Cubulating CAT(0) groups and Property (T) in random groups

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

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This thesis considers two properties important to many areas of mathematics: those of *cubulation* and *Property (T)*. Cubulation played a central role in Agol's proof of the virtual Haken conjecture, while Property (T) has had an impact on areas such as group theory, ergodic theory, and expander graphs. The aim is to cubulate some examples of groups known in the literature, and prove that many 'generic' groups have Property (T). Graphs will be central objects of study throughout this text, and so in Chapter 2 we provide some definitions and note some results. In Chapter 3, we provide a condition on the links of polygonal complexes that allows us to cubulate groups acting properly discontinuously and cocompactly on such complexes. If the group is hyperbolic then this action is also cocompact, hence by Agol's Theorem the group is virtually special (in the sense of Haglund–Wise); in particular it is linear over $\mathbb{Z}$. We consider some applications of this work. Firstly, we consider the groups classified by [KV10] and [CKV12], which act simply transitively on $CAT(0)$ triangular complexes with the minimal generalized quadrangle as their links, proving that these groups are virtually special. We further apply this theorem by considering generalized triangle groups, in particular a subset of those considered by [CCKW20].

To analyse Property (T) in generic groups, we first need to understand the eigenvalues of some random graphs: this is the content of Chapter 4, in which we analyse the eigenvalues of Erdös–Rényi random bipartite graphs. In particular, we consider p satisfying $m_1 p = \Omega(\log m_2)$, and let $G \sim G(m_1, m_2, p)$. We show that with probability tending to 1 as m_1 tends to infinity: $\mu_2(A(G)) \leq O(\sqrt{m_2 p})$.

In Chapter 5 we study Property (T) in the $\Gamma(n, k, d)$ model of random groups: as k tends to infinity this gives the Gromov density model, introduced in [Gro93]. We provide bounds for Property (T) in the k-angular model of random groups, i.e. the $\Gamma(n, k, d)$ model where k is fixed and n tends to infinity. We also prove that for $d > 1/3$, a random group in the $\Gamma(n, k, d)$ model has Property (T) with probability tending to 1 as k tends to infinity, strengthening the results of Żuk and Kotowski–Kotowski, who consider only groups in the $\Gamma(n, 3k, d)$ model.

For my friends; I cannot describe the complexity of feeling for all the people I have met, so will let the following do it for me: *"We cannot tell the precise moment when friendship is formed. As in filling a vessel drop by drop, there is at last a drop which makes it run over; so in a series of kindnesses there is at last one which makes the heart run over." — Ray Bradbury*

For Dr. John Callender, without whom I would not be here.

For Granny Pat and Grandpa George, for love and care, for helping to instill a joy of learning, and countless plates of cheese on toast.

For my parents, for absolutely everything.

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Chapter 1

Introduction

Gromov famously stated that group properties typically fall into two categories: *flexibility* properties and *rigidity* properties. Flexibility properties are those which guarantee that a group has many different attributes, whereas rigidity properties greatly restrict the traits a group can possess. The notions of being *cubulable*, i.e. having a properly discontinuous action on a $CAT(0)$ cube complex, and of having *Property (T)* lie on different ends of this spectrum.

Indeed, being cubulable is an example of a 'flexibility' property. If a group is cubulable, then it is a-(T)-menable (see e.g. [CMV04]), which is a strong negation of having Property (T), and guarantees a proper continuous affine action on a real Hilbert space. If we consider virtually special hyperbolic groups (a subclass of cubulable groups), then these are, for example, large in the sense of Pride and residually finite; they are even linear over $\mathbb{Z}$ and QCERF [HW08]. Famously, virtually special groups played a key role in Agol's proof of the virtual Haken conjecture [Ago13]: the fundamental group of a closed hyperbolic 3-manifold is cubulable (with a cocompact action) [KM12, BW12], and so by [Ago13] is virtually special.

Property (T), however, is an example of a 'rigidity' property: for a countable discrete group it is, for instance, equivalent to every continuous affine isometric action on a real Hilbert space having a fixed point, or the vanishing of $H^1(G, \pi)$ for every unitary representation π. It is well known that many lattices in Lie groups and buildings have Property (T); see e.g. [Kaž67]. Originally introduced by Každan in 1967 [Kaž67], Property (T) has become an extremely important property, not only with regards to group theory, but also geometry, ergodic theory, and many other areas. Indeed, Každan used Property (T) to prove that lattices in many Lie groups are finitely generated [Kaž67], and Margulis used Property (T) in $SL(n, \mathbb{Z})$ to construct families of *expander graphs* [Mar73] (see [GG81] for further analysis of the constants of expansion involved).

Given a property of groups, $\mathcal{P}$, there are two natural questions to ask.

Question 1. *Given a specific group G, can one determine if G has $\mathcal{P}$?*

Note that typically the property $\mathcal{P}$ will be undecidable, and so we will be looking for sufficient conditions to apply in practice.

Question 2. *How common a property is $\mathcal{P}$, i.e. does a generic group have $\mathcal{P}$?*

Both of these questions have some answers in regards to being cubulable or having Property (T). Let us consider the first of the two questions. Given a group G acting properly discontinuously and cocompactly on a simply connected triangular complex X, if $\lambda_1(Lk_X(v)) > 1/2$ for every vertex v of X, then G has Property (T) [Ż96, BS97] (c.f. [Gar73], [Pan98], [Wan98], [Opp15]). If X is instead a $CAT(0)$ polygonal complex, and the edges of the links can be partitioned into 'separated' cutsets, then G is cubulable [HW14, Example 4.3]. Żuk's criterion has been fruitfully applied in many examples: by contrast, Hruska–Wise's criterion is rarely applicable, as we will discover.

The purpose of the first half of this thesis, contained in Chapter 3 and comprising the content of [Ash20], is to improve Hruska–Wise's criterion, and to provide a condition on the links of polygonal complexes that is sufficient to ensure groups acting properly discontinuously and cocompactly on such complexes act properly discontinuously on a $CAT(0)$ cube complex. These conditions recover [HW14, Example 4.3], and are composed of two requirements: firstly, we must be able to find *separated* cut sets in the links of our polygonal complexes, and then we must be able to assign weights to these cutsets that satisfy the *weight equations*. The reader should consult Section 3.2.1 for the full and rigorous definitions.

Theorem A. *Let G be a group acting properly discontinuously and cocompactly on a simply connected $CAT(0)$ polygonal complex X.*

(i) If $G\backslash X$ is gluably weakly π-separated, then G contains a virtually-free codimension-1 subgroup (and therefore does not have Property (T)).

(ii) If $G\backslash X$ is gluably π-separated, then G acts properly discontinuously on a $CAT(0)$ cube complex. If, in addition, G is hyperbolic, then this action is cocompact. In particular, if G is hyperbolic, then it is virtually special, and so linear over $\mathbb{Z}$.

We apply this theorem to several examples in the literature. Firstly, we consider the groups classified by [KV10] and [CKV12], which act simply transitively on $CAT(0)$

triangular complexes with the minimal generalized quadrangle as their links, proving that these groups are virtually special. There is little known about these groups: until now they were not even known to be residually finite. We apply Theorem A to these groups to deduce that they are virtually special.

Corollary B. *Let X be a simply connected polygonal complex such that every face has at least 3 sides, and the link of every vertex is isomorphic to the minimal generalized quadrangle. If a group G acts properly discontinuously and cocompactly on X, then it is virtually special; in particular it is linear over $\mathbb{Z}$.*

The full automorphism groups of Kac–Moody buildings of 2-spherical type of large thickness have Property (T) [DJ02, ER18]: neither [DJ02] nor [ER18] record whether Property (T) fails at small thicknesses. Some of the groups considered in Corollary B are cocompact lattices in a 2-spherical Kac–Moody building with small thickness [CKV12]. Therefore, Corollary B complements [DJ02, ER18], providing an example of the failure of Property (T) in such buildings when the thickness is small: we believe this to be the first such example.

Our final application of Theorem A is to the new census of 252 generalized triangle groups introduced in [CCKW20]. Let $C_{k,2}$ be the cage graph on k edges, i.e. the smallest k regular graph of girth 2. For finite-sheeted covering graphs $\Gamma_i \looparrowright C_{k,2}$, we consider an associated pair of families of triangular complexes of groups $D^j_{0,k}(\Gamma_1, \Gamma_2, \Gamma_3)$, and $D^j_k(\Gamma_1, \Gamma_2, \Gamma_3)$ (see Definitions 3.5.1 and 3.5.3). We remark that these complexes of groups are not necessarily unique for given $\Gamma_1, \Gamma_2, \Gamma_3$.

We consider explicitly the graphs used in [CCKW20]: we refer to them by their Foster Census names (see [Fos88]). The only graph not in the Foster Census is $G54$, the Gray graph, which is edge, but not vertex, transitive. We provide a way to cubulate generalized triangle groups by considering the graph Γ_1 alone in Theorem 3.5.5. Using Theorem A and Theorem 3.5.5, we can deduce the following.

Corollary C. *Let $\Gamma_i \looparrowright C_{k,2}$ be finite-sheeted covers, such that girth$(\Gamma_i) \geq 6$ for each i. Let $G = \pi_1(D^j_{0,k}(\Gamma_1, \Gamma_2, \Gamma_3))$ or $G = \pi_1(D^j_k(\Gamma_1, \Gamma_2, \Gamma_3))$ for some j.*

(i) If $\Gamma_i \in \{F24A,\ F26A,\ F48A\}$ for each i, then G acts properly discontinuously on a $CAT(0)$ cube complex: if G is hyperbolic, then this action is also cocompact and so G is virtually special.

(ii) If $\Gamma_1 \in \{F40A, G54\}$, then G acts properly discontinuously on a $CAT(0)$ cube complex: if G is hyperbolic, then this action is also cocompact and so G is virtually special.

There are 252 groups considered in [CCKW20], of which they show that 168 do not satisfy Property (T). Our method recovers this result for 101 groups, and proves that 30 new groups do not have Property (T). We prove that each of the 131 groups we consider has a proper action on a $CAT(0)$ cube complex, and so, by e.g. [CMV04], is a-(T)-menable. Furthermore, 125 of these groups are hyperbolic and have a proper and cocompact action on a $CAT(0)$ cube complex, and hence by [Ago13] are virtually special.

If we instead consider the second of our two questions, then we must describe what is meant by a 'generic' group: for the purpose of this thesis, a generic group will be described by a choice from a family of models $\{\Gamma(n,k,d)\}_{(n,k,d)}$ depending on 3 parameters: n, the number of generators; k, the length of the relators; and d, the *density* of the relators. Before we discuss the results of the second half of this thesis, perhaps we should stop to ask: why study randomness, or the notion of a 'generic' object, in mathematics? There are two obvious motivating reasons. Firstly, it allows one to construct 'exotic' objects by proving they exist with non-zero probability, removing the need to actually construct them. A classical example is the use of Baire-Category theory to prove the existence of everywhere differentiable, nowhere monotone functions [Wei76]. In the other direction, models of randomness provide both a 'testing ground' for conjectures, as well as give an idea of how common a given property is: for example, almost all d-regular graphs on large sets of vertices are close to optimal expanders [Fri08].

Gromov proposed two models of random groups in [Gro93] to study the notion of a 'generic' finitely presented group. There is some ambiguity in the literature between the two models, and so we provide the full definitions here. Fix $n \geq 2, k \geq 3$, and $0 < d < 1$ (remember that we call d the *density*). The (*strict*) (n,k,d) *model* is obtained as follows. Let $A_n = \{a_1, \ldots, a_n\}$, and let $F_n := \mathbb{F}(A_n)$ be the free group generated by A_n. Let $\mathcal{C}(n,k)$ be the set of cyclically reduced words of length k in F_n (so that $\mathcal{C}(n,k) \approx (2n-1)^k$). Uniformly randomly select a set $R \subseteq \mathcal{C}(n,k)$ of size $|R| = (2n-1)^{kd}$ (we in fact take $|R| = \lfloor (2n-1)^{kd} \rfloor$, but since we are dealing with asymptotics this does not change any of the arguments), and let $\Gamma := \langle A_n \mid R \rangle$. We call Γ a *random group* in the (*strict*) (n,k,d) *model*, and write $\Gamma \sim \Gamma(n,k,d)$. If we keep n fixed and let k tend to infinity, then we obtain the *Gromov density model*, as introduced in [Gro93], whereas if we fix k and let n tend to infinity we obtain the *k-angular model*, as introduced in [ARD20]. The k-angular model was first studied for $k = 3$ (the *triangular model*) by Żuk in [Ż03] and for $k = 4$ (the *square model*) by Odrzygóźdź in [Odr16].

The *lax* (n, k, d) *model* is obtained via the following procedure. Let $\mathcal{C}(n, k, f)$ be the set of cyclically reduced words of length between $k - f(k)$ and $k + f(k)$ in F_n, where $f(k) = o(k)$. Uniformly randomly select a set $R \subseteq \mathcal{C}(n, k, f)$ of size $|R| = (2n-1)^{kd}$, and let $\Gamma := \langle A_n \mid R \rangle$. We call Γ a *random group* in the *lax* (n, k, d, f) *model*, and write $\Gamma \sim \Gamma_{lax}(n, k, d, f)$. We often drop reference to the function f and simply write $\Gamma \sim \Gamma_{lax}(n, k, d)$, as in many applications the choice of function has no effect on the conclusions.

Random groups exhibit many interesting properties, depending on the density chosen. All of the following statements hold asymptotically almost surely, i.e. with probability tending to 1. Firstly, a random group in the density model at density $d < 1/2$ is hyperbolic and torsion-free, while for $d > 1/2$ it is a quotient of $\mathbb{Z}_2$ [Gro93] (c.f. [Oll04]). In fact, for $d < 1/2$ *every* reduced van Kampen diagram for the group satisfies a linear isoperimetric inequality [Oll04]. This argument also transfers to the k-angular model [ARD20]: see [Odr16] for the case of $k = 4$, as well as a generalisation of the argument to a wider class of diagrams.

It is a seminal theorem of Żuk [Ż03] (c.f. [KK13]) that for $d > 1/3$ a random group in the triangular model has Property (T) with probability tending to 1 (see [ALuS15] for a further analysis of $d \to 1/3$), and furthermore, that with probability tending to 1 as k tends to infinity, a random group in the $\Gamma(n, 3k, d)$ model has Property (T) for any $d > 1/3$. For any choice of $f = O(1)$, this immediately implies that a random group in the lax (n, k, d, f) density model at density $d > 1/3$ has Property (T) [Ż03, KK13].

Groups in the density model are virtually special for $d < 1/6$ [OW11] and contain a free codimension-1 subgroup for $d < 5/24$ [MP15]. As observed in [Odr19] this implies that for any $k \geq 3$, a random group in the k-angular model at density $d < 5/24$ does not have Property (T). Groups in the triangular model are free at densities less than 1/3 [ALuS15], groups in the square model are free at densities less than 1/4 [Odr16], and groups in the k-angular model are free for $d < 1/k$ [ARD20]. Furthermore, groups in the square model are virtually special for $d < 1/3$ [Duo17, Odr18] and contain a codimension-1 subgroup for $d < 3/8$ [Odr19]. Finally, groups in the hexagonal model contain a codimension-1 subgroup for $d < 1/3$ and have Property (T) for $d > 1/3$ [Odr19]: in fact any group in the $3k$-angular model has Property (T) for $d > 1/3$ [Mon21].

It is perhaps interesting to note that the above results display one of the most fascinating behaviours of random groups: the *sharp phase transition*. Indeed, for many properties $\mathcal{P}$ there is a density $d_{\mathcal{P}}$ such that for $d < d_{\mathcal{P}}$ a random group has $\mathcal{P}$ with probability tending to 1, while for $d > d_{\mathcal{P}}$ a random group does not have $\mathcal{P}$

with probability tending to 1. This is an example of a more general phenomenon: for *decreasing* properties, i.e. ones preserved by removing relators, a threshold function τ exists. I.e. if $(2n-1)^{kd} = o(\tau)$, then a random group at density d has $\mathcal{P}$, while if $(2n-1)^{kd} = \Omega(\tau)$, then a random group at density d does not have $\mathcal{P}$. The proof of this fact can be deduced similarly to the proof of the existence sharp phase transitions for random graphs: see e.g. [FK16, Theorem 1.7].

The purpose of the second half of this thesis is to analyse Property (T) in random groups: showing that Property (T) is 'generic' in many choices of models.

In order to do this, we take a detour in Chapter 4 to prove a result on the eigenvalues of Erdös–Rényi random bipartite graphs (the below is not the full result, but we omit some further technical results). This chapter is formed from [Ash21a].

Theorem D. *Let $m_1 \geq 1$, $m_2 = m_2(m_1) \geq m_1$, and $p = p(m_1)$ be such that $m_1 p = \Omega(\log m_2)$. Then with probability tending to 1 as m_1 tends to infinity:*

$$\mu_2(A(G(m_1, m_2, p))) \leq O(\sqrt{m_2 p}), \text{ and } \max_{i \neq 1, m_1+m_2} \left| \mu_i(G(m_1, m_2, p)) - 1 \right| = o(1).$$

We will apply this in Chapter 5 to study Property (T) in random groups, the purpose of which is to prove the following two theorems, which originate in [Ash21b].

For $k \geq 3$, let $d_k := [k + (-k \mod 3)]/3k$, i.e.

$$d_k = \begin{cases} \frac{1}{3} \text{ if } k = 0 \mod 3, \\ \frac{k+2}{3k} \text{ if } k = 1 \mod 3, \\ \frac{k+1}{3k} \text{ if } k = 2 \mod 3. \end{cases}$$

Theorem E. *Let $k \geq 8$, $d > d_k$, and let $\Gamma_m \sim \Gamma(m, k, d)$. Then*

$$\lim_{m \to \infty} \mathbb{P}(\Gamma_m \text{ has Property (T)}) = 1.$$

We believe this to be the first result on Property (T) in any k-angular model for any k not divisible by 3, and in fact provides bounds for Property (T) in the k-agonal model for each $k \geq 8$. Indeed the result for $k = 0 \mod 3$ follows by the work of [Ż03, KK13] and [Mon21]. For $k = 6$ this bound agrees with the result of [Odr19]. Unfortunately, for $k = 4$ we have that $d_k = 1/2$; for $d > d_k$ a random group in the square model is trivial [Odr16], and so we obtain no new information. Therefore the only cases we cannot deal with are $k = 5, 7$. Indeed it may be possible to extend the above to simply require $k \geq 3$, but this would require further examination of double edges in certain graphs. The below figure demonstrates various density bounds for the k-angular model (by trivial we mean a quotient of $\mathbb{Z}_k$).

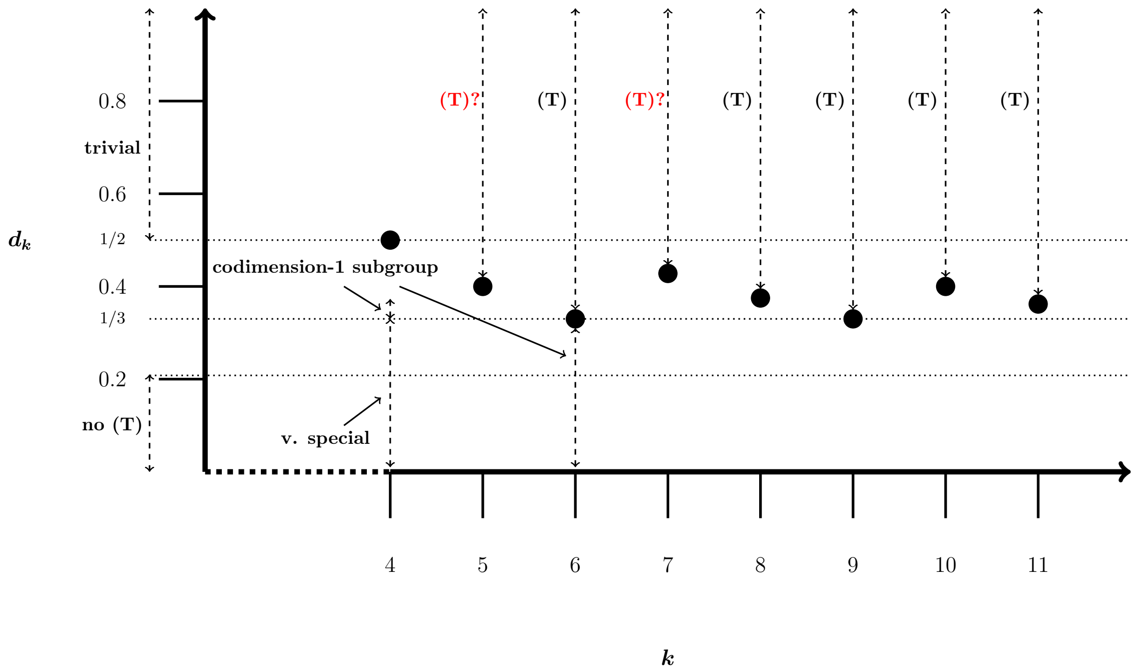

Fig. 1.1 Density bounds for the k-angular model

Secondly, we can consider the density model. The following completes the analysis of Property (T) in $\Gamma(n, k, d)$ for $d > 1/3$.

Theorem F. *Let $n \geq 2$, $d > 1/3$, and let $\Gamma_k \sim \Gamma(n, k, d)$. Then:*

$$\lim_{k \to \infty} \mathbb{P}(\Gamma_k \textit{ has Property } (T)) = 1.$$

Note that this immediately implies for any infinite sequence, $\{k_i\}_i$, of increasing positive integers, and $\Gamma_i \sim \Gamma(n, k_i, d)$, that $\lim_{i\to\infty} \mathbb{P}(\Gamma_i \text{ has Property } (T))$, so that we have strengthened the results of [Ż03, KK13].

Chapter 2

A graph theory primer

Graphs will be central objects of study throughout this text, and so in this chapter we will provide some definitions and note some results: see, e.g. [Ser77], [Chu97], and [Bol98] for further reading. The reader should observe that our notation for graphs changes between Chapter 3 and Chapters 4 and 5. The reason for this is the following. The first of these three chapters discusses discrete (often hyperbolic) finitely generated groups: it is common in the literature to use G to denote such a group, and so to align with the literature and those readers familiar with the area, we use the letter G for a group and Γ for a graph. However, in graph theory it is more common to use G to denote a graph. Furthermore, the study of Property (T) often considers lattices in Lie groups, for which the letter Γ is used: therefore in Chapters 4 and 5 we will use the letter Γ for a group and G for a graph.

2.1 Graphs and metric graphs

We will use Serre's definition of graphs, as introduced in [Ser77].

Definition 2.1.1. A *graph* is a tuple $G = (V, E, \iota, ^{-})$, where:

i) V is the set of *vertices*,

ii) E is the set of *edges*,

iii) $\iota : E \to V$ is the *initial vertex map*: we call $\iota(e)$ the *origin* or *initial vertex* of e,

iv) $^{-} : E \to E$ is a fixed point free involution.

We can define $\tau : E \to V$ by $\tau(e) := \iota(\overline{e})$, the *terminus map.*

An *orientation* of G is a choice of partition $E = E^+ \sqcup E^-$ such that $\overline{E^+} = E^-$: we call an edge e *positively (negatively) orientated* if $e \in E^+$ ($e \in E^-$).

We call a graph *bipartite* if there exists a partition $V = V_1 \sqcup V_2$ such that for any edge e, either $\iota(e) \in V_1,\ \tau(e) \in V_2$ or $\tau(e) \in V_1,\ \iota(e) \in V_2$.

Definition 2.1.2. Given two graphs $G_1 = (V_1, E_1, \iota_1, {}^-)$, $G_1 = (V_2, E_2, \iota_2, {}^=)$, we define the *union* of G and G' as the graph $G \cup G' := (V \cup V', E \sqcup E', \iota, {}^\sim)$; we take the union of vertices, and the disjoint union of edge sets (i.e. we assume that different graphs have disjoint edge sets). Here we define, for $e \in E_i$, $\iota(e) = \iota_i(e)$, and

$$\tilde{e} = \begin{cases} \overline{e};\ e \in E_1, \\ \overline{\overline{e}};\ e \in E_2. \end{cases}$$

Graphs, as set up, appear to be purely combinatorial objects. However, we may apply topological methods to them, using the following.

Definition 2.1.3 (Realisation)**.** Let $G = (V, E, \iota, {}^-)$ be a graph. The *realisation* of G, $|G|$, is defined as the topological space

$$|G| := \left(V \sqcup \bigsqcup_{e \in E} e \times [0,1]\right) \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by $(e, 0) \sim \iota(e)$ and $(e, t) \sim (\overline{e}, 1 - t)$ for $e \in E,\ t \in [0, 1]$.

We can now define concepts like *connectedness*, etc of a graph as connectedness of $|G|$. Of course G uniquely defines $|G|$ and $|G|$ nearly defines G (aside from vertices of degree 2): therefore, we will typically refer to both G and $|G|$ as the graph G. To differentiate between the two, we will refer to an edge as *oriented* if we view it as an edge in G, not as its image in $|G|$. Furthermore, we typically write $e^{-1} := \overline{e}$.

We can also define a *metric graph* as a pair (G, d) where G is a graph, and d is a metric on $|G|$. Typically we will assign a graph G a metric by 'assigning each edge a length': rigorously, for each $e \in E^+$ we identify $e \times [0, 1]$ as being isometric to the interval $[0, \alpha_e]$ for some $\alpha_e > 0$ and then give $|G|$ the inherited path metric.

Importantly, we can also define maps of graphs. Firstly, for a vertex v of G we define the *link* of v as the set $Lk_G(v) = \{e \in E\ :\ \iota(e) = v\}$.

Definition 2.1.4. Let $G = (V, E, \iota, {}^-)$ and $G' = (V', E', \iota', {}^{-\prime})$ be graphs. A *map* from G to G' is a function $\phi : V \sqcup E \to V' \sqcup E'$ such that *i)* $\phi(V) \subseteq V'$, $\phi(E) \subseteq E'$,

ii) for each $e \in E$: $\overline{\phi(e)}' = \phi(\overline{e})$ and $\iota'(\phi(e)) = \phi(\iota(e))$.

For each vertex v, the map ϕ induces a set function $\phi_v^* : Lk_G(v) \to Lk_{G'}(\phi(v))$: we call the map an *immersion* if ϕ_v^* is injective for every v in V. A map ϕ is an *isomorphism* if ϕ is a bijective immersion.

2.2 The eigenvalues of graphs

Let $G = (V, E, \iota, ^-)$ be a graph with vertex set $V = \{v_1, \ldots, v_m\}$. It turns out that many interesting properties of G can be deduced from certain matrices associated to G. The *adjacency matrix* of G, $A(G)$, is the $m \times m$ matrix with $A(G)_{i,j}$ defined to be the number of edges between v_i and v_j, i.e. $A(G)_{i,j} = |\{e \in E \ : \ \iota(e) = v_i, \ \tau(e) = v_j\}|$. The *degree matrix* of G, $D(G)$, is the diagonal matrix with entries $D(G)_{i,i} = deg(v_i) := |\{e \in E \ : \ \iota(e) = v_i\}|$. The *normalised Laplacian* of G, $L(G)$, is defined by

$$L(G) = I - D^{-1/2} A D^{-1/2}.$$

We note that $L(G)$ is symmetric positive semi-definite, with eigenvalues

$$0 \leq \lambda_0(L(G)) \leq \lambda_1(L(G)) \leq \ldots \leq \lambda_{m-1}(L(G)) \leq 2.$$

For $i = 1, \ldots, m$, we define $\lambda_i(G) := \lambda_i(L(G))$. In particular, note that $\lambda_1(G) > 0$ if and only if the graph G is connected.

If M is a symmetric real $m \times m$ matrix, then M has real eigenvalues, which we order by $\lambda_0(M) \leq \lambda_1(M) \leq \ldots \leq \lambda_{m-1}(M)$. We define the reverse ordering of eigenvalues $\mu_1(M) \geq \mu_2(M) \geq \ldots \geq \mu_m(M)$, i.e. $\mu_i(M) = \lambda_{m-i}(M)$. Therefore we may also define $\mu_i(G) = \mu_i(L(G))$. The reason we introduce this ordering is that $\lambda_i(G)$ has a close connection to $\mu_i(A(G))$.

Remark 2.2.1. Let M be a symmetric $m \times m$ matrix. For $i = 1, \ldots, m$:

$$\mu_i(-M) = -\mu_{m+1-i}(M).$$

This follows as $\{\mu_i(-M) \ : \ 1 \leq i \leq m\} = \{-\mu_i(M) \ : \ 1 \leq i \leq m\}$, and $\mu_1(M) \geq \mu_2(M) \geq \ldots \geq \mu_m(M)$, so that $-\mu_1(M) \leq -\mu_2(M) \leq \ldots \leq -\mu_m(M)$.

Remark 2.2.2. Let G be a graph. For $i = 0, \dots, |V(G)| - 1$:

$$\lambda_i(G) = 1 - \mu_{i+1}(D(G)^{-1/2}A(G)D(G)^{-1/2}).$$

This follows as $L(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$, so that

$$\{\lambda_i(L(G)) \,:\, 0 \le i \le |V(G)|-1\} = \{1-\mu_j(D(G)^{-1/2}A(G)D(G)^{-1/2}) \,:\, 1 \le j \le |V(G)|\},$$

and

$$\begin{aligned} 1 - \mu_1(D(G)^{-1/2}A(G)D(G)^{-1/2}) \le 1 - \mu_2(D(G)^{-1/2}A(G)D(G)^{-1/2}) \le \dots \\ \dots \le 1 - \mu_{|V(G)|}(D(G)^{-1/2}A(G)D(G)^{-1/2}). \end{aligned}$$

We can always find an upper bound for $|\mu_i(A(G))|$.

Lemma 2.2.3. *Let G be a graph. Then* $\max_i |\mu_i(A(G))| \le \max_{v \in V(G)} deg(v)$*. If G is bipartite, then*

$$\max_i |\mu_i(A(G))| \le \max_{\substack{v \in V_1(G) \\ w \in V_2(G)}} \sqrt{deg(v)deg(w)}.$$

Proof. The first result follows as $||A(G)||_\infty = \max_{v \in V(G)} deg(v)$, and it is standard that for any square matrix M, $||M||_\infty$ is an upper bound for the absolute values of the eigenvalues of M. The second inequality follows from e.g. [HJ94, 3.7.2].

We have

$$A(G) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

for some matrix B. By definition, the set of eigenvalues of A are the set of *singular values* of B, $\{\sigma_j(B)\}_j$. Therefore, $\max_i |\lambda_i(A(G))| = \max_i |\sigma_i(B)|$. By [HJ94, 3.7.2],

$$\max_i |\sigma_i(B)| \le \sqrt{||B||_\infty ||B||_1} = \max_{\substack{v \in V_1(G) \\ w \in V_2(G)}} \sqrt{deg(v)deg(w)}.$$

□

We note the following invaluable lemma, commonly known as Weyl's inequality.

Lemma (Weyl's inequality, [Wey12]). *Let A and B be symmetric $m \times m$ real matrices. For $i = 1, \dots, m$: $\mu_i(A) + \mu_m(B) \le \mu_i(A + B) \le \mu_i(A) + \mu_1(B)$.*

Chapter 3

Link conditions for cubulation

3.1 Local properties for cubulating groups

Recently, a very fruitful route to understanding groups has been to find an action on a $CAT(0)$ cube complex. Indeed, an action without a global fixed point provides an obstruction to Property (T) [NR97], while a proper action is enough to guarantee a-(T)-menability [CMV04]. Further properties, such as residual finiteness or linearity, can be deduced if the cube complex is *special* [HW08]. Perhaps the most notable recent use of cube complexes was in Agol's proof of the Virtual Haken Conjecture [Ago13].

In this chapter we provide a condition on the links of polygonal complexes (including those with triangular faces) that is sufficient to ensure a group acting properly discontinuously and cocompactly on such a complex contains a virtually free codimension-1 subgroup. We provide stronger conditions that are sufficient to ensure a group acting properly discontinuously and cocompactly on such a complex acts properly discontinuously on a $CAT(0)$ cube complex: in many applications (in particular for hyperbolic groups) this action is also cocompact. We shall see that these conditions can be practically checked in many examples, and can in fact be checked by computer search if desired.

For a polygonal complex X and a vertex v we define the *link* of v, $Lk_X(v)$ (or simply $Lk(v)$ when X is clear from context), as the graph whose vertices are the edges of X incident at v, and two vertices e_1 and e_2 are connected by an edge f in $Lk(v)$ if the edges e_1 and e_2 in X are adjacent to a common face f. We can endow the link graph with the *angular metric*: an edge $f = (e_1, e_2)$ in $Lk(v)$ has length α, where α is the angle between e_1 and e_2 in the shared face f. We refer the reader to Section 3.2.1 for further definitions, such as that of a gluably π-separated complex (this requires a solution to a system of linear equations called the *gluing equations*). We note that in all of our applications, the gluing equations can be solved by considering only the links of vertices of $G\backslash X$.

It is well known that a group containing a codimension-1 subgroup cannot have Property (T) [NR98]. Furthermore, a hyperbolic group acting properly discontinuously and cocompactly on a $CAT(0)$ cube complex is virtually special [Ago13, Theorem 1.1] (see Haglund-Wise [HW08] for a discussion of the notion of specialness): in particular it is linear over $\mathbb{Z}$ and is residually finite.

Theorem A. *Let G be a group acting properly discontinuously and cocompactly on a simply connected $CAT(0)$ polygonal complex X.*

(i) If $G\backslash X$ is gluably weakly π-separated, then G contains a virtually-free codimension-1 subgroup (and therefore does not have Property (T)).

(ii) If $G\backslash X$ is gluably π-separated, then G acts properly discontinuously on a $CAT(0)$ cube complex. If, in addition, G is hyperbolic, then this action is cocompact. In particular, if G is hyperbolic, then it is virtually special, and so linear over $\mathbb{Z}$.

It is commonly far easier to check a local property than a global one, and so local-to-global principles are frequently of great use. When working with complexes, it is often most natural to consider local properties related to the links of vertices. In terms of metric curvature, one of the best-known local-to-global principles is Gromov's Link Condition [Gro87, 4.2A]. Switching to group theoretic properties, Żuk [Ż96] and Ballmann–Świątkowski [BS97] independently provided a condition on the first eigenvalue of the Laplacian of links of simplicial complexes that is sufficient to prove a group acting properly discontinuously and cocompactly on such a complex has Property (T).

If X is a simply connected $CAT(0)$ polygonal complex such that for any vertex v, the edges of $Lk(v)$ can be partitioned into 'π-separated' cutsets, and G acts properly discontinuously and cocompactly on X, then G is cubulable [HW14, Discussion following Example 4.3]. Note that, as opposed to [HW14, Example 4.3], we do not require a partition of the edges of links into cut sets: we can remove this assumption at the expense of requiring that every cutset contains at least two elements, and that the *gluing equations* are satisfied for the cutsets (these equations are trivially satisfied for a collection of proper disjoint edge cutsets). Furthermore, we do not require that the cutsets are two-sided: $\Gamma - C$ is allowed to contain arbitrarily many components. Finally, we allow cutsets to be comprised of vertices or edges. Though we are not always able to cocompactly cubulate non-hyperbolic groups with this method, we can still produce codimension-1 subgroups, and often a proper action on a cube complex.

3.1.1 Applications of the main theorem

We now provide some applications of Theorem A. Firstly, we consider the groups classified by Kangaslampi–Vdovina [KV10] and Carbone–Kangaslampi–Vdovina [CKV12]: these are groups acting simply transitively on simply connected triangular complexes with the link of every vertex isomorphic to the minimal generalized quadrangle. Recall that until now these were not even known to be residually finite, and furthermore, Corollary B complements [DJ02, ER18], providing an example example of the failure of Property (T) in Kac–Moody buildings of 2-spherical type when the thickness is small.

Corollary B. *Let X be a simply connected polygonal complex such that every face has at least 3 sides and the link of every vertex is isomorphic to the minimal generalized quadrangle. If a group G acts properly discontinuously and cocompactly on X, then it is virtually special; in particular it is linear over $\mathbb{Z}$.*

Again, we prove that if X and G are as above, then X can be endowed with a $CAT(0)$ metric such that $G\backslash X$ is gluably π-separated. However, we show that that it is not disjointly π-separated, so that [HW14, Example 4.3] cannot be applied to such a complex.

As a further application of Theorem A, we consider generalized triangle groups: see Definitions 3.5.1 and 3.5.3. We apply Theorem A to polygonal complexes whose links are the graphs used in [CCKW20]: we refer to them by their Foster Census names (see [Fos88]). The only graph not in the Foster Census is $G54$, the Gray graph, which is edge, but not vertex, transitive. Using Theorem A and Theorem 3.5.5 we can deduce the following.

Corollary C. *Let $\Gamma_i \looparrowright C_{k,2}$ be finite-sheeted covers such that girth$(\Gamma_i) \geq 6$ for each i. Let $G = \pi_1(D^j_{0,k}(\Gamma_1, \Gamma_2, \Gamma_3))$ or $G = \pi_1(D^j_k(\Gamma_1, \Gamma_2, \Gamma_3))$ for some j.*

(i) If $\Gamma_i \in \{F24A,\ F26A,\ F48A\}$ for each i, then G acts properly discontinuously on a $CAT(0)$ cube complex: if G is hyperbolic, then this action is also cocompact and so G is virtually special.

(ii) If $\Gamma_1 \in \{F40A, G54\}$, then G acts properly discontinuously on a $CAT(0)$ cube complex: if G is hyperbolic, then this action is also cocompact and so G is virtually special.

Recall that [CCKW20] consider 252 groups, of which they show that 168 do not satisfy Property (T). Our techniques prove that each of the 131 groups we consider has a proper action on a $CAT(0)$ cube complex, and so, by e.g. [CMV04], is a-(T)-menable. Furthermore, 125 of these groups are hyperbolic and have a proper and cocompact action on a $CAT(0)$ cube complex, and hence by [Ago13] are virtually special.

Wise's malnormal special quotient theorem [Wis21] (c.f. [AGM16]) is one of the most important theorems in modern geometric group theory. However, the proof of this theorem is famously complex and so in Section 3.5.3 we apply Theorem A to generalized triangle groups to recover partial consequences of the malnormal special quotient theorem in Corollary 3.5.20. Although this theorem follows from Wise's proof of the MSQT, a far more general theorem, the proof of Corollary 3.5.20 is considerably shorter and simpler, and provides an effective bound on the index of the fillings required.

3.1.2 Structure of the chapter

The main idea of the proof of Theorem A is the following. Since $G\backslash X$ is π-separated, we can find a collection of local geodesics in $G\backslash X$ that are locally separating at vertices of $G\backslash X$. The *gluing equations* provide us with a way to glue these local geodesics together to find a locally geodesic locally separating subcomplex of $G\backslash X$: by lifting we find a geodesic separating subcomplex of X with cocompact stabilizer. We then use the construction of Sageev [Sag95], generalized by Hruska–Wise in [HW14], to construct the desired $CAT(0)$ cube complex.

This chapter is structured as follows. In Section 3.2 we define hypergraphs, which will be separating subspaces constructed in the polygonal complex, and show certain subgroups of their stabilizers are codimension-1. We then prove Theorem A by using Hruska–Wise's [HW14] extension of Sageev's [Sag95] construction of a $CAT(0)$ cube complex, and proving that there are 'enough' hypergraphs to 'separate' the polygonal complex. In Section 3.3, we discuss how to find 'separated' cutsets of a graph by computer search. In Section 3.4 we prove Corollary B by proving that the minimal generalized quadrangle is *weighted edge* 3*-separated* and endowing the polygonal complexes with a suitable $CAT(0)$ metric. In Section 3.5 we prove Theorem 3.5.5 and Corollary C. We again apply Theorem A to prove Corollary 3.5.20 by considering cutsets in covers of graphs.

3.2 Cubulating groups acting on polygonal complexes

This section is structured as follows. We first define the required conditions on graphs and complexes in Section 3.2.1, and in Section 3.2.2 we discuss how to remove cut edges from links. We provide some examples where our conditions can be readily verified for graphs in Section 3.2.3 and for complexes in Section 3.2.4. We use these definitions in Sections 3.2.5, 3.2.6, and 3.2.7 to build separating convex trees in polygonal complexes, and in Section 3.2.8 we use these convex trees, and a construction due to [Sag95] and [HW14], to prove Theorem A. Firstly, we introduce the relevant definitions for links.

3.2.1 Some separation conditions

We now define the notion of 'separatedness' of a graph. The *combinatorial metric* on a graph Γ is the path metric induced by assigning each edge of Γ length 1.

Definition 3.2.1. Let Γ be a finite metric graph.

i) A vertex v (respectively edge e) is a *cut vertex* (respectively *cut edge*) if $\Gamma - \{v\}$ ($\Gamma - \{e\}$) is disconnected as a topological space.

ii) A set $C \subseteq \Gamma$ is a *cutset* if $\Gamma - C$ is disconnected as a topological space.

iii) A cutset C is an *edge cutset* if $C \subseteq E(\Gamma)$ and is a *vertex cutset* if $C \subseteq V(\Gamma)$.

iv) An edge cutset C is *proper* if for any edge $e \in C$, the endpoints of e lie in disjoint components of $\Gamma - C$.

v) A vertex cutset C is *proper* if for any vertex $u \in C$, and any distinct vertices v, w adjacent to u, the vertices v and w lie in distinct components of $\Gamma - C$.

For an edge e in Γ let $m(e)$ be the midpoint of e. For $\sigma > 0$ a set $C \subseteq E(\Gamma)$ is *σ-separated* if for all distinct $e_1, e_2 \in C$, $d_\Gamma(m(e_1), m(e_2)) \geq \sigma$. A set $C \subseteq V(\Gamma)$ is *σ-separated* if for all distinct $v_1, v_2 \in C$, $d_\Gamma(v_1, v_2) \geq \sigma$.

Remark 3.2.2. We note that proper cut sets are very natural to consider. Any minimal edge cut set is proper, and more importantly, proper cutsets are preserved under passing to finite covers. Finding proper edge cutsets is easy, but for a given graph Γ there may not be any proper σ-separated vertex cutsets: see for example the graph $F26A$, considered in Lemma 3.5.15.

Definition 3.2.3 (*Edge separated*)**.** Let Γ be a finite metric graph and let $\sigma > 0$. We will say that Γ is *edge σ-separated* if Γ is connected, contains no vertices of degree 1, and there exists a collection of proper σ-separated edge cutsets $C_i \subseteq E(\Gamma)$ with $\cup_i C_i = E(\Gamma)$ and $|C_i| \geq 2$ for each i. We say the graph is *disjointly edge σ-separated* if the above cutsets form a partition of the edges.

Note that to each edge cutset C we can assign a partition $\mathcal{P}(C)$ to $\pi_0(\Gamma - C)$: we *always require* that such a partition is at least as coarse as connectivity in $\Gamma - C$, and each partition contains at least two elements. The *canonical partition* of C is that induced by connectivity in $\Gamma - C$.

Definition 3.2.4 (*Strongly edge separated*)**.** A graph Γ is *strongly edge σ-separated* if Γ is edge σ-separated and for every pair of points u, v in Γ with $d_\Gamma(u, v) \geq \sigma$ there exists a proper σ-separated edge cutset C_i with u and v lying in distinct components of $\Gamma - C_i$. We say the graph is *disjointly strongly edge σ-separated* if the above cutsets form a partition of the edges.

Definition 3.2.5. For four points u_1, u_2, v_1, v_2 on a graph Γ, we say a cutset C separates $\{u_1, u_2\}$ and $\{v_1, v_2\}$ if each u_i lies in a different component of $\Gamma - C$ to each v_j, i.e $[u_i] \neq [v_j]$ in $H_0(\Gamma - C)$.

Using this, we can see that there is a more combinatorial condition that implies strong edge separation.

Lemma 3.2.6. *Let $n \geq 2$ and let Γ be a graph endowed with the combinatorial metric such that $girth(\Gamma) \geq 2n$. Suppose that Γ is edge n-separated with cutsets $\mathcal{C} = \{C_1, \dots, C_m\}$, and for every pair of vertices u, v in Γ, and any vertices u', v' with $d_\Gamma(u, v) \geq n$ and $d(u, u') = d(v, v') = 1$ there exists an n-separated cutset C_i separating $\{u, u'\}$ and $\{v, v'\}$. Then Γ is strongly edge n-separated with the same cutsets.*

Proof. First note that as Γ is edge n-separated, it is connected and contains no vertices of degree 1. Let u, v be two points in Γ with $d(u, v) \geq n$. If u, v are vertices, then we are done. Suppose u and v both lie on edges: let $e(u), e(v)$ be the respective edges, and u_1, u_2, v_1, v_2 the endpoints of $e(u), e(v)$ respectively. If v is a vertex, take $v = v_1 = v_2$. As $girth(\Gamma) \geq 2n$, without loss of generality $d(u_1, v_1) \geq n$: taking C_i to be the cutset separating u_1, u_2 and v_1, v_2, we see that C_i separates u and v. $\square$

Definition 3.2.7 (*Weakly vertex separated*)**.** Let Γ be a finite metric graph and let $\sigma > 0$. We will say that Γ is *weakly vertex σ-separated* if: Γ is connected and contains no vertices of degree 1, and there exists a collection of σ-separated vertex cutsets $C_i \subseteq V(\Gamma)$ such that $\cup_i C_i = V(\Gamma)$ and $|C_i| \geq 2$ for each i.

Again, to each vertex cutset C we can assign a partition $\mathcal{P}(C)$ to $\pi_0(\Gamma - C)$: we *always require* that such a partition is at least as coarse as connectivity in $\Gamma - C$ and each partition contains at least two elements. The *canonical partition* of C is that induced by connectivity in $\Gamma - C$.

Definition 3.2.8 (*Vertex separated*)**.** Let Γ be a finite metric graph and let $\sigma > 0$. We will say that Γ is *vertex σ-separated* if:

i) Γ is connected and contains no vertices of degree 1,

ii) there exists a collection of σ-separated vertex cutsets $C_i \subseteq V(\Gamma)$ such that $\cup_i C_i = V(\Gamma)$ and $|C_i| \geq 2$ for each i,

iii) for any vertex v and any distinct vertices w, w' adjacent to v there exists a σ-separated vertex cutset C_i such that w and w' lie in distinct components of $\Gamma - C_i$,

iv) and for any points u and v in Γ with $d(u, v) \geq \sigma$, there exists a cutset C_i with u and v lying in distinct components of $\Gamma - C_i$.

Note that, importantly, in general we do not require vertex cutsets to be proper. We say the graph is *disjointly vertex separated* if the above cutsets form a partition of the vertices, and each cutset is proper.

Remark 3.2.9. The reason we do not require vertex cutsets to be proper is the following. For edge cut sets we could weaken the definition of edge separated to require a condition similar to *iii*) above: i.e. that the endpoints of each edge are separated by some cutset. However such a cutset can always be made minimal, and therefore proper, by removing unnecessary edges: the same is not true for vertex cutsets.

Once again, this definition is not as difficult to verify as it may seem.

Lemma 3.2.10. *Let $n \geq 2$, and let Γ be a graph endowed with the combinatorial metric, such that Γ is connected, contains no vertices of degree 1, and girth$(\Gamma) \geq 2n$. Suppose there exists a collection of n-separated vertex cutsets $\mathcal{C} = \{C_1, \dots, C_m\}$ so that*

i) $\cup_i C_i = V(\Gamma)$,

ii) $|C_i| \geq 2$ *for each* i,

iii) for each vertex v and distinct w, w' adjacent to v there exists a n-separated cutset with w and w' lying in distinct components of $\Gamma - C$,

iv) and furthermore that for any pair of vertices u, v with $d_\Gamma(u, v) \geq n$ there exists a cutset C_i with u and v lying in distinct components of $\Gamma - C_i$.

Then Γ is vertex σ-separated with the collection of cutsets $\mathcal{C}$.

Proof. It suffices to show that for any pair of points u, v with $d(u, v) \geq n$ there exists a cutset C_i separating them. If u and v are vertices, then we are finished. Otherwise, let $e(u), e(v)$ be the edges that u and v lie on. Let u_1, u_2 and v_1, v_2 be the endpoints of $e(u), e(v)$ respectively. If v is a vertex simply take $v_1 = v_2 = v$. Then without loss of generality, as $girth(\Gamma) \geq 2n$ and $d(u, v) \geq n$, we have that $d(u_1, v_1) \geq n$. Let C_i be the cutset separating u_1 and v_1: this cutset must also separate u and v. $\square$

Finally, we define weighted σ-separated.

Definition 3.2.11 (Weighted σ-separated)**.** Let $\sigma > 0$ and let Γ be an edge σ-separated graph (respectively strongly edge σ-separated, weakly vertex σ-separated, vertex σ-separated) with σ-separated cutsets $\mathcal{C} = \{C_1, \dots, C_m\}$. We call Γ *weighted edge σ-separated* (respectively *strongly edge σ-separated, weakly vertex σ-separated, vertex σ-separated*) if there exists an assignment of positive integers $n(C_i)$ to the cutsets in $\mathcal{C}$ that solves the *weight equations*: for any edges (respectively edges, vertices, vertices) α, β of Γ,

$$\sum_{C_i \in \mathcal{C} : \alpha \in C_i} n(C_i) = \sum_{C_i \in \mathcal{C} : \beta \in C_i} n(C_i).$$

Note that though the above equations at first appear to be difficult to solve, we can always (after extending $\mathcal{C}$ if necessary) find solutions for a graph with an edge (respectively vertex) transitive automorphism group (see Section 3.2.3).

Next we extend these definitions to $CAT(0)$ polygonal complexes. This requires some care to ensure that the subcomplexes we build will actually be separating. A *polygonal complex* is a 2-dimensional polyhedral complex and is *regular* if all polygonal faces are regular polygons. For a polygonal complex X and a vertex v we define the *link* of v, $Lk_X(v)$ (or simply $Lk(v)$ when X is clear from context), as the graph whose vertices are the edges of X incident at v, and two vertices e_1 and e_2 are connected by an edge f in $Lk(v)$ if the edges e_1 and e_2 in X are adjacent to a common face f. We can endow the link graph with the *angular metric*: an edge $f = (e_1, e_2)$ in $Lk(v)$ has length α, where α is the angle between e_1 and e_2 in the shared face f.

We first define the following graph, which appeared in e.g. [Wis04] and [OW11].

Definition 3.2.12 (*Antipodal graph*)**.** Let Y be a regular non-positively curved polygonal complex. Subdivide edges in Y and add vertices at the midpoints of edges: call these additional vertices *secondary vertices*, and call the other vertices *primary.* Every polygon in Y now contains an even number of edges in its boundary. Construct a graph Δ_Y as follows. Let $V(\Delta_Y) = V(Y)$ and join two vertices v and w by an edge, labelled f, if v and w exist and are antipodal in the boundary of a face f in Y: add as many edges as such faces exist. This is the *antipodal graph* for Y.

Remark 3.2.13. We note that for a secondary vertex s of Y, $Lk_Y(s)$ is a cage graph with edges of length π. Hence, if Y does not contain any free faces, $Lk_Y(s)$ is weighted edge π-separated, with a single π-separated cutset $E(Lk_Y(s))$.

Note that as the complex is regular, the edges of Δ_Y pass through the midpoints of edges in $Lk_Y(v)$ for vertices v. There is a canonical map $\Delta_Y \to Y$; we map a vertex v of Δ_Y to the corresponding vertex of Y, and we map an edge e labelled by f to the local geodesic between the endpoints of e lying in the face f. We note that edge cutsets in $Lk_Y(\iota(e))$ correspond to vertex sets in $Lk_{\Delta_Y}(\iota(e))$. Similarly, vertex cutsets in $Lk_Y(\iota(e))$ correspond to vertex sets in $Lk_{Y^{(1)}}(\iota(e))$.

Definition 3.2.14. Let Y be a non-positively curved polygonal complex, and let Δ be one of $Y^{(1)}$ or Δ_Y. Assign Δ an arbitrary orientation. For each oriented edge e of Δ and each chosen π-separated cutset C in $Lk(\iota(e))$, choose a set of partitions of $\pi_0(Lk(\iota(e))-C)$, $\{P_i(C)\}_i$. For $v \in V(\Delta)$, we define $\mathcal{C}_v = \{C \ : \ C \text{ is a } \pi\text{-separated cutset, } C \subseteq V(Lk_\Delta(v))\}$, for e an edge we define $\mathcal{C}(e) := \{C \in \mathcal{C}_{\iota(e)} \ : \ e \in C\}$, and

$$\mathcal{C} = \bigcup_{e \in E^{\pm}(\Delta)} \mathcal{C}(e).$$

Similarly we can define $\mathcal{CP}(e) := \bigcup_{C \in \mathcal{C}(e)} \{(C, P_i(C))\}_i$, and

$$\mathcal{CP} = \bigcup_{e \in E^{\pm}(\Delta)} \mathcal{CP}(e).$$

For our purposes, given a polygonal complex Y and a point $y \in Y$, the *star* of y, $St_Y(y)$ or just $St(y)$, is the intersection of a closed ϵ ball around y with Y, where $\epsilon > 0$ is any suitably small constant.

Before we introduce the full definition of 'equatable partitions', we provide the following example.

Example 3.2.15. Let Y be a non-positively curved polygonal complex, and let X be the universal cover of Y. Suppose that we wish to build a separating subcomplex of X

by cutting along an edge $e \subseteq X^{(1)}$, which has endpoints v and w. Choose cutsets C containing e in $Lk(v)$ and C' containing e^{-1} in $Lk(w)$, and pick appropriate partitions. Moving in along e to points v' and w', the partitions of $Lk(v) - C$ and $Lk(w) - C'$ give partitions of $St(v') - v'$ and $St(w') - w'$. It is easy to check if these induced partitions match up correctly, as $St(v') \cong St(w')$.

Below we provide an example where these partitions line up correctly. The different elements of the partition are marked by different colours.

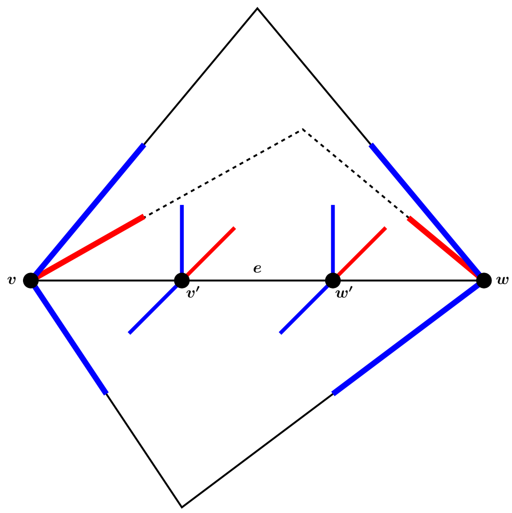

Fig. 3.1 Equatable partitions of C and C'

Now, if the partitions do not match up correctly, then we may be able to move from v to w and back, to find a path between points lying in different elements of $\pi_0(Lk(v) - C)$: this is precisely the problem we wish to avoid.

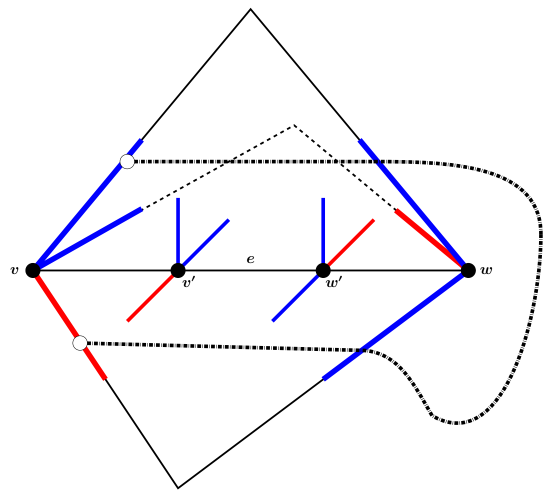

Fig. 3.2 Non-equatable partitions of C and C'

Therefore, we are required to introduce the following. It is extremely similar to the 'splicing' of Manning [Man10]: we will use this for a similar purpose to that of [CM11].

Definition 3.2.16 (Equatable partitions)**.** Let Y be a non-positively curved polygonal complex, and let Δ be one of $Y^{(1)}$ or Δ_Y. Let v, w be two vertices of Δ connected by an oriented edge e, so that $v = \iota(e)$ and $w = \tau(e)$. Let $(C_v, P_v) \in \mathcal{CP}(e)$ and $(C_w, P_w) \in \mathcal{CP}(e^{-1})$.

Let v', w' be points on e in an ϵ-neighbourhood of v, w respectively, so that there are canonical mappings

$$\begin{aligned} i_v &: St(v') \hookrightarrow Lk(v), \\ i_w &: St(w') \hookrightarrow Lk(w), \\ \phi &: St(v') \xrightarrow{\cong} St(w'). \end{aligned}$$

Therefore we have induced mappings

$$\begin{aligned} \bar{i}_v &: St(v') - v' \hookrightarrow Lk(v) - C_v, \\ \bar{i}_w &: St(w') - w' \hookrightarrow Lk(w) - C_w, \\ \overline{\phi} &: St(v') - v' \xrightarrow{\cong} St(w') - w'. \end{aligned}$$

For $u = v, w$ let $\mathcal{P}_u$ be the set of partitions of $\pi_0(Lk(u) - C_u)$, and let $\mathcal{P}_{u'}$ be the set of partitions of $\pi_0(St(u') - u')$. There are induced maps

$$\begin{aligned} \iota_v &: \mathcal{P}_v \to \mathcal{P}_{v'}, \\ \iota_w &: \mathcal{P}_w \to \mathcal{P}_{w'}, \\ \psi &: \mathcal{P}_{v'} \hookrightarrow\!\!\!\!\rightarrow \mathcal{P}_{w'}. \end{aligned}$$

We say that (C_v, P_v) and (C_w, P_w) are *equatable along* e, written

$$(C_v, P_v) \sim_e (C_w, P_w)$$

if $\psi(\iota_v(P_v)) = \iota_w(P_w)$. Note that this also defines an equivalence relation on $\mathcal{CP}(e)$: for $(C, P), (C', P') \in \mathcal{CP}(e)$, we write

$$(C, P) \approx_e (C', P')$$

if $\iota_v(P) = \iota_v(P')$. This defines an equivalence relation on $\mathcal{CP}(e)$, and so defines an equivalence class $[C, P]_e$. We define $[[C, P]]_{e^{-1}}$ to be the equivalence class of cutset partitions in $\mathcal{CP}(e^{-1})$ equatable to (C, P) along e: by definition this is independent of choice of $(C', P') \in [C, P]_e$.

These constructions are designed so that we can 'splice' the local cutsets along each edge. Though this definition is somewhat complicated, note the following remark.

Remark 3.2.17. Let e, v, w, C_v, C_w be as above. If both C_v, C_w are proper with canonical partitions P_v, P_w, then $(C_v, P_v) \sim_e (C_w, P_w)$.

This follows as the induced partitions of $St(v') - v'$ and $St(w') - w'$ are just the partitions induced by connectivity, and by properness every element of the induced partition of $St(v') - v'$ (respectively $St(w') - w'$) contains a unique vertex. Similarly, if $C_1, C_2 \in \mathcal{C}(e)$ are proper, with canonical partitions P_1, P_2, then $(C_1, P_1) \approx_e (C_2, P_2)$.

Definition 3.2.18 (*Gluably σ-separated*)**.** Let Y be a non-positively curved polygonal complex. We call Y *gluably edge σ-separated* (respectively *gluably (weakly) vertex σ-separated*) if :

i) Y is regular (respectively Y is allowed **not to be regular**),

ii) the link of every vertex in Y is edge (respectively (weakly) vertex) σ-separated,

iii) for every π-separated cutset C in $Lk(v)$ there exists a series of partitions $\{P_i(C)\}$ of $\pi_0(Lk(v) - C)$ such that for any distinct pair of points $x, y \in Lk(v)$ separated by C, x and y are separated by some $P_i(C)$,

iv) and there exists a strictly positive integer solution to the *gluing equations*: letting $\Delta = \Delta_Y$ (respectively $\Delta = Y^{(1)}$), we can assign a positive integer $\mu(C, P)$ to every pair

$$(C, P) \in \mathcal{CP} := \bigcup_{e \in E^{\pm}(\Delta)} \mathcal{CP}(e)$$

such that for every oriented edge e of Δ and every $(C, P) \in \mathcal{CP}(e)$,

$$\sum_{(C',P') \in [C,P]_e} \mu(C', P') = \sum_{(C',P') \in [[C,P]]_{e^{-1}}} \mu(C', P').$$

Definition 3.2.19 (*Gluably σ-separated*)**.** Let Y be a non-positively curved polygonal complex. We call Y:

i) *gluably weakly σ-separated* if it is gluably weakly vertex σ-separated,

ii) and *gluably σ-separated* if it is gluably edge or gluably vertex σ-separated.

Remark 3.2.20. Again, note that in the definition of a gluably (weakly) vertex σ-separated complex, **we do not require that the complex Y is regular**. If the link of each vertex in the complex Y is disjointly σ-separated, then we can solve the gluing equations by taking only the canonical partition $P(C)$ for each cutset C, and setting $\mu(C, P(C)) = 1$ for all cutsets C, so that Y is gluably σ-separated.

3.2.2 Removing cut edges

We now show that the existence of cut edges is not too much of an issue.

Lemma 3.2.21. *Let G be a group acting properly discontinuously and cocompactly on a simply connected $CAT(0)$ polygonal complex X, such that the link of every vertex in X is connected. There exists a simply connected $CAT(0)$ polygonal complex X' such that G acts properly discontinuously and cocompactly on X' and the link of any vertex v' in X' is a subgraph of $Lk(v)$ for some vertex v in X. Furthermore for any vertex v of X', either $Lk_{X'}(v)$ is connected and contains no cut edges, or $Lk_{X'}(v)$ is disconnected.*

Proof. First note that we can assume that X contains no vertices of degree 1 in its links. A vertex of degree one corresponds to a free face: pushing in this free face and endowing the resulting complex X' with the inherited path metric, we see that Gromov's Link condition is still satisfied, and so X' is $CAT(0)$.

Now, let $Y = G\backslash X$, and let $v_0, \dots, v_m$ be the vertices of Y. Let v be a vertex in X, and suppose there exists a cut edge f in $Lk(v)$. Let e_1 and e_2 be the endpoints of f: in X e_1, e_2 are edges adjacent to v and occurring consecutively in the boundary of the face f. Suppose that, in X, the endpoints of e_1 are v and w. Construct a new complex X' as follows: let v_1 and v_2 be two copies of v and connect these vertices to w with the edges e_1^1 and e_2^2 respectively. Since f is a cut edge in $Lk(v)$ there is a canonical way to attach edges and faces to v_1 and v_2 that agrees with the connected components of $Lk(v) - f$.

Now, we assume that f is attached to v_1. Then the face f is a free face, which we can push in to remove the vertex of degree 1, e_1^1 in $Lk(v_1)$, so that $Lk(v_1)$ and $Lk(v_2)$ are connected subgraphs of $Lk(v) - f$, and the links of any other vertices x incident to the face f are transformed to a proper subgraph of $Lk(x)$ with the edge f removed.

We can repeat this process finitely many times, applied to the set of vertices Gv_i each time, to find the polygonal complex X' desired.

Finally, metrize each polygon in X' with the metric inherited from X and endow X' with the resulting path metric. Since the link of each vertex in X is a subgraph of the link of some vertex in X, it follows that X' is non-positively curved by Gromov's Link condition. X' is also simply connected, and hence $CAT(0)$. $\square$

3.2.3 Examples of separated graphs

Our definitions of weighted σ-separated graphs required assigning weights to cutsets such that certain equations hold. In this subsection we prove that as long as the automorphism group of a graph is transitive on vertices (or edges, depending on whether cutsets are formed of vertices or edges), then these equations can always be solved. Note that $Aut(\Gamma)$ is the group of automorphisms of Γ as a metric graph.

Lemma 3.2.22. *Let $\sigma > 0$ and let Γ be (weakly) vertex σ-separated. If $Aut(\Gamma)$ is vertex transitive then Γ is weighted (weakly) vertex σ-separated.*

Proof. Assume that Γ is vertex σ-separated, with σ-separated vertex cutsets $\mathcal{C} = \{C_1, \dots, C_n\}$. The proof is similar for weakly separated graphs. Let $H = Aut(\Gamma)$. For each $C \in \mathcal{C}$, let

$$H(C) := \{\gamma C \ : \ \gamma \in H\},$$

counted **with** multiplicity, i.e. if $\gamma_1 C = \gamma_2 C$ and $\gamma_1 \neq_H \gamma_2$, then both $\gamma_1 C, \gamma_2 C$ appear in $H(C)$. Note that for every $C' \in H(C)$, C' is a σ-separated vertex cutset.

Fix some vertex $v \in C$, and let $w \in V(\Gamma)$ be any vertex. Since H acts vertex transitively, there exists $h \in H$ such that $hv = w$. Therefore

$$\{\gamma \in H \ : \ v \in \gamma C\} = \{\gamma \in H \ : \ w \in h\gamma C\} = \{h^{-1}\gamma' \in H \ : \ w \in \gamma' C\},$$

and hence $|\{\gamma \in H \ : \ v \in \gamma C\}| = |\{\gamma \in H \ : \ w \in \gamma C\}|$. Let

$$\tilde{\mathcal{C}}' := \bigsqcup_{C \in \mathcal{C}} H(C),$$

again with multiplicity. By the above, it follows that for any two vertices $v, w \in V(\Gamma)$,

$$|\{C \in \tilde{\mathcal{C}}' \ : \ v \in C\}| = |\{C \in \tilde{\mathcal{C}}' \ : \ w \in C\}|.$$

Let $\mathcal{C}'$ be the underlying set of $\tilde{\mathcal{C}}'$, and for $C \in \mathcal{C}'$, let

$$n(C) = |\{C' \in \tilde{\mathcal{C}}' \ : \ C = C'\}|,$$

i.e. $n(C)$ is the multiplicity of C in $\tilde{\mathcal{C}}'$. It is easily seen that the above weights solve the gluing equations.

As $\mathcal{C} \subseteq \mathcal{C}'$, it follows that Γ is vertex separated with respect to these cutsets: by the above argument it follows that Γ is weighted vertex σ-separated with cutsets $\mathcal{C}'$. $\square$

Similarly, we can prove the following.

Lemma 3.2.23. *Let $\sigma > 0$ and let Γ be (strongly) edge σ-separated. If $Aut(\Gamma)$ is edge transitive then Γ is weighted (strongly) edge σ-separated.*

3.2.4 Examples of solutions of the gluing equations

Recall that we call an edge cutset C *proper* if the endpoints of any edge e in C lie in separate components of $\Gamma - C$, and a vertex cutset C *proper* if any every $v \in C$ the vertices adjacent to v each lie in separate components of $\Gamma - C$.

Lemma 3.2.24. *Let Y be a finite regular non-positively curved polygonal complex and suppose the link of each vertex is weighted edge π-separated. There exists a system of strictly positive weights that solve the gluing equations for Y.*

Proof. Since edge cutsets are proper, any two cutsets are equatable along a shared edge. Therefore we may associate to each cutset C exactly one partition $P(C)$, namely that of connectivity in $\Gamma - C$. In particular for any oriented edge $e \in E^{\pm}(\Delta_Y)$ and any $(C, P(C)) \in \mathcal{CP}(e)$, $[C, P(C)]_e = \mathcal{CP}(e)$.

First, note that for an oriented edge e of Δ_Y, and $v = \iota(e)$, $\mathcal{C}(e) = \mathcal{C}(e) \cap \mathcal{C}_v$. Since the link of each vertex in Y is weighted edge π-separated, for each vertex $v \in Y$ there exists a positive integer $N_v > 0$ and a system of strictly positive weights $n_v(C)$ for $C \in \mathcal{C}_v$ such that for any edge e in $Lk_Y(v)$,

$$\sum_{C \in \mathcal{C}(e)} n_v(C) = \sum_{C \in \mathcal{C}(e) \cap \mathcal{C}_v} n_v(C) = N_v.$$

Let $M = \prod_{v \in V(Y)} N_v$, and for a cutset $C \in \mathcal{C}_v$, define $m(C) = M n_v(C)/N_v$. It follows that for an edge e in $Lk_Y(v)$,

$$\sum_{C \in \mathcal{C}(e)} m(C) = \frac{M}{N_v} \sum_{C \in \mathcal{C}(e)} n_v(C) = \frac{M}{N_v} N_v = M.$$

Finally, taking $\mu(C, P(C)) = m(C)$, these weights immediately solve the gluing equations. $\square$

Similarly, we can prove the following.

Lemma 3.2.25. *Let Y be a finite non-positively curved polygonal complex, such that the link of each vertex is weighted vertex π-separated, and every cutset is proper. There exists a system of strictly positive weights that solve the gluing equations for Y.*

3.2.5 Hypergraphs in π-separated polygonal complexes

We now begin to construct our separating subcomplexes. We provide an explicit example of the construction in Example 3.2.29. Suppose X is a simply connected $CAT(0)$ polygonal complex, and G acts properly discontinuously and cocompactly on X, so that $G\backslash X$ is (weakly) gluably π-separated. If $G\backslash X$ is gluably edge π-separated, let $\Delta = \Delta_{G\backslash X}$, and if it is (weakly) gluably vertex π-separated, let $\Delta = (G\backslash X)^{(1)}$. Assign an arbitrary orientation to Δ. Recall that for an oriented edge e of Δ, we let $\mathcal{C}(e) = \{C \in \mathcal{C} \ : \ e \in C\}$ (note that for any oriented edge e, $\mathcal{C}(e)$ is non-empty, as $G\backslash X$ is gluably π-separated). For every vertex v and π-separated cutset C in $Lk(v)$ let $\{P_i(C)\}$ be the required set of partitions of $\pi_0(Lk(v) - C)$, and let $\mathcal{CP}(e) = \bigcup\limits_{C\in\mathcal{C}(e)} \{(C, P_i(C)\}_i, \mathcal{CP} = \bigcup\limits_{e\in E^{\pm}(\Delta)} \mathcal{CP}(e)$. By assumption, we can assign positive integer weights $\mu(C,P)$ to each cutset $(C,P) \in \mathcal{CP}$ so that for every oriented edge e of Δ:

$$\sum_{(C',P')\in[C,P]_e} \mu(C',P') = \sum_{(C',P')\in[[C,P]]_{e^{-1}}} \mu(C',P').$$

We now construct a second graph Σ as follows. Let

$$V(\Sigma) = \bigsqcup_{(C,P)\in\mathcal{CP}} \{u^1_{(C,P)}, \ldots, u^{\mu(C,P)}_{(C,P)}\}.$$

The gluing equations imply that for each positively oriented edge e of Δ and each equivalence class $[C,P]_e \subseteq \mathcal{CP}(e)$ there exists a bijection

$$\phi_e : \bigsqcup_{(C',P')\in[C,P]_e} \{u^1_{(C',P')}, \ldots, u^{\mu(C',P')}_{(C',P')}\} \to \bigsqcup_{(C',P')\in[[C,P]]_{e^{-1}}} \{u^1_{(C',P')}, \ldots, u^{\mu(C',P')}_{(C',P')}\}.$$

For each positively oriented edge e and each equivalence class $[C,P]_e$ of $\mathcal{CP}(e)$ choose such a bijection, ϕ_e, and add the oriented edges

$$\{(u^i_{(C',P')}, \phi_e(u^i_{(C',P')})) \ : \ (C',P') \in [C,P]_e, 1 \le i \le \mu(C',P')\}.$$

Note that for each $(C,P) \in \mathcal{CP}$, $Lk_{\Sigma}(u^i_{(C,P)})$ is isomorphic to C as labelled oriented graphs. Furthermore, each edge in Σ labelled by e connects two vertices of the form $u^i_{(C,P)}$ $u^j_{(C',P')}$ with $(C,P) \sim_e (C',P')$, i.e. every edge connects vertices with equatable partitions along that edge. There is an immersion $\Sigma \looparrowright \Delta$ that sends $u^i_{(C,P)}$ to the vertex v_C such that $C \subseteq Lk_{\Delta}(v_C)$ and maps an edge labelled by e to the edge e in Δ.

Let $\Sigma_1, \ldots, \Sigma_m$ be the connected components of Σ, and let $\underline{\Lambda}^1, \ldots, \underline{\Lambda}^m$ be the images of these graphs in $G\backslash X$ under the map

$$\Sigma_i \looparrowright \Delta \to G\backslash X.$$

We see that each $\underline{\Lambda}^i$ is locally geodesic as the cut sets are π-separated in $G\backslash X$.

Definition 3.2.26. If $G\backslash X$ is gluably edge π-separated, a lift of $\underline{\Lambda}^i$ from $G\backslash X$ to the $CAT(0)$ complex X is called a *edge hypergraph in* X, and otherwise it is a *vertex hypergraph in* X.

Note that hypergraphs come with two pieces of information at each vertex v in X: the cutset C and partition P. We say Λ *passes through* the above objects.

Remark 3.2.27. In the above construction for every vertex $v \in G\backslash X$, every π-separated edge cutset C in $Lk_{G\backslash X}(v)$ and chosen partition P of $\pi_0(Lk_{G\backslash X}(v) - C)$, and every lift $\tilde{v}$ of v to X, there exists a hypergraph passing through (C,P) in $Lk_X(\tilde{v})$.

Importantly, our construction ensures the following.

Proposition 3.2.28. *Let X satisfy the requirements of Theorem A, and let Λ be a hypergraph in X. Then $Stab_G(\Lambda)$ acts properly discontinuously and cocompactly on Λ.*

Proof. As there are finitely many images $\underline{\Lambda}^i$ in $G\backslash X$, there are finitely many hypergraphs containing a specific edge: since G acts cocompactly on X, the result follows.

□

Let us now consider an example.

Example 3.2.29. Let Y be the genus 2 surface, which can be considered as a polygonal complex with a single vertex, and the fundamental polygon below. In particular the fundamental polygon is an octagon, which is metrized as a unit Euclidean octagon. There is a single vertex v in Y, whose link is an octagon with each edge of length $3\pi/4$.

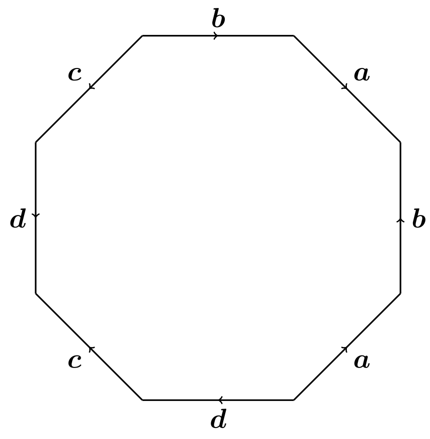

Fig. 3.3 The polygonal complex Y

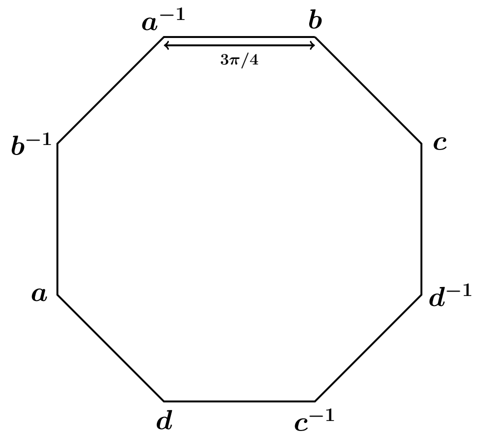

Fig. 3.4 The link of Y, $Lk_Y(v)$

The antipodal graph for Y is then just a rose.

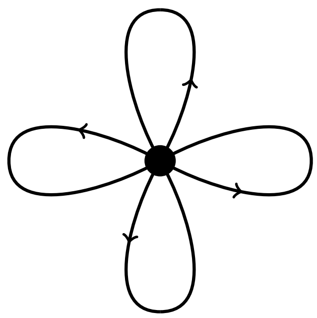

Fig. 3.5 The antipodal graph Δ_Y

There is a partition of the edges of $Lk_Y(v)$ into four π-separated cutsets

$C_1 = \{(a, b^{-1}), (c, d^{-1})\},\ C_2 = \{(a^{-1}, b^{-1}), (c^{-1}, d^{-1})\},\ C_3 = \{(a^{-1}, b), (c^{-1}, d)\},$
$C_4 = \{(a, d), (d, c)\}.$

The weight equations for these cutsets can be solved by giving each cutset weight 1, and so the auxiliary graph that we construct, Σ, is the disjoint union of four circles. In the below figure we also show an example of $\underline{\Lambda_1}$.

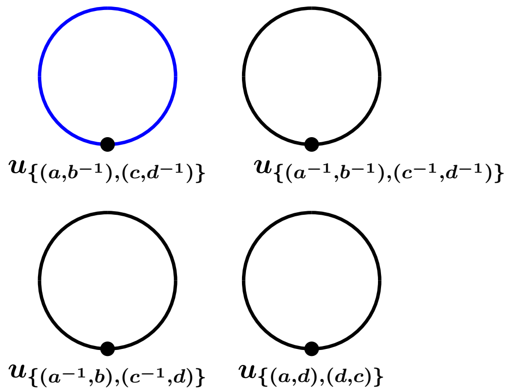

Fig. 3.6 The graph Σ

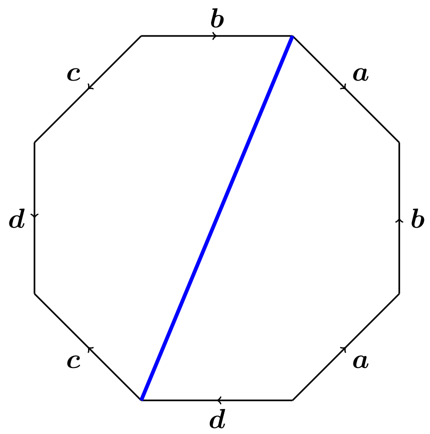

Fig. 3.7 An example of embedding $\underline{\Lambda_1}$

Then any lift Λ_1 of $\underline{\Lambda_1}$ to $\mathbb{H}^2$ is simply an infinite two-ended geodesic line; clearly $\mathbb{H}^2 - \Lambda_1$ consists of two components.

3.2.6 Hypergraphs are separating

We now analyse the structure of the hypergraphs, and show they are in fact separating.

Lemma 3.2.30. *Let $G\backslash X$ be a simply connected (weakly) gluably π-separated $CAT(0)$ polygonal complex, and let Λ be a hypergraph in X. Then Λ is a leafless convex tree.*

Proof. For each i, as the cutsets are π-separated, the image of $\Sigma_i \looparrowright G\backslash X$ is locally geodesic: therefore Λ is locally geodesic in X. As X is $CAT(0)$, local geodesics are geodesic, and geodesics are unique, so that Λ is a convex tree. Since $|C| \geq 2$ for any $v \in V(\Delta_{G\backslash X})$ and $C \in \mathcal{C}_v$, Λ contains no primary vertices of degree 1. If Λ is a vertex hypergraph, it is immediate that Λ is leafless. If Λ is an edge hypergraph, as there are no vertices of degree 1 in the link of a primary vertex in X, there are no cut edges in the link of a secondary vertex and so every edge cut set in the link of a secondary vertex contains at least two elements. It follows that edge hypergraphs are leafless. □

Definition 3.2.31. Let Λ_i be a hypergraph in X and $x, y \in X$ be distinct points in X. We say Λ_i *separates* x and y if x and y lie in distinct components of $X - \Lambda_i$. We write $\#_\Lambda(x, y)$ for the number of edge (or vertex) hypergraphs separating x and y.

We now consider separating points: we prove the following lemma. We call a path γ *transverse* to Λ if $|\gamma \cap \Lambda| = 1$. If x is a point on an edge e of Λ, then there is a canonical partition of $Lk(x) - \Lambda$ obtained from the partitions of $Lk(v) - \Lambda$ and $Lk(w) - \Lambda$, where

v, w are the endpoints of Λ (since these are equatable along e the induced partitions are the same).

Lemma 3.2.32. *Let Λ be a hypergraph in X, and $\gamma = [p, q]$ be a geodesic transverse to Λ. If $\gamma \cap \Lambda = \{x\}$ and p and q lie in different elements of the partition of $Lk(x) - \Lambda$, then p and q lie in different components of $X - \Lambda$.*

Before we prove this, we need to define some technology.

Definition 3.2.33 (Hypergraph retraction)**.** Let X be a $CAT(0)$ space and Λ a hypergraph in X. The projection map

$$\pi_\Lambda : X \to \Lambda$$

maps every point in X to its nearest point in Λ. Since Λ is convex, this map is a deformation retraction. That is, we have a homotopy π_Λ^δ from the identity to π_Λ.

Let Λ be a hypergraph, e an edge of Λ, and $\epsilon > 0$ be such that the length of any edge in Λ is greater than 2ϵ. Define $int_\epsilon(e)$ as the set of points in e lying at distance (in X) at least ϵ from the endpoints of e. Recall that $m(e)$ is the midpoint of e.

Definition 3.2.34 (Λ-balanced paths)**.** Let Λ be a hypergraph in X and $\epsilon > 0$. For points p, q lying in the same component of $X - \Lambda$, a (Λ, ϵ)-*balanced path* from p to q is a path σ starting at p and ending at q such that:

i) $\sigma \subseteq \mathcal{N}_\epsilon(\Lambda) - \Lambda$,

ii) for any edge e of Λ and any $y, y' \in int_\epsilon(e)$, $|(\pi_\Lambda)^{-1}(y) \cap \sigma| = |(\pi_\Lambda)^{-1}(y') \cap \sigma|$ and is even,

iii) and if σ' is a subpath of σ starting and ending in $\mathcal{N}_\epsilon(v)$ for some vertex v of Λ, then for any edge e of Λ and any $y, y' \in int_\epsilon(e)$, $|(\pi_\Lambda)^{-1}(y) \cap \sigma| = |(\pi_\Lambda)^{-1}(y') \cap \sigma|$ and is even.

Given such a path, we define

$$M_{\Lambda,\epsilon}(\sigma) = \frac{1}{2} \sum_{e \in E(\Lambda \cap \pi_\Lambda(\sigma))} \max_{y \in int_\epsilon(e)} |(\pi_\Lambda)^{-1}(y) \cap \sigma| = \frac{1}{2} \sum_{e \in E(\Lambda \cap \pi_\Lambda(\sigma))} |(\pi_\Lambda)^{-1}(m(e)) \cap \sigma|.$$

Note that this is effectively the length of σ.

By considering the retraction map, we see that, under certain circumstances, such paths exist.

Lemma 3.2.35. *Let Λ be a hypergraph in X, v a vertex of Λ, $\epsilon > 0$, and let $p, q \in \mathcal{N}_\epsilon(v) - \Lambda$ be points lying in the same component of $X - \Lambda$. There exists a (Λ, ϵ)-balanced path between them.*

Proof. Let γ be a path from p to q not intersecting Λ. By taking δ close to 1, we have that $\sigma_\delta := \pi^\delta_\Lambda(\gamma) \subseteq \mathcal{N}_\epsilon(\Lambda) - \Lambda$. Since σ_δ maps to a path in Λ, which is a tree, it immediately follows that by taking δ close to 1, and after a small homotopy, for any $y \in \Lambda$, $|(\pi_\Lambda)^{-1}(y) \cap \sigma_\delta|$ is even, and in particular finite. Furthermore, for any edge e of Λ and $y, y' \in int_\epsilon(e)$, we can ensure that $|(\pi_\Lambda)^{-1}(y) \cap \sigma_\delta| = |(\pi_\Lambda)^{-1}(y') \cap \sigma_\delta|$. Finally, after a small homotopy, we can also ensure that if σ' is a subpath of σ_δ starting and ending in $\mathcal{N}_\epsilon(v)$ for some vertex v of Λ, then for any edge e of Λ and any $y, y' \in int_\epsilon(e)$, $|(\pi_\Lambda)^{-1}(y) \cap \sigma| = |(\pi_\Lambda)^{-1}(y') \cap \sigma|$ and is even. $\square$

We can now prove Lemma 3.2.32.

Proof of Lemma 3.2.32. Let κ be the length of the shortest edge in $X^{(1)}$: since G acts properly and cocompactly, $\kappa > 0$. Choose $\epsilon << \kappa/4$. For a path σ, let $l_\epsilon(\sigma) = \lfloor l(\sigma)/\epsilon \rfloor$.

Note that if x is a vertex and u, v lie in two distinct components of a partition of $Lk(x) - \Lambda$, then for any path σ connecting u and v, $l_\epsilon(\sigma) \geq 1$.

We may assume x is a vertex. If not, let x lie on an edge e of Λ with endpoints u, v. We remark that the partitions of $Lk(u) - \Lambda$ and $Lk(v) - \Lambda$ are equatable along e. First, shrink the path σ to lie in $\mathcal{N}_\epsilon(x)$: let this path be σ' with endpoints s, t. Then without loss of generality, s and p lie in the same element of the partition of $Lk(x) - \Lambda$, and t and q lie in the same element of the partition of $Lk(x) - \Lambda$. Now, slide the path σ' along e to $\mathcal{N}_\epsilon(u)$ to find a path σ'' with endpoints s', t' satisfying the requirements of Lemma 3.2.32. We see that p and s' lie in the same element of the partition of $Lk(u) - \Lambda$, and q and t' lie in the same element of the partition of $Lk(u) - \Lambda$. Furthermore, p and s' lie in the same component of $X - \Lambda$, and q and t' lie in the same component of $X - \Lambda$.

Let P be the partition of $Lk(x)$ through which Λ passes. Let P_1 be the element of P containing p and P_2 the element of P containing q. We may assume that $p, q \in \mathcal{N}_\epsilon(\Lambda)$: otherwise, choose w_i lying in P_i such that $w_i \in \mathcal{N}_\epsilon(\Lambda)$. Then w_1 and p lie in the same component of $X - \Lambda$ and w_2 and q lie in the same component of $X - \Lambda$. Suppose p and q lie in the same component of $X - \Lambda$. We first choose $p' \in P_1$, $q' \in P_2$ and σ a (Λ, ϵ)-balanced path between p' and q' so that the pair $(M_{\Lambda,\epsilon}(\sigma), l_\epsilon(\sigma))$ is minimal by lexicographic ordering amongst all such p', q', σ. We induct on $M_{\Lambda,\epsilon}(\sigma)$.

If $M_{\Lambda,\epsilon}(\sigma) = 0$, then p' and q' are connected by a path lying in $\mathcal{N}_\epsilon(x) - \Lambda$, a contradiction. If $M_{\Lambda,\epsilon}(\sigma) = 1$, then σ passes along exactly one edge e of Λ: it follows that the partitions of $X - \Lambda$ at the endpoints of e are not equatable along e, or that $P_1 = P_2$, a contradiction.

Otherwise $M_{\Lambda,\epsilon}(\sigma) = m \geq 2$. Suppose the first and last edges of Λ traversed by σ are the same edge e. Note that we may always travel along σ to put ourselves in the situation assumed above: this is analogous to the classical situation of pushing to a leaf in a tree for graph theory arguments. In particular, if this is not true, then move along σ, starting at q', until we return to $\mathcal{N}_\epsilon(x)$. Let s be the point we reach in $\mathcal{N}_\epsilon(x)$. If s is in the same component as q' in P then $M_{\Lambda,\epsilon}(\sigma|_{[p',s]}) \leq M_{\Lambda,\epsilon}(\sigma)$, and $l(\sigma|_{[p',s]}) < l(\sigma) - \epsilon$, so that

$$(M_{\Lambda,\epsilon}(\sigma|_{[p',s]}), l_\epsilon(\sigma|_{[p',s]})) < (M_{\Lambda,\epsilon}(\sigma), l_\epsilon(\sigma)),$$

a contradiction as p', q', σ were chosen so this pair was minimal. If s is in the same component as p' in P, and is not equal to p', then $M_{\Lambda,\epsilon}(\sigma|_{[s,q']}) \leq M_{\Lambda,\epsilon}(\sigma)$, and $l_\epsilon(\sigma|_{[s,q']}) < l_\epsilon(\sigma)$, so that again

$$(M_{\Lambda,\epsilon}(\sigma|_{[s,q']}), l_\epsilon(\sigma|_{[s,q']})) < (M_{\Lambda,\epsilon}(\sigma), l_\epsilon(\sigma)),$$

a contradiction. Therefore, if $s \neq p'$, then s lies in a different component to q' in P: we have $M_{\Lambda,\epsilon}(\sigma|_{[s,q']}) \leq M_{\Lambda,\epsilon}(\sigma)$. Since s is not in the same component as p', and we have chosen ϵ sufficiently small, we can see that there is no path of length less than ϵ between s and p', so that $l(\sigma|_{[s,q']}) \leq l(\sigma) - \epsilon$. Furthermore, by the definition of $M_{\Lambda,\epsilon}(\sigma)$, $M_{\Lambda,\epsilon}(\sigma|_{[s,q']}) \leq M_{\Lambda,\epsilon}(\sigma) - 1$, and hence by induction s must lie in a separate component of $X - \Lambda$ to q', a contradiction as q' is connected to s by a path not intersecting Λ. Therefore by induction we have that $s = p'$.

Let y be the endpoint of e distinct from x, let α be the the point obtained by pushing p' along σ to $\mathcal{N}_\epsilon(y)$, and similarly β be the the point obtained by pushing q' along σ to $\mathcal{N}_\epsilon(y)$. Let σ' be the subpath of σ connecting α and β.

If α and β lie in the same component of the partition of $Lk(y) - \Lambda$, then the partitions are not equatable along e, a contradiction. Otherwise $M_{\Lambda,\epsilon}(\sigma') < M_{\Lambda,\epsilon}(\sigma)$, and so by induction α and β lie in distinct components of $X - \Lambda$. As p' is connected to α by a path not intersecting Λ, and q' to β, we see that p' and q' lie in distinct components. Since p is connected to p' and q is connected to q' by a path not intersecting Λ, the result follows. □

3.2.7 Hypergraph stabilizers and wallspaces

We now want to use the construction of a cube complex dual to a system of walls, as found in [HW14]. For a group this was first introduced by Sageev [Sag95]: for wallspaces the procedure was first used in [NR03] and [Wis04], before being described in terms of wallspaces in [Nic04] and [CN05].

The definition of a wallspace found in [HW14] is slightly more general than the one we require: we may restrict to the case that X is endowed with a metric.

Definition 3.2.36 (*Walls*)**.** Let X be a metric space. A *wall* is a pair $\{U, V\}$ such that $X = U \cup V$. The *open halfspaces* associated to the wall are $U - (U \cap V)$ and $V - (U \cap V)$. We say a wall *betwixts* a point x if $x \in U \cap V$, and *separates* the points x, y if x and y lie in distinct open halfspaces. If $\mathcal{W}$ is a collection of walls, we write $\#_{\mathcal{W}}(x, y)$ for the number of walls in $\mathcal{W}$ separating x and y.

Definition 3.2.37 (*Wallspace*)**.** A *wallspace* is a pair $(X, \mathcal{W})$, where X is a connected metric space and $\mathcal{W}$ is a collection of walls in X such that;

i) for any $x \in X$, finitely many walls in $\mathcal{W}$ betwixt x,

ii) for any $x, y \in X$, $\#_{\mathcal{W}}(x, y) < \infty$,

iii) and there are no duplicate walls that are genuine partitions.

We say a group G *acts* on a wallspace $(X, \mathcal{W})$ if G acts on X and $G \cdot \mathcal{W} = \mathcal{W}$.

Definition 3.2.38 (Λ *walls*)**.** Let Λ be a vertex or edge hypergraph in X, with disjoint components $X - \Lambda = \{U_\Lambda^i\}_i$. For each U_Λ^i, let $V_\Lambda^i = X - \overline{U_\Lambda^i}$. The *set of* Λ *walls* is the set

$$\mathcal{W}_\Lambda = \left\{ \{\overline{U_\Lambda^i}, \overline{V_\Lambda^i}\} \ : \ U_\Lambda^i \text{ a component of } X - \Lambda \right\}.$$

The *hypergraph wallspace* is the set of walls

$$\mathcal{W} = \cup_\Lambda \mathcal{W}_\Lambda,$$

where we remove any duplicate walls.

We now show that the pair $(X, \mathcal{W})$ is a wallspace. There are several easy but technical steps to this.

Lemma 3.2.39. *Let X be a polygonal complex satisfying the requirements of Theorem A, and let Λ be a hypergraph in X with components $X - \Lambda = \sqcup_i U_\Lambda^i$. Let $H_\Lambda = Stab_G(\Lambda)$, and for any i, let $H_{\Lambda,i} = Stab_{H_\Lambda}(U_\Lambda^i)$. Then $H_{\Lambda,i}$ acts cocompactly on ∂U_Λ^i.*

This Lemma follows immediately from the proof of [HW14, Theorem 2.9]. We include the argument here for completeness. For a set A in a metric space (X, d), we define the *frontier of* A as the set $\partial_f A = \{x \in X \,|0 < d(x, A) \leq 1\}$. The choice of the constant 1 here is not specific: any choice of $\epsilon > 0$ would work.

Proof. Note that H_Λ acts cocompactly on Λ and so on $\partial_f\Lambda$. Furthermore H_Λ preserves the partition of $\partial_f\Lambda$ into $U_\Lambda^i \cap \partial_f\Lambda$. Hence $H_{\Lambda,i}$ acts properly discontinuously and cocompactly on $\partial_f U_\Lambda^i$, and therefore on ∂U_Λ^i. $\square$

Lemma 3.2.40. *There are finitely many G-orbits of walls in $\mathcal{W}$.*

Proof. There are finitely many G-orbits of hypergraphs Λ, and there are finitely many H_Λ orbits of U_Λ^i. The result follows. $\square$

Lemma 3.2.41. *The pair $(X, \mathcal{W})$ is a wallspace.*

Proof. First, we note that, since the set of walls is acted upon cofinitely by G, and each wall has a cocompact stabilizer, for any point x there are finitely many walls betwixting x. In a similar manner we can observe $\#_{\mathcal{W}}(x, y) < \infty$ for any x and y. $\square$

Therefore, we have constructed a wallspace for X: we can understand separation in the wallspace as follows. Recall that we write $\#_\Lambda(x, y)$ for the number of hypergraphs separating x and y, and $\#_{\mathcal{W}}(x, y)$ for the number of walls separating x and y.

Lemma 3.2.42. *Let $(X, \mathcal{W})$ be the wallspace constructed for Lemma 3.2.41. Then $\#_{\mathcal{W}}(x, y) \geq \#_\Lambda(x, y)$.*

Proof. Note that if a hypergraph Λ separates x and y, by taking i such $x \in U_\Lambda^i$, it follows that W_Λ^i separates x and y. The result follows. $\square$

Next, we discuss transverse walls.

Definition 3.2.43 (*Transverse*)**.** Two walls $W = \{U, V\}$ and $W' = \{U', V'\}$ are *transverse* if each of the intersections $U \cap U', U \cap V', V \cap U', V \cap V'$ are nonempty.

There is an easier formulation for this definition.

Lemma 3.2.44. *Two distinct walls $W_\Lambda^i, W_{\Lambda'}^j$ are transverse if and only if $\partial U_\Lambda^i \cap \partial U_{\Lambda'}^j$ is non-empty. In particular the walls are transverse only if $\Lambda \cap \Lambda'$ is non-empty.*

Using this we can now move on to cubulating groups acting on polygonal complexes.

3.2.8 Cubulating groups acting on polygonal complexes

We now understand the structure of the hypergraph stabilisers and the separation in the wallspaces $(X, \mathcal{W})$.

For a metric polygonal complex X, let $D(X)$ be the maximal circumference of a polygonal face in X. We will be considering G acting properly discontinuously and cocompactly on a polygonal complex X so that $D(X) = D(G\backslash X)$ is finite.

Lemma 3.2.45. *Let X be a simply connected $CAT(0)$ polygonal complex with $G\backslash X$ gluably edge π-separated. Let γ be a finite geodesic in X of length at least $4D(X)$. There exists an edge hypergraph Λ that separates the endpoints of any finite geodesic extension of γ.*

Proof. Since γ is of length at least $4D(X)$, we can find a subgeodesic δ of γ of length at least $2D(X)$ that starts at a point $v \in X^{(1)}$ and ends at $w \in X^{(1)}$.

If δ passes through the interior of a 2-cell f then, as δ is of length at least $2D(X)$, it meets the boundary ∂f at two points u_1, u_2. The sides of the polygonal faces are geodesic, and geodesics are unique in $CAT(0)$ spaces, so that there must exist a vertex w in ∂f lying between u_1 and u_2.

Choose a cutset C in $Lk(w)$ containing f, and let P be a chosen partition of $\pi_0(Lk(w) - C)$ so that the endpoints of f lie in different elements of P (this must exist by assumption). Let Λ be any hypergraph passing through (C, P) in $Lk(w)$: by Lemma 3.2.32, Λ separates the endpoints of the subpath of δ between u_1 and u_2: as geodesics in X are unique, it follows that Λ intersects any geodesic extension of δ exactly once, and so separates the endpoints of any geodesic extension of δ.

Otherwise δ lies strictly in $X^{(1)}$: δ is of length at least $2D(X)$ and so it must intersect at least two primary vertices. Therefore δ contains an edge of the form $[u_1, u_2]$ for some primary vertices u_1, u_2: this edge must be geodesic. Furthermore, the geodesic $[u_1, u_2]$ contains a secondary vertex s. Let P be a partition of $Lk(s) - E(Lk(s))$ so that the endpoints of $[u_1, u_2]$ lie in different elements of P (this must exist by assumption). Let Λ be the hypergraph passing through $(E(Lk(s)), P)$ in $Lk(s)$: it follows by Lemma 3.2.32 that Λ separates the endpoints of any finite geodesic extension of δ. $\square$

Similarly, we have the following.

Lemma 3.2.46. *Let X be a simply connected $CAT(0)$ polygonal complex with $G\backslash X$ gluably vertex π-separated. Let γ be a finite geodesic in X of length at least $4D(X)$. There exists a vertex hypergraph Λ that separates the endpoints of any finite geodesic extension of γ.*

Proof. Again, since γ is of length at least $4D(X)$, we can write $\gamma = \gamma_1 \cdot \delta \cdot \gamma_2$, where each γ_i is of length at least $D(X)/2$, and δ is a path of length between $D(X)$ and $2D(X)$ that starts at a point $v \in X^{(1)}$ and ends at $w \in X^{(1)}$.

First suppose that δ contains a nontrivial subpath, δ', which contains exactly one point of $X^{(1)}$, u, in its interior. Let e be the edge of X containing u. Since δ' is geodesic, we see that $\iota(\delta')$ and $\tau(\delta')$ lie in two distinct faces F, F', both adjacent to e. In $Lk(\iota(e))$, F, F' are two edges adjacent to e, and so, as $G\backslash X$ is gluably π-separated. there exists a π-separated cutset $C \ni e$ and partition P of $\pi_0(Lk(\iota(e)) - C)$ with F, F' lying in distinct elements of P. Let Λ be any hypergraph passing through (C, P) in $Lk(\iota(e))$: by Lemma 3.2.32 Λ separates the endpoints of δ', and hence the endpoints of γ.

Otherwise, we may observe that δ contains a subpath lying completely in $X^{(1)}$: as δ is of length at least $D(X)$, it must therefore contain a vertex of X. We have not subdivided X, and so v is a primary vertex of X. Let δ_1, δ_2 be the two subpaths of γ incident to v: as γ is geodesic, $d_{Lk(v)}(\delta_1, \delta_2) \geq \pi$. Let C be the vertex cutset such that γ_1 and γ_2 lie in different components of $Lk(v) - C$ and let P be a chosen partition of $\pi_0(Lk(v) - C)$ separating γ_1 and γ_2 (this exists as $G\backslash X$ is gluably vertex π-separated). Let Λ be any vertex hypergraph passing through (C, P) in $Lk(v)$: by Lemma 3.2.32 this separates γ_1 and γ_2, and so separates the endpoints of γ. $\square$

We now turn our attention to finding codimension-1 subgroups. We first note the following lemma concerning $CAT(0)$ geometry.

Lemma 3.2.47. *Let Y be a $CAT(0)$ space and let γ_1, γ_2 be infinite one-ended geodesics starting from the same point. If there exists $r > 0$ such that $\gamma_1 \subseteq \mathcal{N}_r(\gamma_2)$, then $\gamma_1 = \gamma_2$.*

Proof. Let p be the common start point of γ_1, γ_2 and let $\theta = \angle_p(\gamma_1, \gamma_2)$. Since $\gamma_1 \subseteq \mathcal{N}_r(\gamma_2)$, for all $t > 0$ there exists $t'(t) > 0$ such that $d(\gamma_1(t), \gamma_2(t')) \leq r$. However, $d(\gamma_1(t), p) \to \infty$ as $t \to \infty$, so that $d(\gamma_2(t'(t)), p) \to \infty$ as $t \to \infty$. Consider the Euclidean comparison triangle for the geodesics $\gamma_1(t)$ and $\gamma_2(t'(t))$: this has third side length at most r, and so has angle at p of $\theta(t) \to 0$ as $t \to \infty$. However, $\theta \leq \theta(t)$ for all t, and so $\theta = 0$. It follows that $\gamma_1 = \gamma_2$ in a closed neighbourhood of p, and so the set $\{t \ : \ \gamma_1(t) = \gamma_2(t)\}$ is clopen. The result follows. $\square$

Using this we can prove that hypergraph stabilizers have subgroups that are codimension-1 in G. Let G be a group with finite generating set S and let Γ be the Cayley graph of G with respect to S. A subgroup H of G is *codimension*-1 if the graph $H\backslash\Gamma$ has at least two ends, i.e. for some compact set K, $H\backslash\Gamma - K$ contains at least two infinite components.

Lemma 3.2.48. *Let G be a group acting properly discontinuously and cocompactly on a simply connected $CAT(0)$ polygonal complex X such that $G\backslash X$ is (weakly) gluably π-separated. Let Λ be a hypergraph in X. For any component U_Λ of $X - \Lambda$, the group*

$$H_U = Stab_{Stab(\Lambda)}(U_\Lambda)$$

is virtually free, and is quasi-isometrically embedded and codimension-1 in G.

This again follows by [HW14, Theorem 2.9]: we provide a direct proof for completeness.

Proof. We prove this in the case that Λ is an edge hypergraph: the case for vertex hypergraphs is identical. By Lemma 3.2.30, Λ is a convex tree. Since $\partial U_\Lambda \subseteq \Lambda$, ∂U_Λ is a convex tree. By Lemma 3.2.39, H_U acts properly discontinuously and cocompactly on ∂U_Λ: it follows that H_U is virtually free and quasi-isometrically embedded in G. Furthermore, by Lemma 3.2.32 $X - \Lambda$ consists of at least two path-connected components, $\{U^i_\Lambda\}$. Let $V_\Lambda = X - \overline{U_\Lambda}$.

Let e_1 and e_2 be vertices that lie in distinct components of $Lk(v) - C$ such that, in X, e_1 is an edge lying in $U_\Lambda \cup v$ and e_2 an edge lying in $V_\Lambda \cup v$. We construct two geodesics γ_1 and γ_2: let the first edge of γ_1 be e_1, and let w be the endpoint of e_1 distinct from v. Since the links of vertices have no vertices of degree 1 and have girth at least 2π, it follows that there exists a vertex or edge, a_1, in $\Gamma = Lk(w)$ so that $d_\Gamma(e_1, m(a_1)) \geq \pi$, and so we can extend e_1 to a geodesic $[v, m(a_1)]$. We can continue in this fashion to construct a one-ended geodesic γ_1 that, by Lemma 3.2.32, lies in $U_\Lambda \cup v$ and (as geodesics are unique in X) intersects Λ exactly once. Construct the geodesic γ_2 similarly, with first edge e_2 so that γ_2 intersects Λ exactly once and lies in $V_\Lambda \cup v$.

By Lemma 3.2.47, it follows that for any $r > 0$, $\gamma_1, \gamma_2 \nsubseteq \mathcal{N}_r(\partial U_\Lambda)$. Therefore $H_U\backslash X - H_U\backslash \partial U_\Lambda$ consists of at least two infinite components: $H_U\backslash U_\Lambda$ and $H_U\backslash V_\Lambda$. As G is quasi-isometric to X, and H_U is quasi-isometric to ∂U_Λ, the result follows. $\square$

We will use Hruska–Wise's [HW14] generalisation of Sageev's construction of a $CAT(0)$ cube complex dual to a collection of codimension-1 subgroups, as introduced in [Sag95].

Definition 3.2.49 (*Orientation*)**.** Let $(X, \mathcal{W})$ be a wallspace and $W = \{U, V\}$ a wall. An *orientation* of W is a choice $c(W) = (\overleftarrow{c(W)}, \overrightarrow{c(W)})$ of ordering of the pair W. An *orientation* of $\mathcal{W}$ is an orientation of each wall W in $\mathcal{W}$.

A 0-cube in the dual cube complex $\mathcal{C}(X, \mathcal{W})$ corresponds to a choice of orientation c of $\mathcal{W}$ such that that for any element $x \in X$, x lies in $\overleftarrow{c(W)}$ for all but finitely many $W \in \mathcal{W}$, and $\overleftarrow{c(W)} \cap \overleftarrow{c(W')} \neq \emptyset$ for all $W, W' \in \mathcal{W}$. Two 0-cells are joined by a 1-cell if there exists a unique wall to which they assign opposite orientations. Inductively add an n-cube to $\mathcal{C}(X, \mathcal{W})$ whenever its skeleton exists.

Sageev analysed the properness and cocompactness of the group action on $\mathcal{C}(X, \mathcal{W})$ in [Sag97], and this was generalized by Hruska–Wise in [HW14]. We will use the following, as they are the easiest criteria to verify in our setting.

Theorem. *[HW14, Theorem 1.4] Suppose G acts on a wallspace $(X, \mathcal{W})$, and the action on the underlying metric space (X,d) is metrically proper. If there exists constants $\kappa, \epsilon > 0$ such that for any $x, y \in X$,*

$$\#_{\mathcal{W}}(x, y) \geq \kappa d(x, y) - \epsilon,$$

then G acts metrically properly on $C(X, \mathcal{W})$.

Theorem. *[HW14, Lemma 7.2] Let G act on a wallspace (X, W). Suppose there are finitely many orbits of collections of pairwise transverse walls in X. Then G acts cocompactly on $C(X, \mathcal{W})$.*

This is sufficient to prove Theorem A.

Proof of Theorem A. If $G\backslash X$ is gluably weakly π-separated, then by Lemma 3.2.48, G contains a virtually free codimension-1 subgroup.

Now suppose $G\backslash X$ is a gluably π-separated complex. X is locally finite, and G acts properly discontinuously on X, so acts metrically properly on X. Construct the hypergraph wallspace for X. Then by Lemmas 3.2.45 and 3.2.46,

$$\#_{\Lambda}(p, q) \geq d_X(p, q)/4D(X) - 1.$$

By Lemma 3.2.42, this implies that

$$\#_{\mathcal{W}}(p, q) \geq d_X(p, q)/4D(X) - 1 :$$

by [HW14, Theorem 1.4] it follows that G acts properly discontinuously on the cube complex $C(X, \mathcal{W})$.

Now suppose that G is hyperbolic, so that X is also hyperbolic. As hypergraphs are convex and hypergraph stabilisers are cocompact, by [GMRS98] (c.f. [Sag97]) there is an upper bound on the number of pairwise intersecting hypergraphs. For any point $x \in \Lambda$ there is a finite upper bound on the number of components of $X - \Lambda$ intersecting x, and so by Lemma 3.2.40, we see there is an upper bound on the size of a collection of pairwise transverse walls. As G acts cofinitely on the set of walls, it follows that the hypothesis of [HW14, Lemma 7.2] are met, and so G acts cocompactly on the $CAT(0)$ cube complex $C(X, \mathcal{W})$: by [Ago13, Theorem 1.1], we conclude that G is virtually special. □

3.3 Finding separated cutsets by computer search

In this short section, we discuss how to find separated cutsets by computer search. Let Γ be a finite metric graph, and let $I(\Gamma) = V(\Gamma)$ or $E(\Gamma)$. Define $d_I(x, y) = d_\Gamma(x, y)$ if $x, y \in V(\Gamma)$ and $d_I(x, y) = d_\Gamma(m(x), m(y))$ if $x, y \in E(\Gamma)$. Let $\sigma > 0$. The σ-separated cutsets of Γ that lie in $I(\Gamma)$ can be found in the following way: we can define a dual graph $\bar{\Gamma}$ by $V(\bar{\Gamma}) = I(\Gamma)$, and

$$E(\bar{\Gamma}) = \{(x, y) \in I(\Gamma)^2 \ : \ x \neq y \text{ and } d_I(x, y) < \sigma\}.$$

Finding σ-separated cut sets in Γ then corresponds to finding independent vertex sets in $\bar{\Gamma}$ and checking if they are cut sets in Γ. Importantly, finding independent vertex sets can be done relatively efficiently.

See https://github.com/CJAshcroft/Graph-Cut-Set-Finder for the implementation of the above algorithm, and for the code used to find cutsets in the following sections.

3.4 Triangular buildings

In the following section, we prove Corollary B. In [KV10] and [CKV12] all groups acting simply transitively on triangular buildings whose links are the minimal generalized quadrangle (see Figure 3.8) were classified. We apply Theorem A to these groups, proving they are virtually special by considering the separation of the minimal generalized quadrangle.

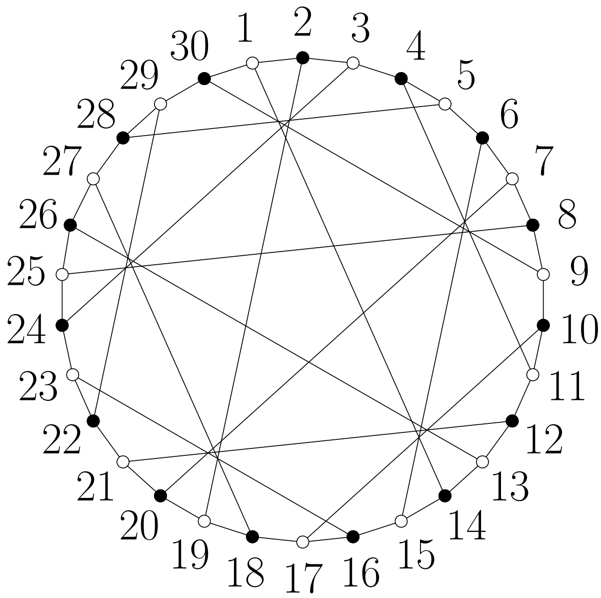

Fig. 3.8 The minimal generalized quadrangle.

x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i		
1	2	14	30	11	4	10	12	21	12	20	22
2	1	3	19	12	11	13	21	22	21	23	29
3	2	4	24	13	12	14	26	23	16	22	24
4	3	5	11	14	1	13	15	24	3	23	25
5	4	6	28	15	6	14	16	25	8	24	26
6	5	7	15	16	15	17	23	26	13	25	27
7	6	8	20	17	10	16	18	27	18	26	28
8	7	9	25	18	17	19	27	28	5	27	29
9	8	10	30	19	2	18	20	29	22	28	30
10	9	11	17	20	7	19	21	30	1	9	29

Table 3.1 Edge incidences for the minimal generalized quadrangle

Lemma 3.4.1. *Let* Γ *be the minimal generalized quadrangle equipped with the combinatorial metric. Then* Γ *is weighted (strongly) edge* 3*-separated.*

Proof. By a computer search, we find the following exhaustive list of 3-separated edge cut sets in Γ:

$$C_1 = \{(x_1, x_2), (x_4, x_5), (x_7, x_{20}), (x_9, x_{10}), (x_{12}, x_{13}), (x_{15}, x_{16}), (x_{18}, x_{27}), \\ (x_{22}, x_{29}), (x_{24}, x_{25})\},$$

$$C_2 = \{(x_1, x_2), (x_4, x_{11}), (x_6, x_{15}), (x_8, x_9), (x_{13}, x_{26}), (x_{17}, x_{18}), (x_{20}, x_{21}), (x_{23}, x_{24}), (x_{28}, x_{29})\},$$

$$C_3 = \{(x_1, x_{14}), (x_3, x_4), (x_6, x_7), (x_9, x_{10}), (x_{12}, x_{21}), (x_{16}, x_{23}), (x_{18}, x_{19}), (x_{25}, x_{26}), (x_{28}, x_{29})\},$$

$$C_4 = \{(x_1, x_{14}), (x_3, x_{24}), (x_6, x_7), (x_9, x_{10}), (x_{12}, x_{21}), (x_{16}, x_{23}), (x_{18}, x_{19}), (x_{25}, x_{26}), (x_{28}, x_{29})\},$$

$$C_5 = \{(x_1, x_{30}), (x_3, x_4), (x_6, x_{15}), (x_8, x_{25}), (x_{10}, x_{17}), (x_{12}, x_{13}), (x_{19}, x_{20}), (x_{22}, x_{23}), (x_{27}, x_{28})\},$$

$$C_6 = \{(x_1, x_{30}), (x_3, x_{24}), (x_5, x_{28}), (x_7, x_8), (x_{10}, x_{11}), (x_{13}, x_{26}), (x_{15}, x_{16}), (x_{18}, x_{19}), (x_{21}, x_{22})\},$$

$$C_7 = \{(x_2, x_3), (x_5, x_6), (x_8, x_{25}), (x_{10}, x_{11}), (x_{13}, x_{14}), (x_{16}, x_{23}), (x_{18}, x_{27}), (x_{20}, x_{21}), (x_{29}, x_{30})\},$$

$$C_8 = \{(x_2, x_3), (x_5, x_{28}), (x_7, x_{20}), (x_9, x_{30}), (x_{11}, x_{12}), (x_{14}, x_{15}), (x_{17}, x_{18}), (x_{22}, x_{23}), (x_{25}, x_{26})\},$$

$$C_9 = \{(x_2, x_{19}), (x_4, x_5), (x_7, x_8), (x_{10}, x_{17}), (x_{12}, x_{21}), (x_{14}, x_{15}), (x_{23}, x_{24}), (x_{26}, x_{27}), (x_{29}, x_{30})\},$$

$$C_{10} = \{(x_2, x_{19}), (x_4, x_{11}), (x_6, x_7), (x_9, x_{30}), (x_{13}, x_{14}), (x_{16}, x_{17}), (x_{21}, x_{22}), (x_{24}, x_{25}), (x_{27}, x_{28})\}.$$

Γ is connected and contains no vertices of degree 1. The cutsets sets are 3-separated, and $\cup_i C_i = E(\Gamma)$. In fact, each cutset is minimal, and so is certainly proper. Furthermore, every edge appears in exactly two cutsets: assigning each cutset weight 1 we see that the weight equations are satisfied, and so Γ is weighted edge 3-separated.

In fact, by a computer search we can see that Γ satisfies the conditions of Lemma 3.2.6, and so is weighted strongly edge 3-separated. □

Note that $C_i \cap C_j$ is nonempty for all i and j, so that we are not able to use [HW14, Example 4.3]. However, we can apply Theorem A to prove groups acting properly discontinuously and cocompactly on triangular buildings with the minimal generalized quadrangle as links are virtually special.

Proof of Corollary B. Let X be a simply connected polygonal complex such that every face has at least 3 sides and the link of every vertex is isomorphic to the minimal generalized quadrangle, Γ, and let G be a group acting properly discontinuously and cocompactly on X. Since Γ has girth 8, X can be endowed with a $CAT(-1)$ metric, so that G is hyperbolic. Endow X with the metric that makes each k-gonal face a regular unit Euclidean k-gon, so that X is regular and the length of each edge in the link of a vertex is at least $\pi/3$. As Γ is weighted edge 3-separated with the combinatorial metric, it follows that the links of X are weighted edge π-separated. Hence by Lemma 3.2.24, $G\backslash X$ is gluably π-separated. Furthermore, by Gromov's link condition, X is $CAT(0)$. Therefore, G is hyperbolic and acts properly discontinuously and cocompactly on a simply connected $CAT(0)$ polygonal complex X with $G\backslash X$ gluably π-separated, so acts properly discontinuously and cocompactly on a $CAT(0)$ cube complex by Theorem A, and hence is virtually special by [Ago13, Theorem 1.1]. □

3.5 Application to generalized triangular groups

In this section we prove Theorem 3.5.5 in Section 3.5.1, Corollary C in Section 3.5.2, and Corollary 3.5.20 in Section 3.5.3.

3.5.1 Cubulating generalized ordinary triangle groups

We now consider generalized ordinary triangle groups, constructed in [LMW19] to answer a question of Agol and Wise: note that the case of $k = 2$ corresponds to classical ordinary triangle groups.

The first complex of groups we define uses the notation from [CCKW20] to more easily align with their work. See e.g. [BH99] for further discussion of complexes of groups.

Definition 3.5.1 (Generalized triangle groups)**.** Consider the following complex of groups over $\mathcal{T}$, the poset of all subsets of $\{1, 2, 3\}$. Let X_1, X_2, X_3 be the vertex groups, and A_1, A_2, A_3 the edge groups, with the face group trivial, and homomorphisms $\phi_{i,i+1} : A_i \to X_{i+1}$, $\phi_{i,i-1} : A_i \to X_{i-1}$ for $i = 1, 2, 3$ taken mod 3. Now, consider the coset graph

$$\Gamma_{X_i}(\phi_{i-1,i}(A_{i-1}), \phi_{i+1,i}(A_{i+1})).$$

Fix $k \geq 2$ and let each $A_i = \mathbb{Z}/k$. For graphs Γ_i, let

$$\{D_k^j(\Gamma_1, \Gamma_2, \Gamma_3)\}_j$$

be the family of complexes of groups obtained by choosing X_i and $\phi_{i,i\pm1}$ such that for each i

$$\Gamma_{X_i}(\phi_{i-1,i}(A_{i-1}), \phi_{i+1,i}A_{i+1}) \cong \Gamma_i.$$

A group

$$G_k^j(\Gamma_1, \Gamma_2, \Gamma_3) = \pi_1(D_k^j(\Gamma_1, \Gamma_2, \Gamma_3))$$

is called a *(k-fold) generalized triangle group.*

Bridson and Haefliger considered the developability of a complex of groups in [BH99, III.$\mathcal{C}$]. The following is well known: see e.g. [CCKW20, Theorem 3.1].

Proposition 3.5.2. *Suppose that girth*$(\Gamma_i) \geq 6$ *for each* i*. Then* $G_k^j(\Gamma_1, \Gamma_2, \Gamma_3)$ *acts properly and cocompactly on a triangular complex* $X_k^j(\Gamma_1, \Gamma_2, \Gamma_3)$ *such that the link of each vertex is isomorphic to* $\Gamma \in \{\Gamma_i\}_i$*. If girth*$(\Gamma_1) > 6$*, then* G *is hyperbolic.*

Definition 3.5.3 (Generalized ordinary triangle groups)**.** Consider the following complex of groups. Fix $k \geq 2$, and identify the boundaries of k 2-simplices to construct a simplicial complex $\mathcal{K}$ with three vertices v_1, v_2, v_3, three edges e_1, e_2, e_3, and k 2-simplices. Then $Lk(v_i) \simeq C_{k,2}$, the cage graph on k edges, i.e. the smallest k regular graph of girth 2.

Let $P_i = \pi_1(Lk(v_i))$, and let $G_{0,k}$ be the free group on $2k-2$ letters. Note that we can view $G_{0,k}$ as the fundamental group of the following complex of groups. The underlying complex is $\mathcal{K}$, the vertex groups are P_1, P_2, P_3, and the edge groups are trivial. Now, let $\Gamma_i \looparrowright Lk(v_i)$ be finite-sheeted normal covering graphs, with associated normal subgroups $Q_i \trianglelefteq P_i$. Let D be a complex of groups with underlying complex $\mathcal{K}$ and (finite) vertex groups $V_i = P_i/Q_i$. Since there are choices for the above complex, we will let $D_{0,k}^j(\Gamma_1, \Gamma_2, \Gamma_3)$, $j = 1, \ldots,$ be the finite exhaustive list of possible complexes of groups achieved by the above construction. Form the *(k-fold) generalized ordinary triangular group*

$$G_{0,k}^j(\Gamma_1, \Gamma_2, \Gamma_3) = \pi_1(D_{0,k}^j(\Gamma_1, \Gamma_2, \Gamma_3)) = G_{0,k}/\langle\langle Q_1 \cup Q_2 \cup Q_3 \rangle\rangle.$$

Note that in this definition the graphs Γ_i are covers of $C_{k,2}$ so that they are connected, contain no cut edges, and have girth at least 2. We use the following proposition, the first part of which is as stated in [LMW19, Proposition 3.2], both parts of which follow by an application of [BH99, Theorem III.$\mathcal{C}$.4.17].

Proposition 3.5.4. *[LMW19, Proposition 3.2][BH99, Theorem III.$\mathcal{C}$.4.17]*

If girth$(\Gamma_i) \geq 6$ for each i, then $G^j_{0,k}(\Gamma_1, \Gamma_2, \Gamma_3)$ acts properly discontinuously and cocompactly on a simply connected simplicial complex $X^j(\Gamma_1, \Gamma_2, \Gamma_3)$ with links isomorphic to Γ, where $\Gamma \in \{\Gamma_1, \Gamma_2, \Gamma_3\}$. If, furthermore, girth$(\Gamma_1) > 6$, then $G_{0,k}(\Gamma_1, \Gamma_2, \Gamma_3)$ is hyperbolic.

Theorem A, along with Proposition 3.5.2 and Proposition 3.5.4 above, allow us to cubulate $G^j_k(\Gamma_1, \Gamma_2, \Gamma_3)$ and $G^j_{0,k}(\Gamma_1, \Gamma_2, \Gamma_3)$ when given enough information about each of $\Gamma_1, \Gamma_2, \Gamma_3$. The purpose of this subsection is to provide a way to prove such a group acts properly discontinuously on a $CAT(0)$ cube complex by considering Γ_1 alone. Again, see Section 3.2.1 for the relevant definitions.

Theorem 3.5.5. *Let $\Gamma_i \looparrowright C_{k,2}$ be finite-sheeted covers such that girth$(\Gamma_i) \geq 6$ for each i, and let $G = G^j_{0,k}(\Gamma_1, \Gamma_2, \Gamma_3)$ or $G = G^j_k(\Gamma_1, \Gamma_2, \Gamma_3)$. If Γ_1 is weighted strongly edge 3-separated, then G acts properly discontinuously on a $CAT(0)$ cube complex. If, in addition, G is hyperbolic, then this action is cocompact.*

Now, fix j, let $G = G^j_k(\Gamma_1, \Gamma_2, \Gamma_3)$ or $G = G^j_{0,k}(\Gamma_1, \Gamma_2, \Gamma_3)$, and let $X = X^j_k(\Gamma_1, \Gamma_2, \Gamma_3)$ be as in Proposition 3.5.2 or Proposition 3.5.4. Note that the antipodal graph $\Delta_{G\backslash X}$ is the disjoint union of three components $\Delta_1, \Delta_2, \Delta_3$, such that for any vertex $v \in \Delta_i$ either v is secondary, or $Lk_{G\backslash X}(v) \cong \Gamma_i$. Suppose Γ_1 is a weighted strongly edge 3-separated graph, and endow X with the metric that turns each triangle into a unit equilateral Euclidean triangle: X is $CAT(0)$ with this metric. Then for each $v \in V(\Delta_1)$, $Lk_{G\backslash X}(v)$ is a strongly edge π-separated graph. Since cutsets are proper, we can assign to every cutset the canonical partition: as discussed in Section 3.2.4 this is sufficient for cubulation, and therefore we omit the reference to partitions for the remainder of this section. As in Section 3.2.5 construct the graphs $\underline{\Lambda}_1, \ldots, \underline{\Lambda}_m$ as images of

$$\Sigma_i \looparrowright \Delta_1 \to G\backslash X.$$

In particular, if a vertex v has $Lk_X(v) = \Gamma_1$, we have a hypergraph passing through every π-separated edge cutset in $Lk_X(v) = \Gamma_1$. As in Section 3.2.7, we can again build the system of hypergraph walls.

We now analyse the separation of this complex by hypergraphs.

Lemma 3.5.6. *Suppose that Γ_1 is weighted strongly edge 3-separated, and let γ be a geodesic in X of length at least 100. There exists a hypergraph Λ that separates the endpoints of any finite geodesic extension of γ.*

Proof. Since γ has length at least 100, we may write $\gamma = \beta \cdot \gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdot \delta$ such that $1 \leq l(\gamma_i) \leq \sqrt{3/2}$, $l(\beta), l(\delta) \geq 40$ and the endpoints of each γ_i lie in $X^{(1)}$. We may assume that either:

case *a*) γ_2 contains an edge of $X^{(1)}$ of the form $[u, v]$,

case *b*) or γ_2 contains a subpath that intersects $X^{(1)}$ at exactly two points x, y in ∂T for some 2-cell T.

Now consider case a). There are two subcases to consider.

Case *a.i*) u and v have links isomorphic to Γ_2 and Γ_3 respectively (or vice versa),

Case *a.ii*) or v has link isomorphic to Γ_1.

In case $a.i$), γ_2 contains a secondary vertex x that is opposite to some w with $Lk_X(w) \cong \Gamma_1$. By Lemma 3.2.32, the hypergraph passing through x and w therefore separates the endpoints of γ', and so the endpoints of any geodesic extension of γ.

In case $a.ii$), consider the path γ_3. If γ_3 is not an edge then γ_3 satisfies the hypothesis of case b) using γ_3 in place of γ_2. Otherwise we may assume that $\gamma_2 \cdot \gamma_3 = [u, v] \cdot [v, w]$ for two edges. Now, $d_{Lk(v)}([u, v], [v, w]) \geq \pi$ as $\gamma_2 \cdot \gamma_3$ is geodesic: let C be the cutset separating $[u, v]$ and $[v, w]$ in $Lk(v) \cong \Gamma_1$ (this exists as Γ_1 is strongly 3-separated): by Lemma 3.2.32 the hypergraph passing through C in $Lk(v)$ separates the endpoints of γ.

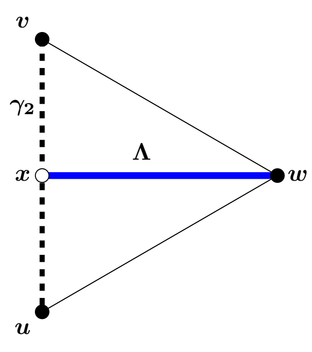

Fig. 3.9 Strongly edge separated analysis: Case a.i)

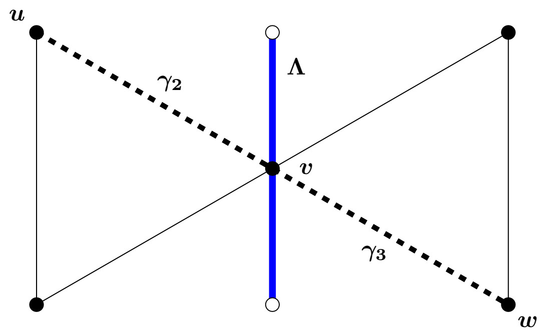

Fig. 3.10 Strongly edge separated analysis: Case a.ii)

For case b) there are three subcases:

case *b.i*) the two paths in ∂T from x to y each contain one of the vertices u, v such that u is primary with $Lk(u) \cong \Gamma_1$, and v is secondary and antipodal to u in ∂T,

case *b.ii*) one of the two paths in ∂T from x to y contains both of the vertices u, v where u is primary with $Lk(u) \cong \Gamma_1$, and v is secondary and opposite to u,

case *b.iii*) or $\gamma_2 = [x, y]$ for some geodesic between $x \in X^{(1)}$ and a primary vertex y that satisfies $Lk(y) \cong \Gamma_1$.

In case b.i), by Lemma 3.2.32 the hypergraph passing through u and v separates the endpoints of γ_2 and so the endpoints of γ.

Consider case $b.ii$). Let T_x, T_y be the two 2-cells adjacent to T containing the vertex x and y respectively, with γ passing through both T_x, T_y. Note that x and y lie on different edges of ∂T. Suppose that γ_1 passes through x and γ_3 passes through y: we may see that by a simple Euclidean geometry argument for angles that either γ_1 or γ_3 satisfies case $b.i$).

In case $b.iii$) extend γ_2 through y until we meet $X^{(1)}$ at a third point z: without loss of generality this can be written $\gamma_2 \cdot \gamma_3 = [x, y] \cdot [y, z]$. Now, as γ is geodesic, we have $d_{Lk(y)}([x, y], [y, z]) \geq 3$. Let C be the cutset in $Lk(y) \cong \Gamma_1$ such that $[x, y]$ and $[y, z]$ are separated by C (this exists as Γ_1 is strongly 3-separated), and let Λ be the hypergraph passing through C in $Lk(y)$. By Lemma 3.2.32 Λ separates the endpoints of γ.

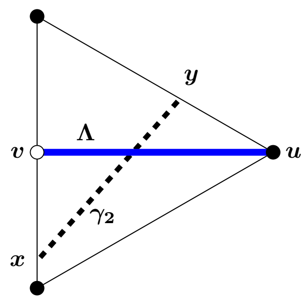

Fig. 3.11 Strongly edge separated analysis: Case b.i)

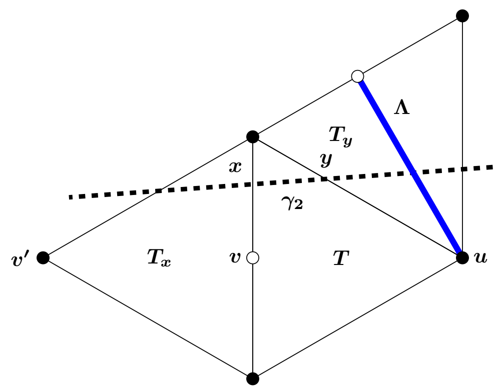

Fig. 3.12 Strongly edge separated analysis: Case b.ii)

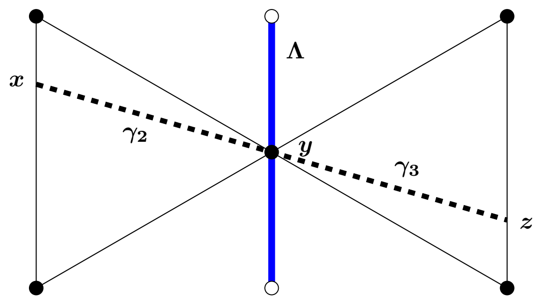

Fig. 3.13 Strongly edge separated analysis: Case b.iii)

$\square$

We can now prove Theorem 3.5.5.

Proof of Theorem 3.5.5. By Proposition 3.5.4, the group G acts properly discontinuously and cocompactly on a simply connected simplicial complex X. Endow this complex with the Euclidean metric: by Gromov's link condition X is $CAT(0)$ and has three types of vertices $\{v_i\}$ where $Lk(v_i) = \Gamma_i$.

If Γ_1 is strongly 3-separated, then by Lemma 3.5.6, we have

$$\#_{\mathcal{W}}(p,q) \geq d_X(p,q)/100 - 1.$$

The results then follow by [HW14, Theorem 5.2] and [HW14, Lemma 7.2] similarly to the proof of Theorem A, using Lemma 3.5.6 in place of Lemma 3.2.45. □

3.5.2 Small girth generalized triangle groups

To prove Corollary C, we now analyse the separation of various small girth graphs considered in [CCKW20]. These graphs arise in the work of [CM95, CD02, CMMP06] and are regular bipartite graphs with girth 6 or 8, diameter 3 or 4, and an edge regular subgroup of the automorphism group. In particular, we have the following.

Lemma. *[CM95] Let* Γ *be one of* $\{F24A, F26A, F40A, F48A\}$. *Then* $Aut(\Gamma)$ *acts vertex transitively.*

Lemma. *[CCKW20, Sections* 4.2, 4.3*] Let* Γ *be one of* $\{F24A, F26A, F40A, F48A, G54\}$*: there exists a subgroup* $H(\Gamma) \leq Aut(\Gamma)$ *that acts freely and transitively on* $E(\Gamma)$ *and preserves the bipartition of* Γ.

We make the following definitions. These are stronger definitions than weighted 3-separated, but allow us to solve the gluing equations more easily, and therefore prove Lemma 3.5.19.

Definition 3.5.7 (Cubic graphs)**.** Let Γ be a finite graph. It is *cubic* if it is connected, bipartite, and trivalent.

Definition 3.5.8 (†-separated graphs)**.** Let Γ be a graph. We say that Γ is *†-separated* if:

i) Γ is cubic,

ii) $girth(\Gamma) = 6$ or 8,

iii) and Γ is disjointly weighted vertex 3-separated by proper cutsets, (so that $\Gamma - C$ consists of exactly three components for each C).

Definition 3.5.9 (Good cubic graphs)**.** A cubic graph is *good* if $girth(\Gamma) = 6$ or 8, $diam(\Gamma) \leq 4$, $Aut(\Gamma)$ acts vertex transitively, and there exists a group $H(\Gamma) \leq Aut(\Gamma)$ that acts freely and transitively on $E(\Gamma)$ and preserves the bipartition of Γ.

In the above definition, for any vertex v of Γ, $H(\Gamma)_v$ is of order three and so cyclically permutes the neighbours of v. Fix a vertex $v_0 \in V(\Gamma)$. For each pair of vertices $v \neq w$, choose an element $\gamma_{v,w} \in Aut(\Gamma)$ with $\gamma_{v,w}v = w$, such that

i) $\gamma_{v,w} = \gamma_{v,v_0}\gamma_{v_0,w}$,

ii) $\gamma_{v,w} = \gamma_{w,v}^{-1}$,

iii) and if $v, w \in V_1$ or $v, w \in V_2$, then $\gamma_{v,w} \in H(\Gamma)$.

For each $v \in V(\Gamma)$ we will let neighbours of v be defined as $w_1(v), w_2(v), w_3(v)$, so that $\gamma_{v_0,v}w_i(v_0) = w_i(v)$. We also assign to $H(\Gamma)_v$ a generator h_v such that $h_v w_1(v) = w_2(v)$, $h_v w_2(v) = w_3(v)$, and so on, i.e. $h_v = \gamma_{v_0,v}h_{v_0}\gamma_{v,v_0}$.

Definition 3.5.10 ($*$-separated cutsets)**.** Let C be a vertex cutset in a graph Γ. We say C is a *$*$-separated cutset* if C is 3-separated, $\Gamma - C$ contains exactly two components, and for any vertex $w \in C$, there are two vertices v, v' adjacent to w such that v and v' lie in distinct components of $\Gamma - C$.

Definition 3.5.11. Let Γ be a good cubical graph, and let $\mathcal{C}$ be a collection of $*$-separated cutsets. For $v \in V(\Gamma)$, we define

$$*(v, i, j)$$

to be the set of all $*$-separated cutsets $C \ni v$ such that $w_i(v)$ and $w_j(v)$ lie in the same connected component of $\Gamma - C$. We further define

$$\mathcal{C}(v, i, j) := \mathcal{C} \cap *(v, i, j).$$

Definition 3.5.12 ($*$-separated graph)**.** Let Γ be a graph. We say that Γ is *$*$-separated* if:

i) Γ is a cubic graph,

ii) Γ is weighted vertex 3-separated by a set $\mathcal{C}$ of $*$-separated cutsets (and hence for any vertex v and any $i \neq j$, $\mathcal{C}(v, i, j)$ is non-empty),

iii) there exists an integer M and positive integers $n(C)$ for each $C \in \mathcal{C}$ such that for any vertex v and any $i \neq j$,

$$\sum_{C \in \mathcal{C}(v,i,j)} n(C) = \frac{M}{3}.$$

For ease, we prove the following lemma.

Lemma 3.5.13. *Let Γ be a good cubic graph. Let $V_1 \sqcup V_2$ be the bipartite partition of vertices, and choose $v_1 \in V_1$. Suppose that there exists a $*$-separated cutset A with $v_1 \in A \in *(v_1, 1, 2)$. Then Γ is $*$-separated.*

Proof. We need to show three separate things. Firstly we show that there exists a collection $\mathcal{C}$ of $*$-separated cutsets so that Γ is vertex 3-separated by $\mathcal{C}$. Recall that we defined the element $\gamma = \gamma_{v_1,v_2}$, the element of $Aut(\Gamma)$ taking v_1 to $v_2 := w_1(v_1) \in V_2$.

Let $H := H(\Gamma)$ be the group acting edge-regularly on Γ and preserving the bipartite partition. Let $B = \gamma \cdot A$, $\mathcal{A} = H \cdot A$, $\mathcal{B} = H \cdot B$, and $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$.

By assumption, $\mathcal{C}(v_1, 1, 2)$ is non-empty. For any vertex v, $\gamma_{v_1,v}\mathcal{C}(v_1, 1, 2) = \mathcal{C}(v, 1, 2)$, and furthermore, $h_v\mathcal{C}(v, 1, 2) = \mathcal{C}(v, 2, 3) = h_v^{-1}\mathcal{C}(v, 1, 3)$. Therefore $\mathcal{C}(v, i, j)$ is non empty for all v and all $i \neq j$. In particular for any vertex v and w, w' adjacent to v there exists a cutset separating w and w'.

Now let u, v be vertices distance at least 3 apart. Note that $d(u, v) \leq 4$ as $diam(\Gamma) \leq 4$. Assume $d(u, v) = 3$, and let

$$p = (u, u_1)(u_1, u_2)(u_2, v)$$

be any edge path between u and v.

Now, suppose without loss of generality that $u = w_1(u_1)$ and $u_2 = w_2(u_1)$. Then choosing a cutset $C \in \mathcal{C}(u_1, 1, 3)$, u and u_2 lie on separate components of $\Gamma - C$. Since C is 3-separated, and $u_1 \in C$, it follows that u, v are not elements of C. As u is adjacent to u_1 and v is adjacent to u_2, it follows that u and v lie in different components of $\Gamma - C$.

If $d(u, w) = 4$, then we repeat the argument for

$$p = (u, u_1)(u_1, u_2)(u_2, u_3)(u_3, v)$$

and for C the cutset containing u_2 and separating u_1 and u_3. It now follows by Lemma 3.2.10 that Γ is vertex 3-separated.

Finally we wish to find the positive integers M and $n(C)$. This immediately implies the weight equations can be solved, and so Γ is weighted vertex 3-separated with respect to $\mathcal{C}$. The proof is similar to the proof of Lemma 3.2.22 concerning vertex transitive automorphism groups.

Let $\tilde{\mathcal{C}} = H \cdot A \cup H \cdot B$ counted *with multiplicity.* Let $u, v \in V_1$. For $i = 1, 2, 3$, we have $\gamma_{u,v}(w_i(u)) = w_i(v)$. It follows that for $C \in \tilde{\mathcal{C}}$,

$$C \in \mathcal{C}(u, i, j) \iff \gamma_{u,v} C \in \mathcal{C}(v, i, j).$$

Similarly

$$C \in \mathcal{C}(u, 1, 2) \iff h_u C \in \mathcal{C}(u, 2, 3) \iff h_u^2 C \in \mathcal{C}(u, 1, 3).$$

Let $n(C) = |\{C' \in \tilde{\mathcal{C}} \ : \ C' = C\}|$, i.e. $n(C)$ is the multiplicity of C in $\tilde{\mathcal{C}}$. By applying $\gamma_{v_1,v}$ and h_{v_1}, we see that for any $v \in V_1$ and $i \neq j, i' \neq j'$:

$$\sum_{C \in \mathcal{C}(v_1,i,j)} n(C) = \sum_{C \in \mathcal{C}(v,i',j')} n(C).$$

Therefore there exists an integer M_1 such that for for any $v \in V_1$ and $i \neq j$:

$$\sum_{C \in \mathcal{C}(v,i,j)} n(C) = \frac{M_1}{3}.$$

Similarly there exists an integer M_2 such that for any $v \in V_2$ and $i \neq j$:

$$\sum_{C \in \mathcal{C}(v,i,j)} n(C) = \frac{M_2}{3}.$$

Now finally we wish to show that $M_1 = M_2$. However, this follows immediately by construction, as $\mathcal{B} = \gamma \cdot \mathcal{A}$, and $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. □

Using this, we investigate the separation of several graphs.

x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i		
0	1	2	3	8	2	11	18	16	6	21	22
1	0	4	5	9	3	10	17	17	9	20	22
2	0	6	8	10	5	9	21	18	8	21	23
3	0	7	9	11	4	8	20	19	12	13	14
4	1	11	14	12	6	7	19	20	11	15	17
5	1	10	13	13	5	19	23	21	10	16	18
6	2	12	16	14	4	19	22	22	14	16	17
7	3	12	15	15	7	20	23	23	13	15	18

Table 3.2 Edge incidences for $F24A$

Lemma 3.5.14. *The graph $F24A$ is †-separated.*

Proof. By a computer search we find all 3-separated vertex cutsets in $F24A$:

$$C_1 = \{x_0, x_{10}, x_{11}, x_{12}, x_{22}, x_{23}\},$$
$$C_2 = \{x_1, x_8, x_9, x_{15}, x_{16}, x_{19}\},$$
$$C_3 = \{x_2, x_4, x_7, x_{13}, x_{17}, x_{21}\},$$
$$C_4 = \{x_3, x_5, x_6, x_{14}, x_{18}, x_{19}\}.$$

We note $diam(F24A) = 4$. As the above are disjoint and proper, it follows easily that $F24A$ is †-separated. □

x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i		
0	1	2	3	9	2	10	18	18	9	20	25
1	0	4	7	10	4	9	22	19	20	21	22
2	0	6	9	11	5	7	20	20	11	18	19
3	0	5	8	12	6	8	21	21	12	16	19
4	1	10	13	13	4	23	24	22	10	17	19
5	3	11	14	14	5	24	25	23	13	15	16
6	2	12	15	15	6	23	25	24	13	14	17
7	1	11	16	16	7	21	23	25	14	15	18
8	3	12	17	17	8	22	24				

Table 3.3 Edge incidences for $F26A$

Lemma 3.5.15. *The graph $F26A$ is $*$-separated.*

Proof. We can take $v_1 = x_0$, $w_i = x_i$. Using the notation as in Lemma 3.5.13 we find $A = \{x_0, x_{10}, x_{12}, x_{14}, x_{20}, x_{23}\}$. The result follows by Lemma 3.5.13. □

We defer the collection of cutsets found for $F40A$ to Section 3.6. The graph $F40A$ has the following edge incidences.

Table 3.4 Edge incidences for $F40A$

x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i		
0	1	2	3	14	7	23	33	27	18	21	34
1	0	4	5	15	6	22	32	28	13	19	36
2	0	6	7	16	9	25	33	29	12	18	36

3	0	8	9	17	8	24	32	30	11	21	37
4	1	10	12	18	6	27	29	31	10	20	37
5	1	11	13	19	7	26	28	32	15	17	38
6	2	15	18	20	8	26	31	33	14	16	38
7	2	14	19	21	9	27	30	34	23	24	27
8	3	17	20	22	11	15	35	35	22	25	26
9	3	16	21	23	10	14	34	36	28	29	39
10	4	23	31	24	13	17	34	37	30	31	39
11	5	22	30	25	12	16	35	38	32	33	39
12	4	25	29	26	19	20	35	39	36	37	38
13	5	24	28								

Lemma 3.5.16. *The graph $F40A$ is weighted strongly edge 3-separated.*

Proof. We require a large number of cutsets for this proof: they can be found in Section 3.6.

In particular, we find a collection of cutsets $\{C_i\}_i$ such that for any vertices w_1, w_2 with $d(x_0, w_1) \geq 3$ and $d(w_1, w_2) = 1$ there exists some C_i separating $\{x_0, x_1\}$ and $\{w_1, w_2\}$ (this can be easily checked by computer). Similarly for any vertices w_1, w_2 with $d(x_1, w_1) \geq 3$ and $d(w_1, w_2) = 1$ there exists some C_i separating $\{x_0, x_1\}$ and $\{w_1, w_2\}$. By passing to subsets of C_i we may assume each of these cutsets are minimal and therefore proper.

As $Aut(F40A)$ acts edge and vertex transitively, it follows by Lemma 3.2.6 that $F40A$ is strongly edge 3-separated. Hence, by Lemma 3.2.23, $F40A$ is weighted disjointly strongly edge 3-separated. □

Table 3.5 Edge incidences for $F48A$

x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i		
0	1	2	3	16	5	27	31	32	21	41	43
1	0	4	5	17	4	26	30	33	20	41	42
2	0	6	8	18	9	25	35	34	19	40	44
3	0	7	9	19	8	24	34	35	18	39	45
4	1	11	17	20	7	28	33	36	43	45	46
5	1	10	16	21	6	29	32	37	42	44	47
6	2	13	21	22	11	13	14	38	39	40	41
7	3	12	20	23	10	12	15	39	31	35	38

8	2	15	19	24	11	19	43	40	30	34	38
9	3	14	18	25	10	18	42	41	32	33	38
10	5	23	25	26	15	17	47	42	25	33	37
11	4	22	24	27	14	16	46	43	24	32	36
12	7	23	29	28	13	20	45	44	29	34	37
13	6	22	28	29	12	21	44	45	28	35	36
14	9	22	27	30	17	40	46	46	27	30	36
15	8	23	26	31	16	39	47	47	26	31	37

Lemma 3.5.17. *The graph $F48A$ is †-separated.*

Proof. By a computer search we find all 3-separated vertex cutsets in $F48A$:

$$\begin{aligned}
C_1 &= \{x_0, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{36}, x_{37}, x_{38}\},\\
C_2 &= \{x_1, x_6, x_7, x_{14}, x_{15}, x_{24}, x_{25}, x_{30}, x_{31}, x_{41}, x_{44}, x_{45}\},\\
C_3 &= \{x_2, x_5, x_9, x_{11}, x_{12}, x_{26}, x_{28}, x_{32}, x_{34}, x_{39}, x_{42}, x_{46}\},\\
C_4 &= \{x_3, x_4, x_8, x_{10}, x_{13}, x_{27}, x_{29}, x_{33}, x_{35}, x_{40}, x_{43}, x_{47}\}.
\end{aligned}$$

The above are disjoint and proper, and it can be seen that $F48A$ is †-separated. □

We defer the collection of cutsets found for $G54$ to Section 3.6.

Table 3.6 Edge incidences for $G54$

x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i			x_i	x_j adjacent to x_i		
0	1	25	53	18	17	19	43	36	7	35	37
1	0	2	30	19	18	20	48	37	12	36	38
2	1	3	15	20	19	21	33	38	37	39	51
3	2	4	44	21	8	20	22	39	26	38	40
4	3	5	11	22	21	23	29	40	39	41	47
5	4	6	52	23	16	22	24	41	34	40	42
6	5	7	31	24	23	25	49	42	13	41	43
7	6	8	36	25	0	24	26	43	18	42	44
8	7	9	21	26	25	27	39	44	3	43	45
9	8	10	50	27	14	26	28	45	32	44	46
10	9	11	17	28	27	29	35	46	45	47	53
11	4	10	12	29	22	28	30	47	40	46	48
12	11	13	37	30	1	29	31	48	19	47	49
13	12	14	42	31	6	30	32	49	24	48	50

14	13	15	27	32	31	33	45	50	9	49	51
15	2	14	16	33	20	32	34	51	38	50	52
16	15	17	23	34	33	35	41	52	5	51	53
17	10	16	18	35	28	34	36	53	0	46	52

Lemma 3.5.18. *The Gray Graph $G54$ is weighted strongly edge 3-separated.*

Proof. We require a large number of cutsets for this proof: they can be found in Section 3.6. In particular, we find a collection of 3-separated cutsets $\{C_i\}_i$ such that each C_i contains one of the edges

$$(x_0, x_1), (x_0, x_{53}), (x_{24}, x_{25}), (x_{25}, x_{26}).$$

Therefore, as each cutset is 3-separated, they cannot contain the edge (x_0, x_{25}), and so for each cutset, x_0 and x_{25} lie in the same component of $G54 - C_i$.

We also show that for any point v with $d(x_0, v) \geq 3$ and any neighbour w of v, there exists some C_i separating $\{x_0, x_{25}\}$ and $\{v, w\}$. Furthermore, for any point v with $d(x_{25}, v) \geq 3$ and any neighbour w of v, there exists some C_i separating $\{x_0, x_{25}\}$ and $\{v, w\}$. By passing to subsets of C_i we may assume each of these cutsets are minimal and therefore proper. Now, let $p = (u, w_1)(w_1, w_2) \ldots (w_n, v)$ be some path with $2 \leq n \leq 5$ of length between 3 and 6. Let u' be adjacent to u and v' be adjacent to v.

Note again that $Aut(G54)$ acts transitively on the set of edges. If we can map u to x_0 by some element $\gamma \in Aut(G54)$, then we may also map u' to x_{25} by γ, and then for some i, C_i separates x_0, x_{25} and $\gamma v, \gamma v'$: $\gamma^{-1} C_i$ then separates u, u' and v, v'. Otherwise, we map u to x_{25} by γ, so that $\gamma u' = x_0$. The result follows similarly.

Therefore, $G54$ is strongly edge 3-separated, and as it has an edge transitive automorphism group, it is weighted strongly edge 3-separated by Lemma 3.2.23. □

We finally need to prove the following.

Lemma 3.5.19. *Let Y be a finite triangle complex such that each triangle is a unit equilateral Euclidean triangle. Suppose that the link of each vertex is either $*$-separated or †-separated with the combinatorial metric (we allow a mixture of these). Then Y is gluably π-separated.*

Proof. It is clear that Y is non-positively curved and regular. By Lemma 3.2.24, if the link of each vertex is †-separated with the combinatorial metric then we are finished.

Otherwise, let $\{v_k\}$ be the vertices such that $Lk(v_k)$ is $*$-separated with the combinatorial metric, and $\{w_l\}$ be the vertices such that $Lk(w_l)$ is $\dagger$-separated with the combinatorial metric. Note that a 3-separated cutset in $Lk(x)$ under the combinatorial metric is a π-separated cutset in $Lk(x)$ under the angular metric.

For each proper π-separated cutset C in $Lk(w_l)$ we may assign the three partitions $P_1(C), P_2(C), P_3(C)$ corresponding to placing two components of $Lk(w_l) - C$ in the same element of the partition. For each cutset C in $Lk(v_k)$ assign the unique partition of connectedness of $Lk(v_k) - C$.

Since the links are $\dagger$-separated and $*$-separated, by assumption for each vertex $x \in Y$ there exists a positive integer $N_x > 0$ and a system of strictly positive weights $n_x(C)$ for $C \in \mathcal{C}_x$ such that for any vertex e in $Lk_Y(x)$,

$$\sum_{C \in \mathcal{C}(e)} n_x(C) = \sum_{C \in \mathcal{C}(e) \cap \mathcal{C}_x} n_x(C) = N_x.$$

Furthermore, if $Lk(v_k)$ is $*$-separated, then for any vertex $y \in V(Lk(v_k))$ and $i \neq j$

$$\sum_{C \in \mathcal{C}_{v_k}(y,i,j)} n_{v_k}(C) = \frac{N_{v_k}}{3}.$$

Let $M = \prod_{x \in V(Y)} N_x$, and for a cutset $C \in \mathcal{C}_x$, define

$$\nu(C) = M n_x(C)/N_x.$$

It follows that for a vertex e in $Lk_{G \backslash X}(x)$,

$$\sum_{C \in \mathcal{C}(e)} \nu(C) = \frac{M}{N_x} \sum_{C \in \mathcal{C}(e)} n_x(C) = \frac{M}{N_x} N_x = M.$$

Now, take $\mu(C, P(C)) = \nu(C)$. It follows that for any oriented edge e of $Y^{(1)}$ starting at some w_l and any partition $(C, P) \in \mathcal{CP}(e)$:

$$\sum_{(C',P') \in [C,P]_e} \mu(C', P') = \sum_{(C',P') \in [C,P]_e} \nu(C') = \frac{1}{3} \sum_{C' \in \mathcal{C}(e)} \nu(C') = \frac{1}{3} M.$$

Similarly, by the definition of $*$-separated, for each v_k, each edge e starting at v_k, and $(C, P(C)) \in \mathcal{C}(e)$,

$$\sum_{(C',P') \in [C,P]_e} \mu(C', P') = \sum_{(C',P') \in [C,P]_e} \nu(C') = \frac{1}{3} \sum_{(C',P') \in \mathcal{C}(e)} \nu(C') = \frac{1}{3} M,$$

and so the gluing equations are solved. $\square$

The results of Corollary C now follow from [CCKW20, Theorem 3.1], Proposition 3.5.4, the above lemmas concerning the separation of the graphs considered, Theorem A, and Theorem 3.5.5.

3.5.3 Cubulating Dehn fillings of generalized ordinary triangle groups

We now apply Theorem A to the generalized triangle groups of [LMW19], in particular retrieving consequences of the malnormal special quotient theorem of Wise [Wis21].

Corollary 3.5.20. *Let $\Gamma_i \looparrowright C_{k,2}$ be finite $n(i)$-sheeted normal covering graphs. There exist finite-sheeted normal covering graphs $\dot{\Gamma}_i \looparrowright \Gamma_i$ of index at most*

$$4^{1+4^{kn(i)}}$$

such that for any collection of finite-sheeted covering graphs $\Delta_i \looparrowright \Gamma_i$ that factor as $\Delta_i \looparrowright \dot{\Gamma}_i \looparrowright \Gamma_i$, and any j, the group $G^j_{0,k}(\Delta_1, \Delta_2, \Delta_3)$ is hyperbolic and acts properly discontinuously and cocompactly on a $CAT(0)$ cube complex.

We consider covers of σ-separated graphs: we restrict our consideration to graphs with the combinatorial metric. We note the following lemma.

Lemma 3.5.21. *Let $p : \tilde{\Gamma} \looparrowright \Gamma$ be a covering graph. Let $e \in E(\Gamma)$ and let $\tilde{e}_1, \tilde{e}_2 \in p^{-1}(e)$ be distinct. Then*

$$d_{\tilde{\Gamma}}(m(\tilde{e}_1), m(\tilde{e}_2)) \geq girth(\Gamma).$$

We now show that covers of σ-separated graphs are also σ-separated.

Lemma 3.5.22. *Let Γ be a weighted (disjointly) edge σ-separated graph with $girth(\Gamma) \geq \sigma$ and $p : \tilde{\Gamma} \looparrowright \Gamma$ a finite-sheeted covering graph. Then $\tilde{\Gamma}$ is also weighted (disjointly) edge σ-separated, and $girth(\tilde{\Gamma}) \geq girth(\Gamma)$.*

Proof. It is clear that $girth(\tilde{\Gamma}) \geq girth(\Gamma)$, and that $\tilde{\Gamma}$ is connected and contains no vertices of degree 1. Let $\mathcal{C}_1, \dots, \mathcal{C}_m \subseteq E(\Gamma)$ be the σ-separated cut sets of Γ. Let $\tilde{\mathcal{C}_i} = p^{-1}(\mathcal{C}_i)$: by Lemma 3.5.21, and by noting that for all $x, y \in \tilde{\Gamma}$ we have $d_{\tilde{\Gamma}}(x, y) \geq d_\Gamma(p(x), p(y))$, we see that $\tilde{\mathcal{C}_i}$ is a collection of proper $\min\{girth(\Gamma), \sigma\}$-separated cut sets. Furthermore $|\tilde{C_i}| \geq |C_i| \geq 2$. As $girth(\Gamma) \geq \sigma$, these are σ-separated and $\cup_i \tilde{\mathcal{C}_i} = E(\tilde{\Gamma})$. Therefore, $\tilde{\Gamma}$ is edge σ-separated.

If Γ is disjointly separated, it is clear that $\tilde{\Gamma}$ is disjointly separated. Finally, defining $n(\tilde{C}_i) = n(C_i)$, it can be seen that the weight equations are satisfied, so that $\tilde{\Gamma}$ is weighted (disjointly) edge σ-separated. □

Using the above, we wish to show that given any graph Γ, there exists a finite-sheeted 3-separated covering graph $\tilde{\Gamma} \looparrowright \Gamma$.

Definition 3.5.23. Let Γ be a graph and $m \geq 0$. The $\mathbb{Z}_m$ *cover of* Γ,

$$p_m : \mathbb{Z}_m(\Gamma) \looparrowright \Gamma,$$

is the $m^{b_1(\Gamma)}$-sheeted normal cover corresponding to the kernel of the canonical map $\pi_1(\Gamma) \to H_1(\Gamma, \mathbb{Z}_m)$.

The use of this is the following.

Lemma 3.5.24. *Let Γ be a finite connected graph with no cut edges and let $m \geq 1$. The covering graph $\mathbb{Z}_{2m}(\Gamma)$ is weighted disjointly edge $girth(\Gamma)$-separated and $girth(\mathbb{Z}_{2m}(\Gamma)) = 2m(girth(\Gamma))$.*

Proof. Let $e \in E(\Gamma)$. We claim $p_{2m}^{-1}(e)$ is a proper $girth(\Gamma)$-separated cut set in $\mathbb{Z}_{2m}(\Gamma)$. By Lemma 3.5.21, $p_{2m}^{-1}(e)$ is $girth(\Gamma)$-separated. It suffices to show that if two points x and y are joined by a path q containing one edge of $p_{2m}^{-1}(e)$, then any path q' between them contains an edge of $p_{2m}^{-1}(e)$. Now suppose not: consider such a path q' not containing any edge of $p_{2m}^{-1}(e)$, and consider the loop qq'. Then $p_{2m}(qq')$ is trivial in the map to $H_1(\Gamma, \mathbb{Z}_{2m})$, so is homotopic to a curve containing e an even number of times, a contradiction.

Therefore the set $\mathcal{C}_e = \{p_{2m}^{-1}(e) \ : \ e \in E(\Gamma)\}$ is a disjoint collection of proper $girth(\Gamma)$-separated edge-cut sets such that any edge in $\mathbb{Z}_{2m}(\Gamma)$ appears exactly one cut set: the weight equations are trivially satisfied and so $\mathbb{Z}_{2m}(\Gamma)$ is weighted disjointly $girth(\Gamma)$-separated.

Any loop in $\mathbb{Z}_{2m}(\Gamma)$ projects to a loop homotopic to a product of loops where each loop is traversed $2m$ times, and so $girth(\mathbb{Z}_{2m}(\Gamma)) = 2m(girth(\Gamma))$. □

Using this, we prove the following.

Proof of Corollary 3.5.20. Let $\Gamma_i \looparrowright C_{k,2}$ be $n(i)$-sheeted normal covering graphs . Let $\dot{\Gamma}_i := \mathbb{Z}_2(\mathbb{Z}_2(\Gamma))$: these are

$$2^{2-(2^{2-2n(i)+kn(i)}+2)n(i)+(2^{1-2n(i)+kn(i)}+1)kn(i)} \leq 4^{1+kn(i)2^{kn(i)}} \leq 4\left(4^{4^{kn(i)}}\right)$$

sheeted covering graphs, which, by Lemma 3.5.24, are weighted disjointly edge 3-separated under the combinatorial metric and have girth at least 8. Furthermore, it is clear that $\dot{\Gamma}_i \looparrowright C_{k,2}$ are normal covers. Suppose $\Delta_i \looparrowright \Gamma_i$ factors as $\Delta_i \looparrowright \dot{\Gamma}_i \looparrowright \Gamma_i$. By Proposition 3.5.4, noting that $girth(\Delta_i) \geq girth(\dot{\Gamma}_i) > 6$, the group $G^j_{0,k}(\Delta_1, \Delta_2, \Delta_3)$ is hyperbolic. The Δ_i are covers of $\dot{\Gamma}_i$, so by Lemma 3.5.22 are also weighted edge 3-separated under the combinatorial metric. The result now follows from Lemma 3.2.24, Proposition 3.5.4, and Theorem A. $\square$

3.6 Large collections of cutsets

In this section we provide the selection of cutsets described in Lemmas 3.5.16 and 3.5.18. Throughout, we use the notation $e_{i,j} = (x_i, x_j)$.

3.6.1 F40A

We find the following cutset for $F40A$.

$\{e_{1,5}, e_{2,7}, e_{3,9}, e_{10,23}, e_{12,25}, e_{15,22}, e_{17,24}, e_{18,27}, e_{20,26}, e_{28,36}, e_{30,37}, e_{33,38}\}$,
$\{e_{1,4}, e_{2,6}, e_{3,8}, e_{11,22}, e_{13,24}, e_{14,23}, e_{16,25}, e_{19,26}, e_{21,27}, e_{29,36}, e_{31,37}, e_{32,38}\}$,
$\{e_{0,3}, e_{4,12}, e_{5,13}, e_{6,18}, e_{7,19}, e_{16,33}, e_{17,32}, e_{20,31}, e_{21,30}, e_{22,35}, e_{23,34}, e_{36,39}\}$,
$\{e_{0,2}, e_{4,10}, e_{5,11}, e_{8,20}, e_{9,21}, e_{14,33}, e_{15,32}, e_{18,29}, e_{19,28}, e_{24,34}, e_{25,35}, e_{37,39}\}$,
$\{e_{0,1}, e_{6,15}, e_{7,14}, e_{8,17}, e_{9,16}, e_{10,31}, e_{11,30}, e_{12,29}, e_{13,28}, e_{26,35}, e_{27,34}, e_{38,39}\}$,
$\{e_{2,7}, e_{3,8}, e_{4,10}, e_{5,13}, e_{15,32}, e_{16,33}, e_{26,35}, e_{27,34}, e_{29,36}, e_{30,37}\}$,
$\{e_{2,6}, e_{3,9}, e_{4,12}, e_{5,11}, e_{14,33}, e_{17,32}, e_{26,35}, e_{27,34}, e_{28,36}, e_{31,37}\}$,
$\{e_{1,5}, e_{3,8}, e_{6,15}, e_{7,19}, e_{10,31}, e_{21,30}, e_{24,34}, e_{25,35}, e_{29,36}, e_{33,38}\}$,
$\{e_{1,5}, e_{2,6}, e_{8,17}, e_{9,21}, e_{12,29}, e_{19,28}, e_{22,35}, e_{23,34}, e_{31,37}, e_{33,38}\}$,
$\{e_{1,4}, e_{3,9}, e_{6,18}, e_{7,14}, e_{11,30}, e_{20,31}, e_{24,34}, e_{25,35}, e_{28,36}, e_{32,38}\}$,
$\{e_{1,4}, e_{2,7}, e_{8,20}, e_{9,16}, e_{13,28}, e_{18,29}, e_{22,35}, e_{23,34}, e_{30,37}, e_{32,38}\}$,
$\{e_{0,3}, e_{5,11}, e_{6,15}, e_{10,31}, e_{12,25}, e_{14,33}, e_{17,24}, e_{19,26}, e_{21,27}, e_{36,39}\}$,
$\{e_{0,3}, e_{4,10}, e_{7,14}, e_{11,30}, e_{13,24}, e_{15,32}, e_{16,25}, e_{18,27}, e_{20,26}, e_{36,39}\}$,
$\{e_{0,2}, e_{5,13}, e_{8,17}, e_{10,23}, e_{12,29}, e_{15,22}, e_{16,33}, e_{19,26}, e_{21,27}, e_{37,39}\}$,
$\{e_{0,2}, e_{4,12}, e_{9,16}, e_{11,22}, e_{13,28}, e_{14,23}, e_{17,32}, e_{18,27}, e_{20,26}, e_{37,39}\}$,
$\{e_{0,1}, e_{7,19}, e_{8,20}, e_{10,23}, e_{13,24}, e_{15,22}, e_{16,25}, e_{18,29}, e_{21,30}, e_{38,39}\}$,
$\{e_{0,1}, e_{6,18}, e_{9,21}, e_{11,22}, e_{12,25}, e_{14,23}, e_{17,24}, e_{19,28}, e_{20,31}, e_{38,39}\}$.

3.6.2 G54

We find the following cutsets for $G54$.

$\{e_{0,53}, e_{2,3}, e_{13,14}, e_{16,17}, e_{21,22}, e_{24,49}, e_{26,39}, e_{28,35}, e_{30,31}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{8,21}, e_{10,17}, e_{12,13}, e_{19,48}, e_{23,24}, e_{26,27}, e_{35,36}, e_{38,51}, e_{40,41}, e_{45,46}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{8,21}, e_{10,17}, e_{12,13}, e_{19,48}, e_{23,24}, e_{26,27}, e_{35,36}, e_{40,41}, e_{45,46}, e_{50,51}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{8,21}, e_{10,17}, e_{12,13}, e_{19,48}, e_{23,24}, e_{26,27}, e_{35,36}, e_{40,41}, e_{45,46}, e_{51,52}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{8,21}, e_{10,17}, e_{14,15}, e_{19,48}, e_{23,24}, e_{28,29}, e_{33,34}, e_{37,38}, e_{42,43}, e_{45,46}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{8,21}, e_{10,17}, e_{14,15}, e_{19,48}, e_{23,24}, e_{28,29}, e_{33,34}, e_{38,39}, e_{42,43}, e_{45,46}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{8,21}, e_{10,17}, e_{14,15}, e_{19,48}, e_{23,24}, e_{28,29}, e_{33,34}, e_{38,51}, e_{42,43}, e_{45,46}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{9,50}, e_{12,13}, e_{15,16}, e_{18,43}, e_{20,33}, e_{22,29}, e_{26,27}, e_{35,36}, e_{40,41}, e_{45,46}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{12,13}, e_{15,16}, e_{18,43}, e_{20,33}, e_{22,29}, e_{26,27}, e_{35,36}, e_{40,41}, e_{45,46}, e_{49,50}\}$,
$\{e_{0,1}, e_{3,4}, e_{6,31}, e_{12,13}, e_{15,16}, e_{18,43}, e_{20,33}, e_{22,29}, e_{26,27}, e_{35,36}, e_{40,41}, e_{45,46}, e_{50,51}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{7,8}, e_{10,11}, e_{13,42}, e_{15,16}, e_{19,48}, e_{22,29}, e_{26,27}, e_{31,32}, e_{34,35}, e_{37,38}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{7,8}, e_{10,11}, e_{13,42}, e_{15,16}, e_{22,29}, e_{26,27}, e_{31,32}, e_{34,35}, e_{37,38}, e_{47,48}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{7,8}, e_{10,11}, e_{13,42}, e_{15,16}, e_{22,29}, e_{26,27}, e_{31,32}, e_{34,35}, e_{37,38}, e_{48,49}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{7,36}, e_{9,50}, e_{11,12}, e_{14,15}, e_{17,18}, e_{20,21}, e_{23,24}, e_{28,29}, e_{31,32}, e_{39,40}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{7,36}, e_{9,50}, e_{11,12}, e_{14,15}, e_{17,18}, e_{20,21}, e_{23,24}, e_{28,29}, e_{31,32}, e_{40,41}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{7,36}, e_{9,50}, e_{11,12}, e_{14,15}, e_{17,18}, e_{20,21}, e_{23,24}, e_{28,29}, e_{31,32}, e_{40,47}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{9,50}, e_{13,42}, e_{17,18}, e_{20,21}, e_{23,24}, e_{26,27}, e_{31,32}, e_{34,35}, e_{37,38}, e_{40,47}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{9,50}, e_{13,42}, e_{17,18}, e_{20,21}, e_{23,24}, e_{26,27}, e_{31,32}, e_{34,35}, e_{37,38}, e_{46,47}\}$,
$\{e_{0,1}, e_{3,44}, e_{5,52}, e_{9,50}, e_{13,42}, e_{17,18}, e_{20,21}, e_{23,24}, e_{26,27}, e_{31,32}, e_{34,35}, e_{37,38}, e_{47,48}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{8,9}, e_{11,12}, e_{16,17}, e_{19,20}, e_{24,49}, e_{27,28}, e_{32,45}, e_{38,51}, e_{40,47}, e_{42,43}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{8,9}, e_{11,12}, e_{16,17}, e_{19,20}, e_{24,49}, e_{28,29}, e_{32,45}, e_{38,51}, e_{40,47}, e_{42,43}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{8,9}, e_{11,12}, e_{16,17}, e_{19,20}, e_{24,49}, e_{28,35}, e_{32,45}, e_{38,51}, e_{40,47}, e_{42,43}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{8,9}, e_{13,14}, e_{16,17}, e_{19,20}, e_{22,29}, e_{24,49}, e_{26,39}, e_{32,45}, e_{34,41}, e_{36,37}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{8,9}, e_{13,14}, e_{16,17}, e_{19,20}, e_{24,49}, e_{26,39}, e_{28,29}, e_{32,45}, e_{34,41}, e_{36,37}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{8,9}, e_{13,14}, e_{16,17}, e_{19,20}, e_{24,49}, e_{26,39}, e_{29,30}, e_{32,45}, e_{34,41}, e_{36,37}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{10,11}, e_{13,14}, e_{18,43}, e_{21,22}, e_{26,39}, e_{32,45}, e_{34,41}, e_{36,37}, e_{47,48}, e_{50,51}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{10,11}, e_{13,14}, e_{18,43}, e_{22,23}, e_{26,39}, e_{32,45}, e_{34,41}, e_{36,37}, e_{47,48}, e_{50,51}\}$,
$\{e_{0,53}, e_{2,3}, e_{5,6}, e_{10,11}, e_{13,14}, e_{18,43}, e_{22,29}, e_{26,39}, e_{32,45}, e_{34,41}, e_{36,37}, e_{47,48}, e_{50,51}\}$,
$\{e_{0,53}, e_{2,3}, e_{7,8}, e_{10,11}, e_{13,14}, e_{16,23}, e_{18,43}, e_{20,33}, e_{26,39}, e_{28,35}, e_{30,31}, e_{47,48}, e_{50,51}\}$,
$\{e_{0,53}, e_{2,3}, e_{7,8}, e_{10,11}, e_{13,14}, e_{18,43}, e_{20,33}, e_{22,23}, e_{26,39}, e_{28,35}, e_{30,31}, e_{47,48}, e_{50,51}\}$,
$\{e_{0,53}, e_{2,3}, e_{7,8}, e_{10,11}, e_{13,14}, e_{18,43}, e_{20,33}, e_{23,24}, e_{26,39}, e_{28,35}, e_{30,31}, e_{47,48}, e_{50,51}\}$,
$\{e_{0,53}, e_{2,3}, e_{7,36}, e_{11,12}, e_{14,27}, e_{16,17}, e_{21,22}, e_{24,49}, e_{30,31}, e_{33,34}, e_{38,51}, e_{40,47}, e_{42,43}\}$,
$\{e_{0,53}, e_{2,3}, e_{7,36}, e_{11,12}, e_{16,17}, e_{21,22}, e_{24,49}, e_{26,27}, e_{30,31}, e_{33,34}, e_{38,51}, e_{40,47}, e_{42,43}\}$,
$\{e_{0,53}, e_{2,3}, e_{7,36}, e_{11,12}, e_{16,17}, e_{21,22}, e_{24,49}, e_{27,28}, e_{30,31}, e_{33,34}, e_{38,51}, e_{40,47}, e_{42,43}\}$,

$\{e_{0,53}, e_{2,15}, e_{4,5}, e_{9,10}, e_{12,37}, e_{18,19}, e_{21,22}, e_{24,49}, e_{26,39}, e_{28,35}, e_{30,31}, e_{41,42}, e_{44,45}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,8}, e_{10,17}, e_{12,37}, e_{14,27}, e_{20,33}, e_{22,29}, e_{24,25}, e_{41,42}, e_{44,45}, e_{47,48}, e_{50,51}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,8}, e_{12,37}, e_{14,27}, e_{16,17}, e_{20,33}, e_{22,29}, e_{24,25}, e_{41,42}, e_{44,45}, e_{47,48}, e_{50,51}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,8}, e_{12,37}, e_{14,27}, e_{17,18}, e_{20,33}, e_{22,29}, e_{24,25}, e_{41,42}, e_{44,45}, e_{47,48}, e_{50,51}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,8}, e_{12,37}, e_{14,27}, e_{17,18}, e_{22,29}, e_{24,25}, e_{31,32}, e_{34,35}, e_{39,40}, e_{46,53}, e_{50,51}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,8}, e_{12,37}, e_{14,27}, e_{18,19}, e_{22,29}, e_{24,25}, e_{31,32}, e_{34,35}, e_{39,40}, e_{46,53}, e_{50,51}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,8}, e_{12,37}, e_{14,27}, e_{18,43}, e_{22,29}, e_{24,25}, e_{31,32}, e_{34,35}, e_{39,40}, e_{46,53}, e_{50,51}\}$,
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$\{e_{1,2}, e_{4,5}, e_{7,36}, e_{9,10}, e_{13,14}, e_{16,23}, e_{18,19}, e_{25,26}, e_{28,29}, e_{33,34}, e_{38,51}, e_{40,47}, e_{44,45}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,36}, e_{9,10}, e_{13,42}, e_{16,23}, e_{18,19}, e_{25,26}, e_{28,29}, e_{33,34}, e_{38,51}, e_{40,47}, e_{44,45}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,36}, e_{9,10}, e_{13,42}, e_{16,23}, e_{20,21}, e_{25,26}, e_{28,29}, e_{31,32}, e_{38,51}, e_{46,53}, e_{48,49}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,36}, e_{9,10}, e_{16,23}, e_{20,21}, e_{25,26}, e_{28,29}, e_{31,32}, e_{38,51}, e_{41,42}, e_{46,53}, e_{48,49}\}$,
$\{e_{1,2}, e_{4,5}, e_{7,36}, e_{9,10}, e_{16,23}, e_{20,21}, e_{25,26}, e_{28,29}, e_{31,32}, e_{38,51}, e_{42,43}, e_{46,53}, e_{48,49}\}$,
$\{e_{1,2}, e_{6,31}, e_{8,21}, e_{11,12}, e_{16,23}, e_{18,19}, e_{25,26}, e_{28,29}, e_{33,34}, e_{40,47}, e_{44,45}, e_{49,50}, e_{52,53}\}$,
$\{e_{1,2}, e_{6,31}, e_{8,21}, e_{12,13}, e_{16,23}, e_{18,19}, e_{25,26}, e_{28,29}, e_{33,34}, e_{40,47}, e_{44,45}, e_{49,50}, e_{52,53}\}$,
$\{e_{1,2}, e_{6,31}, e_{8,21}, e_{12,37}, e_{16,23}, e_{18,19}, e_{25,26}, e_{28,29}, e_{33,34}, e_{40,47}, e_{44,45}, e_{49,50}, e_{52,53}\}$,
$\{e_{1,2}, e_{6,31}, e_{9,10}, e_{14,27}, e_{20,33}, e_{22,29}, e_{24,25}, e_{35,36}, e_{38,39}, e_{41,42}, e_{44,45}, e_{47,48}, e_{52,53}\}$,
$\{e_{1,2}, e_{6,31}, e_{10,11}, e_{14,27}, e_{20,33}, e_{22,29}, e_{24,25}, e_{35,36}, e_{38,39}, e_{41,42}, e_{44,45}, e_{47,48}, e_{52,53}\}$,
$\{e_{1,2}, e_{6,31}, e_{10,17}, e_{14,27}, e_{20,33}, e_{22,29}, e_{24,25}, e_{35,36}, e_{38,39}, e_{41,42}, e_{44,45}, e_{47,48}, e_{52,53}\}$,
$\{e_{1,30}, e_{3,4}, e_{7,8}, e_{12,13}, e_{15,16}, e_{18,43}, e_{24,25}, e_{27,28}, e_{32,45}, e_{34,41}, e_{38,39}, e_{47,48}, e_{52,53}\}$,
$\{e_{1,30}, e_{3,4}, e_{7,36}, e_{10,17}, e_{14,15}, e_{19,20}, e_{22,23}, e_{25,26}, e_{32,45}, e_{40,47}, e_{42,43}, e_{49,50}, e_{52,53}\}$,
$\{e_{1,30}, e_{3,4}, e_{8,9}, e_{12,13}, e_{15,16}, e_{18,43}, e_{24,25}, e_{27,28}, e_{32,45}, e_{34,41}, e_{38,39}, e_{47,48}, e_{52,53}\}$,
$\{e_{1,30}, e_{3,4}, e_{8,21}, e_{12,13}, e_{15,16}, e_{18,43}, e_{24,25}, e_{27,28}, e_{32,45}, e_{34,41}, e_{38,39}, e_{47,48}, e_{52,53}\}$,
$\{e_{1,30}, e_{3,4}, e_{10,17}, e_{14,15}, e_{19,20}, e_{22,23}, e_{25,26}, e_{32,45}, e_{35,36}, e_{40,47}, e_{42,43}, e_{49,50}, e_{52,53}\}$,
$\{e_{1,30}, e_{3,4}, e_{10,17}, e_{14,15}, e_{19,20}, e_{22,23}, e_{25,26}, e_{32,45}, e_{36,37}, e_{40,47}, e_{42,43}, e_{49,50}, e_{52,53}\}$,
$\{e_{1,30}, e_{3,44}, e_{5,6}, e_{8,9}, e_{11,12}, e_{14,15}, e_{17,18}, e_{22,23}, e_{25,26}, e_{33,34}, e_{38,51}, e_{46,53}, e_{48,49}\}$.

Chapter 4

On the eigenvalues of Erdös–Rényi random bipartite graphs

4.1 The spectra of regular graphs and random graphs

It is well known that many models of random graphs are expanders. In particular, if A is the adjacency matrix of a graph G on m vertices, we define the ordering of the eigenvalues of A by $\mu_1(A) \geq \mu_2(A) \geq \ldots \geq \mu_m(A)$ (we keep this ordering convention for the eigenvalues of any symmetric matrix). We further define $\mu(A) = \max\{|\mu_2(A)|, |\mu_m(A)|\}$. We are concerned with how large $\mu(A)$ can be, relative to the average degree of G. A set of strongly related quantities are the eigenvalues $\mu_i(G) := \mu_i(I - D^{-1/2}AD^{-1/2})$, where D is the degree matrix of G.

The Alon–Boppana bound states that for a d-regular graph, $\mu(A) \geq 2\sqrt{d-1} - o(1)$ [Alo86]. A major result due to Friedman is that random d-regular graphs are almost Ramanujan, i.e. for any $\epsilon > 0$, $\mu(A) \leq 2\sqrt{d-1} + \epsilon$ with probability tending to 1 as m tends to infinity [Fri08]. Bordenave provided a new proof of the result, as well as giving a $o(1)$ specific term with $\mu(A) \leq 2\sqrt{d-1} + o(1)$ [Bor15].

We note that for p sufficiently large, and $G \sim G(m,p)$ the Erdös–Rényi random graph, G is almost mp-regular with probability tending to 1 as m tends to infinity. When $mp = \Omega(\log^6 m)$, such a graph also satisfies $\mu(A) \leq 2[1 + o(1)]\sqrt{mp}$ [FK81]: these results were then extended to a more general model of random graphs with given expected degree sequence by [CLV04]. Furthermore, there exists a constant $c > 0$ such that for $p \geq c \log m/m$, almost surely the random graph satisfies $\mu(A) \leq O(\sqrt{mp})$ [FO05] and $\max_{i \neq m} |1 - \mu_i(G(m,p))| \leq O(1/\sqrt{mp})$ [CO07].

Switching to random bipartite graphs, since the eigenvalues of bipartite graphs are symmetric around zero, we need only consider $\mu_2(A)$. The analogue of the Alon–Boppana bound for a (d_L, d_R)-regular bipartite graph is $\mu_2(A) \geq \sqrt{d_L - 1} + \sqrt{d_R - 1} - \epsilon$ for $\epsilon > 0$ and the number of vertices sufficiently large [FL96, LS96]. Furthermore, the bound is almost attained for random (d_L, d_R)-regular graphs: with probability tending to 1 as the number, m, of vertices tends to infinity, for sequences $\epsilon_m, \epsilon'_m \to 0$ and for G a random (d_L, d_R)-regular graph, $\mu_2(A) \leq \sqrt{d_L - 1} + \sqrt{d_R - 1} + \epsilon_m$, and

$$\mu_+(A) = \min_i \{\mu_i(A) > 0\} \geq \sqrt{d_L - 1} + \sqrt{d_R - 1} - \epsilon'_m \text{ [BDH18].}$$

There are many other results concerning the eigenvalues of random bipartite graphs: both [DJ16] and [Tra20] study the spectral distribution of random biregular bipartite graphs, and show it converges to certain laws when $|V_1|/|V_2|$ tends to a limit $\alpha \neq 0, \infty$, though each of the two considers a different range of (d_L, d_R). Finally, the second eigenvalue of the matrix $D - A$ is considered for certain random biregular bipartite graphs in [Zhu20].

For $m_2 = m_2(m_1) \geq m_1$, and $0 \leq p = p(m_1) \leq 1$, we define the Erdös–Rényi random bipartite graph $G(m_1, m_2, p)$ as the graph with vertex partition $V_1 = \{u_1, \dots, u_{m_1}\}$ and $V_2 = \{v_1, \dots, v_{m_2}\}$, and edge set obtained by adding each edge (u_i, v_j) independently with probability p. We write $G \sim G(m_1, m_2, p)$ to indicate that the graph G is obtained by this process.

To our knowledge, there are no results in the literature on the eigenvalues of $G(m_1, m_2, p)$: while we do not show such graphs almost attain the Alon–Boppana bound, we are still able to prove that they are within a multiplicative constant of this bound, and so $\mu_i(G)$ is close to 1 for any $i \neq 1, m_1 + m_2$.

Theorem D. *Let $m_1 \geq 1$ and $m_2 = m_2(m_1) \geq m_1$ and let $G \sim G(m_1, m_2, p)$*

i) There exists constants $c_0, c > 0$ such that if $m_1 p \geq c_0 \log m_1$,

$$m_2 p \geq \frac{c_0 \log m_2}{1 - \log 2 / \log(m_1 + m_2)}, \text{ and } p \leq [(m_1 + m_2)^2 \log^5(m_1 + m_2)]^{-1/3},$$

then with probability tending to 1 as m_1 tends to infinity: $\mu_2(A(G)) \leq c\sqrt{m_2 p}$.

ii) If $m_1 p = \Omega_{m_1}(\log^6(m_1))$ and $m_2 p = \Omega_{m_1}(\log^6(m_2))$, then with probability tending to 1 as m_1 tends to infinity:

$$\mu_2(A(G)) \leq 2[1 + o_{m_1}(1)] \left(\sqrt{(m_1 + m_2)p} + \sqrt{m_1 p} + \sqrt{m_2 p} \right).$$

iii) If p satisfies either of the above and, in addition, $m_1 p = \Omega_{m_1}(\log m_2)$, then with probability tending to 1 as m_1 tends to infinity:

$$\max_{i \neq 1, m_1+m_2} \left| \mu_i(G) - 1 \right| = o_{m_1}(1).$$

4.1.1 Notation and definitions

We now briefly discuss some notation and assumptions. We are dealing with asymptotics, and so we frequently arrive at situations where m is some parameter tending to infinity that is required to be an integer: if m is not integer, we will implicitly replace it by $\lfloor m \rfloor$. Since we are dealing with asymptotics, this does not affect any of our arguments.

Definition 4.1.1. Given $m_1 : \mathbb{N} \to \mathbb{N}$ a function such that $m_1(m) \to \infty$ as $m \to \infty$, we write $m_2 = m_2(m_1)$ to mean that $m_2(m) = f(m_1(m))$ for some function f, and $f(m_1(m)) \to \infty$ as $m \to \infty$, i.e. m_2 only depends on m_1, and tends to infinity as m_1 tends to infinity.

The following are standard.

Definition 4.1.2. Let $f, g : \mathbb{N} \to \mathbb{R}_+$ be two functions. We write

1. $f = o(g)$ if $f(m)/g(m) \to 0$ as $m \to \infty$,

2. $f = O(g)$ if there exists a constant $N \geq 0$ and $M \geq 1$ such that $f(m) \leq N g(m)$ for all $m \geq M$, and

3. $f = \Omega(g)$ if $g = o(f)$.

We write $f = o_m(g)$ etc to indicate that the variable name is m. Typically we will deal with functions $m_2 = m_2(m_1)$, and $f = f(m_1, m_2)$. We will write $f = o_{m_1}(g)$ etc to mean that the function $f'(m_1) = f(m_1, m_2(m_1)) = o_{m_1}(g'(m_1))$, where $g'(m_1) = g(m_1, m_2(m_1))$.

Definition 4.1.3. Let $\mathcal{M}(m)$ be some model of random groups (or graphs) depending on a parameter m, and let $\mathcal{P}$ be a property of groups (or graphs). We say that $\mathcal{P}$ holds *asymptotically almost surely with m (a.a.s.(m))* if

$$\lim_{m \to \infty} \mathbb{P}(G \sim \mathcal{M}(m) \text{ has } \mathcal{P}) = 1.$$

Again, we will regularly have to deal with cases where $m_2 = m_2(m_1)$ is fixed, $\mathcal{M}(m_1, m_2)$ is some model of random groups (or graphs) depending on parameters m_1

and m_2, and $\mathcal{P}$ is a property of groups (or graphs). We say that $\mathcal{P}$ holds *asymptotically almost surely with* (m_1) $(a.a.s.(m_1))$ if

$$\lim_{m_1\to\infty} \mathbb{P}(G \sim \mathcal{M}(m_1, m_2(m_1)) \text{ has } \mathcal{P}) = 1.$$

Typically, we will only use the above in proofs or in the statements of auxiliary technical lemmas.

Finally, we will often be working with bipartite graphs: the vertex partition of a bipartite graph G will always be written $V(G) = V_1(G) \sqcup V_2(G)$.

4.2 Almost regularity of Erdös–Rényi random graphs and their eigenvalues

In this section, we introduce some models of random graphs, and then prove the standard fact that they are almost regular.

Definition 4.2.1 (*Erdös–Rényi random graph*)**.** Let $m \geq 1$ and $0 \leq p := p(m) \leq 1$. The Erdös–Rényi random graph $G(m,p)$ is the random graph model with vertex set $\{u_1, \ldots, u_m\}$ and edge set obtained by adding each edge (u_i, u_j) independently with probability p. For a random graph G we write $G \sim G(m,p)$ to indicate that the distribution of G is that of $G(m,p)$.

Given a model of random graphs $\mathcal{M}$, and a random matrix M, we write $M \sim A(\mathcal{M})$ to indicate that the distribution of M is the same as that obtained by sampling a graph $G \sim \mathcal{M}$ and then taking its adjacency matrix.

Definition 4.2.2 (Almost regular graphs)**.** Let $\{G_m\}_{m=1}^{\infty}$ be a collection of graphs. We say that the graphs G_m are *almost d_m-regular* if for every G_m its minimum and maximum degree are $[1 + o_m(1)]d_m$.

Definition 4.2.3 (Almost regular bipartite graphs)**.** Let $\{G_m\}_{m=1}^{\infty}$ be a collection of bipartite graphs. We say that the graphs G_m are *almost $(d_m^{(1)}, d_m^{(2)})$-regular* if for every G_m the minimum and maximum degree of vertices in $V_1(G_m)$ are $[1 + o_m(1)]d_m^{(1)}$ and the minimum and maximum degree of vertices in $V_2(G_m)$ are $[1 + o_m(1)]d_m^{(2)}$.

We now analyse the regularity of random bipartite graphs. For this we will use the Chernoff bounds: for $X \sim Bin(m,p)$ and $\delta \in [0,1]$,

$$\mathbb{P}(|X - mp| \geq \delta mp) \leq 2\exp(-mp\delta^2/3).$$

Lemma 4.2.4. *Let $m_2 = m_2(m_1) \geq m_1$ and $p = p(m_1)$ be such that $m_1 p = \Omega_{m_1}(\log m_2)$. Then a.a.s.$(m_1)$ $G(m_1, m_2, p)$ is almost $(m_2 p, m_1 p)$-regular.*

Proof. First note we may write $m_1 p = \omega \log m_2$ for some $\omega \to \infty$ as $m_1 \to \infty$ (so that $m_2 p \geq \omega \log m_2$). Let $G \sim G(m_1, m_2, p)$. Let $v \in V_1(G)$, $w \in V_2(G)$. By the Chernoff bounds, for a fixed vertex v in V_1:

$$\mathbb{P}(|deg(v) - m_2 p| \geq \epsilon m_2 p) \leq \exp(-\epsilon^2 m_2 p/3).$$

Let $\epsilon = \omega^{-1/3} = o_{m_1}(1)$. The probability that there exists a vertex in V_1 with degree too large or small is:

$$\begin{aligned}\mathbb{P}(\exists\, v \in V_1 \ : \ |deg(v) - m_2 p| \geq \epsilon m_2 p) &\leq m_1 \exp(-\epsilon^2 m_2 p/3)\\ &\leq m_1 \exp(-\omega^{1/3} \log m_1/3)\\ &= m_1^{-\Omega_{m_1}(1)}.\end{aligned}$$

Similarly:

$$\begin{aligned}\mathbb{P}(\exists\, w \in V_2 \ : \ |deg(w) - m_1 p| \geq \epsilon m_1 p) &\leq m_2 \exp(-\epsilon^2 m_1 p/3)\\ &\leq m_2 \exp(-\omega^{1/3} \log m_2/3)\\ &= m_2^{-\Omega_{m_1}(1)}.\end{aligned}$$

□

We can prove the corresponding result for the Erdös–Rényi random graph in an identical manner.

Lemma 4.2.5. *Let p be such that $mp = \Omega_m(\log m)$. Then a.a.s.(m) $G(m, p)$ is almost mp-regular.*

4.3 The spectra of Erdös–Rényi random bipartite graphs

In this section we now analyse the spectra of Erdös–Rényi random bipartite graphs. We use the following results.

Theorem 4.3.1. *[FO05] There exists constants c_0, c such that for*

$$c_0 \log m/m \leq p \leq (m^2 \log^5 m)^{-1/3}$$

and $G \sim G(m,p)$, *a.a.s.*(m) $\max_{i\neq 1} |\mu_i(A(G))| \leq c\sqrt{mp}$.

The standard reference for the following result is [FK81], while an extension of the result to a more general model of random graphs may be found in [CLV04].

Theorem 4.3.2. *[FK81, Chu97] Let* $p > 0$ *be such that* $mp = \Omega_m(\log^6(m))$, *and let* $G \sim G(m,p)$. *Then a.a.s.*(m), $\max_{i\neq 1} |\mu_i(A(G))| \leq 2[1 + o_m(1)]\sqrt{mp}$.

We now prove Theorem D.

Proof of Theorem D. Let $G \sim G(m_1, m_2, p)$, let $A = A(G)$, and $D = D(G)$. Let G' be the graph obtained by adding to G each (non-loop) edge in V_1^2 with probability p and each (non-loop) edge in V_2^2 with probability p. Then $G' \sim G(m_1 + m_2, p)$. The adjacency matrix of G' is of the form

$$A(G') = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix},$$

where A_1 is the adjacency matrix of a $G(m_1, p)$ graph, A_3 is the adjacency matrix of a $G(m_2, p)$ graph, and the matrix

$$\begin{pmatrix} 0 & A_2 \\ A_2^T & 0 \end{pmatrix}$$

is the adjacency matrix of G.

Let us deal with case i) first, so that we assume

$$m_1 p \geq c_0 \log m_1,\ m_2 p \geq \frac{c_0}{1 - \log 2/\log(m_1 + m_2)} \log m_2,$$

and hence

$$\begin{aligned}(m_1 + m_2)p \geq m_2 p &\geq \frac{c_0}{1 - \log 2/\log(m_1 + m_2)} \log m_2 \\ &\geq \frac{c_0}{1 - \log 2/\log(m_1 + m_2)} \log((m_1 + m_2)/2) \\ &= \frac{c_0}{1 - \log 2/\log(m_1 + m_2)} (\log(m_1 + m_2) - \log 2) \\ &= c_0 \log(m_1 + m_2).\end{aligned}$$

Let

$$A_1' = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix},\ A = A(G(m_1, m_2, p)) = \begin{pmatrix} 0 & A_2 \\ A_2^T & 0 \end{pmatrix},\ A_3' = \begin{pmatrix} 0 & 0 \\ 0 & A_3 \end{pmatrix}.$$

By Theorem 4.3.1, we see that a.a.s.(m_1):

$$\max_{i\neq 1} |\mu_i(A(G'))| \leq c\sqrt{(m_1+m_2)p}.$$

By a further application of Theorem 4.3.1, we see that with probability tending to 1 as m_1 (and hence $m_2 \geq m_1$) tends to infinity:

$$|\max_{i\neq 1} \mu_i(A_1)| \leq c\sqrt{m_1 p},\ |\max_{i\neq 1} \mu_i(A_3)| \leq c\sqrt{m_2 p};$$

this clearly holds for A_1', A_3' also. Hence

$$\mu_1(-A_1') = -\mu_{m_1+m_2}(A_1') \leq c\sqrt{m_1 p},$$

and similarly $\mu_1(-A_3') \leq c\sqrt{m_2 p}$. Therefore, by Weyl's inequality, with probability tending to 1 as m_1 tends to infinity:

$$\begin{aligned}\mu_2(A) = \mu_2\Bigg(A(G') - A_1' - A_3'\Bigg) &\leq \mu_2(A(G')) + \mu_1(-A_1') + \mu_1(-A_3')\\ &\leq c\Bigg(\sqrt{(m_1+m_2)p} + \sqrt{m_1 p} + \sqrt{m_2 p}\Bigg).\end{aligned}$$

In case ii), by assumption, $m_1 p = \Omega_{m_1}(\log^6 m_1)$ and $m_2 p = \Omega_{m_1}(\log^6 m_2)$, so that $(m_1+m_2)p = \Omega_{m_1}(\log^6(m_1+m_2))$. We apply Theorem 4.3.2 to deduce that with probability tending to 1 as m_1 tends to infinity,

$$\max_{i\neq 1} |\mu_i(A(G'))| = 2[1+o_{m_1}(1)]\sqrt{(m_1+m_2)p},$$

and

$$|\max_{i\neq 1} \mu_i(A_1')| = 2[1+o_{m_1}(1)]\sqrt{m_1 p},\ |\max_{i\neq 1} \mu_i(A_3')| = 2[1+o_{m_1}(1)]\sqrt{m_2 p},$$

and the calculation follows similarly.

If, furthermore, $m_1 = \Omega_{m_1}(\log m_2)$, then by Lemma 4.2.4, G is almost $(m_2 p, m_1 p)$-regular with probability tending to 1 as m_1 tends to infinity, i.e. the minimum and maximum degree of vertices in $V_1(G)$ are $[1+o_{m_1}(1)]m_2 p$ and the minimum and maximum degree of vertices in $V_2(G)$ are $[1+o_{m_1}(1)]m_1 p$. Therefore, we see that there exists a matrix

$$K = \begin{pmatrix} 0 & H \\ H^T & 0 \end{pmatrix}$$

with $\sqrt{||H||_\infty ||H||_1} = o_{m_1}(1)$ such that

$$\frac{1}{\sqrt{m_1 m_2 p^2}} A = D^{-1/2} A D^{-1/2} + K.$$

Since $\max_j |\mu_j(K)| \leq \sqrt{||H||_\infty ||H||_1}$, we see that for $i = 1, \ldots, m_1 + m_2$:

$$\mu_i\left(\frac{1}{\sqrt{m_1 m_2 p^2}} A\right) = \mu_i\left(D^{-1/2} A D^{-1/2}\right) + o_{m_1}(1).$$

As $1 - \mu_i(G) = \mu_{m_1+m_2-i+1}(D^{-1/2} A D^{-1/2})$ for $i = 1, \ldots, m_1 + m_2$, the result follows. $\square$

Chapter 5

Property (T) in density-type models of random groups

5.1 Property (T) in random groups

Let us first recall the two models of random groups proposed by Gromov in [Gro93] to study the notion of a ‘generic’ finitely presented group. Fix $n \geq 2, k \geq 3$, and $0 < d < 1$ (the *density*). The (*strict*) (n, k, d) *model* is obtained as followed. Let $A_n = \{a_1, \dots, a_n\}$, and let $F_n := \mathbb{F}(A_n)$ be the free group generated by A_n. Let $\mathcal{C}(n, k)$ be the set of cyclically reduced words of length k in F_n (so that $\mathcal{C}(n, k) \approx (2n-1)^k$). Uniformly randomly select a set $R \subseteq \mathcal{C}(n, k)$ of size $|R| = (2n-1)^{kd}$, and let $\Gamma := \langle A_n \mid R \rangle$. We call Γ a *random group* in the (*strict*) (n, k, d) *model*, and write $\Gamma \sim \Gamma(n, k, d)$. If we keep n fixed and let k tend to infinity, then we obtain the *Gromov density model*, as introduced in [Gro93], whereas if we fix k and let n tend to infinity we obtain the k-*angular model*, as introduced in [Ash21b]. The k-angular model was first studied for $k = 3$ (the *triangular model*) by Żuk in [Ż03] and for $k = 4$ (the *square model*) by Odrzygóźdź in [Odr16].

The *lax* (n, k, d) *model* is obtained via the following procedure. Let $\mathcal{C}(n, k, f)$ be the set of cyclically reduced words of length between $k - f(k)$ and $k + f(k)$ in F_n, where $f(k) = o(k)$. Uniformly randomly select a set $R \subseteq \mathcal{C}(n, k, f)$ of size $|R| = (2n-1)^{kd}$, and let $\Gamma := \langle A_n \mid R \rangle$. We call Γ a *random group* in the *lax* (n, k, d, f) *model*, and write $\Gamma \sim \Gamma_{lax}(n, k, d, f)$. We often drop reference to the function f and simply write $\Gamma \sim \Gamma_{lax}(n, k, d)$, as in many applications the choice of function has no effect on the conclusions.

We first consider the case of the k-angular model. It is a seminal theorem of Żuk [Ż03] (c.f. [KK13]) that for $d > 1/3$ a random group in the triangular model has

Property (T) with probability tending to 1 (see [ALuS15] for the analysis as $d \to 1/3$). As observed in [Odr19] the case of k divisible by 3 is easier, as we may use the work of [Ż03] and [KK13] to observe Property (T) at densities greater than 1/3: see [Mon21] for the proof that $3k$-angular has Property (T) for any $d > 1/3$. This idea was in fact extended in [Mon21] to passing from Property (T) in $\Gamma(n,k,d)$ to $\Gamma(n,lk,d)$ for $l \geq 1$. For $k \geq 3$, let

$$d_k := \frac{k + (-k \mod 3)}{3k},$$

i.e.

$$d_k = \begin{cases} \frac{1}{3} \text{ if } k = 0 \mod 3, \\ \frac{k+2}{3k} \text{ if } k = 1 \mod 3, \\ \frac{k+1}{3k} \text{ if } k = 2 \mod 3. \end{cases}$$

Below, we analyse Property (T) in the k-angular model. We believe this to be the first result on Property (T) in the k-angular model for any k not divisible by 3, and in fact provides bounds for Property (T) in the k-angular model for each $k \geq 8$. Note that we do not get a lower bound than is already known for the Gromov model.

Theorem E. *Let $k \geq 8$, let $d > d_k$, and let $\Gamma_m \sim \Gamma(m,k,d)$. Then*

$$\lim_{m\to\infty} \mathbb{P}(\Gamma_m \textit{ has Property } (T)) = 1.$$

Secondly, we can consider the density model. Again, there is some ambiguity between the strict model and the lax model in the literature. Indeed, many cubulation results, such as those of [OW11] and [MP15], refer to groups in the strict model, whilst results on Property (T) typically refer to groups in the lax model. In particular, the following result is due to Żuk [Ż03] and Kotowski–Kotowski [KK13] (again, see [ALuS15] for finer analysis of $\Gamma(n,3,d)$ as $d \to 1/3$). There is an alternative proof of the below in [DM19, Corollary 12.7]

Theorem. *[Ż03, KK13] (c.f. [DM19, Corollary* 12.7*]) Fix $n \geq 2$, let $d > 1/3$, and let $\Gamma_k \sim \Gamma(n,3k,d)$. Then*

$$\lim_{k\to\infty} \mathbb{P}(\Gamma_k \textit{ has Property } (T)) = 1.$$

Note that the above results *only apply to groups whose relator length is divisible by* 3. However, this result has two important consequences: firstly it provides an infinite number of hyperbolic torsion free groups with Property (T), since such groups are torsion free with probability tending to 1, and the Euler characteristic of such a group is dependent only on n, k, and d [Oll04]. Secondly, it proves that groups in the $\Gamma_{lax}(n, k, d)$ model have Property (T), using the following argument, which is stated in [Ż03, KK13].

Lemma. *Fix $n \geq 2$, let $d > 0$, and let k_i be a sequence of increasing integers such that $|k_{i+1} - k_i|$ is uniformly bounded. If*

$$\lim_{k_i \to \infty} \mathbb{P}(\Gamma \sim \Gamma(n, k_i, d) \textit{ has Property (T)}) = 1,$$

then for any $d' > d$, letting $f(l) := \max_i |k_{i+1} - k_i|$,

$$\lim_{l \to \infty} \mathbb{P}(\Gamma \sim \Gamma_{lax}(n, l, d', f) \textit{ has Property (T)}) = 1.$$

Proof. Let $C = \max_i |k_{i+1} - k_i|$, and let $f(l) = C$. For each l choose $k_{i(l)}$ such that $l - C \leq k_{i(l)} \leq l + C$. Then for sufficiently large l, and for $\Gamma_l = \langle A_n \mid R \rangle \sim \Gamma_{lax}(n, l, d')$, we see that for any $d < d'' < d'$, with probability tending to 1 as l tends to infinity,

$$|R \cap \mathcal{C}(n, k_{i(l)})| \geq (2n-1)^{d'' k_{i(l)}}.$$

Hence, by choosing a random subset $R' \subseteq R \cap \mathcal{C}(n, k_{i(l)})$ of size $(2n-1)^{kd}$, and setting $\Gamma'_{i(l)} := \langle A_n \mid R' \rangle$, we see that there exists an epimorphism $\Gamma'_{i(l)} \twoheadrightarrow \Gamma_l$, and $\Gamma'_{i(l)} \sim \Gamma(n, k_{i(l)}, d)$. Since $\Gamma'_{i(l)}$ has Property (T) with probability tending to 1 as $i(l)$ tends to infinity, and Property (T) is preserved by epimorphisms, the result follows. $\square$

However, we note that the question of Property (T) remains open for the strict model. If $\lim_{k \to \infty} \mathbb{P}(\Gamma \sim \Gamma(n, k, d)$ has Property (T)$) = 1$, then we must also have that $\lim_{i \to \infty} \mathbb{P}(\Gamma \sim \Gamma(n, p_i, d)$ has Property (T)$) = 1$, where p_i denotes the i^{th} prime. Since the results of [Ż03, KK13] do not apply in this regime, we are inspired to further analyse the question of Property (T) for $\Gamma(n, k, d)$.

We now briefly explain the approach taken by [Ż03, KK13] to prove their theorem. Firstly, one takes $n \geq 2$, $d > 1/3$, and considers $\Gamma_m \sim \Gamma(m, 3, d)$ (in fact we require all relators to be positive words, i.e. words containing no inverse letters). It can then be proved that

$$\lim_{m \to \infty} \mathbb{P}(\Gamma_m \text{ has Property (T)}) = 1.$$

The proof of the above is very involved, and requires passing via an alternate model, the *permutation model*: we omit the definition of this model as we do not require it.

One fixes $d' > d$, and finds for each k an integer $m(k, n)$ and a surjection $\Gamma_{m(k,n)} \twoheadrightarrow \Gamma'_k$, where $\Gamma'_k \sim \Gamma(n, 3k(m, n), d')$ (technically this is a surjection onto a finite index subgroup of Γ'_k). The result then follows by preservation of Property (T) under epimorphisms and taking finite index extensions.

A natural approach to extend the results to the strict model using the techniques of Żuk and Kotowski–Kotowski would be to fix $l \geq 3$, let $\Gamma_{(m,l)} \sim \Gamma(m, l, d)$, and consider $m \to \infty$. Then for each $n \geq 2$ and $k \geq 3$ find an integer $m(k, l, n)$ and

$$\Gamma'_k \sim \Gamma(n, lk, d')$$

with $\Gamma_{(m(k,l,n),l)} \twoheadrightarrow \Gamma'_{k(m,n,l)}$, as in [Mon21]. However, if we consider the model $\Gamma(n, p_i, d')$, then we must have in the above that $lk = p_k$ where p_k is the k^{th} prime number, which necessarily forces $m(k, l, n) = n$, and therefore we cannot use statements of the form $\lim_{m\to\infty} \mathbb{P}(\Gamma_m$ has Property (T)), as m must be bounded.

To address this, we therefore must deal with the model $\Gamma(n, k, d)$ directly. The approach is to use the work of Ballmann–Świątkowski [BS97] and Żuk [Ż96] (c.f. [Ż03]), in which a spectral condition for Property (T) was provided independently. This will be used to provide an alternate criterion for Property (T) in terms of the first eigenvalue of a graph we define relative to Γ, $\Delta_k(\Gamma)$. The bulk of this chapter then analyses the eigenvalues of these random graphs.

The following completes the analysis of Property (T) in $\Gamma(n, k, d)$ for $d > 1/3$.

Theorem F. *Let $n \geq 2$, $d > 1/3$, and let $\Gamma_k \sim \Gamma(n, k, d)$. Then*

$$\lim_{k\to\infty} \mathbb{P}(\Gamma_k \textit{ has Property } (T)) = 1.$$

Note that this immediately implies for any infinite sequence, $\{k_i\}_i$, of increasing positive integers, and $\Gamma_i \sim \Gamma(n, k_i, d)$ that:

$$\lim_{i\to\infty} \mathbb{P}(\Gamma_i \text{ has Property } (T)),$$

so that we strengthen the results of [Ż03, KK13].

We could also consider the case of $d \to 1/3$ in a manner similar to that of [ALuS15]. For $n \geq 2$, $k \geq 3$, and $0 < p < 1$, we can define the random group model $\Gamma_p(n, k, p)$: let $\Gamma = \langle A_n \mid R\rangle$, where R is obtained by adding each word in $\mathcal{C}(n, k)$ with probability p. Since Property (T) is an increasing property (one preserved by epimorphisms), it is easy to switch between $\Gamma_p(n, k, p)$ and $\Gamma(n, k, (2n-1)^k p)$ in a manner analogous to switching between the Erdös–Rényi random graph $G(m, p)$ and the random graph $G(m, M)$, since the number of relators in R is $|R| = (1 + o(1))(2n-1)^k p$ almost surely, for p sufficiently large. In fact, we do analyse Property (T) in $\Gamma_p(n, k, p)$ in Theorems

5.5.2 and 5.5.6, and then use these to prove Theorems E and F. However, we believe that the notation and constants involved in the statements of Theorems 5.5.2 and 5.5.6 add unnecessary complexity to the statement of Theorems E and F, and so we leave these to Section 5.5.

5.1.1 Structure of the chapter

Our proof of Theorems E and F are greatly inspired by the work of [Ż03, KK13, ALuS15]. The idea of the proof is the following: for a finitely presented group Γ we find a graph $\Delta(\Gamma)$, and using work of [Ż96], [BS97], we prove that if $\lambda_1(\Delta(\Gamma)) > 1/2$, then Γ has Property (T). This graph loosely corresponds to the 'link of depth $k/3$' of the presentation complex for Γ. For random groups this graph $\Delta(\Gamma)$ can be written as the union of a graph Σ_2 and two graphs Σ_1, Σ_3 (which will be bipartite for $k \neq 0 \bmod 3$). If we allowed all freely reduced words as relators, then these graphs would have the marginal distributions of Erdös–Rényi random graphs. Since we restrict to only having cyclically reduced words as relators, these graphs will not allow some edges, and so will have the marginal distributions of *reduced random graphs*. We need to analyse the eigenvalues of these graphs, and then prove the union of these graphs has high eigenvalue with large probability.

The chapter is structured as follows. In Section 5.2 we provide a spectral criterion for Property (T), related to the graph Δ_k. Sections 5.3 and 5.4 are more geared towards graph theory, and allow us to analyse the eigenvalues of specific random graphs and of unions of graphs. In Section 5.5 we apply these results to prove the main theorems of this chapter.

5.2 A spectral criterion for Property (T)

In this section we deduce a spectral criterion for Property (T): we first remind the reader of some of the relevant definitions. We focus only on finitely generated discrete groups: for a further exposition the reader should see, for example, [BdlHV08].

Let Γ be a finitely generated group with finite generating set S, let $\mathcal{H}$ be a Hilbert space, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. We say that π has *almost-invariant vectors* if for every $\epsilon > 0$ there is some non-zero $u_\epsilon \in \mathcal{H}$ such that for every $s \in S$, $||\pi(s)u_\epsilon - u_\epsilon|| < \epsilon||u_\epsilon||$.

Definition 5.2.1. We say that Γ has *Property (T)* if for every Hilbert space $\mathcal{H}$, and for every unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ with almost-invariant vectors, there exists a non-zero invariant vector for π.

It is standard that the choice of generating set does not matter. We now note the following well known results concerning Property (T): for proofs see, for example, [BdlHV08]. We will use these results implicitly throughout.

Lemma. *Let Γ be a finitely generated group, and let H be a finite index subgroup of Γ: Γ has Property (T) if and only if H has Property (T).*

Lemma. *Let Γ be a finitely generated group with Property (T) and let Γ' be a homomorphic image of Γ. Then Γ' has Property (T).*

5.2.1 The spectral criterion

Now, let $\Gamma = \langle A_n \mid R \rangle$ be a finite presentation of a group and let R_k be the set of words in R of length k. Define the graph $\Delta_3(A_n \mid R)$ by $V(\Delta_3(A_n \mid R)) = A_n \sqcup A_n^{-1}$ and for each relator $r = r_1 r_2 r_3 \in R_3$ add the edges $(r_1, r_3^{-1}), (r_2, r_1^{-1}), (r_3, r_2^{-1})$.

The use of this graph is the following, proved independently by [Ż96] (c.f. [Ż03]) and [BS97]. The result is often stated for a model of Δ_3 without multiple edges, and is often known as *Żuk's criterion for Property (T).*

Theorem 5.2.2. *[Ż96, BS97, Ż03] Let $\Gamma = \langle A_n \mid R \rangle$ be a finite presentation. If $\lambda_1(\Delta_3(A_n \mid R)) > 1/2$, then Γ has Property (T).*

We now apply this to recover an alternate spectral criterion for Property (T). However, before we introduce the graph Δ_k, we first note a result regarding finite index subgroups of free groups. For the free group $F_n := \mathbb{F}(A_n)$ and for $l \geq 1$, we define $\mathcal{W}(n, l)$ to be the set of freely reduced words of length l in F_n. We now prove that these sets always generate finite index subgroups of F_n.

Lemma 5.2.3. *Let $l \geq 1$. Then $[F_n : \langle \mathcal{W}(n, l) \rangle] < \infty$.*

(In fact it is easily seen that for any $l \geq 1$, $[F_n : \langle \mathcal{W}(n, l) \rangle] \leq 2$).

Proof. Note that $\mathcal{W}(n, l) = S_l(F_n)$, the sphere of radius l in F_n. Hence

$$[F_n : \langle \mathcal{W}(n, l) \rangle] \leq |B_{F_n}(id, l-1)| = 2n(2n-1)^{l-2},$$

since $F_n = B_{F_n}(id, l-1)\langle S_l(F_n) \rangle$. $\square$

We now introduce the graph to which our spectral criterion will apply.

Definition 5.2.4. Let $G = \langle A_n \mid R\rangle$ be a finite presentation of a group and let $k \geq 3$. We define the graph $\Delta_k(A_n \mid R)$, as follows, depending on $k \mod 3$.

$\mathbf{k = 0 \mod 3}$: Let $V(\Delta_k(A_n \mid R)) = \mathcal{W}(n, k/3)$. For each relator $r = r_1 \ldots r_k \in R_k$, write $r = r_x r_y r_z$ with $r_x, r_y, r_z \in \mathcal{W}(n, k/3)$, and add the edges (r_x, r_z^{-1}), (r_y, r_x^{-1}), (r_z, r_y^{-1}).

$\mathbf{k = 1 \mod 3}$: Let $\Delta_k(A_n \mid R)$ be the graph with

$$V(\Delta_k(A_n \mid R)) = \mathcal{W}\left(n, \frac{k-1}{3}\right) \bigsqcup \mathcal{W}\left(n, \frac{k+2}{3}\right).$$

For each relator $r = r_1 \ldots r_k \in R_k$ write $r = r_x r_y r_z$ with $r_x, r_y \in \mathcal{W}(n, \frac{k-1}{3})$ and $r_z \in \mathcal{W}(n, \frac{k+2}{3})$, and add the edges (r_x, r_z^{-1}), (r_y, r_x^{-1}), (r_z, r_y^{-1}).

$\mathbf{k = 2 \mod 3}$: Let $\Delta_k(A_n \mid R)$ be the graph with

$$V(\Delta_k(A_n \mid R)) = \mathcal{W}\left(n, \frac{k-2}{3}\right) \bigsqcup \mathcal{W}\left(n, \frac{k+1}{3}\right).$$

For each relator $r = r_1 \ldots r_k \in R_k$ write $r = r_x r_y r_z$ with $r_x, r_y \in \mathcal{W}(n, \frac{k+1}{3})$ and $r_z \in \mathcal{W}(n, \frac{k-2}{3})$, and add the edges (r_x, r_z^{-1}), (r_y, r_x^{-1}), (r_z, r_y^{-1}).

We can prove the following.

Lemma 5.2.5. *Let $\Gamma = \langle A_n \mid R\rangle$ be a finite presentation and let $k \geq 3$. If $\lambda_1(\Delta_k(A_n \mid R)) > 1/2$, then Γ has Property (T).*

We note that this lemma is not particularly effective when given a specific finite presentation of a group: for the above spectral condition to hold, we heuristically require that degree of each vertex is large. Since each relator adds at most three edges to Δ_k, we heuristically require $|R| >> (2n-1)^{(k+(-k \mod 3))/3}$ for the above to be satisfied. However, this is exactly the regime we consider for random groups.

Proof. We prove this for $k = 2 \mod 3$: the other cases are similar. First, for ease, let $\Gamma' = \langle A_n \mid R_k\rangle$. Since Γ is a homomorphic image of Γ', it suffices to prove that Γ' has Property (T). Let $\phi : F_n \twoheadrightarrow \Gamma'$ be the canonical epimorphism induced by the choice of presentation for Γ'. Let $\mathcal{W} = \mathcal{W}(n,(k-2)/3) \sqcup \mathcal{W}(n,(k+1)/3)$, $W = \phi(\mathcal{W})$, and let $H = \langle W\rangle_{\Gamma'}$: by Lemma 5.2.3 we have that $[\Gamma' : H] < \infty$.

For each $r \in R_k$, write $r = r_x r_y r_z$ where $r_x, r_y \in \mathcal{W}(n, (k+1)/3)$ and $r_z \in \mathcal{W}(n, (k-2)/3)$. Let $T = \{r_x r_y r_z \ : \ r \in R_k\}$ and let

$$\tilde{\Gamma} := \mathbb{F}(\mathcal{W}) \Big/ \langle\langle T \rangle\rangle = \langle \mathcal{W} \mid T \rangle.$$

It is clear that there is an epimorphism $\psi : \tilde{\Gamma} \twoheadrightarrow H$, so that Γ' is a finite index extension of a quotient of $\tilde{\Gamma}$. Next, we note that $\Delta_k(A_n \mid R) \cong \Delta_3(\mathcal{W} \mid T)$. By Theorem 5.2.2, if $\lambda_1(\Delta_k(A_n \mid R)) = \lambda_1(\Delta_3(\mathcal{W} \mid T)) > 1/2$, then $\tilde{\Gamma}$ has Property (T). Since Property (T) is preserved under epimorphisms and passing to finite index extensions, it follows that if $\lambda_1(\Delta_k(A_n \mid R)) > 1/2$, then Γ has Property (T). □

5.3 The spectra of almost regular graphs and the unions of regular graphs

In this section we analyse the spectral theory of almost regular graphs, as well as some results on the eigenvalues of Erdös–Rényi random graphs. We also prove a result concerning the eigenvalues of the union of a well connected graph and two bipartite graphs. We first note the following lemmas.

Lemma 5.3.1. *Let $m_2 = m_2(m_1)$ and $p = p(m_1)$ be such that $\min\{m_1, m_2\}p = \Omega_{m_1}(\log \max\{m_1, m_2\})$. Then a.a.s.$(m_1)$*

$$\mu_1(A(G(m_1, m_2, p))) \leq [1 + o_{m_1}(1)]p\sqrt{m_1 m_2}.$$

Proof. By Lemma 4.2.4, a.a.s.(m_1) the maximum degree of a vertex in V_1 is $(1 + o_{m_1}(1))m_2 p$, and the maximum degree of a vertex in V_2 is $(1 + o_{m_1}(1))m_1 p$. By Lemma 2.2.3,

$$\max_i |\mu_i(A(G(m_1, m_2 p)))| \leq \max_{\substack{v \in V_1(G) \\ w \in V_2(G)}} \sqrt{deg(v)deg(w)} \leq [1 + o_{m_1}(1)]\sqrt{m_1 m_2 p^2}$$

with probability tending to 1 as m_1 tends to infinity.

□

Similarly we can deduce the leading value of $G(m, p)$.

Lemma 5.3.2. *Let $m \geq 1$ and $p = p(m)$ be such that $mp = \Omega_m(\log m)$. Then a.a.s.(m) $\mu_1(A(G(m, p))) \leq [1 + o_m(1)]mp$.*

5.3.1 The spectra of almost regular graphs

We now analyse the spectra of almost regular graphs in more detail than in Chapter 4. We note the following results.

Lemma. *[KK13, Lemma 4.4] Let $d_m \to \infty$ and let G_m be almost d_m-regular. Then $\frac{1}{d_m}\mu_2(A(G_m)) = (1 + o_m(1))(1 - \lambda_1(G_m))$. In particular, if $\mu_2(A(G_m)) = o_m(d_m)$ then $\lambda_1(G_m) = 1 - o_m(1)$.*

Lemma. *[KK13, Lemma 4.5] Let G_m be an almost d_m-regular graph and let G'_m be a graph on the same vertex set whose maximum degree is $o_m(d_m)$. Then:*

i) $G_m \cup G'_m$ is almost d_m regular,

ii) and $\lambda_1(G_m) = \lambda_1(G_m \cup G'_m) + o_m(1)$.

Again, recall that $\lambda_1(G) = 1 - \mu_2(D^{-1/2}AD^{-1/2})$. We now prove the corresponding result for bipartite graphs: our proofs are different to [KK13], and rely on Weyl's inequality.

Lemma 5.3.3. *Let $d_m^{(1)}, d_m^{(2)} \to \infty$ and let G_m be almost $(d_m^{(1)}, d_m^{(2)})$-regular. For $i = 1, \ldots, |V(G_m)|$:*

$$\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}}\mu_i(A(G_m)) = \mu_i(D^{-1/2}(G_m)A(G_m)D^{-1/2}(G_m)) + o_m(1).$$

In particular, if $\mu_2(A(G_m)) = o_m\left(\sqrt{d_m^{(1)} d_m^{(2)}}\right)$, then $\lambda_1(G_m) = 1 - o_m(1)$.

Proof. As G_m is almost $(d_m^{(1)}, d_m^{(2)})$-regular, we see that for

$$A = A(G_m) = \begin{pmatrix} 0 & A_1 \\ A_1^T & 0 \end{pmatrix},\ D = D(G_m),$$

there exists a matrix K with norm $||K||_\infty = o_m(1)$ such that

$$\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}}A = D^{-1/2}AD^{-1/2} + K.$$

Since $|\mu_i(K)| \leq ||K||_\infty = o_m(1)$ for all i, the first statement of the Lemma follows easily by Weyl's inequality. The second statement follows from Remark 2.2.2. $\square$

Lemma 5.3.4. *Let $d_m^{(1)}, d_m^{(2)} \to \infty$, and let G_m be almost $(d_m^{(1)}, d_m^{(2)})$-regular. Let G'_m be a bipartite graph on the same vertex set as G_m with the same vertex partitions, such that the maximum degree of $v \in V_i(G'_m)$ is $o_m(d_m^{(i)})$. Then:*

i) *$G_m \cup G'_m$ is almost $(d_m^{(1)}, d_m^{(2)})$ regular,*

ii) *and $\lambda_1(G_m) = \lambda_1(G_m \cup G'_m) + o_m(1)$.*

Proof. Part *i)* is immediate. For part *ii)*, we see that $A(G_m \cup G'_m) = A(G_m) + A(G'_m)$: since the maximum degree of a vertex $v \in V_i(G'_m) = V_i(G_m)$ is $o(d_m^{(i)})$, we have by Lemma 2.2.3 that $\max_i |\mu_i(A(G'_m))| \le o_m\left(\sqrt{d_m^{(1)} d_m^{(2)}}\right)$, and hence

$$\max_i \left| \mu_i\left(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G'_m) \right) \right| = o_m(1).$$

By Weyl's inequality,

$$\begin{aligned}
&\mu_2\left(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} (A(G_m) + A(G'_m)) \right) \\
&\le \mu_2\left(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G_m) \right) + \mu_1\left(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G'_m) \right) \\
&= \mu_2\left(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G_m) \right) + o_m(1).
\end{aligned}$$

Similarly

$$\mu_2\left(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} (A(G_m) + A(G'_m)) \right) \ge \mu_2\left(\frac{1}{\sqrt{d_m^{(1)} d_m^{(2)}}} A(G_m) \right) + o_m(1),$$

and the result follows by Remark 2.2.2 and Lemma 5.3.3. □

5.3.2 The spectra of unions of regular graphs

The purpose of this subsection is to analyse the spectral distribution of unions of three graphs with relatively high first eigenvalue. This is already known when all three graphs share the same vertex set: see e.g. [Ż03].

Lemma 5.3.5. *Let G_1, G_2, G_3 be d-regular graphs on the same vertex set, and suppose $\lambda_1(G_i) \geq 1 - c_i$ for each i. Then*

$$\lambda_1(G_1 \cup G_2 \cup G_3) \geq 1 - \frac{c_1 + c_2 + c_3}{3}.$$

We now wish to extend this to the case where the graphs are relatively well connected, and they do not share the same vertex set. We first recall (a partial consequence of) the Courant-Fischer Theorem, as follows.

Theorem (Courant-Fischer Theorem)**.** *Let M be a symmetric $m \times m$ matrix with first eigenvalue $\mu_1(M)$ and corresponding eigenvector $\underline{e}$. Then*

$$\mu_2(M) = \max_{\substack{\underline{\boldsymbol{x}} \perp \underline{e} \\ ||\underline{\boldsymbol{x}}||=1}} \langle M\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}} \rangle = \max_{\substack{\underline{\boldsymbol{x}} \perp \underline{e} \\ \underline{\boldsymbol{x}} \neq \underline{\boldsymbol{0}}}} \frac{\langle M\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}} \rangle}{\langle \underline{\boldsymbol{x}}, \underline{\boldsymbol{x}} \rangle}.$$

Using this, we can prove the following.

Lemma 5.3.6. *Let G_1, G_2, G_3 be graphs such that:*

i) G_2, G_3 are bipartite, with $V(G_1) = V_1(G_2) = V_1(G_3)$ and $V_2(G_2) = V_2(G_3)$,

ii) G_1 is $2d_1$-regular, and G_2, G_3 are (d_1, d_2)-regular,

iii) and for $i = 1, 2, 3$ there exists $0 \leq c_i \leq 1$ with $\lambda_1(G_i) \geq 1 - c_i$.

Then

$$\lambda_1(G_1 \cup G_2 \cup G_3) \geq 1 - \frac{\sqrt{2}c_1 + c_2 + c_3}{2\sqrt{2}}.$$

Proof. Let $\underline{\mathbf{1}}_l$ be the all 1 vector with l entries, and let $G = G_1 \cup G_2 \cup G_3$. For $i = 1, 2, 3$, let $\Lambda_i = D_i^{-1/2} A_i D_i^{-1/2}$, where $D_i = D(G_i)$ and $A_i = A(G_i)$ (here we view G_1 as a graph on $V_1 \sqcup V_2$). Let $D = D(G)$, $A = A(G)$, and consider $\Lambda = D(G)^{-1/2} A(G) D(G)^{-1/2}$, so that

$$\Lambda = \frac{1}{2}\Lambda_1 + \frac{1}{2\sqrt{2}}\Lambda_2 + \frac{1}{2\sqrt{2}}\Lambda_3 :$$

each of $\Lambda, \Lambda_1, \Lambda_2, \Lambda_3$ is symmetric and hence self-adjoint. We remark again that $\mu_2(\Lambda_i) = 1 - \lambda_1(G_i)$. Recall that $m_2 d_2 = m_1 d_1$, so that $d_2 = m_1 d_1 / m_2$.

Now, we consider the first eigenvalues of the matrices Λ and Λ_i. The eigenvector corresponding to $\mu_1(\Lambda) = 1$ is

$$D^{1/2} \underline{\mathbf{1}}_{m_1+m_2} = \begin{pmatrix} 2\sqrt{d_1}\underline{\mathbf{1}}_{m_1} \\ \sqrt{2d_2}\underline{\mathbf{1}}_{m_2} \end{pmatrix}.$$

The eigenvector corresponding to $\mu_1(\Lambda_2) = 1$ and $\mu_1(\Lambda_3) = 1$ is

$$D_2^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = D_3^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = \begin{pmatrix} \sqrt{d_1}\underline{\mathbf{1}}_{m_1} \\ \sqrt{d_2}\underline{\mathbf{1}}_{m_2} \end{pmatrix}.$$

The eigenvector corresponding to $\mu_1(\Lambda_1) = 1$ is

$$D_1^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = \begin{pmatrix} \sqrt{2d_1}\underline{\mathbf{1}}_{m_1} \\ 0 \end{pmatrix}.$$

Let $\underline{\boldsymbol{\phi}}$ be a vector with $||\underline{\boldsymbol{\phi}}|| = 1$, $\underline{\boldsymbol{\phi}} \cdot D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = 0$, and $\mu_2(\Lambda) = \langle \Lambda\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle$, which exists by the Courant-Fischer Theorem. We may write

$$\underline{\boldsymbol{\phi}} = \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} + \underline{\boldsymbol{u}} \\ \beta\underline{\mathbf{1}}_{m_2} + \underline{\boldsymbol{v}} \end{pmatrix},$$

where $\underline{\boldsymbol{u}} \cdot \underline{\mathbf{1}}_{m_1} = \underline{\boldsymbol{v}} \cdot \underline{\mathbf{1}}_{m_2} = 0$. As

$$\underline{\boldsymbol{\phi}} \cdot D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = 2\sqrt{d_1}\alpha m_1 + \sqrt{2d_2}\beta m_2 = 2\sqrt{d_1}\alpha m_1 + \sqrt{2d_1 m_1/m_2}\beta m_2,$$

we see $\beta = -\sqrt{2m_1}\alpha/\sqrt{m_2}$. Let

$$\underline{\boldsymbol{\phi_1}} = \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ \beta\underline{\mathbf{1}}_{m_2} \end{pmatrix}, \; \underline{\boldsymbol{\phi_2}} = \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\boldsymbol{v}} \end{pmatrix},$$

so that $\underline{\boldsymbol{\phi_1}} \cdot D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = \underline{\boldsymbol{\phi_2}} \cdot D^{1/2}\underline{\mathbf{1}}_{m_1+m_2} = 0$. Write $\gamma = ||\underline{\boldsymbol{\phi_1}}||^2$, with $||\underline{\boldsymbol{\phi_2}}||^2 = 1-\gamma$. Note that $0 \leq \gamma = \alpha^2 m_1 + \beta^2 m_2 = 3\alpha^2 m_1 \leq 1$. We now calculate:

$$\langle \Lambda_1\underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_1}} \rangle = \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ \beta\underline{\mathbf{1}}_{m_2} \end{pmatrix} = \alpha^2 m_1.$$

Secondly

$$\langle \Lambda_1\underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_2}} \rangle = \begin{pmatrix} \alpha\underline{\mathbf{1}}_{m_1} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\boldsymbol{v}} \end{pmatrix} = \alpha\underline{\mathbf{1}}_{m_1} \cdot \underline{\boldsymbol{u}} = 0.$$

Since Λ_1 is self-adjoint, $\langle \underline{\boldsymbol{\phi_1}}, \Lambda_1 \underline{\boldsymbol{\phi_2}} \rangle = \langle \Lambda_1 \underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_2}} \rangle = 0$. Also, since $\underline{\boldsymbol{u}} \cdot D_1^{1/2} \underline{\mathbf{1}}_{m_1} = 0$, we have by the Courant-Fischer Theorem:

$$\langle \Lambda_1 \underline{\boldsymbol{\phi_2}}, \underline{\boldsymbol{\phi_2}} \rangle = \langle \Lambda_1' \underline{\boldsymbol{u}}, \underline{\boldsymbol{u}} \rangle \leq \mu_2(\Lambda_1) ||\underline{\boldsymbol{u}}||^2 = c_1 ||\underline{\boldsymbol{u}}||^2 \leq c_1 ||\underline{\boldsymbol{\phi_2}}||^2 = c_1(1-\gamma),$$

where Λ_1' is $D(G_1)^{-1/2} A(G_1) D(G_1)^{-1/2}$ with G_1 considered as a graph on the vertex set V_1.

We now perform the same calculations for Λ_2. Firstly, for some matrix B_2

$$\Lambda_2 \underline{\boldsymbol{\phi_1}} = \frac{1}{\sqrt{d_1 d_2}} \begin{pmatrix} 0 & B_2 \\ B_2^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \underline{\mathbf{1}}_{m_1} \\ \beta \underline{\mathbf{1}}_{m_2} \end{pmatrix} = \begin{pmatrix} \beta \sqrt{\frac{d_1}{d_2}} \underline{\mathbf{1}}_{m_1} \\ \alpha \sqrt{\frac{d_2}{d_1}} \underline{\mathbf{1}}_{m_2} \end{pmatrix} = \begin{pmatrix} \beta \sqrt{\frac{m_2}{m_1}} \underline{\mathbf{1}}_{m_1} \\ \alpha \sqrt{\frac{m_1}{m_2}} \underline{\mathbf{1}}_{m_2} \end{pmatrix},$$

so that

$$\langle \Lambda_2 \underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_1}} \rangle = \begin{pmatrix} \beta \sqrt{\frac{m_2}{m_1}} \underline{\mathbf{1}}_{m_1} \\ \alpha \sqrt{\frac{m_1}{m_2}} \underline{\mathbf{1}}_{m_2} \end{pmatrix} \cdot \begin{pmatrix} \alpha \underline{\mathbf{1}}_{m_1} \\ \beta \underline{\mathbf{1}}_{m_2} \end{pmatrix} = 2\alpha\beta\sqrt{m_1 m_2}.$$

Next, by the Courant-Fischer Theorem, $\langle \Lambda_2 \underline{\boldsymbol{\phi_2}}, \underline{\boldsymbol{\phi_2}} \rangle \leq c_2 ||\underline{\boldsymbol{\phi_2}}||^2 = c_2(1-\gamma)$ (since $\underline{\boldsymbol{\phi_2}} \cdot D_2^{1/2} \underline{\mathbf{1}}_{m_1+m_2} = 0$). Furthermore,

$$\langle \Lambda_2 \underline{\boldsymbol{\phi_1}}, \underline{\boldsymbol{\phi_2}} \rangle = \begin{pmatrix} \beta \sqrt{\frac{m_2}{m_1}} \underline{\mathbf{1}}_{m_1} \\ \alpha \sqrt{\frac{m_1}{m_2}} \underline{\mathbf{1}}_{m_2} \end{pmatrix} \cdot \begin{pmatrix} \underline{\boldsymbol{u}} \\ \underline{\boldsymbol{v}} \end{pmatrix} = \beta \sqrt{\frac{m_2}{m_1}} \underline{\mathbf{1}}_{m_1} \cdot \underline{\boldsymbol{u}} + \alpha \sqrt{\frac{m_1}{m_2}} \underline{\mathbf{1}}_{m_2} \cdot \underline{\boldsymbol{v}} = 0,$$

since $\underline{\mathbf{1}}_{m_1} \cdot \underline{\boldsymbol{u}} = \underline{\mathbf{1}}_{m_2} \cdot \underline{\boldsymbol{v}} = 0$. Finally, since Λ_2 is symmetric and hence self-adjoint, we see that

$$\langle \Lambda_2 \underline{\boldsymbol{\phi_2}}, \underline{\boldsymbol{\phi_1}} \rangle = \langle \underline{\boldsymbol{\phi_2}}, \Lambda_2 \underline{\boldsymbol{\phi_1}} \rangle = 0.$$

We can perform similar calculations for Λ_3. Putting this all together, we have

$$\langle \Lambda_1 \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle \leq \alpha^2 m_1 + c_1(1-\gamma),$$

$$\langle \Lambda_2 \underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\phi}} \rangle \leq 2\alpha\beta\sqrt{m_1 m_2} + c_2(1-\gamma),$$

$$\langle \Lambda_3 \underline{\phi}, \underline{\phi} \rangle \leq 2\alpha\beta\sqrt{m_1 m_2} + c_3(1-\gamma).$$

We calculate

$$\frac{1}{\sqrt{2}}\alpha\beta\sqrt{m_1 m_2} = -\alpha^2\sqrt{\frac{m_1}{m_2}}\sqrt{m_1 m_2} = -\alpha^2 m_1.$$

Therefore

$$\begin{aligned}
\langle \Lambda \underline{\phi}, \underline{\phi} \rangle &= \frac{1}{2}\langle \Lambda_1 \underline{\phi}, \underline{\phi} \rangle + \frac{1}{2\sqrt{2}}\langle \Lambda_2 \underline{\phi}, \underline{\phi} \rangle + \frac{1}{2\sqrt{2}}\langle \Lambda_3 \underline{\phi}, \underline{\phi} \rangle \\
&\leq \frac{1}{2}c_1(1-\gamma) + \frac{1}{2}\alpha^2 m_1 + \frac{1}{\sqrt{2}}\alpha\beta\sqrt{m_1 m_2} + \frac{1}{2\sqrt{2}}c_2(1-\gamma) \\
&\quad + \frac{1}{\sqrt{2}}\alpha\beta\sqrt{m_1 m_2} + \frac{1}{2\sqrt{2}}c_3(1-\gamma) \\
&= \frac{1}{2}\alpha^2 m_1 - 2\alpha^2 m_1 + \frac{1-\gamma}{2\sqrt{2}}(\sqrt{2}c_1 + c_2 + c_3) \\
&= \frac{-3}{2}\alpha^2 m_1 + \frac{1-\gamma}{2\sqrt{2}}(\sqrt{2}c_1 + c_2 + c_3) \\
&= -\frac{1}{2}\gamma + \frac{1-\gamma}{2\sqrt{2}}(\sqrt{2}c_1 + c_2 + c_3) \\
&\leq \frac{\sqrt{2}c_1 + c_2 + c_3}{2\sqrt{2}},
\end{aligned}$$

since $0 \leq \gamma \leq 1$. As $\underline{\phi}$ was chosen with $\mu_2(\Lambda) = \langle \Lambda \underline{\phi}, \underline{\phi} \rangle$, we see that

$$\mu_2(\Lambda) \leq \frac{\sqrt{2}c_1 + c_2 + c_3}{2\sqrt{2}},$$

and hence

$$\lambda_1(G) = 1 - \mu_2(\Lambda) \geq 1 - \frac{\sqrt{2}c_1 + c_2 + c_3}{2\sqrt{2}}.$$

□

We can apply this in specific cases to obtain an explicit bound.

Lemma 5.3.7. *Let G_i, c_i be as above. Suppose $c_1 = \epsilon, c_2 = c_3 = \epsilon + 1/3$ for some $\epsilon < 1/100$. Then $\lambda_1(G_1 \cup G_2 \cup G_3) \geq 3/4$.*

Proof. We may apply Lemma 5.3.6 to deduce that

$$\lambda_1(G_1 \cup G_2 \cup G_3) \geq 1 - \frac{(\sqrt{2}+2)\epsilon + 2/3}{2\sqrt{2}} \geq 1 - \frac{2/3 + (2+\sqrt{2})/100}{2\sqrt{2}} \geq \frac{3}{4}.$$

□

5.4 The spectra of reduced random graphs

We have almost understood the spectral distribution of $\Delta_k(A_n \mid R)$ for $\langle A_n \mid R\rangle$ in the $\Gamma(n,k,d)$ model. However, there is one small complication which arises from the fact that we insist upon using cyclically reduced words as relators: the random graphs $\Delta_k(A_n \mid R)$ will not allow edges between certain types of words. Therefore we need to introduce a slightly altered model of random graphs.

Some of the results contained within this section are already known. Indeed, [DM19, Section 11,12] provides far more general results concerning the eigenvalues of reduced random graphs: we provide alternate proofs of the results we require (again we stress that the results of [DM19] are far more general than the results we obtain) as the proofs provide an introduction to the proof strategies of alternate results we require that are not covered by [DM19]. We indicate in the text the results already known.

5.4.1 Reduced random graphs

The following subsection analyses the spectra of a certain model of random graphs.

Definition 5.4.1. Fix $n, l \geq 1$, and let $0 < p < 1$. For $i = 1, \dots, n$, let $a_{i+n} := a_i^{-1}$, and for $i = 1, \dots, 2n$ let

$$S_i = \{w_1 \dots w_l \in \mathcal{W}(n,l) \ : \ w_1 = a_i\} = \{(w_1 \dots w_l)^{-1} \in \mathcal{W}(n,l) \ : \ w_l = a_i^{-1}\}.$$

For $v \in \mathcal{W}(n,l)$, let $i(v)$ be the unique integer such that $v \in S_{i(v)}$. The *reduced random graph* $\mathfrak{Red}(n,l,p)$ is the random graph obtained with vertex set $\mathcal{W}(n,l)$, and edge set constructed as follows.

Let $i = 1, \dots, 2n$. For each pair of vertices $v, w \in \mathcal{W}(n,l)$, add (each of) the edges:

- (v,w) labelled by $i(v)$ with probability $p(v,w)$,
- (w,v), labelled by $i(w)$ with probability $p(w,v)$, where:

$$p(s,t) = \begin{cases} p \text{ if } i(s) \neq i(t), \\ 0 \text{ if } i(s) = i(t). \end{cases}$$

Note that $|\mathcal{W}(n,l)| = 2n(2n-1)^{l-1}$. Furthermore, we can break $\mathfrak{Red}(n,l,p)$ into a union of graphs $\mathfrak{R}_i$, where for $i = 1, \dots, 2n$ each $\mathfrak{R}_i$ is a bipartite graph with vertex set $V_1 = S_i$, $V_2 = \mathcal{W}(n,l) \setminus S_i$, and each edge is added with probability p. Note that $\mathfrak{R}_i \sim G((2n-1)^{l-1}, (2n-1)^l, p)$: therefore, for p satisfying $(2n-1)^l p = \Omega_l(l)$, a.a.s. each graph $\mathfrak{R}_i$ is almost $((2n-1)^{l-1}p, (2n-1)^l p)$-regular. Hence for p satisfying

$(2n-1)^l p = \Omega_l(l)$, a.a.s. the graph $\mathfrak{Red}(n,l,p)$ is almost $2(2n-1)^l p$-regular. Next we prove the following.

Lemma 5.4.2. *Let $n, l \geq 1$, let p be such that $(2n-1)^l p = \Omega_l(\log(2n-1)^l)$, and let $G \sim \mathfrak{Red}(n,l,p)$. There exists a random graph*

$$G' \sim G(2n(2n-1)^{l-1}, 2p-p^2)$$

such that a.a.s.(l),

$$\mu_1(A(G)-A(G')) \leq O_l\left(\max\left\{l, (2n-1)^l p^2, \sqrt{(2n-1)^{l-1}p}\right\}\right).$$

Proof. Let Σ_i be the random graph with vertex set S_i and each edge added with probability $2p-p^2$, so that $\Sigma_i \sim G((2n-1)^{l-1}, 2p-p^2)$. By our assumptions on p, we see by Theorems 4.3.1 and 4.3.2 that a.a.s.(l) for all i (there are 2n such i, so we take the intersection of the $2n$ events)

$$\max_{j\neq 1} |\mu_j(A(\Sigma_i))| \leq O_l\left(\sqrt{(2n-1)^{l-1}p}\right).$$

Let $H = \bigcup_i(\mathfrak{R}_i \cup \Sigma_i)$. The probability that at least one edge connects two vertices $v, w \in S_i$ is $2p-p^2$. If $v \in S_i$ and $w \in S_j$ for $i \neq j$ the probability that at least one edge connects v and w is $1-(1-p)^2 = 2p-p^2$. Hence, by collapsing duplicate edges in H we obtain $G' \sim G(2n(2n-1)^{l-1}, 2p-p^2)$. Next, note that

$$A(G') = A(G) + \sum A_i + K,$$

where K takes into account the double edges obtained from the unions, and A_i is the adjacency matrix of the graph G_i which has vertex set $V(G)$ and edge set $E(\Sigma_i)$. Since the edge sets of each Σ_i are pairwise disjoint, one can easily see that $\mu_1(-\sum_i A_i) = \max_i \mu_1(-A_i)$.

Using the Chernoff bounds for the degrees, we can see that if $(2n-1)^l p^2 = \Omega_l(l)$, then a.a.s.$(l)$ $||K||_\infty = O_l((2n-1)^l p^2)$. Otherwise, we may deduce that $||K||_\infty = O_l(\log(2n-1)^l) = O_l(l)$.

Recall that $\mu_1(-A_i) \leq \max_{j\neq 1}|\mu_j(A_i)|$. Hence by Weyl's inequality:

$$\begin{aligned}\mu_1(A(G)-A(G')) &= \mu_1(-K-\sum A_i)\\ &\leq \mu_1(-K)+\mu_1(-\sum A_i)\end{aligned}$$

$$
\begin{aligned}
&= O_l\Bigg(\max \Bigg\{ ||K||_\infty, \mu_1\Bigg(-\sum A_i\Bigg) \Bigg\} \Bigg) \\
&= O_l\Bigg(\max \Bigg\{ ||K||_\infty, \max_i \{\mu_1(-A_i)\} \Bigg\} \Bigg) \\
&\le O_l\Bigg(\max \Bigg\{ l, (2n-1)^l p^2, \sqrt{(2n-1)^{l-1}p} \Bigg\} \Bigg).
\end{aligned}
$$

□

Similarly we define the following.

Definition 5.4.3. Fix $n \geq 1, l \geq 3, 0 < p < 1$. Let $a_{i+n} := a_i^{-1}$. For $i = 1, \ldots, 2n$, let

$$S_i' = \{w_1 \ldots w_l \in \mathcal{W}(n,l) \ : \ w_1 = a_i\},$$

and

$$T_i' = \{(w_1 \ldots w_{l+1})^{-1} \in \mathcal{W}(n, l+1) \ : \ w_{l+1} = a_i^{-1}\}.$$

The *reduced random bipartite graph* $\mathfrak{BRed}(n,l,p)$ is the random graph with vertex set $V_1 = \mathcal{W}(n,l)$, $V_2 = \mathcal{W}(n,l+1)$, and for each $v \in S_i'$ and vertex $w \in V_2 - T_i'$, the edge (v,w) is added with probability p. The graph $\mathfrak{BR}_i$ is the random bipartite graph obtained as a subgraph with vertex set $V_1 = S_i$ and $V_2 = \mathcal{W}(n, l+1) - T_i$.

Again, for large p, i.e. $(2n-1)^l p = \Omega_l(l)$, the graph $\mathfrak{BRed}(n,l,p)$ is almost $((2n-1)^{l+1}p, (2n-1)^l p)$-regular. We can approximate this graph by an Erdös–Rényi random bipartite graph, similarly to the case of $\mathfrak{Red}(n,l,p)$.

Lemma 5.4.4. *Let $G \sim \mathfrak{BRed}(n,l,p)$, where $(2n-1)^l p = \Omega_l(\log(2n-1)^l)$. There exists a random graph $G' \sim G(2n(2n-1)^{l-1}, 2n(2n-1)^l, p)$ such that a.a.s.(l),*

$$\mu_1(A(G) - A(G')) \leq (1 + o_l(1))(2n-1)^{l-1/2}p.$$

Proof. This follows similarly to the proof of Lemma 5.4.2 for $\mathfrak{Red}(n,l,p)$.

For $i = 1, \ldots, 2n$, let Σ_i be the random graph with vertex set $V_1 = S_i, V_2 = T_i$ and each edge added with probability p, so that $\Sigma_i \sim G((2n-1)^{l-1}, (2n-1)^l, p)$.

Then

$$G' = G \cup \bigcup_i \Sigma_i \sim G(2n(2n-1)^{l-1}, 2n(2n-1)^l, p).$$

We see that $\mu_1(A(G) - A(G')) = \mu_1(-\sum_i A_i)$, where A_i is the adjacency matrix of the graph with vertex set $V(G)$ and edge set $E(\Sigma_i)$. Since the edge sets of the Σ_i are pairwise disjoint, (and the graphs are bipartite, so their spectrum is symmetric around 0) we see that

$$\mu_1(-\sum_i A_i) \leq \max_i \mu_1(-A_i) = \max_i \mu_1(A_i) \leq (1+o_l(1))(2n-1)^{l-1/2}p,$$

by Lemmas 2.2.3 and 4.2.4. □

We may analyse the eigenvalues of reduced random graphs, as follows.

Lemma 5.4.5. *[DM19, Theorem 11.8, 11.9] Let $n \geq 2$, and p be such that $p = o_l(1)$ and $(2n-1)^l p = \Omega_l(l)$. Let $G \sim \mathfrak{Red}(n, l, p)$. Then a.a.s.$(l)$ $\lambda_1(G) \geq 1 - o_l(1)$.*

Proof. Let G' be the graph from Lemma 5.4.2, so that $G' \sim G(2n(2n-1)^{l-1}, 2p - p^2)$ and

$$\mu_1(A(G) - A(G')) \leq O_l\left(\max\left\{ l, (2n-1)^l p^2, \sqrt{(2n-1)^{l-1}p} \right\}\right).$$

Let $D' = D(G)$, and $A' = A(G')$. Note that G is almost $2(2n-1)^l p$ regular, and hence,

$$\begin{aligned}\mu_1(D^{-1/2}(A-A')D^{-1/2}) &\leq O_l\left(\frac{1+o_l(1)}{(2n-1)^l p} \max\left\{ l, (2n-1)^l p^2, \sqrt{(2n-1)^{l-1}p} \right\}\right) \\ &= o_l(1).\end{aligned}$$

Next, by our assumption on p, $2n(2n-1)^l p = \Omega_l(l) = \Omega_l\left(\log 2n(2n-1)^{l-1} \right)$, so that by Theorems 4.3.1 and 4.3.2, a.a.s.(l),

$$\mu_2\left(D'^{-1/2} A' D'^{-1/2} \right) = o_l(1).$$

Next, $D(G)^{-1/2} A D(G)^{-1/2} = \frac{(2-p)n}{2n-1} D'^{-1/2} A' D^{-1/2} + K$, where $||K||_\infty = o_l(1)$. Hence $\mu_1(K) = o_l(1)$. Therefore, by Theorems 4.3.1 and 4.3.2, and Weyl's inequality, *a.a.s.*(l)

$$\begin{aligned}\mu_2(D^{-1/2}AD^{-1/2}) &= \mu_2(D^{-1/2}A'D^{-1/2} + D^{-1/2}AD^{-1/2} - D^{-1/2}A'D^{-1/2}) \\ &\leq \mu_2(D^{-1/2}A'D^{-1/2}) + \mu_1(D^{-1/2}(A-A')D^{-1/2}) \\ &= \mu_2\left(\frac{(2-p)n}{2n-1} D'^{-1/2}A'D'^{-1/2} + K \right) + o_l(1) \\ &\leq \frac{(2-p)n}{2n-1} \mu_2\left(D'^{-1/2}A'D'^{-1/2} \right) + \mu_1(K) + o_l(1) \\ &\leq \frac{(2-p)n}{2n-1} \mu_2\left(D'^{-1/2}A'D'^{-1/2} \right) + o_l(1)\end{aligned}$$

$$= o_l(1).$$

The result follows by Remark 2.2.2. □

Lemma 5.4.6. *Let $n \geq 2$, and p be such that $p = o_l(1)$ and $(2n-1)^l p = \Omega_l(l)$. Let $G \sim \mathfrak{BRed}(n, l, p)$. Then a.a.s.$(l)$*

$$\lambda_1(G) \geq 1 - 1/(2n-1) - o_l(1).$$

We note that we cannot prove that the above bound is sharp, but it is sufficient for our needs.

Proof. Let G' be the graph from Lemma 5.4.4, so that $G' \sim G(2n(2n-1)^{l-1}, 2n(2n-1)^l, p)$, and $\mu_1(A(G) - A(G')) \leq O_l\left((2n-1)^{l-1/2}p\right)$. By Lemma 5.4.4,

$$\mu_1(D^{-1/2}(A - A')D^{-1/2}) \leq [1 + o_l(1)]\frac{1}{2n-1}.$$

Next, $D(G)^{-1/2}A'D^{-1/2} = \frac{2n}{2n-1}D'^{-1/2}A'D'^{-1/2} + K$, where

$$K = \begin{pmatrix} 0 & H \\ H^T & 0 \end{pmatrix}$$

and $\sqrt{||H||_\infty ||H||_1} = o_l(1)$. Hence $\mu_1(K) = o_l(1)$. Therefore, by Theorem D, and using Weyl's inequalities similarly to the proof of Lemma 5.4.5,

$$\begin{aligned}
\mu_2(D^{-1/2}AD^{-1/2}) &= \mu_2(D^{-1/2}A'D^{-1/2} + D^{-1/2}AD^{-1/2} - D^{-1/2}A'D^{-1/2}) \\
&\leq \mu_2(D^{-1/2}A'D^{-1/2}) + \mu_1(D^{-1/2}(A - A')D^{-1/2}) \\
&\leq \mu_2\left(\frac{2n}{2n-1}D'^{-1/2}A'D'^{-1/2} + K\right) + \frac{1}{2n-1} + o_l(1) \\
&\leq \frac{2n}{2n-1}\mu_2(D'^{-1/2}A'D'^{-1}) + \mu_1(K) + \frac{1}{2n-1} + o_l(1) \\
&= \frac{1}{2n-1} + o_l(1).
\end{aligned}$$

The result follows by Remark 2.2.2. □

5.4.2 Regular subgraphs of random graphs

We now need an auxiliary result concerning regular subgraphs of random graphs. Recall that a subgraph H of G is *spanning* if $V(H) = V(G)$. We first note the following.

Theorem. *[SU84] Suppose $mp = \omega(m)\log(m)$ for some $\omega(m) \to \infty$. Let $\delta \geq \omega^{-\theta}$ for some $0 < \theta < 1/2$, and let $G \sim G(m,p)$. Then a.a.s.(m), G contains a $(1-\delta)mp$-regular spanning subgraph.*

We wish to prove the analogue for random bipartite graphs. We do this similarly to [FKS16, Theorem 1.4], which proves the result in the regime $m_1 = m_2$. We note that [FKS16, Theorem 1.4] assumes that $\delta > 0$ is constant, however, their techniques also work for suitable choices of $\delta = o_m(1)$.

Theorem. *[FKS16, Theorem 1.4] Let $m \geq 1$ and $p = p(m) > 0$ be such that $mp = \omega(m)\log m$ for some $\omega \to \infty$ as $m \to \infty$. Let $\delta \geq \omega^{-\theta}$ for some $\theta < 1/2$, and $G \sim G(m,m,p)$. Then a.a.s.(m) G contains a $((1-\delta)mp, (1-\delta)mp)$-regular spanning subgraph.*

In the k-angular model, we have $m_1 = m_2/n$, where $n \to \infty$, so we need to extend the above to a more general setting. We will use the following theorem, commonly known as the Ore-Reyser theorem: see for example [Ore59] or Tutte [Tut81]. Recall that for a graph G, and disjoint sets $A, B \subseteq V(G)$, we define $e_G(A,B)$ to be the number of edges in G between the sets A and B.

Theorem (Ore-Reyser Theorem)**.** *Let G be a bipartite graph and let $d_1, d_2 \geq 0$. G contains a (d_1, d_2)-regular spanning subgraph if and only if $d_1|V_1| = d_2|V_2|$, and for all $A \subseteq V_1$ and $B \subseteq V_2$: $d_1|A| \leq e_G(A,B) + d_2(|V_2| - |B|)$.*

Using the above, we can prove the following: this follows almost identically to the proof of [FKS16, Theorem 1.4], with very minor changes.

Theorem 5.4.7. *Let $m_2 = m_2(m_1) \geq m_1$ and let $p = p(m_1) > 0$ be such that $m_1 p = \omega(m_1)\log m_2$ for some $\omega \to \infty$ as $m_1 \to \infty$. Let $\delta \geq \omega^{-\theta}$ for some $\theta < 1/2$, and $G \sim G(m_1, m_2, p)$. Then a.a.s.(m_1) G contains a $((1-\delta)m_2 p, (1-\delta)m_1 p)$-regular spanning subgraph.*

Again, the proof of this follows extremely similarly to the proof of [FKS16, Theorem 1.4]; we include it for completeness.

Proof. Let $d_1 = (1-\delta)m_2 p$ and $d_2 = (1-\delta)m_1 p$. We wish to prove that a.a.s.(m_1) for all $A \subseteq V_1$ and $B \subseteq V_2$:

$$\begin{aligned} 0 &\leq e_G(A,B) + d_2(m_2 - |B|) - d_1|A| \\ &= e_G(A,B) + d_1(m_1 - |A| - m_1|B|/m_2). \end{aligned}$$

If we are able to prove this, then we may conclude the desired result by the Ore-Reyser theorem. Note that if $|A| + m_1|B|/m_2 \leq m_1$ then we are immediately finished. Let us suppose otherwise; we now analyse different cases.

To begin, let $n_1 := m_1/\log\log m_1$. We may now assume that $|A|+m_1|B|/m_2 > m_1$. Suppose first that $|A| \leq n_1$, then $(m_2(m_1 - |A|)/m_1) + 1 \leq |B| \leq m_2$. Note that $e_G(A,B)$ has the distribution $Bin(|A||B|, p)$. We may apply the Chernoff bounds to deduce that

$$\mathbb{P}(e_G(A,B) \leq (1-\delta)|A||B|p) \leq \exp\left(\frac{-\delta^2|A||B|p}{2}\right).$$

For

$$|A| = a \leq n_1, |B| = b \geq \frac{m_2(m_1 - a)}{m_1},$$

and m_1 sufficiently large, this is bounded above by

$$\exp\left(\frac{-\delta^2 a m_2(m_1-a)p/m_1}{2}\right) \leq \exp\left(a m_2 p \frac{-\delta^2(m_1 - n_1)}{2m_1}\right) \leq \exp\left(-\delta^2\frac{m_2 a p}{4}\right).$$

Therefore the probability that there exists such sets with $e_G(A,B) \leq (1-\delta)|A||B|p$ is bounded above by

$$\begin{aligned} \sum_{a=1}^{n_1}\sum_{b=(m_2(m_1-a)/m_1)+1}^{m_2}\binom{m_1}{a}\binom{m_2}{b}e^{-\delta^2 m_2 a p/4} &\leq \sum_{a=1}^{n_1}\sum_{b=1}^{m_2 a/m_1}\binom{m_1}{a}\binom{m_2}{b}e^{-\delta^2 m_2 a p/4} \\ \left(\text{using}\binom{m_2}{b} \leq \binom{m_2}{m_2 a/m_1}\text{ for } b \leq m_2 a/m_1\right) &\leq \sum_{a=1}^{n_1}\frac{m_2 a}{m_1}\binom{m_1}{a}\binom{m_2}{m_2 a/m_1}e^{-\delta^2 m_2 a p/4} \\ \left(\text{using}\binom{m_1}{a} \leq \binom{m_2}{m_2 a/m_1}\text{ as } m_2/m_1 \geq 1\right) &\leq \sum_{a=1}^{n_1}\frac{m_2 a}{m_1}\binom{m_2}{m_2 a/m_1}^2 e^{-\delta^2\frac{m_2}{m_1}a m_1 p/4} \\ &\leq m_2^2\sum_{a=1}^{n_1}\left(\frac{m_2^2 e^2}{m_2^2 a^2/m_1^2}e^{-\Omega(\log m_2)}\right)^{\frac{a m_2}{m_1}} \\ &= m_2^{-\Omega_{m_1}(1)}, \end{aligned}$$

since $\delta^2 m_1 p \geq \omega^{1-2\theta} \log m_2$ for some $\theta < 1/2$. The case is similar for $|B| \leq n_2 := m_2/\log\log m_2$. Next we may assume that $|A| \geq n_1$ and that $|B| \geq n_2$. First assume that $|A| \leq m_1|B|/m_2$, so that $|B| \geq m_2/2$. The probability there exists such A, B with $e_G(A, B) \leq (1-\delta)|A||B|p$ is bounded above by

$$\begin{aligned}\sum_{a=n_1}^{m_1} \sum_{b=m_2/2}^{m_2} \binom{m_1}{a}\binom{m_2}{b} e^{-\delta^2 abp/2} &\leq \sum_{a=n_1}^{m_1} \sum_{b=m_2/2}^{m_2} \binom{m_1}{a}\binom{m_2}{b} e^{-\delta^2 n_1 m_2 p/4} \\ &\leq 2^{m_1+m_2} e^{-\delta^2 m_1 m_2 p/(4 \log\log m_1)} \\ &= o_{m_1}(1),\end{aligned}$$

since $\delta^2 m_1 p/\log\log m_1 \geq \omega^{1-2\theta} \log m_2/\log\log m_1 = \Omega_{m_1}(1)$. Similarly, if $|A| \geq m_1|B|/m_2$, the probability that there exists A, B with $e_G(A, B) \leq (1-\delta)|A||B|p$ is bounded above by

$$\begin{aligned}\sum_{b=n_2}^{m_2} \sum_{a=m_1/2}^{m_1} \binom{m_1}{a}\binom{m_2}{b} e^{-\delta^2 abp/2} &\leq \sum_{b=n_2}^{m_2} \sum_{b=m_1/2}^{m_1} \binom{m_1}{a}\binom{m_2}{b} e^{-\delta^2 n_2 m_1 p/4} \\ &\leq 2^{m_1+m_2} e^{-\delta^2 m_1 m_2 p/(4 \log\log m_2)} \\ &= o_{m_1}(1),\end{aligned}$$

since $\delta^2 m_1 p = \Omega_{m_1}(\log m_2)$.

Now, consider $A \subseteq V_1, B \subseteq V_2$. If $|A| + m_1|B|/m_2 \leq m_1$, then it is immediate that

$$0 \leq e_G(A, B) + d_1(m_1 - |A| - m_1|B|/m_2).$$

Otherwise, we have proved that a.a.s.(m_1) $e_G(A, B) \geq (1-\delta)|A||B|p$, so that a.a.s.$(m_1)$

$$\begin{aligned}&e_G(A, B) + d_1(m_1 - |A| - m_1|B|/m_2) \\ &\geq (1-\delta)|A||B|p + (1-\delta)m_2 p(m_1 - |A| - m_1|B|/m_2) \\ &= (1-\delta)|A||B|p + (1-\delta)m_1 m_2 p - (1-\delta)|A|m_2 p - (1-\delta)m_1|B|p \\ &= (1-\delta)p(|A||B| + m_1 m_2 - |A|m_1 - |B|m_2) \\ &= (1-\delta)p(m_1 - |A|)(m_2 - |B|) \\ &\geq 0,\end{aligned}$$

since $|A| \leq m_1$ and $|B| \leq m_2$. The result now follows by the Ore-Reyser theorem. $\square$

5.4.3 Regular subgraphs in reduced random graphs

Finally, we need to address the issue of vertex degrees: in order to use Lemma 5.3.5 and Theorem 5.3.6, we need our graphs to be regular, and to have large eigenvalue. Therefore we need to show that $\mathfrak{Red}(n,l,p)$, $\mathfrak{BRed}(n,l,p)$ contain regular spanning subgraphs with large first eigenvalue.

Lemma 5.4.8. *Let $n \geq 2$, and let p be such that $(2n-1)^l p = \Omega_l(\log(2n-1)^{l+1}) = \Omega_l(l)$ and $p = o_l(1)$. Let $G_1 \sim \mathfrak{Red}(n,l,p)$ and $G_2 \sim \mathfrak{Bred}(n,l,p)$. There exists $\epsilon = \epsilon(p) = o_l(1)$ such that for all $o_l(1) = \delta \geq \epsilon$, a.a.s.$(l)$ there exist spanning subgraphs $H_i \leq G_i$ such that*

i) H_1 *is* $2(1-\delta)(2n-1)^l p$*-regular, with* $\lambda_1(H_1) \geq 1 - o_l(1)$,

ii) *and* H_2 *is* $((1-\delta)(2n-1)^{l+1}p, (1-\delta)(2n-1)^l p)$*-regular, with* $\lambda_1(H_2) \geq 1 - \dfrac{1}{2n-1} + o_l(1)$.

Proof. The first part of *i)* and *ii)*, i.e. the existence of the regular subgraphs, follows from [SU84] and Lemma 5.4.7. By [KK13, Lemma 4.5] and Lemma 5.3.4, $\lambda_1(H_i) = \lambda_1(G_i) + o_l(1)$, since the G_i is formed from H_i by the addition of graphs of suitably small degrees. The result follows by Lemmas 5.4.5 and 5.4.6. □

Similarly we can prove the following.

Lemma 5.4.9. *Let $n \geq 2$, $k \geq 3$ and let p be such that $(2n-1)^l p = \Omega_n(\log(2n-1)^{l+1})$ and $p = o_n(1)$. Let $G_1 \sim \mathfrak{Red}(n,l,p)$ and $G_2 \sim \mathfrak{Bred}(n,l,p)$. There exists $\epsilon = \epsilon(p) = o_l(1)$ such that for all $o_l(1) = \delta \geq \epsilon$, a.a.s.$(n)$ there exist spanning subgraphs $H_i \leq G_i$ such that*

i) H_1 *is* $2(1-\delta)(2n-1)^l p$*-regular, with* $\lambda_1(H_1) \geq 1 - o_n(1)$,

ii) *and* H_2 *is* $((1-\delta)(2n-1)^{l+1}p, (1-\delta)(2n-1)^l p)$*-regular, with* $\lambda_1(H_2) \geq 1 - o_n(1)$.

5.5 Property (T) in random quotients of free groups

Finally, we may prove Theorems E and F. We in fact provide the full proof for Theorem F, and indicate how to alter the proof of this theorem in order to prove Theorem E. However, we first define a slightly different model of random groups: this model is often called the *binomial model* of random groups.

Definition 5.5.1. Let $n \geq 2$, $k \geq 3$, and let $0 < p = p(n,k) < 1$. The random group model $\Gamma_p(n,k,p)$ is the model obtained as following. We let $\Gamma = \langle A_n \mid R \rangle$, where R is obtained by adding each word in $\mathcal{C}(n,k)$ with probability p.

We in fact prove the following theorem.

Theorem 5.5.2. *Let $n \geq 2$, and let p be such that $(2n-1)^{2k/3}p = \Omega_k(k)$. Let $\Gamma_k \sim \Gamma_p(n,k,p)$. Then $\lim_{k\to\infty} \mathbb{P}(\Gamma_k$ has Property (T)$) = 1$.*

Assuming this, we may prove Theorem F.

Proof of Theorem F. Fix $n \geq 2$ and $d > 1/3$. Choose $1/3 < d' < d$, and let

$$\Gamma'_k = \langle A_n \mid R' \rangle \sim \Gamma_p(n,k,(2n-1)^{kd'-k}).$$

It is easily seen that a.a.s.(k) $|R'| = (1+o_k(1))(2n-1)^{kd'}$. Choose a random subset R with $R' \subseteq R \subseteq \mathcal{W}(n,k)$ and $|R| = (2n-1)^{kd}$, and let $\Gamma_k = \langle A_n \mid R \rangle$. Then $\Gamma_k \sim \Gamma(n,k,d)$, and there is a clear epimorphism $\Gamma'_k \twoheadrightarrow \Gamma_k$. Since Property (T) is preserved under epimorphisms, the result follows by Theorem 5.5.2. □

Let Γ be a random group in the $\Gamma_p(n,k,p)$ model. We consider the three cases.

$\mathbf{k = 0}$ **mod 3**. Let $l_k = L_k = k/3$. We may define the graphs $\Sigma_1, \Sigma_2, \Sigma_3$ where:

$$V(\Sigma_1) = V(\Sigma_2) = V(\Sigma_3) = \mathcal{W}(n,k/3),$$

and for each relator $r = r_x r_y r_z$ with $r_x, r_y, r_z \in \mathcal{W}(n,k/3)$, we add the edge (r_x, r_z^{-1}) to Σ_1, (r_y, r_x^{-1}) to Σ_2 and (r_z, r_y^{-1}) to Σ_3.

$\mathbf{k = 1}$ **mod 3**. Let $l_k = (k-1)/3$ and $L_k = (k+2)/3$. Again, we may write each relator $r = r_x r_y r_z$ for r_x, $r_y \in \mathcal{W}(n,(k-1)/3)$ and $r_z \in \mathcal{W}(n,(k+2)/3)$. We again split the graph $\Delta_k(A_n \mid R)$ into Σ_1, Σ_2, Σ_3, where:

$$V(\Sigma_1) = V(\Sigma_3) = \mathcal{W}(n,(k-1)/3) \sqcup \mathcal{W}(n,(k+2)/3),$$

and $V(\Sigma_2) = \mathcal{W}(n,(k-1)/3)$. For each relator $r = r_x r_y r_z$, we add the edge (r_x, r_z^{-1}) to Σ_1, (r_y, r_x^{-1}) to Σ_2, and (r_z, r_y^{-1}) to Σ_3.

$\mathbf{k = 2}$ **mod 3**. Let $l_k = (k+1)/3$ and $L_k = (k-2)/3$. Again, we may write each relator $r = r_x r_y r_z$ for r_x, $r_y \in \mathcal{W}(n,(k+1)/3)$ and $r_z \in \mathcal{W}(n,(k-2)/3)$. We again split the graph $\Delta_k(A_n \mid R)$ into Σ_1, Σ_2, Σ_3, where:

$$V(\Sigma_1) = V(\Sigma_3) = \mathcal{W}(n,(k-2)/3) \sqcup \mathcal{W}(n,(k+1)/3),$$

and $V(\Sigma_2) = \mathcal{W}(n, (k+1)/3)$. For each relator $r = r_x r_y r_z$, we add the edge (r_x, r_z^{-1}) to Σ_1, (r_y, r_x^{-1}) to Σ_2, and (r_z, r_y^{-1}) to Σ_3.

Next we show there aren't too many double edges in the graphs Σ_i, similarly to [ALuS15].

Lemma 5.5.3. *Let $n \geq 2$, and let p be such that*

i) $(2n-1)^{2k-L_k} p^3 = o_k(1)$,

ii) and $(2n-1)^{2k+l_k} p^4 = o_k(1)$.

Let $G_k \sim \Gamma_p(n, k, p)$, and let Σ_i be described as above. For $i = 1, 2, 3$ a.a.s.(k) there is no pair of vertices u, v with at least three edges between them in Σ_i, and the set of double edges in Σ_i forms a matching, i.e. the endpoints of the double edges are all distinct.

Note that for $1/3 < d < 5/12$, $p(d) = (2n-1)^{kd-k}$ satisfies the above conditions.

Proof. We prove this for $i = 1$. Throughout, note that $k = 2l_k + L_k$. The probability, $\mathbb{P}_3$, that there exists a pair of vertices u, v with at least three edges between u and v is bounded above by

$$\begin{aligned}\mathbb{P}_3 &\leq (2n-1)^{l_k + L_k} (2n-1)^{3l_k} p^3 \\ &= (2n-1)^{2k-L_k} p^3 \\ &= o_k(1).\end{aligned}$$

The probability, $\mathbb{P}_{doub}$ that there are vertices u, v, w with double edges between u and v and u and w is bounded by

$$\begin{aligned}\mathbb{P}_{doub} &= (2n-1)^{l_k} (2n-1)^{2L_k} (2n-1)^{4l_k} p^4 \\ &= (2n-1)^{2k+l_k} p^4 \\ &= o_k(1).\end{aligned}$$

$\square$

This is sufficient to prove our main theorem.

Proof of Theorem 5.5.2. Let $\Gamma_k = \langle A_n \mid R \rangle \sim \Gamma_p(n, k, p)$, and consider $\Delta_k := \Delta_k(A_n \mid R)$. Since Property (T) is preserved by epimorphisms, we may assume that $p \leq (2n-1)^{kd-k}$

for some $d < 4/9$: for any $1/3 < d < 4/9$, $p(n,k,d) = (2n-1)^{kd-k}$ satisfies the conditions of Lemma 5.5.3 and the conditions of Theorem 5.5.2.

As above we may write $\Delta_k = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Now, after collapsing edges, we find Σ_1', Σ_3' with the marginal distribution of

$$\begin{cases} \mathfrak{Red}(n, k/3, (2n-1)^{k/3}p): \ k = 0 \mod 3, \\ \mathfrak{BRed}(n, (k-1)/3, (2n-1)^{(k-1)/3}p): \ k = 1 \mod 3, \\ \mathfrak{BRed}(n, (k-2)/3, (2n-1)^{(k+1)/3}p): \ k = 2 \mod 3. \end{cases}$$

Similarly, by collapsing double edges we find Σ_2' with the marginal distribution of

$$\begin{cases} \mathfrak{Red}(n, k/3, (2n-1)^{k/3}p): \ k = 0 \mod 3, \\ \mathfrak{Red}(n, (k-1)/3, (2n-1)^{(k+2)/3}p): \ k = 1 \mod 3, \\ \mathfrak{Red}(n, (k+1)/3, (2n-1)^{(k-2)/3}p): \ k = 2 \mod 3. \end{cases}$$

Furthermore, letting $\Sigma' = \Sigma_1' \cup \Sigma_2' \cup \Sigma_3'$, then as usual we can see that

$$\mu_1\left(D(\Sigma')^{-1/2} \Big[A(\Delta_k) - A(\Sigma') \Big] D(\Sigma')^{-1/2} \right) = o_k(1).$$

By Lemma 5.4.8, there exists some $\delta = o_k(1)$ such that a.a.s.(k): Σ_2' has a $2(1-\delta)d_2$-regular spanning subgraph, Π_2, with $\lambda_1(\Pi_2) > 1 - o_k(1)$; if $k \neq 0 \mod 3$ then Σ_1', Σ_3' contain $((1-\delta)d_1, (1-\delta)d_2)$-regular spanning subgraphs Π_1, Π_3, with $\lambda_1(\Pi_1), \lambda_1(\Pi_3) \geq 1 - 1/(2n-1) + o_k(1)$; and if $k = 0 \mod 3$ then Σ_1', Σ_3' contain $2(1-\delta)d$-regular spanning subgraphs Π_1, Π_3, with $\lambda_1(\Pi_1), \lambda_1(\Pi_3) \geq 1 - o_k(1)$.

As $n \geq 2$, we may apply Lemmas 5.3.5 and 5.3.7 to deduce that a.a.s.(k):

$$\lambda_1(\Pi_1 \cup \Pi_2 \cup \Pi_3) > 3/4.$$

We see that

$$\begin{aligned} \mu_1(A(\Sigma_1' \cup \Sigma_2' \cup \Sigma_3') - A(\Pi_1 \cup \Pi_2 \cup \Pi_3)) &\leq \frac{\delta + o_k(1)}{1-\delta} ||A(\Pi_1 \cup \Pi_2 \cup \Pi_3)||_\infty \\ &= o_k(1) ||A(\Pi_1 \cup \Pi_2 \cup \Pi_3)||_\infty. \end{aligned}$$

Hence, letting $\Pi = \Pi_1 \cup \Pi_2 \cup \Pi_3$, we see that a.a.s.$(k)$:

$$\begin{aligned} \lambda_1(\Sigma') &= 1 - \mu_2(D^{-1/2}(\Sigma') A(\Sigma') D^{-1/2}(\Sigma')) \\ &= 1 - \mu_2(D^{-1/2}(\Sigma')[A(\Pi) + A(\Sigma') - A(\Pi)] D^{-1/2}(\Sigma')) \end{aligned}$$

$$\begin{aligned}
&\geq 1 - \mu_2(D^{-1/2}(\Sigma')A(\Pi)D^{-1/2}(\Sigma')) \\
&\quad - \mu_1(D^{-1/2}(\Sigma')(A(\Sigma') - A(\Pi))D^{-1/2}(\Sigma')) \\
&= 1 - \left(\frac{1}{1-\delta} + o_k(1)\right)\mu_2(D^{-1/2}(\Pi)A(\Pi)D^{-1/2}(\Pi)) \\
&\quad - \left(\frac{1}{1-\delta} + o_k(1)\right)\mu_1(D^{-1/2}(\Pi)(A(\Sigma') - A(\Pi))D^{-1/2}(\Pi)) \\
&\geq 1 - \frac{1}{4}\left(\frac{1}{1-\delta} + o_k(1)\right) - \left(\frac{1}{1-\delta} + o_k(1)\right)\frac{\delta}{1-\delta} \\
&= \frac{3[1+o_k(1)]}{4}.
\end{aligned}$$

Since $\lambda_1(\Delta_k) = \lambda_1(\Sigma') + o_k(1)$, it follows by Theorem 5.2.5 that a.a.s.(k) G has Property (T). However, as Property (T) is preserved under epimorphisms, it follows immediately that a.a.s.(k) a random group $\Gamma_k \sim \Gamma(n, k, d)$ has Property (T) for any p with

$$(2n-1)^{2k/3}p = \Omega_k(k).$$

□

To prove Theorem E, we wish to prove the corresponding result for the k-angular model: the approach to achieve this is similar.

Lemma 5.5.4. *Let $n \geq 2$, and let p be such that there exists $M \geq 1$ with*

i) $(2n-1)^{(M+1)l_k + L_k}p^M = o_n(1)$,

ii) $(2n-1)^{2l_k + ML_k}p^M = o_n(1)$,

iii) $(2n-1)^{(2M+1)l_k + ML_k}p^{2M} = o_n(1)$,

iv) $(2n-1)^{3Ml_k + L_k}p^{2M} = o_n(1)$,

v) $(2n-1)^{(M+1)l_k + 2ML_k}p^{2M} = o_n(1)$

Let $G_k \sim \Gamma_p(n, k, p)$, and let Σ_i be described as above. For $i = 1, 2, 3$ a.a.s.(n) there is no pair of vertices u, v with at least M edges between them, and no vertex is connected to more than M other vertices by double edges.

Proof. We first prove this for $i = 1, 3$. Throughout, note that $k = 2l_k + L_k$. The probability, $\mathbb{P}_{M,1}$, that there exists a pair of vertices u, v with at least M edges between u and v is bounded above by

$$\mathbb{P}_{M,1} \leq (2n-1)^{l_k + L_k}(2n-1)^{Ml_k}p^M$$

$$= (2n-1)^{(M+1)l_k+L_k}p^M$$
$$= o_k(1).$$

The probability, $\mathbb{P}_{doub,1}$ that there are vertices $u \in V_1$ and $v_1, \ldots, v_M \in V_2$ with double edges between u and each v_i is bounded by

$$\begin{aligned}\mathbb{P}_{doub,1} &= (2n-1)^{l_k}(2n-1)^{ML_k}(2n-1)^{2Ml_k}p^{2M} \\ &= (2n-1)^{(2M+1)l_k+ML_k}p^{2M} \\ &= o_k(1).\end{aligned}$$

The probability, $\mathbb{P}'_{doub,1}$ that there are vertices $u \in V_2$ and $v_1, \ldots, v_M \in V_1$ with double edges between u and each v_i is bounded by

$$\begin{aligned}\mathbb{P}'_{doub,1} &= (2n-1)^{Ml_k}(2n-1)^{L_k}(2n-1)^{2Ml_k}p^{2M} \\ &= (2n-1)^{L_k+3Ml_k}p^{M} \\ &= o_k(1).\end{aligned}$$

Let's now switch to Σ_2. Then the probability, $\mathbb{P}_{M,2}$, that there exists a pair of vertices u, v with at least M edges between u and v is bounded above by

$$\begin{aligned}\mathbb{P}_{M,2} &\leq (2n-1)^{2l_k}(2n-1)^{ML_k}p^M \\ &= o_k(1).\end{aligned}$$

Finally, the probability, $\mathbb{P}_{doub,2}$, that there are vertices u and $v_1, \ldots, v_M$ with double edges between u and each v_i is bounded by

$$\begin{aligned}\mathbb{P}_{doub,2} &= (2n-1)^{(M+1)l_k}(2n-1)^{2ML_k}p^{2M} \\ &= o_k(1).\end{aligned}$$

□

Remark 5.5.5. Let $d > 0$ and $p_d = (2n-1)^{kd-k}$. Then p_d satisfies the conditions above for some M if respectively:

i) $l_k + kd - k < 0$, so that $d < (l_k + L_k)/k$,

ii) $L_k + kd - k < 0$, so that $d < 2l_k/k$,

iii) $2l_k + L_k + 2kd - 2k < 0$, i.e. $d < 1/2$ since $2l_k + L_k = k$, *iv)* $3l_k + 2kd - 2k < 0$, so that $d < (k + L_k - l_k)/2k$, and

v) $l_k + 2L_k + 2kd - 2k < 0$, so that $d < (k + l_k - L_k)/2k$.

This reduces to $d < (k-1)/2k$. For $k \geq 8$, this is satisfied whenever $d < 7/16$. For $k \geq 8$, we have $d_k \leq 10/30 < 7/16$, and so we can find d satisfying the requirements of the above lemma and Theorem 5.5.6.

We can now prove the following.

Theorem 5.5.6. *Let $n \geq 2$, $k \geq 8$. Let p be such that*

$$(2n-1)^{2l_k} p = \Omega_n\bigg(\log(2n-1)^{L_k}\bigg), \ \textit{and} \ (2n-1)^{l_k+L_k} p = \Omega_n\bigg(\log(2n-1)^{l_k}\bigg).$$

Let $\Gamma_k \sim \Gamma_p(n,k,p)$. Then $\lim_{n\to\infty} \mathbb{P}(\Gamma_k$ has Property $(T)) = 1$.

We remark that for $d > d_k$ the above is satisfied.

Proof. This follows similarly to the proof of Theorem 5.5.2. We may assume that p also satisfies the requirements of Lemma 5.5.4 for some M. The main replacement is in the fact that the Σ_i have a very small proportion of disallowed edges so can be treated as having the marginal distribution of an (bipartite) Erdös–Rényi random graph. We then find regular spanning subgraphs, and repeat the above argument, using Lemma 5.5.4 in place of Lemma 5.5.3. This guarantees us that by collapsing double edges, we remove at most M^2 edges adjacent to each vertex, and the argument follows similarly. □

We then apply the above to prove Theorem E, as in the case for the density model.

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