using short exact sequences and split short exact sequences when studying quiver representations. To further use short exact sequences as an algebraic tool, we want to use a short exact sequence of representations to induce a short exact sequence between vector spaces of morphisms. To construct the maps between these Hom spaces we use covariant and contravariant functors. The following constructions are just specific examples of the more general notion of a functor developed in category theory.

- Let X, Y and Z be representations of the quiver Q and let $f : Y \to Z$ and $g : X \to Y$ be morphisms. Consider the map of categories $\text{Hom}\,(X, -) : \mathscr{C} \to \mathscr{C}'$ defined by $\text{Hom}\,(X, -)\,((Y, f)) = (\text{Hom}(X, Y), f_*)$ where $f_* : \text{Hom}(X, Y) \to \text{Hom}(X, Z)$ is the map given by $f_*(g) = f \circ g$. The map $\text{Hom}\,(X, -)$ is called a **covariant functor**. Thus, we have constructed a map f_* between Hom spaces as desired.
- Let X, Y and Z be representations of the quiver Q and let $h : Y \to Z$ and $g : Z \to X$ be morphisms. Consider the map of categories $\text{Hom}\,(-, X) : \mathscr{C} \to \mathscr{C}'$ defined by $\text{Hom}\,(-, X)\,((Y, h)) = (\text{Hom}(Y, X), h_*)$ where $h_* : \text{Hom}(Z, X) \to \text{Hom}(Y, X)$ is the map given by $h_*(g) = g \circ h$. The map $\text{Hom}\,(X, -)$ is called a **contravariant functor**. Thus, we have constructed a map h_* between Hom spaces as desired.

These maps f_* and h_* constructed above allow us to take short exact sequences of quiver representations and produce lots of short exact sequences of Hom spaces as the next theorems shows. For detailed proofs, see [17].

Theorem 2.3.14. *Let*

$$0 \xrightarrow{\quad 0 \quad} L \xrightarrow{\quad g \quad} M \xrightarrow{\quad f \quad} N \xrightarrow{\quad 0 \quad} 0$$

be a sequence of representations in Rep Q. Then this sequence is exact $\Leftrightarrow$ for every representation $X \in$ Rep Q the sequence

$$0 \xrightarrow{\quad 0 \quad} Hom(X, L) \xrightarrow{\quad g_* \quad} Hom(X, M) \xrightarrow{\quad f_* \quad} Hom(X, N) \xrightarrow{\quad 0 \quad} 0$$

is exact.

Theorem 2.3.15. *Let*

$$0 \xrightarrow{\quad 0 \quad} L \xrightarrow{\quad g \quad} M \xrightarrow{\quad f \quad} N \xrightarrow{\quad 0 \quad} 0$$

be a split exact sequence of representations in Rep Q. Then for every representation $X \in$ Rep Q the sequence

$$0 \xrightarrow{\quad 0 \quad} Hom(X, L) \xrightarrow{\quad g_* \quad} Hom(X, M) \xrightarrow{\quad f_* \quad} Hom(X, N) \xrightarrow{\quad 0 \quad} 0$$

is also split exact.

Of course, both of the above theorems hold equally well in terms of the maps between Hom spaces built using the construction of the contravariant functor. We are now in a position to define two additional type of representations, projective and injective representations.

2.4 Projective and Injective Representations

2.4.1 Paths

Projective and injective representations are both defined in terms of paths. Informally, a path can be thought about as a way to "walk" from vertex i to vertex j by moving along the arrows only in the direction in which the arrow is pointing. We provide a formal definition below adapted from [7]. We adopt the convention of paths being read from right to left analogous with composition of functions.

Definition 2.4.1. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A path C from $i \in Q_0$ to $j \in Q_0$ of length l is a (possibly empty) sequence $c = (\alpha_l, \alpha_{l-1}, \cdots, \alpha_1)$ with $\alpha_h \in Q_1$ such that

1. $s\left(\alpha_1\right) = i$
2. $s\left(\alpha_h\right) = t\left(\alpha_{h-1}\right) = i$ for $h = 2, 3, \cdots, l$
3. $t\left(\alpha_1\right) = j$.

One important observation is that for each vertex i of Q we have exactly one path of length zero, denoted e_i which is the path consisting of no arrows. In other words, the individual "walking" along the quiver never leaves the vertex i. This observation leads to two important definitions adapted from [7].

Definition 2.4.2. A **oriented cycle** is a path c from i to i such that $l(c) > 0$.

Definition 2.4.3. Let Q_n be the set of all paths of length n of the quiver Q.

It should be easily seen that Definition 2.4.3 coincides with our use of our notation so far. Each vertex in Q_0 can be thought about as the path of length zero at that vertex and each arrow in Q_1 can simply be thought about as a path of length 1. Our primary goal in introducing paths is to construct a k-vector space, denoted kQ whose basis is the set of all paths in Q. Of course, for this vector space to be well defined, we need a well defined binary operation which we construct as follows. If $c = (\alpha_l, \alpha_{l-1}, \cdots, \alpha_1)$ and $d = (\beta_r, \beta_{r-1}, \cdots, \beta_1)$ are two paths in Q define there product cd as follows: If $t\left(\beta_r\right) = s\left(\alpha_1\right)$ then $cd = \left(\alpha_l, \alpha_{l-1}, \cdots, \alpha_1, \beta_r, \beta_{r-1}, \cdots, \beta_1\right)$. If $t\left(\beta_r\right) \neq s\left(\alpha_1\right)$ then $cd = 0$. In other words, the path cd of length $l(cd) = l(c) + l(d)$ is built by composing the paths c and d. It can be easily shown that the vector spaces kQ_n are subspaces of kQ and that kQ can be expressed as a direct sum of the kQ_i. This shows that kQ is only finite dimensional when Q contains no cycles (otherwise we could construct paths of infinite length). Due to this fact, we assume a quiver Q contains no cycles unless otherwise stated.

Example 2.4.4. Consider the quiver Q

$$\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet}\ .$$

This quiver has exactly three paths e_1, e_2 and $c = (\alpha)$. We compute its multiplication table.

$\bullet$	e_1	e_2	c
e_1	e_1	0	0
e_2	0	e_2	c
c	c	0	0

One natural way to understand this vector space is through the k-linear bijective representation

$$\phi : kQ \to GL_2(k)$$

where $\phi(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A_1$, $\phi(e_2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A_2$ and $\phi(c) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A_3$. Thus, kQ can be understood as a 3 dimensional k-vector space given explicitly as $\mathrm{Span}_k \{A_1, A_2, A_3\} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a, b, c \in k \right\}$. $\diamondsuit$

Example 2.4.5. Let us examine the D_4-type quiver given in Example 2.1.5 and determine the structure of kQ. We will abuse notation slightly and use the same notation for the paths and the associated arrows of the quiver Q. We have the following following multiplication table.

$\cdot$	e_1	e_2	e_3	e_4	α	β	γ
e_1	e_1	0	0	0	0	0	0
e_2	0	e_2	0	0	0	0	0
e_3	0	0	e_3	0	0	0	0
e_4	0	0	0	e_4	α	β	γ
α	α	0	0	0	0	0	0
β	0	β	0	0	0	0	0
γ	0	0	γ	0	0	0	0

Just as above, we want to understand this vector space through a k-linear bijective representation. In this case consider the map $\phi : kQ \to GL_4(k)$ given by

1. $\phi(e_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_1$

2. $\phi(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_2$

3. $\phi(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_3$

4. $\phi(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A_4$

5. $\phi(\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A_5$

6. $\phi(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = A_6$

7. $\phi(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = A_7$

It is easy to verify that this is a k-linear bijective representation since the above set of matricies A_i have the same multiplication table. Thus kQ can be understood as a 7 dimensional k-vector space given explicitly as

$$\text{Span}_k \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ e & f & g & d \end{pmatrix} : a, b, c, d, e, f, g \in k \right\}.$$

♢

2.4.2 Projective and Injective Representations

We are now in the position to define two important families of representations. The projective and injective representations. We will see that for each vertex of a quiver Q we can construct a projective and injective representation. For a more expansive exposition, see [7].

Definition 2.4.6. Let Q be a quiver. The **projective representation at vertex** i is the representation $P(i) = (P(i)_j, \phi_\alpha)_{j \in Q_0, \alpha \in Q_1}$ defined by.

1. $P(i)_j$ is the k-vector space with basis consisting of all paths from i to j.

2. If $j \xrightarrow{\alpha} k$ is an arrow of Q then $\phi_\alpha : P(i)_j \to P(i)_k$ is an injective map which maps a path $c \in P(i)_j$ to $\phi_\alpha(c) = dc$ where d is the path $d = (\alpha)$.

Definition 2.4.7. Let Q be a quiver. The **injective representation at vertex** i is the representation $I(i) = (I(i)_j, \phi_\alpha)_{j \in Q_0, \alpha \in Q_1}$ defined by.

1. $I(i)_j$ is the k-vector space with basis consisting of all paths from j to i.

2. If $j \xrightarrow{\alpha} k$ is an arrow of Q then $\phi_\alpha : I(i)_j \to I(i)_k$ is a surjective map defined as follows. If c is a path from j to i whose 1st term is α, i.e $c = (x_r, x_{r-1}, \cdots, x_1, \alpha)$ then $\phi_\alpha(c) = d = (x_r, x_{r-1}, \cdots, x_1)$. If the first term of c is not α then $\phi_\alpha(c) = 0$. This map is clearly surjective, because for any path z from k to i we can construct a path from j to i as $z\alpha$.

We now have all of the knowledge of quiver representations needed to construct quiver bundles. The next chapter focuses on our second primary object of this thesis, holomorphic vector bundles.

3 Holomorphic Vector Bundles

This chapter aims to introduce the reader to holomorphic vector bundles over Riemann surfaces. We begin by laying the groundwork for holomorphic vector bundles by recalling some basic results from complex analysis followed by formally defining the notion of a Riemann surface. Next, we define the notion of a holomorphic line bundle and give four key examples of line bundles which will be used extensively throughout the chapter. After introducing line bundles, we discuss some of the theory surrounding line bundles including sections and sheaves. After introducing line bundles, we generalize them to introduce vector bundles and introduce two important theorems of holomorphic vector bundles, the Riemann-Roch theorem and Birkhoff-Grothendieck theorem. Lastly, we end this chapter by defining the notion of stability for holomorphic vector bundles and classify stability for vector bundles on $\mathbb{P}^1$.

3.1 Foundations of Holomorphic Vector Bundles

We first recall some foundational material on holomorphic functions and Riemann surfaces. Similar definitions and theorems can be found in any introductory textbook on complex analysis, e.g. [10], and as such we omit their proofs here.

Definition 3.1.1. A function $f : D \to \mathbb{C}$, where $D \subseteq \mathbb{C}$, is called **holomorphic** if for each $a \in D$ there exists a neighbourhood U of a such that f is a complex differentiable at each point in U.

Recall that f is complex differentiable at a point w if and only if the limit $\lim_{z\to w} \frac{f(z)-f(w)}{z-w}$ exists. An equivalent definition of a holomorphic function is defined in terms of a complex analytic function.

Definition 3.1.2. A function $f : D \to \mathbb{C}$, where $D \subseteq \mathbb{C}$ is an open set, is said to be **complex analytic** if for each $x_0 \in D$ there exists a neighbourhood U of x_0 and a power series on U such that

$$f(z) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $a_i \in \mathbb{C}$. In other words, there exists a power series which converges to f on U.

Based on Definition 3.1.2, we see that a complex analytic function is an infinitely differentiable function whose Taylor series at any point z pointwise coverages to f in some neighbourhood U. The connection between holomorphic and complex analytic functions is presented in the following theorem. The proof of the theorem can be found in [10].

Theorem 3.1.3. *A complex valued function is holomorphic if and only if it is complex analytic.*

Theorem 3.1.3 is useful as it allows us to view holomorphic functions locally as Taylor series, a fact that will often be used in upcoming proofs. With the definition of a holomorphic function in place, we are now in a position to define a Riemann surface. The following definition is adapted from [6].

Definition 3.1.4. A **Riemann surface** is a one-dimensional complex manifold (a two-dimensional real smooth manifold) with a maximal set of coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{R}^2 \cong \mathbb{C}$ such that $\phi_\beta \circ \phi_\alpha^{-1}$ is a invertible holomorphic function from $\phi_\alpha(U_\alpha \cap U_\beta)$ to $\phi_\beta(U_\alpha \cap U_\beta)$.

Example 3.1.5 (Riemann sphere)**.** Let $M = S^2$ be the unit sphere in $\mathbb{R}^3$. Our goal is to construct a set of coordinate charts for M satisfying Definition 3.1.4. We do this using stereographic projection. Let N be the north pole of M and S be the south pole of M. Let $p \in M \setminus N$ and let L be the line containing $p = (x, y, z)$ and S. We want to assign p the point p' given by the intersection of L and the plane $z = 1$. L can be defined parametrically by the equation $L(t) = (tx, ty, t(z+1) - 1)$ where $t \in \mathbb{R}$. It is easy to see that p' is the point on L when $t = \frac{2}{z+1}$. So $p' = \left(\frac{2x}{z+1}, \frac{2y}{z+1}, 1\right)$. Thus, we define a coordinate chart $\phi_0 : M \to \mathbb{R}^2$ by $\phi_0(x, y, z) = \left(\frac{2x}{z+1}, \frac{2y}{z+1}\right)$. It is also an easy computation to show that $\phi_0^{-1}(x, y) = \left(\frac{4x}{x^2+y^2+4}, \frac{4y}{x^2+y^2+4}, \frac{8}{x^2+y^2+4} - 1\right)$. We complete a similar construction by taking a point $q \in M \setminus S$ and letting L' be the line containing N and q. As above, we want to assign $q = (x, y, z)$ the point q' given by the intersection of L' and the plane $z = -1$. As expected, solving for q' gives $q' = \left(\frac{-2x}{z+1}, \frac{-2y}{z+1}, -1\right)$. Thus, we define a coordinate chart $\phi_1 : M \to \mathbb{R}^2$ by $\phi_1(x, y, z) = \left(\frac{-2x}{z+1}, \frac{-2y}{z+1}\right)$. Just as above, an easy computation shows that $\phi_1^{-1}(x, y) = \left(\frac{4x}{x^2+y^2+4}, \frac{4y}{x^2+y^2+4}, 1 - \frac{8}{x^2+y^2+4}\right)$. Lastly we observe that $\phi_1 \circ \phi_0^{-1} : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}^2 \setminus (0,0)$ is given by $\phi_1 \circ \phi_0^{-1}(x, y) = \left(\frac{4x}{x^2+y^2}, \frac{4y}{x^2+y^2}\right)$, which is clearly an invertible holomorphic function. ♢

A closer examination of Example 3.1.5 is in order. If $z = a + bi \in \mathbb{C}$ then $z^{-1} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. This of course looks very similar in structure to the map $\phi_1 \circ \phi_0^{-1}$ found above. In fact, given the correct reparametrizations of $\mathbb{C}$ we can transform $\phi_1 \circ \phi_0^{-1}$ into a map of the form $\left(\phi_1 \circ \phi_0^{-1}\right)(z) = z^{-1}$. To see this, observe $\left(\phi_1 \circ \phi_0^{-1}\right)(z) = 4\left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right)$. If we rescale the target space by a factor of $1/4$ in the x coordinate and $-1/4$ in the y coordinate we get a map of the desired form. In other words, with the correct parametrizations, using the coordinate charts U_0 and U_1, a coordinate on the intersection $U_0 \cap U_0$ takes the form z in U_0 and $z^{-1} \in U_1$. From now on, we will denote the Riemann sphere as $\mathbb{P}^1$, which reflects its identification as a complex manifold with 1-dimensional complex projective space, also known as the projective line. As we have seen, holomorphic functions are defined in some open subset of $\mathbb{C}$. This definition can then naturally be extended to the concept of holomorphic functions of Riemann surfaces using the transition functions between coordinate charts. The following definition is adapted from [6].

Definition 3.1.6. A **holomorphic function of Riemann surfaces** is a map $f : M' \to M$ such that for each coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{C}$ on M and $\phi'_\beta : U'_\beta \to \mathbb{C}$ on M' the map $\phi_\alpha \circ f \circ \phi_\beta'^{-1}$ is holomorphic. A **holomorphic function on** M is a holomorphic function $f : M \to \mathbb{C}$.

We end this section with a classic theorem from complex analysis and one that we will make use of throughout this chapter. For the proof, again see [10].

Theorem 3.1.7. *Let M be a Riemann surface. If M is connected and compact, then the only holomorphic functions on M are constants.*

3.2 Holomorphic Line Bundles and their Sections

We now introduce a main character in this chapter, holomorphic line bundles over a Riemann surface. This is an expansive topic. For a more exhaustive and thorough introduction, see chapter two of [6] on which this chapter is based. As explained in the the introduction, holomorphic line bundles are a special type of a two-complex dimensional manifold (or four-real dimensional manifold) which enjoy rich algebraic and topological structure. We begin with a definition.

Definition 3.2.1. A **holomorphic line bundle** L over a Riemann surface M is a two-dimensional complex manifold L with a holomorphic projection $\pi : L \to M$ such that:

1. For each $m \in M$, π^{-1} has the structure of a one-complex dimensional vector space.

2. Each point $m \in M$ has a neighbourhood U and a homeomorphism φ_U such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \cong_{\varphi U} & U \times \mathbb{C} \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & U & \end{array}$$

   is commutative.

3. $\varphi_V \circ {\varphi_U}^{-1}$ is of the form

$$(m, w) \to (m, g_{VU}(m)w)$$

   where g_{VU} is a nonvanishing holomorphic function.

The map g_{VU} in Definition 3.2.1 is refereed to as the **transition function** of the line bundle from U to V. A closer inspection of Definition 3.2.1 has a number of immediate consequences. First, for each point $m \in M$ we assign a copy of the complex vector space $\mathbb{C}$. Second, we see that for some neighbourhood U of m, all of the points in U are assigned the same parametrized vector space $\mathbb{C}$. Finally, if we take a second neighbourhood V which also contains m, then the transition function g_{VU} is really a nonvanishing holomorphic map $M \to \mathbb{C}$. In other words, for a point m in $U \cap V$, the transition function g_{VU} gives us a different parametrization of $\mathbb{C}$ depending on if we are viewing m as contained in U or in V. We now introduce the some of the most important line bundles.

3.2.1 Four Important Line Bundles

There are four important line bundles that appear regularly and are critical to our development of the topic of vector bundles. These four line bundles will be used extensively in our upcoming work. In particular in the proofs of Theorem 3.4.2 and Theorem 3.4.3. For a more thorough treatment, see chapter two of [6].

1. The **trivial bundle** is the bundle $M \times \mathbb{C}$ where the function φ_U is the identity map, $\pi(m, z) = m$ for all $(m, z) \in M \times \mathbb{C}$, and $g_{VU}(m) = 1$ for all $m \in M$. The three axioms of 3.2.1 are immediate.

2. We construct the **point bundle** L_p as follows. Suppose $p \in M$. Choose an open set U_0 containing p such that U_0 is coordinatized by z and $z(p) = 0$. Let $U_1 = M - \{p\}$. Then $U_0 \cap U_1 = U_0 - \{p\}$. Now as z is a holomorphic nonvanishing function on $U_0 \cap U_1$ we see that it defines a transition function g_{01}. In summary, the bundle at a point, which we denote L_p, is the set $M \times \mathbb{C}$ with projection $\pi(m, w) = m$, $\varphi_{U_1} = \text{Id}$ and $\varphi_{U_0}(m, w) = (m, (z(m) + \delta(m))\, w)$ where δ is the complex Dirac delta centered at p (i.e is the function which gives the value zero everywhere except at zero and whose integral over $\mathbb{C}$ is one). Then on the intersection $U_0 \cap U_1 = U_0 - \{p\}$ we have $\varphi_{U_0} \circ \varphi_{U_1}^{-1} = \varphi_{U_0} \circ \text{Id}$ so that $\varphi_{U_0} \circ \varphi_{U_1}^{-1}(m, w) = (m, g_{01}(m)w)$. The three axioms should now be immediate with a simple topological verification that φ_{U_0} does indeed define a homeomorphism (which is apparent in the fact that z is holomorphic and the Dirac delta is continuous).

3. We construct the **canonical bundle** or the **cotangent** bundle ω_X as follows. Let M be a Riemann surface with coordinate charts ϕ_α and ϕ_β. Choose parametrizations z of $\phi_\alpha(U_\alpha)$ and w of $\phi_\beta(U_\beta)$ such that $w(z) = \phi_\beta \circ \phi_\alpha^{-1}(z)$ on $U_\alpha \cap U_\beta$. The canonical bundle is the bundle ω_X where $\varphi_{U_\alpha}(z, q) = (z, dz(q))$ and $\pi(m, z) = m$ for all $(m, z) \in M \times \mathbb{C}$. Then as $dw = w' dz$ we may define a transition function $f = \frac{dz}{dw}$ as $\varphi_{U_\beta} \circ \varphi_{U_\alpha}(z, dz(q)) = \left(z, \frac{dw}{dz}(dz(q))\right) = (z, dw(q))$.

4. Consider the Riemann sphere $\mathbb{P}^1$ given in Example 3.1.5 with the coordinate patches U_0 and U_1. The line bundle $\mathcal{O}(n)$ is the bundle where $\pi(m, z) = m$ for all $(m, z) \in M \times \mathbb{C}$. We want to define the transition function for this line bundle to be the map $g_{01} = w^n$ on the intersection $U_0 \cap U_1 \cong \mathbb{C}^*$ for reasons we will see shortly. Thus, define the the function φ_{U_1} to be the identity map and $\varphi_{U_0}(m, z) = (m, (m^n + \delta(m))\, z)$ where δ is the Dirac delta centred at 0. Thus, on the intersection $U_0 \cap U_1 \cong \mathbb{C}^*$ we have $\varphi_{U_0} \circ \varphi_{U_1}^{-1}(m, z) = (m, m^n z)$. This of course has the form of a nonvanishing holomorphic function since m^n is nonvanishing on the intersection.

3.2.2 Basic Properties of Line Bundles

As we explained earlier, line bundles can really be thought of as a collection of one dimensional vector spaces (which each look like copies of $\mathbb{C}$) parametrized by our Riemann surface M. Therefore it should make sense that we can generalize many of the usual operations on vector spaces studied in linear algebra to line bundles. We summarize these properties here. For a more thorough treatment, we again refer the reader to [6].

1. Suppose L is a line bundle with transition functions $g_{\alpha\beta}$. Then its dual, denoted L^*, is the line bundle with transition functions $g_{\alpha\beta}(L^*) = g_{\alpha\beta}^{-1}(L)$. It should then be immediate that the dual of the tangent bundle is the cotangent bundle and vice versa from our work above.

2. Given two line bundles L and $\tilde{L}$ we can form the line bundle $L \otimes \tilde{L}$ with transition functions $g_{\alpha\beta}(L \otimes \tilde{L}) = g_{\alpha\beta}(L) g_{\alpha\beta}(\tilde{L})$.

3. A homomorphism of line bundles L and $\tilde{L}$ is a pair of smooth continuous maps (f, g) such that the following diagram commutes

$$\begin{array}{ccc} L & \longrightarrow_f & \tilde{L} \\ \downarrow_{\pi_1} & \circlearrowright & \downarrow_{\pi_2} \\ M & \longrightarrow_g & \tilde{M} \end{array}.$$

It can be shown that the space of all homomorphism $\mathrm{Hom}(L, \tilde{L})$ is itself a line bundle and $\mathrm{Hom}(L, \tilde{L}) \cong L^* \otimes \tilde{L}$.

4. Using our above results we see that $\mathrm{Hom}(L, L) \cong L^* \otimes L$ and therefore it has transition functions $g_{\alpha\beta}(L^* \otimes L) = g_{\alpha\beta}(L^*) g_{\alpha\beta}(L) = g_{\alpha\beta}^{-1}(L) g_{\alpha\beta}(L) = 1$. But we have shown above that the line bundle whose transition functions are all equal to the identity is the trivial bundle. Thus, $\mathrm{Hom}(L, L) \cong L^* \otimes L$ is the trivial bundle $M \times \mathbb{C}$. This should make sense to us, because the only endomorphisms of the vector space $\mathbb{C}$ to itself is the underlying field of scalars $\mathbb{C}$.

3.2.3 Sections of a Line Bundle

As we have seen, line bundles have rich algebraic structure and can be classified by studying their transition functions. We now examine a new type of function associated to a line bindle which we will use extensively. As we shall soon see, understanding these functions provide tremendous insight into line bundles.

Definition 3.2.2. A **holomorphic section** of a line bundle L over a Riemann surface M is a holomorphic map $s : M \to L$ such that $\pi \circ s = id_M$.

Consider a holomorphic section s on a Riemann surface M restricted to a neighbourhood U of a point m. Then by the commutativity of the diagram in Definition 3.2.1 we have that $\varphi_U \circ s|_U : U \to U \times \mathbb{C}$ and $\varphi_U \circ s|_U(m) = (m, s_U(m))$ where $s_U : U \to \mathbb{C}$ is a holomorphic function. This structure is easiest to see by noting that $\pi \circ s(m) = \mathrm{Id}_M(m) = m = \mathrm{Id}_U \circ \varphi_U \circ s|_U(m)$ so that s is indeed completely defined locally by a holomorphic map s_U. Applying again the commutativity of Definition 3.2.1 then shows that $s_U = g_{UV} \circ s_V$. This then allows us to consider sections as a collection of holomorphic maps s_U which we can glue together using the transition functions on L. Thus, given two sections s and t we can use the transition functions g_{UV} to see that $s_U = g_{UV} s_V$ and $t_U = g_{UV} t_V$. But this then implies $\frac{s_U}{t_U} = \frac{s_V}{t_V}$. So $\frac{s}{t}$ is a global meromorphic function on M and that for each pair of sections, we can construct one of these global meromorphic functions.

As we know, the only holomorphic functions on a Riemann surface M are constants. Sections, however, give us a means to construct holomorphic objects that only locally are single-valued functions but which enjoy many of the useful properties of functions nonetheless.

For instance, if s and t are two sections of L we can construct the section $s+t$ through pointwise addition, i.e $s + t(r) = s(r) + t(r)$. Furthermore, we can scalar multiply a section via $(\lambda s)(m) = (\lambda)s(m)$. Thus, we see that the sections of L form a vector space, which we denote $H^0(M, L)$.

Example 3.2.3. Consider the line bundle $\mathcal{O}(n)$ constructed previously with transition functions $g_{UV} = z^n$. Then if s is a section on $\mathcal{O}(n)$ we have $s_{U_0} = z^n s_{U_1}$ and so $s_{U_0}(z) = z^n s_{U_1}(z^*)$ for some $z \in U_0 \cong \mathbb{C}$ and $z^* \in U_1 \cong \mathbb{C}$. Since s_{U_0} and s_{U_1} are holomorphic functions, they can be represented locally by power series and therefore we have

$$\sum_0^\infty a_m z^m = z^n \sum_0^\infty a_m^* z^{*m}.$$

But from Example 3.1.5 we know that $z^* = z^{-1}$ and therefore we see the above equality implies $a_m^* = a_m = 0$ for $m > n$ and $a_0^* = a_n, a_1^* = a_{n-1}, \cdots, a_n^* = a_0$. Therefore we see that we have $n+1$ possible choices for $a_0, \cdots, a_n$ and thus the dimension of the vector space $H^0\left(\mathbb{P}^1, \mathcal{O}(n)\right)$ is $n+1$.

$\diamondsuit$

3.3 Line Bundles and Sheaf Theory

We have now seen that line bundles can be understood by studying both their transition functions and sections. Both of these objects, however, are defined locally. Our primary goal is to collect this local information as a global structure. The primary algebraic tool to study these global properties is sheaf theory. The following definitions are adapted from chapter two of [6].

Definition 3.3.1. A **sheaf** S on a topological space X associates to each open set $U \subseteq X$ an abelian group $S(U)$, called the sections over U, such that if V is a open subset of X and $U \subseteq V$ there exists a restriction map $r_{VU} : S(V) \to S(U)$ such that the following hold:

1. If $U \subseteq V \subseteq W$, then $r_{WU} = r_{VU} \circ r_{WV}$.
2. If $\sigma \in S(U)$, $\tau \in S(V)$, and $r_{U(U\cap V)}(\sigma) = r_{V(U\cap V)}(\tau)$ there exits $p \in S(U \cup V)$ such that $r_{(U\cup V)U}(p) = \sigma$ and $r_{(U\cup V)V}(p) = \tau$.
3. If $\sigma \in S(U \cup V)$ is such that $r_{(U\cup V)U}(\sigma) = 0$ and $r_{(U\cup V)V}(\sigma) = 0$ then $\sigma = 0$.

It will primarily be the case that we let $X = M$ in the context of Definition 3.3.1. The remaining question is which abelian group we will assign to the open sets U of M. There are four primary sheaves which will construct depending on need.

1. Assign $S(U)$ the group of holomorphic functions on U via pointwise addition. We denote this group $\mathcal{O}(U)$.

2. Assign $S(U)$ the group of holomorphic sections of the line bundle L over U. We denote this group $\mathcal{O}(L)(U)$.

3. Assign $S(U)$ the group of constant functions of U with values in $\mathbb{C}$ or $\mathbb{Z}$.

4. Assign $S(U)$ the group of nonvanishing holomorphic functions on U via function composition. We denote this group as $\mathcal{O}^*(U)$.

Sheaves naturally come with a notion of cohomology. Our goal is to construct cohomology groups using several of the sheaves listed above and study their algebraic properties to gain insight about algebraic properties of our line bundle. The cohomology groups of a sheaf S are constructed as follows. Suppose the set $\{U_\alpha\}_{\alpha\in A}$ is a locally finite covering of M of open sets. Let $C^0 = \oplus_{\alpha_0\in A}\text{sgn}(\sigma)S(U_{\alpha_0})$ and more generally consider the group

$$C^P = \bigoplus_{\alpha_0\neq\cdots\neq\alpha_p\in A} \text{sgn}(\sigma)S\left(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}\right),$$

where σ is the permutation on $p+1$ letters that transforms $(\alpha_0,\cdots,\alpha_p)$ into the index. In other words, a rearrangement of the order of the U_{α_i} appearing in the intersection results in multiplication by the sign of the permutation used to undertake this reordering. We now define our boundary operator on the groups C^p which will be used in our construction of cohomology groups.

Definition 3.3.2. Suppose that $f\in S\left(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}\right)$. The **boundary operator** is a homomorphism $\delta: C^p\to C^{p+1}$ defined pointwise by

$$(\delta f)_{\alpha_0\cdots\alpha_{p+1}} = \sum_i (-1)^i f_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}} \mid U_{\alpha_0}\cap\cdots\cap U_{\alpha_{p+1}}.$$

Informally, Definition 3.3.2 says that the boundary operator is a map which takes elements of $S\left(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}\right)$ and maps them to linear combinations of components of C^p but restricted to intersections of length $p+2$. It is also easy to check that $\delta^2=0$ or that two applications of this operator becomes the zero map. We have now built a enough machinery to define the cohomology groups for a sheaf S.

Definition 3.3.3. The ***p*-th cohomology** group of a sheaf S, relative to a given covering $\{U_\alpha\}_{\alpha\in A}$, is the group

$$H^p(M,S) = \frac{\ker\delta: C^p\to C^{p+1}}{\text{im}\delta: C^{p-1}\to C^p}.$$

One very important application of the cohomology groups of sheaves is stated and proved in the following theorem. This is a technique that is used extensively in proofs of future results. The proof of the theorem follows that of [6].

Theorem 3.3.4. *If*

$$0 \to \mathcal{S} \xrightarrow{f} \mathcal{T} \xrightarrow{h} \mathcal{U} \to 0$$

is a short exact sequence of sheaves on M, then there is a long exact sequence of cohomology groups

$$0 \to H^0(M,\mathcal{S}) \to H^0(M,\mathcal{T}) \to H^0(M,\mathcal{U}) \xrightarrow{\delta_0} H^1(M,\mathcal{S}) \to H^1(M,\mathcal{T}) \to H^1(M,\mathcal{U}) \xrightarrow{\delta_1} \cdots$$

Proof. We construct the boundary operator δ_0. This proof can then be generalized to construct δ_p using the same method, although with slightly more tedious notation.

Suppose $u = (u_\alpha, u_\beta, u_\gamma, \cdots) \in H^0(M,\mathcal{U}) = \ker(C^0(\mathcal{U}))$. Then by the definition of δ_0 we have $(\delta_0 u)_{\alpha\beta} = u_\alpha - u_\beta$ where $(\delta_0 u)_{\alpha\beta}$ denotes the image of u in the $\alpha\beta$ coordinate. Now by our hypothesis, we have a short exact sequence of sheaves and therefore h is a surjective map. Thus, there exists a $t = (t_\alpha, t_\beta, t_\gamma, \cdots) \in C^0(\mathcal{T})$ such that $h(t) = u$ which sends $t_\alpha \to u_\alpha$. Consider now $\delta_0 t \in C^1(\mathcal{T})$. Then as before $(\delta_0 t)_{\alpha\beta} = t_\alpha - t_\beta$. But as h is a homomorphism we have $h(t_\alpha - t_\beta) = h(t_\alpha) - h(t_\beta) = u_\alpha - u_\beta = 0$ and therefore $\delta_0 t \in \ker(h)$. Thus, by the exactness of our series we know that $\ker(h) = \mathrm{im}(f)$ and therefore there exists a unique (since f is injective) $s \in C^1(S)$ such that $s_{\alpha\beta} \to t_\alpha - t_\beta$. Consider now $\delta_1 s \in C^2(S)$. Then a calculation shows that $(\delta_1 s)_{\alpha\beta\gamma} = -s_{\beta\gamma} + s_{\alpha\gamma} - s_{\alpha\beta}$. But f is a homomorphism so

$$f((\delta_1 s)_{\alpha\beta\gamma}) = -f(s_{\beta\gamma}) + f(s_{\alpha\gamma}) - f(s_{\alpha\beta}) = -(t_\beta - t_\gamma) + (t_\alpha - t_\gamma) - (t_\alpha - t_\beta) = 0,$$

from which $f((\delta_1 s)) = 0$. Using the fact that f is injective allows us to conclude that $\delta_1 s = 0$. Thus, $s \in H^1(M,\mathcal{S})$ and we set $\delta_0 u = s$. □

Theorem 3.3.4 gives us the algebraic tools necessary to develop the notion of the "degree" of a line bundle. Consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{2\pi i f}} \mathcal{O}^* \to 0$$

Where $\mathcal{O}$ is the the sheaf given by holomorphic functions of a line bundle L and $\mathcal{O}^*$ are those holomorphic functions which are nonvanishing. Then by Theorem 3.3.4 we have a long exact sequence of sheaves

$$0 \longrightarrow H^0\left(\mathbb{P}^1, \mathbb{Z}\right) \longrightarrow H^0\left(\mathbb{P}^1, \mathcal{O}\right) \longrightarrow H^0\left(\mathbb{P}^1, \mathcal{O}^*\right) \longrightarrow H^1\left(\mathbb{P}^1, \mathbb{Z}\right) \longrightarrow H^1\left(\mathbb{P}^1, \mathcal{O}\right) \longrightarrow \cdots.$$

As sections are defined locally in terms of holomorphic functions, by Theorem 3.1.7 the only holomorphic functions on compact Riemann surfaces are constants. Therefore we have the long exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow H^1(M,\mathbb{Z}) \longrightarrow H^1(M,\mathcal{O}) \longrightarrow H^1(M,\mathcal{O}^*) \longrightarrow \cdots.$$

Now as the exponential map from $\mathbb{C}$ to $\mathbb{C}^*$ is surjective, the exactness of this series shows that $H^1(M,\mathbb{Z})$ injects into $H^1(M,\mathcal{O})$. Furthermore, from Hitchin [6], we know that $H^1(M,\mathcal{O})$ vanishes for $n > 1$. Thus, combining this information we can produce the short exact sequence

$$0 \longrightarrow \frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \longrightarrow H^1(M,\mathcal{O}^*) \xrightarrow{deg} H^2(M,\mathbb{Z}) \longrightarrow 0.$$

But $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$. It can be shown that $H^1(M,\mathcal{O}^*)$ completely characterizes equivalence classes of line bundles over M [6], and so the map deg gives an integer-valued invariant of such classes, called the *degree* . This notion of degree of a line bundle has a number of important properties which we list below.

1. If L_1 and L_2 are two line bundles, then $\deg(L_1 \otimes L_2) = \deg L_1 + \deg L_2$.
2. If L_p is the line bundle over the point p, then $\deg L_p = 1$.
3. If $s \in H^0(M,L)$ is a section with zeros at points $p_1, \cdots, p_n$ with multiplicities $m_1, \cdots, m_n$ respectively, then $\deg L = \sum_i m_i$.
4. On $M = \mathbb{P}^1$, $\deg \mathcal{O}(n) = n$.

3.4 Vector Bundles and Some Important Results

3.4.1 Vector Bundles

We have now closely examined line bundles and introduced several of their important properties. As expected, the notion of line bundle can be further generalized to higher dimensional complex manifolds through the notion of a vector bundle, which we will now define below.

Definition 3.4.1. A **holomorphic vector bundle** E of rank m over a Riemann surface M is a complex manifold with a holomorphic projection $\pi : E \to M$ such that:

1. For each $m \in M$, π^{-1} has the structure of a m-dimensional complex vector space.
2. Each point $m \in M$ has a neighbourhood U and a homeomorphism φ_U such that the diagram

   $$\begin{array}{ccc} \pi^{-1}(U) & \cong_{\varphi U} & U \times \mathbb{C}^m \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & U & \end{array}$$

   is commutative.
3. $\varphi_V \circ {\varphi_U}^{-1}$ is of the form

   $$(z,w) \to (z, A(z)w)$$

   where $A : U \cap V \to \mathrm{GL}(m,\mathbb{C})$ is a holomorphic map to the space of $m \times m$ invertible matrices with coefficients in $\mathbb{C}$.

Just like in the case of line bundles, most operations that can be preformed on vector spaces can also be preformed on vector bundles, which we describe below. For further explanation, see [6].

1. If E_1 and E_2 are vector bundles, we can construct the vector bundle $E_1 \oplus E_2$ in the natural way (using direct sums of vector spaces) and $\mathrm{rk}E_1 \oplus E_2 = \mathrm{rk}E_1 + \mathrm{rk}E_2$.

2. We can form the tensor product $E_1 \otimes E_2$ that has transition functions $A_1 A_2(z) = A_1(z)A_2(z)$.

3. If E is a vector bundle, then the dual of E is the vector bundle E^* with transition functions A^{-1}.

4. If E is a vector bundle, we can form a line bundle $\mathrm{Det}\,(E)$ with transition functions $\mathrm{Det}\,(A)$. Note here that the transition functions A are invertible, so they have a nonzero determinant which will be a nonvanishing holomorphic function.

5. The degree of a vector bundle E is the degree of $\mathrm{Det}\,(E)$.

Theorem 3.4.2 (Riemann-Roch)**.** *If E is a vector bundle on a compact Riemann surface of genus g, then*

$$\dim H^0(M, E) - \dim H^1(M, E) = \deg E + rkE(1 - g)$$

Proof. The proof proceeds by induction on the rank of E and relies on several results from earlier sections. It is adapted from [6].

Base Case Our basis for induction is when $m = 1$ so that E is a line bundle. We construct this case using three results

1. Suppose that L is the trivial bundle on M and let s be a section of L. Then $s_U = g_{UV} s_V = (1)s_V = s_V$ which shows that the local functions are the same on the intersection. As U and V are arbitrary, we see that we must have that $\mathcal{O}(L) = \mathcal{O}$ or that the sections of L are simply given by the holomorphic functions on M. But the only holomorphic functions on a compact Riemann surface are constants and therefore $\mathcal{O}(L) = \mathcal{O} \cong \mathbb{C}$ and therefore $\dim H^0(M, \mathcal{O}) = 1$. Now as M is compact, we have by definition that the genius g is given by $g = \dim H^0(M, K)$. Applying Serre duality gives $\dim H^1(M, \mathcal{O}) \cong \dim H^0(M, K)^*$. But any finite dimensional vector space has the same dimension as it's dual and therefore $\dim H^1(M, \mathcal{O}) = g$. Thus,

$$\dim H^0(M, \mathcal{O}) - \dim H^1(M, \mathcal{O}) = 1 - g.$$

   Now as L is the trivial bundle we have (almost by definition, since it is defined in terms of a homomorphism to the integers) that $\deg \mathcal{O} = 0$ and of course $\mathrm{rk}\mathcal{O} = 1$. Thus,

$$\dim H^0(M, \mathcal{O}) - \dim H^1(M, \mathcal{O}) = 1 - g = 0 + 1(1 - g) = \deg \mathcal{O} + \mathrm{rk}\mathcal{O}(1 - g).$$

   So the result is shown for the trivial bundle.

2. Suppose that the result holds for a line bundle L. We will show it holds for LL_p^{-1} and LL_p. Consider the short exact sequence

$$0 \to \mathcal{O}(L) \xrightarrow{s_p} \mathcal{O}(LL_p) \to \mathcal{O}_p(LL_p) \to 0.$$

Now for any line bundle $\tilde{L}$ we have that $\dim\mathcal{O}_p(\tilde{L}) = 1$ since $\mathcal{O}_p(\tilde{L}) = \pi^-1(p)$ and therefore $H^0(M, \tilde{L}) \cong \mathbb{C}$. We also will make use of the topological fact that $H^2(M, \mathbb{C}) \cong \mathbb{C}$ and that $H^p(M, \mathbb{C}) = 0$ for $p > 2$. Using the theorem proven above we construct the long exact sequence

$$0 \to H^0(M, L) \to H^0(M, LL_p) \to \mathbb{C} \cong H^2(M, \mathbb{C}) \to H^1(M, L) \to H^1(M, LL_p) \to H^3(M, \mathbb{C}) \cong 0.$$

In earlier sections, we have seen that $\deg(L_p) = 1$ and that $\deg(L\tilde{L}) = \deg(L) +$
$deg(\tilde{L})$.Using these facts along with the algebraic result that the alternating sum of dimensions in any long exact sequence is zero gives

$$\begin{aligned}
\dim H^0(M, LL_p) - \dim H^1(M, LL_p) &= \dim H^0(M, L) - \dim H^1(M, L) + 1 \\
&= \deg(L) + (1 - g) + 1 \text{ (applying our hypothesis)} \\
&= \deg(L) + (1 - g) + \deg(L_p) \\
&= \deg(LL_p) + (1 - g)
\end{aligned}$$

proving the result for LL_p.

We now show the result for LL_p^{-1}. Consider the short exact sequence

$$0 \to \mathcal{O}(LL_p^{-1}) \xrightarrow{s_p} \mathcal{O}(L) \to \mathcal{O}_p(L) \to 0.$$

Using a similar idea to that above we construct the long exact sequence,

$$0 \to H^0(M, LL_p^{-1}) \to H^0(M, L) \to \mathbb{C} \cong H^2(M, \mathbb{C}) \to H^1(M, LL_p^{-1}) \to H^1(M, L) \to H^3(M, \mathbb{C}) \cong 0.$$

Now since $\deg(L_p) = 1$ it must be the case that $\deg(L_p^{-1}) = -1$. Using these facts along with the algebraic result that the alternating sum of dimensions in any long exact sequence is zero gives

$$\begin{aligned}
\dim H^0(M, LL_p^{-1}) - \dim H^1(M, LL_p) &= \dim H^0(M, L) - \dim H^1(M, L) - 1 \\
&= \deg(L) + (1 - g) - 1 \\
&= \deg(L) + (1 - g) + \deg(L_p^{-1}) \\
&= \deg(LL_p^{-1}) + (1 - g)
\end{aligned}$$

proving the result for LL_p^{-1}.

3. We now prove that every line bundle L is isomorphic to some product of line bundles

$$L_{p_1} \cdots L_{p_m} L_{q_1}^{-1} \cdots L_{q_n}^{-1}.$$

Consider the short exact sequence

$$0 \to \mathcal{O}(L) \xrightarrow{s_p^n} \mathcal{O}(LL_p^n) \to \frac{\mathcal{O}(LL_p^n)}{\mathcal{O}(L)} \to 0.$$

if f is a section on L and z is a coordinate that vanishes at p then $\frac{\mathcal{O}(LL_p^n)}{\mathcal{O}(L)}$ is the quotient space induced by the map $f(z) \to z^n f(z)$. By construction, $\frac{\mathcal{O}(LL_p^n)}{\mathcal{O}(L)}$ only sees the first n terms in the Taylor expansion of $f(z)$ at p and therefore $H^0(M, \frac{\mathcal{O}(LL_p^n)}{\mathcal{O}(L)}) \cong \mathbb{C}^n$. Thus we have the long exact sequence.

$$\begin{aligned}0 \to H^0(M,L) \to H^0(M, LL_p^n) \to \mathbb{C}^n \cong H^2(M, \mathbb{C}^n) \to H^1(M,L) \\ \to H^1(M, LL_p) \to H^3(M, \mathbb{C}^n) \cong 0.\end{aligned}$$

Again applying the fact that that the alternating sum of dimensions in any long exact sequence is zero gives

$$\begin{aligned}\dim H^0(M, LL_p^n) &= \dim H^1(M, LL_p) + \dim H^0(M,L) - \dim H^1(M,L) + n \\ &\geq + \dim H^0(M,L) - \dim H^1(M,L) + n\end{aligned}$$

Because $\dim H^1(M, LL_p) \geq 0$.We can choose $n \in \mathbb{N}$ sufficiently large such that that the right hand side becomes positive so that $\dim H^0(M, LL_p^n) > 0$. Thus there exists a holomorphic section s of LL_p^n. Suppose that s vanishes at $p_1, \cdots, p_k$ with multiplicities $m_1, \cdots, m_k$. Then clearly $ss_{p_1}^{-m_1} \cdots s_{p_k}^{-m_k}$ is nonvanishing since we are killing all of the zeros of s using the section $s_{p_1}^{-m_1} \cdots s_{p_k}^{-m_k}$. Thus, we have found a nonvanishing section and therefore it induces a isomorphism between the line bundle $LL_p^n L_{p_1}^{-m_1} \cdots L_{p_k}^{-m_k}$ and the trivial bundle. This then shows that L must be of the form

$$L \cong L_{p_k}^{m_k} \cdots L_{p_1}^{m_1} L_p^{-n}$$

which proves the result.

Induction step. Suppose now that E is a vector bundle of rank m and that the result holds for all bundles of lower rank. Our first objective is to find a subbundle L contained in the vector bundle E. To do this we employ a similar strategy used above by trivializing the vector bundle $L^* \otimes E$ by finding a nonvanishing holomorphic section. If we can show the bundle $L^* \otimes E$ is trivial, then clearly E must equipped with the inverse transition functions of L^*. In other words, for every transition function $g_{\alpha\beta}$ on L^* its inverse function $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$ will be a map on E and therefore it must contain L as a subbundle. Just as above, we construct the short exact sequence

$$0 \to \mathcal{O}(E) \xrightarrow{s_p^n} \mathcal{O}(E \otimes L_p^n) \to \mathcal{S} \to 0.$$

Where $\mathcal{S}$ is simply the quotient $\frac{\mathcal{O}(E \otimes L_p^n)}{\mathcal{O}(E)}$. We argued above that $H^0(M, \frac{\mathcal{O}(LL_p^n)}{\mathcal{O}(L)}) \cong \mathbb{C}^n$. Thus replacing L with E simply results in a dimension increase to mn as E is of rank m. Thus we construct the long exact sequence

$$0 \to H^0(M, E) \to H^0(M, EL_p^n) \to \mathbb{C}^{mn} \cong H^2(M, \mathbb{C}^{mn}) \to H^1(M, E) \to H^1(M, EL_p^n) \to 0.$$

Thus using our now standard technique we show

$$\dim H^0(M, EL_p^n) \geq mn + \dim H^0(M, E) - \dim H^1(M, E).$$

Choosing n sufficiently large we can find a section s of $E \otimes L_p^n$. Suppose s vanishes at $p_1, \cdots, p_k$ with multiplicities $m_1, \cdots, m_k$. Then clearly $ss_{p_1}^{-m_1} \cdots s_{p_k}^{-m_k}$ is nonvanishing since we are killing all of the zeros of s using the section $s_{p_1}^{-m_1} \cdots s_{p_k}^{-m_k}$. Thus we have found a nonvanishing section of $E \otimes L_p^n L_{p_1}^{-m_1} \cdots L_{p_k}^{-m_k}$. Setting $L^* = L_p^n L_{p_1}^{-m_1} \cdots L_{p_k}^{-m_k}$ gives our desired result that $E \otimes L^*$ is trivial and so $L \subset E$. We use this result to construct one last short exact sequence

$$0 \to \mathcal{O}(L) \to \mathcal{O}(E) \to \mathcal{O}(Q) \to 0.$$

Where $Q = \frac{E}{L}$ which is clearly of rank $m - 1$. As is now standard procedure we construct the long exact sequence

$$0 \to H^0(M, L) \to H^0(M, E) \to H^0(M, Q) \to H^1(M, L) \to H^1(M, E) \to H^1(M, Q) \to 0.$$

Again applying the fact that that the alternating sum of dimensions in any long exact sequence is zero gives

$$
\begin{aligned}
\dim H^0(M,E) - \dim H^1(M,E) &= \dim H^0(M,L) - \dim H^1(M,L) + \dim H^0(M,Q) - \dim H^1(M,Q) \\
&= \deg L + (1-g) + \deg Q + (m-1)(1-g) \text{ (using our hypothesis twice)} \\
&= \deg E + m(1-g).
\end{aligned}
$$

We used here that $\deg E = \deg L + \deg Q$ which comes immediately from the fact that the tensor product preserves determinants (from linear algebra) $\det E = \det L \otimes \det Q = L \otimes \det Q$. □

Thus, we have the desired result. We now move to the theorem that most influences our future work as highlighted in the introduction to this thesis. The proof we present here is again adapted from [6].

Theorem 3.4.3 (Birkhoff-Grothendieck)**.** *If E is a rank m holomorphic vector bundle over $\mathbb{P}^1$, then we have the following decomposition of E*

$$E \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_m)$$

Proof. The proof is by induction on m.

Base Case. Since $m = 1$ then E is a line bundle. We want to show E has the desired form. Consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \stackrel{e^{2\pi i f}}{\to} \mathcal{O}^* \to 0$$

Where $\mathcal{O}$ is the the sheaf given by holomorphic functions on E and $\mathcal{O}^*$ are those holomorphic functions which are nonvanishing. Then by Theorem 3.3.4 we have a long exact sequence of sheaves

$$0 \longrightarrow H^0\left(\mathbb{P}^1, \mathbb{Z}\right) \longrightarrow H^0\left(\mathbb{P}^1, \mathcal{O}\right) \longrightarrow H^0\left(\mathbb{P}^1, \mathcal{O}^*\right) \longrightarrow H^1\left(\mathbb{P}^1, \mathbb{Z}\right) \longrightarrow H^1\left(\mathbb{P}^1, \mathcal{O}\right) \longrightarrow \cdots$$

As we know, sections are defined locally in terms of holomorphic functions. But by by Theorem 3.1.7 the only holomorphic functions on compact Riemann surfaces are constants. Therefore we have the long exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow H^1\left(\mathbb{P}^1, \mathbb{Z}\right) \longrightarrow H^1\left(\mathbb{P}^1, \mathcal{O}\right) \longrightarrow H^1\left(\mathbb{P}^1, \mathcal{O}^*\right) \longrightarrow \cdots$$

Now since the exponential map from $\mathbb{C}$ to $\mathbb{C}^*$ is surjective the exactness of this series shows that $H^1(M, \mathbb{Z})$ injects into $H^1(M, \mathcal{O})$. Furthermore we know that $H^1(\mathbb{P}^n, \mathcal{O})$ vanishes for $n > 1$. Thus, combining this information we can produce the short exact sequence

$$0 \longrightarrow \frac{H^1\left(\mathbb{P}^1,\mathcal{O}\right)}{H^1\left(\mathbb{P}^1,\mathbb{Z}\right)} \longrightarrow H^1\left(\mathbb{P}^1,\mathcal{O}^*\right) \xrightarrow{deg} \mathbb{Z} \longrightarrow 0.$$

But $\frac{H^1\left(\mathbb{P}^1,\mathcal{O}\right)}{H^1\left(\mathbb{P}^1,\mathbb{Z}\right)} \cong 0$ since topological this is a zero dimensional complex torus. Thus, the degree map here is a bijection and there exits only one line bundle over $\mathbb{P}^1$ for each degree recalling the fact that the group $H^1\left(\mathbb{P}^1,\mathcal{O}^*\right)$ classifies the line bundles over $\mathbb{P}^1$ completely. Thus since E is a line bundle over $\mathbb{P}^1$ is must be of the form $E \cong \mathcal{O}(n)$.

Induction step. Suppose then that E is a rank m vector bundle and the result holds for all degrees less than m. We begin with s familiar construction to the one used in the proof of Theorem 3.4.2. That is, we construct a short exact sequence using the product of point bundles L_p and its corresponding canonical section s_p which vanishes uniquely at p. Thus, consider the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(E) \xrightarrow{s_{p_1}s_{p_2}\cdots s_{p_n}} \mathcal{O}\left(E \otimes L_{p_1}L_{p_2}\cdots L_{p_n}\right) \longrightarrow \frac{\mathcal{O}\left(E\otimes L_{p_1}L_{p_2}\cdots L_{p_n}\right)}{\mathcal{O}(E)} \longrightarrow 0.$$

But $L_{p_1}L_{p_2}\cdots L_{p_n}$ has a section $s_{p_1}s_{p_2}\cdots s_{p_n}$ that has exactly n zeros each with multiplicity 1 and therefore $L_{p_1}L_{p_2}\cdots L_{p_n}$ has degree n and thus $L_{p_1}L_{p_2}\cdots L_{p_n} \cong \mathcal{O}(n)$. Thus, we have constructed the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{O}\left(E \otimes \mathcal{O}(n)\right) \longrightarrow \frac{\mathcal{O}(E\otimes\mathcal{O}(n))}{\mathcal{O}(E)} \longrightarrow 0.$$

Using our now familiar technique used in the proof of Theorem 3.4.2 we can use this short exact sequence to construct a long exact sequence. This, along with the fact that the alternating sum of dimensions of any long exact sequence is zero gives the relation

$$\dim H^0\left(\mathbb{P}^1, E\otimes\mathcal{O}(n)\right) \geq mn + \dim H^0\left(\mathbb{P}^1, E\right) - \dim H^1\left(\mathbb{P}^1, E\right).$$

Thus, we may chose n sufficiently large so that $\dim H^0\left(\mathbb{P}^1, E\otimes\mathcal{O}(n)\right) > 0$. For such an n, $E\otimes\mathcal{O}(n)$ will have holomorphic sections. We denote $E\otimes\mathcal{O}(n)$ as $E(n)$. Consider now the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}\left(E\left(n-1\right)\right) \xrightarrow{\quad s_p \quad} \mathcal{O}\left(E\left(n\right)\right) \longrightarrow \frac{\mathcal{O}(E(n))}{\mathcal{O}(E(n-1))} \longrightarrow 0.$$

Thus this short exact sequence induces a long exact sequence by Theorem 3.3.4. By the exactness of this long exact sequence we have that the map

$$H^0\left(\mathbb{P}^1, E\left(n-1\right)\right) \xrightarrow{\quad s_p \quad} H^0\left(\mathbb{P}^1, E\left(n\right)\right)$$

is injective. We want to study the space of sections given above. To that end, suppose $\dim H^0\left(\mathbb{P}^1, E\left(n-1\right)\right) = \dim H^0\left(\mathbb{P}^1, E\left(n\right)\right)$. This then would imply that s_p is an isomorphism. Thus, every section of $H^0\left(\mathbb{P}^1, E\left(n\right)\right)$ is of the form ss_p for some $s \in H^0\left(\mathbb{P}^1, E\left(n-1\right)\right)$. But s_p vanishes at p and then so too does ss_p. Thus, every section of $H^0\left(\mathbb{P}^1, E\left(n\right)\right)$ vanishes at p. But p was arbitrary, so every section in $H^0\left(\mathbb{P}^1, E\left(n\right)\right)$ vanishes at every p which implies $E(n)$ has no nontrivial holomorphic sections. This is a contradiction, and therefore s_p is not an isomorphism and $\dim H^0\left(\mathbb{P}^1, E\left(n-1\right)\right) \neq \dim H^0\left(\mathbb{P}^1, E\left(n\right)\right)$. Thus it must be the case that $\dim H^0\left(\mathbb{P}^1, E\left(n-1\right)\right) < \dim H^0\left(\mathbb{P}^1, E\left(n\right)\right)$. Thus choosing n sufficiently large, we can find an n such that $\dim H^0\left(\mathbb{P}^1, E\left(n-1\right)\right) = 0$ and $\dim H^0\left(\mathbb{P}^1, E\left(n\right)\right) \neq 0$. Replacing this sufficiently large n in the long exact sequence constructed above gives the long exact sequence.

$$0 \longrightarrow 0 \longrightarrow H^0\left(\mathbb{P}^1, E(n)\right) \longrightarrow H^0\left(\mathbb{P}^1, \frac{E(n)}{E(n-1)}\right) \longrightarrow H^1\left(\mathbb{P}^1, E(n-1)\right) \longrightarrow \cdots$$

Thus, by the exactness of this sequence, we see that the map $H^0\left(\mathbb{P}^1, E(n)\right) \to H^0\left(\mathbb{P}^1, \frac{E(n)}{E(n-1)}\right)$ is injective. Moreover, this map works by taking a non-zero section of $E(n)$ and evaluating it a point p. So if s is a non-trivial section of $E(n)$ then $s(p) \neq 0$. As this holds for all p, s is a nonvanishing section. Thus, just as in the proof of Theorem 3.4.2, this nonvanishing section trivializes a line bundle contained in $E(n)$ through the map $\phi : \mathbb{P}^1 \times \mathbb{C} \to E(n)$ by $\phi\left(r, \lambda\right) = \lambda s(r)$. Using this inclusion map, we construct a short exact sequence.

$$0 \longrightarrow \mathbb{P}^1 \times \mathbb{C} \xrightarrow{\phi} E(n) \xrightarrow{\varphi} \frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}} \longrightarrow 0$$

where $\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}$ is the quotient bundle induced from ϕ. Since $\mathbb{P}^1 \times \mathbb{C}$ is embedded in $E(n)$ our immediate goal is to decompose $E(n)$ into a product containing $\mathbb{P}^1 \times \mathbb{C}$. This, however, is only possible if $\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}$ is contained in $E(n)$ and is complementary to $\mathbb{P}^1 \times \mathbb{C}$. To show this, we find a homomorphism $\psi : \frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}} \to E(n)$ such that $\varphi \circ \psi = 1$ so that ψ is a bijective map onto a subbundle of $E(n)$. To construct ψ consider the short exact sequence of vector bundles

$$0 \longrightarrow \left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}\right)^{-1} \otimes \left(\mathbb{P}^1 \times \mathbb{C}\right) \longrightarrow \left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}\right)^{-1} \otimes E(n) \longrightarrow \left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}\right)^{-1} \otimes \frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}} \longrightarrow 0.$$

Replacing the associated terms in the sequence with their respective homormophism bundle gives

$$0 \longrightarrow Hom\left(\frac{E(n)}{\mathbb{P}^{-1} \times \mathbb{C}}, \mathbb{P}^1 \times \mathbb{C}\right) \longrightarrow Hom\left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}, E(n)\right) \longrightarrow Hom\left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}, \frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}\right) \longrightarrow 0.$$

But holomorphic homomorphisms between vector bundles are just holomorphic sections, so the above short exact sequence can actually be viewed as a short exact sequence of sheaves. Thus, this short exact sequence induces a long exact sequence.

$$0 \longrightarrow H^0\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}, \mathbb{P}^1 \times \mathbb{C}\right)\right) \longrightarrow H^0\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}, E(n)\right)\right)$$

$$\longrightarrow H^0\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}, \frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}\right)\right) \longrightarrow H^1\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1 \times \mathbb{C}}, \mathbb{P}^1 \times \mathbb{C}\right)\right) \longrightarrow \cdots$$

Consider the identity homomorphism ι from $\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}$ to itself. Then clearly ι is a nonvanishing section, so $\iota \in H^0\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}, \frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}\right)\right)$. Our goal is to show that ι is in the kernel of the map in the long exact sequence. But since this sequence is exact, it is sufficient to show that it would be in the image of the map $H^0\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}, E(n)\right)\right) \to H^0\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}, \frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}\right)\right)$. In other words, this section could be pulled back into
$H^0\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}, E(n)\right)\right)$.

Now of course, $rk\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}} < rkE = m$ and therefore by our induction hypothesis $\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}$ splits into a direct sum of line bundles

$$\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}} = \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_{m-1}).$$

Consider the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}\left(\mathcal{O}\left(-1\right)\right) \longrightarrow \mathcal{O}\left(E\left(n-1\right)\right) \longrightarrow \mathcal{O}\left(\frac{E(-1)}{\mathbb{P}^1\times\mathbb{C}}\right) \longrightarrow 0$$

and the induced long exact sequence

$$0 \longrightarrow H^0\left(\mathbb{P}^1, \mathcal{O}\left(-1\right)\right) \longrightarrow H^0\left(\mathbb{P}^1, E\left(n-1\right)\right) \longrightarrow H^0\left(\mathbb{P}^1, \frac{E(-1)}{\mathbb{P}^1\times\mathbb{C}}\right)$$

$$\longrightarrow H^1\left(\mathbb{P}^1, \mathcal{O}\left(-1\right)\right) \longrightarrow H^1\left(\mathbb{P}^1, E\left(n-1\right)\right) \longrightarrow \cdots.$$

As the bundle $\mathcal{O}(-1)$ has degree -1 then $H^0\left(\mathbb{P}^1, \mathcal{O}\left(-1\right)\right) = 0$. Moreover, with our specific choice of significantly large n, $H^0\left(\mathbb{P}^1, E\left(n-1\right)\right) = 0$ also. Thus, applying Theorem 3.4.2 to $\mathcal{O}(-1)$ shows $\dim H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$ and therefore $H^1(\mathbb{P}^1, \mathcal{O}(-1))$ vanishes also. Therefore our long exact sequence becomes

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow H^0\left(\mathbb{P}^1, \frac{E(-1)}{\mathbb{P}^1\times\mathbb{C}}\right)$$

$$\longrightarrow 0 \longrightarrow H^1\left(\mathbb{P}^1, E\left(n-1\right)\right) \longrightarrow \cdots.$$

But this implies

$$0 = H^0\left(\mathbb{P}^1, \frac{E(-1)}{\mathbb{P}^1\times\mathbb{C}}\right) = \bigoplus_i H^0\left(\mathbb{P}^1, \mathcal{O}\left(b_i - 1\right)\right)$$

using our decomposition found above. Since this direct sum vanishes all the terms in the sum must all vanish, and this can only happen if $\mathcal{O}\left(b_i - 1\right)$ has negative degree. So $b_i \leq 0$. We now apply Theorem 3.4.2 to the line bundle $\mathcal{O}\left(-b_i\right)$ and we see that $\dim H^1\left(\mathbb{P}^1, \mathcal{O}\left(-b_i\right)\right) = 0$. Thus, noting that $Hom\left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}, \mathbb{P}^1\times\mathbb{C}\right) \cong \left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}\right)^{-1}$ and that $\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}$ was defined as a decomposition of $\mathcal{O}(b_i)$ and therefore $\left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}\right)^{-1}$ is defined in terms of a decomposition of $\mathcal{O}(-b_i)$ we see that

$$H^1\left(\mathbb{P}^1, \left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}\right)^{-1}\right) = \bigoplus_i H^1\left(\mathbb{P}^1, \mathcal{O}(-b_i)\right) = 0.$$

Thus, substituting this zero term into our long exact sequence shows that ι must lift to a section of $H^0\left(\mathbb{P}^1, Hom\left(\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}, E(n)\right)\right)$. Therefore $E(n)$ splits into a decomposition of the trivial bundle $\mathbb{P}^1\times\mathbb{C}\cong\mathcal{O}(0)=\mathcal{O}$ and $\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}$. In other words, $E(n)=\mathcal{O}\oplus\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}$. Thus, as $E(n)=E\otimes\mathcal{O}(n)$ we see that

$$\begin{aligned} E &= \left(\mathcal{O}\oplus\frac{E(n)}{\mathbb{P}^1\times\mathbb{C}}\right)\otimes\mathcal{O}(-n) \\ &= (\mathcal{O}\oplus\mathcal{O}(b_1)\oplus\cdots\oplus\mathcal{O}(b_{m-1}))\otimes\mathcal{O}(-n) \\ &= \mathcal{O}(-n)\oplus\mathcal{O}(b_1-n)\oplus\cdots\oplus\mathcal{O}(b_{m-1}-n) \end{aligned}$$

as desired. □

3.4.2 Stable Vector Bundles

We now introduce another topic which will be a primary focus in our work here, namely the notion of a stable vector bundle. This notion of stability was first introduced by Mumford [8].

Definition 3.4.4. Let E be a vector bundle over a Riemann surface X.Then, E is said to be **stable** if for each subundle $0\subsetneq U\subsetneq E$ we have $\frac{degU}{rkU}<\frac{degE}{rkE}$. We say that E is **semistable** if $\frac{degU}{rkU}\leq\frac{degE}{rkE}$.

The next theorem completely classifies stability for the special case when $X=\mathbb{P}^1$.

Theorem 3.4.5. *On $X=\mathbb{P}^1$ there are no stable holomorphic vector bundles of rank $m\geq 2$. There exist semistable bundles if rank $m\geq 2$, which are of the form $E\cong\mathcal{O}(a)\oplus\mathcal{O}(a)\oplus\cdots\oplus\mathcal{O}(a)$ for some $a\in\mathbb{Z}$.*

Proof. Let E be a vector bundle of rank $m\geq 2$ over X. Then by Theorem 3.4.3 we have the following decomposition, $E\cong\mathcal{O}(a_1)\oplus\cdots\oplus\mathcal{O}(a_m)$. We proceed by cases.

1. Suppose that $a_i<0$ for all $i=1,\cdots,m$. Reorder the index as necessary so that $a_1\leq a_2\leq\cdots\leq a_m$. Consider the subbundle $U\cong\mathcal{O}(a_m)$ of E. Then we observe that

   $$\frac{a_m}{1}\geq\frac{a_1+a_2\cdots+a_m}{m}$$

   as $a_m-a_i\geq 0$ for all $i=1,\cdots,m-1$ and therefore $(m-1)a_m-a_1-a_2-\cdots-a_{m-1}\geq 0$ giving the above equality.

2. Suppose that $a_i<0$ for some i and $a_j\geq 0$ for some j. In other words, the decomposition contains line bundles of both negative and non-negative degree. Reorder the index as necessary so that $a_m\leq a_{m-1}\leq\cdots\leq a_{n+1}\leq 0\leq a_n\leq a_{n-1}\leq\cdots\leq a_1$. Consider the subundle $U\cong\mathcal{O}(a_1)\oplus\cdots\oplus\mathcal{O}(a_n)$ of E. Then we observe that

   $$\frac{a_1+\cdots+a_n}{n}\geq\frac{a_1+a_2\cdots+a_m}{m}$$

as $n < m$ and therefore $(m-n)a_1 + \cdots + (m-n)a_n - na_{n+1} - \cdots - na_m \geq 0$.

3. Suppose that $a_i \geq 0$ for all $i = 1, \cdots, m$. Reorder the index as necessary so that $a_1 \leq a_2 \leq \cdots \leq a_m$. Consider the subbundle $U \cong \mathcal{O}(a_m)$ of E. Then we observe that

$$\frac{a_m}{1} \geq \frac{a_1 + a_2 \cdots + a_m}{m}$$

as $a_m - a_i \geq 0$ for all $i = 1, \cdots, m-1$ and therefore $(m-1)a_m - a_1 - a_2 - \cdots - a_{m-1} \geq 0$ giving the above equality.

□

Thus, we see that on $X = \mathbb{P}^1$ there exist only semistable bundles of rank $m \geq 2$ which occur precisely when $a_1 = a_1 = \cdots = a_m$. Of course when $m = 1$, E is a line bundle and as there are no nontrivial proper subbundles E would be trivially stable by definition. Although Theorem 3.4.5 says that there are no stable higher rank bundles on $\mathbb{P}^1$ we feel it is important to note that this is not a general phenomenon. In fact, for most Riemann surfaces it is indeed very easy to find examples of higher rank stable holomorphic vector bundles. We now wish to generalize this notion of stability to twisted quiver representations in the category of bundles on X as presented in [13].

4 Stability Conditions for Twisted Quiver Representations

The primary goal of this chapter is to define a notion of stability for twisted quiver representations on $\mathbb{P}^1$ and give interpretations of this stability in terms of linear programs. We begin with a discussion motivating a suitable definition of stability for these representations and follow this discussion with a formal definition and several examples. Next, we introduce some induced ordinary quiver representations which arise naturally when considering twisted quiver representations of bundles. We then present several standard theorems which characterise stability, with particular focus on connected graphs. Lastly, we end this chapter by interpreting stability conditions for general quiver representations of type $(2,1)$ and $(2,2)$ on $\mathbb{P}^1$ and give interpretations of this stability in terms of linear programs which we then use to find implications for stability in terms of the aforementioned induced ordinary quiver representations.

4.1 Twisted Quiver Representations and Stability

The origin of the study of twisted bundles and their associated twisted quiver representations can be traced to investigations into the topological structure of the moduli space of Higgs bundles, as surveyed for instance in [13]. Twisted quiver representations have become a rich field of mathematical research independent of these underlying motivations, as exhibited in [3] and later works.

To begin, let X be an algebraic variety. We will restrict our attention to those varieties which are non-singular, connected and projective algebraic curves over $\mathbb{C}$. Equivalently, we will want X to be a smooth, connected, compact Riemann surface. We say that a vector bundle F over X is "twisted" by a line bundle L when tensored by L, or $F \otimes L$. Suppose then that E and F are vector bundles over X. If $\phi : E \to F \otimes L$ is a map of vector bundles we also say that the map ϕ is twisted by L. We often refer to ϕ as a length 2 holomorphic chain, for reasons that shall become apparent.

This twisting action is a key notion which allows us to view maps of vector bundles as quiver representations in an enhanced category of bundles on X. Recall that a linear map from a vector space V to a vector space W can be viewed as a representation of the A_2 quiver with nodes labelled m and n, where $m =$ dim V and $n =$ dim W. This map is an element of $\mathrm{Hom}(V, W)$, which is the same as $\mathrm{Hom}(V \otimes W^*, k)$, where k is the field. But $V \otimes W^*$ is an abstract vector space while k is k^1. In other words, we did not write V as k^m and W as k^n (by choosing a basis). To keep everything at the same level of abstraction, one can write Hom($V \otimes W^*, L$) where L is a 1-dimensional k-vector space (isomorphic to k when we take a basis element). Generalizing to vector bundles on X, a representation in this larger category should be a map in Hom($E \otimes F^*, L$), where E and F are rank m and n bundles, respectively, and L is a bundle of rank 1. Equivalently, this is a map $E \to F \otimes L$. The twisting line bundle L naturally appears by generalizing the idea of taking an abstract vector space L instead of k in the ordinary quiver example. To formalize this category, we take morphisms from E to F to be graded by tensor powers of L. This formalization will not be used explicitly, and so we do not pursue it here. Our notion of stability of twisted quiver representations in the category of bundles is defined on the tuple of data (E, F, L, ϕ) as follows.

Definition 4.1.1. Consider a representation (E, F, L, ϕ) as introduced above and the map of vector bundles $\Phi : E \oplus F \to (E \oplus F) \otimes L$ given by $\Phi = \begin{bmatrix} 0 & 0 \\ \phi & 0 \end{bmatrix}$. We say that the tuple (E, F, L, ϕ) is **stable** if for each subbundle $0 \subsetneq U \subsetneq E \oplus F$ such that $\Phi(U) \subseteq U \otimes L$ we have $\frac{degU}{rkU} < \frac{degE \oplus F}{rkE \oplus F}$. We say that the tuple is **semistable** if $\frac{degU}{rkU} \leq \frac{degE \oplus F}{rkE \oplus F}$.

Informally, Definition 4.1.1 says that a tuple is stable if the slope condition holds for all invariant subspaces of Φ. Stability conditions of twisted quiver representations in the category of bundles over $X = \mathbb{P}^1$ are of primary interest to us. The motivation for this particular case comes from work on the moduli spaces of Higgs bundles [11] and from a refinement of this work to argyle quivers [12]. Note that argyle quivers are A_n quivers, with particular labels, for n arbitrarily large. Representations of these quivers consist of a collection of vector bundles and twisted maps ϕ_i linking pairs of them in sequence; hence the "chain" terminology mentioned earlier. Note furthermore our definition of stability is specific to the A_2 case, but can be generalized accordingly to the A_n situation.

Example 4.1.2. Let $X = \mathbb{P}^1$, $E = \mathcal{O}(n)$, $F = \mathcal{O}(-1)$, and $L = \mathcal{O}(n+1)$. So $\phi : E \to F \otimes L$ and therefore $\phi : \mathcal{O}(n) \to \mathcal{O}(n)$ so that $\phi \in \text{Hom}(\mathcal{O}(n), \mathcal{O}(n))$. We want to determine the conditions on ϕ for the tuple to become stable. We proceed by cases.

1. Suppose $\phi = 0$ so that $\Phi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Consider the subbundle E of $\mathbb{E} = E \oplus F$. Then $\Phi(E) = 0 \subseteq E \otimes L$ but $\frac{degE}{rkE} = \frac{n}{1} = n > \frac{n-1}{2} = \frac{deg\mathbb{E}}{rk\mathbb{E}}$. Thus, the tuple is unstable in this case.

2. Suppose $\phi \neq 0$ so that $\Phi = \begin{bmatrix} 0 & 0 \\ \phi & 0 \end{bmatrix}$. It is easy to see that E is not invariant since $\Phi(E) = \begin{bmatrix} 0 \\ \phi(E) \end{bmatrix} \subseteq F \otimes L \nsubseteq E \otimes L$. However, we see that $\Phi(F) = 0 \subseteq E \otimes L$ and that $\frac{degF}{rkF} = \frac{-1}{1} = -1 < \frac{n-1}{2} = \frac{deg\mathbb{E}}{rk\mathbb{E}}$ if and only if $n \geq 0$. Thus, we see that the tuple will have stability under the conditions that $\phi \neq 0$ and $n \geq 0$.

$\diamondsuit$

We also want to consider situations where E and F are of higher rank. When $X = \mathbb{P}^1$, the Birkhoff-Grothendieck theorem gives decompositions $E = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_n)$ and $F = \mathcal{O}(b_1) \oplus \mathcal{O}(b_2) \oplus \cdots \oplus \mathcal{O}(b_m)$, and so ϕ can be represented by an $n \times m$ matrix

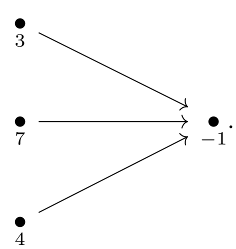
$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1m} \\ \phi_{21} & \ddots & \vdots & \phi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nm} \end{bmatrix}.$$

If $L = \mathcal{O}(t)$, then $\phi_{ij} \in \operatorname{Hom}(\mathcal{O}(a_i), \mathcal{O}(b_j) \otimes \mathcal{O}(t)) \cong H^0(\mathcal{O}(-a_i) \otimes (\mathcal{O}(b_j) \otimes \mathcal{O}(t)))$ and therefore away from ∞, ϕ_{ij} is just a complex polynomial of degree $b_j + t - a_i$. But this representation of ϕ as a linear transformation means the data (E, F, L, ϕ) can simply be viewed as an "expanded" A_2 quiver in which the nodes represented by E and F can be replaced by a collection of n and m nodes, respectively, one for each line bundle in each decomposition. Visually, this quiver is a labelled bipartite graph with two columns of n and m vertices, respectively, constructed by viewing ϕ_{ij} as an adjacency matrix and the labels given by a_i or b_j respectively.

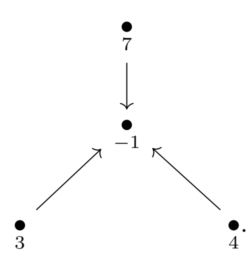
Example 4.1.3. Suppose $E = \mathcal{O}(3) \oplus \mathcal{O}(7) \oplus \mathcal{O}(4)$, $F = \mathcal{O}$, $L = \mathcal{O}(-1)$ and

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \end{bmatrix}.$$

Thus, we have the bipartite graph:



Equivalently, we have



Thus, we see that this quiver resulting from expanding the initial A_2 quiver has the same D_4 type quiver structure we studied extensively in our earlier chapter on representations of quivers.

$\diamondsuit$

4.2 Induced Ordinary Quiver Representations

Consider again the tuple of data (E, F, L, ϕ). This tuple generates several induced ordinary quiver representations which we will continue to explore throughout this chapter. Recall that we have a natural commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \otimes L \\ & {\scriptstyle \pi_E} \searrow \quad \swarrow {\scriptstyle \pi_f \otimes \pi_L} & \\ & X. & \end{array}$$

1. Consider the natural restriction to a point $p \in X$ of the map ϕ. This restriction of ϕ yields

$$\begin{array}{ccc} \mathbb{C}^{rk(E)} \cong E & \xrightarrow{\phi|_p} & \mathbb{C}^{rk(F)} \otimes \mathbb{C} \cong F \otimes L \cong \mathbb{C}^{rk(F)} \\ & {\scriptstyle \pi_E} \searrow \quad \swarrow {\scriptstyle \pi_f \otimes \pi_L} & \\ & \{p\}. & \end{array}$$

   In other words, for each $p \in X$ we a have a natural induced quiver representation

$$\underset{\mathbb{C}^{rk(E)}}{\bullet} \longrightarrow \underset{\mathbb{C}^{rk(F)}}{\bullet}.$$

   Of course, this quiver representation makes sense for all maps ϕ regardless of stability. A natural question which we wish to explore is how the notion of stability translates to conditions on this induced quiver representation.

2. As $\phi : E \to F \otimes L$ is a map vector bundles, we have a induced map of global sections

$$\tilde{\phi} : H^0(X, E) \to H^0(X, F \otimes L).$$

   When $X = \mathbb{P}^1$, we have already seen that $H^0(X, E) \cong \mathbb{C}^n$ and $H^0(X, F \otimes L) \cong \mathbb{C}^m$ for some $m, n \in \mathbb{N}$. Thus, we see ϕ induces a quiver representation $\mathbb{C}^n \to \mathbb{C}^m$. Again, the natural question that arises is how stability is manifested in this induced quiver representation.

Example 4.2.1. Consider again Example 4.1.2. We will examine each of the two induced quiver representations.

1. We have that $\phi \in \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(n)) = H^0(\mathbb{P}^1, \mathcal{O}(-n) \otimes \mathcal{O}(n)) = H^0(\mathbb{P}^1, \mathcal{O})$ where $\mathcal{O}$ is the trivial bundle. Thus, we see that ϕ is constant and then so to is ϕ_p. In this case stability gives no additional information about the induced quiver.

2. We have that ϕ induces a quiver $\tilde{\phi} : H^0(X, \mathcal{O}(n)) \cong \mathbb{C}^{n+1} \to H^0(X, \mathcal{O}(n)) \cong \mathbb{C}^{n+1}$ so that $\tilde{\phi}$ is a endomorphism of $\mathbb{C}^{n+1}$. The stability condition that $\phi \neq 0$ is manifested in the induced quiver as requiring $\tilde{\phi}$ to have positive rank.

$\diamondsuit$

4.3 Argyle Quivers

Our primary goal for this section is prove Theorem 4.3.1 and examine its implications for the induced quivers we introduced in the previous section. Theorem 4.3.1 tries to classify stability for a general type $(2,1)$ quiver over $\mathbb{P}^1$ using linear programming. This type of quiver falls into a family of quivers known as argyle quivers. Argyle quivers are of the form $(n,1)$ for the A_2 case or more generally $(n_1, 1, n_2, 1, \cdots, n_j, 1)$ for longer holomorphic chains of quivers, where each element of the ordered list specifies the rank of the corresponding bundle in the chain. The implications for stability for arglye quivers where first explored by Rayan and Sundbo [16].

Theorem 4.3.1 (General (2,1) case)**.** *Consider the map*

$$\phi : \mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathcal{O}(-1) \otimes \mathcal{O}(t)$$

Where $\phi = (\phi_1, \phi_2)$ *is given by* $\phi_1 : \mathcal{O}(a) \to \mathcal{O}(t-1)$, $\phi_2 : \mathcal{O}(b) \to \mathcal{O}(t-1)$ *and* $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$, $F = \mathcal{O}(-1)$ *and* $L = \mathcal{O}(t)$ *where* $a, b \in \mathbb{Z}$ *and* $t > 0$. *If the tuple* (E, F, L, ϕ) *is stable then* $\phi_1 \neq 0$, $\phi_2 \neq 0$ *and the following system of equations is satisfied.*

$$-a - b - 2 < 0$$
$$a - 2b - 1 < 0$$
$$b - 2a - 1 < 0$$
$$2a + 2b - 3t + 4 < 0.$$

Proof. If $\mathbb{E} = E \oplus F$, then to determine the stability conditions for ϕ we need to consider the map $\Phi : \mathbb{E} \to \mathbb{E} \otimes L$ given by $\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi_1 & \phi_2 & 0 \end{bmatrix}$ and check the slope condition for invariant subbundles observing that

$$\frac{deg\mathbb{E}}{rk\mathbb{E}} = \frac{a+b-1}{3}.$$

We proceed by cases.

1. For case I we complete full calculations, although some may not be strictly necessary, in order for the reader to get a taste for the computations. Suppose $\phi_1 = \phi_2 = 0$, in which case $\Phi = 0$. We consider maximal subbundles of $\mathbb{E}$ to see if they are invariant and gather conditions for stability.

   (a) Consider the fact that $\Phi(E) = 0 \subseteq E \otimes L$. Then in order for E to satisfy the slope condition we require

   $$\frac{degE}{rkE} = \frac{a+b-1}{3} < \frac{a+b-1}{3}$$

   which is only satisfied when $a+b < -2$.

   (b) Next, we see $\Phi(F) = 0 \subseteq F \otimes L$. Then in order for F to satisfy the slope condition we require

   $$\frac{degF}{rkF} = \frac{-1}{1} < \frac{a+b-1}{3}$$

   which is only satisfied when $a+b > -2$.

   (c) If $U_1 = \mathcal{O}(a) \oplus 0 \oplus \mathcal{O}(-1)$, then $\Phi(U_1) = 0 \subseteq U_1 \otimes L$. Then in order for U_1 to satisfy the slope condition we require

   $$\frac{degU_1}{rkU_1} = \frac{a-1}{2} < \frac{a+b-1}{3}$$

   which is only satisfied when $a - 2b - 1 < 0$.

   (d) If $U_2 = 0 \oplus \mathcal{O}(b) \oplus \mathcal{O}(-1)$, then $\Phi(U_2) = 0 \subseteq U_2 \otimes L$. Then in order for U_2 to satisfy the slope condition we require

   $$\frac{degU_2}{rkU_2} = \frac{b-1}{2} < \frac{a+b-1}{3}$$

   which is only satisfied when $b - 2a - 1 < 0$.

   (e) If $U_3 = \mathcal{O}(a) \oplus 0 \oplus 0$, then $\Phi(U_3) = 0 \subseteq U_3 \otimes L$. Then in order for U_3 to satisfy the slope condition we require

   $$\frac{degU_3}{rkU_3} = \frac{a}{1} < \frac{a+b-1}{3}$$

   which is only satisfied when $2a - b + 1 < 0$.

   (f) If $U_4 = 0 \oplus \mathcal{O}(b) \oplus 0$, then $\Phi(U_4) = 0 \subseteq U_4 \otimes L$. Then in order for U_4 to satisfy the slope condition we require

   $$\frac{degU_4}{rkU_4} = \frac{b}{1} < \frac{a+b-1}{3}$$

   which is only satisfied when $2b - a + 1 < 0$.

   Thus, stability requires a solution to the following linear program.

$$a+b+2<0$$
$$-a-b-2<0$$
$$a-2b-1<0$$
$$b-2a-1<0$$
$$2a-b+1<0$$
$$2b-a+1<0.$$

However, no such solution exists.For completeness, we note that $\phi \neq 0$ and $a=b=-1$ result in a semistable but not stable outcome.

2. Without loss of generality, suppose $\phi_1 \neq 0$ and $\phi_2 = 0$, in which case $\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi_1 & 0 & 0 \end{bmatrix}$. We consider maximal subbundles of $\mathbb{E}$ to see if they are invariant and gather conditions for stability.

   (a) If $U_4 = 0 \oplus \mathcal{O}(b) \oplus 0$, then $\Phi(U_4) = 0 \subseteq U_4 \otimes L$. Then in order for U_4 to satisfy the slope condition we require
   $$\frac{degU_4}{rkU_4} = \frac{b}{1} < \frac{a+b-1}{3}$$
   which is only satisfied when $2b-a+1<0$.

   (b) If $U_1 = \mathcal{O}(a) \oplus 0 \oplus \mathcal{O}(-1)$, then $\Phi(U_1) = 0 \subseteq U_1 \otimes L$. Then in order for U_1 to satisfy the slope condition we require
   $$\frac{degU_1}{rkU_1} = \frac{a-1}{2} < \frac{a+b-1}{3}$$
   which is only satisfied when $a-2b-1<0$.

   Thus, stability requires a solution to the following linear program.

   $$a-2b-1<0$$
   $$2b-a+1<0$$

   but no such solution exists. Again, for completeness, we note that $\phi = 0$ and $a=b=-1$ results in a semistable representation in this case.

3. Lastly, suppose $\phi_1 \neq 0$, $\phi_2 \neq 0$, and $\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi_1 & \phi_2 & 0 \end{bmatrix}$. We consider the four (maximal) invariant subbundles of $\mathbb{E}$ and gather conditions for stability.

(a) We have that $\Phi(F) = 0 \subseteq F \otimes L$. Then in order for F to satisfy the slope condition we require

$$\frac{degF}{rkF} = \frac{-1}{1} < \frac{a+b-1}{3}$$

which is only satisfied when $a + b > -2$.

(b) If $U_1 = \mathcal{O}(a) \oplus 0 \oplus \mathcal{O}(-1)$, then $\Phi(U_1) = \begin{bmatrix} 0 \\ 0 \\ \phi_1(a) \end{bmatrix} \subseteq U_1 \otimes L$ as $\phi_1(\mathcal{O}(a)) \subseteq F \otimes L$. Then in order for U_1 to satisfy the slope condition we require

$$\frac{degU_1}{rkU_1} = \frac{a-1}{2} < \frac{a+b-1}{3}$$

which is only satisfied when $a - 2b - 1 < 0$.

(c) If $U_2 = 0 \oplus \mathcal{O}(b) \oplus \mathcal{O}(-1)$, then $\Phi(U_2) = \begin{bmatrix} 0 \\ 0 \\ \phi_2(b) \end{bmatrix} \subseteq U_1 \otimes L$ as $\phi_2(\mathcal{O}(b)) \subseteq F \otimes L$. Then in order for U_2 to satisfy the slope condition we require

$$\frac{degU_2}{rkU_2} = \frac{b-1}{2} < \frac{a+b-1}{3}$$

which is only satisfied when $b - 2a - 1 < 0$.

(d) Of course, $H = \ker\phi$ gives rise to an invariant bundle as $\Phi(H) = 0 \subseteq F \otimes L$. As ϕ is a map from a bundle of rank 2 to one of rank 1 we know it must be the case that $rk(H) = 1$. We now need to determine the degree of H. To do this, consider the short exact sequence

$$0 \longrightarrow H \xrightarrow{1} E \xrightarrow{\phi} F \otimes L \longrightarrow 0,$$

where 1 is the injection of H as a subbundle. Then $degH = degE - deg(F \otimes L) = a + b - (t-1) = a + b - t + 1$. So in order for H to satisfy the slope condition we require

$$\frac{degH}{rkH} = \frac{a+b-t+1}{1} < \frac{a+b-1}{3}$$

which is only satisfied when $2a + 2b - 3t + 4 < 0$.

Thus, stability requires a solution to the following linear program.

$$\begin{aligned} -a - b - 2 &< 0 \\ a - 2b - 1 &< 0 \\ b - 2a - 1 &< 0 \\ 2a + 2b - 3t + 4 &< 0 \end{aligned}$$

which has many solutions depending on our choice of $t > 0$. Python code has been written using Google's OR-tools to provide solutions to this system and is given in Appendix A.

□

Let us examine the induced ordinary quiver representations of the general (2,1) case using the results of Theorem 4.3.1. Consider first the induced map of global sections $\tilde{\phi} : H^0(X, E) \to H^0(X, F \otimes L)$. This induced map $\tilde{\phi}$ corresponds to the ordinary quiver representation

$$\underset{\mathbb{C}^{a+b+2}}{\bullet} \longrightarrow \underset{\mathbb{C}^{t}.}{\bullet}$$

An important consequence of the linear system presented in Theorem 4.3.1 is that it implies $a \geq 0$ and $b \geq 0$. Consider then the last inequality in the theorem given by $2a + 2b - 3t + 4 < 0$. Manipulating this inequality gives $t > \frac{2}{3}(a + b + 2)$, and so stability implies that $\tilde{\phi}$ can at most drop rank by a factor of two thirds. Thus, in the (2,1) case stability is manifested in this induced quiver as a condition on the maximum rank of the kernel of $\tilde{\phi}$.

Next, let us consider the quiver induced by the natural restriction of ϕ to a point $p \in \mathbb{P}^1$. Then

$$\begin{aligned}\phi \in \operatorname{Hom}(\mathcal{O}(a) \oplus \mathcal{O}(b), \mathcal{O}(t-1)) &\cong H^0\left(\mathbb{P}^1, (\mathcal{O}(-a) \oplus \mathcal{O}(-b)) \otimes \mathcal{O}(t-1)\right) \\ &\cong H^0\left(\mathbb{P}^1, \mathcal{O}(t-a-1) \oplus \mathcal{O}(t-b-1)\right).\end{aligned}$$

Away from ∞, ϕ looks like a pair of polynomials (f_1, f_2) where f_1 is of degree $t - a - 1$ and f_2 is of degree $t - b - 1$. Consider the special case where $a = b$. Then, in this case, the inequality $2a + 2b - 3t + 4 < 0$ implies $a + 1 < \frac{3}{4}t$ but then $a + 1 < \frac{3}{4}t < t$ as $t > 0$. So $t - a - 1 > 0$ which means that f_1 and f_2 both have positive degree. Thus, in this quiver, when $a = b$, stability is manifested in the condition that f_1 and f_2 both have positive degree.

4.4 Properties of Stability of Twisted Quiver Representations

In Theorem 4.3.1 we studied the general (2,1) problem. Our primary objective was to determine which labelled quivers admit stable twisted representations. Thus, a natural next step is to develop some characterizations of stability in addition to the direct slope condition given in Definition 4.1.1.

Proposition 4.4.1. *If a representation (E, F, L, ϕ) is stable, then $rkF \deg E - rkE \deg F > 0$.*

Proof. By construction, F is always an invariant subbundle of Φ. Thus, as (E, F, L, ϕ) is stable we must have $\frac{\deg F}{\mathrm{rk} F} < \frac{\deg E+F}{\mathrm{rk} E+F}$ which is easily manipulated to the desired inequality. □

One of the particularly interesting observations of Proposition 4.4.1 is that $E = F$ results in instability.

Theorem 4.4.2. *A representation* (E, F, L, ϕ) *is stable if and only its dual* (E^*, F^*, L^*, ϕ^H) *is stable, where* $\phi^* : F^* \to E^* \otimes L^*$ *is the conjugate transpose of* ϕ.

Proof. We proceed by standard arguments. First, we need to show that an invariant stable subbundle of $\mathbb{E} = E \oplus F$ is also an invariant subundle of $\mathbb{E}^* = E^* \oplus F^*$. Thus, let U be a ϕ invariant subbundle of $\mathbb{E}$. Thus, we can construct the exact sequence

$$0 \longrightarrow U \xrightarrow{1} \mathbb{E} \xrightarrow{\psi} \tfrac{\mathbb{E}}{U} \longrightarrow 0$$

where ψ is just the canonical map into the moduli space, and again 1 is the subbundle injection. Using this exact sequence and Φ, we can form a sequence of commutative squares:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{1} & \mathbb{E} & \xrightarrow{\psi} & \frac{\mathbb{E}}{U} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Psi} & & \\ 0 & \longrightarrow & U \otimes L & \longrightarrow & \mathbb{E} \otimes L & \longrightarrow & \frac{\mathbb{E}}{U} \otimes L & \longrightarrow & 0 \end{array}$$

where Ψ is the induced map constructed from Φ that acts on the moduli space $\frac{\mathbb{E}}{U}$. Dualizing the above sequence of commutative squares gives us a dual sequence of commutative squares

$$\begin{array}{ccccccccc} 0 & \longleftarrow & U^* \otimes L & \longleftarrow & \mathbb{E}^* \otimes L & \longleftarrow & \left(\frac{\mathbb{E}}{U}\right)^* \otimes L & \longleftarrow & 0 \\ & & \uparrow{\scriptstyle \Phi^*} & & \uparrow{\scriptstyle \Phi^*} & & \uparrow{\scriptstyle \Psi^*} & & \\ 0 & \longleftarrow & U^* & \longleftarrow & \mathbb{E}^* & \longleftarrow & \left(\frac{\mathbb{E}}{U}\right)^* & \longleftarrow & 0 \end{array}$$

This shows a natural correspondence between U and $\left(\frac{\mathbb{E}}{U}\right)^* = \frac{\mathbb{E}^*}{U^*}$. In other words, if U is invariant under Φ then $\frac{\mathbb{E}^*}{U^*}$ is also invariant under Φ^*. All that remains to show is that stability of one of these bundles implies stability of the other. Thus, suppose that U is stable subbundle of $\mathbb{E}$. Then,

$$\begin{aligned} \frac{\deg U}{\text{rk } U} < \frac{\deg \mathbb{E}}{\text{rk } \mathbb{E}} &\Leftrightarrow \text{rk } \mathbb{E} \deg U < \text{rk } U \deg \mathbb{E} \\ &\Leftrightarrow \text{rk } \mathbb{E} \deg U - \text{rk } \mathbb{E} \deg \mathbb{E} < \text{rk } U \deg \mathbb{E} - \text{rk } \mathbb{E} \deg \mathbb{E} \\ &\Leftrightarrow (\deg U - \deg \mathbb{E}) \text{ rk } \mathbb{E} < -\deg \mathbb{E} \, (\text{rk } \mathbb{E} - \text{rk } U) \\ &\Leftrightarrow \frac{\deg U - \deg \mathbb{E}}{\text{rk } \mathbb{E} - \text{rk } U} < -\frac{\deg \mathbb{E}}{\text{rk } \mathbb{E}} \\ &\Leftrightarrow \frac{\deg \mathbb{E}^* - \deg U^*}{\text{rk } \mathbb{E}^* - \text{rk } U^*} < \frac{\deg \mathbb{E}^*}{\text{rk } \mathbb{E}^*} \\ &\Leftrightarrow \frac{\deg (\mathbb{E}^*/U^*)}{\text{rk } (\mathbb{E}^*/U^*)} < \frac{\deg \mathbb{E}^*}{\text{rk } \mathbb{E}^*} \end{aligned}$$

which completes the proof.

□

Theorem 4.4.3. *Let $X = \mathbb{P}^1$ and suppose E and F are vector bundles over X. If a representation (E, F, L, ϕ) is stable, then it is simple, meaning that it does not decompose into a direct sum of proper, nonzero subobjects. Equivalently, if (E, F, L, ϕ) is stable then the associated labelled bipartite graph to the adjacency matrix ϕ is connected.*

Proof. We proceed by standard arguments. Thus, suppose that the quiver Q associated to the stable tuple (E, F, L, ϕ) decomposes into two non-trivial quivers Q_1 and Q_2. These quivers then give rise to tuples (E_1, F_1, L, ϕ_1) and (E_2, F_2, L, ϕ_2). If $\deg E_i + \deg F_i = d_i$ and $\mathrm{rk}E_i + \mathrm{rk}F_i = r_i$ then $d_1 + d_2 = \deg E + \deg F = d$ and $\mathrm{rk}E + \mathrm{rk}F = r$. Now as (E, F, L, ϕ) is stable, any proper invariant subbundle of $E \oplus F$ meets the slope condition. But $E_i \oplus F_i$ is a proper subbundle and as the path from E_i to F_i is independent this then implies $E_i \oplus F_i$ will be invariant under Φ (i.e it will map E_i into F_i only). Thus, $E_i \oplus F_i$ also meets the slope condition and therefore $\frac{d_i}{r_i} < \frac{d}{r}$. But $\frac{d}{r} = \frac{d_1+d_2}{r_1+r_2}$. Therefore $\frac{d_1}{r_1} < \frac{d_1+d_2}{r_1+r_2}$ which implies $d_1 r_2 < d_2 r_1$. But $\frac{d_2}{r_2} < \frac{d_1+d_2}{r_1+r_2}$ implies $d_2 r_1 < d_1 r_2$. Combining the previous two inequalities gives a contradiction. □

In Theorem 4.3.1 we classified stability of the general $(2, 1)$ quiver by studying three primary cases. Applying Theorem 4.4.3 to this example shows that in actuality we needed to only consider a single case for stability, and the conditions for that case fully define stability. It is important to see that Theorem 4.4.3 is really a structure theorem, and says something about stability in terms of the general structure of the bipartite graph associated with the representation under consideration.

If M_{nm} is the set of all possible $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1m} \\ \phi_{21} & \ddots & \vdots & \phi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nm} \end{bmatrix}$ for a bundle E of rank n to a bundle F of rank m, then the order of the set of possible underlying graphs is

$$\mid M_{nm} \mid = \sum_{k=0}^{nm} \binom{nm}{k},$$

which is the number of labelled bipartite graphs with $n + m$ and type (n, m). On the other hand, the number of connected bipartite graphs with j nodes is given in [19]. However, this sequence includes all decompositions and is unlabelled. We will only be interested in one specific decomposition (n, m) of the j nodes depending on the rank of E and F respectively. The fact that we consider a sequence constructed by counting unlabelled connected bipartite graphs is perhaps not as daunting as it may appear, since we posses the adjacency matrix ϕ and unlabelled graphs still count the number of edges. The connected unlabelled graphs are each associated to a family of labelled cases as we will soon see.

4.5 Non-Argyle Quivers

In this section our primary goal is to prove Theorem 4.5.1 which involves a non-argyle type (2,2) quiver. These quivers are much more complex in nature primarily as there exists many more possibilities for embedding invariant subbundles in E, F, and $E \oplus F$.

Theorem 4.5.1 (General (2,2) case)**.** *Consider the map*

$$\phi : \mathcal{O}(a) \oplus \mathcal{O}(b) \to (\mathcal{O}(c) \oplus \mathcal{O}) \otimes \mathcal{O}(t)$$

where $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$, $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$, $F = \mathcal{O}(c) \oplus \mathcal{O}$, $L = \mathcal{O}(t)$, $a, b, c \in \mathbb{Z}$, *and* $t > 0$*. If* (E, F, L, ϕ) *is stable, then at most one of the* ϕ_{ij} *is zero. If* $\phi_{ij} = 0$ *for some* $i, j \in \mathbb{N}$ *then the following linear program is satisfied:*

$$\begin{aligned} a + b - 3c &> 0 \\ a + b + c &> 0 \\ a + b - c &> 0 \\ -a + b - c &> 0 \\ -a + 3b - c &> 0 \\ 3a - b - c &> 0. \end{aligned}$$

If $\phi_{ij} \neq 0$ *for all* $i, j \in \mathbb{N}$*, then the following linear program is satisfied:*

$$\begin{aligned} a + b - 3c &> 0 \\ a + b + c &> 0 \\ a + b - c &> 0 \\ -a + 3b - c &> 0 \\ 3a - b - c &> 0. \end{aligned}$$

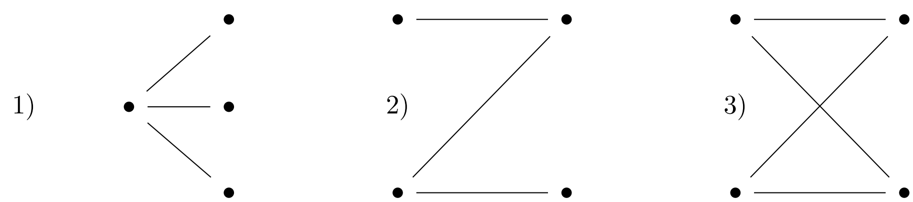
Proof. If $\mathbb{E} = E \oplus F$, then to determine the stability conditions for ϕ we need to consider the map $\Phi : \mathbb{E} \to \mathbb{E} \otimes L$ given by

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \phi_{11} & \phi_{12} & 0 & 0 \\ \phi_{21} & \phi_{22} & 0 & 0 \end{bmatrix}$$

and check the slope condition for invariant subbundles, observing that

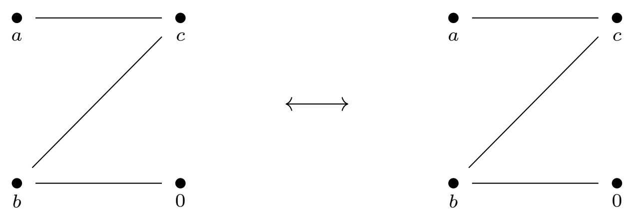
$$\frac{deg\mathbb{E}}{rk\mathbb{E}} = \frac{a+b+c}{4}.$$

Our goal is to again proceed by cases observing that we have a total of $\sum_{k=0}^{4}\binom{4}{k} = 16$ quiver shapes. Importantly, by Theorem 4.4.3 we see immediately that $\binom{4}{0} + \binom{4}{1} + 4 = 11$ of these cases can discarded immediately as admitting no stable representations. We reason as follows. There are exactly three non labelled connected bipartite graphs with 4 nodes [19]:

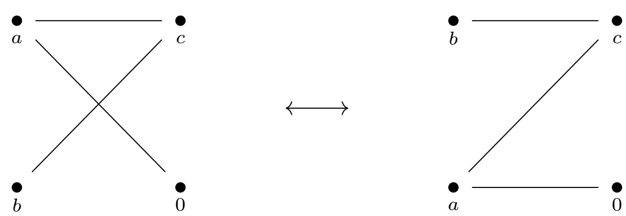


Of these three graphs only two, graphs 2 and 3, are of type (2,2). Thus, the first may be discarded. Graph 2 has a total of three edges, and thus is really represented by a family of four labelled cases as we will illustrate below:

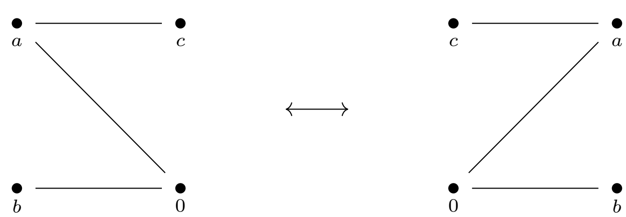
1.



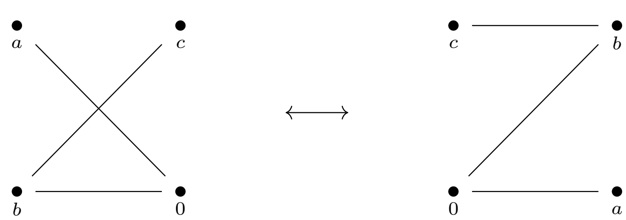
2.



3.



4.



As a, b, and c are arbitrary it should now be apparent that although their are 4 labelled graphs of this type, (in)stability in any one of the cases implies (in)stability in all. Thus, let us simply consider case 1 whose adjacency matrix is given by $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} \end{bmatrix}$ and therefore

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \phi_{11} & \phi_{12} & 0 & 0 \\ 0 & \phi_{22} & 0 & 0 \end{bmatrix}.$$

Let us now examine the invariant bundles in this case and move towards classifying stability.

- If $U_1 = \mathcal{O}(c)$ then $\Phi(U_1) = 0 \subseteq U_1 \otimes L$. Thus, in order for U_1 to satisfy the slope condition we require

$$\frac{\deg U_1}{\text{rk} U_1} = \frac{c}{1} < \frac{a+b+c}{4}$$

  which is only satisfied when $a + b - 3c > 0$.

- If $U_2 = \mathcal{O}$ then $\Phi(U_2) = 0 \subseteq U_2 \otimes L$. Thus, in order for U_2 to satisfy the slope condition we require

$$\frac{\deg U_2}{\text{rk} U_2} = \frac{0}{1} < \frac{a+b+c}{4}$$

  which is only satisfied when $a + b + c > 0$.

- If $U_3 = F$ then $\Phi(U_3) = 0 \subseteq U_3 \otimes L$. Then in order for U_3 to satisfy the slope condition we require

$$\frac{\deg U_3}{\text{rk} U_3} = \frac{c+0}{2} < \frac{a+b+c}{4}$$

  which is only satisfied when $a + b - c > 0$.

- If $U_4 = \mathcal{O}(a) + \mathcal{O}(c)$ then $\Phi(U_4) =\subseteq U_4 \otimes L$ as ϕ_{11} maps $\mathcal{O}(a) \to \mathcal{O}(c) \otimes L$. Thus, in order for U_4 to satisfy the slope condition we require

$$\frac{\deg U_4}{\text{rk} U_4} = \frac{a+c}{2} < \frac{a+b+c}{4}$$

  which is only satisfied when $-a + b - c > 0$.

- If $U_5 = \mathcal{O}(a) + F$ then $\Phi(U_5) =\subseteq U_5 \otimes L$ as ϕ_{11} maps $\mathcal{O}(a) \to \mathcal{O}(c) \otimes L$. Thus, in order for U_5 to satisfy the slope condition we require

$$\frac{\deg U_5}{\text{rk} U_5} = \frac{a+c+0}{3} < \frac{a+b+c}{4}$$

  which is only satisfied when $-a + 3b - c > 0$.

- If $U_6 = \mathcal{O}(b) + F$ then $\Phi(U_6) =\subseteq U_6 \otimes L$ as ϕ_{12} maps $\mathcal{O}(b) \to \mathcal{O}(c) \otimes L$ and ϕ_{22} maps $\mathcal{O}(b) \to \mathcal{O} \otimes L$. Thus, in order for U_6 to satisfy the slope condition we require

$$\frac{\deg U_6}{\text{rk} U_6} = \frac{b+c+0}{3} < \frac{a+b+c}{4}$$

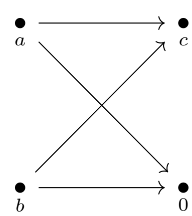
  which is only satisfied when $3a - b - c > 0$.

It is important to recall that $\phi : E \to F$ is mapping a rank two bundle into a rank two bundle. Thus, $\phi_{11} \neq 0$ and $\phi_{22} \neq 0$ implies ϕ embeds a subbundle into both $\mathcal{O}(c)$ and $\mathcal{O}$. However, there still exists the possibility for an embedded subbundle in the kernel of $E \oplus F$ which could be of either rank zero or rank one. Regardless of this ambiguity, it is clear that the following linear system is satisfied:

$$\begin{aligned} a+b-3c &> 0 \\ a+b+c &> 0 \\ a+b-c &> 0 \\ -a+b-c &> 0 \\ -a+3b-c &> 0 \\ 3a-b-c &> 0. \end{aligned}$$

There are of course many solutions to this system. Python code has been written using Google's OR-tools to provide solutions to this system and is given in Appendix B.

We now wish to study the representation with adjacency matrix $\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$ corresponding to



It is perhaps useful to observe that because this bipartite graph is complete and symmetric there is exactly one labelled and one non-labelled graph of this shape. This of course corresponds directly to the fact there is only one 2 by 2 adjacency matrix with four non zero entries.

Let us now examine the invariant bundles in this case and move towards classifying stability.

- If $U_1 = \mathcal{O}(c)$ then $\Phi(U_1) = 0 \subseteq U_1 \otimes L$. Thus, in order for U_1 to satisfy the slope condition we require

$$\frac{\deg U_1}{\mathrm{rk} U_1} = \frac{c}{1} < \frac{a+b+c}{4}$$

  which is only satisfied when $a+b-3c > 0$.

- If $U_2 = \mathcal{O}$ then $\Phi(U_2) = 0 \subseteq U_2 \otimes L$. Thus, in order for U_2 to satisfy the slope condition we require

$$\frac{\deg U_2}{\mathrm{rk} U_2} = \frac{0}{1} < \frac{a+b+c}{4}$$

  which is only satisfied when $a+b+c > 0$.

- If $U_3 = F$ then $\Phi(U_3) = 0 \subseteq U_3 \otimes L$. Then in order for U_3 to satisfy the slope condition we require

$$\frac{\deg U_3}{\mathrm{rk} U_3} = \frac{c+0}{2} < \frac{a+b+c}{4}$$

which is only satisfied when $a + b - c > 0$.

- If $U_4 = \mathcal{O}(a) + F$ then $\Phi(U_4) = \subseteq U_4 \otimes L$ as ϕ_{11} maps $\mathcal{O}(a) \to \mathcal{O}(c) \otimes L$. Thus, in order for U_4 to satisfy the slope condition we require

$$\frac{\deg U_5}{\mathrm{rk} U_5} = \frac{a+c+0}{3} < \frac{a+b+c}{4}$$

which is only satisfied when $-a + 3b - c > 0$.

- If $U_5 = \mathcal{O}(b) + F$ then $\Phi(U_5) = \subseteq U_5 \otimes L$ as ϕ_{12} maps $\mathcal{O}(b) \to \mathcal{O}(c) \otimes L$ and ϕ_{22} maps $\mathcal{O}(b) \to \mathcal{O} \otimes L$. Thus, in order for U_5 to satisfy the slope condition we require

$$\frac{\deg U_6}{\mathrm{rk} U_6} = \frac{b+c+0}{3} < \frac{a+b+c}{4}$$

which is only satisfied when $3a - b - c > 0$.

Then by the same argument above, our linear system is given by:

$$\begin{aligned} a + b - 3c &> 0 \\ a + b + c &> 0 \\ a + b - c &> 0 \\ -a + 3b - c &> 0 \\ 3a - b - c &> 0. \end{aligned}$$

There are again many solutions to this system. The code of Appendix B applies to this system, too. □

Let us examine the induced ordinary quivers of the general (2,2) case using the results of Theorem 4.5.1. Consider first the induced map of global sections $\tilde{\phi} : H^0(X, E) \to H^0(X, F \otimes L)$. This induced map $\tilde{\phi}$ corresponds to the quiver

$$\underset{\mathbb{C}^{a+b+2}}{\bullet} \longrightarrow \underset{\mathbb{C}^{c+2(t+1)}}{\bullet}.$$

Recall that in the (2,1) case we where able to use the stability condition found in Theorem 4.3.1 containing t to provide a meaningful restraint on the kernel of $\tilde{\phi}$. But Theorem 4.5.1 has no conditions on t and therefore is quite different in structure to the system presented in Theorem 4.3.1. Fortunately, the system(s) found in Theorem 4.5.1 does still say something about $\tilde{\phi}$. In both systems we have that $a + b - c > 0$ or equivalently

$a+b+2 > c+2$. An important consequence of the linear system presented in Theorem 4.5.1 is that it implies $a \geq 0$ and $b \geq 0$. Therefore we must have $a+b+2 \geq c+3$. When $t=1$, we have the quiver

$$\underset{\mathbb{C}^{a+b+2}}{\bullet} \longrightarrow \underset{\mathbb{C}^{c+4}}{\bullet}$$

and therefore as $a+b+2 \geq c+3$ we see that $\tilde{\phi}$ can increase in rank by at most one. In general we see this stability condition is manifested as $\tilde{\phi}$ increasing rank by at most $2t-1$.

Next, let us consider the quiver induced by the natural restriction of ϕ to a point $p \in \mathbb{P}^1$. Then

$$\begin{aligned}\phi \in \operatorname{Hom}\left(\mathcal{O}\left(a\right) \oplus \mathcal{O}\left(b\right), \mathcal{O}\left(c+t\right) \oplus \mathcal{O}\left(t\right)\right) &\cong H^0\left(\mathbb{P}^1, \left(\mathcal{O}\left(a\right) \oplus \mathcal{O}\left(b\right)\right) \otimes \left(\mathcal{O}\left(c+t\right) \oplus \mathcal{O}\left(t\right)\right)\right) \\ &\cong H^0\left(\mathbb{P}^1, \left(\mathcal{O}\left(a+c+t\right) \oplus \mathcal{O}\left(a+t\right)\right) \oplus \left(\mathcal{O}\left(b+c+t\right) \oplus \mathcal{O}\left(b+t\right)\right)\right)\end{aligned}$$

So ϕ locally looks like a tuple of polynomials (f_1, f_2, f_3, f_4) where f_1 is of degree $a+c+t$, f_2 is of degree $a+t$, f_3 is of degree $b+c+t$, and f_4 is of degree $b+t$. As, the linear program in Theorem 4.5.1 has no conditions on t it is difficult in this case to say anything concrete about this induced quiver other then stability requires f_2 and f_4 to both have positive degree.

5 Future Directions

We dedicate the final chapter of this thesis to speculating on future directions of this research. We provide a list of both some immediate and longer term objectives and ideas that warrant future study.

1. The ultimate culmination of this research would be a full classification of the stability conditions for a twisted representation of a type (n, m) quiver over $X = \mathbb{P}^1$ including understanding how the stability conditions are manifested in the induced quiver representations. If this can be achieved then, as motivated by [15], a natural next step would be to extend these results to more generalized holomorphic chains of the form $(n_1, m, n_2, m, \cdots, n_j, m)$.

2. Apart from the fact that Theorem 4.4.3 is necessary but insufficient for characterizing stability, it is clear that the techniques employed in this thesis do not scale well with the general problem. Indeed, as an example, there are 4032 unlabelled connected bipartite graphs with 10 nodes [19]. It would be exhaustive to develop unique linear programs for each of these graphs. We speculate that to classify stability conditions completely for a type (m, n) quiver, a much more sophisticated algebraic characterization of stability will need to be developed. In particular, it may an incomplete approach to only consider ϕ as an adjacency matrix (and thus thinking about each entry as zero or nonzero), Instead, we may need to classify these morphisms ϕ in a different manner. This is primarily due to the potential for embedded bundles when E and F are both of higher rank. These embedded bundles will lead to conditions for stability, but they cannot be observed directly by only considering the quiver representation of the bundle. We note that promising motivic methods are developed in [2] for extracting topological invariants of spaces of stable holomorphic chains and are worth investigating here.

3. An obtainable next step towards further understanding the general problem would be to study the general $(3, 2)$ case. Indeed, there are only 5 unlabelled connected bipartite graphs with 5 nodes of which only three are of type $(3, 2)$. Thus, only three linear programs would need to be developed to study this case in general. Furthermore, this situation is advantageous, particularly in the case corresponding to the complete bipartite graph, as in this case ϕ is guaranteed to drop rank allowing us to have a condition on t. The stability conditions for the general $(3, 2)$ quiver would be the first example of a non-argyle representation that in certain cases must drop rank and thus should provide great insight as to whether the results of the general $(2, 1)$ quiver can be extended to more complex cases. In particular, in both the $(2, 1)$ and $(2, 2)$ case we see that stability imposes restrictions on the induced map of global sections $\tilde{\phi}$. By studying the $(3, 2)$ case we hope to understand better how exactly

these conditions are manifested in order to develop a more general theorem.

4. It is not fully understood as to what role the path algebra plays in the stability of twisted quiver representations. In particular, with the appropriate transformations, Theorem 4.4.3 guarantees that a single connected path algebra must exist for each quiver admitting stable twisted representations. In our work here, we have observed instances where the path algebra is not enough to determine stability. This connects to the issue of embedded invariant subbundles that are not "read off" in a simple way from the nodes of the graph, and to the inadequacy of a matrix representative for ϕ. Still, finer properties of the path algebra may interact with stability. For the (2,2) case we examined earlier, it can be shown that the three edge stable bundle has a non-cyclic path algebra. However, in the case of the 4 edge or complete graph, it can be shown that this graph has a cyclic path algebra. We conjecture that the characterizations of the path algebra will play an important role to further understand stability.

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Appendix A

Python Code for (2,1) Quiver on $\mathbb{P}^1$

```
from ortools.sat.python import cp_model


class VarArraySolutionPrinter(cp_model.CpSolverSolutionCallback):
    """Print intermediate solutions."""

    def __init__(self, variables):
        cp_model.CpSolverSolutionCallback.__init__(self)
        self.__variables = variables
        self.__solution_count = 0

    def on_solution_callback(self):
        self.__solution_count += 1
        for v in self.__variables:
            print('%s=%i' % (v, self.Value(v)), end=' ')
        print()

    def solution_count(self):
        return self.__solution_count


def SearchForAllSolutionsSampleSat():
    """Showcases calling the solver to search for all solutions."""
    # Creates the model.
    model = cp_model.CpModel()

    # Creates the variables.
    num_vals = 200
    a = model.NewIntVar(-200, num_vals - 1, 'a')
    b = model.NewIntVar(-200, num_vals - 1, 'b')
    #c = model.NewIntVar(-200, num_vals - 1, 'c')
    #t = model.NewIntVar(-200, num_vals - 1, 't')

    # Creates the constraints.
    model.Add(-a - b- 2 < 0)
    model.Add(a - 2*b - 1 < 0)
    model.Add(b - 2*a - 1 < 0)
    #model.Add(2*a + 2*b - 3*t +4 < 0)
    #model.Add(a < 0)
    #model.Add(b < 0)



    # Create a solver and solve.
    solver = cp_model.CpSolver()
    solution_printer = VarArraySolutionPrinter([a, b])
    status = solver.SearchForAllSolutions(model, solution_printer)

    print('Status = %s' % solver.StatusName(status))
    print('Number of solutions found: %i' % solution_printer.solution_count())


SearchForAllSolutionsSampleSat()
```

Appendix B

Python Code for (2,2) Quiver on $\mathbb{P}^1$

```python
from ortools.sat.python import cp_model


class VarArraySolutionPrinter(cp_model.CpSolverSolutionCallback):
    """Print intermediate solutions."""

    def __init__(self, variables):
        cp_model.CpSolverSolutionCallback.__init__(self)
        self.__variables = variables
        self.__solution_count = 0

    def on_solution_callback(self):
        self.__solution_count += 1
        for v in self.__variables:
            print('%s=%i' % (v, self.Value(v)), end=' ')
        print()

    def solution_count(self):
        return self.__solution_count


def SearchForAllSolutionsSampleSat():
    """Showcases calling the solver to search for all solutions."""
    # Creates the model.
    model = cp_model.CpModel()

    # Creates the variables.
    num_vals = 200
    a = model.NewIntVar(-200, num_vals - 1, 'a')
    b = model.NewIntVar(-200, num_vals - 1, 'b')
    c = model.NewIntVar(-200, num_vals - 1, 'c')
    #t = model.NewIntVar(-20, num_vals - 1, 't')

    # Creates the constraints.
    model.Add(a + b- 3*c > 0)
    model.Add(a + b + c > 0)
    model.Add(a + b - c > 0)
    model.Add(-a + b - c > 0)
    model.Add(-a + 3*b - c > 0)
    model.Add(3*a - b - c > 0)
    #model.Add(a < 0)
    #model.Add(b < 0)
    #model.Add(c < 0)


    # Create a solver and solve.
    solver = cp_model.CpSolver()
    solution_printer = VarArraySolutionPrinter([a, b, c])
    status = solver.SearchForAllSolutions(model, solution_printer)

    print('Status = %s' % solver.StatusName(status))
    print('Number of solutions found: %i' % solution_printer.solution_count())


SearchForAllSolutionsSampleSat()
```