PENALTY METHOD FOR 1D PROBLEM OF TWO RODS WITH BASIC CONCEPTS OF CONTACT MECHANICS OF FRICTIONLESS CONTACT

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UDK: 531:519.6
DOI: 10.14415/konferencijaGFS2021.30

Summary: *Problems involving contact are of great importance in industry related to mechanical and civil engineering, but also in biomechanics and other applications. The contact interaction between surfaces in a bolted splice connection joint or area through which tire interacts with the road is not known a priori, leading to a nonlinear boundary value problem. Due to the rapid improvement of modern computer technology, today one can apply the tools of computational mechanics to analyze contact problems with limited accuracy, depending on design requirements. However, even now most of the standard finite element software is not fully capable of solving contact problems, including friction, using robust algorithms. The aim of this paper is to present some basic concepts of Contact Mechanics. To illustrate the difficulties arising in computational contact mechanics, Newton-Raphson scheme was used to solve simple 1D contact problem using penalty method.*

Keywords: *Contact Mechanics, penalty method, frictionless contact*

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1. INTRODUCTION

Particularly demanding nonlinear problem to analyze is the contact between two or more bodies [1-3]. Contact problems can range from simple approximations of frictionless contact with small displacements to frictional contact with large displacements, large rotations and large strains. The nonlinearity of the analysis with contact included does not depend only on material and geometrical nonlinearity, which is usually considered for deformable bodies, but from contact conditions which are now included in the equations of motion. Without detailed derivation, strong form of the contact problem in the tensor notation can be stated as follows:

$$\begin{cases} \nabla \cdot \underline{\underline{\boldsymbol{\sigma}}} + \underline{\boldsymbol{f_v}} = 0 \quad in \quad \Omega, \\ \underline{\underline{\boldsymbol{\sigma}}} \cdot \underline{\boldsymbol{n}} = \underline{\boldsymbol{\sigma_0}} \quad at \quad F_f, \\ \underline{\boldsymbol{u}} = \underline{\boldsymbol{u_0}} \quad at \quad F_u, \\ g_n \geq 0, \quad \sigma_n \leq 0, \quad \sigma_n g_n = 0 \quad at \quad F_c \end{cases} \tag{1}$$

where: Ω represents volume of the body, F_f is a surface where surface tractions are prescribes, F_u is a surface where displacements are prescribed, F_c is an unknown contact surface. g_n is a normal gap between bodies and σ_n is normal contact pressure, $\boldsymbol{\sigma}$ with the double underscore is Cauchy stress tensor, $\underline{\boldsymbol{n}}$ is unit normal vector to the surface of the body, f_v is body force vector, $\boldsymbol{\sigma_0}$ are prescribed tractions to the surface of the body $\boldsymbol{u_0}$ are prescribed displacements and ∇ is the nabla operator. Double underscore in expression (1) represents tensor of the 2nd rank, while one underscore represents vectors. Last set of expressions in (1) is known as Hertz - Signorini – Moreau conditions in contact mechanics literature or Karush–Kuhn–Tucker (KKT) conditions in mathematical optimization..

2. WEAK FORM OF CONTACT PROBLEM

In this section, brief introduction to discretization of the weak form with special attention to the contact integral will be presented. In order to arrive at a weak form we start from momentum balance of a strong form i.e. first equation of a system given in (1) and take a dot product with any arbitrary vector function $\underline{\boldsymbol{v}}$ (test-function):

$$\nabla \cdot \underline{\underline{\boldsymbol{\sigma}}} + \underline{\boldsymbol{f_v}} = 0 \quad in \quad \Omega \Leftrightarrow \forall \underline{\boldsymbol{v}}, \int_{\Omega} [\nabla \cdot \underline{\underline{\boldsymbol{\sigma}}} + \underline{\boldsymbol{f_v}}] \cdot \underline{\boldsymbol{v}} d\Omega = 0. \tag{2}$$

If we put a constraint that $\underline{\boldsymbol{v}}$ has C^1 continuity, then the first term in (2) can be integrated by parts which yields:

$$\int_{\partial\Omega} \underline{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \underline{\boldsymbol{v}} d\Gamma + \int_{\Omega} [\underline{\boldsymbol{f_v}} \cdot \underline{\boldsymbol{v}} - \underline{\underline{\boldsymbol{\sigma}}} \cdot \cdot \nabla \underline{\boldsymbol{v}}] d\Omega = 0, \tag{3}$$

where $\underline{\boldsymbol{n}}$ is an outward unit normal to the body surface $\partial\Omega$, $\cdot\cdot$ is the double scalar product and $\nabla \boldsymbol{v}$ is the gradient of test functions. With this procedure we removed derivative from the stress tensor which implies that the requirement on smoothness of the stress tensor from strong form has been replaced by a weaker requirement of continuity $\boldsymbol{\sigma} \in C^0$ [3]. We can replace abstract test function $\underline{\boldsymbol{v}}$ by an arbitrary test displacements also called virtual displacements $\delta\boldsymbol{u}$, then weak form (3) becomes the balance of virtual work:

$$\int_{\partial\Omega} \underline{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \delta\underline{\boldsymbol{u}} d\Gamma + \int_{\Omega} [\underline{\boldsymbol{f_v}} \cdot \underline{\boldsymbol{v}} - \underline{\underline{\boldsymbol{\sigma}}} \cdot \cdot \delta\nabla\underline{\boldsymbol{u}}] d\Omega = 0, \tag{4}$$

The stress vector $\boldsymbol{n} \cdot \boldsymbol{\sigma}$ in (4) is not zero on surfaces: F_u, F_f and in the active contact zone $F_c \in (\Gamma_{c1} \cup \Gamma_{c2})$, where Γ_{c1} and Γ_{c2} are contact surfaces of the two contacting bodies respectively. Integral over surface $\partial\Omega$ can be written as a sum of integrals over different surfaces:

$$\int_{\partial\Omega} \underline{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \delta\underline{\boldsymbol{u}} d\Gamma = \int_{\Gamma_{c1}} \underline{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \delta\underline{\boldsymbol{\rho}} d\Gamma_{c1} + \int_{\Gamma_{c2}} \underline{\boldsymbol{\nu}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \delta\underline{\boldsymbol{r}} d\Gamma_{c2} + \int_{\Gamma_{\Gamma_f}} \underline{\boldsymbol{\sigma_0}} \cdot \delta\underline{\boldsymbol{u}} d\Gamma_f \tag{5}$$

where $\boldsymbol{\sigma_0}$ is a prescribed traction vector (Neumann boundary condition), $\underline{\boldsymbol{n}}$ is a unit surface normal at boundary surface Γ_{c1}, $\underline{\boldsymbol{\nu}}$ is a unit surface normal at boundary surface Γ_{c2}, $\underline{\boldsymbol{\rho}}$ is position vector of a point on surface Γ_{c1} (master surface), $\underline{\boldsymbol{r}}$ is position vector of a point on surface Γ_{c2} (slave surface). Newton's 3rd law states that at equilibrium we have:

$$\underline{\boldsymbol{n}} \cdot \underline{\boldsymbol{\sigma}} d\Gamma_{c1} = -\underline{\boldsymbol{\nu}} \cdot \underline{\boldsymbol{\sigma}} d\Gamma_{c1} \tag{6}$$

When two bodies are in contact, these two integrals can be replaced by one integral. We choose the surface Γ_{c1} as surface over which we perform integration (master surface).

$$\int_{\Gamma_{c1}} \underline{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \delta\underline{\boldsymbol{\rho}} d\Gamma_{c1} - \int_{\Gamma_{c2}} \underline{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \delta\underline{\boldsymbol{r}} d\Gamma_{c2} = \int_{\Gamma_{c1}} \underline{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \cdot \delta(\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{r}}) d\Gamma_{c1} \tag{7}$$

where: $\boldsymbol{r} - \boldsymbol{\rho} = \boldsymbol{g}(\boldsymbol{\rho}, \Gamma_{c1})$ is a gap function representing relative position of a point $\underline{\boldsymbol{r}}$ to $\underline{\boldsymbol{\rho}}$. In our case, we concern ourselves with small deformations with only normal components of the displacements as independent variables. In that case we can write function $\boldsymbol{g}$ as follows: $\boldsymbol{g} \rightarrow g_n$ and g_n is normal projection of a gap function. We can now write weak form, including the contact integral as follows:

$$\int_{\Omega} \underline{\underline{\boldsymbol{\sigma}}} \cdot \cdot \delta\nabla\underline{\boldsymbol{u}} d\Omega + \int_{\Gamma_{c1}} \sigma_n \delta g_n d\Gamma_{c1} = \int_{\Gamma_f} \underline{\boldsymbol{\sigma_0}} \cdot \delta\underline{\boldsymbol{u}} d\Gamma + \int_{\Omega} \underline{\boldsymbol{f_v}} \cdot \delta\underline{\boldsymbol{u}} d\Omega \tag{8}$$

3. DISCRETIZATION OF CONTACT INTEGRAL, NODE-TO-NODE APPROACH

Discretization of the weak form using the isoparametric approach is one of the widely used methods in finite element calculations. By isoparametric, it is meant that same functions describing displacement field are used to describe geometry. Since we are going to solve simple contact problem in 1D domain, reader should consult relevant literature regarding weak form discretization [2], [3]. After discretization of the weak form, we arrive at next expression:

$$[K]\underline{\boldsymbol{u}} + \delta W_c = R \tag{9}$$

where: $[K]$ represents a stiffness matrix of assembled structural system, δW_c is contact integral representing contact forces and R is a vector of externally applied tractions and volume forces. Vector of contact forces δW_c depends on the resolution method. In this paper, we are going to use penalty method as a resolution method for calculating contact contribution to the global equilibrium. Consider the problem shown in Figure 1:

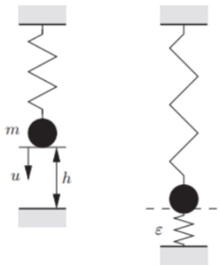

Figure 1. Point mass supported by a spring and a penalty spring due to the penalty term

This 1D system is represented by a mass m suspended on a spring with stiffness k. If we neglect ground constraint, we can write a functional for a mass-spring system in gravitational field as follows:

$$L(u) = \frac{1}{2}ku^2 - mgu \tag{10}$$

where: $L(u)$ is energy functional of the system, k is a spring stiffness, u is displacement of a point mass, mg is gravitational force exerted upon mass m.
If we consider constraint which states that mass m cannot be displaced below certain level h, we can rewrite energy functional by adding a penalty term as follows [1]:

$$L(u) = \frac{1}{2}ku^2 - mgu + \frac{1}{2}\epsilon[c(u)]^2 \tag{11}$$

As can be seen in Figure 1, the penalty parameter ϵ can be interpreted as a spring stiffness in the contact interface between point mass and rigid support. This is due to the fact that the energy of the penalty term has the same structure as the potential energy of a simple spring.
This equation will be solved using variational principle which states that solution to equilibrium equation (11) is achieved if first variation of the functional $L(u)$, with the assumption of contact, is equal to zero:

$$\delta L(u) = ku\delta u - mg\delta u + \epsilon c(u)\delta u = 0 \tag{12}$$

which for arbitrary δu yields the solution:

$$u = \frac{(mg + \epsilon h)}{(k + \epsilon)} \tag{13}$$

The value of constraint equation is then:

$$c(u) = h - u = \frac{kh - mg}{k + \epsilon} \tag{14}$$

Since $mg \geq kh$ in the case of contact, equation (14) means that a penetration of the point mass into the rigid support occurs, which is physically equivalent to a compression of the spring, see Figure 1. This penetration depends on the penalty parameter. One can see that constraint equation is fulfilled only in the limit $\epsilon \rightarrow \infty \Rightarrow c(u) \rightarrow 0$. Because of this fact, we can distinguish two limiting cases:

- $\epsilon \rightarrow \infty \Rightarrow c(u) \rightarrow 0$, which means that one approaches the correct solution for a very large penalty parameters. This means that fictitious contact spring stiffness is very large, hence only small penetration occurs.
- $\epsilon \rightarrow 0$ represents the unconstrained solution, and is only valid for inactive contact constraint. If contact occurs for a small parameter ϵ, large penetration will occur.

The reaction force for a penalty method can be computed from (12) and is equal to:

$$R_N = \epsilon c(u) = \frac{\epsilon}{k + \epsilon}(kh - mg) \tag{15}$$

Now we can turn our attention to contact integral for 1D problems of normal frictionless contact, which takes the following form:

$$\delta W_c = \int_{\Gamma_{c1}} \sigma_n \delta g_n d\Gamma_{c1} \tag{16}$$

Using penalty method as a resolution of contact integral we write the normal pressure as a product of penalty term ϵ_n and penetration g_n. This leads us to:

$$\sigma_n = \epsilon_n g_n \tag{17}$$

Returning to weak form, contact integral can be written as:

$$\delta W_c = \int_{\Gamma_{c1}} \epsilon_n g_n \delta g_n d\Gamma_{c1} \tag{18}$$

In the node-to-node contact discretization [5], Figure 2, contact is enforced trough master and slave nodes. Let's say that node i is the master node, and node j is the slave node. Then, using previously derived expression for normal gap g_n and contact stresses σ_n, we can write the following expression for contact integral in current configuration:

$$\delta W_c = \sum_{i=1}^{s} \epsilon_n (x_i - x_j)(\delta x_i - \delta x_j) \tag{19}$$

where we dropped an integral over unknown surface and replaced it with the sum over all contacting nodes s. This can be done because in Node-to-Node contact approach, unknown surface is replaced by two contacting nodes, master and slave respectively.

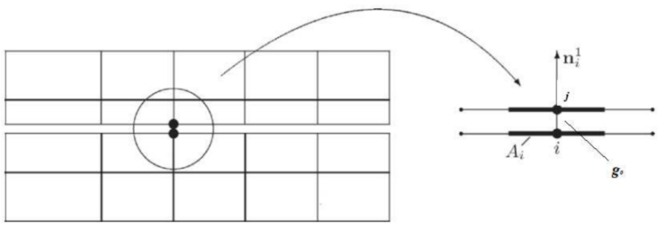

Figure 2. Node-to-Node contact element

Displaced configuration can be written using reference configuration as:

$$x_i = X_i + u_i \tag{20}$$

Now we can rewrite expression for contact integral using reference configuration and displacement as:

$$\varepsilon_n (X_i + u_i - X_j - u_j)(\delta u_i - \delta u_j) = \varepsilon_n (u_i - u_j - g_{ij})(\delta u_i - \delta u_j) \tag{21}$$

where g_{ij} represents initial gap between contacting nodes i and j. Since variations of displacements can be arbitrary, we can rewrite previous expression in matrix notation:

$$\varepsilon_n \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} + \varepsilon_n \begin{bmatrix} -g_{ij} \\ g_{ij} \end{bmatrix} \rightarrow \delta W_c = [K_T]\underline{\mathbf{u}} + R_n \tag{22}$$

where: $[K_T]$ represents tangent stiffness or contact matrix, R_n is residual vector.

4. SOLUTION OF NONLINEAR FINITE ELEMENT EQUATIONS IN STATICS

System of nonlinear algebraic equations can be written in its balance form assuming contact as:

$$F(\underline{\boldsymbol{u}}, R) = [K]\underline{\boldsymbol{u}} + \big([K_T]\underline{\boldsymbol{u}} + R_n\big) - R = 0 \tag{23}$$

This system can be solved using Newton-Raphson iterative procedure. Newton-Raphson procedure is founded on linear approximation of a system at one point in time and marching forward in small steps. Results are accepted once the equilibrium of a system reaches certain “small” value. Let’s assume that we have solved equations and reached converged solution for an n point in time $\boldsymbol{F}(\boldsymbol{u}^n, \boldsymbol{R}^n) = 0$. We are now looking for solution at next point $n + 1$:

$$F(\underline{\boldsymbol{u}}^{n+1}, R^{n+1}) = 0 \tag{24}$$

This system of equations can be expanded in Taylor series around previously known state $\boldsymbol{u}^n$ as:

$$\begin{cases} F(\underline{\boldsymbol{u}}^n + \Delta\underline{\boldsymbol{u}}^{n+1}, R^{n+1}) & = F(\underline{\boldsymbol{u}}^n, R^{n+1}) + \frac{\partial F(\underline{\boldsymbol{u}}^n, R^{n+1})}{\partial \underline{\boldsymbol{u}}}\Delta\underline{\boldsymbol{u}}^{n+1} + r(O) \\ & = [K]\underline{\boldsymbol{u}}^n + \big([K_T]\underline{\boldsymbol{u}}^n + R_n^n\big) - R^{n+1} + \big([K] + [K_T]\big)\Delta\underline{\boldsymbol{u}}^{n+1} \end{cases} \tag{25}$$

where: $r(O)$ represents error of the linear approximation. Last expression in (25) should be equal to zero and can be written as:

$$F_{int}^n + F_{intc}^n + R_n^n - R^{n+1} + [K_T^c]\Delta\underline{\boldsymbol{u}}^{n+1} = 0 \tag{26}$$

where: F_{int}^n represents vector of internal forces at time n, F_{intc}^n is a vector of contact forces at time n, R_n^n is residual vector due to contact at time n, R^{n+1} is a load vector at next time step $n + 1$, $[K_T^c]$ is contact tangent matrix at current time n.
Rearranging terms in (26), we can set up procedure for iterative solution:

$$\begin{aligned} [K_T^c]\Delta\underline{\boldsymbol{u}}_{(i+1)}^{n+1} &= \overbrace{R^{n+1} - \big(F_{int}^{(i)} + F_{intc}^{(i)}\big) - R_n^{(i)}}^{out of balance force vector \Delta R_i^{i+1}}, \\ \underline{\boldsymbol{u}}_{(i+1)}^{n+1} &= \underline{\boldsymbol{u}}_{(i)}^{n+1} + \Delta\underline{\boldsymbol{u}}_{(i)}^{n+1} \\ \underline{\boldsymbol{u}}_{(0)}^{n+1} &= \underline{\boldsymbol{u}}^n \end{aligned} \tag{27}$$

Since we are dealing with nonlinear analysis, calculation of internal forces is not straightforward and needs to be done with care. Best practice would be to update them according to displacements, meaning that if internal forces at iteration i are equal to $\boldsymbol{F}_I^i = \boldsymbol{F}_{int}^i + \boldsymbol{F}_{intc}^i$, then $\boldsymbol{F}_I^{i+1} = \boldsymbol{F}_I^{i+1} + \Delta\boldsymbol{F}$, where $\Delta\boldsymbol{F}$ can be calculated from tangent stiffness matrix $[K_T^c]$ and increment in displacements $\Delta\boldsymbol{u}_i^{n+1}$. Iteration continues until:

$$\left\|\Delta\underline{\boldsymbol{u}}_{(i)}^{n+1}\right\| \leq \epsilon\left\|\underline{\boldsymbol{u}}_{(i)}^{n+1}\right\| \tag{28}$$

where $\|\cdot\|$ is Euclidean norm (i.e. $\|\boldsymbol{x}\| = (\sum x_i^2)^{\frac{1}{2}}$) and ϵ is an error tolerance set as some "small" value.

5. NUMERICAL EXAMPLE

In this example, simple Node-to-Node contact between two truss elements was tested using Newton-Raphson iterative procedure.

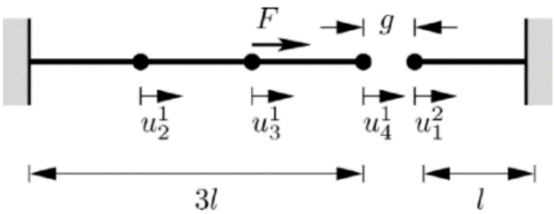

Figure 3. Truss structure with initial gap

Structure consists of four linear truss elements that are supported on the both sides. Initial gap g is assumed between the 3rd and 4th element (Figure 3). Force F is applied to the second degree of freedom of the system. Since we are dealing with nonlinear problem, we apply force F in increments. One can see that before the gap has closed, we are actually solving a linear system that consists of two rods that are not interacting. Once we overcome the gap between rods, normal contact force arises and resists penetration of the rods into each other. When the system is in contact, we are adding penalty stiffness to the structural stiffness matrix and residual vector to the force vector and iterate for equilibrium. Analysis parameters:

$$EA = 1000, \;\; l = 1, \;\; g = 0.01, \;\; F = 20, \;\; \Delta F = 1, \;\; \varepsilon_n = [10^3 \div 10^5]$$

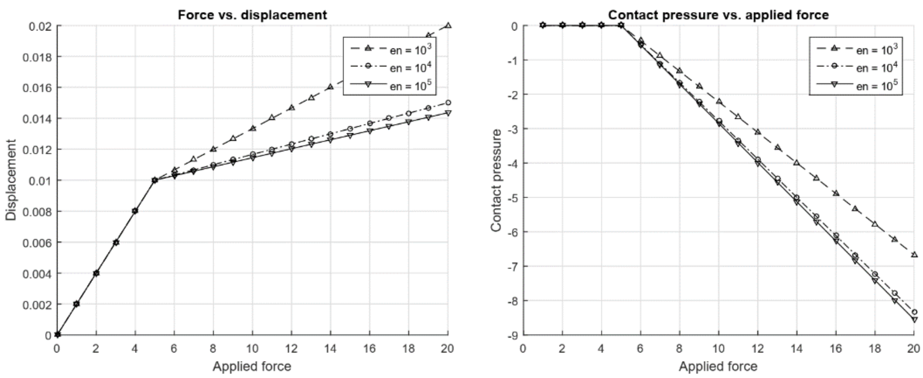

Figure 4. Results of Node-to-Node approach in static analysis

On Figure 4. two graphs that represent displacement and contact pressure with respect to the applied load to the system are shown. On the left graph, it can be seen how system stiffens once contact is achieved. Besides that, one can see how increase in penalty stiffness changes the nature of solution where for higher values of penalty parameter solution converges to the exact solution [1].

6. CONCLUSION

In this paper, we have shown some basic principles of contact mechanics. Strong form of contact problem was presented as well as weak form that is suitable for numerical approximation of the system of equations. Emphasis was placed on the weak form, i.e. contact integral of the virtual work. Using penalty resolution method, contact integral was discretized using Node-to-Node contact element and iterative Newton-Raphson method was set up as a method for the solution of the nonlinear system of equations. At the end, simple numerical example was solved and it was shown how penalty parameter affects the solution of the contact problem.

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PENALTY -МЕТОДА ЗА 1Д ПРОБЛЕМ ДВА ШТАПА СА ОСНОВНИМ КОНЦЕПТИМА КОНТАКТНЕ МЕХАНИКЕ КОНТАКТА БЕЗ ТРЕЊА

Резиме: *Проблематика контакта игра велику улогу како у машинској и грађевинској индустирији тако и у биомеханици и многим другим пољима. Контактна интеракција између површина вијчаних веза или између гуме точка и подлоге није априори позната, што води до нелинеарног граничног услова у механици деформабилног тијела. Усљед великог напретка прорачунске технологије, данас је могуће користити алат нумеричке механике при анализи контактне интеракције. Међутим, већина стандардних софтверских пакета нису у могућности у потпуности да симулирају контактну интеракцију, укључујући* *трење. Циљ овог рада је да представи основе механике контакта. Да би приказали потешкоће које се појављују у нумеричкој анализи контакта, шема Њутн-Рапсона је примијењана у рјешавању 1Д контактног проблема пеналту методом. Представљено је поређење резултата у зависности од пенелту параметра.*

***Кључне речи**: Контактна механика, пеналту-метода, контакт без трења*