Material Design in Physical Modeling Sound Synthesis

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ABSTRACT

This paper deals with designing material parameters for physical models. It is shown that the characteristic relation between modal frequencies and damping factors of a sound object is the acoustic invariant of the material from which the body is made. Thus, such characteristic relation can be used for designing damping models for a conservative physical model to represent a particular material.

1. INTRODUCTION

In recent years, physical modeling has gained interest among computer music researchers and artists. Due to the close interaction with musical acoustics research, this synthesis technique is often understood as a tool for reproducing existing instrument sounds. But, the present work sees the physical model as an open and challenging framework, allowing musicians to explore their own imagination.

However, to retain reference to the physical world, the starting point is observation of traditional instruments and various sound objects. The sound produced by an acoustic instrument can be affected by many factors, including the material from which the instrument is made, its size and shape, and the way it is played. Approximation techniques such as finite elements or finite differences enable us to model different shapes [1,2]. Our aim is to introduce the material as another parameter in the physical model design.

Geometry and Mathematics have trained our mind to conceive pure and abstract forms without any matter. The converse might be more difficult, like imagining a color without any form. There are probably some correlations between the sound perception of material and shape. Anyway, in most situations, it is possible to recognize sound objects as belong to one of the material categories: metal, wood, plastic, etc.

Leaving the above question open, the problem will be treated in a purely physical point of view. The first step is to look for some invariant features in acoustical signals produced by different resonators made of the same material. We are faced to a new version of the famous Kac's problem : "can one hear the *material* of a drum?" [3]. Using the viscoelasticity theory, it is shown that this acoustical invariant is a particular relation between frequencies and damping factors. The presentation is organized as follows. In section 2, constitutive equations and linear viscoelasticity are briefly reviewed. In section 3, it is shown how constitutive laws of linear viscoelasticity are involved in vibrating mechanical systems. In the last section, algorithms for computer simulation and a general framework for physical modeling design are outlined.

2. CONSTITUTIVE EQUATIONS

Mechanical property of a material is expressed in form of a *constitutive* equation, i.e. an equation that is independent of the geometry of the body and depends only on its material nature. This equation involves a pair of intensive and extensive dependent variables stress σ and strain ε. Stress is defined as force per unit area, while strain is the fractional change of size (length, volume, angle, etc.). The simplest constitutive equation is the Hooke's law of pure elasticity, which states that stress is proportional to strain: $\sigma = k\varepsilon$. But in real materials, internal dissipation of energy has also to be considered. When a body of material is subjected to a deformation, its microscopic structure may experience local activities. The microscopic rearrangements, in any real material, necessarily require a finite time. Hence, the constitutive equation must involve the time variable. Throughout this work, we consider only vibrations of small amplitudes, so we may assume linearity. Thus, linear viscoelasticity [4,5] is considered as the fundamental mechanical behavior of materials.

2.1. Linear viscoelasticity

The mathematical formulation of linear viscoelastic behavior considers a material as a causal (fixed) linear system, with σ and ε as input-output variables. Aging phenomenon is then neglected and temperature assumed to be constant. A general linear system can be characterized by one of its system's functions : impulse response, harmonic response, etc. [6]. Considering the strain ε as the input and σ as the output, a constitutive equation has the general form of a convolution integral: $\sigma = k_\delta * \varepsilon$, where k_δ is the impulse response of the material sample.

Expressed in the transform plane, the convolution reduces to multiplication by the *relaxance* $k(s)$, i.e. the Laplace transform of $k_\delta(t)$. The knowledge of viscoelastic properties of materials is based on measurements. The most easily measurable viscoelastic functions are the step and the harmonic responses.

The step response $k_H(t)$, referred to *relaxation*, is the gradual decrease of stress when the material is held at constant strain. The harmonic response expresses the delay of the strain when the material is subjected to a harmonic stress. It is expressed by the complex modulus $k^*(\omega) = k(i\omega) = k'(\omega)+ik''(\omega)$, which is the

ratio σ/ε when the material sample experiences harmonic oscillations at frequency ω. The maximum potential energy stored and the amount of energy dissipated in each cycle are respectively proportional to the storage and loss modulus $k'(\omega)$ and $k''(\omega)$. The loss factor $\eta = k''(\omega)/k'(\omega)$ measures the damping capacity of the material.

The storage modulus is an increasing function of ω. In fact, at high rate strain, fewer relaxation processes have enough time to be completed. A relaxation phenomenon requires time but also kinetic energy. Thus, each relaxation process has best efficiency at an optimal strain rate. Hence, the plot of $k''(\omega)$ may exhibit several peaks at various frequencies. Roughly speaking, the frequency axis can be divided intro three regions:

1. the *rubbery* region, where k' and k'' are low and have slow variations
2. the *transition* region, where k' increases fast and k'' have one or several relaxation peaks.
3. the *glassy* region, k' attains a high stationary value and k'' has again a low value.

2.2. Lumped parameters models

By considering two idealized elements, the pure spring and the pure dashpot and combining them in series-parallel assembly, one obtains a wide variety of viscoelastic behaviors. Relaxance of such model is a rational fraction $k(s) = P(s)/Q(s)$ and the convolution integral may be replaced by a constant coefficient differential equation. The simplest combinations involves a pair of spring and dashpot in series (Maxwell unit) or in parallel (Kelvin unit).

Kelvin unit has an infinite relaxation time, while Maxwell model has no equilibrium elasticity (i.e. $k_H(\infty) = 0$). Relaxation function of a Maxwell unit is a decaying exponential $\exp(-t/\tau)$ where $\tau = z/k$ is the ratio of the dashpot and spring constants. By adding a pure spring k_∞ in parallel one obtains the Zener unit which is the simplest model of linear viscoelastic *solid*. The Zener model is defined by three parameters (k_∞,k,z) or equivalently by (k_∞,τ,χ) where $\chi = (k_\infty+k)/k_\infty = k'(\infty)/k'(0)$ is a measure of the strength of the viscoelastic process. A generalized Maxwell or a n-order Wiechert model [4] is the series assembly of a pure spring k_∞ to n parallel Maxwell units (k_i,z_i) (Fig. 1). Relaxance of a Wiechert model is shown below:

$$k(s) = k_\infty + \sum_{i=1}^{n} k_i - \sum_{i=1}^{n} k_i \frac{\tau_i^{-1}}{s+\tau_i^{-1}} \qquad (1)$$

Relaxed and instantaneous modulus of a Wiechert model are $k'(0) = k_\infty$ and $k'(\infty) = k_\infty+\Sigma k_i$. Here again, the ratio $\chi = k'(\infty)/k'(0)$ measures the overall viscoelastic strength. Relaxation function of a Wiechert model is a sum of decaying exponentials $k(\infty)-\Sigma\exp(-t/\tau_i)/k_i$, where $\tau_i = z_i/k_i$ is the relaxation time of the i-th underlying Maxwell unit. If the relaxation times τ_i are spread enough, the loss modulus $k''(\omega)$ has n peaks located at $\omega_i = 1/\tau_i$.

Every lumped parameter model may be represented as a Wiechert model. The decomposition is close to the partial fraction decomposition of the rational relaxance $k(s) = P(s)/Q(s)$. In real materials, there are many causal mechanisms responsible for viscoelastic behavior: thermoelasticity, relaxation by point defects or solute atoms motions, interface motion and air flow in composite and porous materials, etc. [5,7]. Each relaxation process corresponds to a particular peak of the loss modulus $k''(\omega)$. Wiechert models with $n = 4$ to 10 elements can represent linear viscoelastic properties of solids with a good approximation. Relaxation times τ_i and weighting coefficients k_i may be calculated by model fitting methods so that $\Sigma\exp(-t/\tau_i)/k_i$ approaches an observed relaxation function [4].

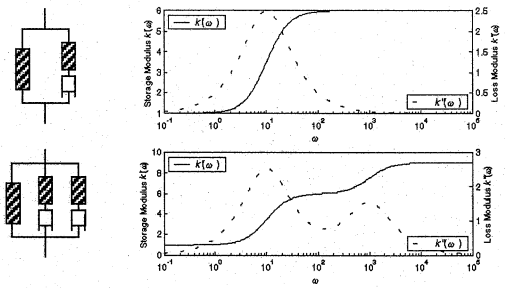

Figure 1 *Plots of storage and loss modulus of a Zener (top) and a 2-order Wiechert (bottom) models.*

3. VIBRATING SYSTEMS

We consider now the influence of the viscoelasticity on vibrations. Let us investigate first the case of a single degree of freedom.

3.1. Single degree of freedom

Let us consider a single mass m, connected to the ground by an n-order Wiechert model. The equation of free oscillations, in the transform plane is

$$ms^2 + k(\infty) - \sum_{i=1}^{n} k_i \frac{\tau_i^{-1}}{s+\tau_i^{-1}} = 0 \qquad (2)$$

Examination of pole-zero positions of equation (2) shows that viscoelastic oscillators have at least n overdamped free solutions $\exp(-\alpha_i t)$ and at most, a single underdamped solution $\exp(-\alpha t)\exp(\pm i\omega t)$, where $-\alpha_i$ and $-\alpha\pm i\omega$ are the $n+2$ zeros of equation (2). Except for the case $n = 1$, we cannot hope to solve analytically this equation. However, some general features of the solutions can be outlined. Let us first introduce the 'natural' frequency ω_0, defined by $\omega_0^2 = k'(\infty)/m$. From (2) we may deduce

$$\sum_{i=1}^{n} \tau_i^{-1} = 2\alpha + \sum_{i=1}^{n} \alpha_i \qquad (3)$$

which entails that the damping factors have an upper bound, in particular $2\alpha < \tau_n^{-1}$, where τ_n is smallest relaxation time. According to the location of ω_0 in respect to relaxation peaks, the following features can be stated (see Fig. 2):

1. If $\omega_0 \ll \tau_1^{-1}$ (i.e. rubbery region), $\omega \approx \omega_0/\sqrt{\chi}$, and $\alpha \propto \omega_0^2$
2. For $\omega_0 \in [\tau_1^{-1}, \tau_n^{-1}]$ (i.e. transition region), α grows by stages
3. If $\tau_n^{-1} \ll \omega_0$ (i.e. glassy region), $\omega \approx \omega_0$ and α reaches a stationary value.

In the case of a Zener model, it can be shown that, in the rubbery region, $\alpha \approx \tau\omega_0^2/2$, while in the glassy region $\alpha \approx (\chi-1)/(2\tau\chi)$. The behavior of α should be compared with the damping factor of a standard harmonic oscillator (i.e. with a Kelvin unit), in which case $\alpha = \tau\omega_0^2/2$, for every ω_0.

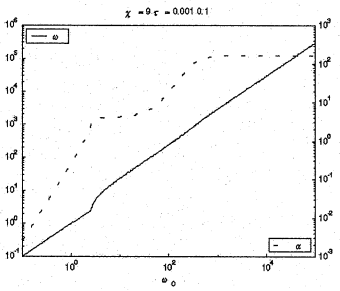

Figure 2 *Plots of α and ω versus ω_0 for a 2-order Wiechert oscillator with τ_i = 0.1 and 0.001, χ_i = 5 and 3.*

3.2. Multiple degrees of freedom

Let us consider a set of p little masses $m_1,\ldots,m_p$ interconnected by a viscoelastic network, defined by the symmetric transfer matrix $[K(s)]$. Let $[M]$ be the mass matrix. The equation of equilibrium in the transform plane is

$$([M]s^2 + [K(s)])X(s) = F(s) \tag{4}$$

Multiplying by an appropriate polynomial $Q(s)$ (i.e. the roots of Q are the n relaxation times of the whole edges), this equation is transformed into the following form

$$[P(s)]X(s) = Q(s)F(s) \tag{5}$$

where the $[P(s)]$ is a matrix polynomial. Thus, the free vibration of the network is a combination of underdamped and overdamped exponentials : $\exp(-\alpha t)\exp(\pm i\omega t)$ and $\exp(-\alpha_i t)$, where $-\alpha_i$ and $-\alpha \pm i\omega$ are the zeros of the $p(n+2)$-order characteristic polynomial $P(s) = \det([P(s)])$. Elementary solutions involving polynomials in t, may appear if the matrix polynomial $[P(s)]$ is defective [8].

A special case of interest is when all of the edges have the same relaxation times and the same viscoelastic strength k_i. In other words all the edges represent the same material. Then, equation (5) reduces to

$$([M]s^2 + k(s)[K])X(s) = F(s) \tag{6}$$

where $k(s)$ is the *normalized* relaxance of the material, i.e. $k(\infty) = 1$ and $[K]$ a constant matrix. The (real) mode shape matrix $[Q]$ of the underlying elastic network is characterized by the following equations [9,10]:

$$[Q]^T[M][Q] = 1 \tag{7}$$

$$[Q]^T[K][Q] = diag\{\omega_{01}^2,\ldots,\omega_{0p}^2\} \tag{8}$$

Thus, in the coordinates system defined by the mode shapes, equation (4) may be expressed as a set of p equations:

$$s^2 + \omega_{0i}^2 k(s) = f_i(s) \tag{9}$$

corresponding each to a single degree of freedom viscoelastic Wiechert oscillator of unitary masse and relaxance $k_i(s) = \omega_{0i}^2 k(s)$. Now the main point of our argument is here: multiplying $k(s)$ by the scalar ω_{0i}^2 does not affect relaxation times and relaxation strengths. Thus, the underdamped homogeneous solutions of (9) obey to the same frequency / damping factor relation defined by the material. This frequency / damping factor relation is what we refer to the *signature* of the material.

The homogeneity condition above may be extended in order to include external viscoelasticity, proportional viscoelasticity and product viscoelastic networks [9]. Extension to continuous viscoelastic systems can be done by replacing matrices by their corresponding differential operators and boundary conditions. The homogeneity condition can be expressed as at each point x, $k(s,x) = k(s)E(x)$, where $E(x)$ is a elastic constant and $k(s)$ the normalized relaxance of the material. Then, underdamped frequencies ω and damping factors α obey to the same frequency / damping factor relation defined by the equation $s^2+\omega_0^2 k(s) = 0$.

4. SOUND SYNTHESIS

Section 3 stated that in the case of a homogeneous vibrating system (i.e. the same strengths $\chi_i = k_i/k_\infty$ and relaxation times $\tau_i = z_i/k_\infty$) all of the vibrating modes $\alpha+i\omega$ verify the same frequency / damping factor relation. Therefore, one can use this property for organizing the physical model design in two steps:

1. Design of the elastic skeleton, i.e. $[M]$ and $[K]$ matrices
2. Wearing the conservative skeleton with a particular material

The first step consists of determining relative frequencies as well as their relative amplitudes. Finite element and finite difference methods as well as experimental methods like modal testing can be used to derive this underlying elastic skeleton [1,2,10]. This conservative model can be obtained also by abstract constructions including assemblies, products, fibrations, etc. [9]. The next step is to choose a material reference for the model. This can be achieved by various ways. One can define time domain attributes (relaxation function, relaxation times and respective strengths) as well as frequency domain attributes (relaxance, storage or loss modulus). This can be done by collecting physical data on real material, as well as designing abstract imaginary material in the form of a Wiechert model. Note that guaranteeing physicality, in particular causality, requires some restrictions on the complex modulus [5].

4.1. State space simulation

The time domain differential equation of a lumped parameter viscoelastic vibrating system has the following general form:

$$[P_0]X^{(n+2)} + \ldots + [P_{n+2}]X = q_0F^{(n)} + \ldots + q_nF \qquad (10)$$

where, X is the nodes displacement vector. Note that the order of the equation is n+2, where n is the number of relaxation processes in the whole system. By a standard transformation, equation (10) can be rewritten in the state space as a first order equation $\dot{Y} = [A]Y + [B]F$, where Y is the state variable containing displacements at the p nodes as well as their first n+1 derivatives. Simulation of this first order equation can be done by standard finite difference methods (Euler forward/backward or Runge-Kutta). Note that for stability, the sampling period must be smaller than the smallest relaxation time in the whole system [2].

4.2. CORDIS-ANIMA simulation

CORDIS-ANIMA system is a mechanical model simulator developed by A.C.R.O.E. based on the two step finite difference simulation of Kelvin (non-linear) oscillators [11]. However, simulation of general viscoelastic behaviors is not possible with the standard version of CORDIS. Nevertheless, Kelvin models (defined by inertia, elasticity and viscosity matrices [M], [K] and [Z]) can however generate all frequency-damping factor relations. Indeed, given a finite set of frequency / damping factor pairs (α_i, ω_i) there exists a unique matrix [Z] having the prescribed spectrum and eigenvectors. Moreover, by polynomial curve fitting methods, one may approximate a given material signature by a polynomial P, in which case the [Z] matrix has to be chosen such that $[M]^{-1/2}[Z][M]^{-1/2} = P([M]^{-1/2}[K]^{1/2}[M]^{-1/2})$. Kelvin models cannot however generate the overdamped components and derivations in the right hand of (10) are ignored. Hence, this method works well as far as only free oscillations are considered.

4.3. Modal synthesis

Classical modal synthesis [9,12,13] enables us to synthesize material sound by a direct control of modal damping factors according to a $\alpha(\omega_0)$ material signature. But, this approach is similar to 4.2 since it neglects overdamped components.

Complex modal synthesis deals with the whole set of solutions, including derivations of the right hand of equation (10). Complex modes are characterized by complex mode-shapes (resulting from non homogeneity) and a set of $p(n+2)$ first-order differential equations with complex coefficients:

$$\dot{y}_j + (-\alpha_j \pm i\omega_j)y = b_0 f_j^{(n)} + \ldots + b_n f_j \qquad (11)$$

Note that every complex data occurs in a conjugate pair corresponding to a single real solution. So, each pair gives rise to two one-order differential equations with real coefficients. Digital simulation of this equation can be done by standard finite difference methods or by digital IIR filters [14]. Modal (complex) displacements weighted by (complex) mode-shapes add then to the real displacements.

For homogeneous materials a third modal synthesis can be considered. According to 3.2, homogeneous material can be decomposed in high order viscoelastic real modes. The synthesis algorithm is then similar to classical modal synthesis. But, here each mode corresponds to a high order differential equation:

$$a_0 y_j^{(n+2)} + a_1 y_j^{(n+1)} + \ldots + a_{n+2} y_j = b_0 f_j^{(n)} + \ldots + b_n f_j \qquad (12)$$

Every modal equation can be simulated by the same techniques as before, i.e. finite differences or IIR digital filters.

5. CONCLUSION

A general framework for introducing materials in physical modeling sound synthesis was presented. It was shown that the acoustic invariant of each material has to be found in some specific frequency/damping factor function. Thus, a physical model may be designed in two step: 1) modeling of geometric data (i.e. strain operator discretization and boundary conditions), 2) wearing this conservative skeleton by a viscoelastic dress to represent a particular material.

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