Sphere Partition Functions and Quantum De Sitter Thermodynamics

Yuk Ting Albert Law

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Abstract

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Driven by a tiny positive cosmological constant, our observable universe asymptotes into a casual patch in de Sitter space in the distant future. Due to the exponential cosmic expansion, a static observer in a de Sitter space is surrounded by a horizon. A semi-classical gravity analysis by Gibbons and Hawking implies that the de Sitter horizon has a temperature and entropy, obeying laws of thermodynamics. Understanding the statistical origin of these thermodynamic quantities requires a precise microscopic model for the de Sitter horizon. With the vision of narrowing the search of such a model with quantum-corrected macroscopic data, we aim to exactly compute the leading quantum (1-loop) corrections to the Gibbons-Hawking entropy, mathematically defined as the logarithm of the effective field theory path integral expanded around the round sphere saddle, i.e. sphere partition functions. This thesis discusses sphere partition functions and their relations to de Sitter (dS) thermodynamics. It consists of three main parts:

The first part addresses the subtleties of 1-loop partition functions for totally symmetric tensor fields on S^{d+1}, and generalizes all known results to arbitrary spin $s \geq 0$ in arbitrary dimensions $d \geq 1$. Starting from a manifestly covariant and local path integral on the sphere, we carry out a detailed analysis for any massive, shift-symmetric, massless, and partially massless fields. For any field with spin $s \geq 1$, we find a finite contribution from longitudinal modes; for any massless and partially massless fields, there is a residual group volume factor due to modes generating constant

gauge transformations; for any massless and partially massless fields with spin $s \geq 2$, we derive the phase factor resulted from Wick-rotating negative conformal modes, generalizing the phase factor first obtained by Polchinski for the case of massless spin 2 to arbitrary spins.

The second part presents a novel formalism for studying 1-loop quantum de Sitter thermodynamics. We first argue that the Harish-Chandra character for the de Sitter group $SO(1, d+1)$ provides a manifestly de Sitter-invariant regularization for normal mode density of states in the static patch, without introducing boundary ambiguities as in the traditional brick wall approach. These characters encode quasinormal mode spectrums in the static patch. With these, we write down a simple integral formula for the thermal (quasi)canonical partition function, which straightforwardly generalizes to arbitrary spin representations. Then, we derive a universal formula for 1-loop sphere partition functions in terms of the $SO(1, d+1)$ characters. We find a precise relation between these and the (quasi)canonical partition function mentioned earlier: they are equal for scalars and spinors; for any fields with spin $s \geq 1$, they differ by "edge" degrees of freedom living on the de Sitter horizon. This formalism allows us to efficiently compute the exact 1-loop corrected de Sitter horizon entropy, which as we argue provides non-trivial constraints on microscopic models for the de Sitter horizon. In three dimensions, higher-spin gravity can be alternatively formulated as an sl(n) Chern-Simons theory, which as we show possesses an exponentially large landscape of de Sitter vacua. For each vacuum, we obtain the all-loop exact sphere partition function, given by the absolute value squared of a topological string partition function. Finally, our formalism elegantly proves the relations between generic dS, AdS, and conformal higher-spin partition functions.

The last part extends our studies in the previous part to grand (quasi)canonical partition functions on the dS static patch, where we generalize the (quasi)canonical partition functions by allowing non-zero chemical potentials in some of the angular directions. For these, we derive a generalized character integral formula in terms of the full $SO(1, d+1)$ characters. In three dimensions, we relate them to path integrals on Lens spaces. Similar to its sphere counterpart, the Lens space path integral exhibits a "bulk-edge" structure.

Table of Contents

List of Figures

List of Conventions and Abbreviations

ΛCDM model	Lambda Cold Dark Matter model
FLRW	Friedmann–Lemaître–Robertson–Walker
QFT	Quantum Field Theory
EFT	Effective Field Theory
CFT	Conformal Field Theory
dS	de Sitter
AdS	Anti-de Sitter
HS	Higher Spin
CHS	Conformal Higher Spin
QNM	Quasinormal Mode
S^{d+1}	The $(d+1)$-dimemsional unit round sphere
IR	Infrared
UV	Ultraviolet
PM	Partially Massless

Acknowledgements

I came to Columbia from the other side of the planet wanting to be a theoretical physicist, with effectively zero idea about what that actually entailed and how difficult it was. Fortunately as always, my worldline has been clustered with amazing people, who gave me tremendous support over the past 6 years.

I am forever indebted to my adviser Frederik Denef, who showed me how to create good physics with the combination of interesting questions, wild ideas and careful calculations. I am inspired to develop my own way of thinking about physics and my own research style. I am grateful for all the patient guidance, constructive criticism, life experience, encouragement and support all along.

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Chapter 1: Introduction and Background

1.1 De Sitter static patch: our ultimate fate

It is well-known that our universe is expanding[1]. What is more, observations of distant supernovae [1–3] strongly indicate that we are entering a phase of accelerating expansion, due to a remarkably small and, crucially, positive cosmological constant Λ appearing in the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{1.1}$$

Fitting the ΛCDM model with current observational data [4–6], we find that today Λ accounts for roughly $\Omega_\Lambda \sim 70\%$ of the critical density, from which we infer $\Lambda \sim 10^{-52}\ \mathrm{m}^{-2} \sim 10^{-122}\ell_{\mathrm{pl}}^{-2}$, $\ell_{\mathrm{pl}} = \sqrt{\hbar G_N/c^3}$ being the Planck length. As tiny as such, however, because of its uniform energy density Λ will increasingly dominate over other forms of matter as the space expands. Eventually it will completely take over, so that our universe becomes indistinguishable from a causal patch of a de Sitter space [7, 8]. See figure 1.1.

[1] Throughout this thesis whenever we say “our universe” we mean the *observable* universe.

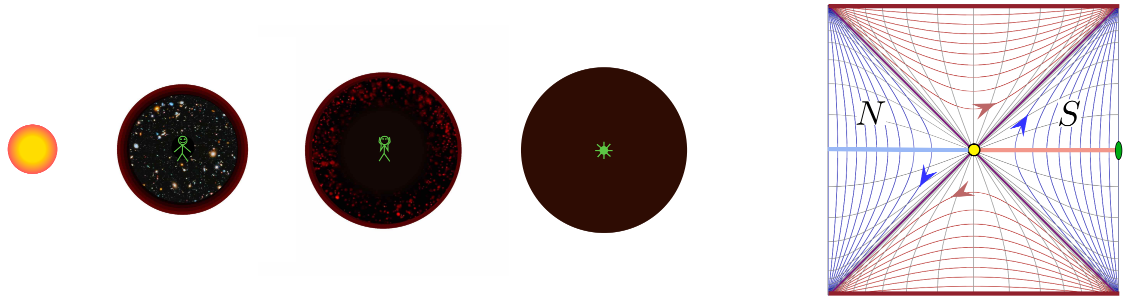


Figure 1.1: From left to right: Due to the exponential expansion driven by Λ, signals from distant galaxies would appear more and more red-shifted. Eventually these galaxies will fall outside of our horizon, and the observable universe will look like a de Sitter static patch [7, 8]. Right: The Penrose diagram of the global de Sitter space. The causal diamond for a static observer sitting at the south pole (the right vertical line) is labeled by "S". This patch is bounded by the observer's past and future horizons (the diagonal lines). The generator H_S for the time flow (shown by the blue arrows) within the southern static patch generates an inverse time flow in the northern patch (labeled by "N"), while it generates spacelike isometries (red arrows) in the past and future triangles.

1.1.1 De Sitter horizon and the Gibbons-Hawking entropy

It is thus desirable to consider an observer siting inside a de Sitter universe (some basic facts of de Sitter space are collected in appendix A). Because of the exponential cosmic expansion, there is only a finite portion of the entire global de Sitter space where the observer can receive signals from and send signals to, namely their causal patch. See figure 1.1. In a $(d+1)$-dimensional de Sitter space dS_{d+1}, this patch is conveniently parametrized by the static coordinates, with metric

$$ds^2 = -\left(1 - \frac{r^2}{\ell_{\rm dS}^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{\ell_{\rm dS}^2}} + r^2 d\Omega_{d-1}^2\,, \qquad 0 < r < \ell_{\rm dS}\,. \tag{1.2}$$

Here $\ell_{\rm dS} = \sqrt{d(d-1)/2\Lambda}$ is the de Sitter length. In these coordinates, the static observer sits at $r = 0$. The de Sitter horizon corresponds to the coordinate singularity at $r = \ell_{\rm dS}$ and has area $A_c = \Omega_{d-1}\ell_{\rm dS}^{d-1}$ with Ω_{d-1} the area of the unit round sphere S^{d-1}. Analogous to black holes, there is a temperature and entropy associated with this horizon [9, 10]. Taking into account quantum effects, one can indeed show [9, 10] that the global de Sitter Euclidean vacuum looks thermal to the static observer, with temperature $T_{\rm dS} = 1/2\pi\ell_{\rm dS}$ and the tree-level Gibbons-Hawking entropy[2]

$$S_{\rm GH} = \frac{A_c}{4G_N} \ . \tag{1.3}$$

What is the microscopic origin of the entropy (1.3)? Can it be computed as the logarithm of the number of microstates?[3] The answers to these questions will require a precise microscopic model of de Sitter quantum gravity or at least of the de Sitter horizon. Despite the enormous efforts such as [11–24], no such model has been constructed at present. For a review of some aspects of the challenges, see for instance [25]. In this thesis, we do not make any assumption about the underlying microscopic theory, and proceed strictly from a macroscopic, low energy EFT perspective.

From an EFT point of view, the tree-level formula (1.3) will be modified by quantum fluctuations of gravitons and matter fields living in this background. Computing quantum corrections to macroscopic data proves to be useful in testing microscopic proposals. On the one hand, when there is a microscopic theory for which the low energy EFT is known, the agreement between the EFT and microscopic computations of the quantum corrections can serve as a consistency check. For instance, for some special black holes in string theory [26–29], the microscopic computation of the logarithmic correction to the Bekenstein-Hawking entropy have been shown to match the prediction from the macroscopic analysis.

On the other hand, quantum corrections can bring further support to or additional constraints on candidate microscopic models. For a holographic example, the matching of AdS bulk HS partition functions with the boundary CFT partition functions at 1-loop or $O(N^0)$ provides strong evidence for the HS/CFT duality [30–33]. Sometimes the constraint from the quantum-corrected EFT data is strong enough to rule out a microscopic model, as demonstrated in [34]: loop quantum gravity proposals for the Schwarzschild black hole horizon are invalidated by their failure to

[2] For an observer in the distant future of our universe, the temperature is roughly $T_{\rm dS} \sim 10^{-30}$K, much lower than the current CMB temperature of 2.73K.

[3] The entropy (1.3) for our universe accounts for a gigantic number $\sim 10^{10^{122}}$ of microstates.

reproduce the correct logarithmic correction to the Bekenstein-Hawking entropy predicted by the macroscopic Euclidean gravity analysis. This last example illustrates the power of EFT quantum corrections: they impose stringent universal constraints on candidate microscopic models, without any knowledge of the true underlying model.

These ideas raise the interesting prospect of narrowing our search of the correct microscopic model for de Sitter horizon using quantum-refined macroscopic data: a candidate model is ruled out if it fails to reproduce the 1-loop corrections to the Gibbons-Hawking entropy (1.3). This motivates the central goal of this thesis: exactly computing the 1-loop corrected de Sitter horizon entropy for any gravity plus matter effective field theory.

1.2 1-loop corrections to the Gibbons-Hawking entropy

The quantum-corrected de Sitter horizon entropy is given by [10]

$$\mathcal{S} = \log \mathcal{Z} \tag{1.4}$$

where $\mathcal{Z}$ is the gravity plus matter EFT Euclidean path integral expanded about the round sphere S^{d+1} saddle. In 1-loop approximation, we keep the metric and matter fluctuations up to quadratic order.

As reviewed in more detail in section (1.2.2), a path integral for a quantum field Φ with Euclidean action $S_E[\Phi]$ on a space of the product form $S^1_\beta \times M_d$, where S^1_β a circle of circumference β and M_d a d-dimensional manifold, is equal to a QFT thermal canonical partition function at inverse temperature β

$$\mathrm{Tr}\; e^{-\beta H} = \int \mathcal{D}\Phi \; e^{-S_E[\Phi]} \quad \text{on} \quad S^1_\beta \times M_d \; . \tag{1.5}$$

Here the Hamiltonian H generates time translation in the spacetime $\mathbb{R} \times M_d$. The trace Tr runs over the entire QFT Hilbert space. In 1-loop approximation, we have a free action in the exponent of the weight in the path integral; in the canonical picture this corresponds to an ideal gas approximation.

The ideal gas canonical partition function (1.5) can then be computed by canonically quantizing the free QFT on $\mathbb{R} \times M_d$ and summing over the multiparticle Fock space, which is computationally easier than evaluating the functional determinants in the path integral picture reviewed in section 1.2.3.

As reviewed in section 1.2.2, while the sphere S^{d+1} does not take the form $S^1_\beta \times M_d$, it is obtained from the static patch (1.2) by periodically identifying the time t in the imaginary direction. One then naturally wonders if a 1-loop sphere path integral can be computed in the canonical framework through the relation (1.5), with H being the free static patch Hamiltonian and trace Tr over the associated Fock space. Because of its computational simplicity, we will first discuss this thermal canonical ideal gas picture in the following. After that we will go back to sphere path integral itself, and point out that in the present case the relation (1.5) should in fact be modified by "edge" modes living on the de Sitter horizon. The last part of this section is devoted to describing the subtleties one encounters when computing sphere path integrals.

1.2.1 Thermal canonical partition function

An initial attempt to obtain the 1-loop quantum corrections to (1.3) is by trying to evaluate the canonical ideal gas partition function

$$Z_{\text{bulk}} = \text{Tr}\; e^{-\beta H}, \tag{1.6}$$

where H is the Hamiltonian for the free static patch QFT and the trace Tr runs over the associated multiparticle Fock space. The subscript "bulk" means that (1.6) captures the quanta living in the bulk of the static patch, as opposed to the "edge" degrees of freedom living on the horizon to be discussed in the next section. With (1.6), one can calculate the entropy using the standard thermodynamic relations at the inverse de Sitter temperature $\beta = 2\pi\ell_{\text{dS}}$. For the rest of this chapter we set $\ell_{\text{dS}} = 1$.

As a simple concrete example, we consider a free massive scalar field ϕ with mass $m^2 =$

$\Delta(d-\Delta)$ on the static patch of dS_{d+1}. Following standard steps for bosonic fields, we write (1.6) as

$$\log\, Z_{\text{bulk}} = -\int_0^\infty d\omega\, \rho(\omega) \left(\log\left(1 - e^{-\beta\omega}\right) + \frac{\beta\omega}{2}\right), \tag{1.7}$$

where

$$\rho(\omega) \equiv \operatorname{tr} \delta(\omega - H) \tag{1.8}$$

is the density of states for the single-particle normal mode spectrum. In (1.8), the trace tr runs over the single-particle Hilbert space.

One then proceeds to solve for the positive-frequency solutions to the equation of motion

$$0 = \left[-\frac{1}{1-r^2}\partial_t^2 + (1-r^2)\partial_r^2 + \left(-(d+1)r + \frac{d-1}{r}\right)\partial_r + \frac{\tilde{\nabla}^2}{r^2} - m^2\right]\phi\,. \tag{1.9}$$

Here $-\tilde{\nabla}^2$ is the Laplacian on the unit round sphere S^{d-1}. Now, for every frequency $\omega > 0$, one obtains the normal modes that are regular everywhere within the static patch (in particular at $r = 0$, the location of the observer):

$$\phi_{\omega\ell\sigma}(t,\Omega,r) \propto e^{-i\omega t}\, Y_{\ell\sigma}(\Omega)\; r^\ell \left(1-r^2\right)^{i\omega/2} {}_2F_1\left(\tfrac{\ell+\Delta+i\omega}{2}, \tfrac{\ell+d-\Delta+i\omega}{2}; \tfrac{d}{2}+\ell; r^2\right), \tag{1.10}$$

where $Y_{\ell\sigma}(\Omega)$ is a basis of spherical harmonics on S^{d-1} labeled by the total $SO(d)$ angular momentum quantum number ℓ and the magnetic quantum numbers collectively denoted as σ. Hence, we have a continuous set of basis $|\omega\ell\sigma)$ labeled by energy ω and $SO(d)$ angular momentum numbers ℓ and σ, which satisfies $(\omega\ell\sigma|\omega'\ell'\sigma') = \delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{\sigma\sigma'}$. Evaluating (1.8) in this basis leads to the divergent result

$$\rho_S(\omega) = \int d\omega' \sum_{\ell\sigma} (\omega'\ell\sigma|\delta(\omega-\omega')|\omega'\ell\sigma) = \sum_{\ell\sigma} \delta(0)\,. \tag{1.11}$$

The appearance of a continuous normal mode spectrum and divergent density of states is ubiquitous for spacetimes with a horizon. The physical reason is that modes with arbitrary energies are allowed to exist because of the unlimited redshift near the horizon. One can cure this by putting a t'Hooft brick wall [35] slight away from the horizon, on which one puts an additional boundary condition so that the spectrum is discretized. In the current case, one can cut off the static patch at $r = 1-\epsilon$ as in figure 1.2. However, this approach suffers from a few unappealing features including broken de Sitter invariance, or dependence on the choice of the boundary condition (for example Dirichlet or Neumann) at the brick wall. Other ideas [36, 37] to get rid of the divergence (1.11) include for instance Pauli-Villars or dimension regularizations. While these avoided some of the problems such as broken general covariance, they are generally difficult to implement, making it challenging to make progress beyond the simplest massive scalar case.

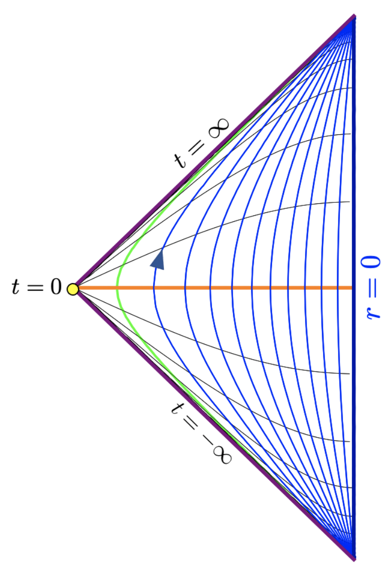


Figure 1.2: We can discretize the continuous normal mode spectrum by imposing an extra boundary condition at a brick wall (green line) put at a location $r = 1 - \epsilon$ slightly away from the horizon. In this figure $\epsilon = 0.03$. The placement of such a wall clearly breaks de Sitter invariance.

Is there an alternative way to cure the divergent density of states (1.11)? Can we do so without breaking de Sitter invariance or introducing boundary ambiguities? In chapter 3, we re-examine the computation leading to (1.11) and introduce a novel approach to make sense of the density of states using the mathematically well-defined Harish-Chandra character for the de Sitter group $SO(1, d + 1)$. Eventually we will derive an integral formula for (1.6) that allows seamless generalization to arbitrary spin representations.

1.2.2 De Sitter static patch and the sphere

Now we go back to the path integral formulation. The standard argument leading to the relation (1.5) goes as follows [10]. First, we recall that an amplitude of the form $\langle\phi_f| e^{-i(t_f-t_i)H} |\phi_i\rangle$ can be expressed as a path integral

$$\langle\phi_f| e^{-i(t_f-t_i)H} |\phi_i\rangle = \langle\phi_f, t_f|\phi_i, t_i\rangle = \int \mathcal{D}\Phi \Big|_{\Phi(t_i,x)=\phi_i(x)}^{\Phi(t_f,x)=\phi_f(x)} e^{iS[\Phi]} , \tag{1.12}$$

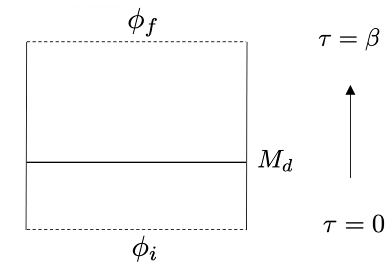
where $S[\Phi]$ is the Lorentzian action on the spacetime $\mathbb{R} \times M_d$, and the path integration over $\Phi(t, x)$ includes all field configurations with initial value $\phi_i(x)$ and final value $\phi_f(x)$.

Now, we view the operator $e^{-\beta H}$ in (1.5) as an evolution in the imaginary time direction $\tau = it$ by amount β. The Euclidean amplitude $\langle\phi_f| e^{-\beta H} |\phi_i\rangle$ then admits a similar path integral representation as (1.12) but on a Euclidean space

$$\langle\phi_f| e^{-\beta H} |\phi_i\rangle = \langle\phi_f, \beta|\phi_i, 0\rangle = \int \mathcal{D}\Phi \Big|_{\Phi(0,x)=\phi_i(x)}^{\Phi(\beta,x)=\phi_f(x)} e^{-S_E[\Phi]} , \tag{1.13}$$

where the field variable Φ now lives on a Euclidean space, and the weight e^{-S_E} of the path integral is conventionally defined so that there is a minus sign in front of the Euclidean action $S_E[\Phi]$. We can depict (1.13) as

$$\langle \phi_f | \, e^{-\beta H} \, |\phi_i \rangle = \qquad (1.14)$$



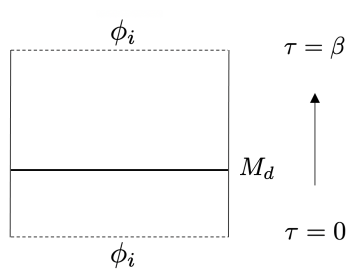
meaning it is a Euclidean path integral over M_d times a Euclidean time interval of width β, with the boundary conditions shown.

Finally, the trace Tr in (1.5) identifies the initial and final field configurations $\phi_i(x) = \phi_f(x)$ in (1.13) and integrates over all possible $\phi_i(x)$. To summarize, the Euclidean path integral for a thermal canonical partition function at inverse temperature β is defined on a geometry related to the original spacetime by

$$t \to -i\tau \qquad \text{with} \qquad \tau \sim \tau + \beta \, , \qquad (1.15)$$

establishing the relation (1.5), pictorially summarized as

$$\mathrm{Tr}\, e^{-\beta H} = \int \mathcal{D}\phi_i$$



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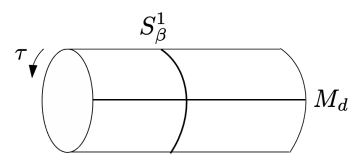

$$. \qquad (1.16)$$

Now let us go back to the de Sitter static patch. After the Wick-rotation and periodic identification (1.15) with $\beta = 2\pi$, the static patch becomes a sphere[4]:

[4]The simplest way to see this is to go to the parametrization (A.22) of the embedding space coordinates. Upon (1.15) with $\beta = 2\pi$ and taking $X^0 \to -iX^0$, the embedding space coordinates X^A cover the entire round sphere $X^2 = 1$.

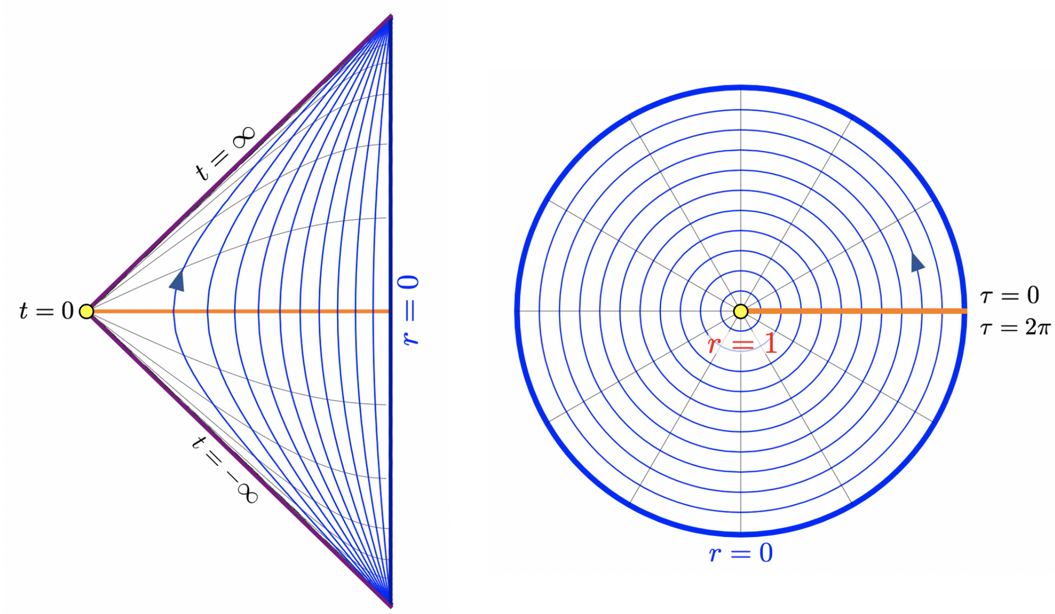

Figure 1.3: After Wick-rotating $t \to -i\tau$ and identifying $\tau \sim \tau + 2\pi$, the de Sitter static patch (left) turns into a sphere (right). The Euclidean time τ becomes an angular variable on the sphere. The horizon (yellow dot) at $r = 1$ is mapped to the origin of the sphere.

Because of this, one might wonder if the following relation is true:

$$Z_{\text{bulk}} \stackrel{?}{=} Z_{\text{PI}} \,. \tag{1.17}$$

where Z_{bulk} is the thermal canonical ideal gas partition function (1.6) on the dS_{d+1} static patch and

$$Z_{\text{PI}} = \int \mathcal{D}\Phi \; e^{-S_E[\Phi]} \,. \tag{1.18}$$

is a 1-loop Euclidean path integral on the round sphere S^{d+1}. While the argument leading to (1.5) is valid for spacetimes of the product form $\mathbb{R} \times M_d$ (Wick-rotated to $S^1_\beta \times M_d$), there are reasons to expect that they fail for a spacetime with a horizon such as the de Sitter static patch. For example, one observation is that the horizon is mapped to the origin of the sphere (the yellow dot in figure 1.3) which is a fixed point of the static patch Hamiltonian H. At that point the interpretation of $e^{-\beta H}$ as a Euclidean time evolution becomes obscure.

In chapter 3, we provide more sophisticated arguments for why the relation (1.17) is expected to break down and requires substantial modifications. Indeed, we will see in chapter 3 that while (1.17) is valid for scalar and spinor fields, it will be corrected by "edge" degrees of freedom living on the horizon for any field with spin $s \geq 1$. In the analogous case of Rindler space, the qualitative string theory picture for these edge contributions was first discussed in [38]. They were later computed explicitly for the case of massless spin-1 fields in [39]. Recent works such as [40–48] interpret the edge modes found in [39] as arising from the non-factorization of the QFT Hilbert space due to the Gauss law constraint satisfied by the vector field. The existing approach [43, 44, 48, 49] of computing these edge modes involves introducing a brick wall near the entangling surface, which again breaks symmetries of the problem and introduces boundary ambiguities. Furthermore, the complexity of such computation grows quickly for fields with higher spins, making it very difficult to generate explicit results.

Quite remarkably, with our approach in chapter 3, it is straightforward to obtain the edge modification to the relation (1.17) for arbitrary field contents on the de Sitter static patch. But before we achieve that, we need to attack head-on the subtleties of the sphere path integral computation itself. We turn to this next.

1.2.3 The sphere is beautiful but subtle

The study of the 1-loop sphere path integral (1.18) is not new. However, as innocuous as (1.18) might look (it is just a free QFT on a sphere), its computation turns out to be quite intricate. One can see for instance the long history [50–59] for the case of pure gravity. In this section we briefly review some of the key subtleties. As a warm-up, let us recall how the computation of 1-loop partition functions such as (1.18) typically proceeds with the simplest example: a free real massive scalar $\phi(x)$, with sphere path integral

$$Z_{\rm PI} = \int \mathcal{D}\phi \ e^{-S[\phi]} \ , \qquad S[\phi] = \frac{1}{2} \int_{S^{d+1}} \phi \, (-\nabla^2 + m^2) \, \phi \ . \tag{1.19}$$

Here $\int \mathcal{D}\phi = \int \prod_x d\phi(x)$ is an infinite product of integrals of the field $\phi(x)$ over every point x on the round sphere S^{d+1}, and we are using the shorthand notation $\int_{S^{d+1}} \equiv \int_{S^{d+1}} \sqrt{g}\, d^{d+1}x$. Varying ϕ in the action results in the equation of motion: $(-\nabla^2 + m^2)\,\phi = 0$. One can think of $\phi(x)$ as an infinite-dimensional vector with a continuous label x and the kinetic operator $-\nabla^2 + m^2$ as an infinite-dimensional matrix acting on this vector. Naturally we expand $\phi(x) = \sum_n c_n \phi_n(x)$ in terms of eigenfunctions ϕ_n of $-\nabla^2 + m^2$ with eigenvalue $\lambda_n + m^2$:

$$\left(-\nabla^2 + m^2\right)\phi_n(x) = \left(\lambda_n + m^2\right)\,\phi_n(x), \qquad \int_{S^{d+1}} \phi_n \phi_{n'} = \delta_{nn'} \tag{1.20}$$

For S^{d+1}, we have $\lambda_n = n(n+d)$ and $n = 0, 1, 2, \cdots$. Upon substituting these, the action becomes a simple sum

$$S[\phi] = \frac{1}{2}\int_{S^{d+1}} \phi\,(-\nabla^2 + m^2)\,\phi = \frac{1}{2}\sum_n \left(\lambda_n + m^2\right)\, c_n^2\,, \tag{1.21}$$

and the integration of the field ϕ over every point x turns into the integration over the expansion coefficients c_n:

$$\int \mathcal{D}\phi = \int \prod_n \frac{dc_n}{\sqrt{2\pi}}\,. \tag{1.22}$$

The path integral (1.19) then becomes an infinite product of Gaussian integrals

$$Z_{\text{PI}} = \int \prod_n \frac{dc_n}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\sum_n \left(\lambda_n + m^2\right) c_n^2} = \prod_n \frac{1}{\sqrt{\lambda_n + m^2}} = \left(\prod_n \left(\lambda_n + m^2\right)\right)^{-1/2}. \tag{1.23}$$

Inside the bracket is the product of all eigenvalues of $-\nabla^2 + m^2$, which we denote formally as the determinant of the latter:

$$\prod_n \left(\lambda_n + m^2\right) = \det\left(-\nabla^2 + m^2\right). \tag{1.24}$$

To summarize we have

$$Z_{\rm PI} = \det\Big(-\nabla^2 + m^2\Big)^{-1/2} \; . \tag{1.25}$$

Since it involves an infinite product of arbitrarily large eigenvalues of $-\nabla^2 + m^2$, this functional determinant is UV-divergent and requires regularization such as heat kernel [60] or zeta function [61] regularization. In chapter 3, after expressing sphere path integrals in terms of a simple integral formula, we provide an efficient algorithm to explicitly evaluate (1.25) and its generalizations to other field contents.

The longitudinal modes of tensor fields

The discussion above for the real massive scalar might lead one to think that the analogous computation for free fields with spins $s \geq 1$ should not be too difficult. After all, the Laplace-type operator in the final answer (1.25) is simply the kinetic operator appearing in the equation of motion. However, this is not the case. For an illustration let us consider a free massive vector field A_μ. The problem for massive higher-spin fields is more complicated but works analogously. The path integral for A_μ is

$$Z_{\rm PI} = \int \mathcal{D}A \; e^{-S[A]} \; , \qquad S[A] = \int_{S^{d+1}} \Big(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu\Big). \tag{1.26}$$

The equations of motion resulted from the Proca action $S[A]$ are

$$\Big(-\nabla^2 + m^2 + d\Big) A_\mu = 0 \quad , \qquad \nabla^\lambda A_\lambda = 0. \tag{1.27}$$

The second statement says that the on-shell degrees of freedom are transverse. One might naively think that the result for path integral (1.26) is simply

$$Z_{\rm PI} = \det\Big(-\nabla^2_{(1)} + m^2 + d\Big)^{-1/2} \qquad \text{(wrong)} \tag{1.28}$$

where $-\nabla^2_{(1)}$ is the Laplacian acting on *transverse* vector fields on S^{d+1}. This turns out to be wrong [62]. Note that for the path integral (1.26) to be manifestly local, in the integration we include all smooth unconstrained vector fields A_μ on S^{d+1}. In particular, the vector fields have both the transverse and longitudinal parts

$$A_\mu = A^T_\mu + A^L_\mu \quad , \qquad \nabla^\lambda A^T_\lambda = 0. \tag{1.29}$$

Upon substituting these into (1.26), the transverse part will result in the functional determinant (1.28), while it is not immediately clear that the longitudinal part will give a trivial contribution. After a closer examination in chapter 2, we will see that it does not. Fortunately, the contribution from the longitudinal part amounts to modifying the naive result (1.28) and its higher-spin generalizations in a simple way.

The residual group volume for massless gauge fields

For massless and partially massless gauge fields, there will be additional complications because of gauge invariance. The simplest example is the free Maxwell theory, with action given by putting $m^2 = 0$ in the Proca action (1.26):

$$S[A] = \int_{S^{d+1}} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{1.30}$$

This action is invariant under the gauge transformations

$$A_\mu \longrightarrow A_\mu + \nabla_\mu \phi \ . \tag{1.31}$$

Therefore, if we simply replace the Proca action in the path integral (1.26) with the Maxwell action, we will be including field configurations related by gauge transformations (1.31) in the integration. To compensate for this over-counting of gauge-equivalent orbits, we divide the path integral by the volume of the space of gauge transformations

$$Z_{\rm PI} = \frac{1}{\mathrm{Vol}(\mathcal{G})} \int \mathcal{D}A \; e^{-S[A]} \,, \qquad \mathrm{Vol}(\mathcal{G}) = \int \mathcal{D}\alpha = \prod_n \frac{d\alpha_n}{\sqrt{2\pi}} \,, \tag{1.32}$$

where α is a scalar field. To proceed we have to gauge fix. There are many approaches, such as the standard Faddeev–Popov or BRST gauge fixing, or the "geometric approach" [63–65] that we will use extensively in chapter 2. Each of these has its own advantages but ultimately they are equivalent. Whichever approach one takes, after gauge fixing there will be an integral over gauge parameters

$$\int \mathcal{D}'\phi. \tag{1.33}$$

On the sphere, there is a normalizable constant scalar mode ϕ_0. The prime in (1.33) means that we are excluding this mode. The reason is that this mode does not generate a non-trivial gauge transformation (1.31), as ϕ_0 is a constant and $\nabla_\mu \phi_0 = 0$. If we take into account the possible interactions of the Maxwell field A_μ with other matter fields, this mode generates global U(1) symmetries on the matter fields. The integration over the mode ϕ_0 is therefore the volume of the global symmetry group.

A puzzle then arises. Usually[5] we expect the factor (1.33) cancels exactly with the factor $\mathrm{Vol}(\mathcal{G})$ in (1.32). Now, should we include the integration over the constant mode ϕ_0 (i.e. $n = 0$ in the product (1.32)) in $\mathrm{Vol}(\mathcal{G})$? If not, $\mathrm{Vol}(\mathcal{G})$ will cancel out (1.33). The price to pay is that $\mathrm{Vol}(\mathcal{G})$ is now a non-local factor, since the field α being integrated over satisfies a non-local constraint: $\int_{S^{d+1}} \alpha = 0$. This apparently contradicts the fact that the path integral (1.32) is defined by a local QFT. However, if we do include this residual group volume factor

$$\mathrm{Vol}(G) = \int \frac{d\alpha_0}{\sqrt{2\pi}} \tag{1.34}$$

[5]for example when computing flat space scattering amplitudes

in Vol($\mathcal{G}$), how do we determine its correct value? What is the correct metric on this space of global symmetries? For Maxwell theory on curved spacetimes, a careful treatment [66] shows that the factor (1.34) has to be included for consistency with locality and unitarity. We wish to generalize these considerations to non-Abelian gauge theories, pure gravity and higher-spin theories.

1.3 Outline and results of the thesis

The previous section shows that computing 1-loop corrections to the Gibbons-Hawking entropy (1.3) is quite challenging: in the canonical approach one encounters the divergent normal mode density of states, while in the path integral approach one is faced with various subtle issues associated with normalizable zero or negative modes on the sphere. What is more, the usual equivalence of the two pictures is expected to be modified. We want to resolve these puzzles, relate the two pictures precisely, and develop an efficient method for exact evaluations. Quite pleasingly, we managed to achieve all of these, and in fact much more. This thesis is the product of this comprehensive study. It consists of three chapters:

1.3.1 A compendium of sphere path integrals (chapter 2)

This chapter is based on the work [67], where we present an extensive analysis for totally symmetric tensor fields on a sphere. Starting from a manifestly covariant and local path integral, we carefully examine the subtleties for free QFTs on S^{d+1}, and make broad generalizations to all massive, shift-symmetric, massless and partially massless totally symmetric tensor fields of arbitrary spins in arbitrary dimensions. To give an example, after a lengthy derivation in gory detail, we obtain the following expression for massless higher-spin theories:

$$Z_{\rm PI}^{\rm HS} = i^{-P} \frac{\gamma^{\dim G}}{\mathrm{Vol}(G)_{\rm can}} \prod_s \left((d+2s-2)(d+2s-4)\right)^{\frac{N_{s-1}^{\rm KT}}{2}} \frac{\det'_{-1}\left|-\nabla^2_{(s-1)} - \lambda_{s-1,s-1}\right|^{1/2}}{\det'_{-1}\left|-\nabla^2_{(s)} - \lambda_{s-2,s}\right|^{1/2}} \qquad (1.35)$$

with

$$P = \sum_s (N^{\text{CKT}}_{s-2} + N^{\text{CKT}}_{s-1} - N^{\text{KT}}_{s-1}), \qquad \gamma = \sqrt{\frac{8\pi G_N}{\text{Vol}(S^{d-1})}}, \qquad \dim G = \sum_s N^{\text{KT}}_{s-1} \tag{1.36}$$

Here N^{CKT}_s and N^{KT}_s denote respectively the numbers of spin-s conformal Killing tensors and Killing tensors on S^{d+1}. The operator $-\nabla^2_{(s)}$ is the Laplacian acting on spin-s symmetric transverse traceless fields on S^{d+1}, with eigenvalues $\lambda_{n,s} = n(n+d) - s$, $n \geq s$. The primes in the functional determinants mean that we exclude the zero modes of the differential operators; absolute values are taken for each eigenvalue so that the functional determinants are positive definite. G denotes the global higher-spin group. Here we comment on several features of (1.35):

1. The subscript -1 in the functional determinants denote an extension of the eigenvalue product to $n = -1, 0, \cdots, s-1$. These correspond to the longitudinal mode contributions for any spinning fields mentioned in section 1.2.3.

2. As mentioned at the end of section 1.2.3, for any massless and partially massless gauge fields, the residual group volume factor $\text{Vol}(G)_{\text{can}}$ has to be included in the path integral for consistency with locality and unitarity. The subscript "can" means that the volume is measured with respect to a theory-independent "canonical" metric that we define building upon the ideas of [68–71]. Closely related are the coupling dependence $\gamma^{\dim G}$ and the factor $((d+2s-2)(d+2s-4))^{\frac{N^{\text{KT}}_{s-1}}{2}}$ for each s, which capture the cubic interaction structures of the parent theory to which we performed the 1-loop approximation.

3. The phase factor i^{-P} is present for any massless and partially massless gauge fields with spin $s \geq 2$. In their actions, all but finite number of the trace modes have negative kinetic terms [72]. A standard procedure [72] is to Wick-rotate these modes in field space so that the path integral is well-defined. Applying this procedure on the round sphere S^{d+1}, Polchinski [59] first obtained an overall phase i^{-d-3} for a massless spin-2 field. Here we generalize it to arbitrary massless higher-spin fields.

Expressions analogous to (1.35) are obtained for all massive, shift-symmetric and partially massless totally symmetric tensor fields of arbitrary spin in arbitrary dimensions.

1.3.2 Quantum de Sitter horizon entropy from quasicanonical bulk, edge, sphere and topological string partition functions (chapter 3)

This long chapter is based on the work [73] with Dionysios Anninos, Frederik Denef and Zimo Sun. We present a novel approach to make sense of the canonical thermal partition function (1.6). We also precisely relate this with the sphere path integral (1.18). What is more, our formalism provides a systematic method for exact evaluations of (1.6) and (1.18). Along the way we uncovers a lot of interesting physics about low-energy EFTs in the de Sitter static patch. Here we highlight a few key results:

$SO(1, d+1)$ **character and density of states** We point out the physics of the Harish-Chandra character [74, 75] for the de Sitter group $SO(1, d+1)$:

$$\chi_{[\Delta,\boldsymbol{s}]}(t) \equiv \mathrm{tr}_G \; e^{-iHt} \; . \tag{1.37}$$

Here Δ labels the $SO(1,1)$ weight and $\boldsymbol{s} = (s_1, s_2, \cdots, s_r)$ with $r = \lfloor \frac{d}{2} \rfloor$ labels the $SO(d)$ highest weight for the unitary irreducible $SO(1, d+1)$ representation [76–78] of interest. The Hamiltonian H generates time flows in the static patch. The subscript G means that we are tracing over the global de Sitter single-particle Hilbert space. For example, a massive scalar has $m^2 = \Delta(d-\Delta)$ and $\boldsymbol{s} = \boldsymbol{0} = (0, 0, \cdots, 0)$, and character [78]

$$\chi_{[\Delta,\boldsymbol{0}]}(t) = \frac{e^{-\Delta t} + e^{-\bar{\Delta} t}}{|1 - e^{-t}|^d} \, , \tag{1.38}$$

where $\bar{\Delta} = d - \Delta$. The character for a massive bosonic spin-s field with $m^2 = \Delta(d-\Delta) - s$ and $\boldsymbol{s} = (s, 0, \cdots, 0)$ is simply given by the scalar character (1.38) multiplied by the $SO(d)$ spin-s degeneracy D^d_s.

Expanding (1.37) in powers of e^{-it},

$$\chi_{[\Delta,s]}(t) = \sum_\lambda N_\lambda \, e^{-i\lambda t} \ , \tag{1.39}$$

the sum is found to run over the quasinormal mode (QNM) spectrum and N_λ is the degeneracy of the QNM of frequency λ. With the mathematically well-defined object (1.37) we make sense of the density of states mentioned in section 1.2.1 and obtain the simple formula for the (quasi)canonical ideal gas partition function on the southern static patch at general inverse temperature β

$$\boxed{\log Z_{\rm bulk}(\beta) \equiv \log \, {\rm Tr}_S \; e^{-\beta H_S} = \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}} \; \chi(t)_{\rm bos} \; - \; \frac{2\, e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}} \; \chi(t)_{\rm fer} \right)} \tag{1.40}$$

where $\chi(t)_{\rm bos}$ and $\chi(t)_{\rm fer}$ are the characters for the bosonic and fermionic fields respectively. The subscripts S emphasize that we are tracing over the southern Hilbert space. Compared to the traditional brick wall approach and its variants [35–37], our approach has numerous advantages such as manifest de Sitter invariance and independence of boundary artifacts. Also, our formalism straightforwardly generalizes to arbitrary field contents in any dimensions.

Canonical vs path integral As discussed in section 1.2.2, the relation (1.17) between the canonical partition functions and Euclidean path integrals is expected to be modified by "edge" modes living on the de Sitter horizon. Without introducing a brick wall [43, 44, 48, 49], we obtain the precise edge contributions for general field contents. For instance, we derive the following universal character integral formula for massive bosonic fields

$$\boxed{\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge} = \int_0^\infty \frac{dt}{2t} \, \frac{1+e^{-t}}{1-e^{-t}} \left(\chi_{\rm bulk}(t) - \chi_{\rm edge}(t) \right)} \ . \tag{1.41}$$

Here $\log Z_{\rm bulk}$ is exactly the bosonic part of the (quasi)canonical partition function (1.40) at $\beta = 2\pi$, whereas $\log Z_{\rm edge}$ is present only for fields with spin $s \geq 1$. To give an example, $\chi_{\rm bulk}$ and $\chi_{\rm edge}$

for a massive spin-s boson are explicitly given by

$$\boxed{\chi_{\text{bulk}} \equiv D^d_s \, \frac{e^{-\Delta t} + e^{-\bar{\Delta} t}}{(1-e^{-t})^d} \,, \qquad \chi_{\text{edge}} \equiv D^{d+2}_{s-1} \, \frac{e^{-(\Delta-1)t} + e^{-(\bar{\Delta}-1)t}}{(1-e^{-t})^{d-2}}} \tag{1.42}$$

where $\bar{\Delta} = d - \Delta$. Notice that the edge character χ_{edge} is a character living in two lower dimensions than the bulk character χ_{bulk}, indicating that these capture the edge degrees of freedom on the de Sitter horizon.

Quantum gravitational thermodynamics Taking into account dynamical gravity and possible matter fields, we compute the exact renormalized 1-loop de Sitter horizon entropy. For instance, with our character formalism we obtain for pure gravity

$$\begin{array}{c|l} d & \mathcal{S} \\ \hline 2 & \mathcal{S}^{(0)} - 3\log\mathcal{S}^{(0)} + 5\log(2\pi) \\ 3 & \mathcal{S}^{(0)} - 5\log\mathcal{S}^{(0)} - \frac{571}{90}\log\left(\frac{\ell_0}{L}\right) - \log\left(\frac{8\pi}{3}\right) + \frac{715}{48} - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3) \\ 4 & \mathcal{S}^{(0)} - \frac{15}{2}\log\mathcal{S}^{(0)} + \log(12) + \frac{27}{2}\log(2\pi) + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4} \end{array} \tag{1.43}$$

where $\mathcal{S}^{(0)}$ is the tree-level entropy. We have absorbed local UV-divergences into gravitational couplings. In the $d = 3$ (S^4) example, ℓ_0 is the tree-level radius of the sphere and L a renormalization scale in the minimal subtraction scheme.

3D higher-spin gravity In three dimensions, higher-spin theory can be alternatively formulated as an sl(n) Chern-Simons theory. We discover an exponentially large landscape of de Sitter vacua, labeled by partitions of $n = \sum_a m_a$. These correspond to different embeddings of of sl(2) into sl(n) as n-dimensional representations $R = \oplus_a \mathbf{m}_a$.

For the vacuum labeled by R, we obtain the all-loop quantum-corrected Euclidean partition function of the dS_3 static patch by analytically continuing the exact $SU(n)_{k_+} \times SU(n)_{k_-}$ partition function on S^3 to $k_\pm \to l \pm i\kappa$ with $l \in \mathbb{N}, \kappa \in \mathbb{R}^+$. Explicitly,[6]

$$Z(R)_0 = \left| \frac{1}{\sqrt{n}} \frac{1}{(n+l+i\kappa)^{\frac{n-1}{2}}} \prod_{p=1}^{n-1} \left(2\sin\frac{\pi p}{n+l+i\kappa} \right)^{(n-p)} \right|^2 \cdot e^{2\pi\kappa T_R}, \tag{1.44}$$

where $T_R = \frac{1}{6}\sum_a m_a(m_a^2-1)$, ℓ/G_N and κ are related as

$$\kappa T_R = \frac{2\pi\ell}{8\pi G_N}. \tag{1.45}$$

Using the large-n duality between $U(n)_k$ Chern-Simons theory on S^3 and closed topological string theory on the resolved conifold [79, 80], the HS gravity partition function (1.44) in vacuum R can be re-cast into the absolute value squared of a weakly-coupled topological string partition function.

Partition functions made easy Our formalism reduces statements about partition functions to statements about characters. This allows us to relate partition functions by manipulating characters algebraically, without evaluating the partition functions themselves. For example, the relation between $\mathrm{AdS}^\pm_{\mathrm{d+1}}$ ($\pm$: standard/alternative boundary conditions) massless spin-s and conformal spin-s sphere partition functions [62, 81, 82]

$$Z_s^{\mathrm{CHS}}(S^d) = \frac{Z_s^{\mathrm{HS}}(AdS^-_{d+1})}{Z_s^{\mathrm{HS}}(AdS^+_{d+1})} \tag{1.46}$$

is neatly proved by noting the algebraic character relation

$$\chi_s\left(\mathrm{CdS_d}\right) = \chi_s\left(\mathrm{AdS^-_{d+1}}\right) - \chi_s\left(\mathrm{AdS^+_{d+1}}\right) \tag{1.47}$$

where $\chi_s\left(\mathrm{CdS_d}\right)$ is the bulk-edge character for a conformal spin-s field in dS_d while $\chi_s\left(\mathrm{AdS^\pm_{d+1}}\right)$ are the bulk-edge characters for a massless spin-s field on AdS_{d+1} [83] with standard and alternative boundary conditions respectively.

[6] As reviewed in chapter 3, the definition of the Chern-Simons theory as a QFT involves a choice of "framing". The result (1.44) assumes the canonical framing, denoted by the subscript 0.

1.3.3 Grand partition functions and Lens space path integrals (chapter 4)

This chapter contains results recently obtained by the author. We make a broad generalization of our formalism in chapter 3 by allowing non-zero chemical potentials. We consider the grand (quasi)canonical partition functions on the dS_{d+1} static patch:

$$Z_{\text{bulk}}(\beta, \boldsymbol{\mu}) = \operatorname{Tr} e^{-\beta(H+i\boldsymbol{\mu}\cdot\boldsymbol{J})} \,. \tag{1.48}$$

where $\boldsymbol{J} = (J_1, \cdots, J_r)$ is a maximal set of commuting angular momenta (i.e. the Cartan generators for $SO(d)$) and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_r)$ are the corresponding chemical potentials. Here $r = \lfloor \frac{d}{2} \rfloor$ is the rank of the subgroup $SO(d)$. For $\beta\boldsymbol{\mu} = \frac{2\pi \boldsymbol{q}}{p}$, $p \in \mathbb{N}$, $\boldsymbol{q} \in \mathbb{Z}^r$, we write down a generalized character formula. For example, the result for bosonic fields is

$$\log Z_{\text{bulk}}\left(\beta, \frac{2\pi\boldsymbol{q}}{p\beta}\right) = \int_0^\infty \frac{dt}{2pt} \sum_{m\in\mathbb{Z}_p} \frac{\sinh\frac{2\pi t}{p\beta}}{\cosh\frac{2\pi t}{p\beta} - \cos\frac{2\pi m}{p}} \chi_{\text{bulk}}\left(t, \frac{2\pi m\boldsymbol{q}}{p}\right) \tag{1.49}$$

where $\chi_{\text{bulk}}(t, \boldsymbol{\theta}) = \chi_{\text{bulk}}(t, \theta_1, \cdots, \theta_r)$ is the full $SO(1, d+1)$ character. For example, a massive spin-s particle has [78]

$$\chi_{[\Delta,s]}(t, \boldsymbol{\theta}) = \chi_s^d(\mathbf{x}) \left(Q^\Delta + Q^{\bar{\Delta}}\right) \mathcal{P}^d(Q, \mathbf{x}) \tag{1.50}$$

where $Q = e^{-t}$, $\mathbf{x} = (x_1, \cdots, x_r) = (e^{i\theta_1}, \cdots, e^{i\theta_r})$, $\chi_s^d(\mathbf{x})$ is the $SO(d)$ spin-s character, and

$$\mathcal{P}^d(Q, \mathbf{x}) = \prod_{i=1}^r \frac{1}{(1-Qx_i)(1-Qx_i^{-1})} \times \begin{cases} 1 & \text{if } d = 2r \\ \frac{1}{1-Q} & \text{if } d = 2r+1 \end{cases} \,. \tag{1.51}$$

Note that (1.50) evaluated at $\boldsymbol{\theta} = \mathbf{0}$ recovers the reduced massive spin-s character (1.42).

Lens space path integrals In three dimensions, we relate the grand (quasi)canonical partition function at $\beta = \frac{2\pi}{p}$ and $\mu = q$

$$Z_{\text{bulk}}\left(\frac{2\pi}{p}, q\right) = \operatorname{Tr} e^{-\frac{2\pi}{p}(H+iqJ)} \tag{1.52}$$

where p and q are two coprime integers: $(p,q) = 1$, to path integrals on Lens spaces $L(p,q)$. These are smooth quotients of S^3 and thus arise as saddle points of the gravitational path integral with a positive cosmological constant. Compared to the S^3 saddle, the contribution from a single Lens space $L(p,q)$ is exponentially suppressed. The relevance of Lens spaces to dS_3 quantum gravity is discussed in [84]. Here we obtain the Lens space generalization of the sphere result (1.41) for a massive spin-s boson

$$\log Z_{\text{PI}} = \int_0^\infty \frac{dt}{2pt}\left[\sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{\cosh t - \cos m\tau_2}\chi_{\text{bulk}}(t, m\tau_1) - \sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{\cosh t - \cos m\tau_1}\chi_{\text{edge}}(t,\tau_1,\tau_2)\right]. \tag{1.53}$$

Here τ_1, τ_2 are related to p, q as

$$\tau_1 = \frac{2\pi q}{p}, \qquad \tau_2 = \frac{2\pi}{p}\,, \tag{1.54}$$

while the bulk and edge characters are

$$\chi_{\text{bulk}}(t,\theta) = 2\cos s\theta\,\frac{e^{-(\Delta-1)t} + e^{-(\bar\Delta-1)t}}{2(\cosh t - \cos\theta)}\,, \quad \chi_{\text{edge}}(t,\tau_1,\tau_2) = \frac{\cos sm\tau_1 - \cos sm\tau_2}{\cos m\tau_1 - \cos m\tau_2}\left(e^{-(\Delta-1)t} + e^{-(\bar\Delta-1)t}\right). \tag{1.55}$$

1.3.4 Other work of the author

It should be mentioned that some of the author's work has not been included in the present thesis for the sake of self-coherence.

This includes the series of work [85–87] with Michael Zlotnikov in celestial holography. Scattering amplitudes in asymptotically flat space can be mapped onto a celestial sphere (known as celestial amplitudes or celestial correlators) at the null infinity, on which they transform as conformal correlators under the Lorentz transformations. The ultimate goal of this program is to construct a holographic Celestial Conformal field theory (CCFT) for flat space quantum gravity. In [85], we study the universal constraints imposed by bulk Poincaré symmetries on massless and massive celestial amplitudes. These results were subsequently extended to amplitudes involving massive spinning bosons in [86]. Analogous to the usual flat space non-gravitational amplitudes, celestial amplitudes can be expanded in terms of relativistic partial waves, making the underlying Poincaré symmetries manifest. In [87], we derive relativistic partial waves directly on the celestial sphere, and perform the partial wave expansions explicitly for scalars, gluons, gravitons and open superstring gluons for demonstration.

Another work [88] with Janna Levin and Kshitij Gupta concerns the study of electromagnetic Penrose process around a charged spinning black hole immersed in a uniform magnetic field. In the traditional Penrose process, particles can attain negative energies within the ergosphere of a spinning black hole, making it possible to extract rotational energy from it. With the inclusion of black hole charge and a magnetic field, the energetics for charged particles becomes more complicated yet very interesting. For example, we find that the region(s) with negative energies can be much larger than the ordinary ergosphere; in some cases they can be totally detached from the horizon.

Chapter 2: A Compendium of Sphere Path Integrals

This chapter is a based on the work [67], where we study 1-loop path integrals on S^{d+1} for general massive, shift-symmetric and (partially) massless totally symmetric tensor fields. After reviewing the cases of massless fields with spin $s = 1, 2$, we provide a detailed derivation for path integrals of massless fields of arbitrary integer spins $s \geq 1$. Following the standard procedure of Wick-rotating the negative conformal modes, we find a higher spin analog of Polchinski's phase for any integer spin $s > 2$. The derivations for low-spin ($s = 0, 1, 2$) massive, shift-symmetric and partially massless fields are also carried out explicitly. Finally, we provide general prescriptions for general massive and shift-symmetric fields of arbitrary integer spins and general partially massless fields of arbitrary integer spins and depths.

2.1 Introduction

Sphere partition functions are of interest in the study of quantum gravity with a positive cosmological constant [10, 50–59, 73]. In a recent work [73], a character formula for 1-loop sphere path integrals has been derived, which for rank-s symmetric tensor fields with generic mass m^2 on S^{d+1} takes the form

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t}\frac{1+e^{-t}}{1-e^{-t}}(\chi_{\rm bulk} - \chi_{\rm edge}). \tag{2.1}$$

Here $\chi_{\rm bulk}$ and $\chi_{\rm edge}$ are

$$\chi_{\rm bulk} = D_s^d \frac{e^{-\Delta t} + e^{-\bar{\Delta} t}}{(1-e^{-t})^d}, \quad \chi_{\rm edge} = D_{s-1}^{d+2} \frac{e^{-(\Delta-1)t} + e^{-(\bar{\Delta}-1)t}}{(1-e^{-t})^{d-2}}, \tag{2.2}$$

characters of the isomatry group $SO(1, d+1)$ of the $(d+1)$-dimensional de Sitter space dS_{d+1}. The scaling dimension Δ is related to the mass m^2 and spin s through $m^2 = (\Delta + s - 2)(\bar{\Delta} + s - 2)$ and $\bar{\Delta} = d - \Delta$. With formula (2.1) we can compute exact 1-loop results for Euclidean de Sitter thermodynamics.

Massless spinning fields are more subtle. Their character formula takes the form [73]

$$\log Z_{\rm PI} = \log Z_G + \log Z_{\rm Char}, \tag{2.3}$$

where the character part $\log Z_{\rm Char}$ is (2.1) but with the characters (2.2) replaced with their massless counterparts, and the first term takes the general form

$$Z_{\rm G} = i^{-P} \frac{(2\pi\gamma)^{\dim G}}{{\rm Vol}(G)_{\rm can}}. \tag{2.4}$$

The second factor is associated with the group G of trivial gauge transformations. γ is related to the coupling constant of the theory, while ${\rm Vol}(G)_{\rm can}$ is what we call canonical group volume in [73]. Later we will define these quantities more precisely. It was emphasized in [66] that the inclusion of this factor is crucial for consistency with locality and unitarity. The phase factor i^{-P} is only present only for fields with spin $s \geq 2$, whose origin is the negative conformal modes in the Euclidean path integral [72] that naively makes the path integral divergent. The standard prescription [72] to cure this problem is to Wick rotate the problematic conformal modes. Polchinski later [59] found that on S^{d+1} this procedure led to a finite number of i factors (with $P = d + 3$ in that case) that could render the Euclidean path integral positive, negative or imaginary depending on the dimensions.

Let us go back to the starting point of the character formula, the left hand sides of (2.1) and (2.3). That is, the 1-loop sphere path integrals

$$Z_{\rm PI} = \int \mathcal{D}\phi e^{-S[\phi]} \tag{2.5}$$

where $S[\phi]$ is the quadratic action of the field ϕ. For massless fields, there will be a division by an infinite gauge group volume factor to compensate for the over-counting of gauge-equivalent field configurations. In this paper, we will perform detailed derivations for the determinant expressions of (2.5) for several classes of fields. We focus on symmetric tensor fields on S^{d+1} with $d \geq 2$.

For massless fields, we will see how the factor Z_{G} arises explicitly from the manifestly local path integral. More generally, such a factor is present for any partially massless gauge fields. We directly check it for the spin-2 depth-0 field, and then provide a prescription for general bosonic partially massless fields. Another class of theories that are of interest involves shift-symmetric fields [89]. These can be thought of as the longitudinal modes decoupled from partially massless gauge fields. Working out explicitly the low-spin cases, we find that their path integrals contain a factor analogous to Z_{G}.

All of our results will be expressed in terms of functional determinants of the symmetric transverse traceless (STT) Laplacians on S^{d+1}. Their relevant properties are summarized in appendix B.2.

Plan of the paper: We first review the computations for massless spin-1 and spin-2 fields in sections 2.2 and 2.3. We then turn to our complete derivation for massless fields of arbitrary integer spins in section 2.4. In section 2.5, we study fields with generic mass. In sections 2.6 and 2.7, we study general shift-symmetric fields and partially massless fields respectively. We conclude in section 2.8. All conventions are summarized in appendix B.1. Relevant properties of the STT Laplacians on S^{d+1} and their eigenfunctions are collected in appendix B.2. The higher spin invariant bilinear form is reviewed in appendix B.3.

2.2 Review of massless vectors

We start with a pedagogical review of the case of massless vectors. The object of interest is the 1-loop approximation to the full Euclidean path integral

$$Z_{\mathrm{PI}} = \frac{1}{\mathrm{Vol}(\mathcal{G})} \int \mathcal{D}A^a \mathcal{D}\Phi e^{-S_E[A^a,\Phi]} \tag{2.6}$$

for a theory that involves a collection of massless vector (for example U(1) or Yang-Mills) gauge fields interacting with some matter fields, denoted as ${A^a}_\mu$ and collectively as Φ respectively, living on S^{d+1}.

$U(1)$ **with a complex scalar** The simplest example involves a single U(1) gauge field A_μ interacting with a complex scalar ϕ (studied in [53]):

$$S_E[A,\phi] = \int_{S^{d+1}} \left[\frac{1}{4\mathrm{g}^2} F_{\mu\nu}F^{\mu\nu} + D_\mu\phi(D^\mu\phi)^* + m^2\phi\phi^* \right], \tag{2.7}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad D_\mu\phi \equiv (\partial_\mu - iA_\mu)\phi \tag{2.8}$$

are the field strength and the covariant derivative of the scalar. This action is invariant under the local U(1) gauge transformations

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x). \tag{2.9}$$

The normalization adopted here is to emphasize the presence of the coupling constant g. In this convention g does not show up in the gauge transformation.

Yang-Mills Another example is Yang-Mills (YM) theory with a Lie algebra

$$[L^a, L^b] = f^{abc}L^c \tag{2.10}$$

generated by some standard basis of anti-hermitian matrices and f^{abc} is real and totally antisymmetric. The YM action is

$$S_E[A,\phi] = \frac{1}{4\mathrm{g}^2}\int_{S^{d+1}} \mathrm{Tr}F^2 = \frac{1}{4\mathrm{g}^2}\int_{S^{d+1}} F^a_{\mu\nu}F^{a,\mu\nu}, \tag{2.11}$$

where the curvature is $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ with $A_\mu = A^a_\mu L^a$. Here the overall normalization for the trace (or Killing form) is *defined* such that the generators L^a are unit normalized:

$$\mathrm{Tr}(L^a L^b) \equiv \delta^{ab}. \tag{2.12}$$

For SU(2) YM, $L^a = -\frac{i\sigma^a}{2}$ satisfying $[L^a, L^b] = \epsilon^{abc} L^c$, and the trace (2.12) would be $\mathrm{Tr} \equiv -2\mathrm{tr}$ with tr being the matrix trace. The YM action is invariant under the non-linear gauge transformations $\alpha = \alpha^a L^a$

$$A_\mu \longrightarrow A_\mu + \partial_\mu \alpha + [A_\mu, \alpha]. \tag{2.13}$$

In both the $U(1)$ and YM examples, the corresponding path integral in (2.6) clearly overcounts gauge equivalent configurations. A factor $\mathrm{Vol}(\mathcal{G})$ is thus inserted in (2.6) to quotient out configurations connected by gauge transformations. This factor is formally the volume of the space of gauge transformations $\mathcal{G}$ (the measure with respect to which the volume is defined will be discussed in later subsection) and is theory dependent. For the U(1) example, $\mathrm{Vol}(\mathcal{G})$ is simply a path integral over a single local scalar field

$$\mathrm{Vol}(\mathcal{G})_{\mathrm{U}(1)} = \int \mathcal{D}\alpha, \tag{2.14}$$

while for SU(2) YM it would be a path integral over 3 local scalar fields

$$\mathrm{Vol}(\mathcal{G})_{\mathrm{SU}(2)} = \int \mathcal{D}\alpha_1 \mathcal{D}\alpha_2 \mathcal{D}\alpha_3. \tag{2.15}$$

More generally, $\mathrm{Vol}(\mathcal{G})$ is an integral over $N = \mathrm{dim}G$ local scalar fields for a gauge group G.

1-loop approximation Now, suppose the equation of motion admits the trivial solution ${A^a}_\mu = 0 = \Phi$, around which we perform a saddle point approximation for (2.6). Then at the quadratic (1 loop) level the vector and matter fields decouple:

$$Z_{\rm PI}^{\text{1-loop}} = Z_{\rm PI}^{\delta A} Z_{\rm PI}^{\delta\Phi}. \tag{2.16}$$

In the following, we focus on the vector part of the 1-loop path integral (with A^a understood as the fluctuations around the background)

$$Z_{\rm PI}^{\delta A} = \frac{1}{\mathrm{Vol}(\mathcal{G})} \int \prod_{a=1}^{\dim G} \mathcal{D}A^a e^{-\sum_{a=1}^{\dim G} S_E[A^a]}. \tag{2.17}$$

where $S_E[A^a]$ is simply a Maxwell action

$$S_E[A^a] = \frac{1}{4\mathrm{g}^2} \int_{S^{d+1}} F^a_{\mu\nu} F^{a,\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu. \tag{2.18}$$

A careful analysis of the Euclidean path integral for the U(1) theory on arbitrary manifolds has been presented in [66], where the authors point out the importance of taking care of zero modes, large gauge transformations and non-trivial bundles for consistency with locality and unitarity. In the following we will express $Z^0_{\rm PI}$ in terms of functional determinants and stress the relevant subtleties in our case of S^{d+1} along the way.

2.2.1 Transverse vector determinant and Jacobian

Geometric approach and change of variables

Since the path integrations over A^a in (2.17) are decoupled, we can focus on one of the factors, and we will suppress the index a. Traditional ways to proceed include Faddeev-Popov or BRST gauge fixing (as done in [66] for example). Here instead we take the "geometric approach" [63–65], which manifests its advantages when we deal with massless higher spin fields later. In this approach one changes the field variables by decomposing

$$A_\mu = A^T_\mu + \partial_\mu \chi \tag{2.19}$$

where A^T_μ is the transverse or on-shell part of A_μ satisfying $\nabla^\mu A^T_\mu = 0$, and χ is the longitudinal or pure gauge part of A_μ. Since S^{d+1} is compact, the scalar Laplacian has a normalizable constant $(0,0)$ mode, which must be excluded from the path integration for the change of variables (2.19) to be unique

$$\mathcal{D}A = J\mathcal{D}A^T\mathcal{D}'\chi \tag{2.20}$$

where prime denotes the exclusion of the $(0,0)$ mode. We will find the Jacobian J for the change of variables (2.19) below.

Action for A^T_μ

Because the gauge invariance of the action, χ simply drops out upon substituting (2.19)

$$S_E[A^T,\chi] = \frac{1}{2g^2}\int_{S^{d+1}}[A^T_\mu(-\nabla^2_{(1)}+d)A^\mu_T] \tag{2.21}$$

where $-\nabla^2_{(1)}$ is the trasnverse Laplacian on S^{d+1}. Now we expand A^T_μ in terms of spin-1 transverse spherical harmonics (see App.B.2 for their basic properties):

$$A^T_\mu = \sum_{n=1}^{\infty} c_{n,1}f_{n,\mu} \tag{2.22}$$

and the integration measure in our convention is

$$\mathcal{D}A^T = \prod_{n=1}^{\infty}\frac{dc_{n,1}}{\sqrt{2\pi}g}. \tag{2.23}$$

Performing the path integration over these modes we have

$$\int \mathcal{D}Ae^{-S_E[A]} = J\det\left(-\nabla^2_{(1)}+d\right)^{-1/2}\int \mathcal{D}'\chi \tag{2.24}$$

Jacobian

We find the Jacobian J by requiring consistency with the normalization condition

$$1 = \int \mathcal{D}A e^{-\frac{1}{2g^2}(A,A)} = \int J \mathcal{D}A^T \mathcal{D}'\chi e^{-\frac{1}{2g^2}(A^T+\nabla\chi, A^T+\nabla\chi)}. \tag{2.25}$$

Since A^T is transverse, we have

$$(A^T + \nabla\chi, A^T + \nabla\chi) = (A^T, A^T) + (\nabla\chi, \nabla\chi). \tag{2.26}$$

We can then path integrate A^T trivially. We expand χ in terms of scalar spherical harmonics:

$$\chi = \sum_{n=1}^{\infty} c_{n,0} f_n \tag{2.27}$$

with path integration measure

$$\mathcal{D}'\chi = \prod_{n=1}^{\infty} \frac{dc_{n,0}}{\sqrt{2\pi} g}. \tag{2.28}$$

Plugging this into (2.25) results in

$$J = \det{}'(-\nabla^2_{(0)})^{1/2}, \tag{2.29}$$

where the prime denotes the omission of the constant $(0,0)$ mode.

2.2.2 Residual group volume

Let us go back to the full 1-loop path integral (2.17). So far we have

$$Z_{\text{PI}}^{\delta A} = \frac{\int \prod_{a=1}^{\dim G} \mathcal{D}'\chi^a}{\text{Vol}(\mathcal{G})} \left(\frac{\det{}'(-\nabla^2_{(0)})^{1/2}}{\det\left(-\nabla^2_{(1)} + d\right)^{1/2}} \right)^{\dim G} \tag{2.30}$$

where we have restored the color index a. Now we focus on the factor

$$\frac{\int \prod_{a=1}^{\dim G} \mathcal{D}'\chi^a}{\mathrm{Vol}(\mathcal{G})}. \tag{2.31}$$

As explained above, the factor $\mathrm{Vol}(\mathcal{G})$ is theory dependent and is formally an integral over $N = \dim G$ **local** scalar fields

$$\mathrm{Vol}(\mathcal{G}) = \int \prod_{n=1}^{\dim G} \mathcal{D}\alpha_n. \tag{2.32}$$

In particular, the integral includes integrations over constant scalar modes. As explained in [66], the inclusion of zero modes is crucial for consistency with locality and unitarity. Thus, this factor does not cancel completely with the integrations over χ, leaving a factor

$$\frac{\int \prod_{a=1}^{\dim G} \mathcal{D}'\chi^a}{\mathrm{Vol}(\mathcal{G})} = \frac{1}{\mathrm{Vol}(G)_{\mathrm{PI}}}, \quad \mathrm{Vol}(G)_{\mathrm{PI}} \equiv \int \prod_{a=1}^{\dim G} \frac{d\alpha_0^a}{\sqrt{2\pi}\mathrm{g}}. \tag{2.33}$$

where α_0^a is the expansion coefficient of the $(0,0)$ mode of α^a (a is the color index)

$$\alpha^a = \sum_{n=0}^{\infty} \alpha_n^a f_n. \tag{2.34}$$

These constant scalar modes correspond to the gauge transformations that leave the background $A^\mu = 0$ invariant, or equivalently whose linear part is trivial. If the original full theory contains matter fields, these act non-trivially on the latter. G is therefore the group of **global** symmetries of the theory and $\mathrm{Vol}(G)_{\mathrm{PI}}$ is the volume of G. Note that the precise value of $\mathrm{Vol}(G)_{\mathrm{PI}}$ depends on the metric on G. We have been using a specific choice of metric

$$ds_{\mathrm{PI}}^2 = \frac{1}{2\pi\mathrm{g}^2} \int_{S^{d+1}} \mathrm{Tr}(\delta\alpha\delta\alpha) \tag{2.35}$$

induced by our convention for the path integral measure. Note that had we normalized the generators L^a in a different way: $L^a \to \lambda L^a$ (or equivalently choosing a different overall normalization for the trace in the action (2.11): $\text{Tr} \to \lambda^2 \text{Tr}$), the path integral describes the same physics if we rescale $\mathrm{g} \to \lambda \mathrm{g}$. In particular, the metric (2.35) remains the same. We want to relate the volume $\text{Vol}(G)_{\text{PI}}$ measured in this metric to a "canonical volume" $\text{Vol}(G)_{\text{can}}$, defined as follows. A general group element in G takes the form

$$e^{\theta \cdot \hat{L}} = e^{\theta^a \hat{L}^a} \tag{2.36}$$

where $\hat{L}^a$ are unit-normalized. We define $\text{Vol}(G)_{\text{can}}$ to be the volume of the space spanned by θ. In our convention, L^a are unit-normalized, and therefore the relation between the metric (2.35) (restricted to the subspace of trivial gauge transformations) and the canonical metric is simply

$$ds^2_{\text{PI}} = \frac{1}{2\pi \mathrm{g}^2} \sum_a (d\alpha_0^a)^2 = \frac{1}{2\pi \mathrm{g}^2} \sum_a \left(\frac{d\theta^a}{f_0}\right)^2 = \frac{\text{Vol}(S^{d+1})}{2\pi \mathrm{g}^2} ds^2_{\text{can}}, \quad ds^2_{\text{can}} \equiv d\theta \cdot d\theta. \tag{2.37}$$

Thus we can express the group volume as

$$\text{Vol}(G)_{\text{PI}} = \left(\frac{\text{Vol}(S^{d+1})}{2\pi \mathrm{g}^2}\right)^{\frac{\dim(G)}{2}} \text{Vol}(G)_{\text{can}} = \left(\frac{\text{Vol}(S^{d-1})}{d \mathrm{g}^2}\right)^{\frac{\dim(G)}{2}} \text{Vol}(G)_{\text{can}}, \tag{2.38}$$

where we have used $\text{Vol}(S^{d+1}) = \frac{2\pi}{d} \text{Vol}(S^{d-1})$ in the last step. The canonical volume $\text{Vol}(G)_{\text{can}}$ so defined is evidently independent of the coupling. To summarize, the full 1-loop path integral is

$$\begin{aligned} Z_{\text{PI}}^{\delta A} &= Z_{\text{G}} Z_{\text{Char}} \\ Z_{\text{G}} &= \frac{\gamma^{\dim G}}{\text{Vol}(G)_{\text{can}}}, \quad \gamma = \frac{\mathrm{g}}{\sqrt{(d-2)\text{Vol}(S^{d-1})}} \\ Z_{\text{Char}} &= (d(d-2))^{\frac{1}{2}\dim G} \left(\frac{\det'(-\nabla^2_{(0)})}{\det\left(-\nabla^2_{(1)} + d\right)} \right)^{\frac{1}{2}\dim G} \end{aligned} \tag{2.39}$$

In retrospect, the coupling dependence of the result is precisely encoded in the group volume factor $\text{Vol}(G)_{\text{PI}}$. In the $G = U(1)$ example, $\text{Vol}(G)_{\text{can}} = \text{Vol}(U(1))_c = 2\pi$, and the full 1 loop vector path integral is therefore

$$Z_{\rm PI}^{U(1)} = \frac{\mathrm{g}}{\sqrt{2\pi \mathrm{Vol}(S^{d+1})}} \frac{\det'(-\nabla_{(0)}^2)^{1/2}}{\det\left(-\nabla_{(1)}^2 + d\right)^{1/2}}, \tag{2.40}$$

which reproduces eq.(2.6) in [90]. For $G = SU(2)$, $\dim G = 3$ and $\mathrm{Vol}(G)_{\rm can} = 16\pi^2$, and thus

$$Z_{\rm PI}^{SU(N)} = \frac{1}{16\pi^2}\left(\frac{2\pi \mathrm{g}^2}{\mathrm{Vol}(S^{d+1})}\right)^{\frac{3}{2}} \left(\frac{\det'(-\nabla_{(0)}^2)}{\det\left(-\nabla_{(1)}^2 + d\right)}\right)^{3/2}. \tag{2.41}$$

Local gauge algebra, global symmetry and invariant bilinear form

For the later discussions on spin 2 and massless higher spin fields, and to make connection with the work in [68], we offer another perspective for the non-abelian case.

Local gauge algebra Recall that the original Yang-Mills action (2.11) is invariant under the full non-linear infinitesimal gauge transformations

$$\begin{aligned} \delta_\alpha A_\mu &= \delta_\alpha^{(0)} A_\mu + \delta_\alpha^{(1)} A_\mu \\ \delta_\alpha^{(0)} A_\mu &= \partial_\mu \alpha, \qquad \delta_\alpha^{(1)} A_\mu = [A_\mu, \alpha]. \end{aligned} \tag{2.42}$$

Here the superscript (n) denotes the power in fields. This generates an algebra

$$\delta_\alpha \delta_{\alpha'} A_\mu - \delta_{\alpha'} \delta_\alpha A_\mu = \delta_{[[\alpha,\alpha']]} A_\mu \tag{2.43}$$

where we have defined a bracket $[[\cdot,\cdot]]$ on the space of gauge parameters, which in our convention is equal to the negative of the matrix commutator[1]

$$[[\alpha,\alpha']] = -[\alpha,\alpha']. \tag{2.44}$$

Global symmetry algebra from the gauge algebra The constant $(0,0)$ modes $\bar{\alpha}$ generate background ($A_\mu = 0$) preserving gauge transformations satisfying

$$\delta^{(0)}_{\bar{\alpha}} = 0, \tag{2.45}$$

which form a subalgebra $\mathfrak{g}$ of the local gauge algebra, with the bracket $[[\cdot,\cdot]]$ naturally inherited from the local gauge algebra

$$[[\bar{\alpha},\bar{\alpha}']] = -[\bar{\alpha},\bar{\alpha}']. \tag{2.46}$$

This global symmetry algebra $\mathfrak{g}$ is clearly isomorphic to the original Lie algebra (2.10). On $\mathfrak{g}$, the path integral metric (2.35) corresponds to the bilinear form with a specific normalization:

$$\langle\bar{\alpha}|\bar{\alpha}'\rangle_{\text{PI}} = \frac{1}{2\pi \mathrm{g}^2}\int_{S^{d+1}} \bar{\alpha}^a\bar{\alpha}'^a = \frac{\text{Vol}(S^{d+1})}{2\pi\mathrm{g}^2}\bar{\alpha}^a\bar{\alpha}'^a. \tag{2.47}$$

We define a theory independent "canonical" invariant bilinear form $\langle\cdot|\cdot\rangle_{\text{c}}$ on $\mathfrak{g}$ as follows.

1. Pick a basis M^a of $\mathfrak{g}$ such that they satisfy the same commutation relation as L^a: $[[M^a, M^b]] = f^{abc}M^c$. This fixes the relative normalizations of M^a.

[1]One should keep in mind that $[[\cdot,\cdot]]$ is defined using the gauge transformations of A_μ, whose precise form depends on the normalization conventions, while the commutator on the right hand side is the matrix commutator $[A,B] = AB - BA$. Had we normalized A_μ canonically, so that the action takes the form $-\frac{1}{4}\int_{S^{d+1}}\text{Tr}F^2$, the gauge transformations will be instead $\delta_\alpha A_\mu = \partial_\mu\alpha + \mathrm{g}[A_\mu,\alpha]$ and the local gauge algebra will become $[\delta_\alpha,\delta_{\alpha'}] = \delta_{-\mathrm{g}[\alpha,\alpha']}$ and the bracket will read $[[\bar{\alpha},\bar{\alpha}']] = -\mathrm{g}[\alpha,\alpha']$.

2. Fix the overall normalization of $\langle\cdot|\cdot\rangle_{\rm c}$ by requiring M^a to be unit-normalized:

$$\langle M^a|M^b\rangle_{\rm c} = \delta^{ab} \tag{2.48}$$

In the current case, this means that we should take $M^a = L^a$ and

$$\langle\alpha|\alpha'\rangle_{\rm c} = \bar{\alpha}^a\bar{\alpha}'^a. \tag{2.49}$$

Comparing this with (2.47), we see that the path integral and canonical metrics are related as in (2.37), leading to the same result (2.38).

2.3 Review of massless spin 2

Next we review the computation for linearized Einstein gravity on S^{d+1}, which has a long and dramatic history [50–59]. The Euclidean path integral for a massless spin-2 particle on S^{d+1} is

$$Z_{\rm PI} = \frac{1}{\mathrm{Vol}(\mathcal{G})}\int \mathcal{D}h e^{-S[h]} \tag{2.50}$$

where the action is[2]

$$S[h] = \frac{1}{2\mathrm{g}^2}\int_{S^{d+1}} h^{\mu\nu}\Big[(-\nabla^2+2)h_{\mu\nu} + 2\nabla_{(\mu}\nabla^\lambda h_{\nu)\lambda} + g_{\mu\nu}(\nabla^2 {h_\lambda}^\lambda - 2\nabla^\sigma\nabla^\lambda h_{\sigma\lambda}) + (D-3)g_{\mu\nu}{h_\lambda}^\lambda\Big], \tag{2.51}$$

where $\mathrm{g} = \sqrt{32\pi G_N}$. (2.51) is invariant under the linearized diffeomorphisms[3]

$$h_{\mu\nu} \to h_{\mu\nu} + \sqrt{2}\nabla_{(\mu}\Lambda_{\nu)} = h_{\mu\nu} + \frac{1}{\sqrt{2}}(\nabla_\mu\Lambda_\nu + \nabla_\nu\Lambda_\mu). \tag{2.52}$$

[2]This is obtained by expanding $g_{\mu\nu} = g^{S^{d+1}}_{\mu\nu} + h_{\mu\nu}$ in the Einstein-Hilbert action $\frac{1}{16\pi G_N}\int_{S^{d+1}}(2\Lambda - R)$.

[3]The insertion of the factor $\frac{1}{\sqrt{2}}$ is for later convenience.

The volume factor Vol($\mathcal{G}$) is the volume of the space of diffeomorphisms inserted to compensate for the over-counting of gauge equivalent orbits connected by (2.52).

Change of variables

As in the case of massless vectors, we decompose $h_{\mu\nu}$ as

$$h_{\mu\nu} = h^{\text{TT}}_{\mu\nu} + \frac{1}{\sqrt{2}}(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) + \frac{g_{\mu\nu}}{\sqrt{d+1}}\tilde{h} \tag{2.53}$$

where $h^{\text{TT}}_{\mu\nu}$ is the transverse-traceless part of $h_{\mu\nu}$ satisfying $\nabla^\lambda h_{\lambda\mu} = 0 = h^\lambda{}_\lambda$, ξ_μ is the pure gauge part of $h_{\mu\nu}$, and $\tilde{h}$ is the trace of $h_{\mu\nu}$. For (2.53) to be unique, we require ξ_ν to be orthogonal to all Killing vectors (KVs) on S^{d+1}

$$(\xi, \xi^{\text{KV}}) = 0, \qquad \nabla_\mu \xi^{\text{KV}}_\nu + \nabla_\nu \xi^{\text{KV}}_\mu = 0 \tag{2.54}$$

and $\tilde{h}$ to be orthogonal to divergence of the rest of all conformal Killing vectors (CKVs)

$$(\tilde{h}, \nabla \cdot \xi^{\text{CKV}}) = 0, \qquad \nabla_\mu \xi^{\text{CKV}}_\nu + \nabla_\nu \xi^{\text{CKV}}_\mu = \frac{1}{2(d+1)} g_{\mu\nu} \nabla^\lambda \xi^{\text{CKV}}_\lambda. \tag{2.55}$$

The path integral measure then becomes

$$\mathcal{D}h = J \mathcal{D}h^{\text{TT}} \mathcal{D}'\xi \mathcal{D}'\tilde{h} \tag{2.56}$$

where the Jacobian J will be found below. The primes indicate that we exclude the integrations over the $(1,1)$ and $(1,0)$ modes excluded due to conditions (2.54) and (2.55).

2.3.1 Transverse tensor and trace mode determinants

Action for $h^{TT}_{\mu\nu}$

Due to the gauge invariance (2.52), we have

$$S[h] = S[h^{\mathrm{TT}} + \tilde{h}] = S[h^{\mathrm{TT}}] + S[\tilde{h}]. \tag{2.57}$$

$S[h^{\mathrm{TT}}]$ can be easily obtained as

$$S[h^{\mathrm{TT}}] = \frac{1}{2\mathrm{g}^2} \int_{S^{d+1}} h^{\mathrm{TT}}_{\mu\nu}(-\nabla^2_{(2)} + 2)h^{\mu\nu}_{\mathrm{TT}}. \tag{2.58}$$

where $-\nabla^2_{(2)}$ is the spin-2 STT Laplacian. The integration over h^{TT} thus gives

$$Z_h^{\mathrm{TT}} = \int \mathcal{D}h^{\mathrm{TT}} e^{-S[h^{\mathrm{TT}}]} = \det\Big(-\nabla^2_{(2)} + 2\Big)^{-1/2}. \tag{2.59}$$

Action for $\tilde{h}$ and the conformal factor problem

Similarly, after a bit more work, the quadratic action for $\tilde{h}$ can be obtained as

$$\begin{aligned} S[\tilde{h}] &= -\frac{d(d-1)}{2(d+1)\mathrm{g}^2} \int_{S^{d+1}} \tilde{h}(-\nabla^2_{(0)} - (d+1))\tilde{h} \\ &= -\frac{d(d-1)}{2(d+1)\mathrm{g}^2} \sum_{n\neq 1} (n(n+d) - (d+1))c^2_{n,0} \end{aligned} \tag{2.60}$$

where in the second line we have inserted the mode expansion

$$\tilde{h} = \sum_{n\neq 1} c_{n,0} f_n, \qquad (f_n, f_m) = \delta_{n,m}. \tag{2.61}$$

Here the sum runs over the spectrum of the scalar Laplacian except the $(1,0)$ modes, which corresponds to the CKVs. Notice that (2.60) has a wrong overall sign for all positive modes of the operator $-\nabla^2_{(0)} - (d+1)$. This is the well-known conformal factor problem [72] in Euclidean gravity method. We follow the standard prescription: we replace $c_{n,0} \to i c_{n,0}$ [4] for all $n \geq 2$, which leads to the change in the path integral measure

$$\mathcal{D}'\tilde{h} = \prod_{n\neq 1} \frac{dc_{n,0}}{\sqrt{2\pi}\mathrm{g}} \to \left(\prod_{n=2}^{\infty} i\right) \prod_{n\neq 1} \frac{dc_{n,0}}{\sqrt{2\pi}\mathrm{g}} = i^{-d-3} \left(\prod_{n=0}^{\infty} i\right) \prod_{n\neq 1} \frac{dc_{n,0}}{\sqrt{2\pi}\mathrm{g}}. \tag{2.62}$$

The factor in the last step runs through the spectrum of $-\nabla^2_{(0)}$ and is thus a local infinite constant that can be absorbed into bare couplings. Doing this the path integral becomes

$$\begin{aligned} Z_{\tilde{h}} &= \int \mathcal{D}'\tilde{h} e^{S[\tilde{h}]} = i^{-d-3} Z^+_{\tilde{h}} Z^-_{\tilde{h}} \\ Z^+_{\tilde{h}} &= \int \mathcal{D}^+\tilde{h} e^{-\frac{d(d-1)}{2(d+1)\mathrm{g}^2}\int_{S^{d+1}} \tilde{h}(-\nabla^2_{(0)}-(d+1))\tilde{h}} \\ Z^-_{\tilde{h}} &= \int \mathcal{D}^-\tilde{h} e^{\frac{d(d-1)}{2(d+1)\mathrm{g}^2}\int_{S^{d+1}} \tilde{h}(-\nabla^2_{(0)}-(d+1))\tilde{h}} \end{aligned} \tag{2.63}$$

where $\pm$ indicate the contribution from positive and negative modes respectively. The overall phase factor i^{-d-3} was first obtained by Polchinski [59]. Later we will see the generalization of this phase factor for all massless higher spin fields.

2.3.2 Jacobian

Again, we find the Jacobian J by requiring consistency with the normalization condition

$$1 = \int \mathcal{D}h e^{-\frac{1}{2\mathrm{g}^2}(h,h)}. \tag{2.64}$$

Since h^{TT} is transverse and traceless, we have

$$(h,h) = (h^{\mathrm{TT}}, h^{\mathrm{TT}}) + (\sqrt{2}\nabla\xi + \frac{g\tilde{h}}{\sqrt{d+1}}, \sqrt{2}\nabla\xi + \frac{g\tilde{h}}{\sqrt{d+1}}). \tag{2.65}$$

To proceed we separate $\xi_\mu = \xi'_\mu + \xi^{\mathrm{CKV}}_\mu$, where ξ^{CKV}_μ is a linear combination of the CKVs and ξ'_μ is the part of ξ_μ that is orthogonal to the CKVs, that is $(\xi', \xi^{\mathrm{CKV}}) = 0$. Note that while $g\tilde{h}$ is

[4] The sign in front of the i is a matter of convention.

orthogonal to ξ_μ^{CKV} because of (2.55), $g\tilde{h}$ and $\nabla\xi'$ are not orthogonal to each other. To remove the off-diagonal terms, we shift

$$\tilde{h}' = \tilde{h} + \sqrt{\frac{2}{d+1}}\nabla^\lambda \xi'_\lambda. \tag{2.66}$$

Since it is just a shift, the Jacobian is trivial. It is then easy to compute

$$(\sqrt{2}\nabla\xi + \frac{g\tilde{h}}{\sqrt{d+1}}, \sqrt{2}\nabla\xi + \frac{g\tilde{h}}{\sqrt{d+1}}) = (\tilde{h}', \tilde{h}') + \frac{1}{2}(K\xi', K\xi') + 2(\nabla\xi^{\mathrm{CKV}}, \nabla\xi^{\mathrm{CKV}}) \tag{2.67}$$

where we have defined the differential operator

$$(K\xi)_{\mu\nu} \equiv \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{2}{d+1} g_{\mu\nu}\nabla^\lambda \xi_\lambda. \tag{2.68}$$

Now the integrations over h^{TT} and $\tilde{h}'$ become trivial. To proceed, we first simplify

$$(K\xi', K\xi') = 2\int_{S^{d+1}} \left[\xi'^\nu\Big(-\nabla^2 - d\Big)\xi'_\nu - \xi'^\nu\Big(\frac{d-1}{d+1}\nabla_\nu\nabla^\lambda\xi'_\lambda\Big)\right]. \tag{2.69}$$

Then we decompose ξ' into its transverse and longitudinal parts: $\xi'_\nu = \xi^T_\nu + \nabla_\nu\sigma$. Once again this change of variables leads to a Jacobian factor which is easily found as before. With this decomposition we can further simplify

$$\frac{1}{2}(K\xi', K\xi') = S[\xi^T] + \frac{2d}{d+1}S[\sigma], \tag{2.70}$$

where

$$S[\xi^T] = \int_{S^{d+1}} \xi^T_\nu(-\nabla^2_{(1)} - d)\xi^\nu_T, \qquad S[\sigma] = \int_{S^{d+1}} \sigma(-\nabla^2_{(0)})(-\nabla^2_{(0)} - (d+1))\sigma. \tag{2.71}$$

We therefore arrive at

$$
\begin{aligned}
J =& \frac{W_\sigma^+}{Y_\xi^{\mathrm{T}} Y_\sigma^+} \frac{1}{Y_\xi^{\mathrm{CKV}}} \\
Y_\xi^{\mathrm{T}} =& \int \mathcal{D}'\xi^T e^{-\frac{1}{2g^2}(\xi^T,(-\nabla^2_{(1)}-d)\xi^T)} \\
Y_\sigma^+ =& \int \mathcal{D}^+\sigma e^{-\frac{1}{2g^2}\frac{2d}{d+1}(\sigma,(-\nabla^2_{(0)})(-\nabla^2_{(0)}-(d+1))\sigma)} \\
W_\sigma^+ =& \int \mathcal{D}^+\sigma e^{-\frac{1}{2g^2}(\sigma,(-\nabla^2_{(0)})\sigma)} \\
Y_\xi^{\mathrm{CKV}} =& \int \mathcal{D}\xi^{\mathrm{CKV}} e^{-\frac{1}{g^2}(\nabla\xi^{\mathrm{CKV}},\nabla\xi^{\mathrm{CKV}})}
\end{aligned}
\tag{2.72}
$$

Here W_σ^+ is the Jacobian corresponding to the change of variables $\{\xi'_\nu\} \to \{\xi^T_\nu + \nabla_\nu\sigma\}$. The +'s denote the positive modes for the operator $(-\nabla^2_{(0)} - (d+1))$ [5].

2.3.3 Residual group volume

As in the massless vector case, we have a factor

$$
\frac{\int \mathcal{D}'\xi}{\mathrm{Vol}(\mathcal{G})}
\tag{2.73}
$$

in the path integral. Here the factor $\mathrm{Vol}(\mathcal{G})$ is a path integral over a local vector field α_μ

$$
\mathrm{Vol}(\mathcal{G}) = \int \mathcal{D}\alpha.
\tag{2.74}
$$

This does not cancel completely with the integration over ξ_μ, and we are left with a factor (restoring the label a for degenerate modes with same quantum number $(1,1)$)

$$
\frac{\int \mathcal{D}'\xi}{\mathrm{Vol}(\mathcal{G})} = \frac{1}{\mathrm{Vol}(G)_{\mathrm{PI}}}, \quad \mathrm{Vol}(G)_{\mathrm{PI}} \equiv \int \prod_{a=1}^{\frac{(d+1)(d+2)}{2}} \frac{d\alpha_{1,1}^{(a)}}{\sqrt{2\pi}g}.
\tag{2.75}
$$

[5]The zero modes of the operator $(-\nabla^2_{(0)} - (d+1))$ are excluded because σ satisfies $(\sigma, f_0) = 0 = (\sigma, \nabla\xi^{\mathrm{CKV}})$.

where $\alpha_{1,1}^{(a)}$ is the expansion coefficient in the expansion

$$\alpha_\mu = \sum_{n=1}^{\infty} \alpha_{n,1} f_{n,\mu} + \sum_{n=1}^{\infty} \alpha_{n,0} \hat{T}^{(0)}_{n,\mu}. \tag{2.76}$$

These $(1,1)$ modes are diffeomorphisms that leave the background S^{d+1} metric invariant, so they in fact correspond to the Killing vectors of S^{d+1}. G is therefore the isometry group $SO(d+2)$ of S^{d+1}. As in the massless vector case, we want to relate $\mathrm{Vol}(G)_{\mathrm{PI}}$ to a canonical volume, following the argument in Sec.2.2.2.

Local gauge algebra Recall that the original Einstein-Hilbert action is invariant under non-linear diffeomorphisms generated by any vector field $\alpha = \frac{1}{\sqrt{2}} \alpha^\mu \partial_\mu$, which reads

$$\begin{aligned} \delta_\alpha h_{\mu\nu} =& \delta_\alpha^{(0)} h_{\mu\nu} + \delta_\alpha^{(1)} h_{\mu\nu} + O(h^2) \\ \delta_\alpha^{(0)} h_{\mu\nu} =& \frac{1}{\sqrt{2}} (\nabla_\mu \alpha_\nu + \nabla_\nu \alpha_\mu) \\ \delta_\alpha^{(1)} h_{\mu\nu} =& \frac{1}{\sqrt{2}} (\alpha^\rho \nabla_\rho h_{\mu\nu} + \nabla_\mu \alpha^\rho h_{\rho\nu} + \nabla_\nu \alpha^\rho h_{\mu\rho}), \end{aligned} \tag{2.77}$$

where the superscript (n) again denotes the power in fields. This generates the algebra

$$[\delta_\alpha, \delta_{\alpha'}] = \delta_{[[\alpha,\alpha']]}. \tag{2.78}$$

In this case, the bracket is proportional to the usual Lie derivative[6]

$$[[\alpha, \alpha']] = -\frac{1}{\sqrt{2}} [\alpha, \alpha']_L, \qquad [\alpha, \alpha']_L = (\alpha^\mu \partial_\mu \alpha'^\nu - \alpha'^\mu \partial_\mu \alpha^\nu) \partial_\nu. \tag{2.79}$$

[6]If we had worked with canonical normalization, obtained by replacing $h_{\mu\nu} \to \mathrm{g} h_{\mu\nu}$, the bracket will read instead $[[\alpha, \alpha']] = -\frac{\mathrm{g}}{\sqrt{2}} [\alpha, \alpha']_L = -\sqrt{16\pi G_N} [\alpha, \alpha']_L$. This relation can be viewed as a *definition* of the Newton constant G_N in any gauge theory with a massless spin 2 field.

Isometry algebra from the local gauge algebra The background (S^{d+1}) preserving gauge transformations or isometries generated by the Killing vectors satisfying

$$\delta^{(0)}_{\bar{\alpha}} = 0 \tag{2.80}$$

and form a subalgebra of the local gauge algebra, which inherits a bracket from the latter

$$[[\bar{\alpha}, \bar{\alpha}']] = -\frac{1}{\sqrt{2}}[\bar{\alpha}, \bar{\alpha}']_L. \tag{2.81}$$

To define the canonical volume, we again first find a set of generators M_{IJ} that satisfy the standard $so(d+2)$ commutation relation under the bracket (2.81):

$$[[M_{IJ}, M_{KL}]] = \eta_{JK}M_{IL} - \eta_{JL}M_{IK} + \eta_{IL}M_{JK} - \eta_{IK}M_{JL}. \tag{2.82}$$

One such basis is $M_{IJ} = -\sqrt{2}(X_I\partial_{X^J} - X_J\partial_{X^I})$ where $X^I X_I = 1, X^I \in \mathbb{R}^{d+2}, I = 1\cdots d+2$ are the coordinates of on S^{d+1} represented in the ambient space. Its norm in the invariant bilinear form induced by the path integral is (it suffices to consider only one of the generators)

$$\langle M_{12}|M_{12}\rangle_{\text{PI}} = \frac{1}{2\pi \text{g}^2}\int_{S^{d+1}}(M_{12})^{IJ}(M_{12})_{IJ} = \frac{2}{2\pi \text{g}^2}\int_{S^{d+1}}(X_1^2 + X_2^2) = \frac{2}{2\pi \text{g}^2}\frac{2}{d+2}\text{Vol}(S^{d+1}). \tag{2.83}$$

Since the canonical bilinear form is defined such that $\langle M_{12}|M_{12}\rangle_{\text{c}} = 1$, the path integral metric on G is related to the canonical metric as

$$ds^2_{\text{PI}} = \frac{2}{2\pi \text{g}^2}\frac{2}{d+2}\text{Vol}(S^{d+1})ds^2_{\text{can}} = \frac{1}{8\pi G_N}\frac{\text{Vol}(S^{d-1})}{d(d+2)}ds^2_{\text{can}} \tag{2.84}$$

where we have used $\mathrm{Vol}(S^{d+1}) = \frac{2\pi}{d}\mathrm{Vol}(S^{d-1})$ and substituted $\mathrm{g} = \sqrt{32\pi G_N}$ in the last step. Therefore

$$\mathrm{Vol}(G)_{\mathrm{PI}} = \left(\frac{1}{8\pi G_N}\frac{\mathrm{Vol}(S^{d-1})}{d(d+2)}\right)^{\frac{(d+1)(d+2)}{4}} \mathrm{Vol}(G)_{\mathrm{can}}. \tag{2.85}$$

The canonical volume $\mathrm{Vol}(G)_{\mathrm{can}} = \mathrm{Vol}(SO(d+2))_{\mathrm{can}}$ is well-known[7]:

$$\mathrm{Vol}(SO(d+2))_c = \prod_{n=1}^{d+1} \mathrm{Vol}(S^n) = \prod_{n=1}^{d+1} \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \tag{2.86}$$

2.3.4 Final result

So far we have

$$Z_{\mathrm{PI}} = \frac{i^{-d-3}}{\mathrm{Vol}(G)_{\mathrm{PI}}}\left(\frac{Z_h^{\mathrm{TT}}}{Y_\xi^{\mathrm{T}}}\right)\left(\frac{Z_{\tilde{h}}^{+} W_\sigma^{+}}{Y_\sigma^{+}}\right)\frac{Z_{\tilde{h}}^{-}}{Y_\xi^{\mathrm{CKV}}}. \tag{2.87}$$

Note that the factor

$$\frac{Z_h^{\mathrm{TT}}}{Y_\xi^{\mathrm{T}}} = \frac{\det'(-\nabla_{(1)}^2 - d)^{1/2}}{\det\left(-\nabla_{(2)}^2 + 2\right)^{1/2}} \tag{2.88}$$

is the usual ratio of determinants. Next, the factors in the second bracket in (2.87) cancel up to an infinite product

$$\frac{Z_{\tilde{h}}^{+} W_\sigma^{+}}{Y_\sigma^{+}} = \int \mathcal{D}^{+}\tilde{h}\, e^{-\frac{d-1}{4\mathrm{g}^2}\int_{S^{d+1}} \tilde{h}^2} = \prod_{n=2}^{\infty}\left(\frac{d-1}{2}\right)^{-\frac{D_{n,0}^{d+2}}{2}} = \left(\frac{d-1}{2}\right)^{\frac{d+3}{2}} \prod_{n=0}^{\infty}\left(\frac{d-1}{2}\right)^{-\frac{D_{n,0}^{d+2}}{2}}, \tag{2.89}$$

where in the last line we have complete the product so that it runs through the spectrum of the scalar Laplacian. The infinite product can then be absorbed into bare couplings. Finally, the factors in

[7]This follows from the fact that $SO(n+1)/SO(n) = S^n$, which implies that $\mathrm{Vol}(SO(n+1)) = \mathrm{Vol}(SO(n))\mathrm{Vol}(S^n)$

the last bracket in (2.87) can be explicitly evaluated to be

$$Z_{\tilde{h}}^{-} = \left(\frac{1}{d(d-1)}\right)^{1/2}, \quad Y_{\xi}^{\rm CKV} = 2^{-\frac{d+2}{2}}. \tag{2.90}$$

Putting everything together, we conclude

$$\begin{aligned}
Z_{\rm PI} =& Z_G Z_{\rm Char}, \\
Z_G =& i^{-d-3} \frac{\gamma^{\frac{(d+1)(d+2)}{2}}}{\mathrm{Vol}(SO(d+2))_c}, \quad \gamma = \sqrt{\frac{8\pi G_N}{\mathrm{Vol}(S^{d-1})}} \\
Z_{\rm Char} =& \Big(d(d+2)\Big)^{\frac{(d+1)(d+2)}{4}} \frac{(d-1)^{\frac{d+2}{2}}}{(2d)^{1/2}} \frac{\det'(-\nabla_{(1)}^2 - d)^{1/2}}{\det\Big(-\nabla_{(2)}^2 + 2\Big)^{1/2}}.
\end{aligned} \tag{2.91}$$

As a check, we note that except for the inclusion of the phase factor i^{-d-3}, for $d = 3$ we agree exactly with the 1-loop part of (5.43) in [58][8].

2.4 Massless higher spin

Now we are ready for the 1-loop path integrals for higher spin (HS) theories on S^{d+1}. Although the equations of motion for this theory have been constructed [91–93], the full action from which these are derived remain elusive[9]. However, since the interactions are at least cubic, their 1-loop partition functions (around the trivial saddle) decouple into a product of free partition functions.

$$Z_{\rm PI}^{\rm HS} = \prod_s Z_{\rm PI}^{(s,m^2=0)}, \tag{2.92}$$

where $Z_{\rm PI}^{(s,m^2=0)}$ is the 1-loop path integral for a massless spin-s field to be described below. The precise range over s in the product depends on the specific higher spin theory we are interested in. In AdS, the determinant expressions for $Z_{\rm PI}^{(s,m^2=0)}$ are obtained in [96] and [97], which are

[8] Note that our expression agrees with the first line of (5.43) in [58], while the authors made an error in evaluating the determinants, so their second line is incorrect, as already noted in [73].

[9] See also [94, 95] for arguments against the existence for consistent interacting HS theories.

subsequently used in 1-loop tests of HS/CFT dualities [30–33]. In the following, we perform a careful computation for $Z_{\rm PI}^{(s,m^2=0)}$ on S^{d+1}, whose early stage has some overlap with [96]. In fact, the following can be viewed as a derivation for the AdS case as well, except that the latter does not contain the subtleties of phases and group volume that appear on S^{d+1}.[10]

2.4.1 Operator formalism

It is much simpler to carry out the entire computation in terms of generating functions, which significantly simplifies tensor manipulations. Here we adopt the convention of [98] but on S^{d+1}. In this formalism, the tensor structure of a totally symmetric spin-s field $\phi_{\mu_1\cdots\mu_s}$ in S^{d+1} is encoded in a constant auxiliary $(d+1)$-dimensional vector u^μ:

$$\phi_{(s)}(x) = \phi_{\mu_1\cdots\mu_s}(x) \longrightarrow \phi_s(x,u) \equiv \frac{1}{s!}\phi_{\mu_1\cdots\mu_s}(x)u^{\mu_1}\cdots u^{\mu_s}. \tag{2.93}$$

In the following we will suppress the position argument x, and interchangeably refer to a rank-s tensor with $\phi_{(s)}$ or its generating function $\phi_s(u)$. Since the original covariant derivative ∇_μ acts on both $\phi_{\mu_1\cdots\mu_s}$ and u^μ, we modify the covariant derivative as

$$\nabla_\mu \longrightarrow \nabla_\mu + \omega_\mu{}^{ab}u_a\frac{\partial}{\partial u^b}, \tag{2.94}$$

where $u^a = e_\mu{}^a u^\mu$ with vielbein $e_\mu{}^a(x)$ and $\omega_\mu{}^{ab}$ is the spin connection. With this modification the actions of covariant derivatives on u^μ offset each other, and we can work as if no derivative is acting on u^μ. In the following we will only work in the contracted variables $u^\mu = e^\mu{}_a u^a$ and the associated derivative $\partial_{u^\mu} = e_\mu{}^a\partial_{u^a}$. As a consequence of vielbein postulate we have

$$[\nabla_\mu, u^\nu] = 0 = [\nabla_\mu, \partial_{u^\nu}]. \tag{2.95}$$

In this formalism all tensor manipulations are translated to an operator calculus. For instance,

[10] This is because the modes that cause these subtleties are non-normalizable in AdS and are excluded from the beginning.

tensor contraction:

$$\phi_{\mu_1\cdots\mu_s}\chi^{\mu_1\cdots\mu_s} = s!\phi_s(\partial_u)\chi_s(u). \tag{2.96}$$

In particular, the inner product (B.7) is represented as

$$(\phi_s, \chi_s) = s!\int_{S^{d+1}} \phi_s(\partial_u)\chi_s(u). \tag{2.97}$$

List of operations:

$$\begin{aligned} &\text{divergence: } \nabla\cdot\partial_u, &&\text{sym. gradient: } u\cdot\nabla, &&\text{Laplacian: } \nabla^2, \\ &\text{sym. metric: } u^2, &&\text{trace: } \partial_u^2, &&\text{spin: } u\cdot\partial_u. \end{aligned} \tag{2.98}$$

One of the biggest advantages of this formalism is that we can work algebraically with these operators without explicitly referring to the tensor. For example, to define the de Donder operator, we can either state explicitly its action on a spin-s field $\phi_{(s)}$

$$\hat{D}\phi_{(s)} = \hat{D}\phi_{\mu_1\cdots\mu_s} = \nabla^\lambda\phi_{\mu_1\cdots\mu_{s-1}\lambda} - \frac{1}{2}\nabla_{(\mu_1}\phi_{\mu_2\cdots\mu_{s-1})\lambda}{}^{\lambda} \tag{2.99}$$

or simply in terms of its generating function

$$\hat{D}(\nabla, u, \partial_u) = \nabla\cdot\partial_u - \frac{1}{2}(u\cdot\nabla)(\partial_u^2). \tag{2.100}$$

In the following we will use these two kinds of notations interchangeably.

On S^{d+1}, the operators (2.98) satisfy the following operator algebra

$$[\nabla_\mu, \nabla_\nu] = u_\mu \partial_{u^\nu} - u_\nu \partial_{u^\mu} \tag{2.101}$$

$$[\nabla^2, u \cdot \nabla] = u \cdot \nabla(2u \cdot \partial_u + d) - 2u^2 \nabla \cdot \partial_u \tag{2.102}$$

$$[\nabla \cdot \partial_u, \nabla^2] = (2u \cdot \partial_u + d)\nabla \cdot \partial_u - 2u \cdot \nabla \partial_u^2 \tag{2.103}$$

$$[\nabla \cdot \partial_u, u \cdot \nabla] = \nabla^2 + u \cdot \partial_u (u \cdot \partial_u + d - 1) - u^2 \partial_u^2 \tag{2.104}$$

$$[\nabla \cdot \partial_u, u^2] = 2u \cdot \nabla \tag{2.105}$$

$$[\partial_u^2, u \cdot \nabla] = 2\nabla \cdot \partial_u \tag{2.106}$$

$$[\partial_u^2, u^2] = 2(d + 1 + 2u \cdot \partial_u) \tag{2.107}$$

where we have denoted $\partial_u^2 \equiv \partial_u \cdot \partial_u, u^2 \equiv u \cdot u$.

2.4.2 Fronsdal action on S^{d+1}

The 1-loop partition function for a free bosonic spin-s massless gauge field on S^{d+1} is

$$Z_{\text{PI}}^{(s)} = \frac{1}{\text{Vol}(\mathcal{G}_s)} \int \mathcal{D}\phi_{(s)} e^{-S[\phi_{(s)}]} \tag{2.108}$$

where the quadratic Fronsdal action [99] in the operator language is given by

$$S[\phi_{(s)}] = \frac{s!}{2\mathrm{g}_s^2} \int_{S^{d+1}} \phi_s(\partial_u) \Big(1 - \frac{1}{4} u^2 \partial_u^2\Big) \hat{\mathcal{F}}_s(\nabla, u, \partial_u) \phi_s(u) \tag{2.109}$$

with $\hat{\mathcal{F}}_s(\nabla, u, \partial_u)$ is the Fronsdal operator

$$\hat{\mathcal{F}}_s(\nabla, u, \partial_u) = -\nabla^2 + M_s^2 - u^2 \partial_u^2 + u \cdot \nabla \hat{D}(\nabla, u, \partial_u) \tag{2.110}$$

$$\hat{D}(\nabla, u, \partial_u) = \nabla \cdot \partial_u - \frac{1}{2}(u \cdot \nabla)(\partial_u^2), \tag{2.111}$$

where

$$M_s^2 = s - (s-2)(s+d-2) \tag{2.112}$$

and $\hat{D}$ is the de Donder operator. An s-dependent factor g_s^2 is inserted as an overall factor. Canonical normalization corresponds to setting $\mathrm{g}_s = 1$. We will choose a particular value for g_s when we discuss the issue of group volume. (2.109) is invariant under the gauge transformations

$$\phi_s(u) \mapsto \phi_s(u) + \frac{1}{\sqrt{s}} u \cdot \nabla \Lambda_{s-1}(u). \tag{2.113}$$

In this off-shell formalism $\phi_s(u)$ satisfies a double-tracelessness condition (trivial for $s \leq 3$)

$$(\partial_u^2)^2 \phi_s(u) = 0, \tag{2.114}$$

which implies that the gauge parameter $\Lambda_{(s-1)}$ must be traceless (imposed even for $s = 3$)

$$\partial_u^2 \Lambda_{s-1}(u) = 0. \tag{2.115}$$

The division by the gauge group volume $\mathrm{Vol}(\mathcal{G}_s)$ in (2.108) compensates for the overcounting of gauge equivalent configurations connected by (2.113).

Change of variables

To proceed, we change field variables

$$\phi_s(u) = \phi_s^{\mathrm{TT}}(u) + \frac{1}{\sqrt{s}} u \cdot \nabla \xi_{s-1}(u) + \frac{1}{\sqrt{2s(s-1)(d+2s-3)}} u^2 \chi_{s-2}(u). \tag{2.116}$$

Here $\phi_{(s)}^{\mathrm{TT}}$ is the transverse traceless piece of $\phi_{(s)}$ for which

$$\nabla \cdot \partial_u \phi_s^{\mathrm{TT}}(u) = 0 = \partial_u^2 \phi_s^{\mathrm{TT}}(u). \tag{2.117}$$

Next, $\xi_{(s-1)}$ is the symmetric traceless spin-$(s-1)$ gauge parameters which are required to be orthogonal to all spin-$(s-1)$ Killing tensors $\epsilon^{KT}_{(s-1)}$ (which generate trivial gauge transformations) so that it is uniquely fixed:

$$\partial_u^2 \xi_{s-1}(u) = 0, \quad (\xi_{(s-1)}, \epsilon^{KT}_{(s-1)}) = 0 \tag{2.118}$$

Finally, $\chi_{(s-2)}$ is the spin-$(s-2)$ piece which carries all the trace information of $\phi_{(s)}$. The double-tracelessness condition (2.114) implies that $\chi_{(s-2)}$ is traceless:

$$\partial_u^2 \chi_{s-2}(u) = 0. \tag{2.119}$$

Note that if $\epsilon^{CKT}_{(s-1)}$ is a conformal Killing tensor (CKT) satisfying

$$\hat{K}_s(\nabla, u, \partial_u)\epsilon^{CKT}_{s-1}(u) \equiv u \cdot \nabla \epsilon^{CKT}_{s-1}(u) - \frac{u^2}{d+2s-3}(\nabla \cdot \partial_u)\epsilon^{CKT}_{s-1}(u) = 0, \tag{2.120}$$

then any new set of variables related by the transformation

$$\xi_{s-1}(u) \longrightarrow \xi_{s-1}(u) + \epsilon^{CKT}_{s-1}(u) \tag{2.121}$$

$$\chi_{s-2}(u) \longrightarrow \chi_{s-2}(u) - \frac{u^2}{d+2s-3}(\nabla \cdot \partial_u)\epsilon^{CKT}_{s-1}(u) \tag{2.122}$$

will result in the same $\phi_{(s)}$. To uniquely fix $\chi_{(s-2)}$, we thus impose

$$(\chi_{(s-2)}, (\nabla \cdot \epsilon^{CKT})_{(s-2)}) = 0 \tag{2.123}$$

for all the spin-$(s-1)$ CKTs $\epsilon^{CKT}_{(s-1)}$. The path integral measure then becomes

$$\mathcal{D}\phi_s = J_{(s)} \mathcal{D}\phi^{\mathrm{TT}}_{(s)} \mathcal{D}'\xi_{(s-1)} \mathcal{D}'\chi_{(s-2)} \tag{2.124}$$

where the Jacobian $J_{(s)}$ will be found below. The primes indicate that we exclude the $(s-1,m)$ $(0\le m\le s-1)$ modes excluded due to conditions (2.118) and (2.123).

2.4.3 Quadratic actions for $\phi^{\rm TT}_{(s)}$ and $\chi_{(s-2)}$

Action for $S[\phi^{\rm TT}_{(s)}]$

The quadratic action for $\phi^{\rm TT}_{(s)}$ is

$$S[\phi^{\rm TT}_{(s)}]=\frac{s!}{2\mathrm{g}_s^2}\int_{S^{d+1}}\phi^{\rm TT}_s(\partial_u)(-\nabla^2_{(s)}+M_s^2)\phi^{\rm TT}_s(u)=\frac{1}{2\mathrm{g}_s^2}(\phi^{\rm TT}_{(s)},(-\nabla^2_{(s)}+M_s^2)\phi^{\rm TT}_{(s)}). \tag{2.125}$$

which leads to the path integral

$$Z^{(s)}_{\phi^{\rm TT}}=\int\mathcal{D}\phi^{\rm TT}_{(s)}e^{-\frac{1}{2\mathrm{g}_s^2}(\phi^{\rm TT}_{(s)},(-\nabla^2_{(s)}+M_s^2)\phi^{\rm TT}_{(s)})}=\det\Big(-\nabla^2_{(s)}+M_s^2\Big)^{-1/2} \tag{2.126}$$

Action for $S[\chi_{(s-2)}]$ and the HS conformal factor problem

From (2.109) we have

$$\begin{aligned}S[\chi_{(s-2)}]=&\frac{(s-2)!}{8\mathrm{g}_s^2}\int_{S^{d+1}}\chi_{s-2}(\partial_u)(\partial_u^2)\Big(1-\frac{1}{4}u^2\partial_u^2\Big)\hat{\mathcal{F}}_s(\nabla,u,\partial_u)u^2\chi_{s-2}(u)\\ =&-\frac{(s-2)!(d+2s-5)}{8(d+2s-3)\mathrm{g}_s^2}\int_{S^{d+1}}\chi_{s-2}(\partial_u)(\partial_u^2)\hat{\mathcal{F}}_s(\nabla,u,\partial_u)u^2\chi_{s-2}(u)\end{aligned} \tag{2.127}$$

where we have used (2.107) and the tracelessness of $\chi_{(s-2)}$ (2.119). Using the operator algebras, one easily finds that

$$\begin{aligned}&\hat{\mathcal{F}}_s(\nabla,u,\partial_u)u^2\\ =&u^2\Big(-\nabla^2-s(-1+d+s)+2\Big)+u^2(u\cdot\nabla)(\nabla\cdot\partial_u)-(d+2s-5)(u\cdot\nabla)^2+\cdots\end{aligned} \tag{2.128}$$

where and henceforth $\cdots$ denotes terms that will not contribute because of the tracelessness condition (2.119): $\partial_u^2 \chi_{s-2}(u) = 0$ or $\chi_{s-2}(\partial_u)u^2 = 0$. Then we have

$$
\begin{aligned}
(\partial_u^2)\hat{\mathcal{F}}_s(\nabla, u, \partial_u)u^2 =& 4(d+2s-4)\Big(-\nabla^2 - (s-1)(s+d-2) - 1\Big) \\
&- 2(d+2s-7)(u\cdot\nabla)(\nabla\cdot\partial_u) + \cdots . \qquad (2.129)
\end{aligned}
$$

Defining the differential operator

$$
\hat{Q}(\nabla, u, \partial_u) \equiv 2\frac{d+2s-4}{d+2s-3}\Big(-\nabla^2 - (s-1)(s+d-2) - 1\Big) - \frac{d+2s-7}{d+2s-3}(u\cdot\nabla)(\nabla\cdot\partial_u), \quad (2.130)
$$

the quadratic action for $\chi_{(s-2)}$ is simply

$$
S[\chi_{(s-2)}] = -\frac{d+2s-5}{4\mathrm{g}_s^2}(\chi_{(s-2)}, \hat{Q}\chi_{(s-2)}). \qquad (2.131)
$$

To proceed, we expand $\chi_{(s-2)}$ (see App.B.2 for the properties of the induced symmetric traceless spherical harmonics)

$$
\chi_{(s-2)} = \sum_{m=0}^{s-2} A_{s-2,m}\hat{T}^{(m)}_{s-2,(s-2)} + \sum_{m=0}^{s-2}\sum_{n=s}^{\infty} A_{n,m}\hat{T}^{(m)}_{n,(s-2)}, \qquad (2.132)
$$

where the modes $(n,m) = (s-1,m), 0 \le m \le s-2$ are excluded because of the condition (2.123). It is easy to verify that $\hat{Q}$ is negative for the modes in the first sum and positive in the second. This is the HS generalization of the conformal factor problem. To make the integrals converge, we replace $A_{n,m} \to iA_{n,m}$ for $0 \le m \le s-2, s \le n < \infty$, leading to the change in the path integral measure

$$
\mathcal{D}'\chi_{(s-2)} = \prod_{m=0}^{s-2}\frac{dA_{s-2,m}}{\sqrt{2\pi}\mathrm{g}_s}\prod_{m=0}^{s-2}\prod_{n=s}^{\infty}\frac{dA_{n,m}}{\sqrt{2\pi}\mathrm{g}_s} \to \left(\prod_{m=0}^{s-2}\prod_{n=s}^{\infty} i^{D^{d+2}_{n,m}}\right)\prod_{m=0}^{s-2}\frac{dA_{s-2,m}}{\sqrt{2\pi}\mathrm{g}_s}\prod_{m=0}^{s-2}\prod_{n=s}^{\infty}\frac{dA_{n,m}}{\sqrt{2\pi}\mathrm{g}_s}. \quad (2.133)
$$

We complete the product so that it runs through the spectrum for the unconstrained spin-$(s-2)$ Laplacian, i.e.

$$\prod_{m=0}^{s-2}\prod_{n=s}^{\infty} i^{D^{d+2}_{n,m}} = i^{-N^{\text{CKT}}_{s-2}-N^{\text{CKT}}_{s-1}+N^{\text{KT}}_{s-1}} \prod_{m=0}^{s-2}\prod_{n=s-2}^{\infty} i^{D^{d+2}_{m,n}} \tag{2.134}$$

where $N^{\text{CKT}}_s = \sum_{m=0}^{s} D^{d+2}_{s,m}$ and $N^{\text{KT}}_s = D^{d+2}_{s,s}$ are the number of spin-s CKTs and spin-s KTs respectively. The local infinite product can then be absorbed into bare couplings. The remaining phase factor is the HS generalization of the Polchinski's phase. We can then write the path integral over $\chi_{(s-2)}$ as

$$\begin{aligned} Z^{(s)}_{\chi} =& i^{-N^{\text{CKT}}_{s-2}-N^{\text{CKT}}_{s-1}+N^{\text{KT}}_{s-1}} Z^{(s)}_{\chi^+} Z^{(s)}_{\chi^-} \\ Z^{(s)}_{\chi^+} =& \int \mathcal{D}^+\chi_{(s-2)} e^{-\frac{d+2s-5}{4g_s^2}(\chi_{(s-2)},\hat{Q}\chi_{(s-2)})} \\ Z^{(s)}_{\chi^-} =& \int \mathcal{D}^-\chi_{(s-2)} e^{\frac{d+2s-5}{4g_s^2}(\chi_{(s-2)},\hat{Q}\chi_{(s-2)})} \end{aligned} \tag{2.135}$$

where the superscripts $\pm$ denotes integrations over the positive (negative) modes of $\hat{Q}$.

2.4.4 Jacobian

Again, we find the Jacobian in (2.124) by the normalization condition

$$\int \mathcal{D}\phi_{(s)} e^{-\frac{1}{2g_s^2}(\phi_{(s)},\phi_{(s)})} = 1. \tag{2.136}$$

We plug in (2.124) and (2.116) to find $J_{(s)}$. Notice that $\phi^{\text{TT}}_{(s)}$ is orthogonal to $g\chi_{(s-2)}$ and $\nabla\xi_{(s-1)}$ with respect to the inner product $(\cdot,\cdot)$; on the other hand, when ξ's are orthogonal to the spin-$(s-1)$ CKTs (denoted as ξ'), $g\chi_{(s-2)}$ and $\nabla\xi'_{(s-1)}$ are not orthogonal, and we remove the off-diagonal terms by shifting

$$\chi'_{s-2}(u) = \chi_{s-2}(u) + \sqrt{\frac{s(s-1)}{2(d+2s-3)}}(\nabla\cdot\partial_u)\xi'_{s-1}(u). \tag{2.137}$$

The Jacobian corresponding to this shift is trivial. We then have

$$(\phi_{(s)},\phi_{(s)}) = (\phi^{\rm TT}_{(s)},\phi^{\rm TT}_{(s)}) + (\chi'_{(s-2)},\chi'_{(s-2)}) + \frac{1}{s}(\hat{K}_s\xi'_{(s-1)},\hat{K}_s\xi'_{(s-1)}) + \frac{1}{s}(\nabla\xi^{CKT}_{(s-1)},\nabla\xi^{CKT}_{(s-1)}). \quad (2.138)$$

where $\hat{K}_s$ is the operator appearing in (2.120). It is useful to note that acting on any symmetric traceless tensor $\epsilon_{(s-1)}$,

$$\partial_u^2\hat{K}_s(\nabla,u,\partial_u)\epsilon_{s-1}(u) = 0 = \hat{K}_s(\nabla,\partial_u,u)\epsilon_{s-1}(\partial_u)u^2. \quad (2.139)$$

The path integrals over $\phi^{\rm TT}_{(s)}$ and $\chi'_{(s-2)}$ are trivial, and therefore $J_{(s)}$ can be expressed as

$$J^{-1}_{(s)} = Y^{(s)}_{\xi'}Y^{(s)}_{\xi^{\rm CKT}} \quad (2.140)$$

$$Y^{(s)}_{\xi'} \equiv \int \mathcal{D}\xi'_{(s-1)}e^{-\frac{1}{2sg_s^2}(K\xi'_{(s-1)},K\xi'_{(s-1)})} \quad (2.141)$$

$$Y^{(s)}_{\xi^{\rm CKT}} \equiv \int \mathcal{D}\xi^{CKT}_{(s-1)}e^{-\frac{1}{2sg_s^2}(\nabla\xi^{CKT}_{(s-1)},\nabla\xi^{CKT}_{(s-1)})}. \quad (2.142)$$

Expressing $Y_{\xi'}$ in terms of functional determinants

To proceed, we use the operator algebra and (2.139) and simplify

$$\begin{aligned}\frac{1}{s}(\hat{K}_s\xi'_{(s-1)},\hat{K}_s\xi'_{(s-1)}) =& (\xi'_{(s-1)},\Big(-\nabla^2_{(s-1)} - (s-1)(s+d-2)\Big)\xi'_{(s-1)}) \\ &+ \frac{d+2s-5}{d+2s-3}(\xi'_{(s-1)},-\nabla\nabla\cdot\xi'_{(s-1)}).\end{aligned} \quad (2.143)$$

We then perform the change of variables

$$\xi'_{(s-1)} = \xi'^{\rm TT}_{(s-1)} + \hat{K}_{s-1}\sigma_{(s-2)}, \quad (2.144)$$

where $\xi'^{\rm TT}_{(s-1)}$ is the transverse traceless part of $\xi'_{(s-1)}$, $\sigma_{(s-2)}$ is a spin-$(s-2)$ symmetric traceless field and the differential operator $\hat{K}_{s-1}(\nabla,u,\partial_u)$ is defined in (2.120). We require $\sigma_{(s-2)}$ to be orthogonal to the kernel of $\hat{K}_{s-1}$, i.e. the spin-$(s-2)$ CKTs. Also, $\xi'^{\rm TT}_{(s-1)}$ and $\sigma_{(s-2)}$ are automatically orthogonal to the spin-$(s-1)$ CKTs.

Plugging in these, we have two decoupled pieces

$$\frac{1}{s}(\hat{K}_s\xi'_{(s-1)}, \hat{K}_s\xi'_{(s-1)}) = S[\xi'^{\mathrm{TT}}_{(s-1)}] + S[\sigma_{(s-2)}]. \tag{2.145}$$

Here the first term is the ghost action

$$S[\xi'^{\mathrm{TT}}_{(s-1)}] = (\xi'^{\mathrm{TT}}_{(s-1)}, \Big(-\nabla^2_{(s-1)} + m^2_{s-1,s} + M^2_{s-1}\Big)\xi'^{\mathrm{TT}}_{(s-1)}) \tag{2.146}$$

with M^2_{s-1} as defined in (2.112) and we have defined

$$m^2_{s,t} = (s-1-t)(d+s+t-3), \tag{2.147}$$

which is exactly the mass for a partially massless field with spin-s and depth t for $0 \le t \le s-1$. The second term in (2.145) is the action of a spin-$(s-2)$ field

$$S[\sigma_{(s-2)}] = (\hat{K}_{s-1}\sigma_{(s-2)}, \hat{\mathcal{P}}\hat{K}_{s-1}\sigma_{(s-2)}) \tag{2.148}$$

$$\hat{\mathcal{P}}(\nabla, u, \partial_u) = -\nabla^2_{(s-1)} - (s-1)(s+d-2) - \frac{d+2s-5}{d+2s-3}(u\cdot\nabla)(\nabla\cdot\partial_u) \tag{2.149}$$

To proceed, we commute $\hat{\mathcal{P}}$ and $\hat{K}_{s-1}$. This requires the relation

$$\nabla^2_{(s-1)}\hat{K}_{s-1}(\nabla, u, \partial_u) - \hat{K}_{s-1}(\nabla, u, \partial_u)\nabla^2_{(s-2)} = (d+2s-4)u\cdot\nabla + \cdots \tag{2.150}$$

and the commutator

$$\begin{aligned}&[(u\cdot\nabla)(\nabla\cdot\partial_u), \hat{K}_{s-1}(\nabla, u, \partial_u)]\\ =&[(u\cdot\nabla)(\nabla\cdot\partial_u), u\cdot\nabla] - \frac{1}{d+2s-5}[(u\cdot\nabla)(\nabla\cdot\partial_u), u^2\nabla\cdot\partial_u]\end{aligned} \tag{2.151}$$

which can be computed using

$$
\begin{aligned}
[(u\cdot\nabla)(\nabla\cdot\partial_u), u\cdot\nabla] =&(u\cdot\nabla)\Big(\nabla^2+(s-2)(s+d-3)\Big)+\cdots && (2.152)\\
[(u\cdot\nabla)(\nabla\cdot\partial_u), u^2\nabla\cdot\partial_u] =&2(u\cdot\nabla)^2(\nabla\cdot\partial_u)+\cdots && (2.153)
\end{aligned}
$$

where and henceforth $\cdots$ denotes terms that will not contribute to (2.148) because of the tracelessness condition (2.139) of the operator $\hat{K}_{s-1}$. We have also used the fact that $u\cdot\partial_u\sigma_{s-2}(u) = (s-2)\sigma_{s-2}(u)$. To briefly summarize,

$$
\begin{aligned}
&\hat{\mathcal{P}}(\nabla,u,\partial_u)\hat{K}_{s-1}(\nabla,u,\partial_u)\\
=&\hat{K}_{s-1}(\nabla,u,\partial_u)\hat{\mathcal{P}}(\nabla,u,\partial_u)-(d+2s-4)u\cdot\nabla\\
&+\frac{d+2s-5}{d+2s-3}(u\cdot\nabla)\left[\Big(-\nabla^2-(s-2)(s+d-3)\Big)+\frac{2}{d+2s-5}(u\cdot\nabla)(\nabla\cdot\partial_u)\right]+\cdots. \qquad (2.154)
\end{aligned}
$$

Now, observe that because of (2.139), $u\cdot\nabla$ can be replaced by the operator $\hat{K}_{s-1}$

$$
u\cdot\nabla = \hat{K}_{s-1}(\nabla,u,\partial_u)+\cdots \qquad (2.155)
$$

up to trace terms that do not contribute to (2.148). Therefore we have

$$
\hat{\mathcal{P}}(\nabla,u,\partial_u)\hat{K}_{s-1}(\nabla,u,\partial_u) = \hat{K}_{s-1}(\nabla,u,\partial_u)\hat{\mathcal{W}}(\nabla,u,\partial_u)+\cdots, \qquad (2.156)
$$

with

$$
\begin{aligned}
\hat{\mathcal{W}}(\nabla,u,\partial_u) =&\hat{\mathcal{P}}(\nabla,u,\partial_u)-(d+2s-4)\\
&+\frac{d+2s-5}{d+2s-3}\left[\Big(-\nabla^2-(s-2)(s+d-3)\Big)+\frac{2}{d+2s-5}(u\cdot\nabla)(\nabla\cdot\partial_u)\right] \qquad (2.157)
\end{aligned}
$$

Amazingly, one can show that this operator is exactly equal to $\hat{Q}$ defined in (2.130), that is

$\hat{\mathcal{W}}(\nabla, u, \partial_u) = \hat{Q}(\nabla, u, \partial_u)$. So we have found

$$S[\sigma_{(s-2)}] = (\hat{K}_{s-1}\sigma_{(s-2)}, \hat{K}_{s-1}\hat{Q}\sigma_{(s-2)}) = (\sigma_{(s-2)}, \hat{K}^\dagger_{s-1}\hat{K}_{s-1}\hat{Q}\sigma_{(s-2)}). \tag{2.158}$$

To conclude, we have

$$Y^{(s)}_{\xi'} = \frac{Y^{(s)}_{\xi^{\mathrm{TT}}} Y^{(s)}_{\sigma^+}}{W^{(s)}_{\sigma^+}} \tag{2.159}$$

$$Y^{(s)}_{\xi^{\mathrm{TT}}} \equiv \int \mathcal{D}\xi'^{\mathrm{TT}}_{(s-1)} e^{-\frac{1}{2g_s^2}\left(\xi'^{\mathrm{TT}}_{(s-1)}, \left(-\nabla^2_{(s-1)} + m^2_{s-1,s} + M^2_{s-1}\right)\xi'^{\mathrm{TT}}_{(s-1)}\right)} \tag{2.160}$$

$$Y^{(s)}_{\sigma^+} \equiv \int \mathcal{D}^+\sigma_{(s-2)} e^{-\frac{1}{2g_s^2}(\sigma_{(s-2)}, K^\dagger K\hat{Q}\sigma_{(s-2)})} \tag{2.161}$$

$$W^{(s)}_{\sigma^+} = \int \mathcal{D}^+\sigma_{(s-2)} e^{-\frac{1}{2g_s^2}(\sigma_{(s-2)}, K^\dagger K\sigma_{(s-2)})} \tag{2.162}$$

Here the superscript + emphasizes the fact that we are integrating over modes orthogonal to the spin-$(s-1)$ and spin-$(s-2)$ CKTs. In particular, this is the part of spectrum that coincides with the "+" integral in (2.135). Here $(W^{(s)}_{\sigma^+})^{-1}$ is the Jacobian associated with the change of variables (2.144).

2.4.5 Residual group volume

Recall that after the change of variables (2.116), the integration over the pure gauge modes ξ decoupled from the $\phi^{\mathrm{TT}}_{(s)}$ and $\chi_{(s-2)}$ path integrals, and we are left with a factor (we have restored the label a for degenerate modes with same quantum number $(s-1, s-1)$)

$$\frac{\int \mathcal{D}'\xi_{(s-1)}}{\mathrm{Vol}(\mathcal{G}_s)} = \frac{1}{\mathrm{Vol}(G_s)}, \quad \mathrm{Vol}(G_s) = \int \prod_{a=1}^{N^{\mathrm{KT}}_{s-1}} \frac{d\alpha^{(a)}_{s-1,s-1}}{\sqrt{2\pi}g_s}. \tag{2.163}$$

due to the integration over the spin-$(s-1)$ Killing tensor modes. This leads to a product in the original path integral (2.92):

$$\mathrm{Vol}(G)_{\mathrm{PI}} = \prod_s \mathrm{Vol}(G_s). \tag{2.164}$$

HS symmetries typically form infinite dimensional groups. Therefore there is an issue of making sense of (2.164), which we are not going to attempt in this paper.

HS invariant bilinear form Instead, we are going to do a more modest task. As in the warm-up examples, the volume $\mathrm{Vol}(G)_{\mathrm{PI}}$ is defined with a particular metric, namely

$$ds^2_{\mathrm{PI}} = \frac{1}{2\pi}\sum_s \frac{1}{\mathrm{g}_s^2} d\alpha^2_{s-1,s-1}. \tag{2.165}$$

Again we want to express this in terms of a canonical metric with respect to which we define a canonical volume $\mathrm{Vol}(G)_{\mathrm{can}}$. There are however complications compared to the massless spin-1 and spin-2 cases:

1. As opposed to the case for Yang-Mills or Einstein gravity, we do not know the full nonlinear actions for Vasiliev theories that give rise to the interacting equations of motion and the full nonlinear gauge transformations in the metric-like formalism. This implies that we do not know the full local HS gauge algebra. Fortunately, the global part of the algebra does not require this knowledge, but only the lowest order ones, which only requires the information of the cubic couplings.

2. Another complication is that since HS symmetries mix different spins, the HS invariant bilinear form depends on the relative normalizations of fields withe different spins in the action. Once this is fixed, the bilinear form is uniquely determined up to an overall normalization.

All of these have been worked out in the case of a negative cosmological constant [68]. To go to the case of a positive cosmological constant is a simple matter of analytic continuation. In App.B.3, we translate the relevant results from [68] to the case of S^{d+1}. The final result is that upon choosing

$$\mathrm{g}_s^2 = s!, \tag{2.166}$$

the HS invariant bilinear form is determined to be

$$\langle \bar{\alpha}_1 | \bar{\alpha}_2 \rangle_{\text{can}} = \frac{8\pi G_N}{\text{Vol}(S^{d-1})} \sum_s (d+2s-2)(d+2s-4) \, \langle \bar{\alpha}_{1,(s-1)} | \bar{\alpha}_{2,(s-1)} \rangle_{\text{PI}} \tag{2.167}$$

where the overall normalization is again fixed by requiring the canonical spin-2 generators to be unit-normalized with respect to (2.167). This implies that the group volume (2.164) is related to the canonical volume as

$$\text{Vol}(G)_{\text{PI}} = \text{Vol}(G)_{\text{can}} \prod_s \left(\frac{\text{Vol}(S^{d-1})}{8\pi G_N} \frac{1}{(d+2s-2)(d+2s-4)} \right)^{\frac{N^{\text{KT}}_{s-1}}{2}} . \tag{2.168}$$

2.4.6 Final result

So far we have

$$Z_{\text{PI}}^{\text{HS}} = \frac{i^{-P}}{\text{Vol}(G)_{\text{PI}}} \prod_s \left(\frac{Z^{(s)}_{\phi^{\text{TT}}}}{Y^{(s)}_{\xi^{\text{TT}}}} \right) \left(\frac{Z^{(s)}_{\chi^+} W^{(s)}_{\sigma^+}}{Y^{(s)}_{\sigma^+}} \right) \left(\frac{Z^{(s)}_{\chi^-}}{Y^{(s)}_{\xi^{\text{CKT}}}} \right) \tag{2.169}$$

where $P = \sum_s (N^{\text{CKT}}_{s-2} + N^{\text{CKT}}_{s-1} - N^{\text{KT}}_{s-1})$. In the infinite product, the first factor is the usual ratio of determinants of physical and ghost operators

$$\frac{Z^{(s)}_{\phi^{\text{TT}}}}{Y^{(s)}_{\xi^{\text{TT}}}} = \frac{\det'(-\nabla^2_{(s-1)} + m^2_{s-1,s} + M^2_{s-1})^{1/2}}{\det\left(-\nabla^2_{(s)} + M^2_s\right)^{1/2}}. \tag{2.170}$$

In the second factor, $Z_\chi^+, W_\sigma^+, Y_\sigma^+$ run over the exact same spectrum and cancel almost completely up to an infinite constant

$$\begin{aligned}\frac{Z_\chi^+ W_\sigma^+}{Y_\sigma^+} &= \int \mathcal{D}\chi_{(s-2)}^+ e^{-\frac{d+2s-5}{4gs}(\chi_{(s-2)}^+,\chi_{(s-2)}^+)} \\ &= \frac{\int \mathcal{D}\chi_{(s-2)} e^{-\frac{d+2s-5}{4gs}(\chi_{(s-2)},\chi_{(s-2)})}}{\int \mathcal{D}\chi_{(s-2)}^0 \mathcal{D}\chi_{(s-2)}^- e^{-\frac{d+2s-5}{4gs}(\chi_{(s-2)},\chi_{(s-2)})}}\end{aligned} \tag{2.171}$$

where in the denominator $\chi_{(s-2)}^0$ denotes the modes excluded due to (2.123). The infinite constant in the numerator is a path integral over the entire spectrum of an unconstrained spin-$(s-2)$ symmetric traceless field and therefore can be absorbed into bare couplings. To proceed, we plug in explicit mode expansions

$$\chi_{(s-2)}^0 = \sum_{m=0}^{s-1} A_{s-1,m}\hat{T}_{s-1,(s-2)}^{(m)}, \quad \chi_{(s-2)}^- = \sum_{m=0}^{s-2} A_{s-2,m}\hat{T}_{s-2,(s-2)}^{(m)}, \quad \xi_{(s-1)}^{CKT} = \sum_{m=0}^{s-2} A_{s-1,m}\hat{T}_{s-1,(s-1)}^{(m)}, \tag{2.172}$$

which lead to

$$\frac{Z_{\chi^+}^{(s)} W_{\sigma^+}^{(s)}}{Y_{\sigma^+}^{(s)}} = \prod_{m=0}^{s-2}\prod_{n=s-2}^{s-1}\left[\frac{2}{d+2s-5}\right]^{D_{n,m}^{d+2}/2} \tag{2.173}$$

$$Z_{\chi^-}^{(s)} = \prod_{m=0}^{s-2}\left[\frac{2}{(d+2s-5)m_{s+1,m}^2}\right]^{\frac{D_{s-2,m}^{d+2}}{2}} \tag{2.174}$$

$$Y_{\xi^{\mathrm{CKT}}}^{(s)} = \prod_{m=0}^{s-2}\left[\frac{2m_{s,m}^2}{d+2s-5}\right]^{-\frac{D_{s-1,m}^{d+2}}{2}}. \tag{2.175}$$

We therefore have

$$\left(\frac{Z_{\chi^+}^{(s)} W_{\sigma^+}^{(s)}}{Y_{\sigma^+}^{(s)}}\right)\left(\frac{Z_{\chi^-}^{(s)}}{Y_{\xi^{\mathrm{CKT}}}^{(s)}}\right) = \prod_{m=0}^{s-2}(m_{s+1,m}^2)^{-\frac{D_{s-2,m}^{d+2}}{2}}\prod_{m=0}^{s-2}(m_{s,m}^2)^{\frac{D_{s-1,m}^{d+2}}{2}}. \tag{2.176}$$

Together with the determinant factor, this can be further written as

$$\left(\frac{Z^{(s)}_{\phi^{\rm TT}}}{Y^{(s)}_{\xi^{\rm TT}}}\right)\left(\frac{Z^{(s)}_{\chi^+}W^{(s)}_{\sigma^+}}{Y^{(s)}_{\sigma^+}}\right)\left(\frac{Z^{(s)}_{\chi^-}}{Y^{(s)}_{\xi^{\rm CKT}}}\right)=\frac{{\det}'_{-1}|-\nabla^2_{(s-1)}-\lambda_{s-1,s-1}|^{1/2}}{{\det}'_{-1}|-\nabla^2_{(s)}-\lambda_{s-2,s}|^{1/2}}. \tag{2.177}$$

Here the subscript -1 means that we extend the eigenvalue product from $n = s$ to $n = -1$. The primes denote omission of the zero modes from the determinants. In the numerator we omitted the $n = s-1$ mode while in the denominator we omitted the $n = s-2$ mode.[11] To obtain this expression we used the relation

$$\lambda_{t-1,s} + M_s^2 = -m_{s,t}^2 \tag{2.178}$$

and the fact that $D^{d+2}_{s-1,t} = -D^{d+2}_{t-1,s}$ (implying $D^{d+2}_{s-1,s} = 0$). This extension of the eigenvalue product from $n = s$ to $n = -1$ is exactly the prescription described in [73]. Putting everything together, we finally obtain the expression

$$\begin{aligned}
Z^{\rm HS}_{\rm PI} =& Z_{\rm G} Z_{\rm Char}\\
Z_{\rm G} =& i^{-P}\frac{\gamma^{\dim G}}{{\rm Vol}(G)_{\rm can}}, \qquad Z_{\rm Char} = \prod_s Z^{(s)}_{\rm Char}\\
Z^{(s)}_{\rm Char} =& \left(\frac{(d+2s-2)(d+2s-4)}{M^4}\right)^{\frac{N^{\rm KT}_{s-1}}{2}} \frac{{\det}'_{-1}\left|\frac{-\nabla^2_{(s-1)}-\lambda_{s-1,s-1}}{M^2}\right|^{1/2}}{{\det}'_{-1}\left|\frac{-\nabla^2_{(s)}-\lambda_{s-2,s}}{M^2}\right|^{1/2}},
\end{aligned} \tag{2.179}$$

with

$$P = \sum_s (N^{\rm CKT}_{s-2} + N^{\rm CKT}_{s-1} - N^{\rm KT}_{s-1}), \qquad \gamma = \sqrt{\frac{8\pi G_N}{{\rm Vol}(S^{d-1})}}, \qquad \dim G = \sum_s N^{\rm KT}_{s-1} \tag{2.180}$$

Note that we have restored the dimensionful parameter M. As noted in [73], the factor $(d + 2s - 2)(d + 2s - 4)$ gets nicely canceled after evaluating the character integrals for the determinants.

[11] Originally $\lambda_{n,s}$ and $D^{d+2}_{n,s}$ were defined only for $n \geq s$, which are now extended to all $n \in \mathbb{Z}$.

2.5 Massive fields

Now let us turn to fields with generic masses. In this case we do not have a group volume factor, and thus no coupling dependence. We will work with canonical normalizations.

2.5.1 Massive scalars and vectors

Massive scalars The path integral for a scalar ϕ with mass $m^2 > 0$ is simply

$$Z_{\rm PI}^{(s=0,m^2)} = \int \mathcal{D}\phi e^{-\frac{1}{2}\int_{S^{d+1}} \phi(-\nabla^2+m^2)\phi} = \det\Big(-\nabla^2 + m^2\Big)^{-1/2} \tag{2.181}$$

Massive vectors Massive vectors are described by the Proca action

$$S[A] = \int_{S^{d+1}} \Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu\Big). \tag{2.182}$$

Similar to the massless case, to proceed we make a change of variables (2.19) with Jacobian (2.29), so that the action becomes

$$\begin{gathered} S[A] = S[A^T] + S[\chi] \\ S[A^T] = \frac{1}{2}(A^T, (-\nabla^2_{(1)} + m^2 + d)A^T), \quad S[\chi] = \frac{m^2}{2}(\chi, (-\nabla^2_{(0)})\chi). \end{gathered} \tag{2.183}$$

For $m^2 > 0$ that corresponds to unitary de Sitter representations, the result is

$$Z_{\rm PI}^{(s=1,m^2)} = \det\Big(-\nabla^2_{(1)} + m^2 + d\Big)^{-1/2}(m^2)^{1/2} = \det{}_{-1}(-\nabla^2_{(1)} + m^2 + d)^{-1/2}. \tag{2.184}$$

The presence of the factor $(m^2)^{1/2}$ originates from the fact that the $(0, 0)$ mode is excluded from the integration over the longitudinal mode. In the last equality we again note that the multiplication of the factor $(m^2)^{1/2}$ is equivalent to extending the product to $n = -1$.

2.5.2 Massive spin 2 and beyond

Massive $s = 2$

The action for a free massive spin-2 field on S^{d+1} is (see for example [100])

$$
\begin{aligned}
S[h] =& \frac{1}{2}\int_{S^{d+1}} h^{\mu\nu}\Big[(-\nabla^2+2)h_{\mu\nu} + 2\nabla_{(\mu}\nabla^{\lambda}h_{\nu)\lambda} + g_{\mu\nu}(\nabla^2 h_{\lambda}{}^{\lambda} - 2\nabla^{\sigma}\nabla^{\lambda}h_{\sigma\lambda}) + (d-2)g_{\mu\nu}h_{\lambda}{}^{\lambda} \\
&+ m^2(h_{\mu\nu} - g_{\mu\nu}h_{\lambda}{}^{\lambda})\Big].
\end{aligned}
\tag{2.185}
$$

If we put $m = 0$ we recover the action (2.51) (with g = 1) for linearized gravity. To proceed, we again change the variables (2.53). It is convenient to further decompose ξ_μ into its transverse and longitudinal parts: $\xi_\mu = \xi^T_\mu + \nabla_\mu\sigma$, so that the full decomposition for $h_{\mu\nu}$ is

$$
h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} + \frac{1}{\sqrt{2}}(\nabla_\mu\xi^T_\nu + \nabla_\nu\xi^T_\mu) + \sqrt{2}\nabla_\mu\nabla_\nu\sigma + \frac{g_{\mu\nu}}{\sqrt{d+1}}\tilde{h}. \tag{2.186}
$$

For this decomposition to be unique, we impose

$$
(\xi^T, f_{1,(1)}) = 0 \quad , \quad (\tilde{h}, f_1) = 0 \quad \text{and} \quad (\sigma, f_0) = 0. \tag{2.187}
$$

The first two constraints are equivalent to (2.54) and (2.55) while the last one ensures $\nabla_\mu\sigma \neq 0$. With a slight modification of the steps in Sec.2.3.2, the Jacobian for the (2.186) is obtained as

$$
\begin{aligned}
\mathcal{D}h =& J\mathcal{D}h^{\mathrm{TT}}\mathcal{D}'\xi^T\mathcal{D}^+\sigma\mathcal{D}'\tilde{h} \\
J =& \frac{1}{Y^{\mathrm{T}}_\xi Y^+_\sigma Y^{\mathrm{CKV}}_\xi} \\
Y^{\mathrm{T}}_\xi =& \int \mathcal{D}'\xi^T e^{-\frac{1}{2}(\xi^T,(-\nabla^2_{(1)}-d)\xi^T)} \\
Y^+_\sigma =& \int \mathcal{D}^+\sigma e^{-\frac{1}{2}\frac{2d}{d+1}(\sigma,(-\nabla^2_{(0)})(-\nabla^2_{(0)}-(d+1))\sigma)} \\
Y^0_\sigma =& \int \mathcal{D}^0\sigma e^{-(\sigma,(-\nabla^2_{(0)})(-\nabla^2_{(0)}-d)\sigma)} = \int \mathcal{D}^0\sigma e^{-(d+1)(\sigma,\sigma)}.
\end{aligned}
\tag{2.188}
$$

Here $\mathcal{D}^+\sigma$ ($\mathcal{D}^0\sigma$) involves integrations over only the positive (zero) modes for the operator $(-\nabla^2_{(0)} - (d+1))$. After substituting (2.186) the action decouples into

$$S[h] = S[h^{TT}] + S[\xi^T] + S[\sigma, \tilde{h}]. \tag{2.189}$$

The quadratic actions for h^{TT} and ξ^T are simply

$$S[h^{\mathrm{TT}}] = \frac{1}{2}\int_{S^{d+1}} h^{\mathrm{TT}}_{\mu\nu}(-\nabla^2_{(2)} + m^2 + 2)h^{\mu\nu}_{\mathrm{TT}}, \tag{2.190}$$

and

$$S[\xi^T] = \frac{m^2}{2}(\xi^T, (-\nabla^2_{(1)} - d)\xi^T) \tag{2.191}$$

respectively. Since σ and $\tilde{h}$ are not orthogonal, they mix in the action

$$\begin{aligned} S[\sigma, \tilde{h}] = &- \frac{(d-1)d}{2(d+1)}(\tilde{h}, (-\nabla^2_{(0)} + (d+1)(\frac{m^2}{d-1} - 1))\tilde{h}) - \sqrt{\frac{2}{d+1}}dm^2(\nabla^2_{(0)}\sigma, \tilde{h}) \\ &+ \frac{m^2}{2}(\nabla\sigma, (-\nabla^2_{(0)} - d)\nabla\sigma) - \frac{m^2}{2}(\nabla^2_{(0)}\sigma, \nabla^2_{(0)}\sigma). \end{aligned} \tag{2.192}$$

To diagonalize $S[\sigma, \tilde{h}]$, we make a shift (with a trivial Jacobian)[12]

$$\sigma' = \sigma - \frac{1}{\sqrt{2(d+1)}}\tilde{h} \tag{2.193}$$

for all scalar modes f_n with $n \geq 2$, so that $S[\sigma, \tilde{h}] = S[\sigma', \tilde{h}] = S[\sigma'] + S[\tilde{h}]$, with

$$S[\sigma'] = -dm^2(\sigma', -\nabla^2_{(0)}\sigma') \quad \text{and} \quad S[\tilde{h}] = \frac{d(m^2 - (d-1))}{2(d+1)}(\tilde{h}, (-\nabla^2_{(0)} - (d+1))\tilde{h}). \tag{2.194}$$

Notice that $S[\sigma']$ and $S[\tilde{h}]$ vanishes identically when $m^2 = 0$ and $m^2 = d-1$ respectively. These are the cases when we have gauge symmetries. The massless case has already been discussed in

[12]Because of the constraints (2.187), the $(0,0)$ and $(1,0)$ modes do not mix in (2.192)

Sec.2.3. The case of $m^2 = d-1$ will be considered in Sec.2.7.

Depending on the precise value of $m^2 > -2(d+2)$[13], some of the modes in (2.191) and (2.194) might acquire an overall negative sign. We Wick rotate the negative modes, absorbing local infinite constants into bare couplings. This will induce a phase factor. Below we give a summary for different cases (n.m. stands for negative modes):

Range of m^2	n.m. in $S[\xi^T]$	n.m. in $S[\sigma']$	n.m. in $S[\tilde{h}]$	Phase
$-2(d+2) < m^2 < 0$	$f_{n,\mu}, n \geq 1$	None	$f_n, n \geq 2$	$i^{-D^{d+2}_{1,1}-D^{d+2}_{1,0}} = i^{-\frac{(d+3)(d+2)}{2}}$
$0 < m^2 < d-1$	None	$f_n, n \geq 1$	$f_n, n \geq 2$	$i^{-2D^{d+2}_{0,0}-D^{d+2}_{1,0}} = i^{-d-4}$
$m^2 > d-1$	None	$f_n, n \geq 1$	f_0	$i^0 = 1$

The last case ($m^2 > d-1$) is precisely the case when the corresponding de Sitter representations are unitary[14]. We will focus on this case from now on.

Putting everything together, we have

$$Z_{\rm PI}^{(s=2,m^2)} = Z_h^{\rm TT}\left(\frac{Z_\xi^{\rm T}}{Y_\xi^{\rm T}}\right)\left(\frac{Z_{\sigma'}^+ Z_{\tilde{h}}^+}{Y_\sigma^+}\right)\left(\frac{Z_{\sigma'}^0}{Y_\sigma^0}\right)Z_{\tilde{h}}^- \tag{2.195}$$

Here $Z_h^{\rm TT}, Z_\xi^{\rm T}, Z_{\sigma'}^\pm, Z_{\sigma'}^0, Z_{\tilde{h}}^\pm$ are the path integrals with actions (2.190), (2.191) and (2.194). The labels $\pm$ and 0 denote the positive (negative) and zero modes for the scalar operator $-\nabla^2_{(0)} - (d+1)$.

[13]This is the range where the kinetic operator in (2.190) is positive definite. The case $m^2 < -2(d+2)$ will be considered when we discuss the shift-symmetric spin-2 fields in Sec.2.6.

[14]Principal series for $m^2 > (\frac{d}{2})^2$ and complementary series for $d-1 < m^2 < (\frac{d}{2})^2$ [101].

Every factor can be easily evaluated:

$$
\begin{aligned}
Z_h^{\mathrm{TT}} &= \det\left(-\nabla_{(2)}^2 + m^2 + 2\right)^{-1/2} \\
\frac{Z_\xi^{\mathrm{T}}}{Y_\xi^{\mathrm{T}}} &= \int \mathcal{D}'\xi^T e^{-\frac{m^2}{2}(\xi^T,\xi^T)} \\
\frac{Z_{\sigma'}^{+} Z_{\tilde{h}}^{+}}{Y_\sigma^{+}} &= \int \mathcal{D}^+\sigma' e^{-\frac{m^2}{2}(\sigma',\sigma')} \int \mathcal{D}^+\tilde{h} e^{-\frac{d(m^2-(d-1))}{2}(\tilde{h},\tilde{h})} \\
\frac{Z_{\sigma'}^{0}}{Y_\sigma^{0}} &= \int \mathcal{D}^0\sigma' e^{-\frac{dm^2}{2}(\sigma',\sigma')} \\
Z_{\tilde{h}}^{-} &= \int \mathcal{D}^-\tilde{h} e^{-\frac{d(m^2-(d-1))}{2}(\tilde{h},\tilde{h})}.
\end{aligned}
\tag{2.196}
$$

Observe that all factors but Z_h^{TT} can be combined in the following way:

$$
\left(\frac{Z_\xi^{\mathrm{T}}}{Y_\xi^{\mathrm{T}}}\right)\left(\frac{Z_{\sigma'}^{+} Z_{\tilde{h}}^{+}}{Y_\sigma^{+}}\right)\left(\frac{Z_{\sigma'}^{0}}{Y_\xi^{\mathrm{CKV}}}\right) Z_{\tilde{h}}^{-} = \frac{\int \mathcal{D}\xi e^{-\frac{m^2}{2}(\xi,\xi)} \int \mathcal{D}\tilde{h} e^{-\frac{d(m^2-(d-1))}{2}(\tilde{h},\tilde{h})}}{\int \mathcal{D}^0\sigma' e^{-\frac{(m^2-(d-1))}{2}(\sigma',\sigma')} \int \mathcal{D}^0\xi^T e^{-\frac{m^2}{2}(\xi^T,\xi^T)}}. \tag{2.197}
$$

In the numerator, the path integrations are over local unconstrained fields and thus can be absorbed into bare couplings. In the denominator $\mathcal{D}^0\xi^T$ denotes integration over the modes $f_{1,\mu}$. The integrals in the denominator can be easily evaluated. To conclude, we have

$$
\begin{aligned}
Z_{\mathrm{PI}}^{(s=2,m^2)} &= \det\left(-\nabla_{(2)}^2 + m^2 + M_2^2\right)^{-1/2} (m^2 - m_{2,0}^2)^{\frac{D_{1,0}^{d+2}}{2}} (m^2 - m_{2,1}^2)^{\frac{D_{1,1}^{d+2}}{2}} \\
&= \det{}_{-1}(-\nabla_{(2)}^2 + m^2 + M_2^2)^{-1/2}
\end{aligned}
\tag{2.198}
$$

where we recall that $m_{s,t}^2$ is defined in (2.147).

Massive arbitrary spin $s \geq 1$

In principle, one starts with the full manifestly local and covariant action [102], which involves a tower of spin $t < s$ Stueckelberg fields, and repeat the derivation above. However, having worked out the cases for $s = 1, 2$, the pattern is clear. For a free massive spin-s field, its path integral is simply

$$Z_{\rm PI}^{(s,m^2)} = \det\nolimits_{-1}\left(\frac{-\nabla^2_{(s)} + m^2 + M_s^2}{M^2}\right)^{-1/2} . \tag{2.199}$$

Note that we have restored the dimensionful parameter M. Recall that the scaling dimension Δ is related to the mass m^2 as

$$m^2 = (\Delta + s - 2)(d + s - 2 - \Delta) \tag{2.200}$$

so that

$$\lambda_{n,s} + m^2 + M_s^2 = (n + \Delta)(d + n - \Delta) = \left(n + \frac{d}{2}\right)^2 - \left(\Delta - \frac{d}{2}\right)^2 . \tag{2.201}$$

The requirement that $\lambda_{n,s} + m^2 + M_s^2$ is positive for all $n \geq -1$ is equivalent to the unitary bounds on Δ [101]:

$$\Delta = \frac{d}{2} + i\nu, \nu \in \mathbb{R} \quad \text{(Principal series)} \quad \text{or} \quad 1 < \Delta < d - 1 \quad \text{(Complementary series)} \tag{2.202}$$

Outside of this bound, a finite number of $\lambda_{n,s} + m^2 + M_s^2$ will become negative, which leads to the presence of some power of i, as we have seen in the $s = 2$ case. Also, as we take $m^2 \to m^2_{s,t}$, (2.199) becomes ill-defined, signaling a gauge symmetry. The case of $t = s - 1$ is the massless case discussed in Sec.2.4. We will comment on the general (s, t) case in Sec.2.7.

2.6 Shift-symmetric fields

In (A)dS space, when massive fields attain certain mass values, they can have shift symmetries [89] that generalize the shift symmetry, galileon symmetry, and special galileon symmetry of massless scalars in flat space. In AdS, these theories are unitary; in dS, these theories do not fall into the classifications of dS UIRs [101]. In the following we study their 1-loop (free) path integrals on S^{d+1}, which contain analogous subtleties as the massless case, namely the phases and group volumes.

2.6.1 Shift-symmetric scalars

Let us start with a free scalar ϕ with generic mass m, with action

$$S[\phi] = \frac{1}{2}\int_{S^{d+1}} \phi(-\nabla^2 + m^2)\phi. \tag{2.203}$$

When m^2 takes values of the negative the eigenvalues of the scalar Laplacian $-\nabla^2_{(0)}$, i.e.

$$m^2 = -\lambda_{k,0} = -k(k+d) = m^2_{0,k+1} + M_0^2 = m^2_{k+2,1} = -m^2_{2,k+1} \leq 0, \quad k \geq 0, \tag{2.204}$$

(recall that $m^2_{s,t}$ is defined in (2.147)), the action is invariant under a shift symmetry (of level k in the terminology of [89])

$$\delta\phi = f_k \tag{2.205}$$

where f_k is the $(k,0)$ eigenmodes of $-\nabla^2_{(0)}$ with eigenvalue λ_k. The case $k = 0$ corresponds to a massless scalar[103, 104]. For any $k \geq 1$, the scalar is tachyonic. See for example [105] and [106] for the study of such tachyonic scalars. The $k = 1$ and $k = 2$ cases are the dS analogs for the Galileon and special Galileon theories in flat space [89] respectively. While these $k \geq 0$ scalars do not fall into the standard classification of dS UIRs [101], there are arguments that they can be cured to become unitary [89]. Note that the action is negative for all $(n,0)$ modes with $n < k$, and vanishes for the $(k,0)$ modes. To make sense of the path integral, we again perform Wick rotations for all $(n,0)$ modes with $n < k$, so that

$$\int \mathcal{D}^{<k}\phi e^{-S_{<k}[\phi]} \to i^{\sum_{n=0}^{k-1} D^{d+2}_{n,0}} \int \mathcal{D}^{<k}\phi e^{S_{<k}[\phi]} = i^{\sum_{n=0}^{k-1} D^{d+2}_{n,0}} \prod_{n=0}^{k-1} |\lambda_{n,0} - \lambda_{k,0}|^{-1/2}, \tag{2.206}$$

and interpret the integration over the $(k,0)$ modes as a residual group volume

$$\mathrm{Vol}(G_k)_{\mathrm{PI}} \equiv \int \prod_{a=1}^{D^{d+2}_{k,0}} \frac{dA^{(a)}_{k,0}}{\sqrt{2\pi}}. \tag{2.207}$$

The modes with $n > k$ can be integrated as usual. The final result is

$$Z_{\mathrm{PI}}^{(s=0,m^2_{k+2,1})} = i^{\sum_{n=0}^{k-1} D^{d+2}_{n,0}} \mathrm{Vol}(G_k)_{\mathrm{PI}} \det{}' \,|-\nabla^2_{(0)} - \lambda_{k,0}|^{-1/2}. \tag{2.208}$$

Note that absolute value is taken in the determinant. The prime denotes the omission of the $(k,0)$ modes from the functional determinant. Unlike the massless case, the residual group volume $\mathrm{Vol}(G_k)_{\mathrm{PI}}$ is multiplying the determinant instead of being divided. As stressed in the massless case, $\mathrm{Vol}(G_k)_{\mathrm{PI}}$ should depend on the non-linear completion of the theory. There will be a problem of relating $\mathrm{Vol}(G_k)_{\mathrm{PI}}$ to a canonical volume $\mathrm{Vol}(G_k)_{\mathrm{can}}$ and the determination of $\mathrm{Vol}(G_k)_{\mathrm{can}}$ itself. Also, we expect there will be a dependence on coupling constants of the interacting theory.[15]

2.6.2 Shift-symmetric vectors

When the mass takes values

$$m^2 = -\lambda_{k+1,1} - d = -(k+2)(k+d) = m^2_{k+3,0} = -m^2_{1,k+2} \le -2d, \quad k \ge 0, \tag{2.211}$$

[15] An example for which we can make sense of these issues is that of a compact scalar. They are scalars subject to the identification

$$\phi \sim \phi + 2\pi R \tag{2.209}$$

so that they take values on a circle of radius R. In this case the integration range for the (0,0) mode is restricted to the fundamental domain $0 < A_{0,0} < 2\pi R\sqrt{\mathrm{Vol}(S^{d+1})}$ and therefore

$$Z_{\mathrm{PI}}^{\text{compact scalar}} = \sqrt{2\pi R^2 \mathrm{Vol}(S^{d+1})}\, \det{}'(-\nabla^2_{(0)})^{-1/2}. \tag{2.210}$$

Here the (inverse of) radius R plays the role of the coupling constant.

the Proca action (2.182) is invariant under a level-k shift symmetry generated by the $(k+1,1)$ modes

$$\delta A_\mu = f_{k+1,\mu}. \tag{2.212}$$

Following analogous steps as for the scalars, it is straightforward to work out the path integral

$$Z_{\rm PI}^{(s=1,m^2_{k+3,0})} = i^{\sum_{n=-1}^{k} D^{d+2}_{n,1}} \mathrm{Vol}(G_{k+1,1})_{\rm PI} \det{}'_{-1} |-\nabla^2_{(1)} - \lambda_{k+1,1}|^{-1/2} \tag{2.213}$$

where the prime denotes the omission of the $(k+1,1)$ modes and

$$\mathrm{Vol}(G_{k+1,1})_{\rm PI} \equiv \int \prod_{a=1}^{D^{d+2}_{k+1,1}} \frac{dA^{(a)}_{k+1,1}}{\sqrt{2\pi}}. \tag{2.214}$$

Note that in the phase factor we have used the fact that $D^{d+2}_{-1,1} = -D^{d+2}_{0,0}$ and $D^{d+2}_{0,1} = 0$.

2.6.3 Shift-symmetric spin $s \geq 2$

Shift-symmetric spin 2 fields

The massive spin-2 action (2.185) with

$$m^2 = -\lambda_{k+2,2} - 2 = m^2_{k+4,1} = -m^2_{2,k+3} \leq 2(d+2), \quad k \geq 0, \tag{2.215}$$

is invariant under a level-k shift symmetry generated by the $(k+2,2)$ modes

$$\delta h_{\mu\nu} = f_{k+2,\mu\nu}. \tag{2.216}$$

It is straightforward to work out the path integral

$$Z_{\rm PI}^{(s=2,m^2_{k+4,1})} = i^{\sum_{n=-1}^{k+1} D^{d+2}_{n,2}} \mathrm{Vol}(G_{k+2,2})_{\rm PI} \det{}'_{-1} |-\nabla^2_{(2)} - \lambda_{k+2,2}|^{-1/2} \tag{2.217}$$

where the prime denotes the omission of the $(k+2,2)$ modes and

$$\mathrm{Vol}(G_{k+2,2})_{\mathrm{PI}} \equiv \int \prod_{a=1}^{D^{d+2}_{k+2,2}} \frac{dA^{(a)}_{k+2,2}}{\sqrt{2\pi}}. \tag{2.218}$$

Note that in the phase factor we have used the fact that $D^{d+2}_{-1,2} = -D^{d+2}_{1,0}$ and $D^{d+2}_{0,2} = -D^{d+2}_{1,1}$.

Shift-symmetric arbitrary spins $s \geq 0$

Now the pattern is clear. When the mass for a spin-s field $\phi_{(s)}$ ($s \geq 0$) reaches the values

$$m^2 = -\lambda^2_{k+s,s} - M^2_s = m^2_{k+s+2,s-1} = -m^2_{s,s+k+1}, \quad k \geq 0, \tag{2.219}$$

there will be a level-k shift symmetry generated by the $(k+s,s)$ modes

$$\delta\phi_{(s)} = f_{k+s,(s)}. \tag{2.220}$$

The path integral is

$$Z_{\mathrm{PI}}^{(s,m^2_{k+s+2,s-1})} = i^{\sum_{n=-1}^{k+s-1} D^{d+2}_{n,s}} \mathrm{Vol}(G_{k+s,s})_{\mathrm{PI}} \det{}'_{-1}\left|\frac{-\nabla^2_{(s)} - \lambda_{k+s,s}}{M^2}\right|^{-1/2} \tag{2.221}$$

where

$$\mathrm{Vol}(G_{k+s,s})_{\mathrm{PI}} \equiv \int \prod_{a=1}^{D^{d+2}_{k+s,s}} \frac{M}{\sqrt{2\pi}} dA^{(a)}_{k+s,s}. \tag{2.222}$$

Note that we have restored the dimensionful parameter M. Such a shift-symmetric field can be thought of as the longitudinal mode decoupled from a massive spin-$(k+s+1)$ field as its mass approaches $m^2_{k+s+1,s}$. Note that for $k=0$, it can be thought of as the ghost part of the spin-$(s+1)$ massless path integral. We will see more connections of shift-symmetric fields with general partially massless fields in the next section.

2.7 Partially massless fields

In (A)dS space, there exist "partial massless" (PM) representations [100, 102, 107–115]. Except for the massless case, they are not unitary in AdS. In dS_{d+1} with $d \geq 4$, they correspond to the unitary exceptional series representations, while for $d = 3$ they correspond to the discrete series representations [101]. A PM spin-s field of depth t has a gauge symmetry[16]

$$\delta\phi_{(s)} = \nabla^{(s-t)}\xi_{(t)} + \cdots \tag{2.223}$$

where $\cdots$ stand for terms with fewer derivatives [115]. The massless case corresponds to $t = s-1$. In the following we first work out the case of spin-2 depth-0 field. Then we will provide a general prescription for general PM fields.

2.7.1 Spin-2 depth-0 field

The action for a spin-2 depth-0 field is (2.185) with mass

$$m^2 = m^2_{2,0} = d - 1, \tag{2.224}$$

in which case there is a gauge symmetry

$$\delta h_{\mu\nu} = \nabla_\mu \nabla_\nu \chi + g_{\mu\nu}\chi. \tag{2.225}$$

This can be seen by first substituting (2.193) into (2.186) so that the decomposition becomes

$$h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} + \frac{1}{\sqrt{2}}(\nabla_\mu \xi^T_\nu + \nabla_\nu \xi^T_\mu) + \sqrt{2}\nabla_\mu\nabla_\nu\sigma' + \frac{1}{\sqrt{d+1}}\left(\nabla_\mu\nabla_\nu\tilde{h} + g_{\mu\nu}\tilde{h}\right) \tag{2.226}$$

[16] We adopt the convention that depth t is equal to the spin of the gauge parameter.

and noting that $S[\tilde{h}]$ defined in (2.194) vanishes identically for $m^2 = d - 1$. Spin-2 field with such a mass was first considered in [100]. This gauge invariance implies that there is an integration

$$\int \mathcal{D}'\tilde{h} \tag{2.227}$$

that must be canceled by a gauge group volume factor $\mathrm{Vol}(\mathcal{G})$ divided by hand. To be consistent with locality, this gauge group factor must take the form of a path integral of a local scalar field α

$$\mathrm{Vol}(\mathcal{G}) = \int \mathcal{D}\alpha. \tag{2.228}$$

Due to mismatch of modes excluded due to (2.187), we have a residual group volume

$$\frac{\int \mathcal{D}'\tilde{h}}{\mathrm{Vol}(\mathcal{G})} = \frac{1}{\mathrm{Vol}(G_{1,0})_{\mathrm{PI}}}, \quad \mathrm{Vol}(G_{1,0})_{\mathrm{PI}} \equiv \int \prod_{a=1}^{D_{1,0}^{d+2}} \frac{dA_{1,0}^{(a)}}{\sqrt{2\pi}}. \tag{2.229}$$

The rest of the computation proceeds as before, and the final result is

$$Z_{\mathrm{PI}}^{(s=2,m_{2,0}^2)} = \frac{i^{-1}}{\mathrm{Vol}(G_{1,0})_{\mathrm{PI}}} \frac{\det'_{-1}|-\nabla_{(0)}^2 - (d+1)|^{1/2}}{\det'_{-1}\left(-\nabla_{(2)}^2 + d + 1\right)^{1/2}}. \tag{2.230}$$

2.7.2 General PM fields

We now provide a prescription to obtain the path integral expression for a general spin-s depth-t field. First, take the spin-s path integral (2.199) with generic mass and take the limit $m^2 \to m_{s,t}^2$ while omitting the $(t-1, s)$ modes:

$$Z_{\mathrm{PI}}^{(s,m^2\to m_{s,t}^2)} \to i^{\sum_{m=-1}^{t-2} D_{m,s}^{d+2}} \det'_{-1}|-\nabla_{(s)}^2 - \lambda_{t-1,s}^2|^{-1/2} \tag{2.231}$$

where we have used (2.178). The phases appear because the mode with $n = -1, 0, \cdots, t-2$ becomes negative. Then we exchange s and t and flip $i \to -i$ to obtain another expression

$$Z_{\rm PI}^{(t, m^2 \to m_{t,s}^2)} \to i^{-\sum_{m=-1}^{s-2} D_{m,t}^{d+2}} \det{}'_{-1} | -\nabla_{(t)}^2 - \lambda_{s-1,t}^2 |^{-1/2}. \tag{2.232}$$

We propose that the final result is simply given by the ratio between these two expressions, divided by a group volume factor:

$$Z_{\rm PI}^{(s, m^2 = m_{s,t}^2)} = \frac{i^{\sum_{m=-1}^{t-2} D_{m,s}^{d+2} + \sum_{m=-1}^{s-2} D_{m,t}^{d+2}}}{\mathrm{Vol}(G_{s-1,t})_{\rm PI}} \frac{\det{}'_{-1} \left| \frac{-\nabla_{(t)}^2 - \lambda_{s-1,t}^2}{M^2} \right|^{1/2}}{\det{}'_{-1} \left| \frac{-\nabla_{(s)}^2 - \lambda_{t-1,s}^2}{M^2} \right|^{1/2}} \tag{2.233}$$

where

$$\mathrm{Vol}(G_{s-1,t})_{\rm PI} \equiv \int \prod_{a=1}^{D_{s-1,t}^{d+2}} \frac{M^2}{\sqrt{2\pi}} dA_{s-1,t} \tag{2.234}$$

Note that we have restored the dimensionful parameter M. One can easily verify that (2.233) reduces to the massless case when $t = s-1$ and the spin-2 depth-0 case when $s = 2, t = 0$. The division by $Z_{\rm PI}^{(t, m^2 \to m_{t,s}^2)}$ can be thought of as the decoupling of the spin-t level-$(s-1-t)$ shift-symmetric field from the massive spin-s field as we take $m^2 \to m_{s,t}^2$. Note that the ratio of determinants (without the extension to $n = -1$ modes) in (2.233) and the relations between PM and conformal higher spin partition functions were first discussed in [82] for S^4 and [62] for S^6.

As we stressed repeatedly, the determination of the group volume factor $\mathrm{Vol}(G_{s-1,t})_{\rm PI}$ requires knowledge of the interactions of the parent theory. In the current case, a natural class of parent theories would be the PM generalizations of higher spin theories [116], which include a tower of PM gauge fields and a finite number of massive fields. These theories gauge the PM algebras studied in [117] and are holographic duals to $\Box^k$ CFTs [118]. Their 1-loop path integrals would take the form

$$Z_{\rm PI}^{\rm PM\ HS}=\frac{i^P}{\mathrm{Vol}(G)_{\rm PI}}\prod_{s,t}\frac{\det'_{-1}\left|\frac{-\nabla^2_{(t)}-\lambda^2_{s-1,t}}{M^2}\right|^{1/2}}{\det'_{-1}\left|\frac{-\nabla^2_{(s)}-\lambda^2_{t-1,s}}{M^2}\right|^{1/2}} \tag{2.235}$$

where

$$P=\sum_{s,t}\left(\sum_{m=-1}^{t-2}D^{d+2}_{m,s}+\sum_{m=-1}^{s-2}D^{d+2}_{m,t}\right),\quad \mathrm{Vol}(G)_{\rm PI}=\prod_{s,t}\mathrm{Vol}(G_{s-1,t})_{\rm PI} \tag{2.236}$$

There will be analogous problem of relating $\mathrm{Vol}(G)_{\rm PI}$ to a canonical volume $\mathrm{Vol}(G)_{\rm can}$ (and making sense of the volume itself) as in the massless case, which will give us the dependence on the Newton's constant G_N. If we demand $\log Z_{\rm PI}^{\rm PM\ HS}$ to be consistent with a universal form as in the massless case [73], we should take

$$\mathrm{Vol}(G)_{\rm PI}=\mathrm{Vol}(G)_{\rm can}\prod_{s,t}\left(\frac{\mathrm{Vol}(S^{d-1})}{8\pi G_N}\frac{M^4}{(d+2s-2)(d+2t-2)}\right)^{\frac{D^{d+2}_{s-1,t}}{2}} \tag{2.237}$$

so that the factor $(d+2s-2)(d+2t-2)$ gets nicely canceled upon evaluating the character integrals for the determinants. To verify this, one has to repeat the analysis of [68] and App.B.3 and express the PM HS invariant bilinear form in terms of the bilinear form induced by the path integral measure. Provided that (2.237) is valid, we note that except the phase and $\mathrm{Vol}(G)_{\rm can}$, the expression (2.235) becomes the inverse of itself upon exchanging s and t. We leave the validation of (2.233), (2.237) and the implication of the suggestive $s\leftrightarrow t$ symmetry for future work.

2.8 Discussion and outlooks

In this work, we derive the determinant expressions of the 1-loop path integrals for massive, shift-symmetric and partially massless fields on S^{d+1}. We conclude with some open problems and generalizations for future investigations:

First is the Polchinski's phase. While we generalize the original massless spin-2 result to other classes of fields, their physical interpretations remain elusive. One is tempted to say perhaps these phases indicate non-unitarity. While this seems to be natural for massive fields with masses outside the unitary bounds (including shift-symmetric fields), the phases are present for PM fields which are perfectly unitary irreducible representations. Without other physical inputs, it is not clear whether we should ignore or retain these phases. However, we stress that these phases deserve our attentions. Perhaps a better understanding of these phases will lead us to the correct statistical interpretation of the path integral[17]. Also, S^{d+1} is only one of the many saddle points of the Euclidean gravitational path integral with a positive cosmological constant. If one considers other saddle points such as $S^2 \times S^{d-1}$, since they have different amount of symmetries, after Wick rotating the conformal modes there will be **relative** phases between different saddle points. In any case, our results provide infinite number of data points for further investigations.

Another mystery is the residual group volume factor present for PM gauge fields and shift-symmetric fields. Such a factor is present for a manifestly local path integral and depends on the non-linear completion of the theory. Higher spin groups are typically infinite-dimensional and there is an issue of making sense of the group volume. The group volume may be more well-defined in theories gauging finite dimensional higher spin algebras studied in [117].

Let us mention the context in which both subtleties of phases and group volume are sharpest, namely in $d+1 = 3$ dimension [73]. In this case one can check that for any PM fields, the determinants for the on-shell kinetic operator and the ghost operator cancel completely, so that the group volume and phases are the **only** non-trivial contributions to the 1-loop path integral. Also, on S^3 there is an alternative formulation of massless HS gravity as a $SU(N) \times SU(N)$ Chern-Simons theory. As noted in [73], one finds that their 1-loop results agree only if we identify the residual group volume with the $SU(N) \times SU(N)$ HS group volume, further supporting the claim that this factor depends on the interactions of the full theory. Also, the phases will match exactly for odd framing.

[17] For example, one might guess that these i's are precisely the i's present in the inverse Laplace transform to extract microcanonical entropies from the partition function.

One natural generalization of this work is to study path integrals involving fermionic PM gauge fields. The free actions for massless fermionic fields are presented in [119]. Since fermionic fields are Grassman-valued, no Wick rotation is needed to make the path integral convergent. However, there is still a group volume factor corresponding to trivial fermionic gauge transformations, whose physical interpretations are even more obscure than their bosonic counterparts, because the Grassman integrals are formally zero. Perhaps we need to combine bosonic and fermionic higher spin fields into a supersymmetric HS theory [120] so that we can make sense of the super-higher-spin group volume.

Chapter 3: Quantum de Sitter horizon entropy from quasicanonical bulk, edge, sphere and topological string partition functions

This chapter is a based on the work [73]. Motivated by the prospect of constraining microscopic models, we calculate the exact one-loop corrected de Sitter entropy (the logarithm of the sphere partition function) for every effective field theory of quantum gravity, with particles in arbitrary spin representations. In doing so, we universally relate the sphere partition function to the quotient of a quasi-canonical bulk and a Euclidean edge partition function, given by integrals of characters encoding the bulk and edge spectrum of the observable universe. Expanding the bulk character splits the bulk (entanglement) entropy into quasinormal mode (quasiqubit) contributions. For 3D higher-spin gravity formulated as an sl(n) Chern-Simons theory, we obtain all-loop exact results. Further to this, we show that the theory has an exponentially large landscape of de Sitter vacua with quantum entropy given by the absolute value squared of a topological string partition function. For generic higher-spin gravity, the formalism succinctly relates dS, AdS$^{\pm}$ and conformal results. Holography is exhibited in quasi-exact bulk-edge cancelation.

3.1 Introduction

As seen by local inhabitants [7–10, 121–123] of a cosmology accelerated by a cosmological constant, the observable universe is evolving towards a semiclassical equilibrium state asymptotically indistinguishable from a de Sitter static patch, enclosed by a horizon of area $A = \Omega_{d-1}\ell^{d-1}$, $\ell \propto 1/\sqrt{\Lambda}$, with the de Sitter universe globally in its Euclidean vacuum state. A picture is shown in fig. 3.1b, and the metric in (3.26)/(C.96)S. The semiclassical equilibrium state locally maximizes

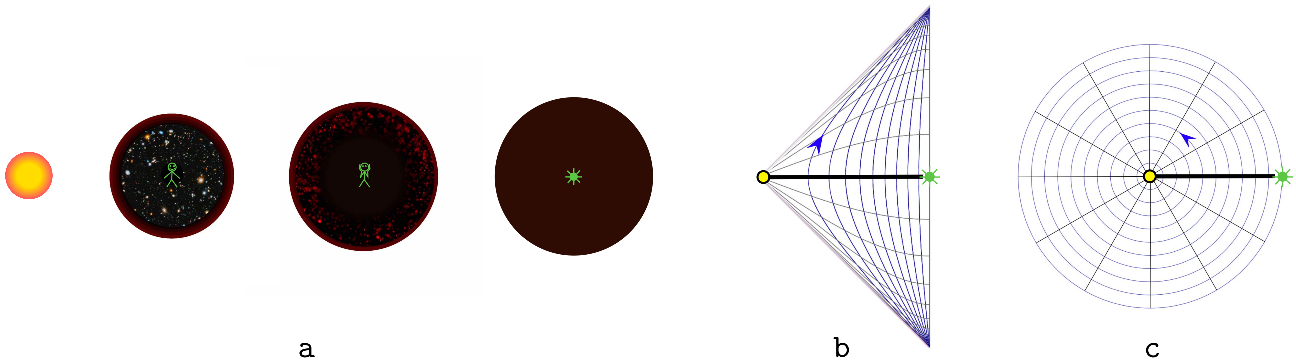

Figure 3.1: `a`: Cartoon of observable universe evolving to its maximal-entropy equilibrium state. The horizon consumes everything once seen, growing until it reaches its de Sitter equilibrium area A. (The spiky dot is a reference point for `b`, `c`; it will ultimately be gone, too.) `b`: Penrose diagram of dS static patch. `c`: Wick-rotated (`b`) = sphere. Metric details are given in appendix C.4.3 + fig. C.5`c`,`d`.

the observable entropy at a value $\mathcal{S}$ semiclassically given by [10]

$$\mathcal{S} = \log \mathcal{Z} \,, \tag{3.1}$$

where $\mathcal{Z} = \int e^{-S_E[g,\cdots]}$ is the effective field theory Euclidean path integral, expanded about the round sphere saddle related by Wick-rotation (C.98) to the de Sitter universe of interest. At tree level in Einstein gravity, the familiar area law is recovered:

$$\mathcal{S}^{(0)} = \frac{A}{4G_{\mathrm{N}}} \,. \tag{3.2}$$

The interpretation of $\mathcal{S}$ as a (metastable) equilibrium entropy begs for a microscopic understanding of its origin. By aspirational analogy with the Euclidean AdS partition function for effective field theories with a CFT dual (see [124] for a pertinent discussion), a natural question is: are there effective field theories for which the semiclassical expansion of $\mathcal{S}$ corresponds to a large-N expansion of a microscopic entropy? Given a proposal, how can it be tested?

In contrast to EAdS, without making any assumptions about the UV completion of the effective field theory, there is no evident extrinsic data constraining the problem. The sphere has no boundary, all symmetries are gauged, and physically meaningful quantities must be gauge and field-redefinition invariant, leaving little. In particular there is no invariant information contained in the tree-level $\mathcal{S}^{(0)}$ other than its value, which in the low-energy effective field theory merely represents a renormalized coupling constant; an input parameter. However, in the spirit of [26–34, 124–126], nonlocal *quantum* corrections to $\mathcal{S}$ do offer unambiguous, intrinsic data, directly constraining models. To give a simple example, discussed in more detail under (3.168), say someone posits that for pure 3D gravity, the sought-after microscopic entropy is $S_{\rm micro} = \log d(N)$, where $d(N)$ is the number of partitions of N. This is readily ruled out. Both macroscopic and microscopic entropy expansions can uniquely be brought to a form

$$\mathcal{S} = \mathcal{S}_0 - a \log \mathcal{S}_0 + b + \textstyle\sum_n c_n \, \mathcal{S}_0^{-2n} \; + \; O(e^{-\mathcal{S}_0/2}) , \tag{3.3}$$

characterized by absence of odd (=local) powers of $1/\mathcal{S}^{(0)}$. The microscopic theory predicts $(a, b) = \big(2, \log\big(\pi^2/6\sqrt{3}\big)\big)$, refuted by the macroscopic one-loop result $(a, b) = \big(3, 5\log(2\pi)\big)$. Some of the models in [11–24] are sufficiently detailed to be tested along these lines.

In this work, we focus exclusively on collecting macroscopic data, more specifically the exact one-loop (in some cases all-loop) corrected $\mathcal{S} = \log \mathcal{Z}$. The problem is old, and computations for $s \leq 1$ are relatively straightforward, but for higher spin $s \geq 2$, sphere-specific complications crop up. Even for pure gravity [50–54, 57–59, 61, 127], virtually no complete, exact results have been obtained at a level brining tests of the above kind to their full potential.

Building on results and ideas from [43, 62, 66, 68–71, 81, 101, 128, 129], we obtain a universal formula solving this problem in general, for all $d \geq 2$ parity-invariant effective field theories, with matter in arbitrary representations, and general gauge symmetries including higher-spin:

$$\boxed{\mathcal{S}^{(1)} \; = \; \log \prod_{a=0}^{K} \frac{(2\pi\gamma_a)^{\dim G_a}}{\mathrm{vol}\, G_a} \; + \int_0^\infty \frac{dt}{2t}\left(\frac{1+q}{1-q}\; \chi_{\rm tot}^{\rm bos} - \frac{2\sqrt{q}}{1-q}\; \chi_{\rm tot}^{\rm fer}\right) + \mathcal{S}_{\rm ct}} \tag{3.4}$$

$q \equiv e^{-t/\ell}$. Below we explain the ingredients in sufficient detail to allow application in practice. A sample of explicit results is listed in (3.12). We then summarize the content of the paper by section, with more emphasis on the physics and other results of independent interest.

G_0 is the subgroup of (possibly higher-spin) gravitational gauge transformations acting trivially on the S^{d+1} saddle. This includes rotations of the sphere. $\mathrm{vol}\, G_0$ is the volume for the invariant metric normalized such that the standard rotation generators have unit norm, implying in particular $\mathrm{vol}\, SO(d+2) =$ (C.93). The other G_i, $i = 1, \ldots, K$ are Yang-Mills group factors, with $\mathrm{vol}\, G_i$ the volume in the metric defined by the trace in the action, as in (C.94). The γ_a are proportional to the (algebraically defined) gauge couplings:

$$\gamma_0 \equiv \sqrt{\frac{8\pi G_{\mathrm{N}}}{A_{d-1}}} = \sqrt{\frac{2\pi}{\mathcal{S}^{(0)}}}, \qquad\qquad \gamma_i \equiv \sqrt{\frac{g_i^2}{2\pi A_{d-3}}}, \tag{3.5}$$

with $A_n \equiv \Omega_n \ell^n$, $\Omega_n =$ (C.92) for $n \geq 0$, and $A_{-1} \equiv 1/2\pi\ell$ for γ_i in $d = 2$.

The functions $\chi_{\mathrm{tot}}(t)$ are determined by the bosonic/fermionic physical particle spectrum of the theory. They take the form of a "bulk" *minus* an "edge" character:

$$\chi_{\mathrm{tot}} = \chi_{\mathrm{bulk}} - \chi_{\mathrm{edge}} \,. \tag{3.6}$$

The bulk character $\chi_{\mathrm{bulk}}(t)$ is defined as follows. Single-particle states on global dS_{d+1} furnish a representation R of the isometry group $SO(1, d+1)$. The content of R is encoded in its Harish-Chandra character $\tilde{\chi}(g) \equiv \mathrm{tr}\, R(g)$ (appendix C.1). Restricted to $SO(1,1)$ isometries $g = e^{-itH}$ acting as time translations on the static patch, $\tilde{\chi}(g)$ becomes $\chi_{\mathrm{bulk}}(t) \equiv \mathrm{tr}\, e^{-itH}$. For example for a massive integer spin-s particle it is given by (C.14):

$$\chi_{\mathrm{bulk,s}} = D_s^d \, \frac{q^{\frac{d}{2}+i\nu} + q^{\frac{d}{2}-i\nu}}{(1-q)^d}, \qquad\qquad q \equiv e^{-|t|/\ell}, \tag{3.7}$$

where D_s^d is the spin degeneracy (C.15), e.g. $D_s^3 = 2s+1$, and ν is related to the mass:

$$s = 0: \; \nu^2 = m^2\ell^2 - \left(\tfrac{d}{2}\right)^2, \qquad\qquad s \geq 1: \; \nu^2 = m^2\ell^2 - \left(\tfrac{d}{2} + s - 2\right)^2. \tag{3.8}$$

For arbitrary massive matter $\chi_{\rm bulk}$ is given by (C.16). Massless spin-s characters are more intricate, but can be obtained by applying a simple "flipping" recipe (3.100) to (3.98), or from the general formulae (C.164) or (C.194) derived from this. Some low (d,s) examples are

(d,s)	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(4,1)$	$(4,2)$
$\chi_{\rm bulk,s}$	$\frac{2\,q}{(1-q)^2}$	0	$\frac{6\,q^2-2\,q^3}{(1-q)^3}$	$\frac{10\,q^3-6\,q^4}{(1-q)^3}$	$\frac{6\,q^2}{(1-q)^4}$	$\frac{10\,q^2}{(1-q)^4}$

(3.9)

The q-expansion of $\chi_{\rm bulk}$ gives the static patch *quasinormal* mode degeneracies, its Fourier transform gives the *normal* mode spectral density, and the bulk part of (3.4) is the quasicanonical ideal gas partition function at $\beta = 2\pi\ell$, as we explain below (3.15).

The edge character $\chi_{\rm edge}(t)$ is inferred from path integral considerations in sections 3.3-3.5. It vanishes for spin $s < 1$. For integer $s \geq 1$ we get (3.85):

$$\chi_{\rm edge,s} = N_s \cdot \frac{q^{\frac{d-2}{2}+i\nu} + q^{\frac{d-2}{2}-i\nu}}{(1-q)^{d-2}}\,, \qquad N_s = D^{d+2}_{s-1}\,, \tag{3.10}$$

e.g. $N_1 = 1$, $N_2 = d+2$. Note this is the bulk character of N_s scalars in *two lower* dimensions. Thus the edge correction effectively *subtracts* the degrees of freedom of N_s scalars living on S^{d-1}, the horizon "edge" of static time slices (yellow dot in fig. 3.1). (3.91) yields analogous results for more general matter; e.g. $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ bulk field $\to$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ edge field, $\square\square$ bulk $\to$ $(d+2)\times\square$ edge. For massless spin-s, use (3.98)-(3.100) or (C.196). The edge companions of (3.9) are

(d,s)	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(4,1)$	$(4,2)$
$\chi_{\rm edge,s}$	0	0	$\frac{2\,q}{1-q}$	$\frac{10\,q^2-2\,q^3}{1-q}$	$\frac{2\,q}{(1-q)^2}$	$\frac{10\,q}{(1-q)^2}$

(3.11)

The edge correction extends observations of [38–43, 45–48, 130–137], reviewed in appendix C.5.5.

The general closed-form evaluation of the integral in (3.4) is given by (C.57) in heat kernel regularization. In even d, the finite part is more easily obtained by summing residues.

Finally, $\mathcal{S}_{\rm ct}$ in (3.4) is a local counterterm contribution fixed by a renormalization condition

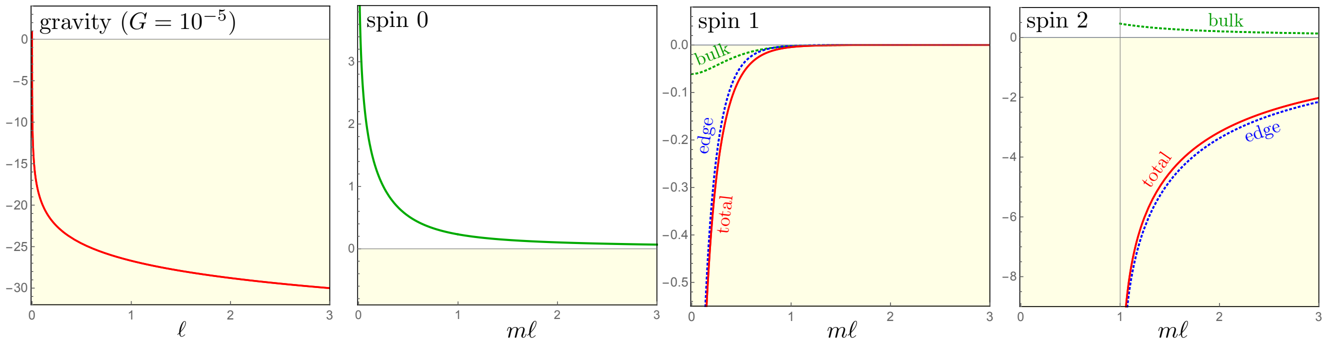

Figure 3.2: Contributions to dS_3 one-loop entropy from gravity and massive $s = 0, 1, 2$.

specified in section 3.8, which in practice boils down to $\mathcal{S}_{\rm ct}(\ell)$ canceling all divergences and finite terms growing polynomially with ℓ in $\mathcal{S}^{(1)}(\ell)$.

For concreteness here are some examples readily obtained from (3.4):

content	$\mathcal{S}^{(1)}$
3D grav	$-3\log\mathcal{S}^{(0)} + 5\log(2\pi)$
3D (s,m)	$\frac{\pi}{3}\big(\nu^3 - (m\ell)^3 + \frac{3(s-1)^2}{2}m\ell\big) - 2\sum_{k=0}^{2}\frac{\nu^k}{k!}\frac{\mathrm{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}} - s^2\big(\pi(m\ell-\nu) - \log\big(1-e^{-2\pi\nu}\big)\big)$
4D grav	$-5\log\mathcal{S}^{(0)} - \frac{571}{45}\log(\ell/L) - \log\frac{8\pi}{3} + \frac{715}{48} - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3)$
5D su(4) ym	$-\frac{15}{2}\log\big(\ell/g^2\big) - \log\frac{256\pi^9}{3} + \frac{75\,\zeta(3)}{16\,\pi^2} + \frac{45\,\zeta(5)}{16\,\pi^4}$
5D $(\boxplus, m)$	$-15\log(2\pi m\ell) + \frac{5\,\zeta(5)}{8\pi^4} + \frac{65\,\zeta(3)}{24\pi^2} \quad (m\ell\to 0), \qquad \frac{5}{12}(m\ell)^4\,e^{-2\pi m\ell} \quad (m\ell\to\infty)$
11D grav	$-33\log\mathcal{S}^{(0)} + \log\big(\frac{4!\,6!\,8!\,10!}{2^4}(2\pi)^{63}\big) + \frac{1998469\,\zeta(3)}{50400\,\pi^2} + \frac{135619\,\zeta(5)}{60480\,\pi^4} - \frac{34463\,\zeta(7)}{3840\,\pi^6} + \frac{11\,\zeta(9)}{6\pi^8} - \frac{11\,\zeta(11)}{256\,\pi^{10}}$
3D HS_n	$-(n^2-1)\log\mathcal{S}^{(0)} + \log\Big[\frac{1}{n}\big(\frac{n(n^2-1)}{6}\big)^{n^2-1}\,\mathrm{G}(n+1)^2\,(2\pi)^{(n-1)(2n+1)}\Big]$

(3.12)

Comparison to previous results for 3D and 4D gravity is discussed under (3.120).[1]

The second line is the contribution of a 3D massive spin-s field, with ν given by (3.8). The term $\propto s^2$ is the edge contribution. It is negative for all $m\ell$ and dominates the bulk contribution (fig. 3.2). It diverges at the unitarity/Higuchi bound $m\ell = s - 1$.

In the 4D gravity example, L is a minimal subtraction scale canceling out of $\mathcal{S}^{(0)} + \mathcal{S}^{(1)}$. In this case, constant terms in $\mathcal{S}^{(1)}$ cannot be distinguished from constants in $\mathcal{S}^{(0)}$ and are as such

[1] In the above and in (3.4) we have dropped Polchinski's phase [59] kept in (3.120) and generalized in (3.112).

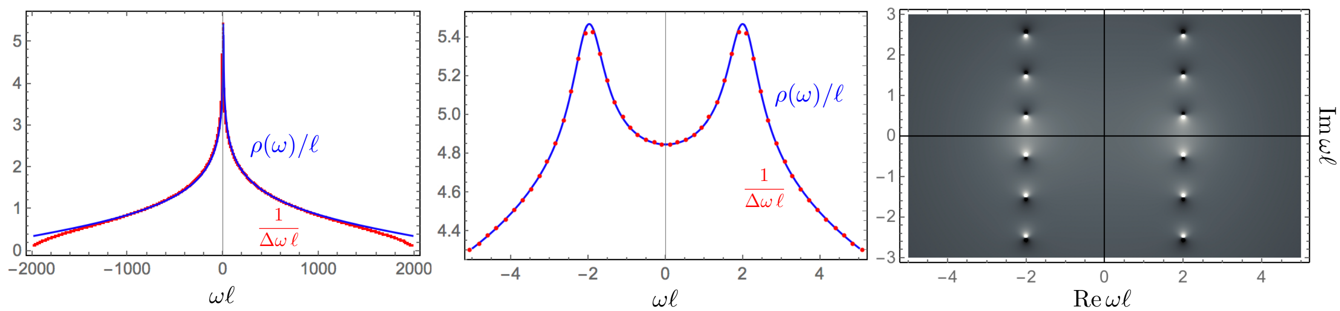

Figure 3.3: Regularized dS_2 scalar mode density with $\nu = 2$, $\Lambda_{\rm uv}\ell \approx 4000$. Blue line = Fourier transform of $\chi_{\rm bulk}$: $\rho(\omega)/\ell = \frac{2}{\pi}\log(\Lambda_{\rm uv}\ell) - \frac{1}{2\pi}\sum\psi\big(\frac{1}{2} \pm i\nu \pm i\omega\ell\big)$. Red dots = inverse eigenvalue spacing of numerically diagonalized 4000×4000 matrix H in globally truncated model (appendix C.2.2). Rightmost panel = $|\rho(\omega)|$ on complex ω-plane, with quasinormal mode poles at $\omega\ell = \pm i(\frac{1}{2} \pm i\nu + n)$.

physically ambiguous.[2] The term $\alpha_4 \log(\ell/L)$ with $\alpha_4 = -\frac{571}{45}$ arises from the log-divergent term $\alpha_4 \log(\ell/\epsilon)$ of the regularized character integral.

For any d, in any theory, the coefficient α_{d+1} of the log-divergent term can simply be read off from the $t \to 0$ expansion of the integrand in (3.4):

$$\text{integrand} = \cdots + \frac{\alpha_{d+1}}{t} + O(t^0) \tag{3.13}$$

For a 4D photon, this gives $\alpha_4 = \alpha_{4,\rm bulk} + \alpha_{4,\rm edge} = -\frac{16}{45} - \frac{1}{3} = -\frac{31}{45}$. The bulk-edge split in this case is the same as the split investigated in [132, 137, 138]. Other illustrations include (partially) massless spin s around (3.116), the superstring in (3.192), and conformal spin s in (3.193).

3D HS_n = higher-spin gravity with $s = 2, 3, \ldots, n$ (section 3.6). $\mathtt{G}$ is the Barnes G-function.

Overview

We summarize the content of sections 3.2-3.9, highlighting other results of interest, beyond (3.4).

Quasicanonical bulk thermodynamics of the static patch (section 3.2)

The global dS bulk character $\chi_{\rm bulk}(t) = \operatorname{tr} e^{-itH}$ locally encodes the quasinormal spectrum and normal mode density of the static patch $ds^2 = -(1 - r^2/\ell^2)dT^2 + (1 - r^2/\ell^2)^{-1}dr^2 + r^2 d\Omega^2$ on which e^{-itH} acts as a time translation $T \to T + t$. Its expansion in powers of $q = e^{-|t|/\ell}$,

$$\chi_{\rm bulk} = \sum_r N_r \, q^r \,, \tag{3.14}$$

yields the number N_r of *quasinormal* modes decaying as $e^{-rT/\ell}$, in resonance with [139–141]. The density of *normal* modes $\propto e^{-i\omega T}$ is formally given by its Fourier transform

$$\rho(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, \chi_{\rm bulk}(t) \, e^{i\omega t} . \tag{3.15}$$

Because $\chi_{\rm bulk}$ is singular at $t = 0$, this is ill-defined as it stands. However, a standard Pauli-Villars regularization of the QFT renders it regular (3.40), yielding a manifestly covariantly regularized mode density, analytically calculable for arbitrary particle content, including gravitons and higher-spin matter. Some simple examples are shown in figs. 3.3, 3.6. Quasinormal modes appear as resonance poles at $\omega = \pm ir$, seen by substituting (3.14) into (3.15).

This effectively solves the problem of making covariant sense of the formally infinite normal mode density universally arising in the presence of a horizon [35]. Motivated by the fact that semiclassical information loss can be traced back to this infinity, [35] introduced a rough model getting rid of it by shielding the horizon by a "brick wall" (reviewed together with variants in C.5.3). Evidently this alters the physics, introduces boundary artifacts, breaks covariance, and is, unsurprisingly, computationally cumbersome. The covariantly regularized density (3.15) suffers none of these problems.

[2] Comparing different saddles, unambiguous linear combinations can however be extracted, cf. (C.345).

In particular it makes sense of the *a priori* ill-defined canonical ideal gas partition function,

$$\log Z_{\rm can}(\beta) = \int_0^\infty d\omega \Big(-\rho_{\rm bos}(\omega)\, \log\big(e^{\beta\omega/2} - e^{-\beta\omega/2}\big) + \rho_{\rm fer}(\omega)\, \log\big(e^{\beta\omega/2} + e^{-\beta\omega/2}\big) \Big) . \tag{3.16}$$

Substituting (3.15) and integrating out ω, this becomes

$$\boxed{\log Z_{\rm bulk}(\beta) = \int_0^\infty \frac{dt}{2t} \left(\frac{1 + e^{-2\pi t/\beta}}{1 - e^{-2\pi t/\beta}}\, \chi_{\rm bulk}^{\rm bos}(t) - \frac{2\, e^{-\pi t/\beta}}{1 - e^{-2\pi t/\beta}}\, \chi_{\rm bulk}^{\rm fer}(t) \right)} \tag{3.17}$$

At the static patch equilibrium $\beta = 2\pi\ell$, this is precisely the *bulk* contribution to the one-loop Euclidean partition function $\log \mathcal{Z}^{(1)}$ in (3.4). Although $Z_{\rm bulk}$ is not quite a standard canonical partition function, calling it a quasicanonical partition function appears apt.

From (3.17), covariantly regularized quasicanonical bulk thermodynamic quantities can be analytically computed for general particle content, as illustrated in section 3.2.3. Substituting the expansion (3.14) expresses these quantities as a sum of quasinormal mode contributions, generalizing and refining [142]. In particular the contribution to the entropy and heat capacity from each physical quasinormal mode is finite and positive (fig. 3.8).

$S_{\rm bulk}$ can alternatively be viewed as a covariantly regularized entanglement entropy between two hemispheres in the global dS Euclidean vacuum (red and blue lines in figs. 3.5, C.5). In the spirit of [140], the quasinormal modes can then be viewed as entangled quasiqubits.

Sphere partition functions (sections 3.3,3.4,3.5)

In sections 3.3-3.5 we obtain character integral formulae computing exact heat-kernel regularized one-loop sphere partition functions $Z_{\rm PI}^{(1)}$ for general field content, leading to (3.4).

For scalars and spinors (section 3.3), this is easy. For massive spin s (section 3.4), the presence of conformal Killing tensors on the sphere imply naive reduction to a spin-s Laplacian determinant is inconsistent with locality [62]. The correct answer can in principle be obtained by path integrating the full off-shell action [102], but this involves an intricate tower of spin $s' < s$ Stueckelberg fields. Guided by intuition from section 3.2, we combine locality and unitary constraints with path

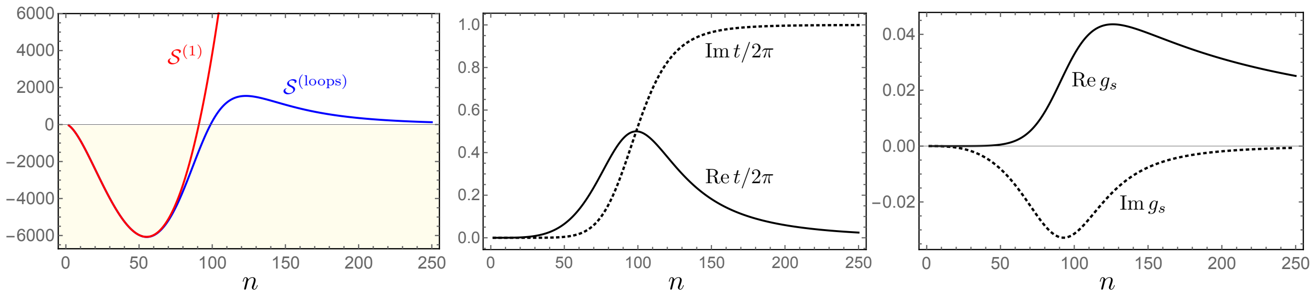

Figure 3.4: One- and all-loop entropy corrections, and dual topological string t, g_s, for 3D HS_n theory in its maximal-entropy de Sitter vacuum, for different values of n at fixed $\mathcal{S}^{(0)} = 10^8$, $l = 0$.

integral considerations to find the terms in $\log Z$ missed by naive reduction. They turn out to be obtained simply by extending the spin-s Laplacian eigenvalue sum to include its "subterranean" levels with formally negative degeneracies, (3.83). The extra terms capture contributions from unmatched spin $s' < s$ conformal Killing tensor ghost modes in the gauge-fixed Stueckelberg path integral. The resulting sum yields the bulk−edge character integral formula (3.84). Locality and unitarity uniquely determine the generalization to arbitrary parity-symmetric matter representations, (3.91).

In the massless case (section 3.5), new subtleties arise: negative modes requiring contour rotations (which translate into the massless character "flipping" recipe mentioned above (3.9)), and ghost zeromodes which must be omitted and compensated by a carefully normalized group volume division. Non-universal factors cancel out, yielding (3.112) modulo renormalization.

3D de Sitter HS_n quantum gravity and the topological string (section 3.6)

The sl(2) Chern-Simons formulation of 3D gravity [143, 144] can be extended to an sl(n) Chern-Simons formulation of $s \leq n$ higher-spin (HS_n) gravity [145–148]. The action for positive cosmological constant is given by (3.121). It has a real coupling constant $\kappa \propto 1/G_{\text{N}}$, and an integer coupling constant $l \in \{0, 1, 2, \ldots\}$ if a gravitational Chern-Simons term is included.

This theory has a landscape of dS_3 vacua, labeled by partitions $\vec{m} = \{m_1, m_2, \ldots\}$ of n. Different vacua have different values of $\ell/G_{\rm N}$, with tree-level entropy

$$\mathcal{S}^{(0)}_{\vec{m}} = \frac{2\pi\ell}{4G_{\rm N}}\bigg|_{\vec{m}} = 2\pi\kappa\cdot T_{\vec{m}}\,, \qquad T_{\vec{m}} = \tfrac{1}{6}\textstyle\sum_a m_a(m_a^2-1)\,. \tag{3.18}$$

The number of vacua grows as $\mathcal{N}_{\rm vac} \sim e^{2\pi\sqrt{n/6}}$. The maximal entropy vacuum is $\vec{m} = \{n\}$.

We obtain the all-loop exact quantum entropy $\mathcal{S}_{\vec{m}} = \log \mathcal{Z}_{\vec{m}}$ by analytic continuation $k_\pm \to l \pm i\kappa$ of the $SU(n)_{k_+} \times SU(n)_{k_-}$ Chern-Simons partition function on S^3, (3.127). In the weak-coupling limit $\kappa \to \infty$, this reproduces $\mathcal{S}^{(1)}$ as computed by (3.4) in the metric-like formulation of the theory, given in (3.12) for the maximal-entropy vacuum $\vec{m} = \{n\}$.

When n grows large and reaches a value $n \sim \kappa$, the 3D higher-spin gravity theory becomes strongly coupled. (In the vacuum $\vec{m} = \{n\}$ this means $n^4 \sim \ell/G_{\rm N}$.) In this regime, Gopakumar-Vafa duality [79, 80] can be used to express the quantum de Sitter entropy $\mathcal{S}$ in terms of a weakly-coupled topological string partition function on the resolved conifold, (3.128):

$$\boxed{\mathcal{S}_{\vec{m}} = \log\left|\tilde{Z}_{\rm top}(g_s,t)\, e^{-\pi T_{\vec{m}}\cdot 2\pi i/g_s}\right|^2} \tag{3.19}$$

where $g_s = \frac{2\pi}{n+l+i\kappa}$, and the conifold Kähler modulus $t \equiv \int_{S^2} J + iB = i g_s n = \frac{2\pi i n}{n+l+i\kappa}$.

Euclidean thermodynamics of the static patch (section 3.7)

In section 3.7 we consider the Euclidean thermodynamics of a QFT on a fixed static patch/sphere background. The partition function $Z_{\rm PI}$ is the Euclidean path integral on the sphere of radius ℓ, the Euclidean energy density is $\rho_{\rm PI} = -\partial_V \log Z_{\rm PI}$, where $V = \Omega_{d+1}\ell^{d+1}$ is the volume of the sphere, and the entropy is $S_{\rm PI} = \log Z_{\rm PI} + 2\pi\ell\, U_{\rm PI} = \log Z_{\rm PI} + V\rho_{\rm PI} = (1 - V\partial_V)\log Z_{\rm PI}$, or

$$\boxed{S_{\rm PI} = \left(1 - \tfrac{1}{d+1}\ell\partial_\ell\right)\log Z_{\rm PI}} \tag{3.20}$$

Using the exact one-loop sphere partition functions obtained in sections 3.3-3.5, this allows general exact computation of the one-loop Euclidean entropy $S^{(1)}_{\rm PI}$, illustrated in section 3.7.2. Euclidean

Rindler results are recovered in the limit $m\ell \to \infty$. The sphere computation avoids introducing the usual conical deficit angle, varying the curvature radius ℓ instead.

For minimally coupled scalars, $S_{\rm PI}^{(1)} = S_{\rm bulk}$, but more generally this is false, due to edge (and other) corrections. Our results thus provide a precise and general version of observations made in the work reviewed in appendix C.5.5. Of note, these "corrections" actually *dominate* the one-loop entropy, rendering it negative, increasingly so as s grows large.

Quantum gravitational thermodynamics (section 3.8)

In section 3.8 (with details in appendix C.9), we specialize to theories with dynamical gravity. Denoting $Z_{\rm PI}$, $\rho_{\rm PI}$ and $S_{\rm PI}$ by $\mathcal{Z}$, ϱ and $\mathcal{S}$ in this case, (3.20) trivially implies $\varrho = 0$, $\mathcal{S} = \log \mathcal{Z}$, reproducing (3.1). All UV-divergences can be absorbed into renormalized coupling constants, rendering the Euclidean thermodynamics well-defined in an effective field theory sense.

Integrating over the geometry is similar in spirit to integrating over the temperature in statistical mechanics, as one does to extract the microcanonical entropy $S(U)$ from the canonical partition function.[3] The analog of this in the case of interest is

$$S(\rho) \equiv \log \int \mathcal{D}g \, \cdots \, e^{-S_E[g,\ldots] + \rho \int \sqrt{g}}\,, \tag{3.21}$$

for some suitable metric path integration contour. In particular $S(0) = \mathcal{S}$. The analog of the microcanonical $\beta \equiv \partial_U S$ is $V \equiv \partial_\rho S$, and the analog of the microcanonical free energy is the Legendre transform $\log Z \equiv S - V\rho$, satisfying $\rho = -\partial_V \log Z$. If we furthermore *define* ℓ by $\Omega_{d+1}\ell^{d+1} \equiv V$, the relation between $\log Z$, ρ and S is by construction identical to (3.20).

Equivalently, the free energy $\Gamma \equiv -\log Z$ can be thought of as a quantum effective action for the volume. At tree level, Γ equals the classical action S_E evaluated on the round sphere of radius

[3]Along the lines of $S(U) = \log\left(\frac{1}{2\pi i}\int \frac{d\beta}{\beta} \operatorname{Tr} e^{-\beta H + \beta U}\right)$, with contour $\beta = \beta_* + iy$, $y \in \mathbb{R}$, for any $\beta_* > 0$.

ℓ. For example for 3D Einstein gravity,

$$\log Z^{(0)} = \frac{2\pi^2}{8\pi G}\big(-\Lambda\,\ell^3 + 3\,\ell\big), \qquad S^{(0)} = \big(1 - \tfrac{1}{3}\ell\partial_\ell\big)\log Z^{(0)} = \frac{2\pi\ell}{4G}\,. \tag{3.22}$$

The tree-level on-shell radius ℓ_0 maximizes $\log Z^{(0)}$, i.e. $\rho^{(0)}(\ell_0) = 0$.

We define renormalized $\Lambda, G, \ldots$ from the $\ell^{d+1}, \ell^{d-1}, \ldots$ coefficients in the $\ell \to \infty$ expansion of the quantum $\log Z$, and fix counterterms by equating tree-level and renormalized couplings for the UV-sensitive subset. For 3D Einstein, the renormalized one-loop correction is

$$\log Z^{(1)} = -3\log\frac{2\pi\ell}{4G} + 5\log(2\pi)\,. \tag{3.23}$$

The *quantum* on-shell radius $\bar\ell = \ell_0 + O(G)$ maximizes $\log Z$, i.e. $\rho(\bar\ell) = 0$. The on-shell entropy can be expressed in two equivalent ways to this order:

$$\boxed{\mathcal{S} = S^{(0)}(\bar\ell) + S^{(1)} = S^{(0)}(\ell_0) + \log Z^{(1)}} \tag{3.24}$$

This clarifies why the one-loop correction $\mathcal{S}^{(1)} \equiv \mathcal{S} - \mathcal{S}^{(0)}$ to the dS entropy is given by $\log Z^{(1)}$ rather than $S^{(1)}$: the extra term $-V\rho^{(1)}$ accounts for the change in entropy of the reservoir (= geometry) due to energy transfer to the system (= quantum fluctuations).

The final result is (3.4). We work out several examples in detail. We consider higher-order curvature corrections and discuss invariance under local field redefinitions, identifying the invariants $\mathcal{S}^{(0)}_M = -S_E[g_M]$ for different saddles M as and their large-ℓ expanded quantum counterparts $\mathcal{S}_M$ as the $\Lambda > 0$ analogs of tree-level and quantum scattering amplitudes, defining invariant couplings and physical observables of the low-energy effective field theory.

dS, AdS$^\pm$, and conformal higher-spin gravity (section 3.9)

Massless $\mathfrak{g} = \mathrm{hs}(\mathrm{so}(d+2))$ higher-spin gravity theories on dS_{d+1} or S^{d+1} [91–93] have infinite spin range and infinite $\dim\mathfrak{g}$, obviously posing problems for the one-loop formula (3.4):

1. Spin sum divergences untempered by the UV cutoff, for example $\dim G = \frac{1}{3}\sum_s s(4s^2 - 1)$ for $d = 3$ and $\chi_{\rm tot} = \sum_s (2s+1)\frac{2q^2}{(1-q)^4} - \sum_s \frac{s(s+1)(2s+1)}{6}\frac{2q}{(1-q)^2}$ for $d = 4$.

2. Unclear how to make sense of $\mathrm{vol}\, G$.

We compare the situation to analogous one-loop expressions [83, 129] for Euclidean AdS with standard (AdS^+_{d+1}) [30–33, 125] and alternate (AdS^-_{d+1}) [81] gauge field boundary conditions, and to the associated conformal higher-spin theory on the boundary S^d (CHS_d) [62, 82]. For AdS^+ the above problems are absent, as $\mathfrak{g}$ is not gauged and $\Delta_s > s$. Like a summed KK tower, the spin-summed bulk character has increased UV dimensionality $d_{\rm eff}^{\rm bulk} = 2d - 2$. However, the edge character almost completely cancels this, leading to a *reduced* $d_{\rm eff} = d - 1$ in (3.181)-(3.183). This realizes a version of a stringy picture painted in [38] repainted in fig. C.9. A HS "swampland" is identified: lacking a holographic dual, characterized by $d_{\rm eff} > d - 1$.

For AdS^- and CHS, the problems listed for dS all reappear. $\mathfrak{g}$ is gauged, and the character spin sum divergences are identical to dS, as implied by the relations (3.187):

$$\boxed{\chi_s(\mathrm{CHS}_d) = \chi_s(\mathrm{AdS}^-_{d+1}) - \chi_s(\mathrm{AdS}^+_{d+1}) = \chi_s(\mathrm{dS}_{d+1}) - 2\,\chi_s(\mathrm{AdS}^+_{d+1})} \tag{3.25}$$

The spin sum divergences are not UV. Their origin lies in low-energy features: an infinite number of quasinormal modes decaying as slowly as $e^{-2T/\ell}$ for $d \geq 4$ (cf. discussion below (C.167)). We see no justification for zeta-regularizing such divergences away. However, in certain supersymmetric extensions, the spin sum divergences cancel in a rather nontrivial way, leaving a finite residual as in (3.194). This eliminates problem 1, but leaves problem 2. Problem 2 might be analogous to $\mathrm{vol}\, G = \infty$ for the bosonic string or $\mathrm{vol}\, G = 0$ for supergroup Chern-Simons: removed by appropriate insertions. This, and more, is left to future work.

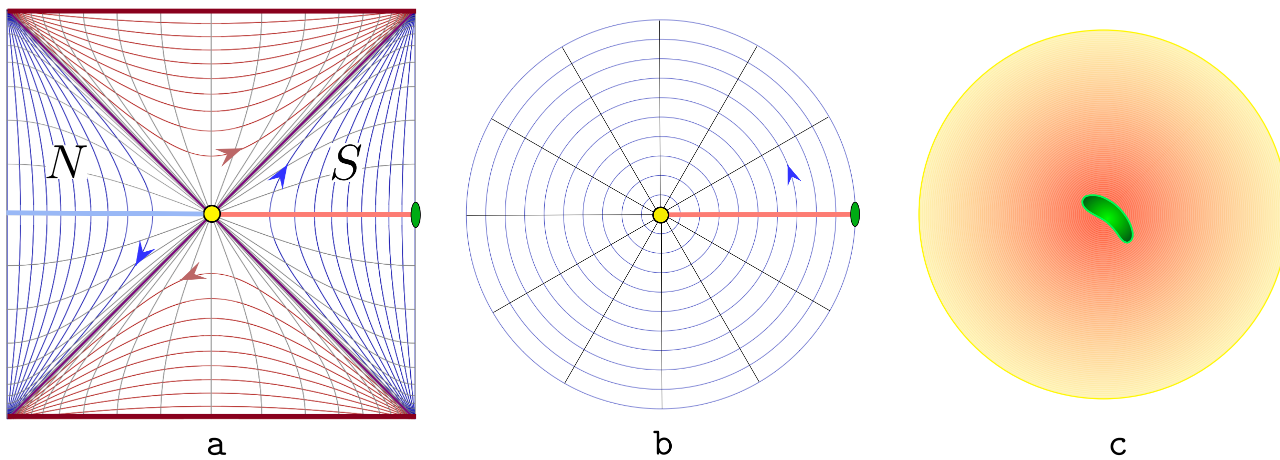

Figure 3.5: a: Penrose diagram of global dS, showing flows of $SO(1,1)$ generator $H = M_{0,d+1}$, S = southern static patch. b: Wick-rotated S = sphere; Euclidean time = angle. c: *Pelagibacter ubique* inertial observer in dS with $\ell = 1.2\,\mu$m finds itself immersed in gas of photons, gravitons and higher-spin particles at a pleasant 30°C. More details are provided in fig. C.5 and appendix C.4.3.

3.2 Quasicanonical bulk thermodynamics

3.2.1 Problem and results

From the point of view of an inertial observer, such as *Pelagibacter ubique* in fig. 3.5c, the global de Sitter vacuum appears thermal [9, 10, 149]: *P. ubique* perceives its universe, the southern static patch (S in fig. 3.5a),

$$ds^2 = -(1 - r^2/\ell^2)dT^2 + (1 - r^2/\ell^2)^{-1}dr^2 + r^2 d\Omega_{d-1}^2, \tag{3.26}$$

as a static ball of finite volume, whose boundary $r = \ell$ is a horizon at temperature $T = 1/2\pi\ell$, and whose bulk is populated by field quanta in thermal equilibrium with the horizon. *P. ubique* wishes to understand its universe, and figures the easiest thing to understand should be the thermodynamics of its thermal environment in the ideal gas approximation. The partition function of an ideal gas is

$$\mathrm{Tr}\, e^{-\beta H} = \exp \int_0^\infty d\omega \Big(-\rho(\omega)_{\rm bos} \log\big(e^{\beta\omega/2} - e^{-\beta\omega/2}\big) + \rho(\omega)_{\rm fer} \log\big(e^{\beta\omega/2} + e^{-\beta\omega/2}\big) \Big), \tag{3.27}$$

where $\rho(\omega) = \rho(\omega)_{\text{bos}} + \rho(\omega)_{\text{fer}}$ is the density of bosonic and fermionic single-particle states at energy ω. However to its dismay, it immediately runs into trouble: the dS static patch mode spectrum is continuous and infinitely degenerate, leading to a pathologically divergent density $\rho(\omega) = \delta(0) \sum_{\ell m \cdots}$. It soon realizes the unbounded redshift is to blame, so it imagines a brick wall excising the horizon, or some variant thereof (appendix C.5.3). Although this allows some progress, it is aware this alters what it is computing and depends on choices. To check to what extent this matters, it tries to work out nontrivial examples. This turns out to be painful. It feels there should be a better way, but its efforts come to an untimely end [4].

Here we will make sense of the density of states and the static patch bulk thermal partition function in a different way, manifestly preserving the underlying symmetries, allowing general exact results for arbitrary particle content. The main ingredient is the Harish-Chandra group character (reviewed in appendix C.1) of the $SO(1, d+1)$ representation R furnished by the physical single-particle Hilbert space of the free QFT quantized on *global* dS_{d+1}. Letting H be the global $SO(1,1)$ generator acting as time translations in the southern static patch and globally as in fig. 3.5a, the character restricted to group elements e^{-itH} is

$$\chi(t) \equiv \mathrm{tr}_G \, e^{-itH} \, . \tag{3.28}$$

Here tr_G traces over the *global* dS single-particle Hilbert space furnishing R. (More generally we denote tr ≡ single-particle trace, Tr ≡ multi-particle trace, G ≡ global, S ≡ static patch. Our default units set the dS radius $\boxed{\ell \equiv 1}$.)

For example for a scalar field of mass $m^2 = (\frac{d}{2})^2 + \nu^2$, as computed in (C.13),

$$\chi(t) = \frac{e^{-t\Delta_+} + e^{-t\Delta_-}}{|1 - e^{-t}|^d} \, , \qquad \Delta_\pm = \tfrac{d}{2} \pm i\nu \, . \tag{3.29}$$

For a massive spin-s field this simply gets an additional spin degeneracy factor D_s^d, (C.14). Mass-

[4] Burdened by broken symmetry and recalcitrant regularization, desperate for help with its cumbersome computation, it decides to engage in self-duplication, unfortunately unaware of a classical equation. Tidal forces trigger tragic disintegration. *P. ubique* is now part of that pesky radiation.

less spin-s characters take a similar but somewhat more intricate form, (C.164)-(C.166).

As mentioned in the introduction, (3.14), the character has a series expansion

$$\chi(t) = \sum_r N_r \, e^{-r|t|} \tag{3.30}$$

encoding the degeneracy N_r of quasinormal modes $\propto e^{-rT}$ of the dS static patch background. For example expanding the scalar character yields two towers of quasnormal modes with $r_{n\pm} = \frac{d}{2} \pm i\nu + n$ and degeneracy $N_{n\pm} = \binom{n+d-1}{n}$.

Our main result, shown in 3.2.2 below, is the observation that

$$\boxed{\log Z_{\rm bulk}(\beta) \equiv \int_0^\infty \frac{dt}{2t} \left(\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}} \, \chi(t)_{\rm bos} \; - \; \frac{2\, e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}} \, \chi(t)_{\rm fer} \right)}\,, \tag{3.31}$$

suitably regularized, provides a physically sensible, manifestly covariant regularization of the static patch bulk thermal partition. The basic idea is that $\rho(\omega)$ can be obtained as a well-defined Fourier transform of the covariantly UV-regularized character $\chi(t)$, which upon substitution in the ideal gas formula (3.27) yields the above character integral formula. Arbitrary thermodynamic quantities at the horizon equilibrium $\beta = 2\pi$ can be extracted from this in the usual way, for example $S_{\rm bulk} = (1-\beta\partial_\beta)\log Z_{\rm bulk}|_{\beta=2\pi}$, which can alternatively be interpreted as the "bulk" entanglement entropy between the northern and southern S^d hemispheres (red and blue lines fig. 3.5a).[5] We work out various examples of such thermodynamic quantities in section 3.2.3. General exact solution are easily obtained. The expansion (3.30) also allows interpreting the results as a sum over quasinormal modes along the lines of [142].

We conclude this part with some comments on the relation with the Euclidean partition function. As reviewed in appendix C.5, general physics considerations, or formal considerations based on Wick-rotating the static patch to the sphere and slicing the sphere path integral along the lines of fig. 3.5b, suggests a relation between the one-loop Euclidean path integral $Z_{\rm PI}^{(1)}$ on S^{d+1} and the

[5] In part because subregion entanglement entropy does not exist in the continuum, an infinity of different notions of it exist in the literature [150]. Based on [138], $S_{\rm bulk}$ appears perhaps most akin to the "extractable"/"distillable" entropy considered there. Either way, our results are nomenclature-independent.

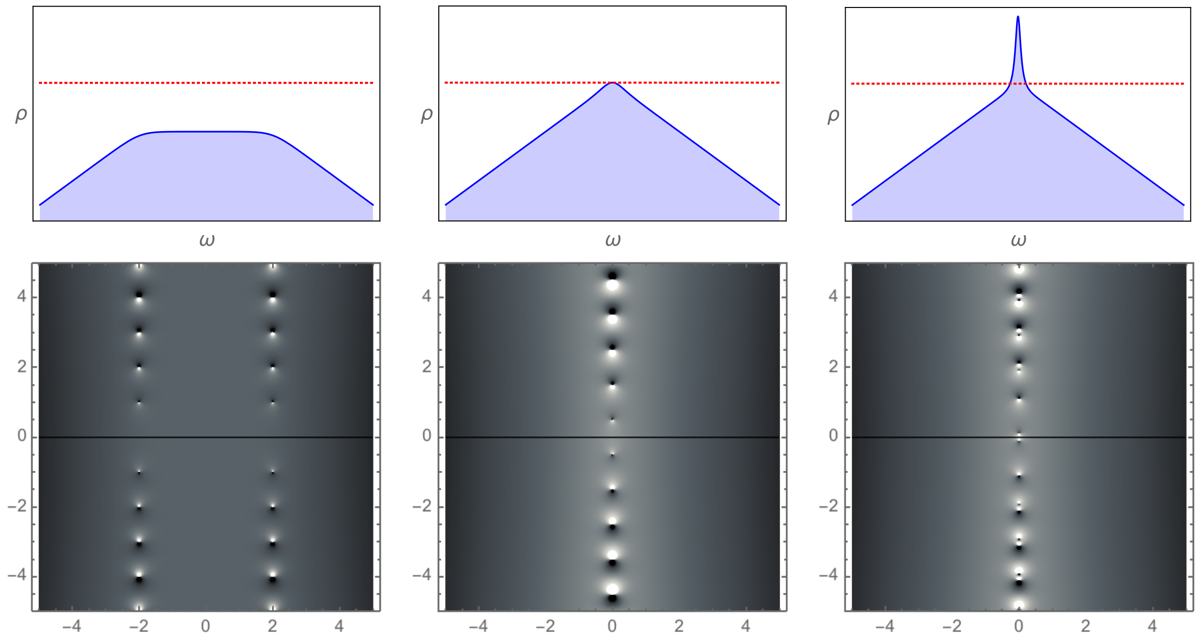

Figure 3.6: Regularized scalar $\rho(\omega)$, $d = 2$, $\nu = 2, i/2, 0.9\,i$; top: $\omega \in \mathbb{R}$; bottom: $\omega \in \mathbb{C}$, showing quasinormal mode poles. See figs. C.1, C.4 for details.

bulk ideal gas thermal partition function $Z_{\rm bulk}$ at $\beta = 2\pi$. More refined considerations suggest

$$\log Z_{\rm PI}^{(1)} = \log Z_{\rm bulk} + \text{edge corrections}\,, \tag{3.32}$$

where the edge corrections are associated with the S^{d-1} horizon edge of the static patch time slices, i.e. the yellow dot in fig. 3.5. The formal slicing argument breaks down here, as does the underlying premise of spatial separability of local field degrees of freedom (for fields of spin $s \geq 1$). Similar considerations apply to other thermodynamic quantities and in other contexts, reviewed in appendix C.5 and more specifically C.5.5.

In sections 3.3-3.5 we will obtain the exact edge corrections by direct computation, logically independent of these considerations, but guided by the physical expectation (3.32) and more generally the intuition developed in this section.

3.2.2 Derivation

We first give a formal derivation and then refine this by showing the objects of interest become rigorously well-defined in a manifestly covariant UV regularization of the QFT.

Formal derivation

Our starting point is the observation that the thermal partition function $\mathrm{Tr}\, e^{-\beta H}$ of a bosonic resp. fermionic oscillator of frequency ω has the integral representation (C.112):

$$\begin{aligned} -\log\big(e^{\beta\omega/2} - e^{-\beta\omega/2}\big) &= +\int_0^\infty \frac{dt}{2t}\,\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}}\,\big(e^{-i\omega t}+e^{i\omega t}\big) \\ \log\big(e^{\beta\omega/2} + e^{-\beta\omega/2}\big) &= -\int_0^\infty \frac{dt}{2t}\,\frac{2\,e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}}\,\big(e^{-i\omega t}+e^{i\omega t}\big)\,. \end{aligned} \tag{3.33}$$

with the pole in the factor $f(t) = c\,t^{-2} + O(t^0)$ multiplying $e^{-i\omega t} + e^{i\omega t}$ resolved by

$$t^{-2} \to \frac{1}{2}\big((t-i\epsilon)^{-2} + (t+i\epsilon)^{-2}\big). \tag{3.34}$$

Now consider a free QFT on some space of finite volume, viewed as a system S of bosonic and/or fermionic oscillator modes of frequencies ω with mode (or single-particle) density $\rho_S(\omega) = \rho_S(\omega)_{\rm bos} + \rho_S(\omega)_{\rm fer}$. The system is in thermal equilibrium at inverse temperature β. Using the above integral representation, we can write its thermal partition function (3.27) as

$$\log \mathrm{Tr}_S\, e^{-\beta H_S} = \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}}\;\chi_S(t)_{\rm bos} - \frac{2\,e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}}\;\chi_S(t)_{\rm fer}\right), \tag{3.35}$$

where we exchanged the order of integration, and we defined

$$\chi_S(t) \equiv \int_0^\infty d\omega\, \rho_S(\omega)\big(e^{-i\omega t} + e^{i\omega t}\big) \tag{3.36}$$

We want to apply (3.35) to a free QFT on the southern static patch at inverse temperature β, with the goal of finding a better way to make sense of it than *P. ubique*'s approach. To this end, we note that the global dS_{d+1} Harish-Chandra character $\chi(t)$ defined in (3.28) can formally be written in a similar form by using the general property (C.4), $\chi(t) = \chi(-t)$:

$$\chi(t) = \mathrm{tr}_G\, e^{-iHt} = \int_{-\infty}^{\infty} d\omega\, \rho_G(\omega)\, e^{-i\omega t} = \int_0^{\infty} d\omega\, \rho_G(\omega)\big(e^{-i\omega t} + e^{i\omega t}\big)\,, \tag{3.37}$$

This looks like (3.36), except $\rho_G(\omega) = \mathrm{tr}_G\, \delta(\omega - H)$ is the density of single-particle excitations of the *global* Euclidean vacuum, while $\rho_S(\omega)$ is the density of single-particle excitations of the *southern* vacuum. The global and southern vacua are very different. Nevertheless, there is a simple kinematic relation between their single-particle creation and annihilation operators: the Bogoliubov transformation (C.108) (suitably generalized to $d > 0$ [149]). This provides an explicit one-to-one, inner-product-preserving map between southern and global single-particle states with $H = \omega > 0$. Hence, formally,

$$\rho_S(\omega) = \rho_G(\omega) \qquad (\omega > 0)\,, \qquad\qquad \rho_S(\omega) = 0 \qquad (\omega < 0)\,. \tag{3.38}$$

While formal in the continuum, this relation becomes precise whenever ρ is rendered effectively finite, e.g. by a brick-wall cutoff or by considering finite resolution projections (say if we restrict to states emitted/absorbed by some apparatus built by *P. ubique*).

At first sight this buys us nothing though, as computing $\rho_G(\omega) = \mathrm{tr}_G\, \delta(\omega - H)$ for say a scalar in dS_4 in a basis $|\omega\ell m\rangle_G$ immediately leads to $\rho_G(\omega) = \delta(0) \sum_{\ell m}$, in reassuring but discouraging agreement with *P. ubique*'s result for $\rho_S(\omega)$. On second thought however, substituting this into (3.37) leads to a nonsensical $\chi(t) = 2\pi\delta(t)\delta(0) \sum_{\ell m}$, not remotely resembling the correct expression (3.29). How could this happen? As explained under (C.17), the root cause is the seemingly natural but actually ill-advised idea of computing $\chi(t) = \mathrm{tr}_G\, e^{-iHt}$ by diagonalizing H: despite its lure of seeming simplicity, $|\omega\ell m\rangle_G$ is in fact the *worst* possible choice of basis to compute the character trace. Its wave functions on the global future boundary S^d of dS_{d+1} are singular at the north and south pole, exactly the fixed points of H at which the correct computation of $\chi(t)$ in appendix C.1.2 localizes. Although $|\omega\ell m\rangle$ is a perfectly fine basis on the cylinder obtained by a conformal map from sphere, the information needed to compute χ is irrecoverably lost by this map.

However we can turn things around, and use the properly computed $\chi(t)$ to extract $\rho_G(\omega)$ as its Fourier transform, inverting (3.37). As it stands, this is not really possible, for (3.29) implies $\chi(t) \sim |t|^{-d}$ as $t \to 0$, so its Fourier transform does not exist. Happily, this problem is automatically resolved by standard UV-regularization of the QFT, as we will show explicitly below. For now let us proceed formally, as at this level we have arrived at our desired result: combining (3.38) with (3.37) and (3.36) implies $\chi_S(t) = \chi(t)$, which by (3.35) yields

$$\mathrm{Tr}_S \, e^{-\beta H_S} = Z_{\mathrm{bulk}}(\beta) \qquad \text{(formal)} \tag{3.39}$$

with $Z_{\mathrm{bulk}}(\beta)$ as defined in (3.31). The above equation formally gives it its claimed thermal interpretation. In what follows we will make this a bit more precise, and spell out the UV regularization explicitly.

Covariant UV regularization of ρ and Z_{bulk}

We begin by showing that $\rho_G(\omega)$ in (3.37) becomes well-defined in a suitable standard UV-regularization of the QFT. As in [36], it is convenient to consider Pauli-Villars regularization, which is manifestly covariant and has a conceptually transparent implementation on both the path integral and canonical sides. For e.g. a scalar of mass $m^2 = (\frac{d}{2})^2 + \nu^2$, a possible implementation is adding $\binom{k}{n}$, $n = 1, \ldots, k \geq \frac{d}{2}$ fictitious particles of mass $m^2 = (\frac{d}{2})^2 + \nu^2 + n\Lambda^2$ and positive/negative norm for even/odd n,[6] turning the character $\chi_{\nu^2}(t)$ of (3.29) into

$$\chi_{\nu^2,\Lambda}(t) = \mathrm{tr}_{G_\Lambda} \, e^{-itH} = \sum_{n=0}^{k} (-1)^n \binom{k}{n} \, \chi_{\nu^2 + n\Lambda^2}(t). \tag{3.40}$$

This effectively replaces $\chi(t) \sim |t|^{-d}$ by $\chi_\Lambda(t) \sim |t|^{2k-d}$ with $2k - d \geq 0$, hence, assuming $\chi(t)$ falls off exponentially at large t, which is always the case for unitary representations [76–78], $\chi_\Lambda(t)$

[6] This is equivalent to inserting a heat kernel regulator $f(\tau\Lambda^2) = \left(1 - e^{-\tau\Lambda^2}\right)^k$ in (3.66), with $k \geq \frac{d}{2} + 1$.

has a well-defined Fourier transform, analytic in ω:

$$\rho_{G,\Lambda}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \chi_\Lambda(t)\, e^{i\omega t} = \frac{1}{2\pi}\int_0^{\infty} dt\, \chi_\Lambda(t)\left(e^{i\omega t} + e^{-i\omega t}\right). \tag{3.41}$$

The above character regularization can immediately be transported to arbitrary massive $SO(1,d+1)$ representations, as their characters χ_{s,ν^2} (C.16) only differ from the scalar one by an overall spin degeneracy factor.[7]

Although we won't need to in practice for computations of thermodynamic quantities (which are most easily extracted directly as character integrals), $\rho_{G,\Lambda}(\omega)$ can be computed explicitly. For the dS_{d+1} scalar, using (3.29) regularized with $k=1$, we get for $\omega \ll \Lambda$

$$\begin{aligned} d=1: &\quad \rho_{G,\Lambda}(\omega) = \frac{2}{\pi}\log\Lambda - \frac{1}{2\pi}\sum_{\pm,\pm}\psi\big(\tfrac{1}{2} \pm i\nu \pm i\omega\big) + O(\Lambda^{-1}) \\ d=2: &\quad \rho_{G,\Lambda}(\omega) = \Lambda - \frac{1}{2}\sum_{\pm}(\omega\pm\nu)\coth\big(\pi(\omega\pm\nu)\big) + O(\Lambda^{-1}) \end{aligned} \tag{3.42}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Denoting the Λ-independent parts of the above $\omega \ll \Lambda$ expansions by $\tilde\rho_{\nu^2}(\omega)$, the exact $\rho_{G,\Lambda}(\omega)$ for general ω and k is $\rho_{G,\Lambda}(\omega) = \sum_{n=0}^{k}(-1)^n\binom{k}{n}\,\tilde\rho_{\nu^2+n\Lambda^2}(\omega)$, illustrated in fig. 3.7 for $k=1,2$. The $\omega \ll \Lambda$ result is independent of k up to rescaling of Λ. The result for massive higher-spin fields is the same up to an overall degeneracy factor D_s^d from (C.14).

To make sense of the southern static patch density $\rho_S(\omega)$ directly in the continuum, we *define* its regularized version by mirroring the formal relation (3.38), thus ensuring all of the well-defined features and physics this relation encapsulates are preserved:

$$\rho_{S,\Lambda}(\omega) \equiv \rho_{G,\Lambda}(\omega) = (3.41) \qquad (\omega > 0)\,. \tag{3.43}$$

This definition of the regularized static patch density evidently inherits all of the desirable properties of $\rho_G(\omega)$: manifest general covariance, independence of arbitrary choices such as brick wall

[7]For massless spin-s, the PV-regulating characters to add to the physical character (e.g. (3.102),(C.165)) are $\hat\chi_{s,n} = \chi_{s,\nu_\phi^2+n\Lambda^2} - \chi_{s-1,\nu_\xi^2+n\Lambda^2}$ where $\nu_\phi^2 = -(s-2+\frac{d}{2})^2$ and $\nu_\xi^2 = -(s-1+\frac{d}{2})^2$, based on (3.95) and (3.97).

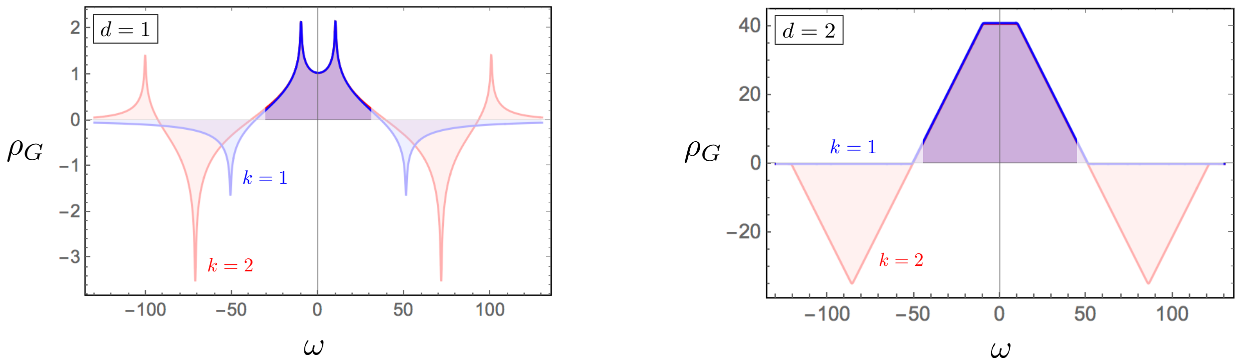

Figure 3.7: $\rho_{G,\Lambda}(\omega)$ for dS_{d+1} scalar of mass $m^2 = (\frac{d}{2})^2 + \nu^2$, $\nu = 10$, for $d = 1, 2$ in $k = 1, 2$ Pauli-Villars regularizations (3.40). Faint part is unphysical UV regime $\omega \gtrsim \Lambda$. The peaks/kinks appearing at $\omega = \sqrt{\nu^2 + n\Lambda^2}$ are related to quasinormal mode resonances $\langle C.2.3 \rangle$.

boundary conditions, and exact analytic computability. The physical sensibility of this identification is also supported by the fact that the quasinormal mode expansion (3.30) of $\chi(t)$ produces the physically expected static patch quasinormal resonance pole structure $\rho_S(\omega) = \frac{1}{2\pi} \sum_{r,\pm} \frac{N_r}{r \pm i\omega}$, cf. appendix C.2.3.

Putting things together in the way we obtained the formal relation (3.39), the correspondingly regularized version of the static patch thermal partition function (3.35) is then

$$\boxed{\log Z_{\mathrm{bulk},\Lambda}(\beta) \equiv \int_0^\infty \frac{dt}{2t} \left(\frac{1 + e^{-2\pi t/\beta}}{1 - e^{-2\pi t/\beta}}\, \chi_\Lambda(t)_{\mathrm{bos}} - \frac{2\, e^{-\pi t/\beta}}{1 - e^{-2\pi t/\beta}}\, \chi_\Lambda(t)_{\mathrm{fer}} \right)} \tag{3.44}$$

Note that if we take $k \geq \frac{d}{2} + 1$, then $\chi_\Lambda(t) \sim t^{2k-d}$ with $2k - d \geq 2$ and we can drop the $i\epsilon$ prescription (3.34). Z_{bulk} (or equivalently χ) can be regularized in other ways, including by cutting off the integral at $t = \Lambda^{-1}$, or as in (C.21), or by dimensional regularization. For most of the paper we will use yet another variant, defined in section 3.3, equivalent, like Pauli-Villars, to a manifestly covariant heat-kernel regularization of the path integral.

In view of the above observations, $Z_{\mathrm{bulk},\Lambda}(\beta)$ is naturally interpreted as a well-defined, covariantly regularized and ambiguity-free *definition* of the static patch ideal gas thermal partition in the continuum. However we refrain from denoting $Z_{\mathrm{bulk}}(\beta)$ as $\mathrm{Tr}_{S,\Lambda}\, e^{-\beta H_S}$, because it is not constructed as an actual sum over states of some definite regularized static patch Hilbert space

$\mathcal{H}_{S,\Lambda}$. This (together with the role of quasinormal modes) is also why we referred to $Z_{\rm bulk}(\beta)$ as a "quasi"-canonical partition function in the introduction.

3.2.3 Example computations

In this section we illustrate the use and usefulness of the character formalism by computing some examples of bulk thermodynamic quantities at the equilibrium inverse temperature $\beta = 2\pi$ of the static patch. The precise relation of these quantities with their Euclidean counterparts will be determined in 3.3-3.5 and 3.7.

Character formulae for bulk thermodynamic quantities at $\beta = 2\pi$

At $\beta = 2\pi$, the bulk free energy, energy, entropy and heat capacity are obtained by taking the appropriate derivatives of (3.31) and putting $\beta = 2\pi$, using the standard thermodynamic relations $F = -\frac{1}{\beta}\log Z$, $U = -\partial_\beta \log Z$, $S = \log Z + \beta U$, $C = -\beta\partial_\beta S$. Denoting $q \equiv e^{-t}$,

$$\log Z_{\rm bulk} = \int_0^\infty \frac{dt}{2t}\left(\frac{1+q}{1-q}\,\chi_{\rm bos} - \frac{2\sqrt{q}}{1-q}\,\chi_{\rm fer}\right), \tag{3.45}$$

$$2\pi U_{\rm bulk} = \int_0^\infty \frac{dt}{2}\left(-\frac{2\sqrt{q}}{1-q}\,\frac{\sqrt{q}}{1-q}\,\chi_{\rm bos} + \frac{1+q}{1-q}\,\frac{\sqrt{q}}{1-q}\,\chi_{\rm fer}\right), \tag{3.46}$$

and similarly for $S_{\rm bulk}$ and $C_{\rm bulk}$. The characters χ for general massive representation are given by (C.16), for massless spin-s representations by (C.164)-(C.166), and for partially massless (s, s') representations by (C.194). Regularization is implicit here.

Leading divergent term

The leading $t \to 0$ divergence of the scalar character (3.29) is $\chi(t) \sim 2/t^d$. For more general representations this becomes $\chi(t) \sim 2n/t^d$ with n the number of on-shell internal (spin) degrees of freedom. The generic leading divergent term of the bulk (free) energy is then given by $F_{\rm bulk}, U_{\rm bulk} \sim -\frac{1}{\pi}(n_{\rm bos} - n_{\rm fer})\int \frac{dt}{t^{d+2}} \sim \pm\Lambda^{d+1}\ell^d$, while for the bulk heat capacity and entropy we get $C_{\rm bulk}, S_{\rm bulk} \sim (\frac{1}{3}n_{\rm bos} + \frac{1}{6}n_{\rm fer})\int \frac{dt}{t^d} \sim +\Lambda^{d-1}\ell^{d-1}$, where we reinstated the dS radius ℓ. In particular $S_{\rm bulk} \sim +\Lambda^{d-1}\times$

horizon area, consistent with an entanglement entropy area law. The energy diverges more strongly because we included the QFT zero point energy term in its definition, which drops out of S and C.

Coefficient of log-divergent term

The coefficient of the logarithmically divergent part of these thermodynamic quantities is universal. A pleasant feature of the character formalism is that this coefficient can be read off trivially as the coefficient of the $1/t$ term in the small-t expansion of the integrand, easily computed for any representation. In odd $d+1$, the integrand is even in t, so log-divergences are absent. In even $d+1$, the integrand is odd in t, so generically we do get a log-divergence $= a \log \Lambda$. For example the $\log Z$ integrand for a dS_2 scalar is expanded as

$$\frac{1}{2t}\frac{1+e^{-t}}{1-e^{-t}}\,\frac{e^{-t(\frac{1}{2}+i\nu)}+e^{-t(\frac{1}{2}-i\nu)}}{1-e^{-t}} = \frac{2}{t^3} + \frac{\frac{1}{12}-\nu^2}{t} + \cdots \quad \Rightarrow \quad a = \tfrac{1}{12} - \nu^2 \,. \tag{3.47}$$

For a $\Delta = \frac{d}{2} + i\nu$ spin-s particle in even $d+1$, the $\log \Lambda$ coefficient for $U_{\rm bulk}$ is similarly read off as $a_{U_{\rm bulk}} = -D_s^d \frac{1}{\pi(d+1)!} \prod_{n=0}^{d} (\Delta - n)$. For a conformally coupled scalar, $\nu = i/2$, so $a_{U_{\rm bulk}} = 0$. Some examples of $a_{S_{\rm bulk}} = a_{\log Z_{\rm bulk}}$ in this case are

$d+1$	2	4	6	8	10	$\cdots$	100	$\cdots$	1000	$\cdots$
$a_{S_{\rm bulk}}$	$\frac{1}{3}$	$-\frac{1}{90}$	$\frac{1}{756}$	$-\frac{23}{113400}$	$\frac{263}{7484400}$	$\cdots$	-8.098×10^{-34}	$\cdots$	-3.001×10^{-306}	$\cdots$

Finite part and exact results

• *Energy:* For future reference (comparison to previously obtained results in section 3.7), we consider dimensional regularization here. The absence of a $1/t$ factor in the integral (3.46) for $U_{\rm bulk}$ then allows straightforward evaluation for general d. For a scalar of mass $m^2 = (\frac{d}{2})^2 + \nu^2$,

$$U_{\rm bulk}^{\rm fin} = \frac{m^2 \cosh(\pi\nu)\, \Gamma(\frac{d}{2}+i\nu)\, \Gamma(\frac{d}{2}-i\nu)}{2\pi\, \Gamma(d+2) \cos\!\left(\frac{\pi d}{2}\right)} \qquad \text{(dim reg)}\,. \tag{3.48}$$

For example for $d = 2$, this becomes

$$U_{\text{bulk}}^{\text{fin}} = -\frac{1}{12}(\nu^2+1)\nu\coth(\pi\nu)\,. \tag{3.49}$$

• *Free energy:* The UV-finite part of the $\log Z_{\text{bulk}}$ integral (3.45) for a massive field in even d can be computed simply by extending the integration contour to the real line avoiding the pole, closing the contour and summing residues. For example for a $d = 2$ scalar this gives

$$\log Z_{\text{bulk}}^{\text{fin}} = \frac{\pi\nu^3}{6} - \sum_{k=0}^{2} \frac{\nu^k}{k!} \frac{\text{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,, \tag{3.50}$$

where Li_n is the polylogarithm, $\text{Li}_n(x) \equiv \sum_{k=1}^{\infty} x^k/k^n$. For future reference, note that

$$\text{Li}_1(e^{-2\pi\nu}) = -\log\Big(1-e^{-2\pi\nu}\Big)\,, \qquad \text{Li}_0(e^{-2\pi\nu}) = \frac{1}{e^{2\pi\nu}-1} = \tfrac{1}{2}\coth(\pi\nu) - \tfrac{1}{2}\,. \tag{3.51}$$

For odd d, the character does not have an even analytic extension to the real line, so a different method is needed to compute $\log Z_{\text{bulk}}$. The exact evaluation of arbitrary character integrals, for any d and any $\chi(t)$ is given in (C.57) in terms of Hurwitz zeta functions. Simple examples are given in (C.59)-(C.60). In (C.57) we use the covariant regularization scheme introduced in section 3.3. Conversion to PV regularization is obtained from the finite part as explained below.

• *Entropy*: Combined with our earlier result for the bulk energy U_{bulk}, the above also gives the finite part of the bulk entropy $S_{\text{bulk}} = \log Z_{\text{bulk}} + 2\pi U_{\text{bulk}}$. In the Pauli-Villars regularization (3.40), the UV-divergent part is obtained from the finite part by mirrorring (3.40). For example for $k = 1$, $S_{\text{bulk},\Lambda} = S_{\text{bulk}}^{\text{fin}}|_{\nu^2} - S_{\text{bulk}}^{\text{fin}}|_{\nu^2+\Lambda^2}$. For the $d = 2$ example this gives for $\nu \ll \Lambda$

$$S_{\text{bulk},\Lambda} = \frac{\pi}{6}(\Lambda-\nu) - \frac{\pi}{3}\,\nu\,\text{Li}_0(e^{-2\pi\nu}) - \sum_{k=0}^{3} \frac{\nu^k}{k!} \frac{\text{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,, \tag{3.52}$$

where we used (3.51). S_{bulk} decreases monotonically with $m^2 = 1+\nu^2$. In the massless limit $m \to 0$, it diverges logarithmically: $S_{\text{bulk}} = -\log m + \cdots$. For $\nu \gg 1$, $S_{\text{bulk}} = \frac{\pi}{6}(\Lambda-\nu)$ up to exponentially small corrections. Thus $S_{\text{bulk}} > 0$ within the regime of validity of the low-energy field theory, consistent with its quasi-canonical/entanglement entropy interpretation. For a conformally coupled scalar $\nu = \frac{i}{2}$, this gives $S^{\text{fin}}_{\text{bulk}} = \frac{3\zeta(3)}{16\pi^2} - \frac{\log(2)}{8}$.

Quasinormal mode expansion

Substituting the quasinormal mode expansion (3.30),

$$\chi(t) = \sum_r N_r e^{-rt} \tag{3.53}$$

in the PV-regularized $\log Z_{\text{bulk}}(\beta)$ (3.44), rescaling $t \to \frac{\beta}{2\pi} t$, and using (C.69) gives

$$\log Z_{\text{bulk}}(\beta) = \sum_r N^{\text{bos}}_r \log \frac{\Gamma(br+1)}{(b\mu)^{br}\sqrt{2\pi br}} - N^{\text{fer}}_r \log \frac{\Gamma(br+\frac{1}{2})}{(b\mu)^{br}\sqrt{2\pi}}, \qquad b \equiv \frac{\beta}{2\pi}. \tag{3.54}$$

Truncating the integral to the IR part (C.69) is justified because the Pauli-Villars sum (3.40) cancels out the UV part. The dependence on μ likewise cancels out, as do some other terms, but it is useful to keep the above form. At the equilibrium $\beta = 2\pi$, $\log Z_{\text{bulk}}$ is given by (3.54) with $b = 1$. This provides a PV-regularized version of the quasinormal mode expansion of [142]. Since it is covariantly regularized, it does not require matching to a local heat kernel expansion. Moreover it applies to general particle content, including spin $s \geq 1$.[8]

QNM expansions of other bulk thermodynamic quantities are readily derived from (3.54) by taking derivatives $\beta\partial_\beta = b\partial_b = \mu\partial_\mu + r\partial_r$. For example $S_{\text{bulk}} = (1 - \beta\partial_\beta) \log Z_{\text{bulk}}|_{\beta=2\pi}$ is

$$S_{\text{bulk}} = \sum_r N^{\text{bos}}_r s_{\text{bos}}(r) + N^{\text{fer}}_r s_{\text{fer}}(r) \tag{3.55}$$

[8]The expansion of [142] pertains to Z_{PI} for $s \leq \frac{1}{2}$. In the following sections we show $Z_{\text{PI}} = Z_{\text{bulk}}$ for $s \leq \frac{1}{2}$ but not for $s \geq 1$. Hence in general the QNM expansion of [142] computes Z_{bulk}, not Z_{PI}.

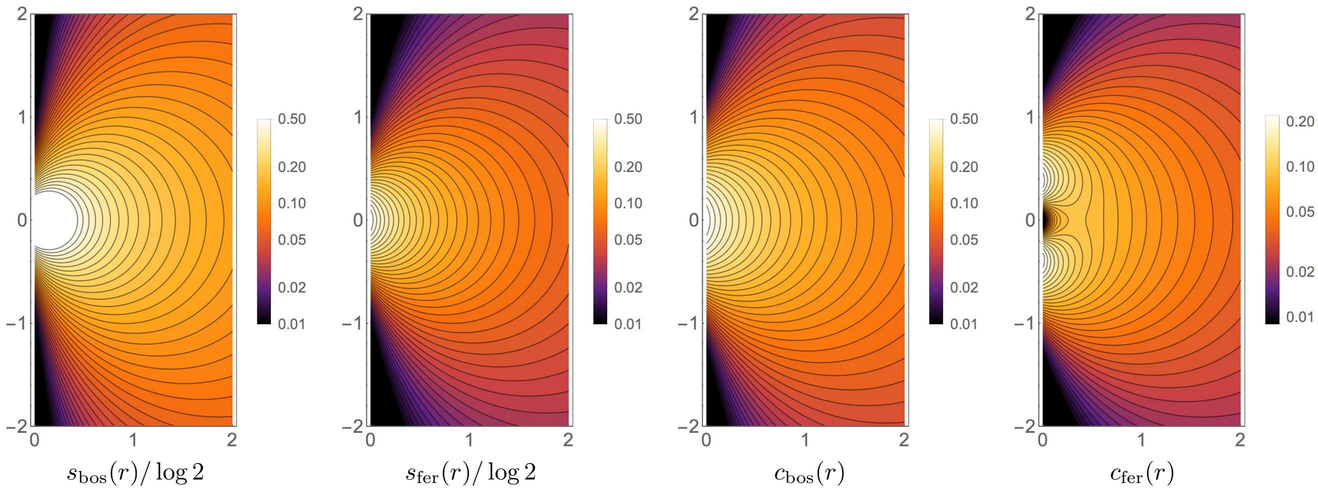

Figure 3.8: Contribution to $\beta = 2\pi$ bulk entropy and heat capacity of a quasinormal mode $\propto e^{-rT}$, $r \in \mathbb{C}$, $\mathrm{Re}\, r > 0$. Only the real part is shown here because complex r come in conjugate pairs $r_{n,\pm} = \frac{d}{2} + n \pm i\nu$. The harmonic oscillator case corresponds to the imaginary axis.

where the entropy $s(r)$ carried by a single QNM $\propto e^{-rT}$ at $\beta = 2\pi$ is given by

$$s_{\mathrm{bos}}(r) = r + (1 - r\partial_r) \log \frac{\Gamma(r+1)}{\sqrt{2\pi r}} , \qquad s_{\mathrm{fer}}(r) = -r - (1 - r\partial_r) \log \frac{\Gamma(r+\frac{1}{2})}{\sqrt{2\pi}} . \tag{3.56}$$

Note the μ-dependence has dropped out, reflecting the fact that the contribution of each *individual* QNM to the entropy is UV-finite, not requiring any regularization. For massive representations, r can be complex, but will always appear in a conjugate pair $r_{n\pm} = \frac{d}{2} + n \pm i\nu$. Taking this into account, all contributions to the entropy are real and positive for the physical part of the PV-extended spectrum. The small and large r asymptotics are

$$r \to 0 : \; s_{\mathrm{bos}} \to \frac{1}{2} \log \frac{e}{2\pi r} , \quad s_{\mathrm{fer}} \to \frac{\log 2}{2} , \qquad r \to \infty : \; s_{\mathrm{bos}} \to \frac{1}{6\, r} , \quad s_{\mathrm{fer}} \to \frac{1}{12\, r} . \tag{3.57}$$

The QNM entropies at general β are obtained simply by replacing

$$r \to \frac{\beta}{2\pi} r . \tag{3.58}$$

The entropy of a normal bosonic mode of frequency ω, $\tilde{s}(\omega) = -\log\left(1 - e^{-\beta\omega}\right) + \frac{\beta\omega}{e^{\beta\omega}-1}$, is recovered for complex conjugate pairs $r_\pm$ in the scaling limit $\beta \to 0$, $\beta\nu = \omega$ fixed, and likewise for fermions. At any finite β, the $n \to \infty$ UV tail of QNM contributions is markedly different however. Instead of falling off exponentially, if falls off as $s \sim 1/n$. PV or any other regularization effectively cuts off the sum at $n \sim \Lambda\ell$, so since $N_n \sim n^{d-1}$, $S_{\rm bulk} \sim \Lambda^{d-1}\ell^{d-1}$.

The bulk heat capacity $C_{\rm bulk} = -\beta\partial_\beta S_{\rm bulk}$, so the heat capacity of a QNM at $\beta = 2\pi$ is

$$c(r) = -r\partial_r s(r)\,. \tag{3.59}$$

The real part of $s(r)$ and $c(r)$ on the complex r-plane are shown in fig. 3.8.

An application of the quasinormal expansion

The above QNM expansions are less useful for exact computations of thermodynamic quantities than the direct integral evaluations discussed earlier, but can be very useful in computations of certain UV-finite quantities. A simple example is the following. In thermal equilibrium with a 4D dS static patch horizon, which set of particle species has the largest bulk heat capacity: (A) six conformally coupled scalars + graviton, (B) four photons? The answer is not obvious, as both have an equal number of local degrees of freedom: $6 + 2 = 4 \times 2 = 8$. One could compute each in full, but the above QNM expansions offers a much easier way to get the answer. From (3.29) and (C.165) we read off the scalar and massless spin-s characters:

$$\chi_0 = \frac{q+q^2}{(1-q)^3}\,, \qquad \chi_s = \frac{2(2s+1)\,q^{s+1} - 2(2s-1)\,q^{s+2}}{(1-q)^3}\,, \tag{3.60}$$

where $q = e^{-|t|}$. We see $\chi_A - \chi_B = \chi_2 + 6\,\chi_0 - 4\,\chi_1 = 6\,q$, so χ_A and χ_B are *almost* exactly equal: A has just 6 more quasinormal modes than B, all with $r = 1$. Thus, using (3.59),

$$C^A_{\rm bulk} - C^B_{\rm bulk} = 6 \cdot c_{\rm bos}(1) = \pi^2 - 9\,. \tag{3.61}$$

Pretty close, but $\pi > 3$ [9], so A wins. The difference is $\Delta C \approx 0.87$. Along similar lines, $\Delta S = 6\, s_{\rm bos}(1) = 3(2\gamma + 1 - \log(2\pi)) \approx 0.95$.

Another UV-finite example: relative entropies of graviton, photon, neutrino

Less trivial to compute but more real-world in flavor is the following UV-finite linear combination of the 4D graviton, photon, and (assumed massless) neutrino bulk entropies:

$$S_{\rm graviton} + \tfrac{60}{7} S_{\rm neutrino} - \tfrac{37}{7} S_{\rm photon} = \tfrac{48}{7}\zeta'(-1) - \tfrac{60}{7}\zeta'(-3) + 6\gamma + \tfrac{149}{56} - \tfrac{33}{14}\log(2\pi) \approx 0.61 \tag{3.62}$$

Finiteness can be checked from the small-t expansion of the total integrand computing this, and the integral can then be evaluated along the lines of (C.53)-(C.54). We omit the details.

Vasiliev higher-spin example

Non-minimal Vasiliev higher-spin gravity on dS_4 has a single conformally coupled scalar and a tower of massless spin-s particles of all spins $s = 1, 2, 3, \ldots$. The prospect of having to compute bulk thermodynamics for this theory by brick wall or other approaches mentioned in appendix C.5.3 would be terrifying. Let us compare this to the character approach. The *total* character obtained by summing the characters of (3.60) takes a remarkably simple form:

$$\chi_{\rm tot} \;=\; \chi_0 + \sum_{s=1}^{\infty} \chi_s \;=\; 2\cdot\left(\frac{q^{1/2}+q^{3/2}}{(1-q)^2}\right)^2 - \frac{q}{(1-q)^2} \;=\; \frac{q+q^3}{(1-q)^4} + 3\cdot\frac{2q^2}{(1-q)^4}\,. \tag{3.63}$$

The first expression is two times the square of the character of a 3D conformally coupled scalar, plus the character of 3D conformal higher-spin gravity (3.188).[10] The second expression equals the character of one $\nu = i$ and three $\nu = 0$ scalars on dS_5. Treating the character integral as such,

[9] As reviewed in D. Bailey, S. Plouffe, P. Borwein and J. Borwein, "The quest for pi," The Mathematical Intelligencer **19**, 1 (1997): According to the Old Testament's π, it's a tie, but the ancient Baylonians and Egyptians measured $\pi = 3.14 \pm 0.02$, and Archimedes proved $\pi > 3 + \frac{10}{71}$. So A wins.

[10] For AdS_4, the analogous $\chi_{\rm tot}$ equals *one* copy of the 3D scalar character squared, reflecting the single-trace spectrum of its holographic dual $U(N)$ model ⟨3.9.2⟩. The dS counterpart thus encodes the single-trace spectrum of two copies of this 3D CFT + 3D CHS gravity, reminiscent of [151]. This is generalized by (3.187).

we immediately get, in $k = 3$ Pauli-Villars regularization (3.40),

$$\begin{aligned} \log Z_{\rm bulk}^{\rm div} &= a_0\Lambda^5 + a_2\Lambda^3 - a_4\Lambda\,, & \log Z_{\rm bulk}^{\rm fin} &= \tfrac{\zeta(5)}{4\pi^4} - \tfrac{\zeta(3)}{24\pi^2} \\ S_{\rm bulk}^{\rm div} &= \tfrac{25}{4}a_2\Lambda^3 - \tfrac{103}{20}a_4\Lambda\,, & S_{\rm bulk}^{\rm fin} &= \tfrac{\zeta(5)}{4\pi^4} - \tfrac{\zeta(3)}{24\pi^2} + \tfrac{1}{20}\,. \end{aligned} \tag{3.64}$$

where $a_0 = \frac{1-4\sqrt{2}+3\sqrt{3}}{10}\pi \approx 0.17$, $a_2 = -\frac{1-2\sqrt{2}+\sqrt{3}}{12}\pi \approx 0.025$, and $a_4 = \frac{3-3\sqrt{2}+\sqrt{3}}{48}\pi \approx 0.032$. The tower of higher-spin particles alters the *bulk* UV dimensionality much like a tower of KK modes would. (We will later see edge "corrections" rather dramatically alter this.)

3.3 Sphere partition function for scalars and spinors

3.3.1 Problem and result

In this section we consider the one-loop Gaussian Euclidean path integral $Z_{\rm PI}^{(1)}$ of scalar and spinor field fluctuations on the round sphere. For a free scalar of mass m^2 on S^{d+1},

$$Z_{\rm PI} = \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int \phi(-\nabla^2+m^2)\phi}\,, \tag{3.65}$$

A convenient UV-regularized version is defined using standard heat kernel methods [60]:

$$\log Z_{{\rm PI},\epsilon} = \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau}\, {\rm Tr}\, e^{-\tau(-\nabla^2+m^2)}\,. \tag{3.66}$$

The insertion $e^{-\epsilon^2/4\tau}$ implements a UV cutoff at length scale $\sim \epsilon$. We picked this regulator for convenience in the derivation below. We could alternatively insert the PV regulator of footnote 6, which would reproduce the PV regularization (3.40). However, being uniformly applicable to all dimensions, the above regulator is more useful for the purpose of deriving general evaluation formulae, as in appendix C.3.

In view of (3.32) we wish to compare $Z_{\rm PI}$ to the corresponding Wick-rotated dS static patch bulk thermal partition function $Z_{\rm bulk}(\beta)$ (3.31), at the equilibrium inverse temperature $\beta = 2\pi$.

Here and henceforth, Z_{bulk} by default means $Z_{\text{bulk}}(2\pi)$:

$$\log Z_{\text{bulk}} \equiv \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-t}}{1-e^{-t}}\,\chi(t)_{\text{bos}} - \frac{2\,e^{-t/2}}{1-e^{-t}}\,\chi(t)_{\text{fer}}\right) \tag{3.67}$$

Below we show that for free scalars and spinors,

$$\boxed{Z_{\text{PI}} = Z_{\text{bulk}}} \tag{3.68}$$

with the specific regularization (3.66) for Z_{PI} mapping to a specific regularization (3.73) for Z_{bulk}. The relation is exact, for any ϵ. This makes the physical expectation (3.32) precise, and shows that for scalars and spinors, there are in fact no edge corrections.

In appendix C.3 we provide a simple recipe for extracting both the UV and IR parts in the $\epsilon \to 0$ limit in the above regularization, directly from the unregularized form of the character formula (3.67). This yields the general closed-form solution $\boxed{(C.57)}$ for the regularized Z_{PI} in terms of Hurwitz zeta functions. The heat kernel coefficient invariants are likewise read off from the character using (C.58). For simple examples see (C.59), (C.60), (C.70)-(C.71).

3.3.2 Derivation

The derivation is straightforward:

Scalars:

The eigenvalues of $-\nabla^2$ on a sphere of radius $\ell \equiv 1$ are $\lambda_n = n(n+d)$, $n \in \mathbb{N}$, with degeneracies D_n^{d+2} given by (C.15), that is $D_n^{d+2} = \binom{n+d+1}{d+1} - \binom{n+d-1}{d+1}$. Thus (3.66) can be written as

$$\log Z_{\text{PI}} = \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau} e^{-\tau\nu^2} \sum_{n=0}^\infty D_n^{d+2}\, e^{-\tau(n+\frac{d}{2})^2}, \qquad \nu \equiv \sqrt{m^2 - \tfrac{d^2}{4}}\,. \tag{3.69}$$

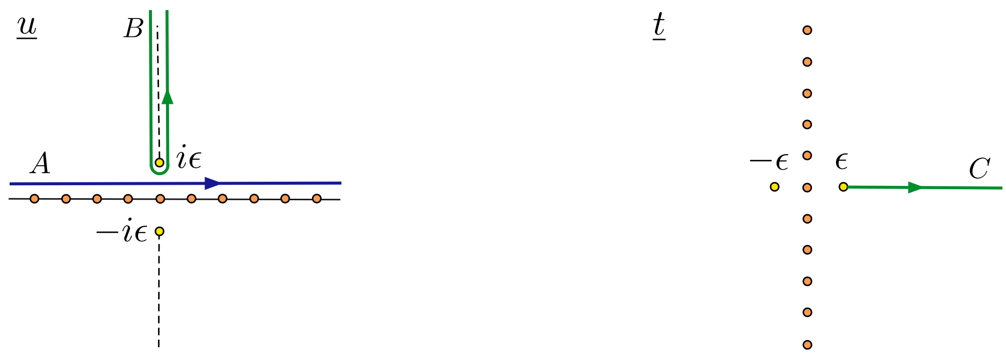

Figure 3.9: Integration contours for $Z_{\rm PI}$. Orange dots are poles, yellow dots branch points.

To perform the sum over n, we use the Hubbard-Stratonovich trick, i.e. we write

$$\sum_{n=0}^{\infty} D_n^{d+2}\, e^{-\tau(n+\frac{d}{2})^2} = \int_A du\, \frac{e^{-u^2/4\tau}}{\sqrt{4\pi\tau}}\, f(u)\,, \qquad f(u) \equiv \sum_{n=0}^{\infty} D_n^{d+2}\, e^{iu(n+\frac{d}{2})}\,. \tag{3.70}$$

with integration contour $A = \mathbb{R} + i\delta$, $\delta > 0$, as shown in fig. 3.9. The sum evaluates to

$$f(u) = \frac{1+e^{iu}}{1-e^{iu}}\, \frac{e^{i\frac{d}{2}u}}{(1-e^{iu})^d}\,, \tag{3.71}$$

We first consider the case $m > \frac{d}{2}$, so ν is real and positive. Then, keeping $\mathrm{Im}\, u = \delta < \epsilon$, we can perform the τ-integral first in (3.69) to get

$$\log Z_{\rm PI} = \int_A \frac{du}{2\sqrt{u^2+\epsilon^2}}\, e^{-\nu\sqrt{u^2+\epsilon^2}}\, f(u)\,. \tag{3.72}$$

Deforming the contour by folding it up along the two sides of the branch cut to contour B in fig. 3.9, changing variables $u = it$ and using that the square root takes opposite signs on both sides of the cut, we transform this to an integral over C in fig. 3.9:

$$\log Z_{\rm PI} = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}}\, \frac{1+e^{-t}}{1-e^{-t}}\, \frac{e^{-\frac{d}{2}t+i\nu\sqrt{t^2-\epsilon^2}} + e^{-\frac{d}{2}t-i\nu\sqrt{t^2-\epsilon^2}}}{(1-e^{-t})^d}\,, \tag{3.73}$$

The result for $0 < m \leq \frac{d}{2}$, i.e. $\nu = i\mu$ with $0 \leq \mu < \frac{d}{2}$ can be obtained from this by analytic continuation. Putting $\epsilon = 0$, this formally becomes

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t}\,\frac{1+e^{-t}}{1-e^{-t}}\,\chi(t)\,, \qquad \chi(t) = \frac{e^{-(\frac{d}{2}-i\nu)t}+e^{-(\frac{d}{2}+i\nu)t}}{(1-e^{-t})^d}\,, \tag{3.74}$$

which we recognize as (3.67) with $\chi(t)$ the scalar character (3.29). Thus we conclude that for scalars, $Z_{\rm PI} = Z_{\rm bulk}$, with $Z_{\rm PI}$ regularized as in (3.66) and $Z_{\rm bulk}$ as in (3.73).

Spinors:

For a Dirac spinor field of mass m we have $Z_{\rm PI} = \int \mathcal{D}\psi\, e^{-\int \bar\psi(\nabla\!\!\!/+m)\psi}$. The relevant formulae for spectrum and degeneracies for general d can be found in appendices C.4.1 and C.6.2. For concreteness we just consider the case $d=3$ here, but the conclusions are valid for Dirac spinors in general. The spectrum of $\nabla\!\!\!/+m$ on S^4 is $\lambda_n = m\pm(n+2)i$, $n\in\mathbb{N}$, with degeneracy $D^5_{n+\frac12,\frac12} = 4\binom{n+3}{3}$, so $Z_{\rm PI}$ regularized as in (3.66) is given by

$$\log Z_{\rm PI} = -\int_0^\infty \frac{d\tau}{\tau} e^{-\epsilon^2/4\tau} \sum_{n=0}^\infty 4\binom{n+3}{3}\, e^{-\tau((n+2)^2+m^2)} \tag{3.75}$$

Following the same steps as for the scalar case, this can be rewritten as

$$\log Z_{\rm PI} = -\int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}}\,\frac{2\,e^{-t/2}}{1-e^{-t}}\cdot 4\cdot \frac{e^{-\frac32 t+im\sqrt{t^2-\epsilon^2}}+e^{-\frac32 t-im\sqrt{t^2-\epsilon^2}}}{(1-e^{-t})^3}\,. \tag{3.76}$$

Putting $\epsilon = 0$, this formally becomes

$$\log Z_{\rm PI} = -\int_0^\infty \frac{dt}{2t}\,\frac{2\,e^{-t/2}}{1-e^{-t}}\;\chi(t)\,, \qquad \chi(t) = 4\cdot\frac{e^{-(\frac32+im)t}+e^{-(\frac32-im)t}}{(1-e^{-t})^3}\,, \tag{3.77}$$

which we recognize as the fermionic (3.67) with $\chi(t)$ the character of the $\Delta = \frac32 + im$ unitary $SO(1,4)$ representation carried by the single-particle Hilbert space of a Dirac spinor quantized on dS_4, given by twice the character (C.16) of the irreducible representation (Δ, S) with $S = (\frac12)$. Thus we conclude $Z_{\rm PI} = Z_{\rm bulk}$. The comment below (3.93) generalizes this to all d.

3.4 Massive higher spins

We first formulate the problem, explaining why it is not nearly as simple as one might have hoped, and then state the result, which turns out to be much simpler than one might have feared. The derivation of the result is detailed in appendix C.6.1.

3.4.1 Problem

Consider a massive spin-$s \geq 1$ field, more specifically a totally symmetric tensor field $\phi_{\mu_1 \cdots \mu_s}$ on dS_{d+1} satisfying the Fierz-Pauli equations of motion:

$$\left(-\nabla^2 + \overline{m}_s^2\right)\phi_{\mu_1\cdots\mu_s} = 0\,, \qquad \nabla^\nu \phi_{\nu\mu_1\cdots\mu_{s-1}} = 0\,, \qquad \phi^\nu{}_{\nu\mu_1\cdots\mu_{s-2}} = 0\,. \tag{3.78}$$

Upon quantization, the global single-particle Hilbert space furnishes a massive spin-s representation of $SO(1, d+1)$ with $\Delta = \frac{d}{2} + i\nu$, related to the effective mass $\overline{m}_s$ appearing above (see e.g. [152]), and to the more commonly used definition of mass m (see e.g. [115]) as

$$\overline{m}_s^2 = (\tfrac{d}{2})^2 + \nu^2 + s\,, \qquad m^2 = (\tfrac{d}{2} + s - 2)^2 + \nu^2 = (\Delta + s - 2)(d + s - 2 - \Delta)\,. \tag{3.79}$$

Then $m = 0$ for the photon, the graviton and their higher-spin generalizations, and for $s = 1$, m is the familiar spin-1 Proca mass.

The massive spin-s *bulk* thermal partition function is immediately obtained by substituting the massive spin-s character (C.14) into the character formula (3.67) for Z_{bulk}. For $d \geq 3$,[11]

$$\log Z_{\text{bulk}} = \int_0^\infty \frac{dt}{2t}\frac{1+q}{1-q}\; D_s^d \cdot \frac{q^{\frac{d}{2}+i\nu} + q^{\frac{d}{2}-i\nu}}{(1-q)^d}\,, \qquad q = e^{-t}\,, \tag{3.80}$$

with spin degeneracy factor read off from (C.15) or (C.89).

[11] For $d = 2$, the single-particle Hilbert space splits into $(\Delta, \pm s)$ with $D_{\pm s}^2 = 1$, so $D_s^2 \to \sum_\pm D_{\pm s}^2 = 2$.

The corresponding free massive spin-s Euclidean path integral on S^{d+1} takes the form

$$Z_{\rm PI} = \int \mathcal{D}\Phi \, e^{-S_E[\Phi]}. \tag{3.81}$$

where Φ includes at least ϕ. However it turns out that in order to write down a local, manifestly covariant action for massive fields of general spin s, one also needs to include a tower of auxiliary Stueckelberg fields of all spins $s' < s$ [102], generalizing the familiar Stueckelberg action (C.146) for massive vector fields. These come with gauge symmetries, which in turn require the introduction of a gauge fixing sector, with ghosts of all spins $s' < s$. The explicit form of the action and gauge symmetries is known, but intricate [102].

Classically, variation of the action with respect the Stueckelberg fields merely enforces the transverse-traceless (TT) constraints in (3.78), after which the gauge symmetries can be used to put the Stueckelberg fields equal to zero. One might therefore hope the intimidating off-shell $Z_{\rm PI}$ (3.81) likewise collapses to just the path integral $Z_{\rm TT}$ over the TT modes of ϕ with kinetic term given by the equations of motion (3.78). This is easy to evaluate. The TT eigenvalue spectrum on the sphere follows from $SO(d+2)$ representation theory. As detailed in eqs. (C.127)-(C.129), we can then follow the same steps as in section 3.3, ending up with[12]

$$\log Z_{\rm TT} = \int_0^\infty \frac{dt}{2t} \left(q^{i\nu} + q^{-i\nu}\right) \sum_{n\geq s} D^{d+2}_{n,s} \, q^{\frac{d}{2}+n} \,. \tag{3.82}$$

Here $D^{d+2}_{n,s}$ is the dimension of the $SO(d+2)$ representation labeled by the two-row Young diagram (n, s), given explicitly by the dimension formulae in appendix C.4.1.

Unfortunately, $Z_{\rm TT}$ is *not* equal to $Z_{\rm PI}$ on the sphere. The easiest way to see this is to consider an example in odd spacetime dimensions, such as (C.130), and observe the result has a logarithmic divergence. A manifestly covariant local QFT path integral on an odd-dimensional sphere cannot possibly have logarithmic divergences. Therefore $Z_{\rm PI} \neq Z_{\rm TT}$. The appearance of such nonlocal divergences in $Z_{\rm TT}$ can be traced to the existence of (normalizable) zeromodes in tensor decom-

[12]For $d = 2$, $D^4_{n,s} \to \sum_\pm D^4_{n,\pm s} = 2D^4_{n,s}$.

positions on the sphere [52, 62]. For example the decomposition $\phi_\mu = \phi^T_\mu + \nabla_\mu\varphi$ has the constant φ mode as a zeromode, $\phi_{\mu\nu} = \phi^{\rm TT}_{\mu\nu} + \nabla_{(\mu}\varphi_{\nu)} + g_{\mu\nu}\varphi$ has conformal Killing vector zeromodes, and $\phi_{\mu_1\cdots\mu_s} = \phi^{\rm TT}_{\mu_1\cdots\mu_s} + \nabla_{(\mu_1}\varphi_{\mu_2\cdots\mu_s)} + g_{(\mu_1\mu_2}\varphi_{\mu_3\cdots\mu_s)}$ has rank $s-1$ conformal Killing tensor zeromodes. As shown in [52, 62], this implies $\log Z_{\rm TT}$ contains a nonlocal UV-divergent term $c_s \log\Lambda$, where c_s is the number of rank $s-1$ conformal Killing tensors. This divergence cannot be canceled by a local counterterm. Instead it must be canceled by contributions from the non-TT part. Thus, in principle, the full off-shell path integral must be carefully evaluated to obtain the correct result. Computing $Z_{\rm PI}$ for general s on the sphere is not as easy as one might have hoped.

3.4.2 Result

Rather than follow a brute-force approach, we obtain $Z_{\rm PI}$ in appendix C.6.1 by a series of relatively simple observations. In fact, upon evaluating the sum in (3.82), writing it in a way that brings out a term $\log Z_{\rm bulk}$ as in (3.80), and observing a conspicuous finite sum of terms bears full responsibility for the inconsistency with locality, the answer suggests itself right away: the non-TT part restores locality simply by canceling this finite sum. This turns out to be equivalent to the non-TT part effectively extending the sum $n \geq s$ in (3.82) to $n \geq -1$:[13]

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t}\left(q^{i\nu} + q^{-i\nu}\right) \sum_{n\geq -1} D^{d+2}_{n,s}\, q^{\frac{d}{2}+n}\,, \tag{3.83}$$

where $D^{d+2}_{n,s}$ is given by the explicit formulae in appendix C.4.1, in particular (C.90). For $n < s$, this is no longer the dimension of an $SO(d+2)$ representation, but it can be rewritten as *minus* the dimension of such a representation, as $D^{d+2}_{n,s} = -D^{d+2}_{s-1,n+1}$. This extension also turns out to be exactly what is needed for consistency with the unitarity bound (C.145) and more refined unitarity considerations. A limited amount of explicit path integral considerations combined with the observation that the coefficients $D^{d+2}_{s-1,n+1}$ count conformal Killing tensor mode mismatches between ghosts and longitudinal modes then suffice to establish this is indeed the correct answer. We refer

[13]For $d=2$ use (3.91): $D^4_{n,s} \to \sum_\pm D^4_{n,\pm s} = 2D^4_{n,s}$ for $n > -1$, and $D^4_{-1,s} \to \sum_\pm \frac{1}{2}D^4_{-1,\pm s} = D^4_{-1,s} = D^4_{s-1}$.

to appendix C.6.1 for details.

Using the identity (C.135), we can write this in a rather suggestive form:

$$\boxed{\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge} = \int_0^\infty \frac{dt}{2t}\,\frac{1+q}{1-q}\big(\chi_{\rm bulk} - \chi_{\rm edge}\big)}\,, \tag{3.84}$$

where $\chi_{\rm bulk}$ and $\chi_{\rm edge}$ are explicitly given by[14]

$$\boxed{\chi_{\rm bulk} \equiv D_s^d\, \frac{q^{\frac{d}{2}+i\nu} + q^{\frac{d}{2}-i\nu}}{(1-q)^d}\,, \qquad \chi_{\rm edge} \equiv D_{s-1}^{d+2}\, \frac{q^{\frac{d-2}{2}+i\nu} + q^{\frac{d-2}{2}-i\nu}}{(1-q)^{d-2}}} \tag{3.85}$$

The $\log Z_{\rm bulk}$ term is the character integral for the bulk partition function (3.80). Strikingly, the correction $\log Z_{\rm edge}$ *also* takes the form a character integral, but with an "edge" character $\chi_{\rm edge}$ in *two lower* dimensions. By our results of section 3.3 for scalars, $Z_{\rm edge}$ effectively equals the Euclidean path integral of D_{s-1}^{d+2} scalars of mass $\tilde m^2 = \big(\frac{d-2}{2}\big)^2 + \nu^2$ on S^{d-1}:

$$Z_{\rm edge} = \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int_{S^{d-1}} \phi^a(-\nabla^2 + \tilde m^2)\phi^a}\,, \qquad a = 1,\ldots,D_{s-1}^{d+2}\,, \tag{3.86}$$

In particular this gives 1 scalar for $s = 1$ and $d+2$ scalars for $s = 2$. The S^{d-1} is naturally identified as the static patch horizon, the edge of the global dS spatial S^d hemisphere at time zero, the yellow dot in fig. 3.5. Thus (3.84) realizes in a precise way the somewhat vague physical expectation (3.32). Notice the relative minus sign here and in (3.84): the edge corrections effectively *subtract* degrees of freedom. We do not have a physical interpretation of these putative edge scalars for general s along the lines of the work reviewed in appendix C.5.5. Some clues are that their multiplicity equals the number of conformal Killing tensor modes of scalar type appearing in the derivation in appendix C.6.1 (the $\square\!\square\!\square$-modes for $s = 4$ in (C.141)), and that they become massless at the unitarity bound $\nu = \pm i(\frac{d}{2} - 1)$, eq. (C.145), where a partially massless field emerges with a scalar gauge parameter.

[14]For $d = 2$, $D_s^2 \to \sum_\pm D_s^2 = 2$ in $\chi_{\rm bulk}$ as in (3.80). $\chi_{\rm edge}$ remains unchanged.

Independent of any interpretation, we can summarize the result (3.84)-(3.85) as

$$\log Z_{\rm PI}^{d+1}(s) = \log Z_{\rm bulk}^{d+1}(s) - D_{s-1}^{d+2}\, \log Z_{\rm PI}^{d-1}(0)\,. \tag{3.87}$$

Examples

For a $d=2$ spin-$s \geq 1$ field of mass $m^2 = (s-1)^2 + \nu^2$, $\log Z_{\rm PI} = \int \frac{dt}{2t}\frac{1+q}{1-q}(\chi_{\rm bulk} - \chi_{\rm edge})$ with

$$\chi_{\rm bulk} = 2\,\frac{q^{1+i\nu} + q^{1-i\nu}}{(1-q)^2}, \qquad \chi_{\rm edge} = s^2(q^{i\nu} + q^{-i\nu})\,. \tag{3.88}$$

That is, $Z_{\rm PI} = Z_{\rm bulk}/Z_{\rm edge}$, with the finite part of $\log Z_{\rm bulk}$ explicitly given by twice (3.50), and with $Z_{\rm edge}$ equal to the Euclidean path integral of $D_{s-1}^4 = s^2$ harmonic oscillators of frequency ν on S^1, naturally identified with the S^1 horizon of the dS_3 static patch, with finite part

$$Z_{\rm edge}^{\rm fin} = \left(\frac{e^{-\pi\nu}}{1-e^{-2\pi\nu}}\right)^{s^2}. \tag{3.89}$$

The heat-kernel regularized $Z_{\rm PI}$ is then, restoring ℓ and recalling $\nu = \sqrt{m^2\ell^2 - (s-1)^2}$,

$$\log Z_{\rm PI} = 2\left(\frac{\pi\nu^3}{6} - \sum_{k=0}^{2}\frac{\nu^k}{k!}\,\frac{\mathrm{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}} - \frac{\pi\nu^2\ell}{4\epsilon} + \frac{\pi\ell^3}{2\epsilon^3}\right) - s^2\left(-\pi\nu - \log\left(1-e^{-2\pi\nu}\right) + \frac{\pi\ell}{\epsilon}\right) \tag{3.90}$$

The $d=3$ spin-s case is worked out as another example in (C.65).

General massive representations

(3.83) has a natural generalization, presented in appendix C.6.2, to arbitrary parity-invariant massive $SO(1,d+1)$ representations $R = \oplus_a(\Delta_a, S_a)$, $\Delta_a = \frac{d}{2} + i\nu_a$, $S_a = (s_{a1},\ldots,s_{ar})$:

$$\boxed{\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t}\sum_a (-1)^{F_a}\left(q^{i\nu_a} + q^{-i\nu_a}\right)\sum_{n\in\frac{F_a}{2}+\mathbb{Z}} \Theta\left(\tfrac{d}{2}+n\right) D_{n,S_a}^{d+2}\, q^{\frac{d}{2}+n}} \tag{3.91}$$

where $\Theta(x)$ is the Heaviside step step function with $\boxed{\Theta(0) \equiv \frac{1}{2}}$, and $F_a = 0, 1$ for bosons resp. fermions. This is the unique TT eigenvalue sum extension consistent with locality and unitarity constraints. As in the $S = (s)$ case, this can be rewritten as a bulk-edge decomposition $\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge}$. For example, using (C.91) and the notation explained above it, the analog of (3.87) for an $S = (s, 1^m)$ field becomes

$$\log Z_{\rm PI}^{d+1}(s, 1^m) = \log Z_{\rm bulk}^{d+1}(s, 1^m) - D_{s-1}^{d+2} \, \log Z_{\rm PI}^{d-1}(1^m), \tag{3.92}$$

so here $Z_{\rm edge}$ is the path integral of D_{s-1}^{d+2} massive *m-form fields* living on the S^{d-1} edge. In particular this implies the recursion relation $\log Z_{\rm PI}^{d+1}(1^p) = \log Z_{\rm bulk}^{d+1}(1^p) - \log Z_{\rm PI}^{d-1}(1^{p-1})$. Similarly for a spin $s = k + \frac{1}{2}$ Dirac fermion, in the notation explained under table (C.89)

$$\log Z_{\rm PI}^{d+1}(s, \tfrac{\mathbf{1}}{\mathbf{2}}) = \log Z_{\rm bulk}^{d+1}(s, \tfrac{\mathbf{1}}{\mathbf{2}}) - \tfrac{1}{2} D_{s-1, \frac{\mathbf{1}}{\mathbf{2}}}^{d+2} \, \log Z_{\rm PI}^{d-1}(\tfrac{\mathbf{1}}{\mathbf{2}}), \tag{3.93}$$

where $Z_{\rm bulk}$ now takes the form of the *fermionic* part of (3.67), with $\chi_{\rm bulk}$ as in (C.16) with $D_S^d = 2\, D_{s,\frac{\mathbf{1}}{\mathbf{2}}}^d$, the factor 2 due to the field being Dirac. The edge fields are Dirac spinors. Note that because $D_{-\frac{1}{2},\frac{\mathbf{1}}{\mathbf{2}}}^{d+2} = 0$, the above implies in particular $Z_{\rm PI}(\frac{\mathbf{1}}{\mathbf{2}}) = Z_{\rm bulk}(\frac{\mathbf{1}}{\mathbf{2}})$.

We do not have a systematic group-theoretic or physical way of identifying the edge field content. For evaluation of $Z_{\rm PI}$ using (C.57), this identification is not needed however. Actually the original expansions (3.83), (3.91) are more useful for this, as illustrated in (C.61)-(C.65).

3.5 Massless higher spins

3.5.1 Problems

Bulk thermal partition function $Z_{\rm bulk}$

Massless spin-s fields on dS_{d+1} are in many ways quite a bit more subtle than their massive spin-s counterparts. This manifests itself already at the level of the characters $\chi_{{\rm bulk},s}$ needed to compute the bulk ideal gas thermodynamics along the lines of section 3.2. The $SO(1, d+1)$

unitary representations furnished by their single-particle Hilbert space belong to the discrete series for $d = 3$ and to the exceptional series for $d \geq 4$ [101]. The corresponding characters, discussed in appendix C.7.1, are more intricate than their massive (principal and complementary series) counterparts. A brief look at the general formula (C.164) or the table of examples (C.165) suffices to make clear they are far from intuitively obvious — as is, for that matter, the identification of the representation itself. Moreover, [101] reported their computation of the exceptional series characters disagrees with the original results in [76–78].

As noted in section 3.2 and appendix C.2.3, the expansion $\chi_{\rm bulk}(q) = \sum_q N_k q^k$ can be interpreted as counting the number N_k of static patch quasinormal modes decaying as $e^{-kT/\ell}$. This gives some useful physics intuition for the peculiar form of these characters, explained in appendix C.7.1. The characters $\chi_{{\rm bulk},s}(q)$ can in principle be computed by explicitly constructing and counting *physical* quasinormal modes of a massless spin-s field. This is a rather nontrivial problem, however.

Thus we see that for massless fields, complications appear already in the computation of $Z_{\rm bulk}$. Computing $Z_{\rm PI}$ adds even more complications, due to the presence of negative and zero modes in the path integral. Happily, as we will see, the complications of the latter turn out to be the key to resolving the complications of the former. Our final result for $Z_{\rm PI}$ confirms the identification of the representation made in [101] and the original results for the corresponding characters in [76–78]. This is explicitly verified by counting quasinormal modes in [141].

Euclidean path integral $Z_{\rm PI}$

We consider massless spin-s fields in the metric-like formalism, that is to say totally symmetric double-traceless fields $\phi_{\mu_1\cdots\mu_s}$, with linearized gauge transformation

$$\delta^{(0)}_\xi \phi_{\mu_1\cdots\mu_s} = \alpha_s \nabla_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}\,, \tag{3.94}$$

with ξ is traceless symmetric in its $s' = s-1$ indices, and α_s picked by convention.[15]

We use the notation $s' \equiv s-1$ as it makes certain formulae more transparent and readily generalizable to the partially massless ($0 \leq s' < s$) case. The dimensions of ϕ_s and $\xi_{s'}$ are

$$\frac{d}{2} + i\nu_\phi = \Delta_\phi = s' + d - 1\,, \qquad \frac{d}{2} + i\nu_\xi = \Delta_\xi = s + d - 1\,. \tag{3.95}$$

Note that this value of ν_ϕ assign a mass $m = 0$ to ϕ according to (3.79). The Euclidean path integral of a collection of (interacting) gauge fields ϕ on S^{d+1} is formally given by

$$Z_{\rm PI} = \frac{\int \mathcal{D}\phi\, e^{-S[\phi]}}{\mathrm{vol}(\mathcal{G})} \tag{3.96}$$

where $\mathcal{G}$ is the group of local gauge transformations. At the one-loop (Gaussian) level $S[\phi]$ is the quadratic Fronsdal action [99]. Several complications arise compared to the massive case:

1. For $s \geq 2$, the Euclidean path integral has negative ("wrong sign" Gaussian) modes, generalizing the well-known issue arising for the conformal factor in Einstein gravity [72]. These can be dealt with by rotating field integration contours. A complication on the sphere is that rotations at the local field level ensuring positivity of short-wavelength modes causes a finite subset of low-lying modes to go negative, requiring these modes to be rotated back [59].

2. The linearized gauge transformations (3.94) have zeromodes: symmetric traceless tensors $\bar{\xi}_{\mu_1\ldots\mu_{s-1}}$ satisfying $\nabla_{(\mu_1}\bar{\xi}_{\mu_2\cdots\mu_s)} = 0$, the Killing tensors of S^{d+1}. This requires omitting associated modes from the BRST gauge fixing sector of the Gaussian path integral. As a result, locality is lost, and with it the flexibility to freely absorb various normalization constants into local counterterms without having to keep track of nonlocal residuals.

3. At the *nonlinear* level, the Killing tensors generate a subalgebra of the gauge algebra. The structure constants of this algebra are determined by the TT cubic couplings of the interacting theory [68]. At least when it is finite-dimensional, as is the case for Yang-Mills,

[15] As explained in appendix C.7.4, for compatibility with certain other conventions we adopt, we will pick $\alpha_s \equiv \sqrt{s}$ with symmetrization conventions such that $\phi_{(\mu_1\cdots\mu_s)} = \phi_{\mu_1\cdots\mu_s}$.

Einstein gravity and the 3D higher-spin gravity theories of section 3.6, the Killing tensor algebra exponentiates to a group G. For example for Einstein gravity, $G = SO(d+2)$. To compensate for the zeromode omissions in the path integral, one has to divide by the volume of G. The appropriate measure determining this volume is inherited from the path integral measure, and depends on the UV cutoff and the coupling constants of the theory. Precisely relating the path integral volume $\mathrm{vol}(G)_{\mathrm{PI}}$ to the "canonical" $\mathrm{vol}(G)_{\mathrm{c}}$ defined by a theory-independent invariant metric on G requires considerable care in defining and keeping track of normalization factors.

Note that these complications do *not* arise for massless spin-s fields on AdS with standard boundary conditions. In particular the algebra generated by the (non-normalizable) Killing tensors in this case is a global symmetry algebra, acting nontrivially on the Hilbert space.

These problems are not insuperable, but they do require some effort. A brute-force path integral computation correctly dealing with all of them for general higher-spin theories is comparable to pulling a molar with a plastic fork: not impossible, but necessitating the sort of stamina some might see as savage and few would wish to witness. The character formalism simplifies the task, and the transparency of the result will make generalization obvious.

3.5.2 Ingredients and outline of derivation

We derive an exact formula for Z_{PI} in appendix C.7.2-C.7.4. In what follows we merely give a rough outline, just to give an idea what the origin is of various ingredients appearing in the final result. To avoid the $d = 2$ footnotes of section 3.4 we assume $\boxed{d \geq 3}$ in what follows.

Naive characters

Naively applying the reasoning of section 3.4 to the massless case, one gets a character formula of the form (3.84), with "naive" bulk and edge characters $\hat{\chi}$ given by

$$\hat{\chi} \equiv \chi_\phi - \chi_\xi \,, \tag{3.97}$$

where χ_ϕ, χ_ξ are the massive bulk/edge characters for the spin-s, $\Delta = s' + d - 1$ field ϕ and the spin-s', $\Delta = s + d - 1$ gauge parameter (or ghost) field ξ, recalling $s' \equiv s - 1$. The subtraction $-\chi_\xi$ arises from the BRST ghost path integral. More explicitly, from (3.85),

$$
\begin{aligned}
\hat{\chi}_{\text{bulk},s} &= D^d_s \, \frac{q^{s'+d-1} + q^{1-s'}}{(1-q)^d} - D^d_{s'} \, \frac{q^{s+d-1} + q^{1-s}}{(1-q)^d} \\
\hat{\chi}_{\text{edge},s} &= D^{d+2}_{s-1} \, \frac{q^{s'+d-2} + q^{-s'}}{(1-q)^{d-2}} - D^{d+2}_{s'-1} \, \frac{q^{s+d-2} + q^{-s}}{(1-q)^{d-2}} \, .
\end{aligned}
\tag{3.98}
$$

For example for $s = 2$ in $d = 3$,

$$
\hat{\chi}_{\text{bulk},2} = \frac{5\,(q^3+1) - 3\,(q^4+q^{-1})}{(1-q)^3} \, , \qquad \hat{\chi}_{\text{edge},2} = \frac{5\,(q^2+q^{-1}) - (q^3+q^{-2})}{1-q} \, . \tag{3.99}
$$

Because of the presence of non-positive powers of q, $\hat{\chi}_{\text{bulk}}$ is manifestly *not* the character of any unitary representation of $SO(1, d+1)$. Indeed, the character integral (3.84) using these naive $\hat{\chi}$ is badly IR-divergent, due to the presence of non-positive powers of q.

Flipped characters

In fact this pathology is nothing but the character integral incarnation of the negative and zeromode mode issues of the path integral mentioned under (3.96). The zeromodes must be omitted, and the negative modes are dealt with by contour rotations. These prescriptions turn out to translate to a certain "flipping" operation at the level of the characters. More specifically the flipped character $[\hat{\chi}]_+$ is obtained from $\hat{\chi} = \sum_k c_k q^k$ by flipping $c_k q^k \to -c_k q^{-k}$ for $k < 0$ and dropping the $k = 0$ terms:

$$
\big[\hat{\chi}\big]_+ = \Big[\sum_k c_k \, q^k\Big]_+ \equiv \sum_{k<0} (-c_k)\, q^{-k} + \sum_{k>0} c_k \, q^k \; = \hat{\chi} - c_0 - \sum_{k<0} c_k \big(q^k + q^{-k}\big) \, . \tag{3.100}
$$

For example for $s = 2$ in $d = 3$, starting from (3.99) and observing $\hat{\chi}_{\rm bulk} = -3\,q^{-1} - 4 + \cdots$ and $\hat{\chi}_{\rm edge} = -q^{-2} + 4\,q^{-1} + 4 + \cdots$, we get

$$\begin{aligned} \big[\hat{\chi}_{\rm bulk,2}\big]_+ &= \hat{\chi}_{\rm bulk,2} + 3(q^{-1}+q) + 4 = \frac{10\,q^3 - 6\,q^4}{(1-q)^3} \\ \big[\hat{\chi}_{\rm edge,2}\big]_+ &= \hat{\chi}_{\rm edge,2} + (q^{-2}+q^2) - 4(q^{-1}+q) - 4 = \frac{10\,q^2 - 2\,q^3}{1-q}\,. \end{aligned} \tag{3.101}$$

Explicit expressions for general d and s are given by $[\hat{\chi}_{\rm bulk}]_+ =$ (C.194) and $[\hat{\chi}_{\rm edge}]_+ =$ (C.196). Some simple examples are

d	s	$\big[\hat{\chi}_{\rm bulk,s}\big]_+ \cdot (1-q)^d$	$\big[\hat{\chi}_{\rm edge,s}\big]_+ \cdot (1-q)^{d-2}$
2	≥ 2	0	0
3	≥ 1	$2(2s+1)\,q^{s+1} - 2(2s-1)\,q^{s+2}$	$\frac{1}{3}s(s+1)(2s+1)\,q^s - \frac{1}{3}(s-1)s(2s-1)\,q^{s+1}$
4	≥ 1	$2(2s+1)\,q^2$	$\frac{1}{3}s(s+1)(2s+1)\,q$
≥ 3	1	$d\,(q^{d-1}+q) - q^d + 1 + (1-q)^d$	$q^{d-2} + 1 - (1-q)^{d-2}$

(3.102)

Contributions to $Z_{\rm PI}$

To be more precise, after implementing the appropriate contour rotations and zeromode subtractions, we get the following expression for the path integral:

$$Z_{\rm PI} = \frac{1}{{\rm vol}(G)_{\rm PI}} \prod_s \Big(\mathcal{A}_s\, i^{-P_s}\, Z_{{\rm char},s}\Big)^{n_s}\,. \tag{3.103}$$

where n_s is the number of massless spin-s fields in the theory, and the different factors appearing here are defined as follows:

1. $Z_{{\rm char},s}$ is defined by the character integral

$$\log Z_{{\rm char},s} \equiv \int_0^\times \frac{dt}{2t}\,\frac{1+q}{1-q}\Big(\big[\hat{\chi}_{{\rm bulk},s}\big]_+ - \big[\hat{\chi}_{{\rm edge},s}\big]_+ - 2\,D^{d+2}_{s-1,s-1}\Big), \tag{3.104}$$

where $\int_0^\times$ means $\int_0^\infty$ with the IR divergence due to the constant term removed:

$$\int_0^\times \frac{dt}{t} f(t) \equiv \lim_{L\to\infty} \int_0^\infty \frac{dt}{t} f(t)\, e^{-t/L} - f(\infty) \log L\,. \tag{3.105}$$

The flipped $\left[\hat\chi_{\text{bulk},s}\right]_+$ turns out to be precisely the massless spin-s exceptional series character $\chi_{\text{bulk},s}$: (C.194) = (C.164). Thus the χ_{bulk} contribution = ideal gas partition function Z_{bulk}, pleasingly consistent with the physics picture. The second term is an edge correction as in the massive case. The third term has no massive counterpart, tied to the presence of gauge zeromodes: $D^{d+2}_{s-1,s-1}$ counts rank $s-1$ Killing tensors on S^{d+1}.

2. $\mathcal{A}_s$ is due to the zeromode omissions. Denoting $M = 2e^{-\gamma}/\epsilon$ as in (C.68),

$$\log \mathcal{A}_s \equiv D^{d+2}_{s-1,s-1} \int_0^\times \frac{dt}{2t}\left(2 + q^{2s+d-4} + q^{2s+d-2}\right) = \frac{1}{2} D^{d+2}_{s-1,s-1} \log \frac{M^4}{(2s+d-4)(2s+d-2)} \tag{3.106}$$

This term looks ugly. Happily, it will drop out of the final result.

3. i^{-P_s} is the spin-s generalization of Polchinski's phase of the one-loop path integral of Einstein gravity on the sphere [59]. It arises because every negative mode contour rotation adds a phase factor $-i$ to the path integral. Explicitly,

$$P_s = \sum_{n=0}^{s-2} D^{d+2}_{s-1,n} + \sum_{n=0}^{s-2} D^{d+2}_{s-2,n} = D^{d+3}_{s-1,s-1} - D^{d+2}_{s-1,s-1} + D^{d+3}_{s-2,s-2}\,. \tag{3.107}$$

In particular $P_1 = 0$, $P_2 = D^{d+2}_1 + D^{d+2}_0 = d+3$ in agreement with [59]. For $d+1 = 4$, $P_s = \frac{1}{3} s\left(s^2-1\right)^2$ and $i^{-P_s} = 1, -1, 1, 1, 1, -1, 1, 1, \ldots$. For $d+1 = 2 \bmod 4$, $i^{-P_s} = 1$.

4. $\text{vol}(G)_{\text{PI}}$ is discussed below.

Volume of G

As mentioned under (3.96), G is the subgroup of gauge transformations generated by the Killing tensors $\bar{\xi}_{s-1}$ in the parent *interacting* theory on the sphere. Equivalently it is the subgroup of gauge transformations leaving the background invariant. For Einstein gravity, we have a single massless $s = 2$ field ϕ_2. The Killing vectors $\bar{\xi}_1$ generate diffeomorphisms rigidly rotating the sphere, hence $G = SO(d + 2)$. For $SU(N)$ Yang-Mills, we have $N^2 - 1$ massless $s = 1$ fields ϕ_1^a. The $N^2 - 1$ Killing scalars $\bar{\xi}_0^a$ generate constant $SU(N)$ gauge transformations, hence $G = SU(N)$.[16] For the 3D higher-spin gravity theories introduced in section 3.6, we have massless fields ϕ_s of spin $s = 2, \ldots, n$. The Killing tensors $\bar{\xi}_{s-1}$ turn out to generate $G = SU(n)_+ \times SU(n)_-$.

$\mathrm{vol}(G)_{\mathrm{PI}}$ is the volume of G according to the QFT path integral measure. We wish to relate it to a "canonical" $\mathrm{vol}(G)_{\mathrm{c}}$. We use the word "canonical" in the sense of defined in a theory-independent way. We determine $\mathrm{vol}(G)_{\mathrm{PI}}/\mathrm{vol}(G)_{\mathrm{c}}$ given our normalization conventions in appendix C.7.4. Below we summarize the most pertinent definitions and results.

For Einstein gravity, the Killing vector Lie algebra is $\mathfrak{g} = \mathrm{so}(d + 2)$. Picking a standard basis M_{IJ} satisfying $[M_{IJ}, M_{KL}] = \delta_{IK} M_{JL} + \delta_{JL} M_{IK} - \delta_{IL} M_{JK} - \delta_{JK} M_{IL}$, we define the "canonical" bilinear form $\langle\cdot|\cdot\rangle_{\mathrm{c}}$ on $\mathfrak{g}$ to be the unique invariant bilinear normalized such that

$$\langle M_{IJ} | M_{IJ} \rangle_{\mathrm{c}} \equiv 1 \qquad (I \neq J, \text{no sum})\,. \tag{3.108}$$

This invariant bilinear on $\mathfrak{g} = \mathrm{so}(d + 2)$ defines an invariant metric ds_{c}^2 on $G = SO(d + 2)$. Closed orbits generated by M_{IJ} then have length $\oint ds_c = 2\pi$, and $\mathrm{vol}(G)_{\mathrm{c}}$ is given by (C.93).

For higher-spin gravity, the Killing tensor Lie algebra $\mathfrak{g}$ contains $\mathrm{so}(d+2)$ as a subalgebra with generators M_{IJ}. We define $\langle\cdot|\cdot\rangle_c$ on $\mathfrak{g}$ to be the unique $\mathfrak{g}$-invariant bilinear form [68, 69] normalized by (3.108). $\mathrm{vol}(G)_{\mathrm{c}}$ is defined using the corresponding metric ds_c^2 on G.

The Killing tensor commutators are determined by the local gauge algebra $[\delta_\xi, \delta_{\xi'}] = \delta_{[\xi,\xi']}$ as in [68]. For Einstein or HS gravity, in our conventions (canonical ϕ + footnote 15), this gives for

[16] or a quotient thereof, such as $SU(N)/\mathbb{Z}_N$, depending on other data such as additional matter content. Here and in other instances, we will not try to be precise about the global structure of G.

the so$(d+2)$ Killing vector (sub)algebra of $\mathfrak{g}$

$$[\bar{\xi}_1, \bar{\xi}'_1] = \sqrt{16\pi G_\mathrm{N}}\, [\bar{\xi}'_1, \bar{\xi}_1]_\mathrm{Lie}\,, \tag{3.109}$$

where $[\cdot,\cdot]_\mathrm{Lie}$ is the standard vector field Lie bracket. In Einstein gravity, G_N is the Newton constant. In Einstein + higher-order curvature corrections (section 3.8) or in higher-spin gravity we take it to *define* the Newton constant. It is related to a "central charge" C in (C.224).

Building on [68, 71], we find the bilinear $\langle\cdot|\cdot\rangle_\mathrm{c}$ determining $\mathrm{vol}(G)_\mathrm{c}$ can then be written as

$$\langle\bar{\xi}|\bar{\xi}\rangle_\mathrm{c} = \frac{4G_\mathrm{N}}{A_{d-1}} \sum_s \sum_{\alpha=1}^{n_s} (2s+d-4)(2s+d-2) \int_{S^{d+1}} \bar{\xi}^{(\alpha)}_{s-1} \cdot \bar{\xi}^{(\alpha)}_{s-1}\,, \tag{3.110}$$

where $A_{d-1} = \mathrm{vol}(S^{d-1})$ is the dS horizon area. On the other hand, the path integral measure computing $\mathrm{vol}(G)_\mathrm{PI}$ is derived from the bilinear $\langle\xi|\bar{\xi}\rangle_\mathrm{PI} = \frac{M^4}{2\pi}\int \bar{\xi}\cdot\bar{\xi}$. From this we can read off the ratio $\mathrm{vol}(G)_\mathrm{c}/\mathrm{vol}(G)_\mathrm{PI}$: an awkward product of factors determined by the HS algebra. This turns out to cancel the awkward eigenvalue product of (3.106), up to a universal factor:

$$\frac{\mathrm{vol}(G)_\mathrm{c}}{\mathrm{vol}(G)_\mathrm{PI}} \prod_s \mathcal{A}_s^{n_s} = \left(\frac{8\pi G_\mathrm{N}}{A_{d-1}}\right)^{\frac{1}{2}\dim G}\,, \tag{3.111}$$

for all theories covered by [68], i.e. all parity-invariant HS theories consistent at cubic level.

For Yang-Mills, $\mathrm{vol}(G)_\mathrm{c}$ is computed using the metric ds_c^2 on $\mathfrak{g}$ defined by the canonically normalized YM action $S =: \frac{1}{4}\int\langle F|F\rangle_\mathrm{c}$. For example for $SU(N)$ YM with $S = -\frac{1}{4}\int \mathrm{Tr}_N F^2$, this gives $\mathrm{vol}(G)_\mathrm{c} = \mathrm{vol}(SU(N))_{\mathrm{Tr}_N} = (C.94)$. A similar but simpler computation gives the analog of (3.111). See appendix C.7.4 for details on all of the above.

3.5.3 Result and examples

Thus we arrive at the following universal formula for the one-loop Euclidean path integral for parity-symmetric (higher-spin) gravity and Yang-Mills gauge theories on S^{d+1}, $d \geq 3$:

$$\boxed{Z_{\rm PI}^{(1)} = i^{-P} \prod_{a=0}^{K} \frac{\gamma_a^{\dim G_a}}{\operatorname{vol} G_a} \cdot \exp \int_0^{\times} \frac{dt}{2t} \frac{1+q}{1-q} \big(\chi_{\rm bulk} - \chi_{\rm edge} - 2 \dim G\big)} \tag{3.112}$$

• $G = G_0 \times G_1 \times \cdots G_K$ is the subgroup of (higher-spin) gravitational and Yang-Mills gauge transformations acting trivially on the background,

$$\gamma_0 \equiv \sqrt{\frac{8\pi G_{\rm N}}{A_{d-1}}}, \qquad \gamma_1 \equiv \sqrt{\frac{g_1^2}{2\pi A_{d-3}}}, \qquad \cdots \tag{3.113}$$

where $A_n \equiv \Omega_n \ell^n$, $\Omega_n = (C.92)$, the gravitational and YM coupling constants $G_{\rm N}$ and $g_1, \ldots, g_K$ are defined by the canonically normalized so$(d+2)$ and YM gauge algebras as explained around (3.109), and $\operatorname{vol} G_a$ is the canonically normalized volume of G_a, defined in the same part.

• For a theory with n_s massless spin-s fields

$$\chi = \sum_s n_s \chi_s \,, \qquad \dim G = \sum_s n_s D^{d+2}_{s-1,s-1} \,, \qquad P = \sum_s n_s P_s \,, \tag{3.114}$$

where $\chi_s = [\hat{\chi}_s]_+$ are the flipped versions (3.100) of the naive characters (3.98), with examples in (3.102) and general formulae in (C.194) and (C.196), and $P_s = (3.107)$ is the spin-s generalization of the $s = 2$ phase $P_2 = d + 3$ found in [59].

• The heat-kernel regularized integral can be evaluated using (C.57), as spelled out in appendix C.3.3. For odd $d + 1$, the finite part can alternatively be obtained by summing residues.

$\int_0^{\times}$ means integration with the IR log-divergence from the constant $-2 \dim G$ term removed as in (3.105). The constant term contribution is then $\dim G \cdot \big(c\, \epsilon^{-1} + \log(2\pi)\big)$, so when keeping track of linearly divergent terms is not needed, one can replace (3.112) by

$$Z_{\rm PI}^{(1)} = i^{-P} \prod_a \frac{(2\pi\gamma_a)^{\dim G_a}}{\operatorname{vol} G_a} \cdot \exp \int_0^\infty \frac{dt}{2t} \frac{1+q}{1-q} \left(\chi_{\rm bulk} - \chi_{\rm edge}\right) \qquad (\text{mod } \epsilon^{-1}) \tag{3.115}$$

• The case $d = 2$ requires some minor amendments, discussed in appendix C.8.1: for $s \geq 2$, nothing changes except P_s, and $\chi = 0$, resulting in (C.229). Yang-Mills gives (C.232), or mod ϵ^{-1} (C.233), equivalent to putting $A_{-1} \equiv 1/2\pi\ell$ in (3.113), and Chern-Simons (C.235).

• The above can be extended to more general theories. For examples (s, s') partially massless gauge fields have characters given by (C.194) and (C.196), and contribute $D^{d+2}_{s-1,s'}$ to $\dim G$. Fermionic counterparts can be derived following the same steps, with $\hat\chi_{\rm edge}$ given by (3.93). Fermionic (s, s') PM fields give negative contributions $-D^{d+2}_{s-1,s',\frac12,\ldots,\frac12}$ to $\dim G$.

Example: coefficient α_{d+1} of log-divergent term

The heat kernel coefficient α_{d+1}, i.e. the coefficient of the log-divergent term of $\log Z$, can be read off simply as the coefficient of the $1/t$ term in the small-t expansion of the integrand. As explained in C.3.3, we can just use the original, naive integrand $\hat F(t) = \frac{1}{2t}(\hat\chi_{\rm bulk} - \hat\chi_{\rm edge})$ for this purpose, obtained from (3.98). For e.g. a massless spin-s field on S^4 this immediately gives $\alpha_4^{(s)} = -\frac{1}{90}\left(75\, s^4 - 15\, s^2 + 2\right)$, in agreement with eq. (2.32) of [62]. For $s = 1, 2$,

d	3	5	7	9	11	13	15
$\alpha^{(1)}_{d+1}$	$-\frac{31}{45}$	$-\frac{1271}{1890}$	$-\frac{4021}{6300}$	$-\frac{456569}{748440}$	$-\frac{1199869961}{2043241200}$	$-\frac{893517041}{1571724000}$	$-\frac{17279945447657}{31261590360000}$
$\alpha^{(2)}_{d+1}$	$-\frac{571}{45}$	$-\frac{3181}{140}$	$-\frac{198851}{5670}$	$-\frac{74203873}{1496880}$	$-\frac{75059846731}{1135134000}$	$-\frac{114040703221}{1347192000}$	$-\frac{821333912103503}{7815397590000}$

(3.116)

Another case of general interest is a partially massless field with $(s, s') = (42, 26)$ on S^{42}:

$$\alpha_{42}^{(42,26)} = -\frac{59257008379951521058183995473963450888216357833051998154446027620215619709911519472215 47}{534886720324851274320276006645566592000000000} \sim -10^{42}$$

Example: $SU(4)$ Yang-Mills on S^5

As a simple illustration and test of (3.112), consider $SU(4)$ YM theory on S^5 of radius ℓ with action $S = \frac{1}{4g^2}\int \mathrm{Tr}_4\, F^2$, so $G = SU(4)$, $n_1 = \dim G = 15$, $\mathrm{vol}(G)_{\mathrm{c}} = \frac{(2\pi)^9}{6}$ as given by (C.94), $\gamma = \sqrt{\frac{g^2}{(2\pi)^2\ell}}$, and $P = 0$. Bulk and edge characters are read off from table (3.102). Thus

$$\log Z_{\mathrm{PI}}^{(1)} = \log \frac{\left(g/\sqrt{\ell}\right)^{15}}{(2\pi)^{15}\cdot\frac{(2\pi)^9}{6}} + 15\cdot\int_0^{\times}\frac{dt}{2t}\frac{1+q}{1-q}\left(\frac{6\,q^2}{(q-1)^4} - \frac{2\,q}{(q-1)^2} - 2\right). \tag{3.117}$$

The finite part can be evaluated by simply summing residues, similar to (3.50):

$$\log Z_{\mathrm{PI}}^{\mathrm{fin}} = \log \frac{\left(g/\sqrt{\ell}\right)^{15}}{\frac{1}{6}(2\pi)^9} + 15\cdot\left(\frac{5\,\zeta(3)}{16\,\pi^2} + \frac{3\,\zeta(5)}{16\,\pi^4}\right). \tag{3.118}$$

The $U(1)$ version of this agrees with [90] eq. (2.27). We could alternatively use (C.57) as in C.3.3, which includes the UV divergent part: $\log Z_{\mathrm{PI}}^{(1)} = \log Z_{\mathrm{PI}}^{\mathrm{fin}} + 15\left(\frac{9\pi}{8}\epsilon^{-5}\ell^5 - \frac{5\pi}{8}\epsilon^{-3}\ell^3 - \frac{7\pi}{16}\epsilon^{-1}\ell\right)$.

Example: Einstein gravity on S^3, S^4 and S^5

The exact one-loop Euclidean path integral for Einstein gravity on the sphere can be worked out similarly. The S^3 case is obtained in (C.231). The S^4 and S^5 cases are detailed in C.3.3, with results including UV-divergent terms given in (C.229), (C.82), (C.85). The finite parts are:

$$Z_{\mathrm{PI}}^{\mathrm{fin}} = i^{-P}\cdot\frac{1}{\mathrm{vol}(G)_{\mathrm{c}}}\left(\frac{8\pi G_{\mathrm{N}}}{A_{d-1}}\right)^{\frac{1}{2}\dim G}\cdot Z_{\mathrm{char}}^{\mathrm{fin}}, \tag{3.119}$$

S^{d+1}	i^{-P}	$\mathrm{vol}(G)_{\mathrm{c}}$	A_{d-1}	$\dim G$	$\log Z_{\mathrm{char}}^{\mathrm{fin}}$
S^3	$-i$	$(2\pi)^4$	$2\pi\ell$	6	$6\log(2\pi)$
S^4	-1	$\frac{2}{3}(2\pi)^6$	$4\pi\ell^2$	10	$-\frac{571}{45}\log(\ell/L) + \frac{715}{48} - \log 2 - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3)$
S^5	i	$\frac{1}{12}(2\pi)^9$	$2\pi^2\ell^3$	15	$15\log(2\pi) + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4}$

(3.120)

Checks: We rederive the S^3 result in the Chern-Simons formulation of 3D gravity [143] in appendix

C.8.2, and find precise agreement, with the phase matching for odd framing of the Chern-Simons partition function (it vanishes for even framing). The coefficient $-\frac{571}{45}$ of the log-divergent term of the S^4 result agrees with [51]. The phases agree with [59]. The powers of $G_{\rm N}$ agree with zeromode counting arguments of [81, 128]. The full one-loop partition function on S^4 was calculated using zeta-function regularization in [58]. Upon correcting an error in the second number of their equation (A.36) we find agreement. As far as we know, the zeta-function regularized $Z_{\rm PI}^{(1)}$ has not been explicitly computed before for S^{d+1}, $d \geq 4$.

Higher-spin theories

Generic Vasiliev higher-spin gravity theories have infinite spin range and $\dim G = \infty$, evidently posing problems for (3.112). We postpone discussion of this case to section 3.9. Below we consider a 3D higher-spin gravity theory with finite spin range $s = 2, \ldots, n$.

3.6 3D HS$_n$ gravity and the topological string

As reviewed in appendix C.8.2, 3D Einstein gravity with positive cosmological constant in Lorentzian or Euclidean signature can be formulated as an $SL(2,\mathbb{C})$ resp. $SU(2) \times SU(2)$ Chern-Simons theory [143].[17] This has a natural extension to an $SL(n,\mathbb{C})$ resp. $SU(n) \times SU(n)$ Chern-Simons theory, discussed in appendix C.8.3, which can be viewed as an $s \leq n$ dS$_3$ higher-spin gravity theory, analogous to the AdS$_3$ theories studied e.g. in [145–148, 153, 154]. The Lorentzian/Euclidean actions S_L/S_E are

$$S_L = iS_E = (l + i\kappa)\, S_{\rm CS}[\mathcal{A}_+] + (l - i\kappa)\, S_{\rm CS}[\mathcal{A}_-]\,, \qquad l \in \mathbb{N}, \quad \kappa \in \mathbb{R}^+\,, \tag{3.121}$$

where $S_{\rm CS}[\mathcal{A}] = \frac{1}{4\pi} \int {\rm Tr}_n\big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\big)$ and $\mathcal{A}_\pm$ are sl(n)-valued connections with reality condition $\mathcal{A}_\pm^* = \mathcal{A}_\mp$ for the Lorentzian theory and $\mathcal{A}_\pm^\dagger = \mathcal{A}_\pm$ for the Euclidean theory.

The Chern-Simons formulation allows all-loop exact results, providing a useful check of our

[17] Or more precisely an $SO(1,3) = SL(2,\mathbb{C})/\mathbb{Z}_2$ or $SO(4) = \big(SU(2) \times SU(2)\big)/\mathbb{Z}_2$ CS theory. For the higher-spin extensions, we could similarly consider quotients. We will use the unquotiented groups here.

result (3.112) for $Z_{\rm PI}^{(1)}$ obtained in the metric-like formulation. Besides this, we observed a number of other interesting features, collected in appendix C.8.3, and summarized below.

Landscape of vacua (C.8.3)

The theory has a set of dS_3 vacua (or round S^3 solutions in the Euclidean theory), corresponding to different embeddings of sl(2) into sl(n), labeled by n-dimensional representations

$$R = \oplus_a \mathbf{m_a} \,, \qquad n = \sum_a m_a \,. \tag{3.122}$$

of su(2), i.e. by partitions of $n = \sum_a m_a$. The radius in Planck units $\ell/G_{\rm N}$ and $Z^{(0)} = e^{-S_E}$ depend on the vacuum R as

$$\log Z^{(0)} = \frac{2\pi\ell}{4G_N} = 2\pi\kappa\, T_R \,, \qquad T_R = \frac{1}{6}\sum_a m_a(m_a^2 - 1)\,. \tag{3.123}$$

Note that $\mathcal{S}^{(0)} = \log Z^{(0)}$ takes the standard Einstein gravity horizon entropy form. The entropy is maximized for the principal embedding, i.e. $R = \mathbf{n}$, for which $T_{\mathbf{n}} = \frac{1}{6}n(n^2-1)$. The number of vacua equals the number of partitions of n:

$$\mathcal{N}_{\rm vac} \sim e^{2\pi\sqrt{n/6}}\,. \tag{3.124}$$

For, say, $n \sim 2\times 10^5$, we get $\mathcal{N}_{\rm vac} \sim 10^{500}$, with maximal entropy $\mathcal{S}^{(0)}|_{R=\mathbf{n}} \sim 10^{15}\kappa$.

Higher-spin algebra and metric-like field content (C.8.3)

As worked out in detail for the AdS analog in [153], the fluctuations of the Chern-Simons connection for the principal embedding vacuum $R = \mathbf{n}$ correspond in a metric-like description to a set of massless spin-s fields with $s = 2, 3, \ldots, n$. The Euclidean higher-spin algebra is $\mathrm{su}(n)_+ \oplus \mathrm{su}(n)_-$, which exponentiates to $G = SU(n)_+ \times SU(n)_-$. The higher-spin field content of the $R = \mathbf{n}$ vacuum can also be inferred from the decomposition of su(n) into irreducible representations of su(2), with $S \in$ su(2) acting on $L \in$ su(n) as $\delta L = \epsilon[R(S), L]$, to wit,

$$(\mathbf{n^2} - \mathbf{1})_{\mathrm{su}(n)} = \sum_{r=1}^{n-1} (\mathbf{2r} + \mathbf{1})_{\mathrm{su}(2)} \,. \tag{3.125}$$

The $(\mathbf{2r}+\mathbf{1},\mathbf{1})$ and $(\mathbf{1},\mathbf{2r}+\mathbf{1})$ of so(4) = su(2)$_+$ ⊕ su(2)$_-$ correspond to rank-r self-dual and anti-self-dual Killing tensors on S^3, the zeromodes of (3.94) for a massless spin-$(r+1)$ field, confirming $R = \mathbf{n}$ has $n_s = 1$ massless spin-s field for $s = 2, \ldots, n$. For different vacua R, one gets decompositions different from (3.125), associated with different field content. For example for $n = 12$ and $R = \mathbf{6} \oplus \mathbf{4} \oplus \mathbf{2}$, we get $n_1 = 2$, $n_2 = 7$, $n_3 = 8$, $n_4 = 6$, $n_5 = 3$, $n_6 = 1$.

One-loop and all-loop partition function (C.8.3-C.8.3)

In view of the above higher-spin interpretation, we can compute the one-loop Euclidean path integral on S^3 for $l = 0$ from our general formula (3.112) for higher-spin gravity theories in the metric-like formalism. The dS$_3$ version of (3.112) is worked out in (C.228)-(C.230), and applied to the case of interest in (C.267), using (3.123) to convert from ℓ/G_{N} to κ. The result is

$$Z_{\mathrm{PI}}^{(1)} = i^{n^2-1} \cdot \frac{\left(2\pi/\sqrt{\kappa}\right)^{\dim G}}{\mathrm{vol}(G)_{\mathrm{Tr}_n}} \,, \tag{3.126}$$

where $\mathrm{vol}(G)_{\mathrm{Tr}_n} = \left(\sqrt{n}\prod_{s=2}^{n}(2\pi)^s/\Gamma(s)\right)^2$ as in (C.94).

This can be compared to the weak-coupling limit of the all-loop expression (C.269)-(C.271), obtained from the known exact partition function of $SU(n)_{k_+} \times SU(n)_{k_-}$ Chern-Simons theory on S^3 by analytic continuation $k_\pm \to l \pm i\kappa$,

$$Z(R)_r = e^{ir\phi} \cdot \left| \frac{1}{\sqrt{n}} \frac{1}{(n+l+i\kappa)^{\frac{n-1}{2}}} \prod_{p=1}^{n-1} \left(2\sin\frac{\pi p}{n+l+i\kappa}\right)^{(n-p)} \right|^2 \cdot e^{2\pi\kappa T_R} \,. \tag{3.127}$$

Here $\phi = \frac{\pi}{4}\sum_\pm c(l \pm i\kappa)$ with $c(k) \equiv (n^2-1)\left(1-\frac{n}{n+k}\right)$, and $r \in \mathbb{Z}$ labels the choice of framing needed to define the Chern-Simons theory as a QFT, discussed in more detail below (C.249). Canonical framing corresponds to $r = 0$. $Z(R)$ is interpreted as the all-loop quantum-corrected Euclidean partition function of the dS_3 static patch in the vacuum R.

The weak-coupling limit $\kappa \to \infty$ of (3.127) precisely reproduces (3.126), with the phase matching for odd framing r. Alternatively this can be seen more directly by a slight variation of the computation leading to (C.235). This provides a check of (3.112), in particular its normalization in the metric-like formalism, and of the interpretation of (3.121) as a higher-spin gravity theory.

Large-n limit and topological string dual (C.8.3)

Vasliev-type hs(so($d+2$)) higher-spin theories (section 3.9) have infinite spin range but finite $\ell^{d-1}/G_{\rm N}$. To mimic this case, consider the $n \to \infty$ limit of the theory at $l = 0$. The semiclassical expansion is reliable only if $n \ll \kappa$. Using $\ell/G_{\rm N} \sim \kappa T_R$, this translates to $n T_R \ll \ell/G_{\rm N}$, which becomes $n^4 \ll \ell/G_{\rm N}$ for the principal vacuum $R = \mathbf{n}$, and $n \ll \ell/G_{\rm N}$ at the other extreme for $R = \mathbf{2} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}$. Either way, the Vasiliev-like limit $n \to \infty$ at fixed $\mathcal{S}^{(0)} = 2\pi\ell/4G_{\rm N}$ is strongly coupled.

However (3.127) continues to make sense in any regime, and in particular *does* have a weak coupling expansion in the $n \to \infty$ 't Hooft limit. Using the large-n duality between $U(n)_k$ Chern-Simons on S^3 and closed topological string theory on the resolved conifold [79, 80], the partition function (3.127) of de Sitter higher-spin quantum gravity in the vacuum R can be expressed in terms of the weakly-coupled topological string partition function $\tilde{Z}_{\rm top}$, (C.274):

$$\boxed{Z(R)_0 = \left|\tilde{Z}_{\rm top}(g_s, t)\, e^{-\pi T_R \cdot 2\pi i/g_s}\right|^2} \tag{3.128}$$

where (in the notation of [80]) the string coupling constant g_s and the resolved conifold Kähler modulus $t \equiv \int_{S^2} J + iB$ are given by

$$g_s = \frac{2\pi}{n + l + i\kappa}, \qquad t = i g_s n = \frac{2\pi i n}{n + l + i\kappa}\,. \tag{3.129}$$

Note that $\left|e^{-\pi T_R \cdot 2\pi i/g_s}\right|^2 = e^{2\pi\kappa T_R} = e^{\mathcal{S}^{(0)}}$, and that $\kappa > 0$ implies $\int_{S^2} J > 0$ and $\mathrm{Im}\, g_s \neq 0$. The dependence on n at fixed $\mathcal{S}^{(0)}$ is illustrated in fig. 3.4. We leave further exploration of the dS quantum gravity - topological string duality suggested by these observations to future work.

3.7 Euclidean thermodynamics

In section 3.2.3 we defined and computed the bulk partition function, energy and entropy of the static patch ideal gas. In this section we define and compute their Euclidean counterparts, building on the results of the previous sections.

3.7.1 Generalities

Consider a QFT on a dS_{d+1} background with curvature radius ℓ. Wick-rotated to the round sphere metric $g_{\mu\nu}$ of radius ℓ $\langle C.4.3\rangle$, we get the Euclidean partition function:

$$Z_{\rm PI}(\ell) \equiv \int \mathcal{D}\Phi \, e^{-S_E[\Phi]} \tag{3.130}$$

where Φ collectively denotes all fields. The quantum field theory is to be thought of here as a (weakly) interacting low-energy effective field theory with a UV cutoff ϵ.

Recalling the path integral definition (C.116) of the Euclidean vacuum $|O\rangle$ paired with its dual $\langle O|$ as $Z_{\rm PI} = \langle O|O\rangle$, the Euclidean expectation value of the stress tensor is

$$\langle T_{\mu\nu}\rangle \equiv \frac{\langle O|T_{\mu\nu}|O\rangle}{\langle O|O\rangle} = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}} \log Z_{\rm PI} = -\rho_{\rm PI}\, g_{\mu\nu} \,, \tag{3.131}$$

The last equality, in which $\rho_{\rm PI}$ is a constant, follows from $SO(d+2)$ invariance of the round sphere background. Denoting the volume of the sphere by $V = {\rm vol}(S_\ell^{d+1}) = \Omega_{d+1}\ell^{d+1}$,

$$\boxed{-\rho_{\rm PI} V = \tfrac{1}{d+1}\int\sqrt{g}\,\left\langle T^\mu_\mu\right\rangle = \tfrac{1}{d+1}\ell\partial_\ell \log Z_{\rm PI} = V\partial_V \log Z_{\rm PI}} \tag{3.132}$$

Reinstating the radius ℓ, the sphere metric in the S coordinates of (C.98) takes the form

$$ds^2 = (1-r^2/\ell^2)d\tau^2 + (1-r^2/\ell^2)^{-1}dr^2 + r^2 d\Omega^2 \,, \tag{3.133}$$

where $\tau \simeq \tau + 2\pi\ell$. Wick rotating $\tau \to iT$ yields the static patch metric. Its horizon at $r = \ell$ has inverse temperature $\beta = 2\pi\ell$. On a constant-T slice, the vacuum expectation value of the Killing energy density corresponding to translations of T equals $\rho_{\rm PI}$ at the location $r = 0$ of the inertial observer. Away from $r = 0$, it is redshifted by a factor $\sqrt{1 - r^2/\ell^2}$. The Euclidean vacuum expectation value $U_{\rm PI}$ of the total static patch energy then equals $\rho_{\rm PI}\sqrt{1 - r^2/\ell^2}$ integrated over a constant-T slice:

$$U_{\rm PI} = \rho_{\rm PI}\,\Omega_{d-1}\int_0^\ell dr\, r^{d-1} = \rho_{\rm PI}\, v\,, \qquad v = \frac{\Omega_{d-1}\ell^d}{d} = \frac{V}{2\pi\ell}\,. \tag{3.134}$$

Note that v is the volume of a d-dimensional ball of radius ℓ in flat space, so effectively we can think of $U_{\rm PI}$ as the energy of an ordinary ball of volume v with energy density $\rho_{\rm PI}$.

Combining (3.132) and (3.134), the Euclidean energy on this background is obtained as

$$\boxed{2\pi\ell\, U_{\rm PI} = V\rho_{\rm PI} = -\tfrac{1}{d+1}\,\ell\partial_\ell \log Z_{\rm PI}} \tag{3.135}$$

and the corresponding Euclidean entropy $S_{\rm PI} \equiv \log Z_{\rm PI} + \beta\, U_{\rm PI}$ is

$$\boxed{S_{\rm PI} = \big(1 - \tfrac{1}{d+1}\ell\partial_\ell\big)\log Z_{\rm PI} = \big(1 - V\partial_V\big)\log Z_{\rm PI}} \tag{3.136}$$

$S_{\rm PI}$ can thus be viewed as the Legendre transform of $\log Z_{\rm PI}$ trading V for $\rho_{\rm PI}$:

$$d\log Z_{\rm PI} = -\rho_{\rm PI}\, dV, \qquad S_{\rm PI} = \log Z_{\rm PI} + V\rho_{\rm PI}\,, \qquad dS_{\rm PI} = V\, d\rho_{\rm PI}\,. \tag{3.137}$$

The above differential relations express the first law of (Euclidean) thermodynamics for the system under consideration: using $V = \beta v$ and $\rho_{\rm PI} = U_{\rm PI}/v$, they can be rewritten as

$$d\log Z_{\rm PI} = -U_{\rm PI}\, d\beta - \beta\rho_{\rm PI}\, dv\,, \qquad dS_{\rm PI} = \beta\, dU_{\rm PI} - \beta\rho_{\rm PI}\, dv\,. \tag{3.138}$$

Viewing v as the effective thermodynamic volume as under (3.134), these take the familiar form of the first law, with pressure $p = -\rho$, the familiar cosmological vacuum equation of state.

The expression (3.136) for the Euclidean entropy and (3.137) naturally generalize to Euclidean partition functions $Z_{\rm PI}(\ell)$ for *arbitrary* background geometries $g_{\mu\nu}(\ell) \equiv \ell^2 \tilde{g}_{\mu\nu}$ with volume $V(\ell) = \ell^{d+1}\tilde{V}$. In contrast, the expression (3.135) for the Euclidean energy is specific to the sphere. A generic geometry has no isometries, so there is no notion of Killing energy to begin with. On the other hand, the density $\rho_{\rm PI}$ appearing in (3.137) does generalize to arbitrary backgrounds. The last equality in (3.131) and the physical interpretation of $\rho_{\rm PI}$ as a Killing energy density no longer apply, but (3.132) remains valid.

3.7.2 Examples

Free $d = 0$ scalar

To connect to the familiar and to demystify the ubiquitous $\mathrm{Li}_n(e^{-2\pi\nu}) = \sum_k e^{-2\pi k\nu}/k^n$ terms encountered later, consider a scalar of mass m on an S^1 of radius ℓ, a.k.a. a harmonic oscillator of frequency m at $\beta = 2\pi\ell = V$. Using (C.70) and applying (3.135)-(3.136) with $\nu(\ell) \equiv m\ell$,

$$\begin{aligned} \log Z_{\rm PI} &= \frac{\pi\ell}{\epsilon} - \pi\nu + \mathrm{Li}_1(e^{-2\pi\nu}) \\ 2\pi\ell\, U_{\rm PI} = V\rho_{\rm PI} &= -\frac{\pi\ell}{\epsilon} + \pi\nu\coth(\pi\nu) \\ S_{\rm PI} &= \mathrm{Li}_1(e^{-2\pi\nu}) + 2\pi\nu\,\mathrm{Li}_0(e^{-2\pi\nu}), \end{aligned} \tag{3.139}$$

Mod $\Delta E_0 \propto -\epsilon^{-1}$, these are the textbook canonical formulae turned into polylogs by (3.51).

Free scalar in general d

The Euclidean action of a free scalar on S^{d+1} is

$$S_E[\phi] = \frac{1}{2}\int \sqrt{g}\,\phi\big(-\nabla^2 + m^2 + \xi R\big)\phi, \tag{3.140}$$

with $R = d(d+1)/\ell^2$ the S^{d+1} Ricci scalar. The total effective mass $m_{\rm eff}^2 = \big((\frac{d}{2})^2 + \nu^2\big)/\ell^2$ is

$$m_{\rm eff}^2 = m^2 + \xi R \qquad \Rightarrow \qquad \nu = \sqrt{(m\ell)^2 - \eta}\,, \quad \eta \equiv \left(\tfrac{d}{2}\right)^2 - d(d+1)\,\xi\,. \tag{3.141}$$

Neither $Z_{\rm PI}$ nor the bulk thermodynamic quantities of section 3.2 distinguish between the m^2 and ξR contributions to $m_{\rm eff}^2$, but $U_{\rm PI}$ and $S_{\rm PI}$ do, due to the ∂_ℓ derivatives in (3.135)-(3.136). This results in an additional explicit dependence on ξ, as

$$\ell\partial_\ell \log Z_{\rm PI} = \big(-\epsilon\partial_\epsilon + J\cdot\nu\partial_\nu\big)\log Z_{\rm PI}\,, \qquad J = \frac{\ell\partial_\ell\nu}{\nu} = \frac{(m\ell)^2}{\nu^2} = \frac{\nu^2+\eta}{\nu^2}\,. \tag{3.142}$$

For the minimally coupled case $\xi = 0$, the Euclidean and bulk thermodynamic quantities agree, but in general not if $\xi \neq 0$. To illustrate this we consider the $d = 2$ example. Using (3.50) and (C.60), restoring ℓ, and putting $\nu \equiv \sqrt{(m\ell)^2 - \eta}$ with $\eta = 1 - 6\xi$,

$$\log Z_{\rm PI} = \frac{\pi\ell^3}{2\epsilon^3} - \frac{\pi\nu^2\ell}{4\epsilon} \; + \frac{\pi\nu^3}{6} - \sum_{k=0}^{2}\frac{\nu^k}{k!}\,\frac{{\rm Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,. \tag{3.143}$$

The corresponding Euclidean energy $U_{\rm PI} = \rho_{\rm PI}\,\pi\ell^2$ (3.135) is given by

$$2\pi\ell\, U_{\rm PI} = V\rho_{\rm PI} = -\frac{\pi\ell^3}{2\epsilon^3} + \frac{\pi(\nu^2+\frac{2}{3}\eta)\ell}{4\epsilon} - \frac{\pi}{6}(\nu^2+\eta)\nu\coth(\pi\nu) \tag{3.144}$$

where $V = {\rm vol}(S^3_\ell) = 2\pi^2\ell^3$. For minimal coupling $\xi = 0$ (i.e. $\eta = 1$), $U_{\rm PI}^{\rm fin}$ equals $U_{\rm bulk}^{\rm fin}$ (3.49), but not for $\xi \neq 0$. For general d, ξ, $U_{\rm PI}^{\rm fin}$ is given by (3.48) with the overall factor m^2 the mass m^2 appearing in the action rather than $m_{\rm eff}^2$, in agreement with [155] or (6.178)-(6.180) of [156]. The entropy $S_{\rm PI} = \log Z_{\rm PI} + 2\pi\ell\, U_{\rm PI}$ (3.136) is

$$S_{\rm PI} = \frac{\pi\eta}{6}\Big(\frac{\ell}{\epsilon} - \nu\coth(\pi\nu)\Big) - \sum_{k=0}^{3}\frac{\nu^k}{k!}\,\frac{{\rm Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,, \tag{3.145}$$

where we used $\coth(\pi\nu) = 1 + 2\,\mathrm{Li}_0(e^{-2\pi\nu})$ (3.51). Since $Z_{\rm PI} = Z_{\rm bulk}$ in general for scalars and $U_{\rm PI} = U_{\rm bulk}$ for minimally coupled scalars, $S_{\rm PI} = S_{\rm bulk}$ for minimally coupled scalars. Indeed, after conversion to Pauli-Villars regularization, (3.145) equals (3.52) if $\eta = 1$. As a check on the results, the first law $dS_{\rm PI} = V d\rho_{\rm PI}$ (3.137) can be verified explicitly.

In the $m\ell \to \infty$ limit, $S_{\rm PI} \to \frac{\pi}{6}\eta(\epsilon^{-1} - m)\ell$, reproducing the well-known scalar one-loop Rindler entropy correction computed by a Euclidean path integral on a conical geometry [38, 39, 130, 131, 135, 157]. Note that $S_{\rm PI} < 0$ when $\eta < 0$. Indeed as reviewed in the Rindler context in appendix C.5.5, $S_{\rm PI}$ does *not* have a statistical mechanical interpretation on its own. Instead it must be interpreted as a correction to the large positive classical gravitational horizon entropy. We discuss this in the de Sitter context in section 3.8.

A pleasant feature of the sphere computation is that it avoids replicated or conical geometries: instead of varying a deficit angle, we vary the sphere radius ℓ, preserving manifest $SO(d+2)$ symmetry, and allowing straightforward exact computation of the Euclidean entropy directly from $Z_{\rm PI}(\ell)$, for arbitrary field content.

Free 3D massive spin s

Recall from (3.90) that for a $d = 2$ massive spin-$s \geq 1$ field of mass m, the bulk part of $\log Z_{\rm PI}$ is twice that of a $d = 2$ scalar (3.143) with $\nu = \sqrt{(m\ell)^2 - \eta}$, $\eta = (s-1)^2$, while the edge part is $-s^2$ times that of a $d = 0$ scalar, as in (3.139), with the important difference however that $\nu = \sqrt{(m\ell)^2 - \eta}$ instead of $\nu = m\ell$. Another important difference with (3.139) is that in the case at hand, (3.135) stipulates $V\rho_{\rm PI} = 2\pi\ell\, U_{\rm PI} = -\frac{1}{d+1}\ell\partial_\ell \log Z_{\rm PI}$ with $d = 2$ instead of $d = 0$. As a result, for the bulk contribution, we can just copy the scalar formulae (3.144) and (3.145) for $U_{\rm PI}$ and $S_{\rm PI}$ setting $\eta = (s-1)^2$, while for the edge contribution we get something rather different from the harmonic oscillator energy and entropy (3.139):

$$V\rho_{\rm PI} = 2 \times (3.144) - s^2\big(-\tfrac{\pi}{3}\tfrac{1}{\epsilon}\ell + \tfrac{\pi}{3}\big(\nu^2 + \eta\big)\nu^{-1}\coth(\pi\nu)\big) \tag{3.146}$$

$$S_{\rm PI} = 2 \times (3.145) - s^2\big(\tfrac{2\pi}{3}(\tfrac{1}{\epsilon}\ell - \nu) + \tfrac{\pi}{3}\eta\nu^{-1}\coth(\pi\nu) + \mathrm{Li}_1(e^{-2\pi\nu}) + \tfrac{2\pi}{3}\nu\,\mathrm{Li}_0(e^{-2\pi\nu})\big) \tag{3.147}$$

The edge contribution renders $S_{\rm PI}$ *negative* for all ℓ. In particular, in the $m\ell \to \infty$ limit, $S_{\rm PI} \to \frac{\pi}{3}\big((s-1)^2 - 2s^2\big)\left(\epsilon^{-1} - m\right)\ell \to -\infty$: although the bulk part gives a large positive contribution for $s \geq 2$, the edge part gives an even larger negative contribution. Going in the opposite direction, to smaller $m\ell$, we hit the $d = 2$, $s \geq 1$ unitarity bound at $\nu = 0$, i.e. at $m\ell = \sqrt{\eta} = s - 1$. Approaching this bound, the bulk contribution remains finite, while the edge part diverges, again negatively. For $s = 1$, $S_{\rm PI} \to \log(m\ell)$, due to the $\mathrm{Li}_1(e^{-2\pi\nu})$ term, while for $s \geq 2$, more dramatically, we get a pole $S_{\rm PI} \to -\frac{s^2(s-1)}{6}\big(m\ell - (s-1)\big)^{-1}$, due to the $\eta\nu^{-1}\coth(\pi\nu)$ term. Below the unitarity bound, i.e. when $\ell < (s-1)/m$, $S_{\rm PI}$ becomes complex. To be consistent as a perturbative low-energy effective field theory valid down to some length scale l_s, massive spin-$s \geq 2$ particles on dS_3 must satisfy $m^2 > (s-1)^2/l_s^2$.

Massless spin 2

From the results and examples in section 3.5.3, $\log Z_{\rm PI}^{(1)} = \log Z_{\rm PI,div}^{(1)} + \log Z_{\rm PI,fin}^{(1)} - \frac{(d+3)\pi}{2}\, i$,

$$\log Z^{(1)}_{\rm PI,fin}(\ell) = -\frac{D_d}{2}\log\frac{A(\ell)}{4G_{\rm N}} + \alpha^{(2)}_{d+1}\log\frac{\ell}{L} + K_{d+1} \tag{3.148}$$

$D_d = \dim \mathrm{so}(d+2) = \frac{(d+2)(d+1)}{2}$, $A(\ell) = \Omega_{d-1}\ell^{d-1}$, $\alpha^{(2)}_{d+1} = 0$ for even d and given by (3.116) for odd d. L is an arbitrary length scale canceling out of the sum of finite and divergent parts, and K_{d+1} an exactly computable numerical constant. Explicitly for $d = 2, 3, 4$, from (3.120):

d	$\log Z^{(1)}_{\rm PI,div}$	$\log Z^{(1)}_{\rm PI,fin}$
2	$0 - \frac{9\pi}{2}\frac{1}{\epsilon}\ell$	$-3\log\Big(\frac{\pi}{2G_{\rm N}}\ell\Big) + 5\log(2\pi)$
3	$\frac{8}{3}\frac{1}{\epsilon^4}\ell^4 - \frac{32}{3}\frac{1}{\epsilon^2}\ell^2 - \frac{571}{45}\log\Big(\frac{2e^{-\gamma}}{\epsilon}L\Big)$	$-5\log\Big(\frac{\pi}{G_{\rm N}}\ell^2\Big) - \frac{571}{45}\log\Big(\frac{1}{L}\ell\Big) - \log\Big(\frac{8\pi}{3}\Big) + \frac{715}{48} - \frac{47\,\zeta'(-1)}{3} + \frac{2\,\zeta'(-3)}{3}$
4	$\frac{15\pi}{8}\frac{1}{\epsilon^5}\ell^5 - \frac{65\pi}{24}\frac{1}{\epsilon^3}\ell^3 - \frac{105\pi}{16}\frac{1}{\epsilon}\ell$	$-\frac{15}{2}\log\Big(\frac{\pi^2}{2G_{\rm N}}\ell^3\Big) + \log(12) + \frac{27}{2}\log(2\pi) + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4}$

(3.149)

The one-loop energy and entropy (3.135)-(3.136) are split accordingly. The finite parts are

$$S^{(1)}_{\rm PI,fin} = \log Z^{(1)}_{\rm PI,fin} + V\rho^{(1)}_{\rm fin}\,, \qquad V\rho^{(1)}_{\rm fin} = \tfrac{1}{2}\tfrac{d-1}{d+1}\,D_d - \tfrac{1}{d+1}\alpha^{(2)}_{d+1}\,, \tag{3.150}$$

where as always $2\pi\ell\, U = V\rho$ with $V = \Omega_{d+1}\ell^{d+1}$. For $d = 2, 3, 4$:

$$
\begin{array}{c|l|l|l}
d & V\rho^{(1)}_{\rm div} & V\rho^{(1)}_{\rm fin} & S^{(1)}_{\rm PI,div} \\
\hline
2 & 0 + \frac{3\pi}{2}\frac{1}{\epsilon}\ell & 1 & -3\pi\frac{1}{\epsilon}\ell \\
3 & -\frac{8}{3}\frac{1}{\epsilon^4}\ell^4 + \frac{16}{3}\frac{1}{\epsilon^2}\ell^2 & \frac{5}{2} + \frac{571}{180} & -\frac{16}{3}\frac{1}{\epsilon^2}\ell^2 - \frac{571}{45}\log\Big(\frac{2e^{-\gamma}}{\epsilon}L\Big) \\
4 & -\frac{15\pi}{8}\frac{1}{\epsilon^5}\ell^5 + \frac{13\pi}{8}\frac{1}{\epsilon^3}\ell^3 + \frac{21\pi}{16}\frac{1}{\epsilon}\ell & \frac{9}{2} & -\frac{13\pi}{12}\frac{1}{\epsilon^3}\ell^3 - \frac{21\pi}{4}\frac{1}{\epsilon}\ell
\end{array}
\tag{3.151}
$$

Like their quasicanonical bulk counterparts, the Euclidean quantities obtained here are UV-divergent, and therefore ill-defined from a low-energy effective field theory point of view. However if the metric itself, i.e. gravity, is dynamical, these the UV-sensitive terms can be absorbed into standard renormalizations of the gravitational coupling constants, rendering the Euclidean thermodynamics finite and physically meaningful. We turn to this next.

3.8 Quantum gravitational thermodynamics

In section 3.7 we considered the Euclidean thermodynamics of effective field theories on a fixed background geometry. In general the Euclidean partition function and entropy depend on the choice of background metric; more specifically on the background sphere radius ℓ. Here we specialize to field theories which include the metric itself as a dynamical field, i.e. we consider gravitational effective field theories. We denote $Z_{\rm PI}$, $\rho_{\rm PI}$ and $S_{\rm PI}$ by $\mathcal{Z}$, ϱ and $\mathcal{S}$ in this case:

$$
\mathcal{Z} = \int \mathcal{D}g \,\cdots\, e^{-S_E[g,\ldots]}\,, \qquad S_E[g,\ldots] = \frac{1}{8\pi G}\int\sqrt{g}\,\big(\Lambda - \tfrac{1}{2}R + \cdots\big)\,. \tag{3.152}
$$

The geometry itself being dynamical, we have $\partial_\ell \mathcal{Z} = 0$, so (3.135)-(3.136) reproduce (3.1):

$$
\varrho = 0\,, \quad \mathcal{S} = \log\mathcal{Z}\,, \tag{3.153}
$$

We will assume $d \geq 2$, but it is instructive to first consider $d = 0$, i.e. 1D quantum gravity coupled to quantum mechanics on a circle. Then $\mathcal{Z} = \int \frac{d\beta}{2\beta}\,\mathrm{Tr}\, e^{-\beta H}$, where β is the circle size and H is the Hamiltonian of the quantum mechanical system shifted by the 1D cosmological constant. To implement the conformal factor contour rotation of [72] implicit in (3.153), we pick an integration contour $\beta = 2\pi\ell + iy$ with $y \in \mathbb{R}$ and $\ell > 0$ the background circle radius. Then $\mathcal{Z} = \pi i\, \mathcal{N}(0)$ where $\mathcal{N}(E)$ is the number of states with $H < E$. This being ℓ-independent implies $\varrho = 0$. A general definition of microcanonical entropy is $S_{\rm mic}(E) = \log \mathcal{N}(E)$. Thus, modulo the content-independent πi factor in $\mathcal{Z}$, $\mathcal{S} = \log \mathcal{Z}$ is the microcanonical entropy at zero energy in this case.

Of course $d = 0$ is very different from the general-d case, as there is no classical saddle of the gravitational action, and no horizon. For $d \geq 2$ and $\Lambda \to 0$, the path integral has a semi-classical expansion about a round sphere saddle or radius $\ell_0 \propto 1/\sqrt{\Lambda}$, and $\mathcal{S}$ is dominated by the leading tree-level horizon entropy (3.2). As in the AdS-Schwarzschild case reviewed in C.5.5, the microscopic degrees of freedom accounting for the horizon entropy, assuming they exist, are invisible in the effective field theory. A natural analog of the dual large-N CFT partition function on $S^1 \times S^{d-1}$ microscopically computing the AdS-Schwarzschild free energy may be some dual large-N quantum mechanics coupled to 1D gravity on S^1 microscopically computing the dS static patch entropy. These considerations suggest interpreting $\mathcal{S} = \log \mathcal{Z}$ as a macroscopic approximation to a microscopic microcanonical entropy, with the semiclassical/low-energy expansion mapping to some large-N expansion.

The one-loop corrected $\mathcal{Z}$ is obtained by expanding the action to quadratic order about its sphere saddle. The Gaussian $Z_{\rm PI}^{(1)}$ was computed in previous sections. Locality and dimensional analysis imply that one-loop divergences are $\propto \int R^n$ with $2n \leq d + 1$. Picking counterterms canceling all (divergent and finite) local contributions of this type in the limit $\ell_0 \propto 1/\sqrt{\Lambda} \to \infty$, we get a well-renormalized $\mathcal{S} = \log \mathcal{Z}$ to this order. Proceeding along these lines would be the most straightforward path to the computational objectives of this section. However, when pondering comparisons to microscopic models, one is naturally led to wondering what the actual physics content is of what has been computed. This in turn leads to small puzzles and bigger questions, such as:

1. A natural guess would have been that the one-loop correction to the entropy $\mathcal{S}$ is given by a

renormalized version of the Euclidean entropy $S_{\rm PI}^{(1)}$ (3.136). However (3.153) says it is given by a renormalized version of the free energy $\log Z_{\rm PI}^{(1)}$. In the examples given earlier, these two look rather different. Can these considerations be reconciled?

2. Besides local UV contributions absorbed into renormalized coupling constants determining the tree-level radius ℓ_0, there will be nonlocal IR vacuum energy contributions (pictorially Hawking radiation in equilibrium with the horizon), shifting the radius from ℓ_0 to $\bar{\ell}$ by gravitational backreaction. The effect would be small, $\bar{\ell} = \ell_0 + O(G)$, but since the leading-order horizon entropy is $S(\ell) \propto \ell^{d-1}/G$, we have $S(\bar{\ell}) = S(\ell_0) + O(1)$, a shift at the one-loop order of interest. The horizon entropy term in (3.153) is $\mathcal{S}^{(0)} = S(\ell_0)$, apparently not taking this shift into account. Can these considerations be reconciled?

3. At any order in the large-ℓ_0 perturbative expansion, UV-divergences can be absorbed into a renormalization of a finite number of renormalized coupling constants, but for the result to be physically meaningful, these must be defined in terms of low-energy physical "observables", invariant under diffeomorphisms and local field redefinitions. In asymptotically flat space, one can use scattering amplitudes for this purpose. These are unavailable in the case at hand. What replaces them?

To address these and other questions, we follow a slghtly less direct path, summarized below, and explained in more detail including examples in appendix C.9.

Free energy/quantum effective action for volume

We define an off-shell free energy/quantum effective action $\Gamma(V) = -\log Z(V)$ for the volume, the Legendre transform of the off-shell entropy/moment-generating function $S(\rho)$:[18]

$$S(\rho) \equiv \log \int \mathcal{D}g \, e^{-S_E[g]+\rho \int \sqrt{g}}, \qquad \log Z(V) \equiv S - V\rho, \qquad V = \partial_\rho S = \left\langle \int \sqrt{g} \right\rangle_\rho . \qquad (3.154)$$

At large V, the geometry semiclassically fluctuates about a round sphere. Parametrizing the mean

[18]Non-metric fields in the path integral are left implicit. Note "off-shell" = on-shell for c.c. $\Lambda' = \Lambda - 8\pi G \, \rho$.

volume V by a corresponding mean radius ℓ as $V(\ell) \equiv \Omega_{d+1}\ell^{d+1}$, we have

$$Z(\ell) = \int_{\text{tree}} d\rho \int \mathcal{D}g \, e^{-S_E[g]+\rho(\int\sqrt{g}-V(\ell))}\,, \tag{3.155}$$

where $\int_{\text{tree}} d\rho$ means saddle point evaluation, i.e. extremization. The Legendre transform (3.154) is the same as (3.137), so we get thermodynamic relations of the same form as (3.135)-(3.137):

$$dS = V d\rho\,, \quad d\log Z = -\rho\, dV\,, \qquad \rho = -\tfrac{1}{d+1}\ell\partial_\ell \log Z \,/\, V\,, \quad S = \big(1 - \tfrac{1}{d+1}\ell\partial_\ell\big)\log Z\,. \tag{3.156}$$

On-shell quantities are obtained at $\rho = 0$, i.e. at the minimum $\bar{\ell}$ of the free energy $-\log Z(\ell)$:

$$\varrho = \rho(\bar{\ell}) = 0\,, \qquad \mathcal{S} = S(\bar{\ell}) = \log Z(\bar{\ell})\,, \qquad \big\langle \textstyle\int\sqrt{g}\, \big\rangle = \Omega_{d+1}\bar{\ell}^{d+1}\,. \tag{3.157}$$

Tree level

At tree level (3.155) evaluates to

$$\log Z^{(0)}(\ell) = -S_E[g_\ell]\,, \qquad g_\ell = \text{round } S^{d+1} \text{ metric of radius } \ell\,, \tag{3.158}$$

readily evaluated for any action using $R_{\mu\nu\rho\sigma} = (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})/\ell^2$, taking the general form

$$\log Z^{(0)} = \frac{\Omega_{d+1}\ell^{d+1}}{8\pi G}\big(-\Lambda + \tfrac{d(d+1)}{2}\,\ell^{-2} + z_1\, l_s^2\,\ell^{-4} + z_2\, l_s^4\,\ell^{-6} + \cdots\big)\,. \tag{3.159}$$

The z_n are R^{n+1} coupling constants and $l_s \ll \ell$ is the length scale of UV-completing physics. The off-shell entropy and energy density are obtained from $\log Z^{(0)}$ as in (3.156).

$$S^{(0)} = \frac{\Omega_{d-1}\ell^{d-1}}{4G}\big(1 + s_1\, l_s^2\,\ell^{-2} + \cdots\big), \qquad \rho^{(0)} = \frac{1}{8\pi G}\big(\Lambda - \tfrac{d(d-1)}{2}\,\ell^{-2} + \rho_1\, l_s^2\,\ell^{-4} + \cdots\big) \tag{3.160}$$

where $s_n, \rho_n \propto z_n$ and we used $\Omega_{d+1} = \frac{2\pi}{d}\Omega_{d-1}$. The on-shell entropy and radius are given by

$$\mathcal{S}^{(0)} = S^{(0)}(\ell_0), \qquad \rho^{(0)}(\ell_0) = 0, \tag{3.161}$$

either solved perturbatively for $\ell_0(\Lambda)$ or, more conveniently, viewed as parametrizing $\Lambda(\ell_0)$.

One loop

The one-loop order, (3.155) is a by construction tadpole-free Gaussian path integral, (C.306):

$$\log Z = \log Z^{(0)} + \log Z^{(1)}, \qquad \log Z^{(1)} = \log Z_{\rm PI}^{(1)} + \log Z_{\rm ct}, \tag{3.162}$$

with $Z_{\rm PI}^{(1)}$ as computed in sections 3.4-3.5 and $\log Z_{\rm ct}(\ell) = -S_{E,\rm ct}[g_\ell]$ a polynomial counterterm. We define renormalized coupling constants as the coefficients of the ℓ^{d+1-2n} terms in the $\ell \to \infty$ expansion of $\log Z$, and fix $\log Z_{\rm ct}$ by equating tree-level and renormalized coefficients of the polynomial part, which amounts to the renormalization condition

$$\lim_{\ell\to\infty} \partial_\ell \log Z^{(1)} = 0, \tag{3.163}$$

in even $d+1$ supplemented by $\log Z_{\rm ct}(0) \equiv -\alpha_{d+1}\log(2e^{-\gamma}L/\epsilon)$, implying $L\partial_L \log Z^{(0)} = \alpha_{d+1}$.
Example: 3D Einstein gravity + minimally coupled scalar (C.9.4), putting $\nu \equiv \sqrt{m^2\ell^2-1}$,

$$\log Z^{(1)} = -3\log\frac{2\pi\ell}{4G} + 5\log(2\pi) \;-\; \sum_{k=0}^{2}\frac{\nu^k}{k!}\frac{\mathrm{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}} + \frac{\pi\nu^3}{6} \;-\; \frac{\pi m^3\ell^3}{6} + \frac{\pi m\ell}{4}\,. \tag{3.164}$$

The last two terms are counterterms. The first two are nonlocal graviton terms. The scalar part is $O(1/m\ell)$ for $m\ell \gg 1$ but goes nonlocal at $m\ell \sim 1$, approaching $-\log(m\ell)$ for $m\ell \ll 1$.

Defining $\rho^{(1)}$ and $S^{(1)}$ from $\log Z^{(1)}$ as in (3.156), and the quantum on-shell $\bar\ell = \ell_0 + O(G)$ as in

(3.157), the quantum entropy can be expressed in two equivalent ways, (C.313)-(C.314):

$$A: \ \mathcal{S} = S^{(0)}(\bar{\ell}) + S^{(1)}(\bar{\ell}) + \cdots, \qquad B: \ \mathcal{S} = S^{(0)}(\ell_0) + \log Z^{(1)}(\ell_0) + \cdots \tag{3.165}$$

where the dots denote terms neglected in the one-loop approximation. This simultaneously answers questions 1 and 2 on our list, reconciling intuitive (A) and (3.153)-based (B) expectations. To make this physically obvious, consider the quantum static patch as two subsystems, geometry (horizon) + quantum fluctuations (radiation), with total energy $\propto \rho = \rho^{(0)} + \rho^{(1)} = 0$. If $\rho^{(0)} = 0$, the horizon entropy is $S^{(0)}(\ell_0)$. But here we have $\rho = 0$, so the horizon entropy is actually $S^{(0)}(\bar{\ell}) = S^{(0)}(\ell_0) + \delta S^{(0)}$, where by the first law (3.156), $\delta S^{(0)} = V \delta \rho^{(0)} = -V\rho^{(1)}$. Adding the radiation entropy $S^{(1)}$ and recalling $\log Z^{(1)} = S^{(1)} - V\rho^{(1)}$ yields $\mathcal{S} = A = B$. Thus $A = B$ is just the usual small+large = system+reservoir approximation, the horizon being the reservoir, and the Boltzmann factor $e^{-V\rho^{(1)}} = e^{-\beta U^{(1)}}$ in $Z^{(1)}$ accounting for the reservoir's entropy change due to energy transfer to the system.

Viewing the quantum contributions as (Hawking) radiation has its picturesque merits and correctly conveys their nonlocal/thermal character, e.g. $\mathrm{Li}(e^{-2\pi\nu}) \sim e^{-\beta m}$ for $m\ell \gg 1$ in (3.164), but might incorrectly convey a presumption of positivity of $\rho^{(1)}$ and $S^{(1)}$. Though positive for minimally coupled scalars (fig. C.12), they are in fact *negative* for higher spins (figs. C.13, C.14), due to edge and group volume contributions. Moreover, although the negative-energy backreaction causes the horizon to grow, partially compensating the negative $S^{(1)}$ by a positive $\delta S^{(0)} = -V\rho^{(1)}$, the former still wins: $\mathcal{S}^{(1)} \equiv \mathcal{S} - \mathcal{S}^{(0)} = S^{(1)} - V\rho^{(1)} = \log Z^{(1)} < 0$.

Computational recipe and examples

For practical purposes, (B) is the more useful expression in (3.165). Together with (3.161) computing $\mathcal{S}^{(0)}$, the exact results for $Z_{\mathrm{PI}}^{(1)}$ obtained in previous sections (with $\gamma_0 = \sqrt{2\pi/\mathcal{S}^{(0)}}$, see (3.167) below), and the renormalization prescription outlined above, it immediately gives

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \cdots, \qquad \mathcal{S}^{(0)} = S^{(0)}(\ell_0), \qquad \mathcal{S}^{(1)} = \log Z^{(1)}(\ell_0) \tag{3.166}$$

in terms of the renormalized coupling constants, for general effective field theories of gravity coupled to arbitrary matter and gauge fields.

For 3D gravity, this gives $\mathcal{S} = \mathcal{S}^{(0)} - 3\log\mathcal{S}^{(0)} + 5\log(2\pi) + O(1/\mathcal{S}^{(0)})$. We work out and plot several other concrete examples in appendix C.9.4: 3D Einstein gravity + scalar (C.9.4, fig. C.12), 3D massive spin s (C.9.4, fig. C.13), 2D scalar (C.9.4), 4D massive spin s (C.9.4, fig. C.14), and 3D,4D,5D gravity (including higher-order curvature corrections) (C.9.4). Table 3.12 in the introduction lists a few more sample results.

Local field redefinitions, invariant coupling constants and physical observables

Although the higher-order curvature corrections to the tree-level dS entropy $\mathcal{S}^{(0)} = S^{(0)}(\ell_0)$ (3.160) seem superficially similar to curvature corrections to the entropy of black holes in asymptotically flat space [158], there are no charges or other asymptotic observables available here to endow them with physical meaning. Indeed, they have no intrinsic low-energy physical meaning at all, as they can be removed order by order in the l_s/ℓ expansion by a metric field redefinition, bringing the entropy to pure Einstein form (3.2). In $Z^{(0)}(\ell)$ (3.159), this amounts to setting all $z_n \equiv 0$ by a redefinition $\ell \to \ell \sum_n c_n \ell^{-2n}$ (C.296). The value of $\mathcal{S}^{(0)} = \max_{\ell \gg l_s} \log Z^{(0)}(\ell)$ remains of course unchanged, providing the unique field-redefinition invariant combination of the coupling constants $G, \Lambda(\text{or } \ell_0), z_1, z_2, \ldots$.

Related to this, as discussed in C.9.4, caution must be exercised when porting the one-loop graviton contribution in (3.112) or (3.148): $G_{\rm N}$ appearing in $\gamma_0 = \sqrt{8\pi G_{\rm N}/A}$ is the algebraically defined Newton constant (3.109), as opposed to G defined by the Ricci scalar coefficient $\frac{1}{8\pi G}$ in the low-energy effective action. The former is field-redefinition invariant; the latter is not. In Einstein frame ($z_n = 0$) the two definitions coincide, hence in a general frame

$$\gamma_0 = \sqrt{2\pi/\mathcal{S}^{(0)}}\,. \tag{3.167}$$

Since $\log\mathcal{S}^{(0)} = \log\frac{A}{4G} + \log\Big(1 + O(l_s^2/\ell_0^2)\Big)$, this distinction matters only at $O(l_s^2/\ell_0^2)$, however.

In $d = 2$, $\mathcal{S}^{(0)}$ is in fact the only invariant gravitational coupling: because the Weyl tensor vanishes identically, any 3D parity-invariant effective gravitational action can be brought to Einstein form by a field redefinition. In the Chern-Simons formulation of C.8.2, $\mathcal{S}^{(0)} = 2\pi\kappa$. In $d \geq 3$, the Weyl tensor vanishes on the sphere, but not identically. As a result, there are coupling constants not picked up by the sphere's $\mathcal{S}^{(0)} = -S_E[g_{\ell_0}]$. Analogous $\mathcal{S}_M^{(0)} \equiv -S_E[g_M]$ for different saddle geometries g_M, approaching Einstein metrics in the limit $\Lambda \propto \ell_0^{-2} \to 0$, can be used instead to probe them, and analogous $\mathcal{S}_M \equiv \log \mathcal{Z}_M$ expanded about g_M provide quantum observables. Section C.9.5 provides a few more details, and illustrates extraction of unambiguous linear combinations of the 4D one-loop correction for 3 different M.

This provides the general picture we have in mind as the answer, in principle, to question 3 on our list below (3.153): the tree-level $\mathcal{S}_M^{(0)}$ are the analog of tree-level scattering amplitudes, and the analog of quantum scattering amplitudes are the quantum $\mathcal{S}_M$.

Constraints on microscopic models

For pure 3D gravity $\mathcal{S}^{(0)} = \frac{2\pi}{4G}\big(\ell_0 + s_1\ell_0^{-1} + s_2\,\ell_0^{-3} + \cdots\big)$, and to one-loop order we have (C.337):

$$\mathcal{S} = \mathcal{S}^{(0)} - 3\log\mathcal{S}^{(0)} + 5\log(2\pi) + \cdots\,. \tag{3.168}$$

Granting[19] (C.248) with $l = 0$ gives the all-loop expansion of pure 3D gravity, taking into account $G \equiv SO(4)$ here while $G \equiv SU(2)\times SU(2)$ there, to all-loop order,

$$\mathcal{S} = \mathcal{S}_0 + \log\left|\sqrt{\tfrac{4}{2+i\,\mathcal{S}_0/2\pi}}\,\sin\!\big(\tfrac{\pi}{2+i\,\mathcal{S}_0/2\pi}\big)\right|^2 = \mathcal{S}_0 - 3\log\mathcal{S}_0 + 5\log(2\pi) + \textstyle\sum_n c_n\,\mathcal{S}_0^{-2n} \tag{3.169}$$

where $\mathcal{S}_0 \equiv \mathcal{S}^{(0)}$ to declutter notation. Note all quantum corrections are strictly nonlocal, i.e. no odd powers of ℓ_0 appear, reflected in the absence of odd powers of $1/\mathcal{S}_0$.

Though outside the scope of this paper, let us illustrate how such results may be used to constrain microscopic models identifying large-ℓ_0 and large-N expansions in some way. Say a modeler posits a model consisting of $2N$ spins $\sigma_i = \pm 1$ with $H \equiv \sum_i \sigma_i = 0$. The microscopic

[19]This does not affect the 1-loop based conclusions below, but does affect the c_n. One could leave l general.

entropy is $S_{\rm mic} = \log \binom{2N}{N} = 2\log 2 \cdot N - \frac{1}{2}\log(\pi N) + \sum_n c'_n N^{1-2n}$. There is a unique identification of $\mathcal{S}_0$ bringing this in a form with the same analytic/locality structure as (3.169), to wit, $\mathcal{S}_0 = \log 4 \cdot N + \sum_n c'_n N^{1-2n}$, resulting in

$$S_{\rm mic} \;=\; \mathcal{S}_0 - \tfrac{1}{2}\log\mathcal{S}_0 + \log\Big(\tfrac{\pi}{2\log 2}\Big) + \textstyle\sum_n c''_n\, \mathcal{S}_0^{-2n}\,, \tag{3.170}$$

where $c''_1 = -\frac{1}{8}\log 2$, $c''_2 = \frac{3}{64}(\log 2)^2 + \frac{1}{48}(\log 2)^3, \ldots$, fully failing to match (3.169), starting at one loop. The model is ruled out.

A slightly more sophisticated modeler might posit $S_{\rm mic} = \log d(N)$, where $d(N)$ is the N-th level degeneracy of a chiral boson on S^1. To leading order $S_{\rm mic} \approx 2\pi\sqrt{N/6} \equiv K$. Beyond, $S_{\rm mic} = K - a'\log K + b' + \sum_n c'_n K^{-n} + O(e^{-K/2})$, where $a' = 2$, $b' = \log\big(\pi^2/6\sqrt{3}\big)$ and c'_n given by [159]. Identifying $\mathcal{S}_0 = K + \sum_n c'_{2n-1} K^{-(2n-1)}$ brings this to the form (3.169), yielding $S_{\rm mic} = \mathcal{S}_0 - a'\log\mathcal{S}_0 + b' + \sum_n c''_n \mathcal{S}_0^{-2n} + O(e^{-\mathcal{S}_0/2})$, with $c''_1 = -\frac{5}{2}$, $c''_2 = \frac{37}{12}, \ldots$ — ruled out.

We actually did not need the higher-loop corrections at all to rule out the above models. In higher dimensions, or coupled to more fields, one-loop constraints moreover become increasingly nontrivial, evident in (3.12). For pure 5D gravity (C.337),

$$\mathcal{S} = \mathcal{S}^{(0)} - \frac{15}{2}\log\mathcal{S}^{(0)} + \log(12) + \frac{27}{2}\log(2\pi) + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4}\,. \tag{3.171}$$

It would be quite a miracle if a microscopic model managed to match this.

3.9 dS, AdS±, and conformal higher-spin gravity

Vasiliev higher-spin gravity theories [91–93] have infinite spin range and an infinite-dimensional higher-spin algebra, $\mathfrak{g} = {\rm hs}({\rm so}(d+2))$, leading to divergences in the one-loop sphere partition function formula (3.112) untempered by the UV cutoff. In this section we take a closer look at these divergences. We contrast the situation to AdS with standard boundary conditions (AdS+), where the issue is entirely absent, and we point out that, on the other hand, for AdS with alternate HS boundary conditions (AdS−) as well as conformal higher-spin (CHS) theories, similar issues arise.

We end with a discussion of their significance.

3.9.1 dS higher-spin gravity

Nonminimal type A Vasiliev gravity on dS_{d+1} has a tower of massless spin-s fields for all $s \geq 1$ and a $\Delta = d-2$ scalar. We first consider $d = 3$. The total bulk and edge characters are obtained by summing (3.102) and adding the scalar, as we did for the bulk part in (3.63):

$$\chi_{\text{bulk}} \;=\; 2 \cdot \left(\frac{q^{1/2}+q^{3/2}}{(1-q)^2}\right)^2 - \frac{q}{(1-q)^2}\,, \qquad \chi_{\text{edge}} = 2 \cdot \left(\frac{q^{1/2}+q^{3/2}}{(1-q)^2}\right)^2 . \tag{3.172}$$

Quite remarkably, the bulk and edge contributions almost exactly cancel:

$$\chi_{\text{bulk}} - \chi_{\text{edge}} = -\frac{q}{(1-q)^2} \,. \tag{3.173}$$

For $d = 4$ however, we see from (3.102) that due to the absence of overall q^s suppression factors, the total bulk and edge characters each diverge separately by an overall multiplicative factor:

$$\chi_{\text{bulk}} = \sum_s (2s+1) \cdot \frac{2\,q^2}{(1-q)^4}\,, \qquad \chi_{\text{edge}} = \sum_s \tfrac{1}{6} s(s+1)(2s+1) \cdot \frac{2\,q}{(1-q)^2} \,. \tag{3.174}$$

This pattern persists for all $d \geq 4$, as can be seen from the explicit form of bulk and edge characters in (C.165), (C.194), (C.196). For any d, there is moreover an infinite-dimensional group volume factor in (3.112) to make sense of, involving a divergent factor $(\ell^{d-1}/G_{\mathrm{N}})^{\dim G/2}$ and the volume of an object of unclear mathematical existence [160].

Before we continue the discussion of what, if anything, to make of this, we consider AdS± and CHS theories within the same formalism. Besides independent interest, this will make clear the issue is neither intrinsic to the formalism, nor to de Sitter.

3.9.2 AdS± higher-spin gravity

AdS characters for standard and alternate HS boundary conditions

Standard boundary conditions on massless higher spin fields φ in AdS_{d+1} lead to quantization such that spin-s single-particle states transform in a UIR of $\mathrm{so}(2,d)$ with primary dimension $\Delta_\varphi = \Delta_+ = s + d - 2$. Higher-spin Euclidean AdS one-loop partition functions with these boundary conditions were computed in [30–33, 125]. In [81], the Euclidean one-loop partition function for alternate boundary conditions ($\Delta_\varphi = \Delta_- = 2 - s$) was considered. In the EAdS+ case, the complications listed under (3.96) are absent, but for EAdS− close analogs do appear.

EAdS path integrals can be expressed as character integrals [83, 129], in a form exactly paralleling the formulae and bulk/edge picture of the present work [83].[20] The AdS analog of the dS bulk and edge characters (3.85) for a *massive* spin-s field φ with $\Delta_\varphi = \Delta_\pm$ is [83]

$$\chi^{\mathrm{AdS}\pm}_{\mathrm{bulk},\varphi} \equiv D^d_s \, \frac{q^{\Delta_\pm}}{(1-q)^d}\,, \qquad \chi^{\mathrm{AdS}\pm}_{\mathrm{edge},\varphi} \equiv D^{d+2}_{s-1} \, \frac{q^{\Delta_\pm - 1}}{(1-q)^{d-2}}\,, \tag{3.175}$$

where $\Delta_- = d - \Delta_+$. Thus, as functions of q,

$$\chi^{\mathrm{dS}}_\varphi = \chi^{\mathrm{AdS}+}_\varphi + \chi^{\mathrm{AdS}-}_\varphi\,. \tag{3.176}$$

The AdS analog of (3.97) for a massless spin-s field ϕ_s with gauge parameter field $\xi_{s'}$ is

$$\hat\chi^{\mathrm{AdS}\pm}_s \equiv \chi^{\mathrm{AdS}\pm}_\phi - \chi^{\mathrm{AdS}\pm}_\xi\,, \tag{3.177}$$

[20] In this picture, EAdS is viewed as the Wick-rotated AdS-Rindler wedge, with dS_d static patch boundary metric, as in [161, 162]. The bulk character is $\chi \equiv \mathrm{tr}_G\, q^{iH}$, with H the *Rindler* Hamiltonian, *not* the global AdS Hamiltonian. Its q-expansion counts *quasinormal modes* of the Rindler wedge. The one-loop results are interpreted as corrections to the gravitational thermodynamics of the AdS-Rindler horizon [83, 161, 162].

where $\Delta_{\phi,+} = s' + d - 1$, $\Delta_{\xi,+} = s + d - 1$, $s' \equiv s - 1$. More explicitly, analogous to (3.98),

$$\hat{\chi}^{\text{AdS}+}_{\text{bulk},s} = \frac{D^d_s\, q^{s'+d-1} - D^d_{s'}\, q^{s+d-1}}{(1-q)^d}, \qquad \hat{\chi}^{\text{AdS}+}_{\text{edge},s} = \frac{D^{d+2}_{s-1}\, q^{s'+d-2} - D^{d+2}_{s'-1}\, q^{s+d-2}}{(1-q)^{d-2}} \tag{3.178}$$

$$\hat{\chi}^{\text{AdS}-}_{\text{bulk},s} = \frac{D^d_s\, q^{1-s'} - D^d_{s'}\, q^{1-s}}{(1-q)^d}, \qquad \hat{\chi}^{\text{AdS}-}_{\text{edge},s} = \frac{D^{d+2}_{s-1}\, q^{-s'} - D^{d+2}_{s'-1}\, q^{-s}}{(1-q)^{d-2}}\,. \tag{3.179}$$

The presence of non-positive powers of q in $\chi^{\text{AdS}-}$ has a similar path integral interpretation as in the dS case summarized in section 3.5.2. The necessary negative mode contour rotation and zeromode subtractions are again implemented at the character level by flipping characters. In particular the proper χ_s to be used in the character formulae for EAdS$\pm$ are

$$\chi^{\text{AdS}-}_s = \left[\hat{\chi}^{\text{AdS}-}_s\right]_+, \qquad \chi^{\text{AdS}+}_s = \left[\hat{\chi}^{\text{AdS}+}_s\right]_+ = \hat{\chi}^{\text{AdS}+}_s\,, \tag{3.180}$$

with $[\hat{\chi}]_+$ defined as in (3.100). The omission of Killing tensor zeromodes for alternate boundary conditions must be compensated by a a division by the volume of the residual *gauge* group G generated by the Killing tensors. Standard boundary conditions on the other hand kill these Killing tensor zeromodes: they are not part of the dynamical, fluctuating degrees of freedom. The group G they generate acts nontrivially on the Hilbert space as a global symmetry group.

AdS+

For standard boundary conditions, the character formalism reproduces the original results of [30–33, 125] by two-line computations [83]. We consider some examples:

For nonmimimal type A Vasiliev with $\Delta_0 = d - 2$ scalar boundary conditions, dual to the free $U(N)$ model, using (3.178) and the scalar $\chi_0 = q^{d-2}/(1-q)^d$, the following total bulk and edge characters are readily obtained:

$$\chi^{\text{AdS}+}_{\text{bulk}} = \sum_{s=0}^{\infty} \chi^{\text{AdS}+}_{\text{bulk},s} = \left(\frac{q^{\frac{d}{2}-1} + q^{\frac{d}{2}}}{(1-q)^{d-1}}\right)^2, \quad \chi^{\text{AdS}+}_{\text{edge}} = \sum_{s=0}^{\infty} \chi^{\text{AdS}+}_{\text{edge},s} = \left(\frac{q^{\frac{d}{2}-1} + q^{\frac{d}{2}}}{(1-q)^{d-1}}\right)^2. \tag{3.181}$$

The total bulk character takes the singleton-squared form expected from the Flato-Fronsdal theorem [163]. More interestingly, the edge characters sum up to exactly the same. Thus the generally negative nature of edge "corrections" takes on a rather dramatic form here:

$$\chi_{\rm tot}^{\rm AdS+} = \chi_{\rm bulk}^{\rm AdS+} - \chi_{\rm edge}^{\rm AdS+} = 0 \qquad \Rightarrow \qquad \log Z_{\rm PI}^{\rm AdS+} = 0\,. \tag{3.182}$$

As $Z_{\rm bulk}^{\rm AdS+}$ has an Rindler bulk ideal gas interpretation analogous to the static patch ideal gas of section 3.2 [83], the exact bulk-edge cancelation on display here is reminiscent of analogous one-loop bulk-edge cancelations expected in string theory according to the qualitative picture reviewed in appendix C.5.5.

For minimal type A, dual to the free $O(N)$ model, the sum yields an expression which after rescaling of integration variables $t \to t/2$ is effectively equivalent to the so$(2,d)$ singleton character, which is also the so$(1,d)$ character of a conformally coupled ($\nu = i/2$) scalar on S^d. Using (3.74), this means $Z_{\rm PI}^{\rm AdS+}$ equals the sphere partition function on S^d, immediately implying the $N \to N-1$ interpretation of [30–33, 125].

For nonminimal type A with $\Delta_0 = 2$ scalar boundary conditions, dual to an interacting U(N) CFT, the cancelation is *almost* exact but not quite:

$$\chi_{\rm tot}^{\rm AdS+} = \frac{\sum_{k=2}^{d-3} q^k}{(1-q)^{d-1}}\,. \tag{3.183}$$

AdS+ higher-spin swampland

In the above examples it is apparent that although the spin-summed $\chi_{\rm bulk}$ has increased effective UV-dimensionality $d_{\rm eff}^{\rm bulk} = 2d-2$, as if we summed KK modes of a compactification manifold of dimension $d-2$, the edge subtraction collapses this back down to a net $d_{\rm eff} = d-1$, *decreasing* the original d. Correspondingly, the UV-divergences of $Z_{\rm PI}^{(1)}$ are not those of a $d+1$ dimensional bulk-local theory, but rather of a d-dimensional *boundary*-local theory. In fact this peculiar property appears necessary for quantum consistency, in view of the non-existence of a nontrivially interacting local *bulk* action [95]. It appears to be true for all AdS+ higher spin theories with a known holographic dual [83], but not for all classically consistent higher-spin theories. Thus it appears to be some kind of AdS higher-spin "swampland" criterion:

$$\text{AdS}_{d+1} \text{ HS theory has holographic dual} \qquad \Rightarrow \qquad d_{\text{eff}} = d - 1\,. \tag{3.184}$$

Higher-spin theories violating this criterion do exist. Theories with a tower of massless spins $s \geq 2$ and an a priori undetermined number n of real scalars can be constructed in AdS_3 [164, 165]. Assuming all integer spins $s \geq 2$ are present, the total character sums up to

$$\chi_{\text{tot}} = \frac{2\,q^2}{(1-q)^2} - \frac{4\,q}{(1-q)^2} + \sum_{i=1}^{n} \frac{q^{\Delta_i}}{(1-q)^3}\,. \tag{3.185}$$

For $t \to 0$ diverges as $\chi_{\text{HS}} \sim (n-2)/t^2 + O(1/t)$. To satisfy (3.184), the number of scalars must be $n = 2$. This is inconsistent with the $n = 4$ AdS_3 theory originally conjectured in [165] to be dual to a minimal model CFT_2, but consistent with the amended conjecture of [166, 167].

AdS−

For alternate boundary conditions, one ends up with a massless higher-spin character formula similar to (3.112). The factor $\gamma^{\dim G}$ in (3.112) is consistent with $\log Z_{\text{PI}}^{\text{AdS}-} \propto (G_{\text{N}})^{\frac{1}{2}\sum_s N_{s-1}^{\text{KT}}}$ found in [81]. (3.176) implies the massless AdS± and dS bulk and edge characters are related as

$$\boxed{\chi_s^{\text{AdS}-} = \chi_s^{\text{dS}} - \chi_s^{\text{AdS}+}} \tag{3.186}$$

hence we can read off the appropriate flipped $\chi_s^{\text{AdS}-} = [\hat{\chi}_s^{\text{AdS}-}]_+$ characters from our earlier explicit results (C.194) and (C.196) for χ_s^{dS}. Just like in the dS case, the final result involves divergent spin sums when the spin range is infinite.

3.9.3 Conformal higher-spin gravity

Conformal HS characters

Conformal (higher-spin) gravity theories [168] have (higher-spin extensions of) diffeomorphisms and local Weyl rescalings as gauge symmetries. If one does not insist on a local action, a general way to construct such theories is to view them as induced theories, obtained by integrating out the degrees of freedom of a conformal field theory coupled to a general background metric and other background fields. In particular one can consider a free $U(N)$ CFT$_d$ in a general metric and higher-spin source background. For even d, this results in a local action, which at least at the free level can be rewritten as a theory of towers of partially massless fields with standard kinetic terms [62, 82]. Starting from this formulation of CHS theory on S^d (or equivalently dS$_d$), using our general explicit formulae for partially massless higher-spin field characters (C.194) and (C.196), and summing up the results, we find

$$\boxed{\chi_s^{\mathrm{CdS_d}} = \chi_s^{\mathrm{AdS_{d+1}}-} - \chi_s^{\mathrm{AdS_{d+1}}+} = \chi_s^{\mathrm{dS_{d+1}}} - 2\,\chi_s^{\mathrm{AdS_{d+1}}+}} \tag{3.187}$$

where $\chi_s^{\mathrm{CdS_d}}$ are the CHS bulk and edge characters and the second equality uses (3.186). Since we already know the explicit dS and AdS HS bulk and edge characters, this relation also provides the explicit CHS bulk and edge characters. For example

d	s	$\chi_{\mathrm{bulk},s}^{\mathrm{CdS_d}} \cdot (1-q)^d$	$\chi_{\mathrm{edge},s}^{\mathrm{CdS_d}} \cdot (1-q)^{d-2}$
2	≥ 2	$-4q^s(1-q)$	$-2\big(s^2q^{s-1} - (s-1)^2q^s\big)$
3	≥ 1	0	0
3	0	$-q(1-q)$	0
4	≥ 0	$2(2s+1)\,q^2 + 2s^2q^{s+3} - 2(s+1)^2q^{s+2}$	$\frac{s(s+1)(2s+1)}{3}\,q + \frac{(s-1)s^2(s+1)}{6}\,q^{s+2} - \frac{s(s+1)^2(s+2)}{6}\,q^{s+1}$
5	≥ 0	$\frac{(s+1)(2s+1)(2s+3)}{3}\,q^2(1-q)$	$\frac{s(s+1)(s+2)(2s+1)(2s+3)}{30}\,q(1-q)$

(3.188)

The bulk $SO(1,d)$ q-characters $\chi^{\mathrm{CdS_d}}_{\mathrm{bulk},s}$ computed from (3.187) agree with the so$(2,d)$ q-characters obtained in [169]. Edge characters were not derived in [169], as they have no role in the thermal $S^1 \times S^{d-1}$ CHS partition functions studied there.[21]

The one-loop Euclidean path integral of the CHS theory on S^d is given by (3.112) using the bulk and edge CHS characters $\chi^{\mathrm{CdS_d}}_s$ and with G the CHS symmetry group generated by the conformal Killing tensors on S^d (counted by $D^{d+3}_{s-1,s-1}$). The coefficient of the log-divergent term, the Weyl anomaly of the CHS theory, is extracted as usual, by reading off the coefficient of the $1/t$ term in the small-t expansion of the integrand in (3.112), or more directly from the "naive" integrand $\frac{1}{2t}\frac{1+q}{1-q}\,\hat{\chi}$. For example for conformal $s=2$ gravity on S^2 coupled to D massless scalars, also known as bosonic string theory in D spacetime dimensions, we have $\dim G = \sum_\pm D^4_{1,\pm 1} = 6$, generating $G = SO(1,3)$, and from the above table (3.188),

$$\chi_{\mathrm{tot}} = D \cdot \frac{1+q}{1-q} - \frac{4q^2}{1-q} + 2(4q - q^2)\,. \tag{3.189}$$

The small-t expansion of the integrand in (3.112) for this case is

$$\frac{1}{2t}\frac{1+q}{1-q}\big(\chi_{\mathrm{tot}} - 12\big) \to \frac{2(D-2)}{t^3} + \frac{D-26}{3\,t} + \cdots\,, \tag{3.190}$$

reassuringly informing us the critical dimension for the bosonic string is $D = 26$. Adding a massless $s = \frac{3}{2}$ field, we get 2D conformal supergravity. For half-integer conformal spin s, $\chi_{\mathrm{bulk}} = -4q^s/(1-q)$ and $\chi_{\mathrm{edge}} = -2\big((s-\frac{1}{2})(s+\frac{1}{2})q^{s-1} - (s-\frac{3}{2})(s-\frac{1}{2})q^s\big)$. Furthermore adding D' massless Dirac spinors, the total fermionic character is

$$\chi^{\mathrm{fer}}_{\mathrm{tot}} = D' \cdot \frac{2\,q^{1/2}}{1-q} - \frac{4\,q^{3/2}}{1-q} + 4\,q^{1/2}\,. \tag{3.191}$$

The symmetry algebra has $\sum_\pm D^4_{\frac{1}{2},\pm\frac{1}{2}} = 4$ fermionic generators, contributing negatively to $\dim G$

[21] A priori the interpretation of the bulk characters in (3.188) and those in [169] is different. Their mathematical equality is a consequence of the enhanced so$(2,d)$ symmetry allowing to map $S^d \to \mathbb{R} \times S^{d-1}$.

in (3.112). Putting everything together,

$$\frac{1}{2t}\,\frac{1+q}{1-q}\big(\chi^{\rm bos}_{\rm tot}-2(6-4)\big)-\frac{1}{2t}\,\frac{\sqrt{q}}{1-q}\,\chi^{\rm fer}_{\rm tot}\to\frac{2(D-D')}{t^3}+\frac{2D+D'-30}{6\,t}+\cdots\,, \tag{3.192}$$

from which we read off supersymmetry + conformal symmetry requires $D'=D=10$.

More systematically, the Weyl anomaly $\alpha_{d,s}$ can be read off by expanding $\frac{1}{2t}\frac{1+q}{1-q}\,\hat\chi^{\rm CS^d}$ with $\hat\chi^{\rm CS^d}=\hat\chi^{\rm AdS_{d+1}-}-\hat\chi^{\rm AdS_{d+1}+}$ given by (3.178)-(3.179) for integer s. For example,

d	$-\alpha_{d,s}$
2	$\frac{2(6s^2-6s+1)}{3}$
4	$\frac{s^2(s+1)^2(14s^2+14s+3)}{180}$
6	$\frac{(s+1)^2(s+2)^2(22s^6+198s^5+671s^4+1056s^3+733s^2+120s-50)}{151200}$
8	$\frac{(s+1)(s+2)^2(s+3)^2(s+4)(150s^8+3000s^7+24615s^6+106725s^5+261123s^4+351855s^3+225042s^2+31710s-14560)}{2286144000}$

(3.193)

This reproduces the $d=2,4,6$ results of [62, 82] and generalizes them to any d.

Physics pictures

Cartoonishly speaking, the character relation (3.187) translates to one-loop partition function relations of the form $Z^{{\rm CS}^d}\sim Z^{\rm EAdS_{d+1}-}/Z^{\rm EAdS_{d+1}+}$ and $Z^{S^{d+1}}\sim Z^{{\rm CS}^d}\big(Z^{\rm EAdS_{d+1}+}\big)^2$. The first relation can then be understood as a consequence of the holographic duality between AdS$_{d+1}$ higher-spin theories and free CFT$_d$ vector models [62, 81, 82], while the second relation can be understood as an expression at the Gaussian/one-loop level of $Z^{S^{d+1}}\sim\int\mathcal{D}\sigma\,\big|\psi_{\rm HH}(\sigma)\big|^2$, where $\psi_{\rm HH}(\sigma)=\psi_{\rm HH}(0)\,e^{-\frac{1}{2}\sigma K\sigma+\cdots}$ is the late-time dS Hartle-Hawking wave function, related by analytic continuation to the EAdS partition function with boundary conditions σ [170]. The factor $\big(Z^{{\rm EAdS}_{d+1}+}\big)^2$ can then be identified with the bulk one-loop contribution to $|\psi_{\rm HH}(0)|^2$, and $Z^{{\rm cnf}\,S^d}$ with $\int\mathcal{D}\sigma\,e^{-\sigma K\sigma}$, along the lines of [81]. Along the lines of footnote 10, perhaps another interpretation of the spin-summed relation (3.187) exists within the picture of [151].

3.9.4 Comments on infinite spin range divergences

Let us return now to the discussion of section 3.9.1. Above we have seen that for EAdS+, summing spin characters leads to clean and effortless computation of the one-loop partition function. The group volume factor is absent because the global higher-spin symmetry algebra $\mathfrak{g}$ generated by the Killing tensors is not gauged. The character spin sum converges, and no additional regularization is required beyond the UV cutoff at $t \sim \epsilon$ we already had in place. The underlying reason for this is that in AdS+, the minimal energy of a particle is bounded below by its spin, hence a UV cutoff is effectively also a spin cutoff. In contrast, for dS, AdS− and CHS theories alike, $\mathfrak{g}$ is gauged, leading to the group volume division factor, and moreover, for $d \geq 4$, the quasinormal mode levels (or energy levels for CHS on $\mathbb{R} \times S^{d-1}$) are infinitely degenerate, not bounded below by spin, leading to character spin sum divergences untempered by the UV cutoff. The geometric origin of quasinormal modes decaying as slowly as $e^{-2T/\ell}$ for every spin s in $d \geq 4$ was explained below (C.167).

One might be tempted to use some form of zeta function regularization to deal with divergent sums $\sum_s \chi_s$ such as (3.174), which amounts to inserting a convergence factor $\propto e^{-\delta s}$ and discarding the divergent terms in the limit $\delta \to 0$. This might be justified if the discarded divergences were UV, absorbable into local counterterms, but that is not the case here. The divergence is due to *low*-energy features, the infinite multiplicity of slow-decaying quasinormal modes, analogous to the divergent thermodynamics of an ideal gas in a box with an infinite number of different massless particle species. Zeta function regularization would give a finite result, but the result would be meaningless.

As discussed at the end of section 3.6, the Vasiliev-like[22] limit of the 3D HS_n higher-spin gravity theory, $n \to \infty$ with $l = 0$ and $\mathcal{S}^{(0)}$ fixed, is strongly coupled as a 3D QFT. Unsurprisingly, the one-loop entropy "correction" $\mathcal{S}^{(1)} = \log Z^{(1)}$ diverges in this limit: writing the explicit expression for the maximal-entropy vacuum $R = \mathbf{n}$ in (3.12) as a function of $\dim G = 2(n^2 - 1)$, one gets $\mathcal{S}^{(1)} = \dim G \cdot \log\bigl(\dim G/\sqrt{\mathcal{S}^{(0)}}\bigr) + \cdots \to \infty$. The higher-spin decomposition (C.265)

[22]"Vasiliev-like" is meant only in a superficial sense here. The higher-spin algebras are rather different [69].

might inspire an ill-advised zeta function regularization along the lines of $\dim G = 2\sum_{r=1}^{\infty} 2r+1 = 4\,\zeta(-1) + 2\,\zeta(0) = -\frac{4}{3}$. This gives $\mathcal{S}^{(1)} = \frac{2}{3}\log\mathcal{S}^{(0)} + c$ with c a computable constant — a finite but meaningless answer. In fact, using (3.127), the all-loop quantum correction to the entropy can be seen to *vanish* in the limit under consideration, as illustrated in fig. 3.4. As discussed around (3.128), there are more interesting $n \to \infty$ limits one can consider, taking $\mathcal{S}^{(0)} \to \infty$ together with n. In these cases, the weakly-coupled description is not a 3D QFT, but a topological string theory.

Although these and other considerations suggest massless higher-spin theories with infinite spin range cannot be viewed as weakly-coupled field theories on the sphere, one might wonder whether certain quantities might nonetheless be computable in certain (twisted) supersymmetric versions. We did observe some hints in that direction. One example, with details omitted, is the following. First consider the supersymmetric AdS_5 higher-spin theory dual to the 4D $\mathcal{N} = 2$ supersymmetric free $U(N)$ model, i.e. the $U(N)$ singlet sector of N massless hypermultiplets, each consisting of two complex scalars and a Dirac spinor. The AdS_5 bulk field content is obtained from this following [171]. In their notation, the hypermultiplet corresponds to the so(2, 4) representation $\mathrm{Di} + 2\,\mathrm{Rac}$. Decomposing $(\mathrm{Di} + 2\,\mathrm{Rac}) \otimes (\mathrm{Di} + 2\,\mathrm{Rac})$ into irreducible so(2, 4) representations gives the AdS_5 free field content: four $\Delta = 2$ and two $\Delta = 3$ scalars, one $\Delta = 3$, $S = (1, \pm 1)$ 2-form field, six towers of massless spin-s fields for all $s \geq 1$, one tower of massless $S = (s, \pm 1)$ fields for all $s \geq 2$, one $\Delta = \frac{5}{2}$ Dirac spinor, and four towers of massless spin $s = k + \frac{1}{2}$ fermionic gauge fields for all $k \geq 1$. Consider now the same field content on S^5. The bulk and edge characters are obtained paralleling the steps summarized in section 3.5.2, generalized to the present field content using (3.92) and (3.93). Each individual spin tower gives rise to a badly divergent spin sum similar to (3.174). However, a remarkable conspiracy of cancelations between various bosonic and fermionic bulk and edge contributions in the end leads to a finite, unambiguous net integrand:[23]

$$\int \frac{dt}{2t}\left(\frac{1+q}{1-q}\,\chi_{\rm tot}^{\rm bos} - \frac{2\sqrt{q}}{1-q}\,\chi_{\rm tot}^{\rm fer}\right) = -\frac{3}{4}\int \frac{dt}{2t}\,\frac{1+q}{1-q}\,\frac{q}{(1-q)^2}\,. \tag{3.194}$$

[23]The spin sums are performed by inserting a convergence factor such as $e^{-\delta s}$, but the end result is finite and unambiguous when taking $\delta \to 0$, along the lines of $\lim_{\delta\to 0}\sum_{s\in\frac{1}{2}\mathbb{N}}(-1)^{2s}(2s+1)\,e^{-\delta s} = \frac{1}{4}$.

Note that the effective UV dimensionality is reduced by *two* in this case.

An analogous construction for S^4 starting from the 3D $\mathcal{N} = 2$ $U(N)$ model, gives two $\Delta_\pm = 1, 2$ scalars, a $\Delta = \frac{3}{2}$ Dirac spinor and two massless spin-$1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ towers, as in [120, 172]. The fermionic bulk and edge characters cancel and the bosonic part is twice (3.173). In this case we moreover get a finite and unambiguous $\dim G = \lim_{\delta\to 0} \sum_{s\in\frac{1}{2}\mathbb{N}}^{\infty} (-1)^{2s}\, 2\, D^5_{s-1,s-1}\, e^{-\delta s} = \frac{1}{4}$.

The above observations are tantalizing, but leave several problems unresolved, including what to make of the supergroup volume $\mathrm{vol}\, G$. Actually supergroups present an issue of this kind already with a finite number of generators, as their volume is generically zero. In the context of supergroup Chern-Simons theory this leads to indeterminate $0/0$ Wilson loop expectation values [173]. In this case the indeterminacy is resolved by a construction replacing the Wilson loop by an auxiliary worldline quantum mechanics [173]. Perhaps in this spirit, getting a meaningful path integral on the sphere in the present context may require inserting an auxiliary "observer" worldline quantum mechanics, with a natural action of the higher-spin algebra on its phase space, allowing to soak up the residual gauge symmetries.

One could consider other options, such as breaking the background isometries, models with a finite-dimensional higher-spin algebra [117, 118, 174, 175], models with an α'-like parameter breaking the higher-spin symmetries, or models of a different nature, perhaps along the lines of [176], or bootstrapped bottom-up. We leave this, and more, to future work.

Chapter 4: Grand Partition Functions and Lens Space Path Integrals

In most of this thesis, we have been studying (quasi)canonical partition functions for the southern static patch in dS_{d+1}, which at the inverse de Sitter temperature $\beta = 2\pi$ (with the de Sitter length ℓ_{dS} set to 1) is related to the Euclidean path integral on S^{d+1}. In this section, we would like to generalize our considerations in chapter 3 by including non-zero chemical potentials. We will focus on $d \geq 2$. If we turn on the chemical potential μ in one of the angular directions J, the grand canonical partition function at general inverse temperature β is

$$Z_{\mathrm{bulk}}(\beta, \mu) = \mathrm{Tr}_S \, e^{-\beta(H+i\mu J)} \; . \tag{4.1}$$

Here the trace Tr_S is over the southern QFT Hilbert space. The factor i is inserted so that the corresponding Euclidean path integral is defined on a space with a real metric. In the first part of this chapter, we write down a generalized character formula for (4.1) and its generalization to the case of multiple non-zero chemical potentials. In the second half of this chapter, we restrict our attention to the case of three dimensions, where we relate the grand canonical partition function (4.1) with a Euclidean path integral on the Lens spaces, which are smooth quotients of S^3. We find that the Lens space path integral for a massive spinning field exhibits a bulk-edge structure as its S^3 counterpart.

4.1 Grand quasicanonical bulk thermodynamics

In the following we will formally derive a character formula for both bosonic and fermionic grand quasicanonical partition functions.

4.1.1 Bosons

We first focus on the bosonic case. Analogous to the formal derivation in section 3.2.2, our starting point is the grand canonical partition function for a single bosonic oscillator of frequency ω and angular momentum m at inverse temperature β and chemical potential μ:

$$-\log\left(e^{\beta\omega/2} - e^{-\beta(\omega/2+i\mu m)}\right) . \tag{4.2}$$

Due to the extra factor $e^{-i\beta\mu m}$, there is no simple integral representation for this expression as (3.33). However, we can still proceed as such and write for a free QFT on the southern static patch

$$\log Z_{\text{bulk}}(\beta,\mu) = \sum_{m\in\mathbb{Z}} \int_0^\infty d\omega \rho_m^S(\omega) \left[-\log\left(1 - e^{-\beta(\omega+i\mu m)}\right) - \frac{\beta\omega}{2}\right] \tag{4.3}$$

where $\rho_m^S(\omega)$ is the density of states of frequency ω and angular momentum m.

Density of states and the full $SO(1,d+1)$ character Following the discussions in appendix C.2.1, we argue that the density of states $\rho_m^S(\omega)$ can be regularized by the Harish-Chandra character of the de Sitter group $SO(1,d+1)$. This time we will need the full character

$$\chi_{\text{dS}}(t,\boldsymbol{\theta}) \equiv \text{tr}_G\, e^{-iHt+i\boldsymbol{\theta}\cdot\boldsymbol{J}} . \tag{4.4}$$

Again G means that we are tracing over the global de Sitter Hilbert space. For example, a massive spin-s particle has [78]

$$\chi_{[\Delta,s]}(t,\boldsymbol{\theta}) = \chi_s^d(\mathbf{x}) \left(Q^\Delta + Q^{\bar{\Delta}}\right) \mathcal{P}^d(Q,\mathbf{x}) \tag{4.5}$$

where $Q = e^{-t}$, $\mathbf{x} = (x_1, \cdots, x_r) = (e^{i\theta_1}, \cdots, e^{i\theta_r})$, $\chi_s^d(\mathbf{x})$ is the $SO(d)$ spin-s character, and

$$\mathcal{P}^d(Q, \mathbf{x}) = \prod_{i=1}^{r} \frac{1}{(1 - Qx_i)(1 - Qx_i^{-1})} \times \begin{cases} 1, & \text{if } d = 2r \\ \frac{1}{1-Q} & \text{if } d = 2r + 1 \end{cases} . \tag{4.6}$$

Note that (4.5) evaluated at $\boldsymbol{\theta} = \mathbf{0}$ recovers the reduced massive spin-s character (C.14). Analogous to (C.27), we then regularize the density of states $\rho_m^S(\omega)$ through

$$\rho_m^S(\omega) = \int_0^\infty \frac{dt}{\pi} \cos \omega t \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-im\theta} \chi_S(t, \theta) \tag{4.7}$$

where θ is the angular variable in the chosen angular direction and the other suppressed angular variables are set to zero. Note that we have used the properties for bosonic characters

$$\chi_S(-t, \theta) = \chi_S(t, \theta) , \qquad \chi_S(t, \theta + 2\pi) = \chi_S(t, \theta) \qquad \text{(bosons)} . \tag{4.8}$$

Character formula for the grand partition function We substitute (4.7) into (4.3) and Taylor-expand the logarithm $-\log(1 - x) = \sum_{k=1}^{\infty} x^k/k$, so that the right hand side of (4.3) becomes

$$\int_0^\infty \frac{dt}{\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \chi_S(t, \theta) \sum_{m\in\mathbb{Z}} \int_0^\infty d\omega \cos \omega t e^{-im\theta} \left[\sum_{k=1}^{\infty} \frac{1}{k} e^{-k\beta(\omega + i\mu m)} - \frac{\beta\omega}{2} \right] . \tag{4.9}$$

For a fixed k, the sum over m leads to a Dirac delta function $\delta(\theta + k\beta\mu)$ which collapses the θ integral. The ω integral is also easy to compute. Putting these together we have

$$\log Z_{\text{bulk}}(\beta, \mu) = \int_0^\infty \frac{dt}{2\pi} \sum_{k\in\mathbb{Z}} \frac{\beta}{t^2 + k^2\beta^2} \chi_S(t, k\beta\mu) . \tag{4.10}$$

Note that we have used the property $\chi_S(t, -\theta) = \chi_S(t, \theta)$ to extend the sum to all $k \in \mathbb{Z}$. Also, the $k = 0$ term corresponding to zero-point energy has a pole at $t = 0$ for which we resolve by

$$-\frac{1}{t^2} \to -\frac{1}{2}\left(\frac{1}{(t+i\epsilon)^2} + \frac{1}{(t-i\epsilon)^2}\right). \tag{4.11}$$

For generic real values of β and μ, the sum (4.10) is difficult to evaluate. However, if the combination $\beta\mu$ equals 2π times a rational number, i.e.

$$\beta\mu = \frac{2\pi q}{p}\,, \qquad p \in \mathbb{N} \quad q \in \mathbb{Z}\,, \tag{4.12}$$

it becomes possible to perform the sum (4.10), thanks to the periodic condition in (4.8). In such a special case we can write $k = pn + m, n \in \mathbb{Z}, m \in \mathbb{Z}_p$, and the sum in (4.10) is equivalent to a sum over n and m. The key simplification is

$$\chi_S\left(t, \frac{2\pi q(pn+m)}{p}\right) = \chi_S\left(t, \frac{2\pi qm}{p}\right), \tag{4.13}$$

allowing an exact evaluation of the sum over n:

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{\beta}{t^2 + (pn+m)^2\beta^2} = \frac{1}{2pt}\frac{\sinh\frac{2\pi t}{p\beta}}{\cosh\frac{2\pi t}{p\beta} - \cos\frac{2\pi m}{p}}\,. \tag{4.14}$$

Putting everything together, the twisted character formula for bosonic fields at inverse temperature β and chemical potentials (4.12) is thus

$$\text{Boson:} \qquad \log Z_{\text{bulk}}\left(\beta, \frac{2\pi q}{p\beta}\right) = \int_0^\infty \frac{dt}{2pt}\sum_{m\in\mathbb{Z}_p}\frac{\sinh\frac{2\pi t}{p\beta}}{\cosh\frac{2\pi t}{p\beta} - \cos\frac{2\pi m}{p}}\chi_S\left(t, \frac{2\pi qm}{p}\right)\,. \tag{4.15}$$

It is easy to check that for $q = 0$, (4.15) reduces to the bosonic part of the untwisted formula (3.35).

4.1.2 Fermion

For a free fermionic QFT on the southern static patch, we have

$$\log Z_{\text{bulk}}(\beta,\mu)=\sum_{m\in\mathbb{Z}}\int_0^\infty d\omega\rho_m^S(\omega)\left[\log\left(1+e^{-\beta(\omega+i\mu m)}\right)+\frac{\beta\omega}{2}\right]\ . \tag{4.16}$$

The derivation proceeds similarly as in the previous section. That is, we express the density of states in terms of $SO(1,d+1)$ characters and expand the logarithm $\log(1+x)=-\sum_{k=1}^\infty(-x)^k/k$. After doing the sum over m and performing the ω integral, we obtain

$$\log Z_{\text{bulk}}(\beta,\mu)=-\int_0^\infty\frac{dt}{2\pi}\sum_{k\in\mathbb{Z}}(-)^k\frac{\beta}{t^2+k^2\beta^2}\chi_S(t,k\beta\mu)\ . \tag{4.17}$$

Note that the crucial extra factor $(-)^k$ compared to the bosonic case. Again, we can perform the sum over k for the special case (4.12). However, since fermionic characters are 4π-periodic instead of 2π-periodic: $\chi_S(t,\theta+4\pi)=\chi_S(t,\theta)$, we will write instead $k=2pn+m,,n\in\mathbb{Z},m\in\mathbb{Z}_{2p}$ and perform the sum as in (4.14) with $p\to 2p$. The final result is

$$\text{Fermion:}\qquad \log Z_{\text{bulk}}\left(\beta,\frac{2\pi q}{p\beta}\right)=-\int_0^\infty\frac{dt}{4pt}\sum_{m\in\mathbb{Z}_{2p}}(-)^m\frac{\sinh\frac{\pi t}{p\beta}}{\cosh\frac{\pi t}{p\beta}-\cos\frac{\pi m}{p}}\chi_S\left(t,\frac{2\pi qm}{p}\right). \tag{4.18}$$

It is easy to check that for $q=0$, this reduces to the fermionic part of the untwisted formula (3.35). In fact, since the bosonic characters are automatically 4π-periodic, we can slightly modify our bosonic derivation and summarize our results at inverse temperature β and chemical potentials (4.12) in a single formula

$$\log Z_{\text{bulk}}=\int_0^\infty\frac{dt}{4pt}\sum_{m\in\mathbb{Z}_{2p}}\frac{\sinh\frac{\pi t}{p\beta}}{\cosh\frac{\pi t}{p\beta}-\cos\frac{\pi m}{p}}\left[\chi_S\left(t,\frac{2\pi qm}{p}\right)_{\text{bos}}+(-)^{m+1}\chi_S\left(t,\frac{2\pi qm}{p}\right)_{\text{fer}}\right]. \tag{4.19}$$

4.1.3 Turning on all chemical potentials

For a static patch in dS_{d+1}, the maximum number of independent chemical potentials equals the rank $r = \lfloor \frac{d}{2} \rfloor$ of the subgroup $SO(d)$, so that the most general canonical partition function takes the form

$$\log Z_{\text{bulk}}(\beta, \boldsymbol{\mu}) = \operatorname{Tr} e^{-\beta(H + i\boldsymbol{\mu}\cdot\boldsymbol{J})} , \tag{4.20}$$

where $\boldsymbol{J} = (J_1, \cdots, J_r)$ is a maximal set of commuting angular momenta (i.e. the Cartan generators for $SO(d)$) and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_r)$ are the corresponding chemical potentials. It turns out that the previous formal derivation straightforwardly extends to the more general special case

$$\beta\boldsymbol{\mu} = \frac{2\pi\boldsymbol{q}}{p} , \qquad p \in \mathbb{N} \quad \boldsymbol{q} \in \mathbb{Z}^r . \tag{4.21}$$

The result is that at inverse temperature β and chemical potentials (4.21)

$$\log Z_{\text{bulk}} = \int_0^\infty \frac{dt}{4pt} \sum_{m\in\mathbb{Z}_{2p}} \frac{\sinh \frac{\pi t}{p\beta}}{\cosh \frac{\pi t}{p\beta} - \cos \frac{\pi m}{p}} \left[\chi_S\left(t, \frac{2\pi m\boldsymbol{q}}{p}\right)_{\text{bos}} + (-)^{m+1} \chi_S\left(t, \frac{2\pi m\boldsymbol{q}}{p}\right)_{\text{fer}} \right] \tag{4.22}$$

where $\chi_S(t, \theta_1, \cdots, \theta_r)$ is the full $SO(1, d+1)$ character (4.4). We conclude this part with some comments on the relation of (4.1) with Euclidean partition functions. Recall that the canonical partition function at $\beta = 2\pi$ is related to the path integral on a sphere (with edge mode corrections for spinning fields). Here the Euclidean space relevant to the grand canonical partition function (4.1) is not the sphere, but a sphere subject to further quotients. In general, such a quotient does not lead to a smooth manifold, and the resulting space generically contains conical singularities. In the next section we will see a special case in three dimensions where (4.1) is related to a path integral on a smooth quotient of S^3.

4.2 Euclidean path integrals on Lens spaces

In three dimensions, it turns out that the static patch grand partition function[1]

$$Z_{\text{bulk}} = \text{Tr}_S\, e^{-\frac{2\pi}{p}(H+iqJ)} \tag{4.23}$$

at $\beta = \frac{2\pi}{p}$ and $\mu = q$ is related to a Euclidean path integral on a smooth manifold for some specific integer values of p and q. In particular, when q and p satisfies a coprime condition $(q, p) = 1$, a class of smooth manifolds called Lens spaces $L(p, q)$ can be obtained by quotienting S^3. Looking locally the same as S^3, Lens spaces arise as saddle points of the full gravitational path integrals other than S^3. Compared to the S^3 saddle, the contribution from a single Lens space $L(p, q)$ is exponentially suppressed. The relevance of Lens spaces to dS_3 quantum gravity is discussed in [84]. Here our goal is simply to make a precise connection between Lens space path integrals to the grand partition function (4.23), generalizing our considerations in chapter 3. We expect the following discussions have straightforward generalizations to higher dimensions.

4.2.1 The Lens spaces $L(p, q)$

The simplest way to describe a Lens space $L(p, q)$ is to make use of the fact that S^3 is the $SU(2)$ group manifold. Recall that there is a 1-1 mapping between points on S^3 and elements of $SU(2)$:

$$X_A \in S^3 \mapsto g(X) = X_0 I + iX_i\sigma^i = \begin{pmatrix} X_0 + X_3 & iX_1 + X_2 \\ iX_1 - X_2 & X_0 - X_3 \end{pmatrix} \in SU(2)\,. \tag{4.24}$$

where $\det(g(X)) = X_A X_A = X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1$. Here we are setting the radius of the 3-sphere to 1. Isometries correspond to the left and right matrix multiplications

$$g(X) \mapsto Mg(X)N \text{ with } Mg(X)N \in SU(2)\ . \tag{4.25}$$

[1] In three dimensions, $d = 2$ and the rank of $SO(2)$ is 1. Therefore we can turn on the chemical potential only in one angular direction.

preserving the determinant $\det(g(X)) = 1$. Then we see that $M, N \in SU(2)$ and thus the isometry group is $SU(2) \times SU(2)/\mathbb{Z}_2$. The $\mathbb{Z}_2$ quotient arises because the element $(M, N) = (-1, -1)$ acts trivially.

Now we can give a description of the Lens spaces. A Lens space $L(p, q)$ is given by quotienting S^3 through the identification

$$g(X) \sim Lg(X)R \,, \quad L = \begin{pmatrix} \omega_p^{\frac{1+q}{2}} & 0 \\ 0 & \omega_p^{-\frac{1+q}{2}} \end{pmatrix} , \quad R = \begin{pmatrix} \omega_p^{\frac{1-q}{2}} & 0 \\ 0 & \omega_p^{-\frac{1-q}{2}} \end{pmatrix} \tag{4.26}$$

where $\omega_p = e^{2\pi i/p}$. As discrete isometries, (4.26) act freely on S^3 and thus the identification results in a smooth manifold. Clearly (4.26) generates a $\mathbb{Z}_p$ quotient since $(L, R) \in SU(2) \times SU(2)/\mathbb{Z}_2$ is a p-th root of unity. The isometry groups of $L(p, q)$ can be seen as the subgroup of matrix multiplications (4.25) that commute with the identification (4.26). Therefore, the isometry group is $SU(2) \times SU(2)/\mathbb{Z}_2$ for $(p, q) = (2, 1)$, $U(1) \times SU(2)/\mathbb{Z}_2$ for $q = \pm 1 \bmod p$, and $U(1) \times U(1)/\mathbb{Z}_2$ for $q \neq \pm 1 \bmod p$.

The Hopf parametrization To make connection to static patch physics, we parametrize S^3 with the Hopf coordinates (τ, ψ, φ), which are related to the embedding space coordinates by

$$X_0 = \cos\psi \sin\tau \,, \quad X_1 = \sin\psi \cos\varphi \,, \quad X_2 = \sin\psi \sin\varphi \,, \quad X_3 = \cos\psi \cos\tau \,, \tag{4.27}$$

where $0 < \psi < \pi/2$. For the 3-sphere, we have the periodicity condition

$$(\tau, \varphi) \sim (\tau, \varphi) + 2\pi(m, n) \,, \quad m, n \in \mathbb{Z} \,. \tag{4.28}$$

The Hopf metric is

$$ds^2 = \cos^2\psi \, d\tau^2 + d\psi^2 + \sin^2\psi \, d\varphi^2 \,. \tag{4.29}$$

We recover the dS_3 static patch metric if we Wick-rotate to real time $\tau \to it$ and take $r = \sin\psi$. Now, we can parametrize a Lens space $L(p, q)$ with the same coordinates, as long as we modify the identification (4.28) to

$$(\tau, \varphi) \sim (\tau, \varphi) + 2\pi \left(\frac{n}{p}, \frac{nq}{p} + m \right) , \quad \forall n, m \in \mathbb{Z} \; . \tag{4.30}$$

It is straightforward to check that this identification is equivalent to (4.26).

4.2.2 Massive scalar

We now consider Euclidean path integrals on Lens spaces. The simplest case is a free real scalar with mass $m^2 = \Delta(2-\Delta) = \Delta\bar{\Delta}$ on $L(p, q)$:[2]

$$Z_{\rm PI} = \int \mathcal{D}\phi \; e^{-\frac{1}{2}\int \phi(-\nabla^2+m^2)\phi} = \det\left(-\nabla^2 + m^2\right)^{-1/2} \; . \tag{4.31}$$

As usual, the logarithm of this functional determinant can be expressed in terms of the heat kernel of the Laplace operator $-\nabla^2 + m^2$. Since $L(p, q)$ is simply a quotient of S^3, the heat kernel on the former can be obtained from the latter using the method of images [177]. We simple employ the result obtained in [177] and write

$$\log Z_{\rm PI} = \int_0^\infty \frac{d\lambda}{2\lambda} e^{-\epsilon^2/4\lambda} K_\Delta^{(0)}(\lambda) \tag{4.32}$$

where the scalar heat kernel is[3]

$$K_\Delta^{(0)}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{p} d_n^{(0)} e^{-(n-1+\Delta)(n-1+\bar{\Delta})\lambda} \; , \quad d_n^{(0)} = \sum_{m\in\mathbb{Z}_p} \frac{\cos(nm\tau_2) - \cos(nm\tau_1)}{\cos(m\tau_2) - \cos(m\tau_1)} \; . \tag{4.33}$$

[2] As in section 3.3.2, we first consider $\Delta = 1 + i\nu$, $\nu \in \mathbb{R}$ and extend our final formula by analytic continuation.

[3] Note that the expressions in this chapter are related to those in previous chapters by a $n \to n+1$ shift. This will turn out to be a more natural convention in the current context.

Here τ_1, τ_2 are related to p, q as

$$\tau_1 = \frac{2\pi q}{p}\,, \qquad \tau_2 = \frac{2\pi}{p}\ . \tag{4.34}$$

Following the steps in the derivation in section 3.3.2, we have

$$\log Z_{\rm PI} = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}}\left(e^{-(\Delta-1)\sqrt{t^2-\epsilon^2}} + e^{-(\bar\Delta-1)\sqrt{t^2-\epsilon^2}}\right)\sum_{n=1}^\infty \frac{1}{p} d_n^{(0)} e^{-nt}\,. \tag{4.35}$$

The sum over n in (4.35) can be performed explicitly:

$$\sum_{n=1}^\infty \frac{1}{p} d_n^{(0)} e^{-nt} = \frac{1}{2p}\sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{(\cosh t - \cos m\tau_2)\,(\cosh t - \cos m\tau_1)}\ . \tag{4.36}$$

Plugging this into (4.35) yields

$$\log Z_{\rm PI} = \int_\epsilon^\infty \frac{dt}{2p\sqrt{t^2-\epsilon^2}} \sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{\cosh t - \cos m\tau_2} \frac{e^{-(\Delta-1)\sqrt{t^2-\epsilon^2}} + e^{-(\bar\Delta-1)\sqrt{t^2-\epsilon^2}}}{2(\cosh t - \cos m\tau_1)}\ . \tag{4.37}$$

Putting $\epsilon = 0$, this formally becomes

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2pt} \sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{\cosh t - \cos m\tau_2}\chi(t, m\tau_1)\,, \quad \chi(t,\theta) = \frac{e^{-(\Delta-1)t} + e^{-(\bar\Delta-1)t}}{2(\cosh t - \cos\theta)}\,. \tag{4.38}$$

which we recognize as (4.15) with $\beta = \frac{2\pi}{p}$ and the scalar character $\chi(t,\theta)$. Thus we conclude that for scalars, $Z_{\rm PI} = Z_{\rm bulk}$.

4.2.3 Massive higher spins

Now we consider a massive field with integer spin $s \geq 1$ and mass $m^2 = (\Delta + s - 2)(s - \Delta)$ on $L(p,q)$. We first note that the discussions in appendix C.6.1 remains valid. In particular, we expect the full, manifestly covariant, local path integral takes the form

$$Z_{\text{PI}} = Z_{\text{TT}} \cdot Z_{\text{non-TT}} = Z_{\text{bulk}} \cdot Z_{\text{edge}}^{-1} \,. \tag{4.39}$$

Again, the heat kernel for the TT part has been computed in [177] using the method of images, so that we have

$$\log Z_{\text{TT}} = \int_0^\infty \frac{d\lambda}{2\lambda} e^{-\epsilon^2/4\lambda} K^{(s)}_{\Delta,\text{TT}}(\lambda) \,, \quad K^{(s)}_{\Delta,\text{TT}}(\lambda) = \sum_{n=s+1}^{\infty} \frac{1}{p} d_n^{(s)} e^{-(n-1+\Delta)(n-1+\bar{\Delta})\lambda} \tag{4.40}$$

where for $n \geq s+1$

$$d_n^{(s)} = 2 \sum_{m \in \mathbb{Z}_p} \frac{\cos(sm\tau_1)\cos(nm\tau_2) - \cos(sm\tau_2)\cos(nm\tau_1)}{\cos(m\tau_2) - \cos(m\tau_1)} \,, \tag{4.41}$$

with τ_1, τ_2 defined as in (4.34). To figure out the non-TT part, we are again guided by the locality constraint that there cannot be any logarithmic divergence in odd dimensions. From what we saw in the sphere case, we expect the non-TT part amounts to extending the eigenvalue sum (4.40) of the TT heat kernel appropriately as in (C.144). This turns out to be the case, as long as we modify the definition of $d_n^{(s)}$ for all $s, n \geq 0$:

$$d_n^{(s)} = 2\left(1 - \frac{\delta_{s,0}}{2}\right)\left(1 - \frac{\delta_{n,0}}{2}\right) \sum_{m \in \mathbb{Z}_p} \frac{\cos(sm\tau_1)\cos(nm\tau_2) - \cos(sm\tau_2)\cos(nm\tau_1)}{\cos(m\tau_2) - \cos(m\tau_1)} \,. \tag{4.42}$$

For $s = 0, n \geq 1$, this matches the degeneracies (4.33) for the scalar harmonics. Now, we simply state the result, namely that the full path integral takes the form

$$\log Z_{\text{PI}} = \int_0^\infty \frac{d\lambda}{2\lambda} e^{-\epsilon^2/4\lambda} K^{(s)}_{\Delta}(\lambda) \,, \quad K^{(s)}_{\Delta}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{p} d_n^{(s)} e^{-(n-1+\Delta)(n-1+\bar{\Delta})\lambda} \,. \tag{4.43}$$

The non-TT part corresponds to the terms with $n = 0, 1, \cdots, s$.

Bulk and edge part Starting from (4.43), we again follow the steps in the derivation in section 3.3.2, leading to

$$\log Z_{\rm PI} = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}} \left(e^{-(\Delta-1)\sqrt{t^2-\epsilon^2}} + e^{-(\bar\Delta-1)\sqrt{t^2-\epsilon^2}}\right) \sum_{n=0}^\infty \frac{1}{p} d_n^{(s)} e^{-nt} \,. \tag{4.44}$$

At this point, we recall that there was a key relation (C.135) in our derivation for the sphere, which eventually led to the bulk-edge split. The analog for (C.135) in the present case is, for $s \geq 1$:

$$d_n^{(s)} = \sum_{m\in\mathbb{Z}_p} 2\cos(sm\tau_1)\frac{\cos(nm\tau_2)-\cos(nm\tau_1)}{\cos(m\tau_2)-\cos(m\tau_1)} + \sum_{m\in\mathbb{Z}_p} \frac{\cos(sm\tau_1)-\cos(sm\tau_2)}{\cos(m\tau_2)-\cos(m\tau_1)} \left(2-\delta_{n,0}\right)\cos(nm\tau_1)\,. \tag{4.45}$$

Putting this back in (4.44) and summing over n, we have the final result for the 1-loop partition function for a massive spin-s field on $L(p,q)$

$$\begin{aligned}
\log Z_{\rm PI} &= \log Z_{\rm bulk} - \log Z_{\rm edge} \\
\log Z_{\rm bulk} &= \int_\epsilon^\infty \frac{dt}{2p\sqrt{t^2-\epsilon^2}} \sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{\cosh t - \cos m\tau_2} 2\cos s\theta \frac{e^{-(\Delta-1)\sqrt{t^2-\epsilon^2}} + e^{-(\bar\Delta-1)\sqrt{t^2-\epsilon^2}}}{2(\cosh t - \cos m\tau_1)} \\
\log Z_{\rm edge} &= \int_\epsilon^\infty \frac{dt}{2p\sqrt{t^2-\epsilon^2}} \sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{\cosh t - \cos m\tau_1} \frac{\cos sm\tau_1 - \cos sm\tau_2}{\cos m\tau_1 - \cos m\tau_2} \left(e^{-(\Delta-1)\sqrt{t^2-\epsilon^2}} + e^{-(\bar\Delta-1)\sqrt{t^2-\epsilon^2}}\right) .
\end{aligned} \tag{4.46}$$

If we put $\epsilon = 0$, this takes the form of a character formula

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2pt} \left[\sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{\cosh t - \cos m\tau_2} \chi_{\rm bulk}(t, m\tau_1) - \sum_{m\in\mathbb{Z}_p} \frac{\sinh t}{\cosh t - \cos m\tau_1} \chi_{\rm edge}(t, \tau_1, \tau_2)\right] \tag{4.47}$$

where the bulk and edge characters are respectively

$$\chi_{\text{bulk}}(t,\theta) = 2\cos s\theta \frac{e^{-(\Delta-1)t} + e^{-(\bar{\Delta}-1)t}}{2(\cosh t - \cos\theta)}, \quad \chi_{\text{edge}}(t,\tau_1,\tau_2) = \frac{\cos sm\tau_1 - \cos sm\tau_2}{\cos m\tau_1 - \cos m\tau_2}(e^{-(\Delta-1)t} + e^{-(\bar{\Delta}-1)t}). \tag{4.48}$$

As a check, let us consider (4.47) for the case $(p,q) = (1,0)$. In this case the sum collapses into one single term with $m = 0$ and the ratio appearing in the edge character seems not well-defined. However, if we think of this as a limit $m \to 0$, then applying the L' Hospital rule twice reproduces (3.84) with the massive spin-s characters (3.88) in dS_3. Again, at this point we do not have a systemic group-theoretic or physical way of identifying the edge field content, but such an identification is not needed for the evaluation of (4.46).

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Appendix A: Basics of de Sitter space and spheres

In this appendix we collect some basic facts about de Sitter space and spheres.

A.1 Classical de Sitter geometry

A.1.1 Definition and basic geometric properties

A $(d+1)$-dimensional de Sitter (dS) space dS_{d+1} is a maximally symmetric solution to Einstein equation with a positive cosmological constant,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \qquad \Lambda > 0 \ . \tag{A.1}$$

The simplest way to describe the geometry of dS_{d+1} is as a $(d+1)$-dimensional hyperbloid embedded in a $(d+2)$-dimensional Minkowski space $\mathbb{R}^{1,d+1}$:

$$\eta_{AB} X^A X^B = -(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 = \ell_{\rm dS}^2 \ , \tag{A.2}$$

where $\ell_{\rm dS}$ is the de Sitter length related to the cosmological constant through

$$\Lambda = \frac{d(d-1)}{2\ell_{\rm dS}^2} \ . \tag{A.3}$$

We will call the line with $X^i = 0, i = 1, \ldots, d$ and $X^{d+1} < 0$ ($X^{d+1} > 0$) the south (north) pole. The de Sitter metric is induced from the embedding space one

$$ds^2 = \eta_{AB} dX^A dX^B \ . \tag{A.4}$$

Some basic geometric properties:

1. It is clear that dS_{d+1} has an $O(1, d+1)$ isometry, which comprises

   (a) the $SO(1, d+1)$ component connected to the identity, generated by $\frac{(d+1)(d+2)}{2}$ generators $L_{AB} = X_A \partial_B - X_B \partial_A$, $A, B = 0, 1, \cdots, d+1$, which satisfy the commutation relation

$$[L_{AB}, L_{CD}] = \eta_{BC} L_{AD} - \eta_{BD} L_{AC} + \eta_{AD} L_{BC} - \eta_{AC} L_{BD}. \tag{A.5}$$

   (b) discrete symmetries: $T = \text{diag}(-1, 1, \cdots, 1)$, $S_I = \text{diag}(1, 1, \cdots, -1, \cdots, 1)$

   (c) antipodal transformation $A = (-1, -1, \cdots, -1)$. For a given point X, we denote X_A as its antipodal counterpart.

2. A bit abstractly dS_{d+1} can be thought of as the homogeneous space $SO(1, d+1)/SO(1, d)$.

3. There is no Killing vector that is timelike everywhere. For example, a generator

$$L_{10} = X_1 \frac{\partial}{\partial X^0} - X_0 \frac{\partial}{\partial X^1} \tag{A.6}$$

   can move us towards increasing or decreasing X^0 depending on the sign of X_1. This implies that there is no globally conserved positive energy. Consequences of this seemingly innocuous fact includes the non-existence of unbroken supersymmetry in de Sitter space, and that there is no unique dS-invariant vacuum in a global dS QFT.

4. A geodesic on de Sitter can be visualized in the embedding space as the intersection of the hyperbloid (A.2) and a plane passing through the origin. Note that not all pairs of points can be connected through a geodesic. In particular, a point X cannot be connected with a geodesic to any point lying the future or past light cone of its antipodal counterpart X_A.

5. It is useful to express dS-invariant quantities (such as the Wightman function) in terms of the

$O(1, d+1)$-invariant distance

$$P(X, Y) = \eta_{AB} X^A Y^B \ . \tag{A.7}$$

This is related to the geodesic distance through

$$d(X, Y) = \ell_{\mathrm{dS}} \cos^{-1} P(X, Y) \ , \tag{A.8}$$

which is real for spacelike geodesics and imaginary for timelike geodesics.

6. We can obtain AdS_{d+1} from (A.2) by a double Wick rotation[1]

$$X_{d+1} \to i X_{d+1}, \quad \ell_{\mathrm{dS}} \to i \ell_{\mathrm{AdS}} \ . \tag{A.9}$$

A.2 Various coordinate systems and the Penrose diagram

In this section we collect some common coordinate systems that parametrize de Sitter space. We will set $\ell_{\mathrm{dS}} = 1$ throughout.

A.2.1 Global sphere slicing

These are coordinates that slice dS_{d+1} with global d-dimensional spatial spheres S^d, so that the maximal compact subgroup $SO(d+1)$ of the de Sitter group $SO(1, d+1)$ becomes manifest. These will cover the entire global de Sitter space.

Global proper time coordinates

These are given by the following intrinsic parametrization of embedding coordinates

$$X^0 = \sinh \tau, \qquad X^I = \omega_d^I \cosh \tau, \qquad -\infty < \tau < \infty, \qquad \omega_d^2 = 1 \tag{A.10}$$

[1]Strictly speaking AdS is the universal covering of the Wick-rotated hyperbloid.

where $I = 1, \cdots, d+1$. The coordinates ω_d^I parametrize the global spatial S^d, which can be chosen to be the standard angular coordinates or the stereographic coordinates, reviewed in App. A.3. The metric in these coordinates is

$$ds^2 = -d\tau^2 + \cosh^2 \tau d\Omega_d^2 \ . \tag{A.11}$$

Note that under the Wick rotation $\tau \to -i\tau_E$ and demanding $\tau_E \sim \tau_E + 2\pi$, this becomes the metric on S^{d+1}. This observation allows us to interpret the path integral on sphere as computing the normal of wave function in *global* de Sitter, at least for simple enough theories like a massive scalar.

Global conformal time coordinates and the Penrose diagram

Next, we introduce the global conformal time $\bar{T}$, related to the proper time τ through

$$\cosh \tau = \frac{1}{\cos \bar{T}}, \qquad -\frac{\pi}{2} < \bar{T} < \frac{\pi}{2} \ . \tag{A.12}$$

This has metric

$$ds^2 = \frac{-d\bar{T}^2 + d\omega_d^2}{\cos^2 \bar{T}} \ , \tag{A.13}$$

which is related by a Weyl rescaling to the metric $ds^2 = -d\bar{T}^2 + d\omega_d^2$ on a cylinder with a finite length. This provides a mapping of the entire de Sitter space onto a finite diagram while preserving its casual structures (since light rays are preserved by a Weyl rescaling). This is the so-called Penrose diagram, illustrated in figure C.5.

A.2.2 Planar (inflationary) slicing

Next, we can slice a de Sitter space with Euclidean planes $\mathbb{R}^d$, so that the $ISO(d)$ subgroup of the full isometry group $SO(1, d+1)$ becomes manifest.

Poincaré time

These are given by the following intrinsic parametrization of embedding coordinates

$$X^0 = -\frac{1 + x^2 - \eta^2}{2\eta}, \qquad X^i = -\frac{x^i}{\eta}, \qquad X^{d+1} = \frac{1 - x^2 + \eta^2}{2\eta}, \qquad \eta < 0, \quad x^i \in \mathbb{R}^d \tag{A.14}$$

for which

$$ds^2 = \frac{-d\eta^2 + dx^2}{\eta^2} \ . \tag{A.15}$$

Note that this coordinate system does not cover the entire de Sitter space. Specifically, it only covers the region with $X^{d+1} < X^0$, the causal future of an observer sitting at the south pole. Other than the Poincaré symmstries that are trivially manifest, the metric is also clearly invariant under the rescaling

$$\eta \longrightarrow \lambda\eta, \qquad x^i \longrightarrow \lambda x^i \ . \tag{A.16}$$

This is why η is sometimes referred to as conformal time (not to be confused with the global conformal time in (A.12)).

FLRW time

Changing the Poincaré time η to the proper time t

$$\eta = -e^{-t}, \qquad -\infty < t < \infty \ , \tag{A.17}$$

we can put the metric into the FLRW form

$$ds^2 = -dt^2 + e^{2t} dx^2 \tag{A.18}$$

with Hubble parameter $H = 1/l_{dS} = 1$. This is why it is also called inflationary slicing. Note that the rescaling (A.16) of η is mapped to a time translation in t: $t \to t - \log \lambda$.

A.2.3 Hyperbolic slicing

We can also slice a de Sitter space with hyperbolic planes $\mathbb{H}^d$, so that the $SO(1,d)$ subgroup of the full isometry group $SO(1, d+1)$ becomes manifest. These are given by the following intrinsic parametrization of embedding coordinates

$$X^0 = \sinh\bar{\tau}\cosh\psi, \qquad X^i = \omega^i_{d-1}\sinh\bar{\tau}\sinh\psi, \qquad X^{d+1} = \cosh\bar{\tau} \tag{A.19}$$

where

$$0 < \bar{\tau} < \infty, \qquad 0 < \psi < \infty, \qquad \omega^2_{d-1} = 1 \ . \tag{A.20}$$

The metric reads

$$ds^2 = -d\bar{\tau}^2 + \sinh^2\bar{\tau}\left(d\psi^2 + \sinh^2\psi\, d\Omega^2_{d-1}\right) \ . \tag{A.21}$$

Surfaces of constant $\bar{\tau}$ are d-dimensional hyperbolic planes $\mathbb{H}^d$. Note that these coordinates cover only a portion of the entire de Sitter space.

A.2.4 The static patch

To understand the physics from the perspective of a local inertial observer, it is natural to introduce coordinates that adapt to the causal diamond of such an observer.

Static coordinates

The static coordinates parametrize the embedding coordinates as follows

$$X^0 = \sqrt{1-r^2}\sinh t, \quad X^i = r\omega^i_{d-1}, \quad X^{d+1} = \sqrt{1-r^2}\cosh t \tag{A.22}$$

where

$$-\infty < t < \infty, \qquad 0 < r < 1, \qquad \omega^2_{d-1} = 1 \ . \tag{A.23}$$

From this parametrization it is clear that it only cover part of the global de Sitter space. Specifically, it covers only the causal diamond of the observer sitting at the north pole ($r = 0$), as shown in figure C.5. In these coordinates the metric takes a static form

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega^2_{d-1} \ . \tag{A.24}$$

Analogous to static black holes, the coordinate singularity at $r = 1$ corresponds to a horizon. However, unlike the black hole case, this cosmological horizon is observer-dependent. There is no way the observer can "fall into" it. Another notable feature of the static patch is that there is a timelike Killing vector ∂_t, which means we can define a conserved positive energy within this patch. The last comment crucial to this thesis is that under the Wick rotation $t \to -it_E$ and demanding t_E to be periodic in 2π, (A.24) becomes the metric on S^{d+1}. The simplest way to see this is to note that after this procedure and taking $X^0 \to -iX^0$, the embedding space coordinates X^A cover the entire round sphere $X^2 = 1$. This allows us to interpret a sphere path integral as computing a thermal ensemble for a static patch observer.

Eddington-Finkelstein coordinates

Analogous to black holes, we can construct coordinates that are convenient for studying light rays. For outgoing light rays we construct the outgoing Eddington-Finkelstein coordinates through[2]

$$dt = dx^+ + \frac{dr}{1-r^2} \tag{A.25}$$

solving which gives

$$x^+ = t + \frac{1}{2}\ln\frac{1+r}{1-r} \ . \tag{A.26}$$

Outgoing light rays are characterized by $x^+ =$ constant. In these coordinates the metric is

$$ds^2 = -(1-r^2)(dx^+)^2 - 2dx^+dr + r^2 d\Omega_{d-1}^2 \ . \tag{A.27}$$

We can also define the ingoing Eddington-Finkelstein coordinates

$$x^- = t - \frac{1}{2}\ln\frac{1+r}{1-r} \tag{A.28}$$

so that ingoing light rays are characterized by $x^- =$ constant. We can put the metric into a more symmetric form:

$$ds^2 = -\left(1 - \tanh^2\left(\frac{x^+ - x^-}{2}\right)\right) dx^+dx^- + r^2 d\Omega_{d-1}^2 \ . \tag{A.29}$$

Kruskal coordinates

We can also define the Kruskal coordinates

$$x^- = \ln U, \qquad x^+ - \ln(-V) \tag{A.30}$$

[2]Note that outgoing (ingoing) direction corresponds to decreasing (increasing) r.

so that

$$r = \frac{1 + UV}{1 - UV} \ . \tag{A.31}$$

These coordinates cover the entire de Sitter space. The north and south poles correspond to $UV = -1$, the horizons $UV = 0$ and the past and future infinities $UV = 1$. In these coordinates the metric is

$$ds^2 = \frac{1}{(1 - UV)^2} \left(-4dUdV + (1 + UV)^2 d\Omega_{d-1}^2 \right) \ . \tag{A.32}$$

de Sitter slicing

Finally, a static patch of dS_{d+1} can be foliated with de Sitter dS_d slices, which makes the $SO(1, d)$ subgroup of the full isometry group $SO(1, d + 1)$ manifest. These are given by the following intrinsic parametrization of embedding coordinates

$$X^0 = \sin w \sinh \tilde{\tau}, \qquad X^i = \omega_{d-1}^i \sin w \cosh \tilde{\tau}, \qquad X^{d+1} = \cos w \tag{A.33}$$

where

$$-\infty < \tilde{\tau} < \infty, \qquad 0 < w < \pi, \qquad \omega_{d-1}^2 = 1 \ . \tag{A.34}$$

The metric takes the form

$$ds^2 = dw^2 + \sin^2 w \left(-d\tilde{\tau}^2 + \cosh^2 \tilde{\tau} d\Omega_{d-1}^2 \right) \ . \tag{A.35}$$

Surfaces of constant w are d-dimensional de Sitter spaces dS_d. In these coordinates the horizon is at $w = 0$ and $w = \pi$, where the warp factor $\sin^2 w$ vanishes. This metric (A.35) takes exactly the same form as a Rindler-AdS metric. To obtain the latter one simply replaces the warp factor

$\sin^2 w \to \sinh^2 w$ and extend the range of w to $0 < w < \infty$. This observation is one of the starting points of the dS/dS correspondence [151].

A.3 Coordinates on round spheres

A.3.1 Angular coordinates

In $\mathbb{R}^{d+1}$ the unit sphere S^d is given by the submanifold $\omega^2 = \omega_i \omega^i = 1$. The standard angular parametrization

$$\begin{aligned} \omega^1 &= \cos\theta_1, \\ \omega^2 &= \sin\theta_1 \cos\theta_2, \\ &\vdots \\ \omega^d &= \sin\theta_1 \cdots \sin\theta_{d-1} \sin\theta_d, \\ \omega^{d+1} &= \sin\theta_1 \cdots \sin\theta_{d-1} \cos\theta_d \ , \end{aligned} \tag{A.36}$$

where $0 \leq \theta_i < \pi$ for $1 \leq i \leq d-1$ and $0 \leq \theta_d < 2\pi$, leads to the metric

$$d\Omega_d^2 = d\omega_i d\omega^i = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{d-1} d\theta_d^2 \ . \tag{A.37}$$

A.3.2 Stereographic projection

Another common parametrization of S^d is by mapping its points to a plane $v^a \in \mathbb{R}^d$ through

$$\omega^a = \frac{2v^a}{1+v^2}, \qquad \omega^{d+1} = -\frac{1-v^2}{1+v^2} \tag{A.38}$$

where $v^2 = v_a v^a$. Note that the point $\omega^i = (0, \cdots, 0, -1)$ is mapped to the origin while the point $\omega^i = (0, \cdots, 0, 1)$ is mapped to infinity. We can also define another patch

$$\omega^a = \frac{2\tilde{v}^a}{1+\tilde{v}^2}, \qquad \omega^{d+1} = \frac{1-\tilde{v}^2}{1+\tilde{v}^2} \tag{A.39}$$

so that the point $\omega^i = (0, \cdots, 0, 1)$ is mapped to the origin while the point $\omega^i = (0, \cdots, 0, -1)$ is mapped to infinity. The metric in these coordinates reads

$$d\Omega_d^2 = \frac{4dv^2}{(1+v^2)^2} = \frac{4d\tilde{v}^2}{(1+\tilde{v}^2)^2} \ . \tag{A.40}$$

Appendix B: Appendix for chapter 2

B.1 Conventions and definitions

Symmetrization We symmetrize a rank-s tensor $\phi_{\mu_1\cdots\mu_s}$ by summing all the permutations followed by a division by $s!$. That is

$$\phi_{(\mu_1\cdots\mu_s)} = \frac{1}{s!}\sum_{\sigma:\text{perm}} \phi_{\mu_{\sigma(1)}\cdots\mu_{\sigma(s)}} \tag{B.1}$$

Shorthand notations Throughout this paper we denote

$$\int_{S^{d+1}} \equiv \int_{S^{d+1}} d^{d+1}x\sqrt{g}. \tag{B.2}$$

When dealing with a rank-s totally symmetric tensor, we sometimes use the notations:

$$A_{(s)} \equiv A_{\mu_1\cdots\mu_s} \tag{B.3}$$

$$g^k\nabla^{(n-2k)}A_{(s)} \equiv g_{(\mu_1\mu_2}\cdots g_{\mu_{2k-1}\mu_{2k}}\nabla_{\mu_{2k+1}}\cdots\nabla_{\mu_n}A_{\mu_{n+1}\cdots\mu_{s+n})} \tag{B.4}$$

$$\nabla\cdot A_{(s)} \equiv \nabla^\lambda A_{\mu_1\cdots\mu_{s-1}\lambda} \tag{B.5}$$

$$\mathrm{Tr}A_{(s)} \equiv g^{\lambda\rho}A_{\lambda\rho\mu_1\cdots\mu_{s-2}} \tag{B.6}$$

For two spin-s fields $\psi_{(s)}$ and $\psi'_{(s)}$, we define the inner product

$$(\psi_{(s)},\psi'_{(s)}) \equiv \int_{S^{d+1}} \psi^{\mu_1\cdots\mu_s}\psi'_{\mu_1\cdots\mu_s}. \tag{B.7}$$

Path integral measure Path integrals for a spin-s bosonic field $\phi_{(s)}$ take the form

$$\int \mathcal{D}\phi_{(s)} e^{-\frac{1}{2g^2}(\phi_{(s)},-Q\phi_{(s)})} \tag{B.8}$$

where Q is a Laplace type operator. The measure $\mathcal{D}\phi_{(s)}$ is defined as follows. Suppose $\phi_{(s)}$ has mass dimension $\frac{d-2p}{2}$ and an expansion in terms of orthonormal modes $f_{n,(s)}$, i.e.

$$\phi_{(s)} = \sum_n a_{n,s} f_{n,(s)}, \quad (f_{n,(s)}, f_{m,(s)}) = \delta_{nm}. \tag{B.9}$$

We define the path integration measure for ϕ to be

$$\mathcal{D}\phi_{(s)} \equiv \prod_n \frac{M^p}{\sqrt{2\pi}g} da_{n,s}. \tag{B.10}$$

Here are some comments:

- M is a parameter with mass dimension 1, and the power p is determined by dimension analysis so that the partition function remains dimensionless. In most of this paper we will set $M = 1$ and restore it by dimension analysis when necessary.

- The factors of $\sqrt{2\pi}g$ are inserted such that the path integration results in a determinant without any extra factor other than the dimensionful parameter M:

$$\int \mathcal{D}\phi e^{-S[\phi]} = \det\left(-\frac{Q}{M^{2p}}\right)^{-1/2}. \tag{B.11}$$

- The multiplication of the factor $\frac{M^p}{\sqrt{2\pi}g}$ only affects UV divergent terms of the resulting free energy and thus can be absorbed into the bare couplings of the local curvature densities. This can be seen as follows. In heat kernel regularization, the path integral is expressed as an integral transform

$$\log Z_{\rm PI} = \int_0^\infty \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} {\rm Tr} e^{-D\tau} \tag{B.12}$$

of the trace heat kernel for an unconstrained differential operator D. Here for concreteness we have chosen a specific UV regulator $e^{-\frac{\epsilon^2}{4\tau}}$. The result takes the form

$$\log Z_{\text{PI}} = \frac{1}{2}\zeta'(0) + \alpha_{d+1}\log\left(\frac{2}{e^{\gamma_E}\epsilon}\right) + \frac{1}{2}\sum_{k=0}\alpha_k \Gamma\left(\frac{d+1-k}{2}\right)\left(\frac{2}{\epsilon}\right)^{d+1-k}. \tag{B.13}$$

Here $\zeta(z)$ is the spectral zeta function for the operator D. The heat kernel coefficients α_i are given by integrals of local curvature densities (see for example [60] for explicit formulas). The term α_{d+1} is present only for odd d. Now, the multiplication by a local infinite constant is equivalent to rescaling the differential operator by a constant,

$$\log Z_{\text{PI}} \to \log Z'_{\text{PI}} = \int_0^\infty \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} \text{Tr} e^{-\tau\frac{(-\nabla^2+\sigma)}{\text{g}}} = \int_0^\infty \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\text{g}\tau}} \text{Tr} e^{-\tau(-\nabla^2+\sigma)}, \tag{B.14}$$

which alters only the divergent terms as $\epsilon \to 0$.

- With these conventions we also see that the field ϕ satisfies the normalization condition

$$\int \mathcal{D}\phi e^{-\frac{1}{2\text{g}^2}(\phi,\phi)} = 1. \tag{B.15}$$

- We can think of the measure (B.10) as putting the following metric on the field space

$$ds^2 = \frac{M^{2p}}{2\pi \text{g}^2}\int_{S^{d+1}}(\delta\phi)^2 = \frac{M^{2p}}{2\pi \text{g}^2}\sum_n da_n^2 \tag{B.16}$$

Commutator In our convention the commutator of two covariant derivatives acts on a totally symmetric rank-s tensor as

$$[\nabla_\mu, \nabla_\nu]\phi^{\rho_1\cdots\rho_s} = \sum_{j=1}^{s} R^{\rho_j}{}_{\lambda\mu\nu}\phi^{\rho_1\cdots\hat{\rho}_j\cdots\rho_s}, \quad R_{\lambda\rho\mu\nu} = \frac{g_{\lambda\mu}g_{\rho\nu} - g_{\lambda\nu}g_{\rho\mu}}{l^2}. \tag{B.17}$$

where $\hat{\rho}_j$ means that ρ_j is excluded. l is the radius of the sphere and will be set to 1 for most of this paper.

B.2 Symmetric transverse traceless Laplacians and symmetric tensor spherical harmonics on S^{d+1}

Here we collect some useful facts from [178] and [179] about spin-s symmetric transverse traceless (STT) Laplacians and symmetric tensor spherical harmonics (STSH) on S^{d+1}.

Definition, eigenvalues and degeneracies

STSHs $f_{n,(s)} \equiv f_{n,\mu_1\cdots\mu_s}$ are labeled by their spin s and angular momentum number $n \geq s$. These are the STT eigenfunctions of the STT Laplacian $-\nabla^2_{(s)}$ on S^{d+1}

$$-\nabla^2_{(s)} f_{n,(s)} = \lambda_{n,s} f_{n,(s)}, \quad \nabla \cdot f_{n,(s)} = 0, \quad \mathrm{Tr} f_{n,(s)} = 0 \tag{B.18}$$

with eigenvalues and degeneracies

$$\lambda_{n,s} = n(n+d) - s, \quad n \geq s \tag{B.19}$$

$$D^{d+2}_{n,s} = g_s \frac{(n-s+1)(n+s+d-1)(2n+d)(n+d-2)!}{d!(n+1)!} \tag{B.20}$$

$$g_s = \frac{(2s+d-2)(s+d-3)!}{(d-2)!s!}. \tag{B.21}$$

These furnish $SO(d+2)$ irreducible representations corresponding to two-row Young diagrams with n boxes in the first row and s boxes in the second row. We sometimes call them (n, s) modes in the paper. We normalize them with respect to (B.7), i.e.

$$(f_{n,(s)}, f_{m,(s)}) = \delta_{nm} \tag{B.22}$$

When we use a double labeling such as $f_{n,(s)}$ for the spin-s STSHs or $\lambda_{n,s}$ for its eigenvalues, the n automatically labels the spectrum of $-\nabla^2_{(s)}$. Also, when we write $\sum_n$ or $\prod_n$, there is an implied sum or product over degenerate spin-s STSHs with the same label n.

Killing tensors A spin-s Killing tensor (KT) $\epsilon_{(s)}$ is a totally symmetric traceless tensor satisfying the Killing equation

$$\nabla_{(\mu_1}\epsilon_{\mu_2\cdots\mu_{s+1})} = 0. \tag{B.23}$$

Taking the trace of this equation shows that they are divergenceless, while taking the divergence we recover (B.18) with $n = s$ and thus they are in fact spanned by the (s, s) modes.

Induced spin-s symmetric traceless spherical harmonics

Given a STSH $f_{n,(s)}$, one can construct the m-th induced symmetric traceless tensors

$$T^{(s)}_{n,(s+m)} = \nabla_{(\mu_1}\cdots\nabla_{\mu_m} f_{n,\mu_{m+1}\cdots\mu_{m+s})} - \text{trace terms}, \tag{B.24}$$

where the subtraction of trace terms is such that the expression is traceless. From its definition, it is clear that $T^{(s)}_{n,(s)} = f_{n,(s)}$. There are two important facts to note:

1. $T^{(m)}_{n,(s)}$ satisfy an orthogonality condition under the inner product (B.7).
2. $T^{(m)}_{n,(s)}$ vanishes identically for $s > n$.

The more familiar lower spin examples include the orthonormal modes for the longitudinal part of a vector field

$$T^{(0)}_{n,\mu} = \nabla_\mu f_n$$

or the orthonormal modes for the symmetric traceless part of a spin-2 tensor constructed from scalar spherical harmonics

$$T^{(0)}_{n,\mu\nu} = \nabla_\mu\nabla_\nu f_n + \frac{\lambda_{n,0}}{d+1} g_{\mu\nu} f_n.$$

We might use the notation (n, s) to refer to a spin-$(s+m)$ symmetric traceless spherical harmonics $T^{(s)}_{n,(s+m)}$ induced from the STSH $f_{n,(s)}$ when the context is clear.

Mode expansions for symmetric traceless tensors In general, a spin-s symmetric traceless (not necessarily transverse) field $V_{(s)}$ on S^{d+1} has the mode expansion

$$V_{(s)} = \sum_{m=0}^{s}\sum_{n=s}^{\infty} A_{n,m}\hat{T}^{(m)}_{n,(s)}, \tag{B.25}$$

where $\hat{T}^{(m)}_{n,(s)}$ is the normalized version of $T^{(m)}_{n,(s)}$, i.e.

$$\hat{T}^{(m)}_{n,(s)} \equiv \frac{T^{(m)}_{n,(s)}}{||T^{(m)}_{n,(s)}||} \tag{B.26}$$

where the norm $||\cdot,\cdot|| \equiv \sqrt{(\cdot,\cdot)}$ is defined with respect to (B.7).

Useful identities In this work we make use of the following identities for $T^{(m)}_{n,(s)}$

$$-\nabla^2 T^{(m)}_{n,(s)} = a^{(m)}_{s,n} T^{(m)}_{n,(s)} \tag{B.27}$$

$$\nabla\cdot T^{(m)}_{n,(s)} = b^{(m)}_{s,n} T^{(m)}_{n,(s-1)} \tag{B.28}$$

where

$$a^{(m)}_{s,n} = \lambda_{n,m} - (s-m)(s+m+d-1) \tag{B.29}$$

$$b^{(m)}_{s,n} = \frac{(s-m)(d+s+m-2)}{2}(\lambda_{s-1,s} - \lambda_{n,s}). \tag{B.30}$$

Norms For our purpose, we do not need to know the norm of $T^{(m)}_{n,(s)}$ (with respect to (B.7)), but we need the relative normalizations between $T^{(m)}_{n,(s)}$, $T^{(m)}_{n,(s-1)}$ and $\nabla\cdot T^{(m)}_{n,(s)}$, which can be easily computed:

$$(\nabla\cdot T^{(m)}_{n,(s)}, \nabla\cdot T^{(m)}_{n,(s)}) = (b^{(m)}_{s,n})^2 (T^{(m)}_{n,(s-1)}, T^{(m)}_{n,(s-1)}) = -b^{(m)}_{s,n}(T^{(m)}_{n,(s)}, T^{(m)}_{n,(s)}). \tag{B.31}$$

Conformal Killing tensors A spin-s conformal Killing tensor (CKT) $\epsilon_{(s)}$ is a totally symmetric traceless tensor satisfying the conformal Killing equation

$$\nabla_{(\mu_1}\epsilon_{\mu_2\cdots\mu_{s+1})} - \frac{s}{d+2s-1} g_{(\mu_1\mu_2}\nabla^\lambda\epsilon_{\mu_3\cdots\mu_{s+1})\lambda} = 0. \tag{B.32}$$

The solution space to this equation is spanned by $T^{(m)}_{s,(s)}$ with $m = 0, 1, \cdots, s$. Notice that the modes $T^{(s)}_{s,(s)} = f_{s,(s)}$ correspond to spin-s KTs.

B.3 Higher spin invariant bilinear form

In this appendix, we relate the HS invariant bilinear form in [68] to the one induced by our path integral measure.

The Noether approach

Suppose we have a quadratic action $S^{(2)}$ of a collection of fields φ that is invariant under the linear gauge symmetries $\delta^{(0)}_\xi\varphi$, which we want to deform into an interacting action

$$S = S^{(2)} + S^{(3)} + S^{(4)} + \cdots \tag{B.33}$$

invariant under the non-linear gauge symmetries

$$\delta_\xi\varphi = \delta^{(0)}_\xi\varphi + \delta^{(1)}_\xi\varphi + \delta^{(2)}_\xi\varphi + \cdots . \tag{B.34}$$

Here the superscript (n) denotes the power in fields (or coupling constants). Requiring full gauge invariance, i.e.

$$\delta_\xi S = 0, \tag{B.35}$$

we have a system of equations relating deformations and the gauge transformations at particular orders:

$$\begin{aligned}
\delta_\xi^{(0)} S^{(2)} &= 0 \\
\delta_\xi^{(0)} S^{(3)} + \delta_\xi^{(1)} S^{(2)} &= 0 \\
\delta_\xi^{(0)} S^{(4)} + \delta_\xi^{(1)} S^{(3)} + \delta_\xi^{(2)} S^{(2)} &= 0 \\
&\ldots
\end{aligned} \tag{B.36}$$

This can be solved as follows:

1. We solve the second equation on the solutions of the first equation $\delta S^{(2)} = 0$ to infer the cubic interaction $S^{(3)}$.

2. From this we can infer $\delta_\xi^{(1)}$ by solving the second equation again without imposing the first equation $\delta S^{(2)} = 0$.

3. Proceed in a similar fashion for all higher order $S^{(n\geq 3)}$ and the field-dependent part of the gauge transformations $\delta_\xi^{(n\geq 1)}$. That is, we solve for $S^{(n)}$ using by the $(n-1)$-th constraint with the $(n-1)$-th order equation of motion imposed, and then for the deformation $\delta_\xi^{(n-1)}$ by solving the same equation without imposing equations of motion.

Local gauge algebra The full non-linear gauge transformations (B.34) are required to form an (open) algebra

$$\delta_{\xi_1}\delta_{\xi_2} - \delta_{\xi_2}\delta_{\xi_1} = \delta_{[[\xi_1,\xi_2]]} + (\text{on-shell trivial}) \tag{B.37}$$

where (as illustrated in the Yang Mills and Einstein gravity case) the precise form of the bracket $[[\cdot,\cdot]]$ depends on how gauge transformations act on φ. In particular, it can be field dependent and can be expanded as

$$[[\cdot,\cdot]] = [[\cdot,\cdot]]^{(0)} + [[\cdot,\cdot]]^{(1)} + [[\cdot,\cdot]]^{(2)} + \cdots . \tag{B.38}$$

The full algebra (B.37) can then be perturbatively expanded in powers of fields.

Global symmetry algebra We are interested in the global symmetry algebra, the subalgebra of the full local gauge algebra satisfying

$$\delta^{(0)} = 0. \tag{B.39}$$

To determine this, it suffices to consider the lowest order:

$$\delta^{(1)}_{\xi_1}\delta^{(0)}_{\xi_2} - \delta^{(1)}_{\xi_2}\delta^{(0)}_{\xi_1} = \delta^{(0)}_{[[\xi_1,\xi_2]]^{(0)}}. \tag{B.40}$$

To summarize, the idea is that once the cubic interaction $S^{(3)}$ is determined, we can deduce the deformation of the gauge symmetry $\delta^{(1)}_{\xi}\varphi$ and the gauge algebra $[[\xi_1,\xi_2]]^{(0)}$:

$$S^{(3)} \implies \delta^{(1)}_{\xi}\varphi \implies [[\xi_1,\xi_2]]^{(0)}, \tag{B.41}$$

which then completely fixes the global symmetry algebra and the invariant bilinear form on the algebra (up to an overall normalization). In Sec.2.2 and 2.3 we see how it works for Yang-Mills and Einstein theories. Following a similar line of reasoning, the global HS algebra and the HS invariant form has been determined in [68] for massless higher spin gauge theories. To correctly apply their results in our setting, we are going to make suitable identifications carefully.

Embedding space formalism

The relevant results in [68] are expressed in the embedding space formalism. The starting point is to realize S^{d+1} as a $(d+1)$-dimensional hypersurface embedded in an ambient Euclidean space

$\mathbb{R}^{d+2}$:

$$X^2 = (X^1)^2 + \cdots + (X^{d+2})^2 = l_{S^{d+1}}^2 \tag{B.42}$$

with $l_{S^{d+1}}$ being the radius of the sphere. Symmetric spin-s fields $\phi_{\mu_1 \cdots \mu_s}(x)$ intrinsic to this submanifold are described by an ambient avatar $\Phi_{I_1 \cdots I_s}(X)$ subject to homogeneity and tangentiality constraints

$$(X \cdot \partial_X - U \cdot \partial_U + 2 + \mu)\Phi(X, U) = 0, \quad X \cdot \partial_U \Phi(X, U) = 0, \tag{B.43}$$

where we have packaged all the $\Phi_{(s)}(X)$ into a generating function

$$\Phi(X, U) = \sum_s \frac{1}{s!} \Phi_{I_1 \cdots I_s}(X) U^{I_1} \cdots U^{I_s} \tag{B.44}$$

with an ambient auxiliary vector U^A. The homogeneity degree μ in (B.43) is related to the mass of the field. The massless case of interest corresponds to $\mu = 0$, in which case we have a gauge symmetry

$$\delta_E \Phi(U) = U \cdot \partial_X E(X, U) + O(\Phi) \tag{B.45}$$

where the field-dependent part is to be determined by the cubic couplings. The gauge parameter[1]

$$E(X, U) = \sum_s \frac{\sqrt{s}}{s!} E_{I_1 \cdots I_{s-1}}(X) U^{I_1} \cdots U^{I_{s-1}} \tag{B.46}$$

satisfies the homogeneity and tangentiality conditions to be consistent with (B.43):

$$(X \cdot \partial_X - U \cdot \partial_U)E = 0, \quad X \cdot \partial_U E = 0. \tag{B.47}$$

[1]There is an extra factor of $\frac{1}{\sqrt{s}}$ compared to [68], so that we can identify $E_{I_1 \cdots I_{s-1}}(X) = \Lambda_{I_1 \cdots I_{s-1}}(X)$, with $\Lambda_{I_1 \cdots I_{s-1}}(X)$ being the embedding space representative of $\Lambda_{\mu_1 \cdots \mu_{s-1}}(x)$.

In this framework, the quadratic action invariant under the linear gauge symmetries (B.45) is given by

$$S^{(2)} \overset{\mathrm{TT}}{=} -\frac{1}{2}\int_{S^{d+1}} e^{\partial_{U_1}\cdot\partial_{U_2}}\Phi(U_1)\partial_X^2\Phi(U_2)\Big|_{U_i=0}, \tag{B.48}$$

where the notation $\overset{\mathrm{TT}}{=}$ means equivalence up to trace and divergence terms. and we are going to construct the cubic vertices following the program described in App.B.3. Note that this normalization is equivalent to choosing

$$\mathrm{g}_s^2 = s! \tag{B.49}$$

in (2.109).

Killing tensors and global HS algebra

Killing Tensors Global HS symmetries are generated by traceless gauge parameters satisfying the Killing equation

$$U\cdot\partial_X\bar{E}(X,U) = 0, \quad \partial_U^2\bar{E}(X,U) = 0. \tag{B.50}$$

Together with the homogeneity and tangentiality conditions (B.47) on the gauge parameter, one can also conclude that the Killing tensors satisfy

$$\partial_U\cdot\partial_X\bar{E}(X,U) = 0, \quad \partial_X^2\bar{E}(X,U) = 0. \tag{B.51}$$

It is straightforward to write down the general solution to these equations:

$$\begin{gathered}\bar{E}(X,U) = \sum_r \frac{1}{\sqrt{r+1}}\bar{E}^{(r+1)}(X,U)\\ \bar{E}^{(r+1)}(X,U) = \frac{1}{r!}\bar{E}^{(r+1)}_{I_1\cdots I_r}(X)U^{I_1}\cdots U^{I_r} = \frac{1}{(r!)^2}\bar{E}_{I_1J_1,\cdots,I_rJ_r}X^{[I_1}U^{J_1]}\cdots X^{[I_r}U^{J_r]}.\end{gathered} \tag{B.52}$$

The HS generators are the duals of the parameters $\bar{E}_{I_1 J_1, \cdots, I_r J_r}$, which due to the complete tracelessness of the latter are defined as equivalence classes

$$T^{I_1 \cdots I_r, J_1 \cdots J_r} = X^{[I_1} U^{J_1]} \cdots X^{[I_r} U^{J_r]} + \cdots \tag{B.53}$$

modulo trace terms $X^2, X \cdot U, U^2$ denoted by $\cdots$ in the equation.

Global HS algebra Following the framework described in sec.B.3, one can determine the full global HS algebra. What is most relevant to us is the bracket for the spin-2 generators (i.e. Killing vectors), which generate the isometry subalgebra $so(d+2)$:[2]

$$[[\bar{E}^{(2)}, \bar{E}'^{(2)}]] = -\frac{g}{\sqrt{2}}(\bar{E}^I \partial_I \bar{E}'_J - \bar{E}'^I \partial_I \bar{E}_J)U^J, \tag{B.54}$$

where g is the coupling constant of the theory, which can be identified with the Newton's constant through

$$g^2 = 32\pi G_N. \tag{B.55}$$

To obtain (B.54), one can recall the footnote around (2.79), and note that there is an extra factor of $\frac{1}{\sqrt{2}}$ because of the non-canonical normalization due to the identification (B.49). Canonical generators are those satisfying the standard $so(d+2)$ commutation relation under the bracket (B.54):

$$[[M_{IJ}, M_{KL}]] = \eta_{JK} M_{IL} - \eta_{JL} M_{IK} + \eta_{IL} M_{JK} - \eta_{IK} M_{JL}. \tag{B.56}$$

One such basis is $M_{IJ} = -\frac{\sqrt{2}}{g}(X_I U_J - X_J U_I)$ with $I, J = 1, \cdots, d+2$, with which we will fix the overall normalization of the canonical metric. In general, the higher spin commutators mix Killing tensors with different spins. For example, a commutator of two spin-3 generators is a

[2] We omit the superscript (0) because it is the complete bracket for the global $so(d+2)$ algebra.

linear combination of a spin-2 and a spin-4 generator

$$[[\bar{E}^{(3)}, \bar{E}'^{(3)}]] \sim \bar{E}^{(2)} + \bar{E}^{(4)}. \tag{B.57}$$

Fortunately, upon the identifications (B.49), the HS invariant bilinear form obtained in [68] is uniquely related to our path integral metric (up to an overall normalization), and therefore the knowledge of the brackets for all higher spin generators is not needed.

HS invariant bilinear form

The HS bilinear form takes the general form[3]

$$\begin{aligned}\langle \bar{E}_1 | \bar{E}_2 \rangle &= \sum_s \frac{b_s}{s} \frac{(\partial_{U_1} \cdot \partial_{U_2})^{s-1}}{(s-1)!} \frac{(\partial_{X_1} \cdot \partial_{X_2})^{s-1}}{(s-1)!} \bar{E}_1(X_1, U_1) \bar{E}_2(X_2, U_2) \Big|_{X_i = U_i = 0} \\ &= \sum_s \frac{b_s}{s!} \frac{(\partial_{X_1} \cdot \partial_{X_2})^{s-1}}{(s-1)!} \bar{E}_1^{(s)}(X_1) \bar{E}_2^{(s)}(X_2) \Big|_{X_i = 0}\end{aligned} \tag{B.58}$$

where the constant b_s is fixed by requiring the cyclic property

$$\langle \bar{E}_1 | [[\bar{E}_2, \bar{E}_3]] \rangle = \langle \bar{E}_2 | [[\bar{E}_3, \bar{E}_1]] \rangle = \langle \bar{E}_3 | [[\bar{E}_1, \bar{E}_2]] \rangle . \tag{B.59}$$

For AdS_{d+1} it was determined to be [68][4]

$$b_s^{AdS_{d+1}} = \frac{b_2^{AdS_{d+1}} (-l_{AdS}^2)^{s-2} \Gamma(\frac{d}{2})}{2^{s-2} \Gamma(\frac{d}{2} + s - 2)} = \frac{b_2^{AdS_{d+1}} (-l_{AdS}^2)^{s-2}}{d(d+2) \cdots (d+2s-8)(d+2s-6)} \tag{B.60}$$

where $b_2^{AdS_{d+1}}$ is an overall s-independent normalization constant and we have restored the AdS length l_{AdS}. Wick rotating this to S^{d+1} mounts to replacing $l_{AdS} = i l_{S^{d+1}}$ and thus

$$b_s^{S^{d+1}} = \frac{b_2^{S^{d+1}} (l_{S^{d+1}})^{s-2} \Gamma(\frac{d}{2})}{2^{s-2} \Gamma(\frac{d}{2} + s - 2)} = \frac{b_2^{S^{d+1}} (l_{S^{d+1}}^2)^{s-2}}{d(d+2) \cdots (d+2s-8)(d+2s-6)}. \tag{B.61}$$

[3]The factor of $\frac{1}{\sqrt{s}}$ came from (B.46).

[4]As noted in [73], we have corrected what we believe to be a typo in [68].

From now on we set $l_{S^{d+1}} = 1$.

Relation to path integral metric In the current notations, the bilinear form for a particular spin induced by the path integral measure is simply

$$\langle \bar{E}_1^{(s)} | \bar{E}_2^{(s)} \rangle_{\mathrm{PI}} = \frac{(s-1)!}{2\pi g_s^2} \int_{S^{d+1}} \bar{E}_1^{(s)}(X, \partial_U) \bar{E}_2^{(s)}(X, U). \tag{B.62}$$

The HS invariant form (B.58) is a linear combination of these

$$\langle \bar{E}_1 | \bar{E}_2 \rangle = \sum_s B_s \, \langle \bar{E}_1^{(s)} | \bar{E}_2^{(s)} \rangle_{\mathrm{PI}} \, . \tag{B.63}$$

We want to determine the s-dependence of the coefficient B_s. To that end we note that the contraction in (B.58) can be written as[5]

$$\frac{(\partial_{X_1} \cdot \partial_{X_2})^{s-1}}{(s-1)!} \bar{E}_1^{(s)}(X_1) \bar{E}_2^{(s)}(X_2) \bigg|_{X_i = 0} = \frac{\int_{\mathbb{R}^{d+2}} e^{-X^2/2} \bar{E}_1^{(s)}(X, \partial_U) \bar{E}_2^{(s)}(X, U)}{\int_{\mathbb{R}^{d+2}} e^{-X^2/2}} . \tag{B.64}$$

Computing the integral on the right hand side in the radial coordinates, we have

$$\int_{\mathbb{R}^{d+2}} e^{-X^2/2} \bar{E}_1^{(s)}(X, \partial_U) \bar{E}_2^{(s)}(X, U) = 2^{s + \frac{d}{2} - 1} \Gamma\left(\frac{d}{2} + s\right) \int_{S^{d+1}} \bar{E}_1^{(s)}(X, \partial_U) \bar{E}_2^{(s)}(X, U) \tag{B.65}$$

Now, with the identification (B.49), comparing (B.58) with (B.63), we conclude

$$B_s \propto (d + 2s - 2)(d + 2s - 4) \tag{B.66}$$

up to a s-independent overall normalization constant.

[5]To see this, note that

$$\int_{\mathbb{R}^{d+2}} e^{-X^2/2} X_{I_1} \cdots X_{I_{s-1}} X_{J_1} \cdots X_{J_{s-1}} = \frac{(2\pi)^{\frac{d+2}{2}}}{2^{s-1}(s-1)!} \Big(\delta_{I_1 J_1} \cdots \delta_{I_{s-1} J_{s-1}} + \text{perm} \Big).$$

Here "perm" includes all permutations among $\{I_1, J_1, I_2, J_2, \cdots, I_{s-1}, J_{s-1}\}$. In particular, it includes terms like $\delta_{I_1 I_2} \cdots$, which do not contribute in the inner product (B.64) since $\bar{E}_1$ and $\bar{E}_2$ are traceless. Therefore, among the $(2s-2)!$ permutations, only $2^{s-1}((s-1)!)^2$ of them gives non-zero contributions in (B.64).

Canonical isometry generators What we have so far is the HS invariant bilinear form up to an overall normalization factor

$$\langle \bar{E}_1 | \bar{E}_2 \rangle_{\text{can}} = C \sum_s (d+2s-2)(d+2s-4) \, \langle \bar{E}_1^{(s)} | \bar{E}_2^{(s)} \rangle_{\text{PI}} \, . \tag{B.67}$$

We fix C by requiring the canonical isometry generators $M_{IJ} = -\frac{\sqrt{2}}{\mathrm{g}}(X_I U_J - X_J U_I)$ to be unit-normalized. Evaluating

$$1 = \langle M_{12} | M_{12} \rangle_{\text{can}} = 2Cd(d+2) \, \langle M_{12} | M_{12} \rangle_{\text{PI}} = \frac{4C}{\mathrm{g}^2} \text{Vol}(S^{d-1}), \tag{B.68}$$

we fix

$$C = \frac{8\pi G_N}{\text{Vol}(S^{d-1})} \tag{B.69}$$

upon the identification (B.55). To conclude, we have found

$$\langle \bar{E}_1 | \bar{E}_2 \rangle_{\text{can}} = \frac{8\pi G_N}{\text{Vol}(S^{d-1})} \sum_s (d+2s-2)(d+2s-4) \, \langle \bar{E}_1^{(s)} | \bar{E}_2^{(s)} \rangle_{\text{PI}} \, , \tag{B.70}$$

which leads to the relation (2.168).

Appendix C: Appendix for chapter 3

C.1 Harish-Chandra characters

C.1.1 Definition of χ

A central ingredient in this work is the Harish-Chandra group character for unitary representations R of Lie groups G,

$$\tilde{\chi}_R(g) \equiv \mathrm{tr}\, R(g)\,, \qquad g \in G\,. \tag{C.1}$$

More rigorously this should be viewed as a distribution to be integrated against smooth test functions $f(g)$ on G. The smeared operators $\int [dg]\, f(g) R(g)$ are trace-class operators, and $\tilde{\chi}_R(g)$ is always a locally integrable function on G, analytic away from its poles [74, 75].

The group of interest to us is $SO(1, d+1)$, the isometry group of global dS_{d+1}, generated by M_{IJ} as defined under (C.95). The representations of $SO(1, d+1)$ were classified and their characters explicitly computed in [76–78]. For a recent review and an extensive dictionary between fields and representations, see [101].

For our purposes in this work we only need to consider characters restricted to group elements of the form $g = e^{-itH}$, where $H = M_{0,d+1}$ generates global $SO(1, 1)$ transformations acting as time translations $T \to T + t$ on the southern static patch (fig. 3.5):

$$\chi(t) \equiv \mathrm{tr}\, e^{-itH}\,. \tag{C.2}$$

For example for a spin-0 UIR corresponding to a scalar field of mass $m^2 = \Delta(d - \Delta)$, as we will explicitly compute below, this takes the form

$$\chi(t) = \operatorname{tr} e^{-itH} = \frac{e^{-t\Delta} + e^{-t\bar{\Delta}}}{|1-e^{-t}|^d}\,, \qquad \bar{\Delta} \equiv d - \Delta\,. \tag{C.3}$$

Putting $\Delta = \frac{d}{2} + i\nu$, we get $m^2 = (\frac{d}{2})^2 + \nu^2$, so $m^2 > 0$ if $\nu \in \mathbb{R}$ (principal series) or $\nu = i\mu$ with $|\mu| < \frac{d}{2}$ (complementary series). Since $\bar{\Delta} = d - \Delta = \frac{d}{2} - i\nu$, this implies $\chi(t) = \chi(t)^*$, as follows more generally from $H^\dagger = H$. The absolute value signs moreover ensure $\chi(t) = \chi(-t)$ for all d. The latter property holds for any $SO(1, d+1)$ representation:

$$\chi(-t) = \chi(t)\,. \tag{C.4}$$

This follows from the fact that the $SO(1,1)$ boost generator $H = M_{0,d+1}$ can be conjugated to a boost $-H$ in the opposite direction by a 180-degree rotation: $-H = uHu^{-1}$ for e.g. $u = e^{i\pi M_{d,d+1}}$, implying $\chi(-t) = \operatorname{tr} e^{iHt} = \operatorname{tr} u\, e^{-iHt} u^{-1} = \operatorname{tr} e^{-iHt} = \chi(t)$.

C.1.2 Computation of χ

Here we show how characters $\chi(t) = \operatorname{tr} e^{-itH}$ can be computed by elementary means. The full characters $\chi(t, \phi) = \operatorname{tr} e^{-itH + i\phi\cdot J}$ can be computed similarly, but we will focus on the former.

Simplest example: $d = 1$, $s = 0$

We first consider a $d = 1$, spin-0 principal series representation with $\Delta = \frac{1}{2} + i\nu$, $\nu \in \mathbb{R}$. This corresponds to a massive scalar field on dS_2 with mass $m^2 = \frac{1}{4} + \nu^2$. This unitary irreducible representation of $SO(1,2)$ can be realized on the Hilbert space of square-integrable wave functions $\psi(\varphi)$ on S^1, with standard inner product. The circle can be thought of as the future conformal boundary of global dS_2 in global coordinates (cf. (C.96)), which for dS_2 becomes $ds^2 = (\cos\vartheta)^{-2}(-d\vartheta^2 + d\varphi^2)$. Kets $|\varphi\rangle$ can be thought of as states produced by a boundary conformal field[1] $\mathcal{O}(\varphi)$ of dimension $\Delta = \frac{1}{2} + i\nu$ acting on an $SO(1,2)$-invariant global vacuum state $|\text{vac}\rangle$ such as the global Euclidean

[1] $\mathcal{O}(\varphi)$ arises from the bulk scalar $\phi(\vartheta, \varphi)$ as $\phi(\frac{\pi}{2} - \epsilon, \varphi) \sim \mathcal{O}(\varphi)\, \epsilon^{\Delta} + \bar{\mathcal{O}}(\varphi)\, \epsilon^{\bar{\Delta}}$ in the infinite future $\epsilon \to 0$

vacuum:

$$|\varphi\rangle \equiv \mathcal{O}(\varphi)|\text{vac}\rangle\,, \qquad \langle\varphi|\varphi'\rangle = \delta(\varphi-\varphi')\,. \tag{C.5}$$

This pairing is $SO(1,d+1)$ invariant. Normalizable states $|\psi\rangle$ are then superpositions

$$|\psi\rangle = \int_{-\pi}^{\pi} d\varphi\,\psi(\varphi)\,|\varphi\rangle\,, \qquad \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} d\varphi\,|\psi(\varphi)|^2 < \infty\,. \tag{C.6}$$

In conventions in which H, P and K are hermitian, the Lie algebra of so$(1,2)$ is

$$[H,P] = iP\,, \qquad [H,K] = -iK\,, \qquad [K,P] = 2iH\,, \tag{C.7}$$

the action of these generators on kets $|\varphi\rangle$ in the above representation is

$$\begin{aligned} H|\varphi\rangle &= i\big(\sin\varphi\,\partial_\varphi + \Delta\cos\varphi\big)|\varphi\rangle \\ P|\varphi\rangle &= i\big((1+\cos\varphi)\partial_\varphi - \Delta\sin\varphi\big)|\varphi\rangle \\ K|\varphi\rangle &= i\big((1-\cos\varphi)\partial_\varphi + \Delta\sin\varphi\big)|\varphi\rangle\,. \end{aligned} \tag{C.8}$$

Note that his implies that the action of for example H on wave functions $\psi(\varphi)$ is given by $H|\psi\rangle = \int d\varphi\,\mathcal{H}\psi(\varphi)\,|\varphi\rangle$ where $\mathcal{H}\psi(\varphi) = -i\big(\sin\varphi\,\partial_\varphi + \bar\Delta\cos\varphi\big)\psi(\varphi)$, with $\bar\Delta = 1-\Delta = \frac{1}{2} - i\nu$. One gets simpler expressions after conformally mapping this to planar boundary coordinates $x = \tan\frac{\varphi}{2}$, that is to say changing basis from $|\varphi\rangle_{S^1}$ to $|x\rangle_{\mathbb{R}}$, $x \in \mathbb{R}$, where

$$|x\rangle_{\mathbb{R}} \equiv \big(\tfrac{\partial\varphi}{\partial x}\big)^{\Delta}\big|\varphi(x)\big\rangle_{S^1} = \big(\tfrac{2}{1+x^2}\big)^{\Delta}\big|2\arctan x\big\rangle_{S^1}\,, \qquad \langle x|x'\rangle = \delta(x-x')\,. \tag{C.9}$$

Then H, P, K take the familiar planar dilatation, translation and special conformal form:

$$H|x\rangle = i(x\partial_x + \Delta)|x\rangle \qquad P|x\rangle = i\partial_x|x\rangle\,, \qquad K|x\rangle = i(x^2\partial_x + 2\Delta x)|x\rangle\,. \tag{C.10}$$

In particular this makes exponentiation of H easy:

$$e^{-itH}|x\rangle = e^{t\Delta}|e^t x\rangle\,. \tag{C.11}$$

However one has to keep in mind that planar coordinates miss a point of the global boundary, here the point $\varphi = \pi$. This will actually turn out to be important in the computation of the character. Let us first ignore this though and compute

$$\chi(t)|_{\text{planar}} = \int dx\,\langle x|e^{-itH}|x\rangle = e^{t\Delta}\int dx\,\delta\big(x - e^t x\big) = e^{t\Delta}\int dx\,\frac{\delta(x)}{|1-e^t|} = \frac{e^{t\Delta}}{|1-e^t|}\,.$$

We see that the computation localizes at the point $x = 0$, singled out because it is a fixed point of H. Actually there is another fixed point, which we missed here because it is exactly the point at infinity in planar coordinates. This is clear from the global version (C.8): one fixed point of H is at $\varphi = 0$, which maps to $x = 0$ and was picked up in the above computation, while the other fixed point is at $\varphi = \pi$, which maps to $x = \infty$ and so was missed.

This is easily remedied though. The most straightforward way is to repeat the computation in the global boundary basis $|\varphi\rangle$, which is sure not to miss any fixed points. It suffices to consider an infinitesimally small neighborhood of the fixed points. For $\varphi = y \to 0$, we get $H \approx i(y\partial_y + \Delta)$, which coincides with the planar expression, while for $\varphi = \pi + y$ with $y \to 0$, we get $H \approx -i(y\partial_y + \Delta)$, which is the same except with the opposite sign. Thus we obtain

$$\chi(t) = \int d\varphi\,\langle\varphi|e^{-itH}|\varphi\rangle = \frac{e^{t\Delta}}{|1-e^t|} + \frac{e^{-t\Delta}}{|1-e^{-t}|} = \frac{e^{-t\Delta} + e^{-t\bar{\Delta}}}{|1-e^{-t}|}\,, \tag{C.12}$$

where $\bar{\Delta} = 1 - \Delta = \frac{1}{2} - i\nu$, reproducing (C.3) for $d = 1$.

For the complementary series $0 < \Delta < 1$, we have $\Delta^* = \Delta$ instead of $\Delta^* = \bar{\Delta} \equiv 1 - \Delta$, so the conjugation properties of H, D, K are different. As a result they are no longer hermitian with respect to the inner product (C.5), but rather with respect to $\langle\varphi|\varphi'\rangle \propto \big(1-\cos(\varphi-\varphi')\big)^{-\Delta}$. However we can now define a "shadow" bra $(\varphi| \propto \int d\varphi'\big(1-\cos(\varphi-\varphi')\big)^{-\bar{\Delta}}\langle\varphi'|$ satisfying $(\varphi|\varphi'\rangle = \delta(\varphi-\varphi')$

and compute the trace as $\chi(t) = \int d\varphi \, (\varphi|e^{-itH}|\varphi\rangle$. The computation then proceeds in exactly the same way, with the same result (C.12).

General dimension and spin

The generalization to $d > 1$ is straightforward. Again the trace only picks up contributions from fixed points of H. The fixed point at the origin in planar coordinates contributes $e^{t\Delta} \int d^d x \, \delta^d(x - e^t x) = \frac{e^{t\Delta}}{|1-e^t|^d}$, while the fixed point at the other pole of the global boundary sphere gives a contribution of the same form but with $t \to -t$. Together we get

$$\chi_{0,\Delta}(t) = \frac{e^{-t\Delta} + e^{-t\bar{\Delta}}}{|1 - e^{-t}|^d}, \tag{C.13}$$

where $\bar{\Delta} = d - \Delta$.

For massive spin-s representations, the basis merely gets some additional $SO(d)$ spin labels, and the trace picks up a corresponding degeneracy factor, so[2]

$$\chi_{s,\Delta}(t) = D_s^d \, \frac{e^{-t\Delta} + e^{-t\bar{\Delta}}}{|1 - e^{-t}|^d}, \tag{C.14}$$

where $\bar{\Delta} = d - \Delta$ as before, and D_s^d is the spin degeneracy, for example $D_s^3 = 2s+1$. More generally for $d > 2$ it is the number of totally symmetric traceless tensors of rank s:

$$D_s^d = \binom{s+d-1}{d-1} - \binom{s+d-3}{d-1} \tag{C.15}$$

(For $d = 2$ we get spin $\pm s$ irreps of $SO(2)$ with $D_{\pm s}^2 = 1$. However both of these appear when quantizing a Fierz-Pauli spin-s field.) Explicit low-d spin-s degeneracies are listed in (C.89).

The most general massive unitary representation of $SO(1, d+1)$ is labeled by an irrep $S = (s_1, \ldots, s_r)$ of $SO(d)$ (cf. appendix C.4.1) and $\Delta = \frac{d}{2} + i\nu$, $\nu \in \mathbb{R}$ (principal series) or $\Delta = \frac{d}{2} + \mu$, $|\mu| <$

[2] Here $\Delta = \frac{d}{2} + i\nu$ with either $\nu \in \mathbb{R}$ (principal series) or $\nu = i\mu$ with $|\mu| < \frac{d}{2}$ for $s = 0$ and $|\mu| < \frac{d}{2} - 1$ for $s \geq 1$ (complementary series). For $s = 0$ the mass is $m^2 = (\frac{d}{2})^2 + \nu^2 = \Delta(d - \Delta)$ while for $s \geq 1$ it is given by (3.79): $m^2 = (\frac{d}{2} + s - 2)^2 + \nu^2 = (\Delta + s - 2)(d - \Delta + s - 2)$.

$\mu_{\max}(S) \leq \frac{d}{2}$ (complementary series) [76–78, 101]. The spin-s case discussed above corresponds to $S = (s_1, 0, \ldots, 0)$. The character for general S is

$$\chi_{S,\Delta}(t) = D_S^d \, \frac{e^{-t\Delta} + e^{-t\bar{\Delta}}}{|1 - e^{-t}|^d} \,, \tag{C.16}$$

where the generalized spin degeneracy factor D_S^d is the dimension of the $SO(d)$ irrep S, explicitly given for general S in appendix C.4.1.

Massless and partially massless representations

(Partially) massless representations correspond to higher-spin gauge fields and are in the exceptional or discrete series. These representations and their characters $\chi(t)$ are considerably more intricate. We give the general expression and examples in appendix C.7.1 for the massless case. Guided by our path integral results of section 3.5, we are led to a simple recipe for constructing these characters from their much simpler "naive" counterparts, spelled out in (3.100). This generalizes straightforwardly to the partially massless case, leading to the explicit general-d formula (C.194).

C.1.3 Importance of picking a globally regular basis

Naive evaluation of the character trace $\chi(t) = \operatorname{tr} e^{-itH}$ by diagonalization of H results in nonsense. In this section we explain why: emphasizing the importance of using a basis on which finite $SO(1, d+1)$ transformations act in a globally regular way.

Failure of computation by diagonalization of H

Naively, one might have thought the easiest way to compute $\chi(t) = \operatorname{tr} e^{-itH}$ would be to diagonalize H and sum over its eigenstates. The latter are given by $|\omega\sigma\rangle$, where $H = \omega \in \mathbb{R}$, σ labels $SO(d)$ angular momentum quantum numbers, and $\langle \omega\sigma | \omega'\sigma' \rangle = \delta(\omega - \omega')\, \delta_{\sigma\sigma'}$. However this produces a nonsensical result,

$$\chi(t) = \operatorname{tr} e^{-itH} \stackrel{\text{naive}}{=} \int d\omega \sum_\sigma \langle \omega\sigma | e^{-itH} | \omega\sigma \rangle = 2\pi \sum_\sigma \delta(0)\,\delta(t) \qquad \text{(naive)}\,, \tag{C.17}$$

not even remotely resembling the correct $\chi(t)$ as computed earlier in C.1.2.

Our method of computation there also illuminates why this naive computation fails. To make this concrete, let us go back to the $d = 1$ scalar example with $\Delta = \frac{1}{2} + i\nu$, $\nu \in \mathbb{R}$. Recalling the action of H on wave functions $\psi(\varphi)$ mentioned below (C.8), it is straightforward to find the wave functions $\psi_{\omega\sigma}(\varphi)$ of the eigenkets $|\omega\sigma\rangle = \int_{-\pi}^{\pi} d\varphi\, \psi_{\omega\sigma}(\varphi)\,|\varphi\rangle$ of H:

$$\psi_{\omega\sigma}(\varphi) = \frac{\Theta(\sigma \sin\varphi)}{\sqrt{2\pi}}\,|\sin\varphi|^{-\bar{\Delta}} \left|\tan\frac{\varphi}{2}\right|^{i\omega}, \qquad \omega \in \mathbb{R}, \quad \sigma = \pm 1\,. \tag{C.18}$$

where Θ is the step function. Alternatively we can first conformally map S^1 to the "cylinder" $\mathbb{R} \times S^{d-1} = \mathbb{R} \times S^0$ parametrized by (T, Ω), $T \in \mathbb{R}$, $\Omega \in \{-1, +1\} = S^0$, that is to say change basis $|\varphi\rangle_{S^1} \to |T\Omega\rangle_{\mathbb{R}\times S^0}$.[3] Then H generates translations of T, so the wave functions of $|\omega\sigma\rangle$ in this basis are simply

$$\psi_{\omega\sigma}(T, \Omega) = \frac{1}{\sqrt{2\pi}}\,\delta_{\Omega,\sigma}\, e^{i\omega T}\,. \tag{C.19}$$

The cylinder is the conformal boundary of the future wedge, F in fig. C.5 (which actually splits in two wedges at $\Omega = \pm 1$ in the case of dS_2), and the $|\omega\pm\rangle$ are the states obtained by the usual free field quantization corresponding to the natural modes $\phi_{\omega\pm}(T, r)$ in this patch.

It is now clear why the naive computation (C.17) of $\chi(t)$ in the basis $|\omega\sigma\rangle$ fails to produce the correct result: the wave functions $\psi_{\omega\sigma}(\varphi)$ are singular precisely at the fixed points $\varphi = 0, \pi$ of H (top corners of Penrose diagram in fig. 3.5), which are exactly the points at which the character trace computation of section C.1.2 localizes. Closely related failure would be met in the basis $|T\Omega\rangle$: H acts as by translating T, seemingly without fixed points, oblivious to their actual presence

[3] Explicitly $T = \log|\tan\frac{\varphi}{2}|$, $\Omega = \operatorname{sign}\varphi$, which analogously to the global $\to$ planar map (C.9) yields $\big|T\pm\big\rangle_{\mathbb{R}\times S^0} = (\cosh T)^{-\Delta}\big|\pm 2\arctan e^T\big\rangle_{S^1}$, satisfying $\langle T\Omega|T'\Omega'\rangle = \delta(T - T')\,\delta_{\Omega\Omega'}$ and $H|T\Omega\rangle = i\partial_T|T\Omega\rangle$.

at $T = \pm\infty$. In other words, despite their lure as being the bases in which the action of H is maximally simple, $|T\Omega\rangle$ or its Fourier dual $|\omega\sigma\rangle$ are in fact the worst possible choice one could make to compute the trace.

Similar observations hold for in higher dimensions. The wave functions diagonalizing H take the form $\psi_{\omega\sigma}(T,\Omega) \propto e^{i\omega T} Y_\sigma(\Omega)$ in $\mathbb{R} \times S^{d-1}$ cylinder coordinates. Transformed to global S^d coordinates, these are singular precisely at the fixed points of H, excluded from the cylinder, making this frame particularly ill-suited for computing $\operatorname{tr} e^{-itH}$.

Globally regular bases

More generally, to ensure correct computation of the full Harish-Chandra group character $\chi_R(g) = \operatorname{tr} R(g)$, $g \in SO(1,d+1)$, we must use a basis on which finite $SO(1,d+1)$ transformations g act in a globally nonsingular way. This is the case for a global dS_{d+1} boundary basis $|\bar\Omega\rangle_{S^d}$, $\bar\Omega \in S^d$, generalizing the $d = 1$ global S^1 basis $|\varphi\rangle_{S^1}$, but not for a planar basis $|x\rangle_{\mathbb{R}^d}$ or a cylinder basis $|T\Omega\rangle_{\mathbb{R}\times S^{d-1}}$. Indeed generic $SO(d+1)$ rotations of the global S^d move the poles of the sphere, thus mapping finite points to infinity in planar or cylindrical coordinates. This singular behavior is inherited by the corresponding Fourier dual bases $|p\rangle \propto \int d^d x\, e^{ipx} |x\rangle$ and $|\omega\sigma\rangle \propto \int dT\, d\Omega\, e^{i\omega T} Y_\sigma(\Omega)\, |T\Omega\rangle$. From a bulk point of view these are the states obtained by standard mode quantization in the planar patch resp. future wedge. The singular behavior is evident here from the fact that these patches have horizons that are moved around by global $SO(d+1)$ rotations. Naively computing $\chi(g)$ in these frames will in general give incorrect results. More precisely the result will be wrong unless the fixed points of g lie at finite position on the corresponding conformal boundary patch.

On the other hand the normalizable dual basis $|\bar\sigma\rangle = \int d\bar\Omega\, Y_{\bar\sigma}(\bar\Omega)\, |\bar\Omega\rangle$ inherits the global regularity of $|\bar\Omega\rangle_{S^d}$. Here $Y_{\bar\sigma}(\Omega)$ is a basis of spherical harmonics on S^d, with $\bar\sigma$ labeling the global $SO(d+1)$ angular momentum quantum numbers, and $\langle\bar\sigma|\bar\sigma'\rangle = \delta_{\bar\sigma\bar\sigma'}$. (From the bulk point of view this is essentially the basis obtained by quantizing the natural mode functions of the global dS_{d+1} metric in table C.96.) Although in practice much harder than computing $\chi(t) = \int d\bar\Omega\, \langle\bar\Omega|e^{-iHt}|\bar\Omega\rangle$

as in section C.1.2, computing

$$\chi(t) = \operatorname{tr} e^{-itH} = \sum_{\bar{\sigma}} \langle \bar{\sigma} | e^{-itH} | \bar{\sigma} \rangle \tag{C.20}$$

gives in principle the correct result. Note that this suggests a natural UV regularization of $\chi(t)$ for $t \to 0$, by cutting off the global $SO(d+1)$ angular momentum. For example for a scalar on dS_3 with $SO(3)$ angular momentum cutoff L, this would be

$$\chi_L(t) \equiv \sum_{\ell=0}^{L} \langle \ell m | e^{-itH} | \ell m \rangle \,. \tag{C.21}$$

C.2 Density of states and quasinormal mode resonances

The review in appendix C.1 focuses mostly on mathematical and computational aspects of the Harish-Chandra character $\chi(t) = \operatorname{tr} e^{-itH}$. Here we focus on its physics interpretation, in particular the density of states $\rho(\omega)$ obtained as its Fourier transform. We define this in a general and manifestly covariant way using Pauli-Villars regularization in section 3.2. Here we will not be particularly concerned with general definitions or manifest covariance, taking a more pedestrian approach. At the end we briefly comment on an "S-matrix" interpretation and a possible generalization of the formalism including interactions.

In C.2.1, we contrast the spectral features encoded in the characters of unitary representations of the $\mathrm{so}(1, d+1)$ isometry algebra of global dS_{d+1} with the perhaps more familiar characters of unitary representations of the $\mathrm{so}(2, d)$ isometry algebra of AdS_{d+1}: in a sentence, the latter encodes bound states, while the former encodes scattering states. In C.2.2 we explicitly compare $\rho(\omega)$ obtained as the Fourier transform of $\chi(t)$ for dS_2 to the coarse-grained eigenvalue density obtained by numerical diagonalization of a model discretized by global angular momentum truncation, and confirm the results match at large N. In C.2.3 we identify the poles of $\rho(\omega)$ in the complex ω plane as scattering resonances/quasinormal modes, counted by the power series expansion of the character. As a corollary this implies the relation $Z_{\mathrm{PI}} = Z_{\mathrm{bulk}}$ of (3.68) can be viewed as a precise version of the formal quasinormal mode expansion of $\log Z_{\rm PI}$ proposed in [142].

C.2.1 Characters and the density of states: dS vs AdS

We begin by highlight some important differences in the spectrum encoded in the characters of unitary $\mathrm{so}(1,d+1)$ representations furnished by global dS_{d+1} single-particle Hilbert spaces and the characters of unitary $\mathrm{so}(2,d)$ representations furnished by global AdS_{d+1} single-particle Hilbert spaces. Although the discussion applies to arbitrary representations, for concreteness we consider the example of a scalar of mass $m^2 = (\frac{d}{2})^2 + \nu^2$ on dS_{d+1}. Its character as computed in (C.13) is

$$\chi_{\rm dS}(t) \equiv \operatorname{tr} e^{-itH} = \frac{e^{-\Delta_+ t} + e^{-\Delta_- t}}{|1-e^{-t}|^d}, \qquad \Delta_\pm = \tfrac{d}{2} \pm i\nu, \qquad t \in \mathbb{R}. \tag{C.22}$$

where tr traces over the *global* single-particle Hilbert space and we recall $H = M_{0,d+1}$ is a global $SO(1,1)$ boost generator, which becomes a spatial momentum operator in the future wedge and the energy operator in the southern static patch (cf. fig. C.5c). This is to be contrasted with the familiar character of the unitary lowest-weight representation of a scalar of mass $m^2 = -(\frac{d}{2})^2 + \mu^2$ on global AdS_{d+1} with standard boundary conditions:

$$\chi_{\rm AdS}(t) \equiv \operatorname{tr} e^{-itH} = \frac{e^{-i\Delta_+ t}}{(1-e^{-it})^d}, \qquad \Delta_+ = \tfrac{d}{2} + \mu, \qquad \operatorname{Im} t < 0\,. \tag{C.23}$$

Here the $\mathrm{so}(2)$ generator H is the energy operator in global AdS_{d+1}. Besides the occurrence of both $\Delta_\pm$ in (C.22), another notable difference is the absence of factors of i in the exponents.

The physics content of $\chi_{\rm AdS}$ is clear: $\chi_{\rm AdS}(-i\beta) = \operatorname{tr} e^{-\beta H}$ is the single-particle partition function at inverse temperature β for a scalar particle trapped in the global AdS gravitational potential well. Equivalently for $\operatorname{Im} t < 0$, the expansion

$$\chi_{\rm AdS}(t) = \sum_\lambda N_\lambda \, e^{-it\lambda}, \qquad \lambda = \Delta_+ + n, \quad n \in \mathbb{N}, \tag{C.24}$$

counts normalizable single-particle states of energy $H = \lambda$, or equivalently global normal modes

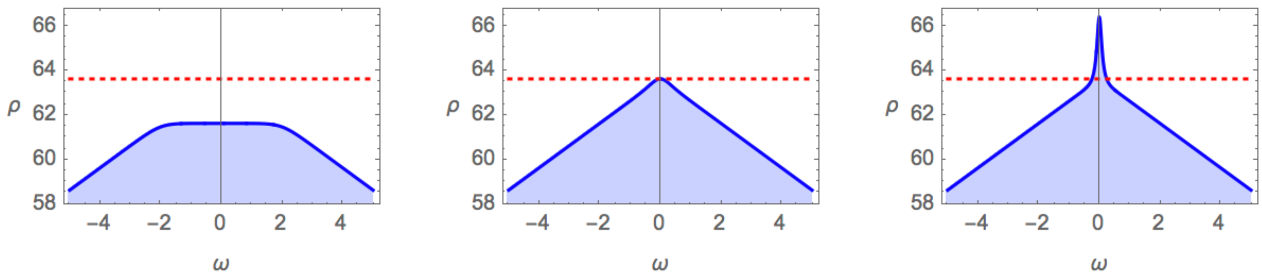


Figure C.1: Density of states $\rho_\Lambda(\omega)$ for dS_3 scalars with $\Delta = 1+2i$, $\Delta = \frac{1}{2}$, $\Delta = \frac{1}{10}$, and UV cutoff $\Lambda = 100$, according to (C.28). The red dotted line represents the term $2\Lambda/\pi$. The peak visible at $\Delta = \frac{1}{10}$ is due to a resonance approaching the real axis, as explained in section C.2.3.

of frequency λ. The corresponding density of single-particle states is

$$\rho_{\rm AdS}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, \chi_{\rm AdS}(t)\, e^{i\omega t} = \sum_\lambda N_\lambda\, \delta(\omega - \lambda)\,. \tag{C.25}$$

For dS, we can likewise expand the character as in (C.24). For $t > 0$,

$$\chi_{\rm dS}(t) = \sum_\lambda N_\lambda\, e^{-it\lambda}\,, \qquad \lambda = -i(\Delta_\pm + n) = -i(\tfrac{d}{2} + n) \pm \nu\,, \quad n \in \mathbb{N}\,. \tag{C.26}$$

However λ is now *complex*, so evidently N_λ does not count physical eigenstates of the hermitian operator H. Rather, as further discussed in section C.2.3, it counts *resonances*, or *quasinormal* modes. The density of *physical* states with $H = \omega \in \mathbb{R}$ is formally given by

$$\rho_{\rm dS}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, \chi_{\rm dS}(t)\, e^{i\omega t} = \int_0^{\infty} \frac{dt}{2\pi}\, \chi_{\rm dS}(t)\big(e^{i\omega t} + e^{-i\omega t}\big)\,, \tag{C.27}$$

where ω can be interpreted as the momentum along the T-direction of the future wedge (F in fig. C.5 and table C.96). Alternatively for $\omega > 0$ it can be interpreted as the energy in the southern static patch, as discussed in section 3.2.2. A manifestly covariant Pauli-Villars regularization of the above integral is given by (3.41). For our purposes here a simple $t > \Lambda^{-1}$ cutoff suffices. For

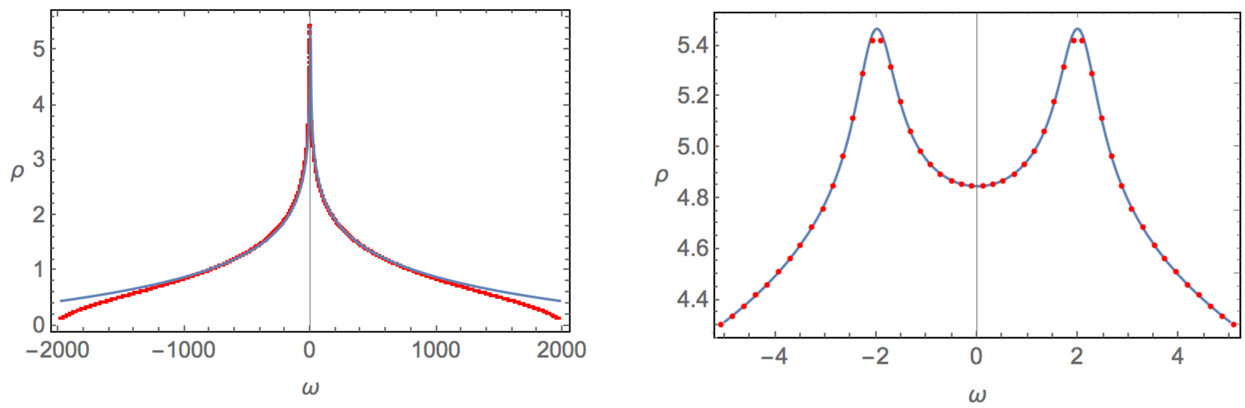


Figure C.2: Density of states for a $\Delta = \frac{1}{2} + i\nu$ scalar with $\nu = 2$ in dS_2. The red dots show the local eigenvalue density $\bar{\rho}_N(\omega)$, (C.31), of the truncated model with global angular momentum cutoff $N = 2000$, obtained by numerical diagonalization. The blue line shows $\rho(\omega)$ obtained as the Fourier transform of $\chi(t)$, explicitly (C.29) with $e^{-\gamma}\Lambda \approx 4000$. The plot on the right zooms in on the IR region. The peaks are due to the proximity of quasinormal mode poles in $\rho(\omega)$, discussed in C.2.3.

example for dS_3,

$$\begin{aligned}\rho_{dS_3,\Lambda}(\omega) &\equiv \int_{\Lambda^{-1}}^{\infty} \frac{dt}{2\pi} \, \frac{e^{-(1+i\nu)t} + e^{-(1-i\nu)t}}{(1-e^{-t})^2} \big(e^{i\omega t} + e^{-i\omega t}\big) \\ &= \frac{2\Lambda}{\pi} - \frac{1}{2} \sum_{\pm} (\omega \pm \nu) \coth\big(\pi(\omega \pm \nu)\big) \, .\end{aligned} \tag{C.28}$$

Some examples are illustrated in fig. C.1. In contrast to AdS, $\rho_{dS}(\omega)$ is continuous. Indeed energy eigenkets $|\omega\sigma\rangle$ of the static patch form a continuum of scattering states, coming out of and going into the horizon, instead of the discrete set of bound states one gets in the global AdS potential well. Note that although the above $\rho_{dS_3,\Lambda}(\omega)$ formally goes negative in the large-ω limit, it is positive within its regime of validity, that is to say for $\omega, \nu \ll \Lambda$.

C.2.2 Coarse-grained density of states in globally truncated model

For a $\Delta = \frac{1}{2} + i\nu$ scalar on dS_2, the density of states regularized by as in (C.28) is

$$\rho(\omega) = \frac{2}{\pi} \log(e^{-\gamma}\Lambda) - \frac{1}{2\pi} \sum_{\pm,\pm} \psi\big(\tfrac{1}{2} \pm i\nu \pm i\omega\big)\big) \, , \tag{C.29}$$

where γ is the Euler constant, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, and the sum is over the four different combinations of signs. To ascertain it makes physical sense to identify this as the density of states, we would like to compare this to a model with discretized spectrum of eigenvalues ω.

An efficient discretization — which does not require solving bulk equations of motion and is quite natural from the point of view of dS-CFT approaches to de Sitter quantum gravity [170, 180, 181] — is obtained by truncating the *global* dS_{d+1} angular momentum $SO(d+1)$ of the single-particle Hilbert space, considering instead of H a *finite*-dimensional matrix

$$h_{\bar{\sigma}\bar{\sigma}'} \equiv \langle \bar{\sigma}|H|\bar{\sigma}'\rangle , \tag{C.30}$$

where $\bar{\sigma}$ are $SO(d+1)$ quantum numbers, as in (C.20). For dS_2 this is $SO(2)$ and $\bar{\sigma} = n \in \mathbb{Z}$, truncated e.g. by $|n| \leq N$. The matrix h is sparse and can be computed either directly using $|n\rangle \propto \int d\varphi\, e^{in\varphi}|\varphi\rangle$ and the explicit form of H given in (C.8), or algebraically.

The algebraic way goes as follows. A normalizable basis $|n\rangle$ of the global dS_2 scalar single-particle Hilbert space can be constructed from the $SO(1,2)$ conformal algebra (C.7), using a basis of generators L_0, $L_\pm$ related to H, K and P as $L_0 = \frac{1}{2}(P+K)$, $L_\pm = \frac{1}{2}(P-K) \pm iH$. Then L_0 is the global angular momentum generator $i\partial_\phi$ along the future boundary S^1 and $L_\pm$ are its raising and lowering operators. In some suitable normalization of the L_0 eigenstates $|n\rangle$, we have $L_0|n\rangle = n|n\rangle$, $L_\pm|n\rangle = (n \pm \Delta)|n \pm 1\rangle$. Cutting off the single-particle Hilbert space at $-N < n \leq N$,[4] the operator $H = \frac{i}{2}(L_- - L_+)$ acts as a sparse $2N \times 2N$ matrix on the truncated basis $|n\rangle$.

A minimally coarse-grained density of states can then be defined as the inverse spacing of its

[4]The asymmetric choice here allows us to use the simple coarse graining prescription (C.31) and keep this discussion short. A symmetric choice $|n| \leq N$ would lead to an enhanced $\mathbb{Z}_2$ and two families of eigenvalues distinguished by their $\mathbb{Z}_2$ parity, inducing persistent microstructure in the level spacing. The most efficient way to proceed then is to compute $\bar{\rho}_{N,\pm}(\omega)$ as the inverse level spacing for these two families separately and then add the contributions together as interpolated functions. For dS_3 with $SO(3)$ cutoff $\ell \leq N$ one similarly gets $2N+1$ families of eigenvalues, labeled by $SO(2)$ angular momentum m, and one can proceed analogously. Alternatively, one can compute $\bar{\rho}_N(\omega)$ directly by binning and counting, but this requires larger N.

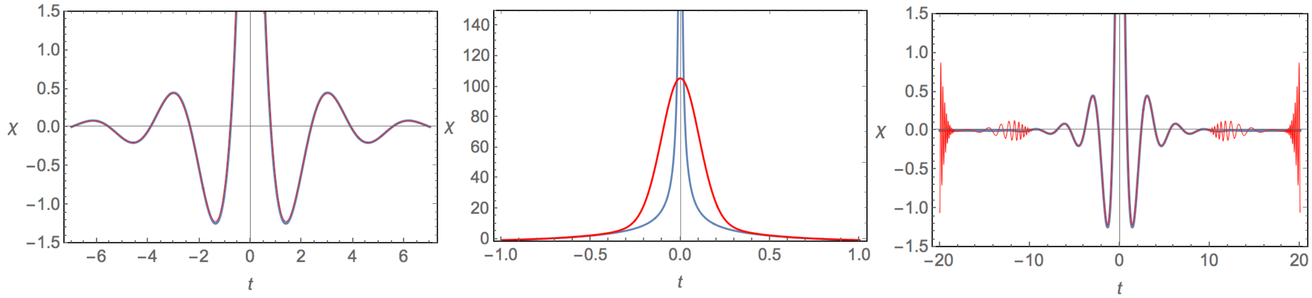


Figure C.3: Comparison of $d = 1$ character $\chi(t)$ defined in (C.22) (blue) to the coarse-grained discretized character $\bar{\chi}_{N,\delta}(t)$ defined in (C.32) (red), with $\delta = 0.1$ and other parameters as in fig. C.2. Plot on the right shows wider range of t. Plot in the middle smaller range of t, but larger χ.

eigenvalues ω_i, $i = 1, \ldots, 2N$, obtained by numerical diagonalization:

$$\bar{\rho}_N(\omega_i) \equiv \frac{2}{\omega_{i+1} - \omega_{i-1}}. \tag{C.31}$$

The continuum limit corresponds to $N \to \infty$ in the discretized model, and to $\Lambda \to \infty$ in (C.29). To compare to (C.29), we adjust Λ, in the spirit of renormalization, to match the density of states at some scale ω, say $\omega = 0$. The results of this comparison for $\nu = 2$, $N = 2000$ are shown in fig. C.2. Clearly they match remarkably well indeed in the regime where they should, i.e. well below the UV cutoff scale.

We can make a similar comparison directly at the (UV-finite) character level. The discrete character is $\sum_i e^{-i\omega_i t}$, which is a wildly oscillating function. At first sight this seems very different from the character $\chi(t) = \operatorname{tr} e^{-iHt}$ in (C.27). However to properly compare the two, we should coarse grain this at a small but finite resolution δ. We do this by convolution with a Gaussian kernel, that is to say we consider

$$\bar{\chi}_{N,\delta}(t) \equiv \frac{1}{\sqrt{2\pi}\delta} \int_{-\infty}^{\infty} dt' \, e^{-(t-t')^2/2\delta^2} \sum_i e^{-i\omega_i t'} = \sum_i e^{-it\omega_i - \delta^2 \omega_i^2/2} . \tag{C.32}$$

A comparison of $\bar{\chi}_{N,\delta}$ to χ is shown in fig. C.3 for $\delta = 0.1$. The match is nearly perfect for $|t|$ not too large and not too small. For small t, the $\bar{\chi}_{N,\delta}(t)$ caps off at a finite value, the number

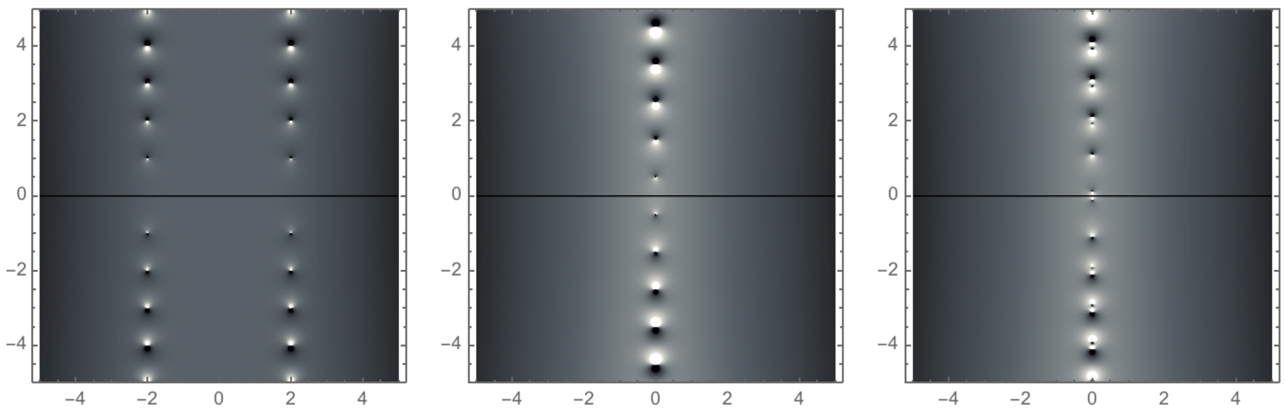


Figure C.4: Plot of $|\rho(\omega)|$ in complex ω-plane corresponding to the dS_3 examples of fig. C.1, that is $\Delta_\pm = \{1+2i, 1-2i\}$, $\{\frac{1}{2}, \frac{3}{2}\}$, $\{0.1, 1.9\}$, and $2\Lambda/\pi \approx 64$. Lighter is larger with plot range 58 (black) $< |\rho| <$ 67 (white). Resonance poles are visible at $\omega = \mp i(\Delta_\pm + n)$, $n \in \mathbb{N}$.

of eigenvalues $|\omega_i| \lesssim 1/\delta$, while $\chi(t) \sim 1/|t| \to \infty$. The approximation gets better here when δ is made smaller. For larger values of t, $\bar{\chi}_{N,\delta}(t)$ starts showing some oscillations again. These can be eliminated by increasing δ, at the cost of accuracy at smaller t. In the $N \to \infty$ limit, the discretized approximation gets increasingly better over increasingly large intervals of t, with $\lim_{\delta\to 0} \lim_{N\to\infty} \bar{\chi}_{N,\delta}(t) = \chi(t)$.

Note that there is no reason to expect *any* discretization scheme will converge to $\chi(t)$ or $\rho(\omega)$. For example it is not clear a brick wall discretization along the lines described in section C.5.3 would. On the other hand, the convergence of the above global angular momentum cutoff scheme to the continuum $\chi(t)$ was perhaps to be expected, given (C.20) and the discussion preceding it.

C.2.3 Resonances and quasinormal mode expansion

Substituting the expansion (C.26) of the dS character,

$$\chi(t) = \sum_\lambda N_\lambda \, e^{-it\lambda} \qquad (t > 0), \tag{C.33}$$

into (C.27), $\rho(\omega) = \frac{1}{2\pi}\int_0^\infty dt\, \chi(t)\, (e^{i\omega t} + e^{-i\omega t})$, we can formally express the density of states as

$$\rho(\omega) = \frac{1}{2\pi i}\sum_\lambda N_\lambda\Big(\frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega}\Big)\,, \tag{C.34}$$

From this we read off that $\rho(\omega)$ analytically continued to the complex plane has poles at $\omega = \pm\lambda$ which for massive representations means $\omega = \mp i(\Delta_\pm + n)$. This can also be checked from explicit expressions such as the dS_3 scalar density of states (C.28), illustrated in fig. C.4. These values of ω are precisely the frequencies of the (anti-)quasinormal field modes in the static patch, that is to say modes with purely ingoing/outgoing boundary conditions at the horizon, regular in the interior. If we think of the normal modes as scattering states, the quasinormal modes are to be thought of as scattering resonances. Indeed the poles of $\rho(\omega)$ are related to the poles/zeros of the static patch S-matrix $S(\omega)$, cf. (C.35) below. Thus we see the coefficients N_λ in (C.33) count resonances (or quasinormal modes), rather than states (or normal modes) as in AdS. This expresses at the level of characters the observations made in [139]. It holds for any $SO(1, d+1)$ representation, including massless representations, as explored in more depth in [141] (see also appendix C.7.1). Some corresponding quasinormal mode expansions of bulk thermodynamic quantities are given in (3.56) and (3.59), and related there to the quasinormal mode expansion of [142] for scalar and spinor path integrals.

"S-matrix" formulation

The appearance of resonance poles in the analytically continued density of states is well-known in quantum mechanical scattering off a fixed potential V. They are directly related to the poles/zeros in the S-matrix $S(\omega)$ at energy ω through the relation [182]

$$\rho(\omega) - \rho_0(\omega) = \frac{1}{2\pi i}\,\frac{d}{d\omega}\,\mathrm{tr}\log S(\omega)\,, \tag{C.35}$$

where $\rho_0(\omega)$ is the density of states at $V = 0$.

Using the explicit form of the dS_2 dimension-Δ scalar static patch mode functions (C.122)

$\phi^{\Delta}_{\omega\ell}(r,T)$, expanding these for $r =: \tanh X \to 1$ as

$$\phi^{\Delta}_{\omega\ell}(r) \to A^{\Delta}_{\ell}(\omega)\, e^{-i\omega(T+X)} + B^{\Delta}_{\ell}(\omega)\, e^{-i\omega(T-X)}\,, \tag{C.36}$$

and defining $S^{\Delta}_{\ell}(\omega) \equiv B^{\Delta}_{\ell}(\omega)/A^{\Delta}_{\ell}(\omega)$, one can check that $\rho^{\Delta}(\omega)$ as obtained in (C.29) satisfies

$$\rho^{\Delta}(\omega) - \rho_0(\omega) = \frac{1}{2\pi i}\frac{d}{d\omega}\sum_{\ell=0,1}\log S^{\Delta}_{\ell}(\omega)\,, \tag{C.37}$$

where $\rho_0(\omega) = \frac{1}{\pi}(\psi(i\omega) + \psi(-i\omega)) + \text{const.}$ does not depend on Δ. This can be viewed as a rough analog of (C.35), although the interpretation of $\rho_0(\omega)$ in the present setting is not clear to us. Similar observations can be made in higher dimensions.

In [183], a general (flat space) S-matrix formulation of statistical mechanics for interacting QFTs was developed. In this formulation, the canonical partition function is expressed as

$$\log Z - \log Z_0 = \frac{1}{2\pi i}\int dE\, e^{-\beta E}\,\frac{d}{dE}\left[\operatorname{Tr}\log S(E)\right]_c\,, \tag{C.38}$$

where the subscript c indicates restriction to connected diagrams (where "connected" is defined with the rule that particle permutations are interpreted as interactions [183]). Combined with the above observations, this hints at a possible generalization of our free QFT results to interacting theories.

C.3 Evaluation of character integrals

The most straightforward way of UV-regularizing character integrals is to simply cut off the t-integral at some small $t = \epsilon$. However to compare to the standard heat kernel (or spectral zeta function) regularization for Gaussian Euclidean path integrals [60], it is useful to have explicit results in the latter scheme. In this appendix we give an efficient and general recipe to compute the exact heat kernel-regularized one-loop Euclidean path integral, with regulator $e^{-\epsilon^2/4\tau}$ as in (3.66), requiring only the unregulated character formula as input. For concreteness we consider the scalar case in the derivation, but because the scalar character $\chi_0(t)$ provides the basic building block for all other characters $\chi_S(t)$, the final result will be applicable in general. We spell out the derivation is some detail, and summarize the final result together with some examples in section C.3.2. Application to the massless higher-spin case is discussed in section C.3.3, where we work out the exact one-loop Euclidean path integral for Einstein gravity on S^4 as an example. In section C.3.4 we consider different regularizations, such as the simple $t > \epsilon$ cutoff.

C.3.1 Derivation

As shown in section 3.3, the scalar Euclidean path integral regularized as

$$\log Z_\epsilon = \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\frac{\epsilon^2}{4\tau}}\, F_D(\tau)\,, \qquad F_D(\tau) \equiv \operatorname{Tr} e^{-\tau D} = \sum_n D_n^{d+2}\, e^{-\left(n+\frac{d}{2}+i\nu\right)\left(n+\frac{d}{2}-i\nu\right)}\,, \tag{C.39}$$

where $D = -\nabla^2 + \frac{d^2}{4} + \nu^2$, can be written in character integral form as

$$\log Z_\epsilon = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}} \sum_n D_n^{d+2}\left(e^{-(n+\frac{d}{2})t-i\nu\sqrt{t^2-\epsilon^2}} + e^{-(n+\frac{d}{2})t+i\nu\sqrt{t^2-\epsilon^2}}\right) \tag{C.40}$$

$$= \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}}\, \frac{1+e^{-t}}{1-e^{-t}}\, \frac{e^{-\frac{d}{2}t-i\nu\sqrt{t^2-\epsilon^2}} + e^{-\frac{d}{2}t+i\nu\sqrt{t^2-\epsilon^2}}}{(1-e^{-t})^d}\,, \tag{C.41}$$

Putting $\epsilon = 0$ we recover the formal (UV-divergent) character formula

$$\log Z_{\epsilon=0} = \int_0^\infty \frac{dt}{2t}\, F_\nu(t)\,,$$

$$F_\nu(t) \equiv \sum_n D_n^{d+2}\left(e^{-(n+\frac{d}{2}+i\nu)t} + e^{-(n+\frac{d}{2}-i\nu n)t}\right) = \frac{1+e^{-t}}{1-e^{-t}}\, \frac{e^{-(\frac{d}{2}+i\nu)t} + e^{-(\frac{d}{2}-i\nu)t}}{(1-e^{-t})^d}\,. \tag{C.42}$$

To evaluate (C.41), we split the integral into UV and IR parts, each of which can be evaluated in closed form in the limit $\epsilon \to 0$.

Separation into UV and IR parts

The separation of the integral in UV and IR parts is analogous to the usual procedure in heat kernel regularization, where one similarly separates out the UV part of the τ integral by isolating the leading terms in the $\tau \to 0$ heat kernel expansion

$$F_D(\tau) := \mathrm{Tr}\, e^{-\tau D} \;\to\; \sum_{k=0}^{d+1} \alpha_k\, \tau^{-(d+1-k)/2} =: F_D^{\mathrm{uv}}(\tau)\,. \tag{C.43}$$

Introducing an infinitesimal IR cutoff $\mu \to 0$, we may write $\log Z_\epsilon = \log Z_\epsilon^{\mathrm{uv}} + \log Z^{\mathrm{ir}}$ where

$$\log Z_\epsilon^{\mathrm{uv}} \equiv \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\frac{\epsilon^2}{4\tau}}\, F_D^{\mathrm{uv}}(\tau)\, e^{-\mu^2\tau}\,, \quad \log Z^{\mathrm{ir}} \equiv \int_0^\infty \frac{d\tau}{2\tau}\big(F_D(\tau) - F_D^{\mathrm{uv}}(\tau)\big)\, e^{-\mu^2\tau}\,. \tag{C.44}$$

Dropping the UV regulator in the IR integral is allowed because all UV divergences have been removed by the subtraction. The factor $e^{-\mu^2\tau}$ serves as an IR regulator needed for the separate integrals when F^{uv} has a term $\frac{\alpha_{d+1}}{2\tau} \neq 0$, that is to say when $d+1$ is even. The resulting $\log\mu$ terms cancel out of the sum at the end. Evaluating this using the specific UV regulator of (C.39) gives

$$\log Z_\epsilon = \frac{1}{2}\zeta_D'(0) + \alpha_{d+1}\log\big(\tfrac{2}{e^\gamma \epsilon}\big) + \frac{1}{2}\sum_{k=0}^{d} \alpha_k\, \Gamma\big(\tfrac{d+1-k}{2}\big)\left(\tfrac{2}{\epsilon}\right)^{d+1-k}\,, \tag{C.45}$$

where $\zeta_D(z) = \mathrm{Tr}\, D^{-z} = \frac{1}{\Gamma(z)}\int \frac{d\tau}{\tau}\,\tau^z\, \mathrm{Tr}\, e^{-\tau D}$ is the zeta function of D and $\alpha_{d+1} = \zeta_D(0)$.

We can apply the same idea to the square-root regulated character formula (C.41) for Z_ϵ. The latter is obtained from the simpler integrand of the formal character formula (C.42) for $Z_{\epsilon=0}$ by dividing it by $r(\epsilon,t) \equiv \sqrt{t^2-\epsilon^2}/t$ and replacing ν by $\nu r(\epsilon,t)$:

$$\log Z_{\epsilon=0} = \int_0^\infty \frac{dt}{2t}\, F_\nu(t) \qquad \Rightarrow \qquad \log Z_\epsilon = \int_\epsilon^\infty \frac{dt}{2rt}\, F_{r\nu}(t)\,, \qquad r \equiv \frac{\sqrt{t^2-\epsilon^2}}{t}\,. \tag{C.46}$$

Note that $0 < r < 1$ for all $t > \epsilon$, $r \sim O(1)$ for $t \sim \epsilon$ and $r \to 1$ for $t \gg \epsilon$. Therefore, given the

$t \to 0$ behavior of the integrand in the formal character formula for $Z_{\epsilon=0}$,

$$\frac{1}{2t}F_\nu(t) \to \frac{1}{t}\sum_{k=0}^{d+1} b_k(\nu)\, t^{-(d+1-k)} =: \frac{1}{2t}F_\nu^{\rm uv}(t)\,, \qquad b_k(\nu) = \sum_{\ell=0}^{k} b_{k\ell}\,\nu^\ell\,, \tag{C.47}$$

we get the $t \sim \epsilon \to 0$ behavior of the integrand for the exact Z_ϵ:

$$\frac{1}{2rt}\,F_{r\nu}(t) \to \frac{1}{2rt}\,F_{r\nu}^{\rm uv}(t) = \frac{1}{rt}\sum_{k,\ell} b_{k\ell}\,\nu^\ell\, r^\ell\, t^{-(d+1-k)}\,. \tag{C.48}$$

Thus we can separate $\log Z_\epsilon = \log \tilde{Z}_\epsilon^{\rm uv} + \log \tilde{Z}^{\rm ir}$, with

$$\log \tilde{Z}_\epsilon^{\rm uv} \equiv \int_\epsilon^\infty \frac{dt}{2rt}\,F_{r\nu}^{\rm uv}(t)\,e^{-\mu t}\,, \qquad \log \tilde{Z}^{\rm ir} \equiv \int_0^\infty \frac{dt}{2t}\,\big(F_\nu(t) - F_\nu^{\rm uv}(t)\big)e^{-\mu t}\,. \tag{C.49}$$

Again the limit $\mu \to 0$ is understood. We were allowed to put $\epsilon = 0$ in the IR part because it is UV finite.

Evaluation of UV part

Using the expansion (C.48), the UV part can be evaluated explicitly as

$$\log \tilde{Z}_\epsilon^{\rm uv} = \frac{1}{2}\sum_{\ell,k\le d} b_{k\ell}\,B\big(\tfrac{d+1-k}{2},\tfrac{\ell+1}{2}\big)\,\nu^\ell\,\epsilon^{-(d+1-k)} - \sum_\ell b_{d+1,\ell}\big(H_\ell - \tfrac{1}{2}H_{\ell/2} + \log\big(\tfrac{e^\gamma\,\epsilon\,\mu}{2}\big)\big)\nu^\ell \tag{C.50}$$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler beta function and $H_x = \gamma + \frac{\Gamma'(1+x)}{\Gamma(1+x)}$ which for integer x is the x-th harmonic number $H_x = 1 + \frac{1}{2} + \cdots + \frac{1}{x}$. For example for $d = 3$, we get

$$\log \tilde{Z}_\epsilon^{\rm uv} = \tfrac{4}{3}\,\epsilon^{-4} - \tfrac{4\nu^2+1}{12}\,\epsilon^{-2} - \big(\tfrac{\nu^4}{9} + \tfrac{\nu^2}{24}\big) - \big(\tfrac{\nu^4}{12} + \tfrac{\nu^2}{24} - \tfrac{17}{2880}\big)\log\big(\tfrac{e^\gamma\epsilon\mu}{2}\big)\,. \tag{C.51}$$

This gives an explicit expression for the part of $\log Z$ denoted Pol(Δ) in [142], without having to invoke an independent computation of the heat kernel coefficients. Indeed, turning this around, by comparing (C.50) to (C.45), we can express the heat kernel coefficients α_k explicitly in terms of the character coefficients $b_{k,\ell}$. In particular the Weyl anomaly coefficient is simply given by the coefficient $b_{d+1} = \sum_\ell b_{d+1,\ell}\nu^\ell$ of the $1/t$ term in the integrand of the formal character formula (C.42). More generally,

$$\alpha_k = \sum_\ell \frac{\Gamma(\frac{\ell+1}{2})}{2^{d+1-k}\Gamma(\frac{d+1-k+\ell+1}{2})}\, b_{k\ell}\, \nu^\ell\,. \tag{C.52}$$

For example for $d = 3$, this becomes $\alpha_0 = \frac{1}{12}b_{00}$, $\alpha_2 = \frac{1}{2}b_{20} + \frac{\nu^2}{6}b_{22}$ and $\alpha_4 = b_4$. From the small-t expansion $\frac{1}{2t}F_\nu(t) \to \sum_k b_k t^{3-k}$ in (C.42) we read off $b_0 = 2$, $b_2 = -\frac{1}{12} - \nu^2$ and $b_4 = -\frac{17}{2880} + \frac{1}{24}\nu^2 + \frac{1}{12}\nu^4$. Thus $\alpha_0 = \frac{1}{6}$, $\alpha_2 = -\frac{1}{24} - \frac{1}{6}\nu^2$ and $\alpha_4 = -\frac{17}{2880} + \frac{1}{24}\nu^2 + \frac{1}{12}\nu^4$.

Evaluation of IR part

As we explain momentarily, the IR part can be evaluated as

$$\log \tilde{Z}^{\rm ir} = \frac{1}{2}\zeta_\nu'(0) + b_{d+1}\log\mu\,, \qquad \zeta_\nu(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{t}\, t^z\, F_\nu(t)\,, \tag{C.53}$$

where like for the spectral zeta function $\zeta_D(z)$, the "character zeta function" $\zeta_\nu(z)$ is defined by the above integral for z sufficiently large and by analytic continuation for $z \to 0$. This zeta function representation of $\log Z^{\rm ir}$ follows from the following observations. If we define $\zeta_\nu^{\rm ir}(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{t}\, t^z \big(F_\nu(t) - F_\nu^{\rm uv}(t)\big)\, e^{-\mu t}$, then since the integral remains finite for $z \to 0$, while $\Gamma(z) \sim 1/z$ and $\partial_z(1/\Gamma(z)) \to 1$, we trivially have $\frac{1}{2}\partial_z\zeta_\nu^{\rm ir}(z)|_{z=0} = \log\tilde{Z}^{\rm ir}$. Moreover for z sufficiently large we have in the limit $\mu \to 0$ that $\frac{1}{2}\zeta^{\rm uv}(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{2t}\, t^z F_\nu^{\rm uv}(t)\, e^{-\mu t} = b_{d+1}\mu^{-z}$, so upon analytic continuation we have $1/2\partial_z\zeta^{\rm uv}(z)|_{z=0} = -b_{d+1}\log\mu$, and (C.53) follows.

In contrast to the spectral zeta function, the character zeta function can straightforwardly be evaluated in terms of Hurwitz zeta functions. Indeed, denoting $\Delta_\pm = \frac{d}{2} \pm i\nu$, we have $F_D(t) = \sum_n Q(n)\, e^{-t(n+\Delta_+)(n+\Delta_-)}$ where the spectral degeneracy $Q(n)$ is some polynomial in n, and $\zeta_D(z) = \sum_{n=0}^\infty Q(n)\, \big((n+\Delta_+)(n+\Delta_-)\big)^{-z}$, which is quite tricky to evaluate, whereas $F_\nu(t) = \sum_n Q(n)\big(e^{-t(n+\Delta_+)} + e^{-t(n+\Delta_-)}\big)$, and we can immediately express the associated character zeta function as a finite sum of Hurwitz zeta functions $\zeta(z,\Delta) = \sum_{n=0}^{\infty}(n+\Delta)^{-z}$:

$$\zeta_\nu(z) = \sum_{\pm}\sum_{n=0}^{\infty} Q(n)(n+\Delta_\pm)^{-z} = \sum_{\pm} Q(\hat{\delta}-\Delta_\pm)\,\zeta(z,\Delta_\pm)\,. \tag{C.54}$$

Here $\hat{\delta}$ is the unit z-shift operator acting as $\hat{\delta}^n\zeta(z,\Delta) = \zeta(z-n,\Delta)$; for example if $Q(n) = n^2$ we have $Q(\hat{\delta}-\Delta)\,\zeta(z,\Delta) = (\hat{\delta}^2 - 2\Delta\hat{\delta} + \Delta^2)\zeta(z,\Delta) = \zeta(z-2,\Delta) - 2\Delta\zeta(z-1,\Delta) + \Delta^2\zeta(z,\Delta)$.

C.3.2 Result and examples

Result

Altogether we conclude that given a formal character integral formula

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t}\,F_\nu(t)\,, \tag{C.55}$$

for a field corresponding to a dS_{d+1} irrep of dimension $\frac{d}{2}+i\nu$, with IR and UV expansions

$$F_\nu(t) = \sum_{\Delta}\sum_{n=0}^{\infty} P_\Delta(n)\,e^{-(n+\Delta)t}\,, \qquad \frac{1}{2t}F_\nu(t) = \frac{1}{t}\sum_{k=0}^{d+1} b_k(\nu)\,t^{-(d+1-k)} + O(t^0)\,, \tag{C.56}$$

where $b_k(\nu) = \sum_\ell b_{k\ell}\,\nu^\ell$, we obtain the exact $Z_{\rm PI}$ with heat kernel regulator $e^{-\epsilon^2/4\tau}$ as

$$\boxed{\begin{aligned}\log Z_{\mathrm{PI},\epsilon} =& \frac{1}{2}\sum_{\Delta} P_\Delta(\hat{\delta}-\Delta)\,\zeta'(0,\Delta) - \sum_{\ell=0}^{d+1} b_{d+1,\ell}\big(H_\ell - \tfrac{1}{2}H_{\ell/2}\big)\nu^\ell + b_{d+1}(\nu)\log(2e^{-\gamma}/\epsilon)\\ &+ \frac{1}{2}\sum_{k=0}^{d}\sum_{\ell=0}^{k} b_{k\ell}\,B\big(\tfrac{d+1-k}{2},\tfrac{\ell+1}{2}\big)\,\nu^\ell\,\epsilon^{-(d+1-k)}\,.\end{aligned}} \tag{C.57}$$

Here $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $H_x = \gamma + \frac{\Gamma'(1+x)}{\Gamma(1+x)}$, which for integer x is the x-th harmonic number $H_x = 1+\frac{1}{2}+\cdots+\frac{1}{x}$, and $\hat{\delta}$ is the unit shift operator acting on the first argument of the Hurwitz zeta function $\zeta(z,\Delta)$: the polynomial $P_\Delta(\hat{\delta}-\Delta)$ is to be expanded in powers of $\hat{\delta}$, setting $\hat{\delta}^n\zeta'(0,\Delta) \equiv \zeta'(-n,\Delta)$.

Finally the heat kernel coefficients are

$$\alpha_k = \sum_\ell \frac{\Gamma(\frac{\ell+1}{2})}{2^{d+1-k}\Gamma(\frac{d+1-k+\ell+1}{2})}\, b_{k\ell}\, \nu^\ell\,. \tag{C.58}$$

If we are only interested in the finite part of $\log Z$, only the first three terms in (C.57) matter. Note that the third and the second term $\mathcal{M}_\nu \equiv \sum_\ell b_{d+1,\ell}\big(H_\ell - \frac{1}{2}H_{\ell/2}\big)$ is in general nonvanishing for even $d+1$. By comparing (C.57) to (C.45), say in the scalar case discussed earlier, we see that $\zeta_D'(0) = \zeta_\nu'(0) + 2\mathcal{M}_\nu$. Thus $2\mathcal{M}_\nu$ can be thought of as correcting the formal factorization $\sum_n \log(n+\Delta_+)(n+\Delta_-) = \sum_n \log(n+\Delta_+) + \sum_n \log(n+\Delta_-)$ in zeta function regularization. For this reason $\mathcal{M}_\nu$ is called the multiplicative "anomaly", as reviewed in [184]. The above thus generalizes the explicit formulae in [184] for $\mathcal{M}_\nu$ to fields of arbitrary representation content.

Examples

1. A scalar on S^2 ($d=1$) with $\Delta_\pm = \frac{1}{2} \pm i\nu$ has $F_\nu(t) = \frac{1+e^{-t}}{1-e^{-t}}\frac{e^{-\Delta_+ t}+e^{-\Delta_- t}}{1-e^{-t}}$ so the IR and UV expansions are $F_\nu(t) = \sum_\pm \sum_{n=0}^\infty (2n+1)e^{-(\Delta_\pm+n)t}$ and $\frac{1}{2t}F_\nu(t) = \frac{2}{t^3} + \frac{\frac{1}{12}-\nu^2}{t} + O(t^0)$. Therefore according to (C.57)

$$\log Z_{\mathrm{PI},\epsilon} = \sum_{\Delta=\frac{1}{2}\pm i\nu} \Big(\zeta'(-1,\Delta) - (\Delta - \tfrac{1}{2})\zeta'(0,\Delta)\Big) + \nu^2 + \big(\tfrac{1}{12} - \nu^2\big)\log\big(2\,e^{-\gamma}/\epsilon\big) + \frac{2}{\epsilon^2}\,. \tag{C.59}$$

The heat kernel coefficients are obtained from (C.58) as $\alpha_0 = 1$ and $\alpha_2 = \frac{1}{12} - \nu^2$.

2. For a scalar on S^3, $F_\nu(t) = \sum_\pm \sum_{n=0}^\infty (n+1)^2 e^{-(\Delta_\pm+n)t}$, $\frac{1}{2t}F_\nu(t) \to \frac{2}{t^4} - \frac{\nu^2}{t^2} + O(t^0)$, so

$$\log Z_{\mathrm{PI},\epsilon} = \sum_\pm \Big(\tfrac{1}{2}\zeta'(-2, 1\pm i\nu) \mp i\nu\zeta'(-1,1\pm i\nu) - \tfrac{1}{2}\nu^2\zeta'(0,1\pm i\nu)\Big) - \frac{\pi\nu^2}{4\epsilon} + \frac{\pi}{2\epsilon^3}\,. \tag{C.60}$$

The heat kernel coefficients are $\alpha_0 = \frac{\sqrt{\pi}}{4}$, $\alpha_2 = -\frac{\sqrt{\pi}}{4}\nu^2$. In particular for a conformally coupled scalar, i.e. $\Delta = \frac{1}{2}, \frac{3}{2}$ or equivalently $\nu = i/2$, we get for the finite part the familiar result $\log Z_{\mathrm{PI}} = \frac{3\zeta(3)}{16\pi^2} - \frac{\log(2)}{8}$. For $\Delta = 1$, i.e. $\nu = 0$, we get $\log Z_{\mathrm{PI}} = -\frac{\zeta(3)}{4\pi^2}$. Notice that the finite part looks quite different from (3.50) obtained by contour integration. Nevertheless they are in fact the same function.

3. A more interesting example is the massive spin-s field on S^4 with $\Delta_\pm = \frac{3}{2} \pm i\nu$. In this case, (3.83) combined with (C.135) or equivalently (3.84) gives $F_\nu = F_{\rm bulk} - F_{\rm edge}$ with

$$F_{\rm bulk}(t) = \sum_{\Delta=\frac{3}{2}\pm i\nu} \sum_{n=-1}^{\infty} D_s^3 D_n^5 \, e^{-(n+\Delta)t} = D_s^3 \, \frac{1+e^{-t}}{1-e^{-t}} \frac{e^{-(\frac{3}{2}+i\nu)t} + e^{-(\frac{3}{2}-i\nu)t}}{(1-e^{-t})^3} \,, \tag{C.61}$$

$$F_{\rm edge}(t) = \sum_{\Delta=\frac{1}{2}\pm i\nu} \sum_{n=-1}^{\infty} D_{s-1}^5 D_{n+1}^3 \, e^{-(n+\Delta)t} = D_{s-1}^5 \, \frac{1+e^{-t}}{1-e^{-t}} \frac{e^{-(\frac{1}{2}+i\nu)t} + e^{-(\frac{1}{2}-i\nu)t}}{(1-e^{-t})} \,, \tag{C.62}$$

where $D_p^3 = 2p+1$, $D_p^5 = \frac{1}{6}(2p+3)(p+2)(p+1)$. In particular note that with $g_s \equiv D_s^3 = 2s+1$, we have $D_{s-1}^5 = \frac{1}{24} g_s (g_s^2 - 1)$. The small-$t$ expansions are

$$\tfrac{1}{2t} F_{\rm bulk}(t) \to g_s \Big(2\, t^{-5} - \big(\nu^2 + \tfrac{1}{12}\big) t^{-3} + \big(\tfrac{\nu^4}{12} + \tfrac{\nu^2}{24} - \tfrac{17}{2880}\big) t^{-1} + O(t^0) \Big) \tag{C.63}$$

$$\tfrac{1}{2t} F_{\rm edge}(t) \to \tfrac{1}{24} g_s (g_s^2 - 1) \Big(2\, t^{-3} + \big(\tfrac{1}{12} - \nu^2\big) t^{-1} + O(t^0) \Big) \,. \tag{C.64}$$

Thus the exact partition function for a massive spin-s field is

$$\begin{aligned} \log Z_{{\rm PI},\epsilon} = g_s & \sum_{\Delta=\frac{3}{2}\pm i\nu} \Big(\tfrac{1}{6}\zeta'(-3,\Delta) \mp \tfrac{1}{2} i\nu \zeta'(-2,\Delta) - \big(\tfrac{1}{2}\nu^2 + \tfrac{1}{24}\big)\zeta'(-1,\Delta) \pm i\big(\tfrac{1}{24}\nu + \tfrac{1}{6}\nu^3\big)\zeta'(0,\Delta) \Big) \\ & - \tfrac{1}{24} g_s (g_s^2 - 1) \sum_{\Delta=\frac{1}{2}\pm i\nu} \Big(\zeta'(-1,\Delta) \mp i\nu \zeta'(0,\Delta) \Big) - \tfrac{1}{24} g_s^3 \nu^2 - \tfrac{1}{9} g_s \nu^4 \\ & + \Big(g_s^3 \big(\tfrac{1}{24}\nu^2 - \tfrac{1}{288}\big) + g_s \big(\tfrac{1}{12}\nu^4 - \tfrac{7}{2880}\big) \Big) \log(2\, e^{-\gamma}/\epsilon) - \big(\tfrac{1}{12} g_s^3 + \tfrac{1}{3} g_s \nu^2\big) \epsilon^{-2} + \tfrac{4}{3} g_s \epsilon^{-4} \,. \end{aligned} \tag{C.65}$$

Finally the heat kernel coefficients are

$$\alpha_0 = \tfrac{1}{6} g_s \,, \qquad \alpha_2 = -\tfrac{1}{24} g_s^3 - \tfrac{1}{6} g_s \nu^2 \,, \qquad \alpha_4 = g_s^3 \big(\tfrac{1}{24}\nu^2 - \tfrac{1}{288}\big) + g_s \big(\tfrac{1}{12}\nu^4 - \tfrac{7}{2880}\big) \,. \tag{C.66}$$

Single-mode contributions

Contributions from single path integral modes and contributions of single quasinormal modes are of use in some of our derivations and applications. These are essentially special cases of the above general results, but for convenience we collect some explicit formulae here:

• **Path integral single-mode contributions**: For our choice of heat-kernel regulator $e^{-\epsilon^2/4\tau}$, the contribution to $\log Z_{\mathrm{PI},\epsilon}$ from a single bosonic eigenmode with eigenvalue λ is

$$I_\lambda = \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau}\, e^{-\tau\lambda} = K_0(\epsilon\sqrt{\lambda}) \to -\frac{1}{2}\log\frac{\lambda}{M^2}\,, \qquad M \equiv \frac{2e^{-\gamma}}{\epsilon}\,, \tag{C.67}$$

Different regulator insertions lead to a similar result in the limit $\epsilon \to 0$, with $M = c/\epsilon$ for some regulator-dependent constant c. A closely related formula is obtained for the contribution from an individual term in the sum (C.40) or equivalently in the IR expansion of (C.56), which amounts to computing (C.55) with $F_\nu(t) \equiv e^{-\rho t}$, $\rho = a \pm i\nu$. The small-t expansion is $\frac{1}{2t}F_\nu(t) = \frac{1}{2t} + O(t^0)$, so the UV part is given by the log term in (C.57) with coefficient $\frac{1}{2}$, and the IR part is $\frac{1}{2}\zeta_\nu'(0) = -\frac{1}{2}\log\rho$ as in (C.53). Thus

$$I'_\rho = \int_0^\infty \frac{dt}{2t}\, e^{-\rho t} \to -\frac{1}{2}\log\frac{\rho}{M}\,, \qquad M = \frac{2e^{-\gamma}}{\epsilon}\,, \tag{C.68}$$

where the integral is understood to be regularized as in (C.40), $I'_\rho = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}}\, e^{-ta - i\nu\sqrt{t^2-\epsilon^2}}$, left implicit here. The similarities between (C.67) and (C.68) are of course no accident, since in our setup, the former splits into the sum of two integrals of the latter type: writing $\lambda = a^2 + \nu^2 = (a+i\nu)(a-i\nu)$, we have $I_\lambda = I'_{a+i\nu} + I'_{a-i\nu}$.

• **Quasinormal mode contributions**: Considering a character quasinormal mode expansion $\chi(t) = \sum_r N_r\, e^{-r|t|}$ as in (3.14), the IR contribution from a single bosonic/fermionic QNM is

$$\int_0^\infty \frac{dt}{2t}\frac{1+e^{-t}}{1-e^{-t}}\, e^{-r\,t}\bigg|_{\mathrm{IR}} = \log\frac{\Gamma(r+1)}{\mu^r\sqrt{2\pi r}}\,, \qquad -\int_0^\infty \frac{dt}{2t}\frac{2e^{-t/2}}{1-e^{-t}}\, e^{-r\,t}\bigg|_{\mathrm{IR}} = -\log\frac{\Gamma(r+\frac{1}{2})}{\mu^r\sqrt{2\pi}} \tag{C.69}$$

• **Harmonic oscillator**: The character of a $d = 0$ scalar of mass ν is $\chi(t) = e^{-i\nu t} + e^{i\nu t}$, hence

$$\log Z_{\mathrm{PI},\epsilon} = \int_0^\infty \frac{dt}{2t}\,\frac{1+e^{-t}}{1-e^{-t}}\left(e^{-i\nu t}+e^{i\nu t}\right) = \frac{\pi}{\epsilon} - \log\!\left(e^{\pi\nu}-e^{-\pi\nu}\right). \tag{C.70}$$

The finite part gives the canonical bosonic harmonic oscillator thermal partition function $\mathrm{Tr}\, e^{-\beta H} = \sum_n e^{-\beta\nu(n+\frac{1}{2})} = \left(e^{\beta\nu/2}-e^{-\beta\nu/2}\right)^{-1}$ at $\beta = 2\pi$. The fermionic version is

$$\log Z_{\mathrm{PI},\epsilon} = -\int_0^\infty \frac{dt}{2t}\,\frac{2e^{-t/2}}{1-e^{-t}}\left(e^{-i\nu t}+e^{i\nu t}\right) = -\frac{\pi}{\epsilon} + \log\!\left(e^{\pi\nu}+e^{-\pi\nu}\right). \tag{C.71}$$

C.3.3 Massless case

Here we give a few more details on how to use (C.57) to explicitly evaluate Z_{PI} in the massless case, and work out the exact Z_{PI} for Einstein gravity on S^4 as an example.

Our final result for the massless one-loop $Z_{\mathrm{PI}} = Z_G \cdot Z_{\mathrm{char}}$ is given by (3.112):

$$Z_{\mathrm{PI}} = i^{-P}\,\frac{\gamma^{\dim G}}{\mathrm{vol}(G)_{\mathrm{c}}}\cdot \exp\int^{\times}\frac{dt}{2t}\,F\,, \qquad F = \frac{1+q}{1-q}\Big(\big[\hat\chi_{\mathrm{bulk}}\big]_+ - \big[\hat\chi_{\mathrm{edge}}\big]_+ - 2\dim G\Big), \tag{C.72}$$

where for $s = 2$ gravity $\gamma = \sqrt{\frac{8\pi G_{\mathrm{N}}}{A_{d-1}}}$, $P = d+3$, $G = SO(d+2)$ and $\mathrm{vol}(G)_{\mathrm{c}} = (C.93)$.

• **UV part:** As always, the coefficient of the log-divergent term simply equals the coefficient of the $1/t$ term in the small-t expansion of the integrand in (C.72). For the other UV terms in (C.57) (including the "multiplicative anomaly"), a problem might seem to be that we need a continuously variable dimension parameter $\Delta = \frac{d}{2} + i\nu$, whereas massless fields, and our explicit formulae for $\hat\chi \to [\hat\chi]_+$, require fixed integer dimensions. This problem is easily solved, as the *UV part* can actually be computed from the original *naive* character formula (C.170):

$$\log Z_{\mathrm{PI}}\big|_{\mathrm{UV}} = \int \frac{dt}{2t}\,\hat F\Big|_{\mathrm{UV}}, \qquad \hat F = \frac{1+q}{1-q}\left(\hat\chi_{\mathrm{bulk}} - \hat\chi_{\mathrm{edge}}\right), \tag{C.73}$$

Indeed since $\hat F \to F = \{\hat F\}_+$ in (C.172) affects just a finite number of terms $c_k q^k \to c_k q^{-k}$, it does not alter the small-t (UV) part of the integral. Moreover $\hat\chi_s = \hat\chi_{s,\nu_\phi} - \hat\chi_{s,\nu_\xi}$, where $\hat\chi_{s,\nu}$ is a massive spin-s character. Thus the UV part may be obtained simply by combining the results of (C.57) for general ν and s, substituting the values ν_ϕ, ν_ξ set by (3.95).

• **IR part:** The *IR part* is the ζ' part of (C.57), obtained from the q-expansion of $F(q)$ in (C.72). This can be found in general by using

$$\frac{1+q}{1-q}\,\frac{q^\Delta}{(1-q)^k} = \sum_{n=0}^{\infty} P(n)\,q^{n+\Delta}\,, \qquad P(n) = D_n^{k+2}\,, \tag{C.74}$$

with D_n^{k+2} the polynomial given in (C.15). For $k = 0$, (C.69) is useful. In particular, using the $\int^\times$ prescription (C.178), the IR contribution from the last term in (C.72) is obtained by considering the $r \to 0$ limit of the bosonic formula in (C.69):

$$\int^\times \frac{dt}{2t}\,\frac{1+q}{1-q}\big(-2\dim G\big)\Big|_{\rm IR} = \dim G\cdot\log(2\pi)\,. \tag{C.75}$$

Example: Einstein gravity on S^4

As a simple application, let us compute the exact one-loop Euclidean path integral for pure gravity on S^4. In this case $G = SO(5)$, $\dim G = 10$, $d = 3$ and $s = 2$. From (3.95) we read off $i\nu_\phi = \frac{3}{2}$, $i\nu_\xi = \frac{5}{2}$, and from (3.102) we get

$$\chi_{\rm bulk} = \big[\hat\chi_{\rm bulk}\big]_+ = \frac{10\,q^3 - 6\,q^4}{(1-q)^3}\,, \qquad \chi_{\rm edge} = \big[\hat\chi_{\rm edge}\big]_+ = \frac{10\,q^2 - 2\,q^3}{1-q}\,. \tag{C.76}$$

The small-t expansion of the integrand in (C.72) is $\frac{1}{2t}F = 4\,t^{-5} - \frac{47}{3}\,t^{-3} - \frac{571}{45}\,t^{-1} + O(t^0)$. The coefficient of the log-divergent part of $\log Z_{\rm PI}$ is the coefficient of t^{-1}:

$$\log Z_{\rm PI}|_{\rm log\,div} = -\frac{571}{45}\,\log\big(2e^{-\gamma}\epsilon^{-1}\big)\,, \tag{C.77}$$

in agreement with [51]. The complete heat-kernel regularized UV part of (C.57) can be read off directly from our earlier results for massive spin-s in $d = 3$ as

$$\begin{aligned}\log Z_{\rm PI}\big|_{\rm UV} &= \log Z_{\rm PI}(s=2, \nu=\tfrac{3}{2}i)\big|_{\rm UV} - \log Z_{\rm PI}(s=1, \nu=\tfrac{5}{2}i)\big|_{\rm UV} \\ &= \frac{8}{3}\,\epsilon^{-4} - \frac{32}{3}\,\epsilon^{-2} - \frac{571}{45}\log\!\big(2e^{-\gamma}\epsilon^{-1}\big) + \frac{715}{48}\,. \end{aligned} \tag{C.78}$$

Here $\mathcal{M} = \frac{715}{48}$ is the "multiplicative anomaly" term. The integrated heat kernel coefficients are similarly obtained from (C.66): $\alpha_0 = \frac{1}{3}$, $\alpha_2 = -\frac{16}{3}$, $\alpha_4 = -\frac{571}{45}$.

The IR (ζ') contributions from bulk and edge characters are obtained from the expansions

$$\frac{1+q}{1-q}\big(\chi_{\rm bulk} - \chi_{\rm edge}\big) = \sum_n P_{\rm b}(n)\,\big(10\,q^{3+n} - 6\,q^{4+n}\big) - \sum_n P_{\rm e}(n)\,\big(10\,q^{2+n} - 2\,q^{3+n}\big)\,, \tag{C.79}$$

where $P_{\rm b}(n) = D^5_n = \frac{1}{6}(n+1)(n+2)(2n+3)$, $P_{\rm e}(n) = D^3_n = 2n+1$. According to (C.57) this gives a contribution to $\log Z_{\rm char}|_{\rm IR}$ equal to

$$5\,P_{\rm b}(\hat\delta - 3)\,\zeta'(0,3) - 3\,P_{\rm b}(\hat\delta-4)\,\zeta'(0,4) - 5\,P_{\rm e}(\hat\delta-2)\,\zeta'(0,2) + P_{\rm e}(\hat\delta-3)\,\zeta'(0,3)\,, \tag{C.80}$$

where the polynomials are to be expanded in powers of $\hat\delta$, putting $\hat\delta^n\zeta'(0,\Delta) \equiv \zeta'(-n,\Delta)$. Working this out and adding the contribution (C.75), we find

$$\log Z_{\rm char}\big|_{\rm IR} = -\log 2 - \frac{47}{3}\,\zeta'(-1) + \frac{2}{3}\,\zeta'(-3)\,. \tag{C.81}$$

Combining this with the UV part and reinstating ℓ, we get[5]

$$\begin{aligned}\log Z_{\rm char} =& \frac{8}{3}\,\frac{\ell^4}{\epsilon^4} - \frac{32}{3}\,\frac{\ell^2}{\epsilon^2} - \frac{571}{45}\log\frac{2e^{-\gamma}L}{\epsilon} \\ & - \frac{571}{45}\log\frac{\ell}{L} + \frac{715}{48} - \log 2 - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3)\,, \end{aligned} \tag{C.82}$$

[5]This splits as $\log Z_{\rm char} = 10\,\log(2\pi) + \log Z_{\rm bulk} - \log Z_{\rm edge}$ where $\log Z_{\rm bulk} = \frac{8\ell^4}{3\epsilon^4} - \frac{8\ell^2}{3\epsilon^2} - \frac{331}{45}\log\frac{2e^{-\gamma}\ell}{\epsilon} + \frac{475}{48} - \frac{23}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3) - 5\log(2\pi)$ and $\log Z_{\rm edge} = \frac{8\ell^2}{\epsilon^2} + \frac{16}{3}\log\frac{2e^{-\gamma}\ell}{\epsilon} - 5 + 8\zeta'(-1) + \log 2 + 5\log(2\pi)$.

where L is an arbitrary length scale introduced to split off a finite part:

$$\log Z_{\text{char}}^{\text{fin}} = -\frac{571}{45}\log(\ell/L) + \frac{715}{48} - \log 2 - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3)\,, \tag{C.83}$$

To compute the group volume factor Z_G in (C.72), we use (C.93) for $G = SO(5)$ to get $\text{vol}(G)_{\text{c}} = \frac{2}{3}(2\pi)^6$, and $\gamma = \sqrt{8\pi G_{\text{N}}/4\pi\ell^2}$. Finally, $i^{-P} = i^{-(d+3)} = -1$. Thus we conclude that the one-loop Euclidean path integral for Einstein gravity on S^4 is

$$Z_{\text{PI}} = -\frac{(8\pi G_{\text{N}}/4\pi\ell^2)^5\, Z_{\text{char}}}{\frac{2}{3}(2\pi)^6}\,, \tag{C.84}$$

where Z_{char} is given by (C.82).

Example: Einstein gravity on S^5

For S^5 an analogous (actually simpler) computation gives $Z_{\text{PI}} = i^{-7} Z_G Z_{\text{char}}$ with

$$\begin{aligned} \log Z_{\text{char}} &= \frac{15\,\pi}{8}\frac{\ell^5}{\epsilon^5} - \frac{65\,\pi}{24}\frac{\ell^3}{\epsilon^3} - \frac{105\,\pi}{16}\frac{\ell}{\epsilon} + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4} + 15\log(2\pi) \\ \log Z_G &= \frac{15}{2}\log\frac{8\pi G_{\text{N}}}{2\pi^2\ell^3} - \log\frac{(2\pi)^9}{12}\,. \end{aligned} \tag{C.85}$$

C.3.4 Different regularization schemes

If we simply cut off the character integral at $t = \epsilon$, we get the following instead of (C.57):

$$\log Z_\epsilon = \frac{1}{2}\sum_\Delta P_\Delta(\hat\delta - \Delta)\,\zeta'(0,\Delta) + b_{d+1}(\nu)\log(e^{-\gamma}/\epsilon) + \sum_{k=0}^{d}\frac{b_k(\nu)}{d+1-k}\,\epsilon^{-(d+1-k)}\,, \tag{C.86}$$

with $b_k(\nu)$ defined as before, $\frac{1}{2t}F_\nu(t) = \sum_{k=0}^{d+1} b_k(\nu)\,t^{-(d+2-k)} + O(t^0)$. Unsurprisingly, this differs from (C.57) only in its UV part, more specifically in the terms polynomial in ν, including the "multiplicative anomaly" term discussed below (C.58). The transcendental (ζ') part and the $\log\epsilon$ coefficient remain unchanged. This remains true in any other regularization.

If we stick with heat-kernel regularization but pick a different regulator $f(\tau/\epsilon^2)$ instead of

$e^{-\epsilon^2/4\tau}$ (e.g. the $f = (1 - e^{-\tau\Lambda^2})^k$ PV regularization of section 3.2) or use zeta function regularization, more is true: the same finite part is obtained for any choice of f provided logarithmically divergent terms (arising in even $d+1$) are expressed in terms of M defined as in (C.67) with $e^{-\epsilon^2/4\tau} \to f$. The relation $M(\epsilon)$ will depend on f, but nothing else.

In dimensional regularization, some polynomial terms in ν will be different, including the "multiplicative anomaly" term. Of course no physical quantity will be affected by this, as long as self-consistency is maintained. In fact any regularization scheme (even (C.86)) will lead to the same physically unambiguous part of the one-loop corrected dS entropy/sphere partition function of section 3.8. However to go beyond this, e.g. to extract more physically unambiguous data by comparing different saddles along the lines of (C.342) and (C.345), a portable covariant regularization scheme, like heat-kernel regularization, must be applied consistently to each saddle. A sphere-specific ad-hoc regularization as in (C.86) is not suitable for such purposes.

C.4 Some useful dimensions, volumes and metrics

C.4.1 Dimensions of representations of $SO(K)$

General irreducible representations of $SO(K)$ with $K = 2r$ or $K = 2r+1$ are labeled by r-row Young diagrams or more precisely a set $S = (s_1, \ldots, s_r)$ of highest weights ordered from large to small, which are either all integer (bosons) or all half-integer (fermions). When $K = 2r$, s_r can be either positive of negative, distinguishing the chirality of the representation. For various applications in this paper we need the dimensions D^K_S of these $SO(K)$ representations S. The Weyl dimension formula gives a general expression for the dimensions of irreducible representations of simple Lie groups. For the $SO(K)$ this is

- $K = 2r$:

$$D^K_S = \mathcal{N}_K^{-1} \prod_{1 \le i < j \le r} \big(\ell_i + \ell_j\big)\big(\ell_i - \ell_j\big), \qquad \ell_i \equiv s_i + \tfrac{K}{2} - i \tag{C.87}$$

with $\mathcal{N}_K$ independent of S, hence fixed by $D^K_0 = 1$, i.e. $\mathcal{N}_K = \prod_{1 \le i < j \le r}(K - i - j)(j - i)$.

• $K = 2r + 1$:

$$D_S^K = \mathcal{N}_K^{-1} \prod_{1\leq i\leq r} (2\ell_i) \prod_{1\leq i<j\leq r} \big(\ell_i + \ell_j\big)\big(\ell_i - \ell_j\big)\,, \qquad \ell_i \equiv s_i + \tfrac{K}{2} - i\,, \tag{C.88}$$

where $\mathcal{N}_K$ is fixed as above: $\mathcal{N}_K = \prod_{1\leq i\leq r}(K-2i)\prod_{1\leq i<j\leq r}(K-i-j)(j-i)$.

For convenience we list here some low-dimensional explicit expressions:

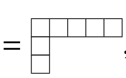
K	D_s^K	$D_{n,s}^K$	$D^K_{k+\frac{1}{2},\boldsymbol{\frac{1}{2}}}$
2	1		1
3	$2s+1$		$2\binom{k+1}{1}$
4	$(s+1)^2$	$(n-s+1)\,(n+s+1)$	$2\binom{k+2}{2}$
5	$\frac{(s+1)(s+2)(2s+3)}{6}$	$\frac{(2n+3)(n-s+1)(n+s+2)(2s+1)}{6}$	$4\binom{k+3}{3}$
6	$\frac{(s+1)(s+2)^2(s+3)}{12}$	$\frac{(n+2)^2(n-s+1)(n+s+3)(s+1)^2}{12}$	$4\binom{k+4}{4}$
7	$\frac{(s+1)(s+2)(s+3)(s+4)(2s+5)}{120}$	$\frac{(n+2)(n+3)(2n+5)(n-s+1)(n+s+4)(s+1)(s+2)(2s+3)}{720}$	$8\binom{k+5}{5}$
8	$\frac{(s+1)(s+2)(s+3)^2(s+4)(s+5)}{360}$	$\frac{(n+2)(n+3)^2(n+4)(n-s+1)(n+s+5)(s+1)(s+2)^2(s+3)}{4320}$	$8\binom{k+6}{6}$

(C.89)

Here $(k+\frac{1}{2},\boldsymbol{\frac{1}{2}})$ means $(s_1,\ldots,s_r) = (k+\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$, i.e. the spin $s = k+\frac{1}{2}$ representation.

For general $d \geq 3$, we can use (C.15) and (C.135) to compute

$$D_s^K = \binom{s+K-1}{K-1} - \binom{s+K-3}{K-1}\,, \qquad D_{n,s}^K = D_n^K D_s^{K-2} - D_{s-1}^K D_{n+1}^{K-2}\,. \tag{C.90}$$

Denoting 1 repeated m times by 1^m, e.g. $(5,1^2) = (5,1,1) =$ , we furthermore have

$$D_{1^p}^d = \binom{d}{p} \quad (p < \tfrac{d}{2}), \qquad D_{1^{p-1},\pm 1}^{2p} = \frac{1}{2}\binom{2p}{p}\,, \qquad D_{n,s,1^m}^{d+2} = D_n^{d+2} D_{s,1^m}^d - D_{s-1}^{d+2} D_{n+1,1^m}^d\,. \tag{C.91}$$

C.4.2 Volumes

The volume of the unit sphere S^n is

$$\Omega_n \equiv \mathrm{vol}(S^n) = \frac{2\,\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \;=\; \frac{2\pi}{n-1} \cdot \Omega_{n-2} \tag{C.92}$$

The volume of $SO(d+2)$ with respect to the invariant group metric normalized such that minimal $SO(2)$ orbits have length 2π is

$$\mathrm{vol}\big(SO(d+2)\big)_{\mathrm{c}} = \prod_{k=2}^{d+2} \mathrm{vol}(S^{k-1}) = \prod_{k=2}^{d+2} \frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})} \,. \tag{C.93}$$

This follows from the fact that the unit sphere $S^{n-1} = SO(n)/SO(n-1)$, which implies $\mathrm{vol}(SO(n))_{\mathrm{c}} = \mathrm{vol}(S^{n-1})\,\mathrm{vol}(SO(n-1))_{\mathrm{c}}$ in the assumed normalization.

The volume of $SU(N)$ with respect to the invariant metric derived from the matrix trace norm on the Lie algebra su(N) viewed as traceless $N \times N$ matrices is (see e.g. [185])

$$\mathrm{vol}\big(SU(N)\big)_{\mathrm{Tr}_N} \;=\; \sqrt{N} \prod_{k=2}^{N} \frac{(2\pi)^k}{\Gamma(k)} \;=\; \sqrt{N}\, \frac{(2\pi)^{\frac{1}{2}(N-1)(N+2)}}{\mathrm{G}(N+1)} \,. \tag{C.94}$$

C.4.3 de Sitter and its Wick rotations to the sphere

Global dS_{d+1} has a convenient description as a hyperboloid embedded in $\mathbb{R}^{1,d+1}$,

$$X^I X_I \equiv \eta_{IJ} X^I X^J \equiv -X_0^2 + X_1^2 + \cdots + X_{d+1}^2 = \ell^2 \,, \qquad ds^2 = \eta_{IJ}\, dX^I dX^J \,. \tag{C.95}$$

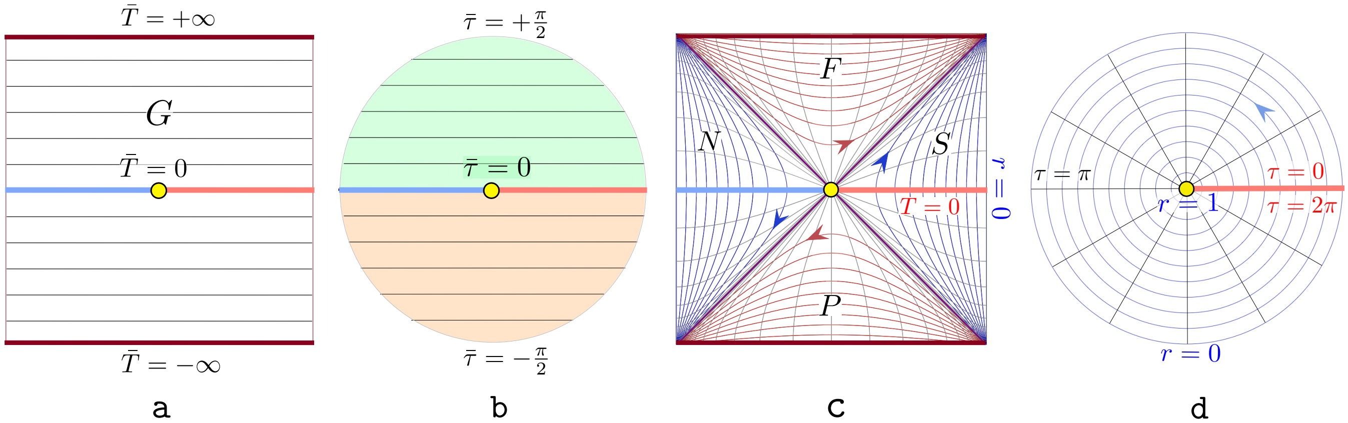


Figure C.5: Penrose diagrams of dS_{d+1} and S^{d+1} with coordinates C.96, C.98. Each point corresponds to an S^{d-1}, contracted to zero size at thin-line boundaries. `a`: Global dS_{d+1} in slices of constant $\bar{T}$. `b`: Wick rotation of global dS_{d+1} to S^{d+1}. `c`: S/N = southern/northern static patch, F/P = future/past wedge; slices of constant T (gray) and r (blue/red) = flows generated by H. Yellow dot = horizon $r = 1$. `d`: Wick-rotation of static patch S to S^{d+1}; slices of constant τ and constant r.

Below we set $\boxed{\ell \equiv 1}$. The isometry group is $SO(1, d+1)$, with generators $M_{IJ} = X_I \partial_J - X_J \partial_I$. Various coordinate patches are shown in fig. C.5`a`, `c`, with coordinates and metric given by

co	embedding $(X^0, \ldots, X^{d+1})$	coordinate range	metric $ds^2 = \eta_{IJ} dX^I dX^J$
G	$(\sinh \bar{T}, \cosh \bar{T}\, \bar{\Omega})$	$\bar{T} \in \mathbb{R}$, $\bar{\Omega} \in S^d$	$-d\bar{T}^2 + \cosh^2 \bar{T}\, d\bar{\Omega}^2$
S	$(\sqrt{1-r^2} \sinh T, r\Omega, \sqrt{1-r^2} \cosh T)$	$T \in \mathbb{R}, 0 \le r < 1$, $\Omega \in S^{d-1}$	$-(1-r^2) dT^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega^2$
F	$(\sqrt{r^2-1} \cosh T, r\Omega, \sqrt{r^2-1} \sinh T)$	$T \in \mathbb{R}$, $r > 1$, $\Omega \in S^{d-1}$	$-\frac{dr^2}{r^2-1} + (r^2-1) dT^2 + r^2 d\Omega^2$

(C.96)

illustrated in fig. C.5`a`, `c`. N is obtained from S by $X^{d+1} \to -X^{d+1}$, and P from F by $X^0 \to -X^0$. The southern static patch S is the part of de Sitter causally accessible to an inertial observer at the south pole of the global spatial S^d. The metric in this patch is static, with the observer at $r = 0$ and a horizon at $r = 1$. The $SO(1,1)$ generator $H = M_{0,d+1}$ acts by translation of the coordinate T, which is timelike in S, N and spacelike in F, P. From the direction of the flow lines in fig. C.5`c`, it can be seen that the positive energy operator is H in S, whereas it is $-H$ in N. In F/P, r is the time coordinate, and H is the operator corresponding to spatial momentum along the T-axis of the $\mathbb{R} \times S^{d-1}$ spatial slices.

A Wick rotation $X^0 \to -iX^0$ maps (C.95) to the round sphere S^{d+1}:

$$\delta_{IJ} X^I X^J = \ell^2 \,, \qquad ds^2 = \delta_{IJ} dX^I dX^J \,. \tag{C.97}$$

The full S^{d+1} can be obtained either from global dS G by Wick rotating global time $\bar{T} \to -i\bar{\tau}$, or from a single static patch S by Wick rotating static time $T \to -i\tau$, as illustrated in fig. C.5`b,d`. The corresponding sphere coordinates and metric are, again setting $\boxed{\ell \equiv 1}$

co	embedding $(X^0, X^1, \ldots, X^{d+1})$	coordinate range	metric $ds^2 = \delta_{IJ} dX^I dX^J$
G	$(\sin\bar{\tau}, \cos\bar{\tau}, \bar{\Omega})$	$-\frac{\pi}{2} \leq \bar{\tau} \leq \frac{\pi}{2}, \bar{\Omega} \in S^d$	$d\bar{\tau}^2 + \cos^2\bar{\tau}\, d\bar{\Omega}^2$
S	$(\sqrt{1-r^2}\sin\tau, r\Omega, \sqrt{1-r^2}\cos\tau)$	$0 \leq r < 1, \tau \simeq \tau + 2\pi,\ \Omega \in S^{d-1}$	$(1-r^2)d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega^2$

(C.98)

C.5 Euclidean vs canonical: formal & physics expectations

Given a QFT on a static spacetime $\mathbb{R} \times M$ with metric $ds^2 = -dt^2 + ds_M^2$, Wick rotating $t \to -i\tau$ yields a Euclidean QFT on a space with metric $ds^2 = d\tau^2 + ds_M^2$. The Euclidean path integral $Z_{\rm PI}(\beta) = \int \mathcal{D}\Phi\, e^{-S[\Phi]}$ on $S^1_\beta \times M$ obtained by identifying $\tau \simeq \tau + \beta$ equals the thermal partition function: $Z_{\rm PI}(\beta) = \mathrm{Tr}\, e^{-\beta H}$, as follows from cutting the path integral along constant-τ slices and viewing $e^{-\tau H}$ as the Euclidean time evolution operator.

At least for noninteracting theories, it is in practice much more straightforward to compute the partition function as the state sum $\mathrm{Tr}\, e^{-\beta H}$ of an ideal gas in a box M than as a one-loop path integral $Z_{\rm PI} = \int \mathcal{D}\Phi\, e^{-S[\Phi]}$ on $S^1_\beta \times M$, in particular for higher-spin fields. In view of this, it is reasonable to wonder if a free QFT path integral on the sphere could perhaps similarly be computed as a simple state sum, by viewing the sphere as the Wick-rotated static patch (fig. C.5`d`), with inverse temperature $\beta = 2\pi$ given by the period of the angular coordinate τ:

$$Z_{\rm PI} \overset{?}{=} \mathrm{Tr}_S\, e^{-2\pi H} \,. \tag{C.99}$$

Below we review the formal path integral slicing argument suggesting the above relation and why it fails, emphasizing the culprit is the presence of a fixed-point locus of H, the yellow dot in fig. C.5. At the same formal level, we show the above relation is equivalent to $Z_{\rm PI} \stackrel{?}{=} Z_{\rm bulk}$, with $Z_{\rm bulk}$ defined as a character integral as in section 3.2. This improves the situation, but is still incorrect for spin $s \geq 1$. In more detail, the content is as follows:

In C.5.1 we consider the $d = 0$ case: a scalar of mass ω on dS_1 in its Euclidean vacuum state, i.e. an entangled pair of harmonic oscillators. Though surely superfluous to most readers, we use the occasion to provide a pedagogical introduction to some standard constructions.

In C.5.2 we formally apply the same template to general d, ignoring yellow-dot issues, leading to the standard formal "thermofield double" description of the static patch of de Sitter [149], and more specifically to $Z_{\rm PI} \simeq \mathrm{Tr}\, e^{-2\pi H} \simeq Z_{\rm bulk}$. We review the pathological divergences that ensue when one attempts to evaluate the trace, and some of its proposed fixes such as the "brick-wall" cutoff [35] and refinements thereof. We contrast these to $Z_{\rm bulk}$ defined as a character integral.

In C.5.5, we turn to the edge corrections missed by such formal arguments, explaining from various points of view why they are to be expected.

C.5.1 S^1

Though slightly silly, it is instructive to first consider the $d = 0$ case: a free scalar field of mass ω on dS_1 (fig. C.6). Global dS_1 is the hyperbola $X_0^2 - X_1^2 = 1$ according to (C.95), which consists of two causally disconnected lines, globally parametrized according to table C.96 by $(\bar{T}, \bar{\Omega})$ where $\bar{\Omega} \in S^0 = \{-1, +1\} \equiv \{N, S\}$. The pictures of fig. C.5 still apply, except there are no interior points, resulting in fig. C.6. Putting a free scalar of mass ω on this space just means we consider two harmonic oscillators ϕ_S and ϕ_N, with action

$$S_L = \frac{1}{2} \int_{-\infty}^{\infty} d\bar{T} \big(\dot{\phi}_S^2 - \omega^2 \phi_S^2 + \dot{\phi}_N^2 - \omega^2 \phi_N^2 \big) \,. \tag{C.100}$$

The dS_1 isometry group is $SO(1, d + 1) = SO(1, 1)$, generated by $H \equiv M_{01}$, which acts as forward/backward time translations on ϕ_S/ϕ_N, to be contrasted with the global *Hamiltonian* H', which

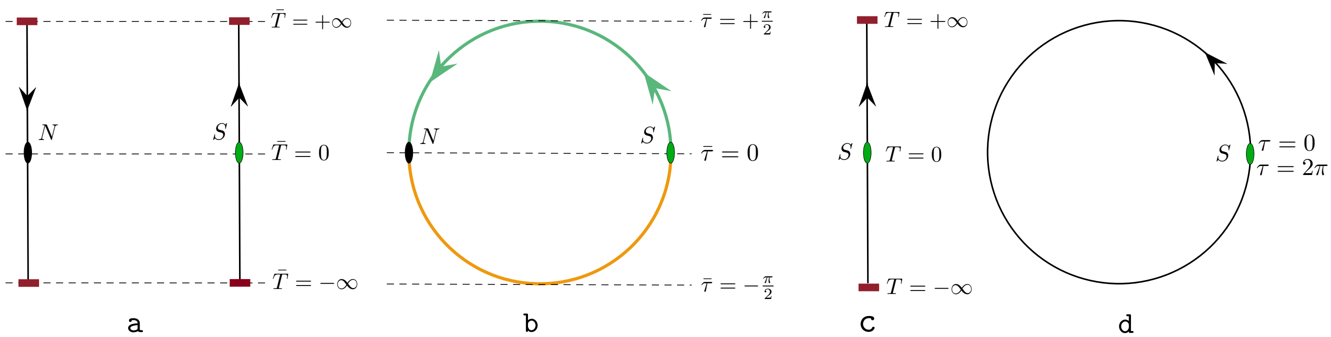


Figure C.6: dS_1 version of fig. C.5 (in c we only show S here). Wick rotation of global time $\bar{T} \to -i\bar{\tau}$ maps a $\to$ b while wick rotation of static patch time $T \to -i\tau$ maps c $\to$ d. Coordinates are as defined in tables C.96 and C.98 with $d = 0$.

acts as forward time translations on both. The southern and northern static patch are parametrized by T, and each contains one harmonic oscillator, respectively ϕ_S and ϕ_N. Introducing creation and annihilation operators $a^S_\omega, a^{S\dagger}_\omega, a^N_{-\omega}, a^{N\dagger}_{-\omega}$ satisfying $[a, a^\dagger] = 1$, we have

$$H = H_S - H_N\,, \quad H' = H_S + H_N\,, \qquad H_S = \omega\big(a^{S\dagger}_\omega a^S_\omega + \tfrac{1}{2}\big)\,, \quad H_N = \omega\big(a^{N\dagger}_{-\omega} a^N_{-\omega} + \tfrac{1}{2}\big)\,. \quad \text{(C.101)}$$

The subscript $\pm\omega$ refers to the H eigenvalue: $[H, a^\dagger_{\pm\omega}] = \pm\omega\, a^\dagger_{\pm\omega}$, $[H, a_{\pm\omega}] = \mp\omega\, a_{\pm\omega}$. The southern and northern Hilbert spaces $\mathcal{H}_S$, $\mathcal{H}_N$ each have a positive energy eigenbasis $|n)$ with energies $E_n = (n + \frac{1}{2})\,\omega$. In QFT language, $|0)$ is the static patch "vacuum", and each patch has one "single-particle" state, $|1) = a^\dagger|0)$. The global Hilbert space is $\mathcal{H}_G = \mathcal{H}_S \otimes \mathcal{H}_N$, with basis $|n_S, n_N\rangle = |n_S) \otimes |n_N)$ satisfying $H|n_S, n_N\rangle = \omega(n_S - n_N)|n_S, n_N\rangle$.

Wick-rotating dS_1 produces an S^1 of radius $\ell = 1$. If we consider this as the Wick rotation of the static patch as in fig. C.5d/C.6d, S in table (C.98), the S^1 is parametrized by the periodic Euclidean time coordinate $\tau \simeq \tau + 2\pi$. The corresponding Euclidean action for the scalar is

$$S_E = \frac{1}{2}\int_0^{2\pi} d\tau\big(\dot{\phi}^2 + \omega^2\phi^2\big) \qquad \phi(2\pi) = \phi(0)\,. \qquad \text{(C.102)}$$

The Euclidean path integral $Z_{\rm PI}$ on S^1 is most easily computed by reverting to the canonical formalism with $e^{-\tau H_S} = e^{-\tau H}$ as the Euclidean time evolution operator, which maps it to the harmonic oscillator thermal partition function at inverse temperature $\beta = 2\pi$:

$$Z_{\rm PI} = \int \mathcal{D}\phi \, e^{-S_E[\phi]} = \mathrm{Tr}_{\mathcal{H}_S} \, e^{-2\pi H} = \sum_n e^{-2\pi\omega(n+\frac{1}{2})} = \frac{e^{-2\pi\omega/2}}{1 - e^{-2\pi\omega}} \,. \tag{C.103}$$

We can alternatively consider the S^1 to be obtained as the Wick rotation of *global* dS_1 as in fig. C.5b/C.6b, G in (C.98), parametrizing the S^1 by $(\bar\tau, \bar\Omega)$, $-\frac{\pi}{2} \leq \bar\tau \leq \frac{\pi}{2}$, $\bar\Omega \in S^0 = \{S, N\}$, identifying $(\pm\frac{\pi}{2}, S) = (\pm\frac{\pi}{2}, N)$. The global action (C.100) then Wick rotates to

$$S_E = \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\bar\tau \big(\dot\phi_S^2 + \omega^2\phi_S^2 + \dot\phi_N^2 + \omega^2\phi_N^2\big) , \qquad \phi_S(\pm\tfrac{\pi}{2}) = \phi_N(\pm\tfrac{\pi}{2}) , \tag{C.104}$$

which is identical to (C.102), just written in a slightly more awkward form. This form naturally leads to an interpretation of $Z_{\rm PI}$ as computing the norm squared of the Euclidean vacuum state $|O\rangle$ of the scalar on the *global* dS_1 Hilbert space $\mathcal{H}_G$, by cutting the path integral at the $S^0 = \{N, S\}$ equator $\bar\tau = 0$ of the S^1 (cf. fig. C.6b):

$$Z_{\rm PI} = \int d^2\phi_0 \, \langle O|\phi_0\rangle\langle\phi_0|O\rangle \equiv \langle O|O\rangle \, , \qquad \langle\phi_0|O\rangle \equiv \int_{\bar\tau\leq 0} \mathcal{D}\phi|_{\phi_0} \, e^{-S_E[\phi]} , \tag{C.105}$$

where $\phi_0 = (\phi_{S,0}, \phi_{N,0})$. The notation $\int_{\bar\tau\leq 0}\mathcal{D}\phi|_{\phi_0}$ means the path integral of $\phi = (\phi_S, \phi_N)$ is performed on the lower hemicircle $\bar\tau \leq 0$ (orange part in fig. C.6b), with boundary conditions $\phi|_{\bar\tau=0} = \phi_0$. $\langle O|\phi_0\rangle$ is similarly defined as a path integral on the upper hemicircle (green part). It is not too difficult to explicitly compute $|O\rangle$ in the $|\phi_{S,0}, \phi_{N,0}\rangle$ basis, but it is easier to compute it in the oscillator basis $|n_S, n_N\rangle$, noticing that slicing the path integral defining $|O\rangle$ allows us to write it as $\langle n_S, n_N|O\rangle = (n_S|e^{-\pi H}|n_N) = e^{-\pi\omega(n_S+\frac{1}{2})}\,\delta_{n_S,n_N}$. Thus

$$|O\rangle = \sum_n e^{-\pi\omega(n+\frac{1}{2})}|n, n\rangle = e^{-\pi\omega/2}\exp\big(e^{-\pi\omega} a_\omega^{S\dagger} a_{-\omega}^{N\dagger}\big)|0, 0\rangle \, . \tag{C.106}$$

In the Schrödinger picture, $|O\rangle$ is to be thought of as an initial state at $\bar T = 0$ for global dS_1: pictorially, we are gluing the bottom half of fig. C.6b to the top half of fig. C.6a. This state evolves

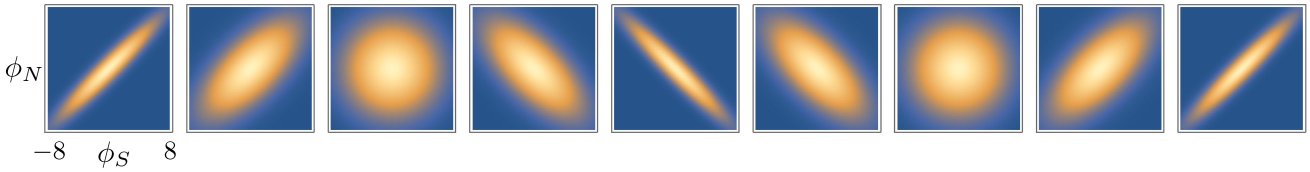


Figure C.7: Global time evolution of $P_{\bar{T}}(\phi_S, \phi_N) = \left|\langle \phi_S, \phi_N| e^{-iH'\bar{T}} |O\rangle\right|^2$ for free $\omega = 0.1$ scalar on dS_1, from $\bar{T} = 0$ to $\bar{T} = \pi/\omega$. $P(\phi_S) = \int d\phi_N \, P_{\bar{T}}(\phi_S, \phi_N)$ is thermal and time-independent.

nontrivially in global time $\bar{T}$: though invariant under $SO(1,1)$ generated by $H = H_S - H_N$, it is *not* invariant under forward global time translations generated by the global Hamiltonian $H' = H_S + H_N$. For viewing pleasure this is illustrated in fig. C.7, which also visually exhibits the north-south entangled nature of $|O\rangle$.

Note that $Z_{\rm PI} = \langle O|O\rangle = \sum_n e^{-2\pi\omega(n+\frac{1}{2})}$, reproducing the dS_1 static patch thermal partition function (C.116). Indeed from the point of view of the static patch, the global Euclidean vacuum state looks thermal with inverse temperature $\beta = 2\pi$: the southern reduced density matrix $\hat{\varrho}_S$ obtained by tracing out the northern degree of freedom ϕ_N in the global Euclidean vacuum $|O\rangle$ is $\hat{\varrho}_S = \sum_n e^{-2\pi\omega(n+\frac{1}{2})} |n)(n| = e^{-2\pi H_S}$. In contrast to the global $|O\rangle$, the reduced density matrix is time-*independent*.

The path integral slicing arguments we used did not rely on the precise form of the action. In particular the conclusions remain valid when we add interactions:

$$|O\rangle = \sum_n e^{-\beta E_n/2} |n,n\rangle \,, \qquad \hat{\varrho}_S = e^{-\beta H_S} \,, \qquad Z_{\rm PI} = \langle O|O\rangle = \mathrm{Tr}_S \, e^{-\beta H} \qquad (\beta = 2\pi) \quad \text{(C.107)}$$

Actually in the $d = 0$ case at hand, we can generalize all of the above to arbitrary values of β. (For $d > 0$ this would create a conical singularity at $r = 1$ on S^{d+1}, but for S^1 the point $r = 1$ does not exist.) Note that since the reduced density matrix is thermal, the north-south entanglement entropy in the Euclidean vacuum $|O\rangle$ equals the thermal entropy: $S_{\rm ent} = -\mathrm{tr}_S \, \varrho_S \log \varrho_S = S_{\rm th} = (1 - \beta\partial_\beta) \log Z$, where $\varrho_S \equiv \hat{\varrho}_S / Z$, $Z = \mathrm{Tr}_S \, \hat{\varrho}_S$.

Despite appearing distinctly non-vacuous from the point of view of a local observer, and being globally time-dependent, the state $|O\rangle$ does deserve its "vacuum" epithet. As already mentioned, it is invariant under the global $SO(1,1)$ isometry group: $H|O\rangle = 0$. Moreover, for the free scalar, (C.106) implies $|O\rangle$ is itself annihilated by a pair of *global* annihilation operators a^G related related to a^S, $a^{S\dagger}$, a^N and $a^{N\dagger}$ by a Bogoliubov transformation:

$$a^G_{\pm\omega}|O\rangle = 0\,, \qquad a^G_{\omega} \equiv \frac{a^S_{\omega} - e^{-\pi\omega} a^{N\dagger}_{-\omega}}{\sqrt{1-e^{-2\pi\omega}}}\,, \quad a^G_{-\omega} \equiv \frac{a^N_{-\omega} - e^{-\pi\omega} a^{S\dagger}_{\omega}}{\sqrt{1-e^{-2\pi\omega}}}\,, \tag{C.108}$$

normalized such that $[a^G_{\pm\omega}, a^{G\dagger}_{\pm\omega}] = \delta_{\pm,\pm}$. From (C.101) we get $H = \omega\, a^{G\dagger}_{\omega} a^G_{\omega} - \omega\, a^{G\dagger}_{-\omega} a^G_{-\omega}$. Thus we can construct the global Hilbert space $\mathcal{H}_G$ as a Fock space built on the Fock vacuum $|O\rangle$, by acting with the global creation operators $a^{G\dagger}_{\pm\omega}$. The Hilbert space $\mathcal{H}^{(1)}_G$ of "single-particle" excitations of the global Euclidean vacuum is two-dimensional, spanned by

$$|\pm\omega\rangle \equiv a^{G\dagger}_{\pm\omega}|O\rangle, \qquad H|\pm\omega\rangle = \pm\omega\,|\pm\omega\rangle\,. \tag{C.109}$$

The character $\chi(t)$ of the $SO(1,1)$ representation furnished by $\mathcal{H}^{(1)}_G$ is

$$\chi(t) \equiv \mathrm{tr}_G\, e^{-itH} = e^{-it\omega} + e^{it\omega}\,. \tag{C.110}$$

The above constructions are straightforwardly generalized to fermionic oscillators. The character of a collection of bosonic and fermionic oscillators of frequencies ω_i and ω'_j is

$$\chi(t) = \mathrm{tr}_G\, e^{-iHt} = \chi(t)_{\mathrm{bos}} + \chi(t)_{\mathrm{fer}} = \sum_{i,\pm} e^{\pm i\omega_i} + \sum_{j,\pm} e^{\pm i\omega'_j}\,. \tag{C.111}$$

Character formula

For a single bosonic resp. fermionic oscillator of frequency ω, $\log \mathrm{Tr}\, e^{-\beta H}$ has the following integral representation:[6]

$$
\begin{aligned}
\log\left(e^{-\beta\omega/2}\big(1-e^{-\beta\omega}\big)^{-1}\right) &= +\int_0^\infty \frac{dt}{2t}\,\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}}\,\big(e^{-i\omega t}+e^{i\omega t}\big) \\
\log\Big(e^{+\beta\omega/2}\big(1+e^{-\beta\omega}\big)\Big) &= -\int_0^\infty \frac{dt}{2t}\,\frac{2\,e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}}\,\big(e^{-i\omega t}+e^{i\omega t}\big)\,.
\end{aligned}
\tag{C.112}
$$

Combining this with (C.111) expresses the thermal partition function of a collection of bosonic and harmonic oscillators as an integral transform of its $SO(1,1)$ character:

$$
\log \operatorname{Tr} e^{-\beta H} = \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}}\;\chi(t)_{\rm bos} - \frac{2\,e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}}\;\chi(t)_{\rm fer}\right). \tag{C.113}
$$

The Euclidean path integral on an S^1 of radius $\ell = 1$ for a collection of free bosons and fermions (the latter with thermal, i.e. antiperiodic, boundary conditions) is then given by putting $\beta = 2\pi$ in the above:

$$
\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-t}}{1-e^{-t}}\;\chi(t)_{\rm bos} - \frac{2\,e^{-t/2}}{1-e^{-t}}\;\chi(t)_{\rm fer}\right). \tag{C.114}
$$

C.5.2 S^{d+1}

The arguments in this section will be formal, following the template of section C.5.1 while glossing over some important subtleties, the consequence of which we discuss in section C.5.5.

Wick-rotating a QFT on dS_{d+1} to S^{d+1}, we get the Euclidean path integral

$$
Z_{\rm PI} = \int \mathcal{D}\Phi\, e^{-S_E[\Phi]}, \tag{C.115}
$$

where Φ collects all fields in the theory. Just like in the $d=0$ case, the two different paths from dS_{d+1} to S^{d+1}, i.e. Wick-rotating global time $\bar{T}$ or static patch time T (cf. fig. C.5 and table C.98), naturally give rise to two different dS Hilbert space interpretations: one involving the global Hilbert

[6] The t^{-2} pole of the integrand is resolved by the $i\epsilon$-prescription $t^{-2} \to \frac{1}{2}\left((t-i\epsilon)^{-2}+(t+i\epsilon)^{-2}\right)$, left implicit here and in the formulae below. The integral formula can be checked by observing the integrand is even in t, extending the integration contour to the real line, closing the contour, and summing residues.

space $\mathcal{H}_G$ and one involving the static patch Hilbert space $\mathcal{H}_S$.

The global Wick rotation of fig. C.5b leads to an interpretation of $Z_{\rm PI}$ as computing $\langle O|O\rangle$, analogous to (C.105), by cutting the path integral on the globlal S^d equator $\bar\tau = 0$:

$$Z_{\rm PI} = \int_{\bar\tau=0} d\Phi_0 \, \langle O|\Phi_0\rangle\langle\Phi_0|O\rangle \equiv \langle O|O\rangle \, , \qquad \langle\Phi_0|O\rangle \equiv \int_{\bar\tau\leq 0} \mathcal{D}\Phi|_{\Phi_0} \, e^{-S_E[\Phi]} \, , \tag{C.116}$$

where $\int_{\bar\tau\leq 0}\mathcal{D}\phi|_{\phi_0}$ means the path integral is performed on the lower hemisphere $\bar\tau \leq 0$ of S^{d+1} (orange region in fig. C.5b) with boundary conditions $\Phi|_{\bar\tau=0} = \Phi_0$. $\langle O|\Phi_0\rangle$ is similarly defined as a path integral on the upper hemisphere $\bar\tau \geq 0$ (green region). This defines the Hartle-Hawking/Euclidean vacuum state $|O\rangle$ [186] of global dS_{d+1}, with $Z_{\rm PI}$ computing the natural pairing of $|O\rangle$ with $\langle O|$.[7]

The static patch Wick rotation of fig. C.5d on the other hand leads to an interpretation of $Z_{\rm PI}$ as a thermal partition function at inverse temperature $\beta = 2\pi$, analogous to (C.103): slicing the path integral along constant-τ slices as in fig. C.5d, and viewing $e^{-\tau H}$ with $H = M_{0,d+1}$ as the Euclidean time evolution operator acting on $\mathcal{H}_S$, we formally get[8]

$$Z_{\rm PI} \simeq \mathrm{Tr}_{\mathcal{H}_S} \, e^{-\beta H} \qquad (\beta = 2\pi) \, . \tag{C.117}$$

Like in the $d = 0$ case, this interpretation can be related to the global interpretation (C.116). Picking suitable bases of $\mathcal{H}_S$ and $\mathcal{H}_N$ diagonalizing H, and applying a similar slicing argument,

[7]For kind enough theories, such as a scalar field theory, this pairing can be identified with the Hilbert space inner product. However not all theories are kind enough, as is evident from the negative-mode rotation phase $i^{-(d+3)}$ in the one-loop graviton contribution to $Z_{\rm PI} = \langle O|O\rangle$ according to (3.112) and [59]. Indeed for gravity this pairing is not in an obvious way related to the semiclassical inner product of [187]. On the other hand, in the CS formulation of 3D gravity it appears to be framing-dependent, vanishing in particular for canonical framing (cf. (C.252) and discussion below it). The phase also drops out of $\langle A\rangle \equiv \langle O|A|O\rangle/\langle O|O\rangle$.

[8] The notation $\simeq$ means "equal according to these formal arguments". Besides the default deferment of dealing with divergences, we are ignoring some additional important points here, including in particular the fixed points of H: the S^{d-1} at $r = 1$ (yellow dot in fig. C.5), where the equal-τ slicing of (C.117) degenerates, and the $\mathcal{H}_G = \mathcal{H}_N \otimes \mathcal{H}_S$ factorization implicit in (C.118) breaks down. We return to these points in section C.5.5.

we formally get the analog of (C.107):

$$|O\rangle \simeq \text{``}\sum_n\text{''} e^{-\beta E_n/2}|E_n, E_n\rangle\,, \qquad \hat{\varrho}_S \simeq e^{-\beta H_S} \qquad (\beta = 2\pi)\,, \tag{C.118}$$

where we have put the sum in quotation marks because the spectrum is actually continuous, as we will describe more precisely for free QFTs below. Granting this, we conclude that an inertial observer in de Sitter space sees the global Euclidean vacuum as a thermal state at inverse temperature $\beta = 2\pi$, the Hawking temperature of the observer's horizon [9, 10, 149].

Applying (C.117) to a free QFT on dS_{d+1}, we can write the corresponding Gaussian Z_{PI} on S^{d+1} as the thermal partition function of an ideal gas in the southern static patch:

$$\log Z_{\mathrm{PI}} \simeq \log \mathrm{Tr}_S\, e^{-2\pi H} = \sum_{\pm} \mp \int_0^\infty d\omega\, \rho_S(\omega)_\pm \big(\log\Big(1 \mp e^{-2\pi\omega}\Big) + 2\pi\omega/2\big)\,, \tag{C.119}$$

where $\rho_S(\omega) \equiv \mathrm{tr}_S\, \delta(\omega - H)$ is the density of single-particle states at energy $\omega > 0$ above the vacuum energy in the static patch, split into bosonic and fermionic parts as $\rho_S = \rho_{S+} + \rho_{S-}$. Using (C.4), we can write the character for arbitrary $SO(1, d+1)$ representations as

$$\chi(t) = \mathrm{tr}_G\, e^{-itH} = \int_0^\infty \rho_G(\omega)\left(e^{-i\omega t} + e^{i\omega t}\right) \tag{C.120}$$

where $\rho_G(\omega) \equiv \mathrm{tr}_G\, \delta(\omega - H)$. The Bogoliubov map (C.108) formally implies $\rho_G(\omega) \simeq \rho_S(\omega)$ for $\omega > 0$, hence, following the reasoning leading to (C.114),

$$\log Z_{\mathrm{PI}} \simeq \log Z_{\mathrm{bulk}} \equiv \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-t}}{1-e^{-t}}\, \chi(t)_{\mathrm{bos}} - \frac{2\,e^{-t/2}}{1-e^{-t}}\, \chi(t)_{\mathrm{fer}}\right). \tag{C.121}$$

C.5.3 Brick wall regularization

Here we review how attempts at evaluating the ideal gas partition function (C.119) directly hit a brick wall. Consider for example a scalar field of mass m^2 on dS_{d+1}. Denoting $\Delta_\pm = \frac{d}{2} \pm \big((\frac{d}{2})^2 -$

$m^2\big)^{1/2}$, the positive frequency solutions on the static patch are of the form

$$\phi_{\omega\sigma}(T,\Omega,r) \propto e^{-i\omega T}\, Y_\sigma(\Omega)\; r^\ell \left(1-r^2\right)^{i\omega/2} {}_2F_1\big(\tfrac{\ell+\Delta_+ +i\omega}{2}, \tfrac{\ell+\Delta_- +i\omega}{2}; \tfrac{d}{2}+\ell; r^2\big), \qquad \text{(C.122)}$$

where $\omega > 0$, and $Y_\sigma(\Omega)$ is a basis of spherical harmonics on S^{d-1} labeled by σ, which includes the total $SO(d)$ angular momentum quantum number ℓ. A basis of energy and $SO(d)$ angular momentum eigenkets is therefore given by $|\omega\sigma)$ satisfying $(\omega\sigma|\omega'\sigma') = \delta(\omega-\omega')\,\delta_{\sigma\sigma'}$. Naive evaluation of the density of states in this basis gives a pathologically divergent result $\rho_S(\omega) = \int d\omega' \sum_\sigma (\omega'\sigma|\delta(\omega-\omega')|\omega'\sigma) = \sum_\sigma \delta(0)$, and commensurate nonsense in (C.119).

Pathological divergences of this type are generic in the presence of a horizon. Physically they can be thought of as arising from the fact that the infinite horizon redshift enables the existence of field modes with arbitrary angular momentum and energy localized in the vicinity of the horizon. One way one therefore tries to deal with this is to replace the horizon by a "brick wall" at a distance δ away from the horizon [35], with some choice of boundary conditions, say $\phi(T,\Omega,1-\frac{1}{2}\delta^2) = 0$ in the example above. This discretizes the energy spectrum and lifts the infinite angular momentum degeneracy, allowing in principle to control the divergences as $\delta \to 0$. However, inserting a brick wall alters what one is actually computing, introduces ambiguities (e.g. Dirichlet/Neumann), potentially leads to new pathologies (e.g. Dirichlet boundary conditions for the graviton are not elliptic [188]), and breaks most of the symmetries in the problem.

A more refined version of the idea considers the QFT in Pauli-Villars regularization [36]. This eliminates the dependence on δ in the limit $\delta \to 0$ at fixed PV-regulator scale Λ. It was shown in [36] that for scalar fields the remaining divergences for $\Lambda \to \infty$ agree with those of the PV-regulated path integral.[9] A somewhat different approach, reviewed in [37, 135], first maps the equations of motion in the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ by a (singular) Weyl transformation to formally equivalent equations of motion in the "optical" metric $d\bar{s}^2 = |g_{00}|^{-1} ds^2$. In the case at hand this would be $d\bar{s}^2 = -dT^2 + (1-r^2)^{-2}dr^2 + (1-r^2)^{-1}r^2 d\Omega^2$, corresponding to $\mathbb{R}\times$ hyperbolic d-ball. The thermal trace is then mapped to a path integral on the Euclidean optical geometry with an

[9] This work directly inspired the use of Pauli-Villars regularization in section 3.2.

S^1 of constant radius β and a Weyl-transformed action. (This is not a standard covariant path integral. In the case at hand, unless the theory happens to be conformal, non-metric r-dependent terms break the $SO(1,d)$ symmetry of the hyperbolic ball to $SO(d)$.) This path integral can be expressed in terms of a heat kernel trace $\int_x \langle x|e^{-\tau\bar{D}}|x\rangle$. The divergences encountered earlier now arise from the fact that the optical metric $d\bar{s}^2$ has infinite volume near $r = 1$. This is regularized by cutting the $\int_x$ integral off at $r = 1 - \delta$, analogous to the brick wall cutoff, though computationally more convenient. For scalars and spinors, Pauli-Villars or dimensional regularization again allows trading the $\delta \to 0$ divergences for the standard UV divergences [37].

Unfortunately, certainly for general field content and in the absence of conformal invariance, none of these variants offers any simplification compared to conventional Euclidean path integral methods. In the case of interest to us, the large underlying $SO(1, d+1)$ symmetry is broken, and with it one's hope for easy access to exact results. Generalization to higher-spin fields, or even just the graviton, appears challenging at best.

C.5.4 Character regularization

The character formula (C.121) is formally equivalent to the ideal gas partition function (C.119), and indeed at first sight, naive evaluation in a global single-particle basis $|\omega\sigma\rangle = a^{G\dagger}_{\omega\sigma}|0\rangle$ diagonalizing $H = \omega \in \mathbb{R}$, obtained e.g. by quantization of the natural cylindrical mode functions of the future wedge (F in fig. C.5 and table C.96), gives a similarly pathological $\chi(t) = \mathrm{tr}_G\, e^{-iHt} = \int_{-\infty}^{\infty} d\omega \sum_\sigma \langle\omega\sigma|e^{-i\omega t}|\omega\sigma\rangle = 2\pi\delta(t)\sum_\sigma \delta(0)$; hardly a surprise in view of the Bogoliubov relation $\rho_G(\omega) \simeq \rho_S(\omega)$ and our earlier result $\rho_S(\omega) = \sum_\sigma \delta(0)$. Thus the conclusion would appear to be that the situation is as bad, if not worse, than it was before.

However this is very much the wrong conclusion. As reviewed in appendix C.1, $\chi(t)$, properly defined as a Harish-Chandra character, is in fact rigorously well-defined, analytic in t for $t \neq 0$, and moreover easily computed. For example for a scalar of mass m^2 on dS_{d+1}, we get (C.3):

$$\chi(t) = \frac{e^{-t\Delta_+} + e^{-t\Delta_-}}{|1 - e^{-t}|^d} \qquad \Delta_\pm \equiv \tfrac{d}{2} \pm \sqrt{\left(\tfrac{d}{2}\right)^2 - m^2} \tag{C.123}$$

as explicitly computed in appendix C.1.2. The reason why naive computation by diagonalization of H fails so badly is explained in detail in appendix C.1.3: it is not the trace itself that is sick, but rather the *basis* $|\omega\sigma\rangle$ used in the naive computation.

Substituting the explicit $\chi(t)$ into the character integral (C.121), we still get a UV-divergent result, but this divergence is now easily regularized in a standard, manifestly covariant way, as explained in section 3.2.2. Keeping the large underlying symmetry manifest allows exact evaluation, for arbitrary particle content.

In section 3.3 we show that for scalars and spinors, the Euclidean path integral $Z_{\rm PI}$ on S^{d+1}, regularized as in (3.66), exactly equals $Z_{\rm bulk}$ as defined in (C.121), regularized as in (3.73):

$$Z_{{\rm PI},\epsilon} = Z_{{\rm bulk},\epsilon} \qquad \text{(scalars and spinors)}\,, \tag{C.124}$$

One might wonder how it is possible the switch to characters makes such a dramatic difference. After all, (C.119) and (C.121) are formally equal. Yet the former first evaluates to nonsense and then hits a brick wall, while the latter somehow ends up effortlessly producing sensible results upon standard UV regularization. The discussion in C.1.3, in particular (C.20), provides some clues: character regularization can be thought of, roughly speaking, as being akin to a regularization cutting off *global* $SO(d+1)$ angular momentum.

This goes some way towards explaining why the character formalism fits naturally with the Euclidean path integral formalism on S^{d+1}, as covariant (e.g. heat kernel) regularization of the latter effectively cuts off the $SO(d+2) \supset SO(d+1)$ angular momentum.

It also goes some way towards explaining what happened above. One way of thinking about the origin of the pathological divergences encountered in section C.5.3 is that, as mentioned in footnote 8, the formal argument implicitly starts from the premise that the QFT Hilbert space can be factorized as $\mathcal{H}_G = \mathcal{H}_S \otimes \mathcal{H}_N$, like in the S^1 toy model. However this cannot be done in the continuum limit of QFT: locally factorized states, such as the formal state $|O) \otimes |O)$ in which both the southern and the northern static patch are in their minimal energy state, are violently singular objects [189]. Cutting off the global $SO(d+1)$ angular momentum does indeed smooth out the sharp north-south divide: $SO(d+1)$ is the isometry group of the global spatial slice at $\bar{T}=0$ (fig. C.5b). The angular momentum cutoff means we only have a finite number of spherical harmonics available to build our field modes. This makes it impossible in particular to build field modes sharply localized in the southern or northern hemisphere: the harmonic expansion of a localized mode always has infinitely many terms. Cutting off this expansion will necessarily leave some support on the other hemisphere. Quite similar in this way again to the Euclidean path integral, this offers some intuition on why the UV-regularized character integral avoids the pathological divergences induced by sharply cutting space.

C.5.5 Edge corrections

In view of all this and (C.124), one might be tempted at this point to jump to the conclusion that the arguments of section C.5.2, while formal and glossing over some subtle points, are apparently good enough to give the right answer provided we use the character formulation, and that likewise $Z_{\rm PI}^{(1)}$ on the sphere for a field of arbitrary spin s, despite its off-shell baroqueness, is just the ideal gas partition function $Z_{\rm bulk}$ on the dS static patch, calculable with on-shell ease: mission accomplished. As further evidence in favor of declaring footnote 8 overly cautious, one might point to the fact that in the context of theories of *quantum gravity*, identifying $Z_{\rm PI}^{\rm grav}={\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ elegantly reproduces the thermodynamics of horizons inferred by other means [10], and that such identifications are moreover known to be valid in a quantitatively precise way in many well-understood cases in string theory and AdS-CFT. If the formal argument is good enough for quantum gravity, then surely it is good enough for field theory, one might think.

These naive considerations are wrong: the formal relation $Z_{\rm PI}\simeq Z_{\rm bulk}$ for fields of spin $s\geq 1$ receives "edge" corrections. In sections 3.4 and 3.5, we determine these for massive resp. massless spin-s fields on S^{d+1} by direct computation. The results are eqs. (3.84) and (3.112). The corrections we find exhibit a concise and suggestive structure: again taking the form of a character formula like (C.121), but encoding instead a path integral on a sphere in *two lower* dimensions, i.e. on S^{d-1}

rather than S^{d+1}. This S^{d-1} is naturally identified with the horizon $r = 1$, i.e. the edge of the static patch hemisphere, the yellow dot in fig. C.5. The results of section 3.7 then imply $S_{\rm PI} \simeq S_{\rm bulk}$ likewise receives edge corrections (besides corrections due to nonminimal coupling to curvature, which arise already for scalars).

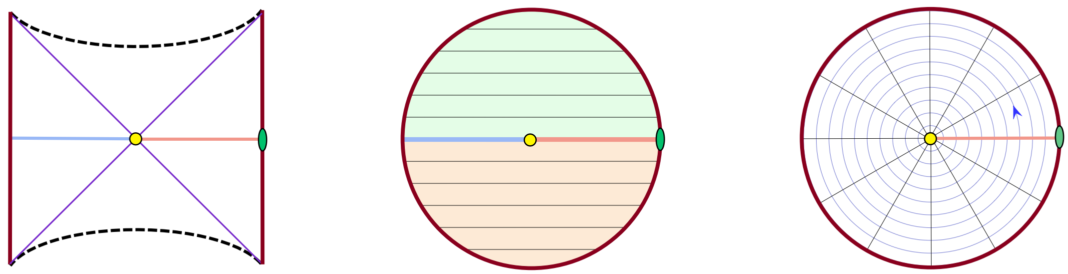
Similar edge corrections, to the entropy $S_{\rm PI} \simeq S_{\rm bulk}$ in the conceptually analogous case of Rindler space, were anticipated long ago in [38] and explicitly computed shortly thereafter for massless spin-1 fields in [39]. The result of [39] was more recently revisited in several works including [43, 48], relating it to the local factorization problem of constrained QFT Hilbert spaces [40–42, 45–47] and given an interpretation in terms of the edge modes arising in this context.

We leave the precise physical interpretation of the explicit edge corrections we obtain in this paper to future work. Below we will review why they were to be expected, and how related corrections can be interpreted in analogous, better-understood contexts in quantum gravity and QFT. We begin by explaining why the quantum gravity argument was misleading and what its correct version actually suggests, first from a boundary CFT point of view in the precise framework of AdS-CFT, then from a bulk point of view in a qualitative picture based on string theory on Rindler space. Finally we return to interpretations within QFT itself, clarifying more directly why the caution expressed in footnote 8 was warranted indeed.

AdS-CFT considerations

As mentioned above, there are reasons to believe that in theories of quantum gravity, the identification $Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ is exact as a semiclassical (small-$G_{\rm N}$) expansion.

However, the key point here is that $\mathcal{H}$ is the Hilbert space of the *fundamental* microscopic degrees of freedom, *not* the Hilbert space of the low energy effective field theory. This can be made very concrete in the context of AdS-CFT, where $\mathcal{H}$ has a precise boundary CFT definition. For example for asymptotically Euclidean AdS$_{d+1}$ geometries with $S^1_\beta \times S^{d-1}$ conformal boundary, certain analogs of the formal relations (C.117) and (C.118) then become exact in the semiclassical/large-N



c   b   d

Figure C.8: AdS-Schwarzschild analogs of c, b, d in fig. C.5. Black dotted line = singularity. Thick brown line = conformal boundary.

expansion [190, 191]:

$$
Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H} = \langle O|O\rangle\,, \qquad |O\rangle = \sum_n e^{-\beta E_n/2} |E_n)_{\mathcal{H}} \otimes |E_n)_{\mathcal{H}}\,. \tag{C.125}
$$

Crucially, $\mathcal{H}$ here is the complete boundary CFT Hilbert space, and $|O\rangle$ is the Euclidean vacuum state of two disconnected copies of the boundary CFT, constructed exactly like in the dS_1 toy model of section C.5.1, but with the hemicircle $\frac{1}{2}S^1$ replaced by $\frac{1}{2}S^1 \times S^{d-1}$. From a semiclassical bulk dual point of view this can be viewed as the Euclidean vacuum of two disconnected copies of global AdS or of the eternal AdS-Schwarzschild geometry [191], depending on whether β lies above or below the Hawking-Page phase transition point β_c [192].

When $\beta > \beta_c$, where $\beta_c \sim O(1)$ assuming the low-energy gravity theory is approximately Einstein with $G_{\rm N} \ll \ell^{d-1} = 1$, $Z_{\rm PI}^{\rm grav}$ is dominated by the thermal EAdS saddle [192], with on-shell action $S_E \equiv 0$, so in the limit $G_{\rm N} \to 0$, $Z_{\rm PI}^{\rm grav} = Z_{\rm PI}^{(1)}$. Thus in this case, the relation $Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ of (C.125) indeed implies $Z_{\rm PI}^{(1)}$ equals a statistical mechanical partition function. There is no need to invoke quantum gravity to see this, of course: the thermal S^1 is noncontractible in the bulk geometry, so the bulk path integral slicing argument is free of subtleties, directly implying $Z_{\rm PI}^{(1)}$ equals the partition function ${\rm Tr}\, e^{-\beta H}$ of an ideal gas in global AdS.

On the other hand if $\beta < \beta_{\rm crit}$, the dominant saddle is the Euclidean Schwarzschild geometry (fig. C.8), with on-shell action $\tilde{S}_E \propto -\frac{1}{G_{\rm N}}$, so in the limit $G_{\rm N} \to 0$, $Z_{\rm PI}^{\rm grav} = Z_{\rm PI}^{(0)} = e^{-\tilde{S}_E}$. In this case the identification $Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ of (C.125) no longer implies the one-loop correction

$Z^{(1)}_{\rm PI}$ can be identified as a statistical mechanical partition function. In particular the bulk one-loop contributions $S^{(1)} = (1 - \beta\partial_\beta) \log Z^{(1)}_{\rm PI}$ to the entropy need not be positive. (More specifically its leading divergent term, which in a UV-complete description of the bulk theory would become finite but generically still dominant, need not be positive.) From the CFT point of view, these are just O(1) corrections in the large-N expansion of the statistical entropy. Although the total entropy must of course be positive, corrections can come with either sign. From the bulk point of view, since the Euclidean geometry is the Wick-rotated exterior of a black hole, the thermal circle is contractible, shrinking to a point analogous to the yellow dot in fig. C.5d, leading to the same issues as those mentioned in footnote 8.

Strings on Rindler considerations

To gain some insight from a bulk point of view, we consider the simplest example of a space-time with a horizon: the Rindler wedge $ds^2 = -\rho^2 dt^2 + d\rho^2 + dx_\perp^2$ of Minkowski space. While not quite at the level of AdS-CFT, we do have a perturbative theory of quantum gravity in Minkowski space: string theory. In fact, that $Z^{(1)}_{\rm PI}$ on a Euclidean geometry with a contractible thermal circle cannot be interpreted as a statistical mechanical partition function in general, even if the full $Z^{\rm grav}_{\rm PI}$ has such an interpretation, was anticipated long ago in [38], in an influential attempt at developing a string theoretic understanding of the thermodynamics of the Rindler horizon. Rindler space Wick rotates to

$$ds^2 = \rho^2 d\tau^2 + d\rho^2 + dx_\perp^2, \qquad \tau \simeq \tau + \beta, \qquad \beta = 2\pi - \epsilon\,. \tag{C.126}$$

with the conical defect $\epsilon = 0$ on-shell. The argument given in [38] is based on the point of view developed in their work that loop corrections in the semiclassical expansion of the Rindler entropy $S_{\rm PI} \equiv (1 - \beta\partial_\beta) \log Z^{\rm grav}_{\rm PI}|_{\beta=2\pi}$ are equivalent to loop corrections to the Newton constant, ensuring the entropy $S = A/4G_{\rm N}$ involves the physically measured $G_{\rm N}$ rather than than the bare $G_{\rm N}$. In $\mathcal{N} = 4$ compactifications of string theory to 4D Minkowski space (and in $\mathcal{N} = 4$ supergravity theories

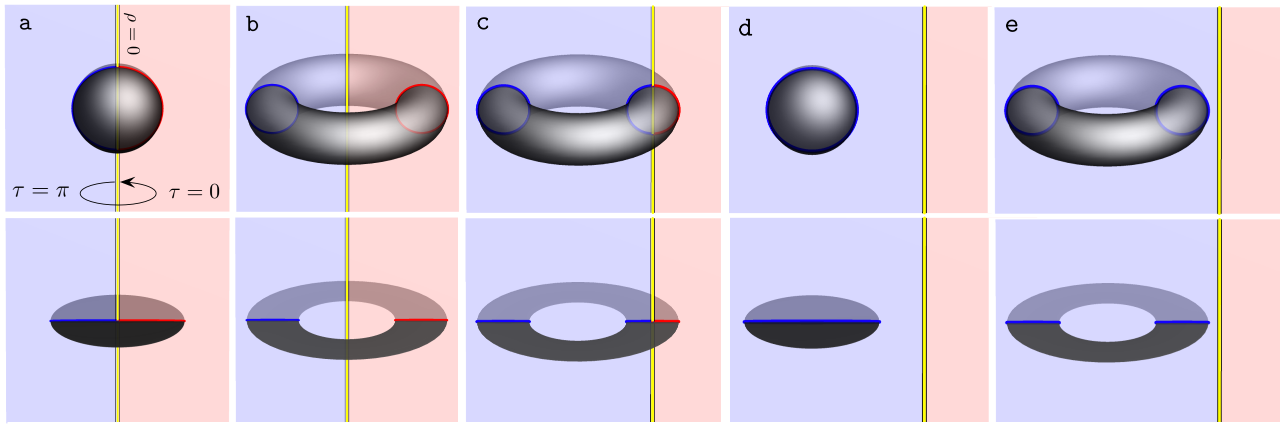


Figure C.9: Closed/open string contributions to the total Euclidean Rindler ($ds^2 = \rho^2 d\tau^2 + d\rho^2 + dx^2$) partition function according to the picture of [38]. τ = angle around yellow axis $\rho = 0$; blue|red plane is $\tau = \pi|0$. a, b, c contribute to the entropy. Sliced along Euclidean time τ, a and b can be viewed as free bulk resp. edge string thermal traces contributing positively to the entropy, while c can be viewed as an edge string emitting and reabsorbing a bulk string, contributing a (negative) interaction term.

more generally), loop corrections to the Newton constant vanish. By the above observation, this implies loop corrections to $S_{\rm PI}$ vanish as well. Hence there must be cancelations between different particle species, and in particular the one-loop contribution to the entropy of some fields in the supergravity theory must be *negative*. Since statistical entropy is always positive, the one-loop $Z^{(1)}_{\rm PI}$ of such fields cannot be equal to a statistical mechanical partition function.

In the same work [38], a qualitative stringy picture was sketched giving some bulk intuition about the nature of such negative contributions to $S_{\rm PI}$ when the total $S_{\rm PI}$ is a statistical entropy. In this picture, all relevant microscopic fundamental degrees of freedom are presumed to be realized in the bulk quantum gravity theory as weakly coupled strings. More specifically it is presumed that $Z^{\rm grav}_{\rm PI} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ where $\mathcal{H}$ is the string Hilbert space on Rindler space and H is the Rindler Hamiltonian, so $S_{\rm PI} = S$, the statistical entropy. Tree level and one-loop contributions to $\log Z_{\rm PI}$ are shown in fig. C.9. Diagrams d, e do not contribute to the entropy $S_{\rm PI} = (1 - \beta\partial_\beta) \log Z_{\rm PI}$ as their $\log Z_{\rm PI} \propto \beta$. Cutting b along constant-τ slices gives it an interpretation as a thermal trace over "bulk" string states away from $\rho = 0$ (closed strings in top row).[10] Similarly, a can be viewed

[10]As a simple analog of what is meant here, consider a free scalar field on S^1 parametized by $\tau \simeq \tau + \beta$. Then $\log Z_{\rm PI} = \int_0^\infty \frac{ds}{2s}\, {\rm Tr}\, e^{-s\frac{1}{2}(-\partial_\tau^2 + m^2)} = \sum_n \int \frac{ds}{2s} \int \mathcal{D}\tau|_n \, \exp\big[-\frac{1}{2}\int_0^s (\dot\tau^2 + m^2)\big] = \sum_n \frac{1}{2|n|}\, e^{-|n|\beta m}$. Here n labels the wind-

as a thermal trace over "edge" string states stuck to $\rho = 0$ (*open* strings in top row). On the other hand $\mathtt{c}$ represents an *interaction* between bulk and edge strings, with no thermal or state counting interpretation on its own. Being statistical mechanical partition functions, $\mathtt{a}$ and $\mathtt{b}$ contribute positively to $S_{\rm PI}$, whereas $\mathtt{c}$ may contribute negatively. In fact in the $\mathcal{N} = 4$ case discussed above, $\mathtt{c}$ *must* be negative, canceling $\mathtt{b}$ to render $S_{\rm PI}^{(1)} = 0$. From an effective field theory point of view, $\mathtt{b}$ and $\mathtt{e}$ correspond to the bulk ideal gas partition function inferred from formal arguments along the lines of section C.5.2, while $\mathtt{c}$ *represents "edge" corrections missed by such arguments*.

This picture is qualitative, as the individual contributions corresponding to a sharp split of the worldsheet path integrals along these lines are likely ill-defined/divergent [193]. Moreover, even without any splitting, an actual string theory calculation of $S_{\rm PI} = (1 - \beta\partial_\beta) \log Z_{\rm PI}|_{\beta=2\pi}$ is problematic, as Euclidean Rindler with a generic conical defect $\epsilon = 2\pi - \beta$ is off-shell. Shortly after [38], [194] proposed to compute $Z_{\rm PI}$ on the orbifold $\mathbb{R}^2/\mathbb{Z}_N$ for general integer N and then analytically continue the result to $N \to 1 + \epsilon$. Unfortunately such orbifolds have closed string tachyons leading to befuddling IR-divergences [194, 195]. Recently, progress was made in resolving some of these issues: in an open string version of the idea, arranged in type II string theory by adding a sufficiently low-dimensional D-brane, it was shown in [196] that upon careful analytic continuation, the tachyon appears to disappear at $N = 1 + \epsilon$.

QFT considerations

The problem of interest to us is really just a problem involving Gaussian path integrals in free quantum field theory, so there should be no need to invoke quantum gravity to gain some insight in what kind of corrections we should expect to the naive $Z_{\rm PI} \simeq Z_{\rm bulk}$. Indeed the above stringy Rindler considerations have much more straightforwardly computable low-energy counterparts in QFT.

Motivated by [38], [39] computed $Z_{\rm PI}^{(1)}$ for scalars, spinors and Maxwell fields on Rindler space.

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ing number sector of the particle worldline path integral with target space S^1. Discarding the UV-divergent $\frac{1}{0}$ term, this sums to $\log Z_{\rm PI} = -\log\left(1 - e^{-\beta m}\right) - \frac{1}{2}\beta m = \log \operatorname{Tr} e^{-\beta H}$ as in (C.103). $\mathtt{b}$ is analogous to the $|n| = 1$ contribution $e^{-\beta m}$, $\mathtt{e}$ is analogous to $n = 0$, and higher winding versions of $\mathtt{b}$ correspond to $|n| > 1$.

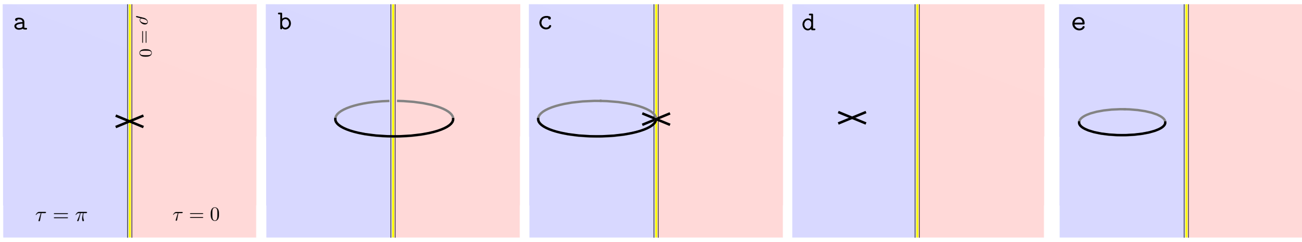

Figure C.10: Tree-level and one-loop contributions to $\log Z_{\text{PI}}$ for massless vector field in Euclidean Rindler (C.126). These can be viewed as field theory limits of fig. C.9, with verbatim the same comments applicable to a-e. The worldline path integral c appears with a sign opposite to b in $\log Z_{\text{PI}}^{(1)}$ [39].

For scalars and spinors, this was found to coincide with the ideal gas partition function, whereas for Maxwell an additional contact term was found, expressible in terms of a "edge" worldline path integral with coincident start and end points at $\rho = 0$, fig. C.10c. This term contributes negatively to $S_{\text{PI}}^{(1)} = (1 - \beta\partial_\beta) \log Z_{\text{PI}}|_{\beta=2\pi}$ and thus has no thermal interpretation on its own. In fact it causes the total $S_{\text{PI}}^{(1)}$ to be negative in less than 8 dimensions. The results of [39] and more generally the picture of [38] were further clarified by low-energy effective field theory analogs in [130], emphasizing in particular that whereas S_{PI} remains invariant under Wilsonian RG, the division between contributions with or without a low-energy statistical interpretation does not, the former gradually turning into the latter as the UV-cutoff Λ is lowered. At $\Lambda = 0$, only the tree-level contribution $S = A/4G_{\text{N}}$ of fig. C.10a is left.

The contact/edge correction of fig. C.10c to $\log Z_{\text{PI}}$ can be traced to the presence of a curvature coupling X linear in the Riemann tensor in $S_E = \int A(-\nabla^2 + X)A + \cdots$ [130, 131, 135]. Such terms appear for any spin $s \geq 1$ field, massless or not. Hence, as one might have anticipated from the stringy picture of fig. C.9, they are the norm rather than the exception.

The result of [39] was more recently revisited in [43], relating the appearance of edge corrections to the local factorization problem of QFT Hilbert spaces with gauge constraints [40–42, 45–47] like Gauss' law $\nabla \cdot E = 0$ in Maxwell theory. This problem arises more generally when contemplating the definition of entanglement entropy $S_R = -\text{Tr}\, \varrho_R \log \varrho_R$ of a spatial subregion R in gauge theories. In principle ϱ_R is obtained by factoring the global Hilbert space $\mathcal{H}_G = \mathcal{H}_R \otimes \mathcal{H}_{R^c}$

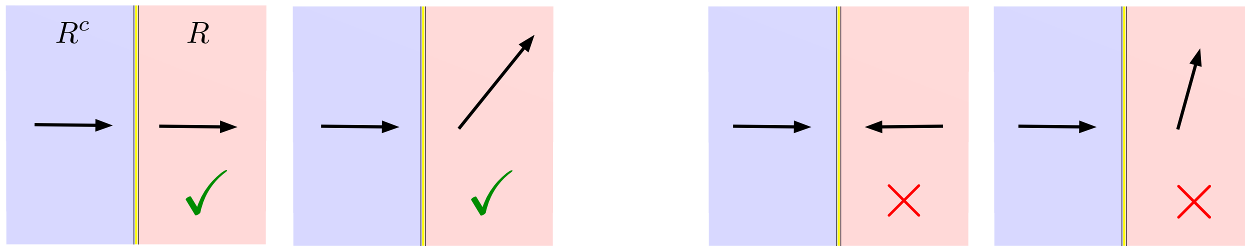

Figure C.11: Candidate classical initial electromagnetic field configurations (phase space points), with $A_0 = 0$, $A_i = 0$, showing electric field $E_i = \Pi_i = \dot{A}_i$. Gauss' law requires continuity $E_\perp$ across the boundary, disqualifying the two candidates on the right.

and tracing out $\mathcal{H}_{R^c}$. As mentioned at the end of C.5.4, local factorization is impossible in the continuum limit of any QFT, including scalar field theories, but the issue raised there can be dealt with by a suitable regularization. However for a gauge theory such as free Maxwell theory, there is an additional obstruction to local factorization, which persists after regularization, and indeed is present already at the classical phase space level: the Gauss law constraint $\nabla \cdot E = 0$ prevents us from picking independent initial conditions in both R and R^c (fig. C.11), unless the boundary is a physical object that can accommodate compensating surface charges — but this is not the case here. One way to resolve this is to decompose the global phase space into sectors labeled by "center" variables located at the boundary surface [40–42, 45–47], for example the normal component $E_\perp$ of the electric field. The center variables Poisson-commute with all local observables inside R and R^c. In any given sector, factorization then becomes possible.

Building on this framework it was shown in [43] that in a suitable brick wall-like regularization scheme and for some choice of measure $\mathcal{D}E_\perp$, the edge correction of [39] arises as a classical contribution $\int \mathcal{D}E_\perp \, e^{-S_E[E_\perp]}$ to the thermal statistical partition function. Here $S_E[E_\perp]$ is the on-shell action for static electromagnetic field modes in Euclidean Rindler space with prescribed $E_\perp$, localized vanishingly close to $\rho = 0$ when the brick-wall cutoff is taken to zero, and thus interpreted as edge modes. They also find a more precise form for the result of [39] for Rindler with its transverse dimensions compactified on a torus, which is identical in form to our de Sitter result (3.112) for $s = 1$, $G = U(1)$.

Similar results for massive vector fields were obtained in [48]. (The Stueckelberg action for a massive vector has a $U(1)$ gauge symmetry, so from that point of view it may fit into the above considerations.) An open string realization of the above ideas was proposed in [197]. It has been suggested that edge modes and "soft hair" might be related [198].

C.6 Derivations for massive higher spins

C.6.1 Massive spin-s fields

Here we derive (3.84) and (3.83). The starting point is the path integral (3.81). To get a result guaranteed to be consistent with QFT locality and general covariance, we should in principle start with the full off-shell system [102] involving auxiliary Stueckelberg fields of all spin $s' < s$.

Transverse-traceless part Z_{TT}

One's initial hope might be that Z_{PI} ends up being equal to the path integral Z_{TT} restricted to the propagating degrees of freedom, the transverse traceless modes of ϕ, with kinetic operator given by the second-order equation of motion in (3.78). Regularized as in (3.66), this is

$$\log Z_{\mathrm{TT}} \equiv \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau}\, \mathrm{Tr}_{\mathrm{TT}}\, e^{-\tau\left(-\nabla^2_{s,\mathrm{TT}}+\overline{m}_s^2\right)}\,. \tag{C.127}$$

The index TT indicates the object is defined on the restricted space of transverse traceless modes. This turns out to be correct for Euclidean AdS with standard boundary conditions [81]. However, this is not quite true for the sphere, related to the presence of normalizable tensor decomposition zeromodes.

The easiest way to convince oneself that $Z_{\mathrm{PI}} \neq Z_{\mathrm{TT}}$ on the sphere is to just compute Z_{TT} and observe it is inconsistent with locality, in a sense made clear below. To evaluate Z_{TT}, all we need is the spectrum of $-\nabla^2_{\mathrm{TT}} + \overline{m}_s^2$ [178, 199]. The eigenvalues are $\lambda_n = \left(n + \frac{d}{2}\right)^2 + \nu^2$, $n \geq s$ with degeneracy given by the dimension $D^{d+2}_{n,s}$ of the so$(d+2)$ representation corresponding to the two-row Young diagram (n, s), for example for $d = 3$, $(n, s) = (7, 3)$,

$$D^5_{7,3} = \dim^{\mathrm{so}(5)} \ydiagram{7,3} = 1190\,. \tag{C.128}$$

Explicit dimension formulae and tables can be found in appendix C.4.1.

Following the same steps as for the scalar case in section 3.3, we end up with

$$\log Z_{\mathrm{TT}} = \int_0^\infty \frac{dt}{2t}\left(q^{i\nu} + q^{-i\nu}\right) f_{\mathrm{TT}}(q)\,, \qquad f_{\mathrm{TT}}(q) \equiv \sum_{n \geq s} D^{d+2}_{n,s}\, q^{\frac{d}{2}+n}\,. \tag{C.129}$$

Now let us evaluate this explicitly for the example of a massive vector on S^5, i.e. $d = 4$, $s = 1$. From (C.89) we read off $D^6_{n,1} = \frac{1}{3} n(n+2)^2(n+4)$. Performing the sum we end up with

$$f_{\mathrm{TT}}(q) = \frac{1+q}{1-q}\left(\frac{4\,q^2}{(1-q)^4} - \frac{q}{(1-q)^2}\right) + q \qquad (d = 4, s = 1)\,. \tag{C.130}$$

The first term inside the brackets can be recognized as the $d = 4$ massive spin-1 bulk character.

The small-t expansion of the integrand in (C.129) contains a term $1/t$. This term arises from the term $+q$ in the above expression, as the other parts give contributions to the integrand that are manifestly even under $t \to -t$. The presence of this $1/t$ term in the small-t expansion implies $\log Z_{\mathrm{TT}}$ has a logarithmic UV divergence $\log Z_{\mathrm{TT}}|_{\log \mathrm{div}} = \log M$ where M is the UV cutoff scale. More precisely in the heat-kernel regularization under consideration, the contribution of the term $+q$ to $\log Z_{\mathrm{TT}}$ is, according to (C.68),

$$\int_0^\infty \frac{dt}{2t}\left(q^{1+i\nu} + q^{1-i\nu}\right) = \log \frac{M}{\sqrt{1+\nu^2}}\,, \qquad M \equiv \frac{2e^{-\gamma}}{\epsilon}\,. \tag{C.131}$$

Note that $m = \sqrt{1+\nu^2}$ is the Proca mass (3.79) of the vector field. The presence of a logarithmic divergence means $Z_{\mathrm{PI}} \neq Z_{\mathrm{TT}}$, for $\log Z_{\mathrm{PI}}$ itself is defined as a manifestly covariant, local QFT path integral on S^5, which cannot have any logarithmic UV divergences, as there are no local curvature invariants of mass dimension 5.

For $s = 2$ and $d = 4$ we get similarly

$$f_{\mathrm{TT}}(q) = \frac{1+q}{1-q}\left(\frac{9\,q^2}{(1-q)^4} - \frac{6\,q}{(1-q)^2}\right) + 6\,q + 15\,q^2 \qquad (d=4, s=2)\,. \tag{C.132}$$

The terms $6\,q + 15\,q^2$ produce a nonlocal logarithmic divergence $\log Z_{\mathrm{TT}}|_{\log\mathrm{div}} = c\ \log M$, where $c = 6 + 15 = 21$, so again $Z_{\mathrm{PI}} \neq Z_{\mathrm{TT}}$. Note that $21 = \frac{7\times 6}{2} = \dim \mathrm{so}(1,6)$, the number of conformal Killing vectors on S^5. That this is no coincidence can be ascertained by repeating the same exercise for general $d \geq 3$ and $s = 2$:

$$f_{\mathrm{TT}}^{(s=2)}(q) = \frac{1+q}{1-q}\left(D_2^d \cdot \frac{q^{\frac{d}{2}}}{(1-q)^d} - D_1^{d+2} \cdot \frac{q^{\frac{d-2}{2}}}{(1-q)^{d-2}}\right) + D_1^{d+2}\,q + D_{1,1}^{d+2}\,q^2\ . \tag{C.133}$$

The q, q^2 terms generate a log-divergence $c_2 \log M$, $c_2 = D_1^{d+2} + D_{1,1}^{d+2} = (d+2) + \frac{1}{2}(d+2)(d+1) = \frac{1}{2}(d+3)(d+2) = D_{1,1}^{d+3} = \dim \mathrm{so}(1, d+2)$, the number of conformal Killing vectors on S^{d+1}. The identity $N_{\mathrm{CKV}} = D_{1,1}^{d+3} = D_{1,1}^{d+2} + D_{1,0}^{d+2}$ and its generalization to the spin-s case will be a crucial ingredient in establishing our claims. It has a simple group theoretic origin. As a complex Lie algebra, the conformal algebra $\mathrm{so}(1, d+2)$ generated by the conformal Killing vectors is the same as $\mathrm{so}(d+3)$, which is generated by antisymmetric matrices and therefore forms the irreducible representation with Young diagram $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ of $\mathrm{so}(d+3)$. This decomposes into irreps of $\mathrm{so}(d+2)$ by the branching rule

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \quad \rightarrow \quad \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} + \begin{array}{|c|}\hline \\ \hline\end{array}, \tag{C.134}$$

implying in particular $D_{1,1}^{d+3} = D_{1,1}^{d+2} + D_{1,0}^{d+2}$. Geometrically this reflects the fact that the conformal Killing modes split into two types: (i) transversal vector modes $\varphi_\mu^{i_1}$, $i_1 = 1, \ldots, D_{1,1}^{d+2}$, satisfying the ordinary Killing equation $\nabla_{(\mu}\varphi_{\nu)}^{i_1} = 0$, spanning the $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ eigenspace of the transversal vector Laplacian, and (ii) longitudinal modes $\varphi_\mu^{i_0} = \nabla_\mu \varphi^i$, $i_0 = 1, \ldots, D_1^{d+2}$, satisfying $\nabla_\mu \nabla_\nu \varphi^{i_0} + g_{\mu\nu}\varphi^{i_0} = 0$, with the scalar φ^{i_0} modes spanning the $\begin{array}{|c|}\hline \\ \hline\end{array}$ eigenspace of the scalar Laplacian on S^{d+2}.

We can extend the above to general s, d by observing the following key relation:[11]

$$\boxed{D^{d+2}_{n,s} = D^{d+2}_n D^d_s - D^{d+2}_{s-1} D^d_{n+1}} \tag{C.135}$$

which together with the explicit expression (C.15) for D^d_s with $d \geq 3$ immediately leads to

$$f_{\rm TT}(q) = \sum_{n\geq -1} D^{d+2}_{n,s}\, q^{\frac{d}{2}+n} - \sum_{n=-1}^{s-1} D^{d+2}_{n,s}\, q^{\frac{d}{2}+n} \tag{C.136}$$

$$= \frac{1+q}{1-q}\left(D^d_s \cdot \frac{q^{\frac{d}{2}}}{(1-q)^d} - D^{d+2}_{s-1} \cdot \frac{q^{\frac{d-2}{2}}}{(1-q)^{d-2}}\right) + \sum_{n=-1}^{s-2} D^{d+2}_{s-1,n+1}\, q^{\frac{d}{2}+n}\,. \tag{C.137}$$

To rewrite the finite sum we used $D^{d+2}_{n,s} = -D^{d+2}_{s-1,n+1}$ and $D^{d+2}_{s-1,s} = 0$, both of which follow from (C.135). Substituting this into the integral (C.129), we get

$$\log Z_{\rm TT} = \log Z_{\rm bulk} - \log Z_{\rm edge} + \log Z_{\rm res}\,, \tag{C.138}$$

where $\log Z_{\rm bulk}$ and $\log Z_{\rm edge}$ are the character integrals defined in (3.84)-(3.85), and, evaluating the integral of the remaining finite sum as in (C.131),

$$\log Z_{\rm res} = \sum_{n=-1}^{s-2} D^{d+2}_{s-1,n+1} \int \frac{dt}{2t}\left(q^{\frac{d}{2}+n+i\nu} + q^{\frac{d}{2}+n-i\nu}\right) = \sum_{n=-1}^{s-2} D^{d+2}_{s-1,n+1} \log \frac{M}{\sqrt{(\frac{d}{2}+n)^2+\nu^2}} \tag{C.139}$$

The term $\log Z_{\rm res}$ has a logarithmic UV-divergence:

$$\log Z_{\rm res} = c_s \log M + \cdots\,, \qquad c_s = \sum_{n=-1}^{s-2} D^{d+2}_{s-1,n+1} = D^{d+3}_{s-1,s-1} = N_{\rm CKT}\,, \tag{C.140}$$

where $N_{\rm CKT} = D^{d+3}_{s-1,s-1}$ is the number of rank $s-1$ conformal Killing tensors on S^{d+2} [200]. This identity has a group theoretic origin as an $\mathrm{so}(d+3) \to \mathrm{so}(d+2)$ branching rule generalizing

[11] This can be checked for any given d from the Weyl dimension formula of appendix C.4.1, or from the general d formula given in e.g. [178, 199], or proven directly (with some effort) by an $SO(d+2) \to SO(2) \times SO(d)$ reduction. It has a stronger version as an $\mathrm{so}(d+2)$ character relation: $\chi^{\mathrm{so}(d+2)}_{n,s}(x) = D^d_s\, \chi^{\mathrm{so}(d+2)}_n(x) - D^d_{n+1}\, \chi^{\mathrm{so}(d+2)}_{s-1}(x)$.

(C.134). For example for $s = 4$:

$$\ydiagram{3,3} \quad \to \quad \ydiagram{3,3} + \ydiagram{3,2} + \ydiagram{3,1} + \ydiagram{3} \,. \tag{C.141}$$

Geometrically this reflects the fact that the rank $s-1 = 3$ Killing tensor modes split up into 4 types: Schematically $\varphi^{i_3}_{\mu_1\mu_2\mu_3}$, $i_3 = 1, \ldots, D_{3,3}$; $\varphi^{i_2}_{\mu_1\mu_2\mu_3} \sim \nabla_{(\mu_1}\varphi^{i_2}_{\mu_2\mu_3)}$, $i_2 = 1, \ldots, D_{3,2}$; $\varphi^{i_1}_{\mu_1\mu_2\mu_3} \sim \nabla_{(\mu_1}\nabla_{\mu_2}\varphi^{i_1}_{\mu_3)}$, $i_1 = 1, \ldots, D_{3,1}$; $\varphi^{i_0}_{\mu_1\mu_2\mu_3} \sim \nabla_{(\mu_1}\nabla_{\mu_2}\nabla_{\mu_3)}\varphi^{i_0}$, $i_0 = 1, \ldots, D_3$, where the $\varphi^{i_r}_{\mu_1\cdots\mu_r}$ span the eigenspace of the TT spin-r Laplacian labeled by the above Young diagrams.

As pointed out in examples above and discussed in more detail below, the log-divergence of $\log Z_{\rm res}$ is inconsistent with locality, hence $Z_{\rm PI} \neq Z_{\rm TT}$: locality must be restored by the non-TT part of the path integral. Below we argue this part in fact *exactly cancels* the $\log Z_{\rm res}$ term, thus ending up with $\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge}$, i.e. the character formula (3.84).

Full path integral $Z_{\rm PI}$: locality constraints

The full, manifestly covariant, local path integral takes the form (a simple example is (C.148)):

$$Z_{\rm PI} = Z_{\rm TT} \cdot Z_{\text{non-TT}} = Z_{\rm bulk} \cdot Z_{\rm edge}^{-1} \cdot Z_{\rm res} \cdot Z_{\text{non-TT}} \,. \tag{C.142}$$

All UV-divergences of $\log Z_{\rm PI}$ are *local*, in the sense they can be canceled by local counterterms, more specifically local curvature invariants of the background metric. In particular for odd $d+1$, this implies there cannot be any logarithmic divergences at all, as there are no curvature invariants of odd mass dimension. Recall from (C.140) that the term $\log Z_{\rm res}$ is logarithmically divergent. For odd $d+1$, this is clearly the only log-divergent contribution to $\log Z_{\rm TT}$, as the integrands of both $\log Z_{\rm bulk}$ and $\log Z_{\rm edge}$ are even in t in this case. More generally, for even or odd $d+1$, $\log Z_{\rm res}$ is the only *nonlocal* log-divergent contribution to $\log Z_{\rm TT}$, as follows from the result of [52, 62] mentioned below (3.81), combined with the observation in (C.140) that $c_s = N_{\rm CKT}$. Therefore the log-divergence of $\log Z_{\rm res}$ must be canceled by an equal log-divergence in $\log Z_{\text{non-TT}}$ of the opposite sign.

The simplest way this could come about is if $Z_{\text{non-TT}}$ exactly cancels Z_{res}, that is if

$$Z_{\text{non-TT}} = Z_{\text{res}}^{-1} = \prod_{n=-1}^{s-2} \left(M^{-1} \sqrt{(\tfrac{d}{2}+n)^2 + \nu^2} \right)^{D^{d+2}_{s-1,n+1}} \qquad \Rightarrow \qquad Z_{\text{PI}} = \frac{Z_{\text{bulk}}}{Z_{\text{edge}}} . \tag{C.143}$$

Note furthermore that from (C.136), or from (C.139) and $D^{d+2}_{s-1,n+1} = -D^{d+2}_{n,s}$, it follows this identification is equivalent to the following simple prescription: *The full Z_{PI} is obtained from Z_{TT} by extending the TT eigenvalue sum $\sum_{n \geq s}$ in (C.129) down to $\sum_{n \geq -1}$*:

$$\log Z_{\text{PI}} = \int_0^\infty \frac{dt}{2t} \left(q^{i\nu} + q^{-i\nu} \right) \sum_{n \geq -1} D^{d+2}_{n,s} \, q^{\frac{d}{2}+n} , \tag{C.144}$$

i.e. (3.83). In what follows we establish this is indeed the correct identification. We start by showing it precisely leads to the correct spin-s unitarity bound, and that it moreover exactly reproduces the critical mass ("partially massless") thresholds at which a new set of terms in the action defining the path integral $Z_{\text{non-TT}}$ fails to be positive definite. Assisted by those insights, it will then be rather clear how (C.143) arises from explicit path integral computations.

Unitarity constraints

A significant additional piece of evidence beyond consistency with locality is consistency with unitarity. It is clear that both the above integral (C.144) for $\log Z_{\text{PI}}$ and the integral (C.129) for $\log Z_{\text{TT}}$ are real provided ν is either real or imaginary. Real ν corresponds to the principal series $\Delta = \frac{d}{2} + i\nu$, while imaginary $\nu = i\mu$ corresponds to the complementary series $\Delta = \frac{d}{2} - \mu \in \mathbb{R}$. In the latter case there is in addition a bound on $|\mu|$ beyond which the integrals cease to make sense, due to the appearance of negative powers of $q = e^{-t}$ and the integrand blowing up at $t \to \infty$. The bound can be read off from the term with the smallest value of n in the sum. In the Z_{TT} integral (C.129) this is the $n = s$ term $\propto q^{\frac{d}{2}+s\pm\mu}$, yielding a bound $|\mu| < \frac{d}{2} + s$. In the Z_{PI} integral (C.144), assuming $s \geq 1$, this is the $n = -1$ term $\propto q^{\frac{d}{2}-1\pm\mu}$, so the bound becomes much tighter:

$$|\mu| < \frac{d}{2} - 1 \qquad (s \geq 1)\,. \tag{C.145}$$

This is exactly the correct unitarity bound for the spin-$s \geq 1$ complementary series representations of $SO(1, d+1)$ [76–78, 101]. In terms of the mass $m^2 = (\frac{d}{2} + s - 2)^2 - \mu^2$ in (3.79), this becomes $m^2 > (s-1)(d-3+s)$, also known as the Higuchi bound [100] (a convenient concise summary is given in [201] s.a. [202]). From a path integral perspective, this bound can be understood as the requirement that the full off-shell action is positive definite [102], so indeed $\log Z_{\rm PI}$ should diverge exactly when the bound is violated. Moreover, we get new divergences in the integral formula for $\log Z_{\text{non-TT}}$, according to the above identifications, each time $|\mu|$ crosses a critical value $\mu_{*n} = \frac{d}{2}+n$, where $n = -1, 0, 1, 2, \ldots, s-2$. These correspond to critical masses $m_{*n}^2 = (\frac{d}{2} + s - 2)^2 - (\frac{d}{2} + n)^2 = (s-2-n)(d+s-2+n)$, which on the path integral side precisely correspond to the points where a new set of terms in the action fails to be positive definite. [102].

This establishes the terms in the integrand of (C.139), or equivalently the extra terms $n = -1, \ldots, s-2$ in (C.144), have exactly the correct powers of q to match with $\log Z_{\text{non-TT}}$. It does not yet confirm the precise values of the coefficients $D^{d+2}_{n,s}$ — except for their sum (C.140), which was fixed earlier by the locality constraint. To complete the argument, we determine the origin of these coefficients from the path integral point of view in what follows.

Explicit path integral considerations

Complementary to but guided by the above general considerations, we now turn to more concrete path integral calculations to confirm the expression (C.143) for $Z_{\text{non}-\text{TT}}$, focusing in particular on the origin of the coefficients $D^{d+2}_{n,s}$.

Spin 1:

We first consider the familar $s = 1$ case, a vector field of mass m, related to ν by (3.79) as $m = \sqrt{(\frac{d}{2} - 1)^2 + \nu^2}$. The local field content in the Stueckelberg description consists of a vector ϕ_μ and a scalar χ, with action and gauge symmetry given by

$$S_0 = \int \nabla_{[\mu}\phi_{\nu]}\nabla^{[\mu}\phi^{\nu]} + \tfrac{1}{2}(\nabla_\mu \chi - m\,\phi_\mu)(\nabla^\mu \chi - m\,\phi^\mu)\,; \qquad \delta\chi = m\xi\,, \quad \delta\phi_\mu = \nabla_\mu \xi\,. \quad \text{(C.146)}$$

Gauge fixing the path integral by putting $\chi \equiv 0$, we get the gauge-fixed action

$$S = \int \nabla_{[\mu}\phi_{\nu]}\nabla^{[\mu}\phi^{\nu]} + \tfrac{1}{2}m^2\phi_\mu\phi^\mu + m\bar{c}c\,, \tag{C.147}$$

with BRST ghosts c, $\bar{c}$. Decomposing ϕ_μ into a transversal and longitudinal part, $\phi_\mu = \phi^T_\mu + \phi'_\mu$, we can decompose the path integral as $Z_{\text{PI}} = Z_{\text{TT}} \cdot Z_{\text{non-TT}}$ with

$$Z_{\text{TT}} = \int \mathcal{D}\phi^T \, e^{-\frac{1}{2}\int \phi^T(-\nabla^2+\bar{m}_1^2)\phi^T}\,, \qquad Z_{\text{non-TT}} = \int \mathcal{D}\phi' \mathcal{D}c \mathcal{D}\bar{c}\, e^{-\int \frac{1}{2}m^2\phi'^2 + m\bar{c}c}\,, \tag{C.148}$$

Both the ghosts and the longitudinal vectors $\phi'_\mu = \nabla_\mu \varphi$ have an mode decomposition in terms of orthonormal real scalar spherical harmonics Y_i.[12] In our heat kernel regularization scheme, each longitudinal vector mode integral gives a factor M/m, which is exactly canceled by a factor m/M from integrating out the corresponding ghost mode.[13] However there is one ghost mode which remains unmatched: the constant mode. A constant scalar does not map to a longitudinal vector mode, because $\phi'_\mu = \nabla_\mu \varphi = 0$ for constant φ. Thus we end up with a ghost factor m/M in excess, and

$$Z_{\text{non-TT}} = m/M = M^{-1}\sqrt{(\tfrac{d}{2}-1)^2 + \nu^2}\,, \tag{C.149}$$

in agreement with (C.143) for $s = 1$.

Spin 2:

For $s = 2$, the analogous Stueckelberg action involves a symmetric tensor $\phi_{\mu\nu}$, a vector χ_μ, and a

[12] Explicitly, $c = \sum_i c_i Y_i$, $\bar{c} = \sum_i \bar{c}_i Y_i$, $\phi'_\mu = \sum_{i:\lambda_i \neq 0} \phi'_i \nabla_\mu Y_i/\sqrt{\lambda_i}$ where $\nabla^2 Y_i = -\lambda_i Y_i$, $\int Y_i Y_j = \delta_{ij}$.

[13] A priori there might be a relative numerical factor κ between ghost and longitudinal factors, depending on the so far unspecified normalization of the measure $\mathcal{D}c$. But because c is local, *unconstrained*, rescaling $\mathcal{D}c = \prod_i dc_i \to \prod_i(\lambda dc_i)$ merely amounts to a trivial constant shift of the bare cc. So we are free to take $\kappa = 1$.

scalar χ, subject to the gauge transformations [102]

$$\delta\chi = a_{-1}\,\xi\,, \qquad \delta\chi_\mu = a_0\,\xi_\mu + \sqrt{\tfrac{d-1}{2d}}\,\nabla_\mu\xi\,, \qquad \delta\phi_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu + \sqrt{\tfrac{2}{d(d-1)}}\,a_0\,\xi\,, \quad \text{(C.150)}$$

where $a_0 \equiv m$ and $a_{-1} \equiv \sqrt{m^2-(d-1)}$. Equivalently, recalling (3.79), $a_n = \sqrt{(\frac{d}{2}+n)^2+\nu^2}$. Gauge fixing by putting $\chi = 0$, $\chi_\mu = 0$, we get a ghost action

$$S_{\rm gh} = \int a_{-1}\,\bar{c}c + a_0\,\bar{c}^\mu c_\mu\,. \quad \text{(C.151)}$$

We can decompose $\phi_{\mu\nu}$ into a TT part and a non-TT part orthogonal to it as $\phi_{\mu\nu} = \phi^{\rm TT}_{\mu\nu} + \phi'_{\mu\nu}$, where $\phi'_{\mu\nu}$ can be decomposed into vector and scalar modes as $\phi'_{\mu\nu} = \nabla_{(\mu}\varphi_{\nu)} + g_{\mu\nu}\varphi$. Analogous to the $s = 1$ example, we should expect that integrating out ϕ' cancels against integrating out the ghosts, up to unmatched modes of the latter. The unmatched modes correspond to mixed vector-scalar modes solving $\nabla_{(\mu}\varphi_{\nu)} + g_{\mu\nu}\varphi = 0$. This is equivalent to the conformal Killing equation. Hence the unmatched modes are the conformal Killing modes. As discussed below (C.134), the conformal Killing modes split according to $\yng(1,1) \to \yng(1,1) + \square$ into $D_{1,1}$ vector $\yng(1,1)$-modes and $D_{1,0}$ scalar $\square$-modes. Integrating out the $\yng(1,1)$-modes of the vector ghost c_μ then yields an unmatched factor $(a_0/M)^{D_{1,1}}$, while integrating out the $\square$-modes of the scalar ghost c yields an unmatched factor $(a_{-1}/M)^{D_1}$. All in all, we get

$$Z_{\text{non-TT}} = (a_{-1}/M)^{D_{1,0}}(a_0/M)^{D_{1,1}} = \Big(M^{-1}\sqrt{(\tfrac{d}{2}-1)^2+\nu^2}\Big)^{D_{1,0}}\Big(M^{-1}\sqrt{(\tfrac{d}{2})^2+\nu^2}\Big)^{D_{1,1}} \quad \text{(C.152)}$$

in agreement with (C.143) for $s = 2$.

Spin s:

The pattern is now clear: according to [102], the Stueckelberg system for a massive spin-s field consists of an unconstrained symmetric s-index tensor $\phi^{(s)}$ and of a tower of unconstrained symmetric s'-index auxiliary Stueckelberg fields $\chi^{(s')}$ with $s' = 0, 1, \ldots, s-1$, with gauge symmetries of the form

$$\delta\chi^{(s')} = a_{s'-1}\,\xi^{(s')} + \cdots\,, \qquad \delta\phi^{(s)} = \cdots\,, \qquad a_n \equiv \sqrt{(\tfrac{d}{2}+n)^2+\nu^2}\,, \tag{C.153}$$

where the dots indicate terms we won't technically need — which is to say, as transpired from $s = 1, 2$ already, we need very little indeed. The ghost action is $S = \sum_{s'=0}^{s-1} a_{s'-1}\bar{c}^{(s')}c^{(s')}$. The unmatched modes correspond to the conformal Killing tensors modes on S^{d+1}, decomposed for say $s = 4$ as in (C.141) into $D_{3,3}$ $\ydiagram{3,3}$-modes, $D_{3,2}$ $\ydiagram{3,2}$-modes, $D_{3,1}$ $\ydiagram{3,1}$-modes, and $D_{3,0}$ $\ydiagram{3}$-modes. The corresponding unmatched modes of respectively $c^{(3)}$, $c^{(2)}$, $c^{(1)}$ and $c^{(0)}$ then integrate to unmatched factors $(a_2/M)^{D_{3,3}}(a_1/M)^{D_{3,2}}(a_0/M)^{D_{3,1}}(a_{-1}/M)^{D_{3,0}}$. For general s:

$$Z_{\text{non-TT}} = \prod_{s'=0}^{s-1}\big(a_{s'-1}/M\big)^{D_{s-1,s'}} = \prod_{n=-1}^{s-2}\left(M^{-1}\sqrt{(\tfrac{d}{2}+n)^2+\nu^2}\right)^{D^{d+2}_{s-1,n+1}}\,, \tag{C.154}$$

in agreement with (C.143) for general s. This establishes our claims.

The above computation was somewhat schematic of course, and one could perhaps still worry about missed purely numerical factors independent of ν, perhaps leading to an additional finite constant term being added to our final formulae (3.84) -(3.83) for $\log Z_{\text{PI}}$. However at fixed UV-regulator scale, the limit $\nu \to \infty$ of these final expressions manifestly approaches zero, as should be the case for particles much heavier than the UV cutoff scale. This would not be true if there was an additional constant term. Finally, we carefully checked the analogous result in the massless case (which has a more compact off-shell formulation [99]), discussed in section 3.5, by direct path integral computations in complete gory detail [67], for all s.

Also, the result is pretty.

C.6.2 General massive representations

Here we give a generalization of (3.83) for arbitrary massive representations of the dS_{d+1} isometry group $SO(1, d+1)$.

Massive irreducible representations of $SO(1, d+1)$ are labeled by a dimension $\Delta = \frac{d}{2} + i\nu$ and an so(d) highest weight $S = (s_1, \ldots, s_r)$ [76–78]. The massive spin-s case considered in (3.83) corresponds to $S = (s, 0, \ldots, 0)$, a totally symmetric tensor field. More general irreps correspond to more general mixed-symmetry fields. The analog of (C.127) in this generalized setup is

$$\log Z_{\rm PI}^{\text{“TT”}} = \pm \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau} \sum_{n \geq s_1} D^{d+2}_{n,S}\, e^{-\tau((n+\frac{d}{2})^2+\nu^2)}\,, \tag{C.155}$$

where for bosons the sum runs over integer n with an overall + sign and for fermions the sum runs over half-integer n with an overall − sign. The dimensions of the so($d+2$) irreps (n, S) are given explicitly as polynomials in $(n, s_1, \ldots, s_r)$ by the Weyl dimension formulae (C.87)-(C.88). From this it can be seen that $D^{d+2}_{n,S}$ is (anti-)symmetric under reflections about $n = -\frac{d}{2}$, more precisely

$$D^{d+2}_{n,S} = (-1)^d D^{d+2}_{-d-n,S}. \tag{C.156}$$

Moreover the exponent in (C.155) is symmetric under the same reflection. The most natural extension of the sum is therefore to *all* (half-)integer n, taking into account the sign in (C.156) for odd d, and adding an overall factor $\frac{1}{2}$ to correct for double counting, suggesting

$$\log Z_{\rm PI} = \pm\frac{1}{2} \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau} \sum_{n} \sigma_d(\tfrac{d}{2}+n)\, D^{d+2}_{n,S}\, e^{-\tau((n+\frac{d}{2})^2+\nu^2)}\,, \tag{C.157}$$

where $\sigma_d(x) \equiv 1$ for even d and $\sigma_d(x) \equiv \text{sign}(x)$ for odd d. Equivalently, in view of (C.156)

$$\boxed{\log Z_{\rm PI} = \pm \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau} \sum_{n} \Theta(\tfrac{d}{2}+n)\, D^{d+2}_{n,S}\, e^{-\tau((n+\frac{d}{2})^2+\nu^2)}} \tag{C.158}$$

where $n \in \mathbb{Z}$ for bosons and $n \in \frac{1}{2} + \mathbb{Z}$ for fermions, and

$$\Theta(x) = 1 \text{ for } x > 0, \quad \Theta(0) = \frac{1}{2}, \quad \Theta(x) = 0 \text{ for } x < 0. \tag{C.159}$$

At first sight this seems to be different from the extension to $n \geq -1$ in (3.83) for the spin-s case $S = (s, 0, \ldots, 0)$. However it is actually the same, as (C.87)-(C.88) imply that $D_{n,S}$ vanishes for

$2 - d \leq n \leq -2$ when $S = (s, 0, \ldots, 0)$.

The obvious conjecture is then that (C.158) is true for general massive representations. Here are some consistency checks, which are satisfied precisely for the sum range in (C.158):

• *Locality:* For even d, the summand in (C.157) is analytic in n. Applying the Euler-Maclaurin formula to extract the $\tau \to 0$ asymptotic expansion of the sum gives in this case

$$\sum_n D^{d+2}_{n,S} \, e^{-\tau(n+\frac{d}{2})^2} \sim \int_{-\infty}^{\infty} dn \, D^{d+2}_{n,S} \, e^{-\tau(n+\frac{d}{2})^2} \, . \tag{C.160}$$

The symmetry (C.156) tells us that the integrand on the right hand side is even in $x \equiv n + \frac{d}{2}$. Since $\int dx \, x^{2k} \, e^{-\tau x^2} \propto \tau^{-k-1/2}$, this implies the absence of $1/\tau$ terms in the $\tau \to 0$ expansion of the integrand in (C.157), and therefore, in contrast to (C.155), the absence of nonlocal log-divergences, as required by locality of $Z_{\rm PI}$ in odd spacetime dimension $d+1$.

• *Bulk – edge structure:* By following the usual steps, we can rewrite (C.158) as

$$\boxed{\log Z_{\rm PI} = \int \frac{dt}{2t} \, F(e^{-t}) \, , \qquad F(q) = \pm\big(q^{i\nu} + q^{-i\nu}\big) \sum_n \Theta\big(\tfrac{d}{2} + n\big) \, D^{d+2}_{n,S} \, q^{\frac{d}{2}+n}} \tag{C.161}$$

Using (C.87)-(C.88), this can be seen to sum up to the form $\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge}$, where $Z_{\rm bulk}$ is the physically expected bulk character formula for an ideal gas in the dS_{d+1} static patch consisting of massive particles in the (Δ, S) UIR of $SO(1, d+1)$, and $Z_{\rm edge}$ can be interpreted as a Euclidean path integral of local fields living on the S^{d-1} edge/horizon.

• *Unitarity:* Note that for $\Delta = \frac{d}{2} + \mu$ with $\mu \equiv i\nu$ real, we get a bound on μ from requiring $t \to \infty$ (IR) convergence of the integral (C.161), generalizing (C.145), namely

$$|\mu| < \frac{d}{2} + n_*(S) \, , \tag{C.162}$$

where $n_*(S)$ is the lowest value of n in the sum for which $D^{d+2}_{n,S}$ is nonvanishing. This coincides again with the unitarity bound on μ for massive representations of $SO(1, d+1)$ [76–78, 101]. Recalling the discussion below (C.145), this can be viewed as a generalization of the Higuchi bound to arbitrary representations.

Combining (C.158) with (C.57), we thus arrive at an exact closed-form solution for the Euclidean path integral on the sphere for arbitrary massive field content.

C.7 Derivations for massless higher spins

In this appendix we derive (3.112) and provide details of various other points summarized in section 3.5.

C.7.1 Bulk partition function: $Z_{\rm bulk}$

The bulk partition function $Z_{\rm bulk}$ as defined in (3.67) for a massless spin-s field is given by

$$Z_{\rm bulk} = \int \frac{dt}{2t} \, \frac{1+q}{1-q} \, \chi_{{\rm bulk},s}(q) \,, \tag{C.163}$$

where $q = e^{-t}$, and $\chi_{{\rm bulk},s}(q) = {\rm tr}\, q^{iH}$ in the case at hand is the (restricted) q-character of the massless spin-s $SO(1,d+1)$ UIR. For generic d, this UIR is part of the exceptional series [101]. More precisely in the notation of [76–78] it is the $D^{j}_{S;p}$ representation, with $p = 0$, $j = (d-4)/2$ for even d, $j = (d-3)/2$ for odd d, and $S = (s,s,0,\ldots,0)$. In the notation of [101] this is the exceptional series with $\Delta = p = 2$, $S = (s,s,0,\ldots,0)$.[14] The characters $\chi_{{\rm bulk},s}(q)$ for these irreps are quite a bit more intricate than their massive counterparts. The full $SO(1,d+1)$ characters $\tilde{\chi}(g)$ were obtained in [76–78]. Restricting to $g = e^{-itH}$ gives $\chi_{{\rm bulk},s}(t)$,[15]

$$\begin{aligned}(1-q)^d \, \chi_{{\rm bulk},s}(q) = & \big(1-(-1)^d\big)\big(D^d_s \, q^{s+d-2} - D^d_{s-1} \, q^{s+d-1}\big) \\ & + \sum_{m=0}^{r-2} (-1)^m D^d_{s,s,1^m} \big(q^{2+m} + (-1)^d q^{d-2-m}\big) \,, \qquad r \equiv {\rm rank}\ {\rm so}(d) = \lfloor \tfrac{d}{2} \rfloor \,,\end{aligned} \tag{C.164}$$

[14] For $d = 3$, it is in the discrete series, but (C.164) still applies.

[15] Actually we obtained the formula from (C.194), then Mathematica checked agreement with [76–78].

Here we used the notation of [101]: the so(d) irrep $(s,s,1^m) \equiv (s,s,1,\ldots,1,0,\ldots,0)$ with 1 repeated m times. The degeneracies $D^d_{s,s,1^m}$ can be read off from (C.87)-(C.88). Some explicit low-dimensional examples are

d	r	$(1-q)^d\,\chi_{\text{bulk},s}(q)$
3	1	$2D^3_s\,q^{s+1} - 2D^3_{s-1}\,q^{s+2}$
4	2	$D^4_{s,s}\,2\,q^2$
5	2	$D^5_{s,s}\left(q^2 - q^3\right) + 2D^5_s q^{s+3} - 2D^5_{s-1} q^{s+4}$
6	3	$D^6_{s,s}\left(q^2 + q^4\right) - D^6_{s,s,1}\,2\,q^3$
7	3	$D^7_{s,s}\left(q^2 - q^5\right) - D^7_{s,s,1}\left(q^3 - q^4\right) + 2D^7_s\,q^{s+5} - 2D^7_{s-1}\,q^{s+6}$,

(C.165)

where $D^3_s = 2s+1$, $D^4_{s,s} = 2s+1$, $D^5_s = \frac{1}{6}(s+1)(s+2)(2s+3)$, $D^5_{s,s} = \frac{1}{3}(2s+1)(s+1)(2s+3)$, $D^6_{s,s} = \frac{1}{12}(s+1)^2(s+2)^2(2s+3)$, $D^6_{s,s,1} = \frac{1}{12}s(s+1)(s+2)(s+3)(2s+3)$, etc. For $s=1$, the character can be expressed more succinctly as

$$\chi_{\text{bulk},1}(q) = d \cdot \frac{q^{d-1}+q}{(1-q)^d} - \frac{q^d+1}{(1-q)^d} + 1\,. \tag{C.166}$$

With the exception of the $d=3$ case, the above so$(1,d+1)$ q-characters encoding the H-spectrum of massless spin-s fields in dS$_{d+1}$ are very different from the so$(2,d)$ characters encoding the energy spectrum of massless spin-s fields in AdS$_{d+1}$ with standard boundary conditions, the latter being $\chi^{\text{AdS}_{d+1}}_{\text{bulk},s} = (D^d_s\,q^{s+d-2} - D^d_{s-1}\,q^{s+d-1})/(1-q)^d$. In particular for $d \geq 4$, the lowest power q^Δ appearing in the q-expansion of the character is $\Delta = 2$, and is associated with the so(d) representation $S=(s,s)$, i.e. $\ydiagram{1,1}$, $\ydiagram{2,2}$, $\ydiagram{3,3}$, $\ldots$ for $s = 1,2,3,\ldots$, whereas for the so$(2,d)$ character this is $\Delta = s+d-2$ and $S=(s)$. An explanation for this was given in [101]: in dS, S should be thought of as associated with the higher-spin Weyl curvature tensor of the gauge field rather than the gauge field itself.

This fits well with the interpretation of the expansion

$$\chi(q) = \sum_r N_r q^r \,, \tag{C.167}$$

as counting the number N_r of *physical* static patch quasinormal modes decaying as e^{-rT} (cf. section 3.2 and appendix C.2.3). Indeed for $d \geq 4$, the longest-lived physical quasinormal modes of a massless spin-s field in the static patch of dS_{d+1} always decay as e^{-2T} [141], which can be understood as follows. Physical quasinormal modes of the southern static patch can be thought of as sourced by insertions of gauge-invariant[16] local operators on the past conformal boundary $T = -\infty$ of the static patch, or equivalently at the south pole of the past conformal boundary (or alternatively the north pole of future boundary) of global dS_{d+1} [139–141]. By construction, the dimension r of the operator maps to the decay rate r of the quasinormal mode $\propto e^{-rT}$. For $s = 1$, the gauge-invariant operator with the smallest dimension $r = \Delta$ is the magnetic field strength $F_{ij} = \partial_i A_j - \partial_j A_i$ of the boundary gauge field A_i, which has $\Delta = \dim \partial + \dim A = 1 + 1 = 2$. For $s = 2$ in $d \geq 4$, the gauge-invariant operator with smallest dimension is the Weyl tensor of the boundary metric: $\Delta = 2+0 = 2$. Similarly for higher-spin fields we get the spin-s Weyl tensor, with $\Delta = s + 2 - s = 2$. The reason $d = 3$ is special is that the Weyl tensor vanishes identically in this case. To get a nonvanishing gauge-invariant tensor, one has to act with at least $2s - 1$ derivatives (spin-s Cotton tensor), yielding $\Delta = (2s - 1) + (2 - s) = s + 1$. An extensive analysis is given in [141].

Note on a literature disagreements: The characters (C.164) agree with the characters listed in the original work [76–78], computed by undisclosed methods. They do not agree with those listed in the more recent work [101], computed by Bernstein-Gelfand-Gelfand resolutions. Indeed [101] emphasized they disagreed with [76–78] for even d. More precisely, in their eq. (2.14) applied to $p = 2$, $Y_p = (s, s, 0, \ldots, 0)$, $\vec{x} = 0$, they find a factor $2 = (1 + (-1)^d)$ instead of the factor $(1 - (-1)^d) = 0$ in (C.164). It is stated in [101] that on the other hand their results do agree with

[16]More precisely, invariant under linearized gauge transformations acting on the conformal boundary.

[76–78] for *odd d*. Actually we find this is not quite true either, as in that case eq. (2.13) in [101] applied to $p = 2$, $Y_p = (s, s, 0, \ldots, 0)$, $\vec{x} = 0$) has a factor 1 instead of the factor $(1 - (-1)^d) = 2$ in (C.164). Our Euclidean path integral result (C.194) coupled with the physics of section 3.2 strongly suggests the original results in [76–78] and (C.164) are the correct versions. Further support is provided in [141] by direct construction of higher-spin quasinormal modes.

C.7.2 Euclidean path integral: $Z_{\rm PI} = Z_G Z_{\rm char}$

The Euclidean path integral of a collection of gauge fields ϕ on S^{d+1} is formally given by

$$Z_{\rm PI} = \frac{\int \mathcal{D}\phi \, e^{-S[\phi]}}{\mathrm{vol}(\mathcal{G})} \tag{C.168}$$

where $\mathcal{G}$ is the local gauge group generated by the local field ξ appearing in (3.94). This ill-defined formal expression is turned into something well-defined by BRST gauge fixing. A convenient gauge for higher-spin fields is the de Donder gauge. At the Gaussian level, the resulting analog of (C.127) is[17]

$$\log Z_{\rm TT} \equiv \sum_s \int_0^\infty \frac{d\tau}{2\tau} \, e^{-\epsilon^2/4\tau} \Big(\mathrm{Tr}_{\rm TT} \, e^{-\tau\left(-\nabla^2_{s,\rm TT} + m^2_{\phi,s}\right)} - \mathrm{Tr}_{\rm TT} \, e^{-\tau\left(-\nabla^2_{s-1,\rm TT} + m^2_{\xi,s}\right)} \Big), \tag{C.169}$$

where $\sum_s$ sums over the spin-s gauge fields in the theory (possibly with multiplicities) and $m^2_{\phi,s}$ and $m^2_{\xi,s}$ are obtained from the relations below (3.78) using (3.95). The first term arises from the path integral over the TT modes of ϕ, while the second arises from the TT part of the gauge fixing sector in de Donder gauge — a combination of integrating out the TT part of the spin-$(s-1)$ ghost fields and the corresponding longitudinal degrees of freedom of the spin-s gauge fields. The above (C.169) is the difference of two expressions of the form (C.127). Naively applying the formula

[17] A detailed discussion of normalization conventions left implicit here is given above and below (C.202).

(C.161) or (3.83) for the corresponding full $Z_{\rm PI}$, we get

$$Z_{\rm PI} = \exp \sum_s \int \frac{dt}{2t}\, \hat{F}_s(e^{-t}) \qquad \text{(naive)} \tag{C.170}$$

where (assuming $d \geq 3$)

$$\hat{F}_s(q) = \sum_{n\geq -1} D^{d+2}_{n,s}\big(q^{s+d-2+n} + q^{2-s+n}\big) - \sum_{n\geq -1} D^{d+2}_{n,s-1}\big(q^{s+d-1+n} + q^{1-s+n}\big) \tag{C.171}$$

However this is clearly problematic. One problem is that for $s \geq 2$, the above $\hat{F}_s(q)$ contains *negative* powers of $q = e^{-t}$, making (C.170) exponentially divergent at $t \to \infty$. The appearance of such "wrong-sign" powers of q is directly related to the appearance of "wrong-sign" Gaussian integrals in the path integral, as can be seen for instance from the relation between (C.161) and the heat kernel integral (C.158). In the path integral framework, one deals with this problem by analytic continuation, generalizing the familiar contour rotation prescription for negative modes in the gravitational Euclidean path integral [61]. Thus one defines $\int dx\, e^{-\lambda x^2/2}$ for $\lambda < 0$ by rotating $x \to ix$, or equivalently by rotating $\tau \to -\tau$ in the heat kernel integral. Essentially this just boils down to flipping any $\lambda < 0$ to $-\lambda > 0$. Since the Laplacian eigenvalues are equal to the products of the exponents appearing in the pairs $\big(q^{\Delta+n} + q^{d-\Delta+n}\big)$ in (C.171), the implementation of this prescription in our setup is to flip the negative powers q^k in $\hat{F}_s(q) = \sum_k c_k\, q^k$ to positive powers q^{-k}, that is to say replace

$$\hat{F}_s(q) \to F_s(q) \equiv \big\{\hat{F}_s(q)\big\}_+ = \Big\{\sum_k c_k\, q^k\Big\}_+ \equiv \sum_{k<0} c_k\, q^{-k} + \sum_{k\geq 0} c_k\, q^k\,. \tag{C.172}$$

In addition, each negative mode path integral contour rotation produces a phase $\pm i$, resulting in a definite, finite overall phase in $Z_{\rm PI}$ [59]. The analysis of [59] translates to each corresponding flip in (C.172) contributing with the *same* sign,[18] hence to an overall phase i^{-P_s} with P_s the total

[18]This can be seen in a more careful path integral analysis [67].

degeneracy of negative modes in (C.171). Using $D^{d+2}_{n,s} = -D^{d+2}_{s-1,n+1}$:

$$Z_{\rm PI} \to i^{-P_s} Z_{\rm PI}\,, \qquad P_s = \sum_{n'=0}^{s-2} D^{d+2}_{s-1,n'} + \sum_{n'=0}^{s-2} D^{d+2}_{s-2,n'} = D^{d+3}_{s-1,s-1} - D^{d+2}_{s-1,s-1} + D^{d+3}_{s-2,s-2} \tag{C.173}$$

In particular this implies $P_1 = 0$ and $P_2 = d+3$ in agreement with [59].

After having taken care of the *negative* powers of q, the resulting amended formula $Z_{\rm PI} = \int \frac{dt}{2t} F_s(q)$ is still problematic, however, as $F_s(q)$ still contains terms proportional to q^0, causing the integral to diverges (logarithmically) for $t \to \infty$. These correspond to zeromodes in the original path integral. Indeed such zermodes were to be expected: they are due to the existence of normalizable rank $s-1$ traceless Killing tensors $\bar\xi^{(s-1)}$, which by definition satisfy $\nabla_{(\mu_1}\bar\xi_{\mu_2\cdots\mu_s)} = 0$, and therefore correspond to vanishing gauge transformations (3.94), leading in particular to ghost zeromodes. Zeromodes of this kind must be omitted from the Gaussian path integral. They are easily identified in (C.171) as the values of n for which a term proportional to q^0 appears. Since we are assuming $d > 2$, this is $n = s-2$ in the first sum and $n = s-1$ in the second. Thus we should refine (C.172) to

$$\hat F_s \to F_s - F^0_s\,, \tag{C.174}$$

where $F^0_s = D^{d+2}_{s-2,s}(q^{2s+d-4}+1) - D^{d+2}_{s-1,s-1}(q^{2s+d-2}+1)$. Noting that $D^{d+2}_{s-2,s} = -D^{d+2}_{s-1,s-1}$ and $D^{d+2}_{s-1,s-1}$ is the number $N^{\rm KT}_{s-1}$ of rank $s-1$ traceless Killing tensors on S^{d+1}, we can rewrite this as

$$F^0_s = -N^{\rm KT}_{s-1}\big(q^{2s+d-4} + 1 + q^{2s+d-2} + 1\big)\,, \qquad N^{\rm KT}_{s-1} = D^{d+2}_{s-1,s-1}\,, \tag{C.175}$$

making the relation to the existence of normalizable Killing tensors manifest. For example $N^{\rm KT}_0 = 1$, corresponding to constant $U(1)$ gauge transformations; $N^{\rm KT}_1 = \frac{1}{2}(d+2)(d+1) = \dim SO(d+2)$, corresponding to the Killing vectors of the sphere; and $N^{\rm KT}_{s-1} \propto s^{2d-3}$ for $s \to \infty$, corresponding to large-spin generalizations thereof.

We cannot just drop the zeromodes and move on, however. The original formal path integral expression (C.168) is local by construction, as both numerator and denominator are defined with a local measure on local fields. In principle BRST gauge fixing is designed to maintain manifest locality, but if we remove any finite subset of modes by hand, including in particular zeromodes, locality is lost. Indeed the $-F_s^{(0)}$ subtraction results in nonlocal log-divergences in the character integral, i.e. divergences which cannot be canceled by local counterterms. From the point of view of (C.168), the loss of locality is due the fact that we are no longer dividing by the volume of the local gauge group $\mathcal{G}$, since we are effectively omitting the subgroup G generated by the Killing tensors. To restore locality, and to correctly implement the idea embodied in (C.168), we must divide by the volume of G by hand. This volume must be computed using the same local measure defining $\mathrm{vol}(\mathcal{G})$, i.e. the invariant measure on $\mathcal{G}$ normalized such that integrating the gauge fixing insertion in the path integral over the gauge orbits results in a factor 1. Hence the appropriate measure defining the volume of G in this context is inherited from the BRST path integral measure. As such we will denote it by $\mathrm{vol}(G)_{\mathrm{PI}}$. A detailed general discussion of the importance of these specifications for consistency with locality and unitarity in the case of Maxwell theory can be found in [66]. Relating $\mathrm{vol}(G)_{\mathrm{PI}}$ to a "canonical", theory-independent definition of the group volume $\mathrm{vol}(G)_{\mathrm{c}}$ (such as for example $\mathrm{vol}(U(1))_{\mathrm{c}} \equiv 2\pi$) is not trivial, requiring considerable care in keeping track of various normalization factors and conventions. Moreover $\mathrm{vol}(G)_{\mathrm{PI}}$ depends on the nonlinear interaction structure of the theory, as this determines the Lie algebra of G. We postpone further analysis of $\mathrm{vol}(G)_{\mathrm{PI}}$ to section C.7.4.

Conclusion

To summarize, instead of the naive (C.170), we get the following formula for the 1-loop Euclidean path integral on S^{d+1} for a collection of massless spin-s gauge fields:

$$Z_{\mathrm{PI}} = i^{-P_s} \left(\mathrm{vol}(G)_{\mathrm{PI}}\right)^{-1} \exp \sum_s \int \frac{dt}{2t}\left(F_s - F_s^0\right), \qquad \text{(C.176)}$$

where $F_s = \{\hat{F}_s\}_+$ and F_s^0 were defined in (C.171), (C.172) and (C.175); G is the subgroup of gauge transformations generated by the Killing tensors $\bar{\xi}^{(s-1)}$, i.e. the zeromodes of (3.94); and i^{-P_s} is the phase (C.173). We can split up the integrals by introducing an IR regulator:

$$\boxed{Z_{\text{PI}} = i^{-P_s}\, Z_G\, Z_{\text{char}}\,, \qquad Z_G \equiv \frac{\exp\left(-\sum_s \int^{\times} \frac{dt}{2t} F_s^0\right)}{\text{vol}(G)_{\text{PI}}}\,, \qquad Z_{\text{char}} \equiv \exp \sum_s \int^{\times} \frac{dt}{2t}\, F_s} \tag{C.177}$$

where the notation $\int^{\times}$ means we IR regulate by introducing a factor $e^{-\mu t}$, take $\mu \to 0$, and subtract the $\log \mu$ divergent term. For a function $f(t)$ such that $\lim_{t\to\infty} f(t) = c$, this means

$$\int^{\times} \frac{dt}{t}\, f(t) \equiv \lim_{\mu\to 0}\left(c \log \mu + \int \frac{dt}{t}\, f(t)\, e^{-\mu t}\right) \tag{C.178}$$

For example for $f(t) = \frac{t}{t+1}$ this gives $\int_0^{\times} \frac{dt}{t} \frac{t}{t+1} = \log \mu - \log(e^{\gamma}\mu) = -\gamma$, and for $f(t) = 1$ with the integral UV-regularized as in (C.68) we get $\int^{\times} \frac{dt}{t} = \log(2e^{-\gamma}/\epsilon)$.

In section C.7.3 we recast Z_{char} as a character integral formula. In section C.7.4 we express Z_G in terms of the canonical group volume $\text{vol}(G)_{\text{c}}$ and the coupling constant of the theory.

C.7.3 Character formula: $Z_{\text{char}} = Z_{\text{bulk}}/Z_{\text{edge}} Z_{\text{KT}}$

In this section we derive a character formula for Z_{char} in (C.177). If we start from the naive $\hat{F}_s$ given by (C.171) and follow the same steps as those bringing (3.83) to the form (3.84), we get

$$\hat{F}_s = \frac{1+q}{1-q}\, \hat{\chi}_s\,, \qquad \hat{\chi}_s = \hat{\chi}_{\text{bulk},s} - \hat{\chi}_{\text{edge},s}\,, \tag{C.179}$$

where

$$\hat{\chi}_{\text{bulk},s} = D_s^d\, \frac{q^{s+d-2} + q^{2-s}}{(1-q)^d} - D_{s-1}^d\, \frac{q^{s+d-1} + q^{1-s}}{(1-q)^d} \tag{C.180}$$

$$\hat{\chi}_{\text{edge},s} = D_{s-1}^{d+2}\, \frac{q^{s+d-3} + q^{1-s}}{(1-q)^{d-2}} - D_{s-2}^{d+2}\, \frac{q^{s+d-2} + q^{-s}}{(1-q)^{d-2}}\,. \tag{C.181}$$

Note that these take the form of "field − ghost" characters obtained respectively by substituting the values of ν_ϕ and ν_ξ given by (3.95) into the massive spin s and spin $s-1$ characters (3.85). The naive bulk characters $\hat{\chi}_{\text{bulk,s}}$ thus obtained cannot possibly be the character of any UIR of $SO(1,d+1)$, as is obvious from the presence of negative powers of q. In particular it is certainly not equal to the physical exceptional series bulk character (C.164). Now let us consider the actual $F_s = \{\hat{F}_s\}_+$ appearing in (C.177). Then we find[19]

$$F_s = \left\{\frac{1+q}{1-q}\,\hat{\chi}_s\right\}_+ = \frac{1+q}{1-q}\Big(\big[\hat{\chi}_s\big]_+ - 2\,N^{\text{KT}}_{s-1}\Big)\,, \tag{C.182}$$

where the "flipped" character $[\hat{\chi}]_+$ is obtained from $\hat{\chi} = \sum_k c_k q^k$ by flipping $c_k q^k \to -c_k q^{-k}$ for $k<0$ and dropping the $k=0$ terms:

$$\big[\hat{\chi}\big]_+ = \Big[\sum_k c_k\, q^k\Big]_+ \equiv \sum_{k<0}(-c_k)\, q^{-k} + \sum_{k>0} c_k\, q^k \;= \hat{\chi} - c_0 - \sum_{k<0} c_k\big(q^k + q^{-k}\big)\,. \tag{C.183}$$

Thus this flipping prescription can be thought of as the character analog of contour rotations for "wrong-sign" Gaussians in the path integral. Notice the slight differences in the map $\hat{\chi} \to [\hat{\chi}]_+$ and the related but different map $\hat{F} \to \{\hat{F}\}_+$ defined in (C.172).

Substituting (C.182) into (C.177), we conclude

$$\boxed{\log Z_{\text{char}} = \sum_s \int^\times \frac{dt}{2t}\,\frac{1+q}{1-q}\Big(\big[\hat{\chi}_{\text{bulk},s}\big]_+ - \big[\hat{\chi}_{\text{edge},s}\big]_+ - 2N^{\text{KT}}_{s-1}\Big)} \tag{C.184}$$

[19]To check (C.182) starting from (C.172), observe that $\left\{\frac{1+q}{1-q}\big(q^k + q^{-k} - 2\big)\right\}_+ = 0$ for any integer k, so $\left\{\frac{1+q}{1-q}\,q^k\right\}_+ = \frac{1+q}{1-q}\big(-q^{-k}+2\big)$ for $k<0$, while of course $\left\{\frac{1+q}{1-q}\,q^k\right\}_+ = \frac{1+q}{1-q}\,q^k$ for $k \geq 0$, . This accounts for the $k<0$ and $k>0$ terms in the expansion $\sum_k c_k\, q^k$ of (C.182). The coefficient $2N^{\text{KT}}_{s-1}$ of the q^0 term is most easily checked by comparing the q^0 terms on the left and right hand sides of (C.182), taking into account that, by definition, $[\chi]_+$ has no q^0 term, and that the q^0 terms of the left hand side are given by (C.175).

Consistency with ideal gas bulk character formula

Consistency with the physical ideal gas picture used to derive $Z_{\rm bulk}$ in section C.7.1 requires the bulk part of $\log Z_{\rm char}$ as given in (C.184) agrees with (C.163), that is to say it requires

$$\chi_{{\rm bulk},s} = [\hat{\chi}_{{\rm bulk},s}]_+ \qquad (?) \tag{C.185}$$

where $\chi_{{\rm bulk},s}$ is one of the intricate exceptional series characters (C.164), while $[\hat{\chi}_{{\rm bulk},s}]_+$ is obtained from the simple naive bulk character (C.180) just by flipping some polar terms as in (C.183). At first sight this might seems rather unlikely. Nevertheless, quite remarkably, it turns out to be true. Let us first check this in some simple examples:

- $s = 2$ *in* $d = 3$: The naive character (C.180) is

$$\hat{\chi}_{\rm bulk} = 5 \cdot \frac{q^3+1}{(1-q)^3} - 3 \cdot \frac{q^4+q^{-1}}{(1-q)^3} \,. \tag{C.186}$$

  The polar and q^0 terms are obtained by expanding $\hat{\chi}_{\rm bulk} = -\frac{3}{q} - 4 + O(q)$. Thus

$$\big[\hat{\chi}_{\rm bulk}\big]_+ = \hat{\chi}_{\rm bulk} + 4 + \frac{3}{q} + 3\,q = \frac{2\cdot 5\cdot q^3 - 2\cdot 3\cdot q^4}{(1-q)^3} \,, \tag{C.187}$$

  correctly reproducing the $d = 3, s = 2$ character in (C.165).

- $s = 1$, *general* $d \geq 3$: In this case the map $[\ldots]_+$ merely eliminates the q^0 term in the naive character (C.180) by adding $+1$:

$$\big[\hat{\chi}_{\rm bulk}\big]_+ = \hat{\chi}_{\rm bulk} + 1 = d \cdot \frac{q^{d-1}+q}{(1-q)^d} - \frac{q^d+1}{(1-q)^d} + 1 \,, \tag{C.188}$$

  correctly reproducing (C.166).

Using Mathematica, it is straightforward to check an arbitrary large number of examples in this way. Below we will derive a general explicit formula for the character flip map $\hat{\chi} \to [\hat{\chi}]_+$.

This will provide a general proof of (C.185), and more generally will enable efficient closed-form computation of the proper bulk and edge characters for general s and d. Generalizations are implemented with equal ease: as an illustration thereof we compute the bulk and edge characters for partially massless fields.

Flipping formula

We wish to derive a general explicit formula for $\big[\frac{q^\Delta}{(1-q)^d}\big]_+$ defined in (C.183), for

$$\Delta \in \{0, -1, -2, -3, \ldots\} \tag{C.189}$$

which suffices to obtain an explicit expression for $[\hat{\chi}]_+$ for any bosonic character of interest, including in particular (C.180)-(C.181). This is achieved by expanding $(1-q)^{-d} = \sum_k \binom{d-1+k}{k} q^k$, splitting the sum in its polar and nonpolar part, incorporating the appropriate sign flips, and resumming in terms of hypergeometric functions ${}_2F_1(a,b,c;q) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{q^n}{n!}$. After a bit more sanding and polishing we find

$$\left[\frac{q^\Delta}{(1-q)^d}\right]_+ = \frac{(-1)^{d+1} q^{d-\Delta} + p_\Delta(q) + (-1)^d p_{d-\Delta}(q)}{(1-q)^d}, \tag{C.190}$$

where $p_\Delta(q) = \binom{d-\Delta}{d-1} \cdot q \cdot {}_2F_1\big(1-d, 1-\Delta, 2-\Delta, q\big) = \binom{d-\Delta}{d-1} \cdot q \cdot \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} \frac{1-\Delta}{k+1-\Delta}\, q^k$. The hypergeometric series terminates because $1-d \le 0$. A more interesting version is

$$\begin{aligned} &\left[\frac{q^\Delta}{(1-q)^d}\right]_+ = \frac{\mathcal{P}_\Delta(q)}{(1-q)^d} \\ &\mathcal{P}_\Delta(q) \equiv (-1)^{d+1} q^{d-\Delta} + \sum_{m=0}^{r-1} (-1)^m\, D^d_{1-\Delta,1^m} \big(q^{1+m} + (-1)^d q^{d-1-m}\big) \qquad r = \left\lfloor \tfrac{d}{2} \right\rfloor . \end{aligned} \tag{C.191}$$

Here $D^d_{1-\Delta,1^m}$ is the dimension of the irrep of so(d) corresponding to the Young diagram $S = (1-\Delta, 1, \ldots, 1, 0, \ldots, 0)$, with 1 repeated m times, explicitly given in (C.87)-(C.88). We obtained this formula using Mathematica and also obtained a proof of (C.191) by expressing (C.87)-(C.88) in terms of gamma functions and comparing to (C.190). This is somewhat tedious and not especially illuminating, so we omit it here.

Bulk characters for (partially) massless fields

Now let us apply this to a slight generalization of the massless $\hat{\chi}_{\text{bulk,s}}$ given in (C.180),

$$\hat{\chi}^{\text{bulk}}_{ss'}(q) \equiv D^d_s \, \frac{q^{1-s'} + q^{d-1+s'}}{(1-q)^d} - D^d_{s'} \, \frac{q^{1-s} + q^{d-1+s}}{(1-q)^d} \,. \tag{C.192}$$

This is the naive bulk character for a *partially massless* spin-s field $\phi_{\mu_1 \cdots \mu_s}$ with a spin-s' ($0 \le s' \le s-1$) gauge parameter field $\xi_{\mu_1 \cdots \mu_{s'}}$ [203]. The massless case (C.180) corresponds to

$$s' = s - 1 \qquad \text{(massless case).} \tag{C.193}$$

We consider the more general partially massless case here to illustrate the versatility of (C.191), and because in a sense the resulting formulae are more elegant than in the massless case, due to the neat $s \leftrightarrow s'$ symmetry already evident in (C.192). Applying (C.191), still with $r = \left\lfloor \frac{d}{2} \right\rfloor$,

$$\boxed{\begin{aligned}(1-q)^d \big[\hat{\chi}^{\text{bulk}}_{ss'}(q)\big]_+ = &\big(1 + (-1)^{d+1}\big)\big(D^d_s \, q^{d-1+s'} - D^d_{s'} \, q^{d-1+s}\big) \\ &+ \sum_{m=1}^{r-1} (-1)^{m-1} \, D^d_{s,s'+1,1^{m-1}} \big(q^{1+m} + (-1)^d q^{d-1-m}\big) \,.\end{aligned}} \tag{C.194}$$

We used $D^d_s \, \mathcal{P}_{1-s'}(q) - D^d_{s'} \mathcal{P}_{1-s}(q) = \sum_{m=0}^{r-1} (-1)^m \left(D^d_s \, D^d_{s',1^m} - D^d_{s'} \, D^d_{s,1^m}\right)\big(q^{1+m} + (-1)^d q^{d-1-m}\big)$ $= \sum_{m=1}^{r-1} (-1)^{m-1} \, D^d_{s,s'+1,1^{m-1}} \big(q^{1+m} + (-1)^d q^{d-1-m}\big)$ with $\mathcal{P}_\Delta$ as defined in (C.191) to get the second term. Like for (C.191), we obtained this formula using Mathematica. It can be proven starting from (C.87)-(C.88).

Remarkably, (C.194) precisely reproduces the massless exceptional series characters (C.164) for $s' = s - 1$, further strengthening our physical picture, adding evidence for (C.161), proving (C.185), and generalizing it moreover to partially massless gauge fields. Comparing to [76–78], the partially massless gauge field characters we find here coincide with those of the unitary exceptional series $D^j_{S,p}$ with $p = 0$, $S = (s, s' + 1)$, and $j = (d-3)/2$ for odd d, $j = (d-4)/2$ for even d. In the notation of [101] this is the exceptional series with $\Delta = p = 2$, $S = (s, s' + 1)$, which was indeed identified in [101] as the so$(1, d+1)$ UIR for partially massless fields.

Edge characters for (partially) massless fields

For the edge correction we proceed analogously. The naive PM edge character is

$$\hat{\chi}^{\rm edge}_{ss'}(q) = D^{d+2}_{s-1} \, \frac{q^{-s'} + q^{d-2+s'}}{(1-q)^{d-2}} - D^{d+2}_{s'-1} \, \frac{q^{-s} + q^{d-2+s}}{(1-q)^{d-2}} \,, \tag{C.195}$$

reducing to the massless case (C.181) for $s' = s - 1$. Applying (C.191) gives, still with $r = \lfloor \frac{d}{2} \rfloor$,

$$\boxed{\begin{aligned}(1-q)^{d-2}\big[\hat{\chi}^{\rm edge}_{ss'}(q)\big]_+ = {}& \big(1 + (-1)^{d+1}\big)\big(D^{d+2}_{s-1} \, q^{d-2+s'} - D^{d+2}_{s'-1} \, q^{d-2+s}\big) \\ & + \sum_{m=0}^{r-2} (-1)^m \, \tilde{D}_m \big(q^{1+m} + (-1)^d q^{d-3-m}\big)\end{aligned}} \tag{C.196}$$

where $\tilde{D}_m \equiv D^{d+2}_{s-1} \, D^{d-2}_{s'+1,1^m} - D^{d+2}_{s'-1} \, D^{d-2}_{s+1,1^m}$.

Note that in the massless spin-1 case

$$\big[\hat{\chi}^{\rm edge}_1(q)\big]_+ = \frac{q^{d-2} + 1}{(1-q)^{d-2}} - 1 \,. \tag{C.197}$$

In the notation of [76–78], this equals the character for the unitary $SO(1, d-1)$ irrep in the exceptional series $D^j_{S;p=0}$ with $S = (1)$ and $j = (d-4)/2$ for even d and $j = (d-3)/2$ for odd d — the irreducible representation indeed of a massless scalar on dS_{d-1} with its zeromode removed. The fact that $S = (1)$ is analogous to what happens in the 2D CFT of a massless free scalar X: the actual CFT primary operators are the spin ± 1 derivatives $\partial_\pm X(0)$.

In contrast to (C.194), we did not find a way of rewriting $\tilde{D}_m$ for general spin to suggest an interpretation along these lines in general. Indeed unlike (C.194), $\big[\hat{\chi}^{\rm edge}_{ss'}(q)\big]_+$ in general does not appear to be proportional to the character of a *single* exceptional series irrep of $SO(1, d-1)$. This is not in conflict with the picture of edge corrections as a Euclidean path integral of some collection of local fields on S^{d-1}, since if the fields have nontrivial spin / so$(d-2)$ weights, the corresponding character integrals will have a complicated structure, involving sums of iterations of $SO(1, d-1-2k)$ characters with $k = 0, 1, 2, \ldots$, exhibiting patterns that might be hard to discern without knowing what to look for. It should also be kept in mind we have not identified a reason the edge correction *must* have a local QFT path integral interpretation. On the other hand, the coefficients of the q-expansion of the effective edge character *do* turn out to be positive, consistent with an interpretation in terms of some collection of fields corresponding to unitary representations of dS_{d-1}. A more fundamental group-theoretic or physics understanding of the edge correction would evidently be desirable.

For practical purposes, the interpretation does not matter of course. The formula (C.196) gives a general formula for $\chi_{\rm edge}$, which is all we need. For example for $d = 3$, this gives $\left[\hat{\chi}_s^{\rm edge}(q)\right]_+ = 2\,\frac{D^5_{s-1}\,q^s - D^5_{s-2}\,q^{s+1}}{1-q} = 2D^5_{s-1}\,q^s + 2D^4_{s-1}\frac{q^{s+1}}{1-q}$, where $D^5_{s-1} = \frac{1}{6}s(s+1)(2s+1)$ and $D^4_{s-1} = D^5_{s-1} - D^5_{s-2} = s^2$. The second form makes positivity of coefficients manifest. For $d = 4$ we get $\left[\hat{\chi}_s^{\rm edge}(q)\right]_+ = D^5_{s-1}\frac{2\,q}{(1-q)^2}$.

Conclusion

We conclude that (C.184) can be written as

$$\log Z_{\rm char} = \log Z_{\rm bulk} - \log Z_{\rm edge} - \log Z_{\rm KT}\,, \tag{C.198}$$

where the bulk and edge contributions are explicitly given by (C.194)-(C.196) with $s' = s - 1$,[20] and

$$\log Z_{\rm KT} = \dim G \int^{\times} \frac{dt}{2t}\,\frac{1+q}{1-q}\cdot 2\,, \qquad \dim G = \sum_s N^{\rm KT}_{s-1} = \sum_s D^{d+2}_{s-1,s-1}\,. \tag{C.199}$$

[20]In the partially massless case $\log Z_{\rm KT}$ takes the same form, but with $D^{d+2}_{s-1,s-1}$ replaced by $D^{d+2}_{s-1,s'}$.

The finite (IR) part of $Z_{\rm KT}$ is given by (C.75): $Z_{\rm KT}|_{\rm IR} = (2\pi)^{-\dim G}$.

C.7.4 Group volume factor: Z_G

The remaining task is to compute the factor Z_G defined in (C.177), that is

$$Z_G = \big(\mathrm{vol}(G)_{\rm PI}\big)^{-1} \exp \sum_s N^{\rm KT}_{s-1} \int^{\times} \frac{dt}{2t} \big(q^{2s+d-4} + q^{2s+d-2} + 2\big) \tag{C.200}$$

We imagine the spin range to be finite, or cut off in some way. (The infinite spin range case is discussed in section 3.9.) In the heat kernel regularization scheme of appendix C.3, we can then evaluate the integral using (C.68):

$$Z_G = \big(\mathrm{vol}(G)_{\rm PI}\big)^{-1} \prod_s \left(\frac{M^4}{(2s+d-4)(2s+d-2)} \right)^{\frac{1}{2} N^{\rm KT}_{s-1}}, \qquad M \equiv \frac{2e^{-\gamma}}{\epsilon}, \tag{C.201}$$

On general grounds, the nonlocal UV-divergent factors M appearing here in Z_G should cancel against factors of M in $\mathrm{vol}(G)_{\rm PI}$, as we will explicitly confirm below.

Generalities

Recall that G is the group of gauge transformations generated by the Killing tensors. Equivalently it is the subgroup of gauge transformations leaving the background invariant. $\mathrm{vol}(G)_{\rm PI}$ is the volume of G with respect to the path integral induced measure. This is different from what we shall call the "canonical" volume $\mathrm{vol}(G)_{\rm c}$, defined with respect to the invariant metric normalized such that the generators of some standard basis of the Lie algebra have unit norm. (In the case of Yang-Mills, this coincides with the metric defined by the canonically normalized Yang-Mills action, providing some justification for the (ab)use of the word canonical.) In particular, in contrast to $\mathrm{vol}(G)_{\rm c}$, $\mathrm{vol}(G)_{\rm PI}$ depends on the coupling constants and UV cutoff of the field theory.

As mentioned at the end of section C.7.2, the computation of Z_G brings in a series of new complications. One reason is that the Lie algebra structure constants defining G are not determined by the free part of the action, but by its interactions, thus requiring data going beyond the usual one-loop Gaussian level. Another reason is that due to the omission of zeromodes and the ensuing loss of locality in the path integral, a precise computation of $\mathrm{vol}(G)_{\rm PI}$ requires keeping track of an unpleasantly large number of normalization factors, such as for instance constants multiplying kinetic operators, as these can no longer be automatically discarded by adjusting local counterterms. Consequently, exact, direct path integral computationz of Z_G for general higher-spin theories requires great care and considerable persistence, although it can be done [67]. Below we obtain an exact expression for Z_G in terms of $\mathrm{vol}(G)_{\rm c}$ and the Newton constant in a comparatively painless way, by combining results and ideas from [66, 68–71, 81, 128], together with the observation that the form of (C.169) actually determines all the normalization factors we need. Although the expressions at intermediate stages are still a bit unpleasant, the end result takes a strikingly simple and universal form.

If G is finite-dimensional, as is the case for example for Yang-Mills, Einstein gravity and certain (topological) higher-spin theories [117, 118, 145–148, 153, 174, 175] including the dS_3 higher-spin theory analyzed in section 3.6, we can then proceed to compute $\mathrm{vol}(G)_{\rm c}$, and we are done. If G is infinite-dimensional, as is the case in generic higher-spin theories, one faces the remaining problem of making sense of $\mathrm{vol}(G)_{\rm c}$ itself. Glossing over the already nontrivial problem of exponentiating the higher-spin algebra to an actual group [160], the obvious issue is that $\mathrm{vol}(G)_{\rm c}$ is going to be divergent. We discuss and interpret this and other infinite spin range issues in section 3.9. In what follows we will continue to assume the spin range is finite or cut off in some way as before, so G is finite-dimensional.

We begin by determining the path integral measure to be used to compute $\mathrm{vol}(G)_{\rm PI}$ in (C.201). Then we compute Z_G in terms of $\mathrm{vol}(G)_{\rm c}$ and the coupling constant of the theory, first for Yang-Mills, then for Einstein gravity, and finally for general higher-spin theories.

Path integral measure

To determine $\mathrm{vol}(G)_{\rm PI}$ we have to take a quick look at the path integral measure. This is fixed by locality and consistency with the regularized heat kernel definition of Gaussian path integrals we have been using throughout. For example for a scalar field as in (3.66), we have $\int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi(-\nabla^2+m^2)\phi} \equiv \exp \int \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau}\, \mathrm{Tr}\, e^{-\tau(-\nabla^2+m^2)}$. An eigenmode of $-\nabla^2+m^2$ with eigenvalue λ_i contributes a factor $M/\sqrt{\lambda_i}$ to the right hand side of this equation, with $M = \exp \int \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau} e^{-\tau} = 2e^{-\gamma}/\epsilon$, the same parameter as in (C.201) (essentially by definition). To ensure the left hand side matches this, we must use a path integral measure derived from the local metric $ds^2_\phi = \frac{M^2}{2\pi} \int (\delta\phi)^2$. To see this, expand $\phi(x) = \sum_i \varphi_i \psi_i(x)$ with $\psi_i(x)$ an orthonormal basis of eigenmodes of $-\nabla^2+m^2$ on S^{d+1}. The metric in this basis becomes $ds^2 = \sum_i \frac{M^2}{2\pi} d\varphi_i^2$, so a mode with eigenvalue λ_i contributes a factor $\int d\varphi_i\, \frac{M}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\lambda_i \varphi_i^2} = M/\sqrt{\lambda_i}$ to the left hand side, as required.

We work with canonically normalized fields. For a spin-s field ϕ this means the quadratic part of the action evaluated on its transverse-traceless part ϕ^{TT} takes the form

$$S\big[\phi^{\mathrm{TT}}\big] = \frac{1}{2} \int \phi^{\mathrm{TT}}(-\nabla^2 + \overline{m}^2)\phi^{\mathrm{TT}}. \tag{C.202}$$

Consistency with (C.127) or (C.169) then requires the measure for ϕ to be derived again from the metric $ds^2_\phi = \frac{M^2}{2\pi} \int (\delta\phi)^2$. If ϕ has a gauge symmetry, the formal division by the volume of the gauge group $\mathcal{G}$ is conveniently implemented by BRST gauge fixing. For example for a spin-1 field with gauge symmetry $\delta\phi_\mu = \partial_\mu \xi$, we can gauge fix in Lorenz gauge by adding the BRST-exact action $S_{\mathrm{BRST}} = \int iB\, \nabla^\mu \phi_\mu - \bar{c}\nabla^2 c$. This requires specifying a measure for the Lagrange multiplier field B and the ghosts $c, \bar{c}$. It is straightforward to check that a ghost measure derived from $ds^2_{\bar{c}c} = M^2 \int \delta\bar{c}\, \delta c$ (which translates to a mode measure $\prod_i \frac{1}{M^2} d\bar{c}_i\, dc_i$) combined with a B-measure derived from $ds^2_B = \frac{1}{2\pi} \int (\delta B)^2$, reproduces precisely the second term in (C.169) upon integrating out B, c, $\bar{c}$ and the longitudinal modes of ϕ. It is likewise straightforward to check that BRST gauge fixing is then formally equivalent to dividing by the volume of the local gauge group $\mathcal{G}$ with respect to the measure derived from the following metric on the algebra of local gauge transformations:

$$ds^2_\xi = \frac{M^4}{2\pi} \int (\delta\xi)^2\,. \tag{C.203}$$

Note that all of these metrics take the same form, with the powers of M fixed by dimensional analysis. An important constraint in the above was that the second term in (C.169) is exactly reproduced, without some extra factor multiplying the Laplacian. This matters when we omit zeromodes. For this to be the case with the above measure prescriptions, it was important that the gauge transformation took the form $\delta\phi_\mu = \alpha_1 \partial_\mu \xi$ with $\alpha_1 = 1$ as opposed to some different value of α_1, as we a priori allowed in (3.94). For a general α_1, we would have obtained an additional factor α_1 in the ghost action, and a corresponding factor α_1^2 in the kinetic term in the second term of (C.169). To avoid having to keep track of this, we picked $\alpha_1 \equiv 1$. For Yang-Mills theories, everything remains the same, with internal index contractions understood, e.g. $S[\phi^{\rm TT}] = \frac{1}{2}\int \phi^{a\rm TT}(-\nabla^2 + \overline{m}^2)\phi^{a\rm TT}$, $ds_\phi^2 = \frac{M^2}{2\pi}\int(\delta\phi^a)^2$, $ds_\xi^2 = \frac{M^4}{2\pi}\int(\delta\xi^a)^2$.

For higher-spin fields, we gauge fix in the de Donder gauge. All metrics remain unchanged, except for the obvious additional spacetime index contractions. The second term of (C.169) is exactly reproduced upon integrating out the TT sector of the BRST fields together with the corresponding longitudinal modes of ϕ, provided we pick

$$\alpha_s = \sqrt{s} \tag{C.204}$$

in (3.94), with symmetrization conventions such that $\phi_{(\mu_1\cdots\mu_s)} = \phi_{\mu_1\cdots\mu_s}$. (Technically the origin of the factor s can be traced to the fact that if $\phi_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}$ for a TT ξ, we have $\int\phi^2 = s^{-1}\int\xi(-\nabla^2 + c_s)\xi$.) Equation (C.204) fixes the normalization of ξ, and (C.203) then determines unambiguously the measure to be used to compute $\mathrm{vol}(G)_{\rm PI}$ in (C.201). We will see more concretely how this works in what follows, first spelling out the basic idea in detail in the familiar YM and GR examples, and then moving on to the general higher-spin gauge theory case considered in [68].

Yang-Mills

Consider a Yang-Mills theory with with a simple Lie algebra

$$[L^a, L^b] = f^{abc} L^c\,, \tag{C.205}$$

with the L^a some standard basis of anti-hermitian matrices and f^{abc} real and totally antisymmetric. For example for su(2) Yang-Mills, $L^a = -\frac{1}{2} i\sigma^a$ and $[L^a, L^b] = \epsilon^{abc} L^c$. Consistent with our general conventions, we take the gauge fields $\phi_\mu = \phi_\mu^a L^a$ to be canonically normalized: the curvature takes the form $F_{\mu\nu}^a L^a = F_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu + g[\phi_\mu, \phi_\nu]$, and the action is

$$S = \frac{1}{4} \int F^a \cdot F^a \tag{C.206}$$

The quadratic part of S is invariant under the linearized gauge transformations $\delta_\xi^{(0)} \phi_\mu = \partial_\mu \xi$, where $\xi = \xi^a L^a$, taking the form (3.94) with $\alpha_1 = 1$ as required. The full S is invariant under local gauge transformations $\delta_\xi \phi_\mu = \partial_\mu \xi + g[\phi_\mu, \xi]$, generating the local gauge algebra

$$[\delta_\xi, \delta_{\xi'}] = \delta_{g[\xi', \xi]}\,. \tag{C.207}$$

The rank-0 Killing tensors $\bar\xi$ satisfy $\partial_\mu \bar\xi = 0$: they are the constant gauge transformations $\bar\xi = \bar\xi^a L^a$ on the sphere, forming the subalgebra $\mathfrak{g}$ of local gauge transformations acting trivially on the background $\phi_\mu = 0$, generating the group G whose volume we have to divide by. The bracket of $\mathfrak{g}$, denoted $[\![\cdot, \cdot]\!]$ in [68], is inherited from the local gauge algebra (C.207):

$$[\![\bar\xi, \bar\xi']\!] = g[\bar\xi', \bar\xi]\,. \tag{C.208}$$

Evidently this is isomorphic to the original YM Lie algebra. Being a simple Lie algebra, $\mathfrak{g}$ has an up to normalization unique invariant bilinear form/metric. The path integral metric ds^2_{PI} of (C.203) corresponds to such an invariant bilinear form with a specific normalization:

$$\langle\bar{\xi}|\bar{\xi}'\rangle_{\rm PI} = \frac{M^4}{2\pi}\int \bar{\xi}^a\bar{\xi}'^a = \frac{M^4}{2\pi}\,\mathrm{vol}(S^{d+1})\,\bar{\xi}^a\bar{\xi}'^a\,. \tag{C.209}$$

We define the theory-independent "canonical" invariant bilinear form $\langle\cdot|\cdot\rangle_{\rm c}$ on $\mathfrak{g}$ as follows. First pick a "standard" basis M^a of $\mathfrak{g}$, i.e. a basis satisfying the same commutation relations as (C.205): $[\![M^a, M^b]\!] = f^{abc}M^c$. This fixes the normalization of the M^a. Then we fix the normalization of $\langle\cdot|\cdot\rangle_{\rm c}$ by requiring these standard generators have unit norm, i.e.

$$\langle M^a|M^b\rangle_c \equiv \delta^{ab}\,. \tag{C.210}$$

The explicit form of (C.208) implies such a basis is given by the constant functions $M^a = -L^a/g$ on the sphere. Thus we have $\langle L^a|L^b\rangle_{\rm c} = g^2\delta^{ab}$ and

$$\langle\bar{\xi}|\bar{\xi}'\rangle_{\rm c} = g^2\bar{\xi}^a\bar{\xi}'^a \tag{C.211}$$

Comparing (C.211) and (C.209), we see the path integral and canonical metrics on G and their corresponding volumes are related by

$$ds^2_{\rm PI} = \frac{M^4}{2\pi}\frac{\mathrm{vol}(S^{d+1})}{g^2}\,ds^2_c \qquad \Rightarrow \qquad \frac{\mathrm{vol}(G)_{\rm PI}}{\mathrm{vol}(G)_{\rm c}} = \left(\frac{M^4}{2\pi}\frac{\mathrm{vol}(S^{d+1})}{g^2}\right)^{\frac{1}{2}\dim G}. \tag{C.212}$$

From (C.201), we get $Z_G = \mathrm{vol}(G)^{-1}_{\rm PI}\big(\frac{M^4}{(d-2)d}\big)^{\frac{1}{2}\dim G}$, hence

$$Z_G = \frac{\gamma^{\dim G}}{\mathrm{vol}(G)_{\rm c}}, \qquad \gamma \equiv \frac{g}{\sqrt{(d-2)A_{d-1}}}, \qquad A_{d-1} \equiv \mathrm{vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}, \tag{C.213}$$

where we used $\mathrm{vol}(S^{d+1}) = \frac{2\pi}{d}\mathrm{vol}(S^{d-1})$. (Recall we have been assuming $d > 2$. The case $d = 2$ is discussed in appendix C.8.1.) The quantity γ may look familiar: the Coulomb potential energy for two unit charges at a distance r in flat space is $V(r) = \gamma^2/r^{d-2}$.

Practically speaking, the upshot is that Z_G is given by (C.213), with $\mathrm{vol}(G)_\mathrm{c}$ the volume of the Yang-Mills gauge group with respect to the metric defined by the Yang-Mills action (C.206). For example for $G = SU(2)$ with $f^{abc} = \epsilon^{abc}$ as before, $\mathrm{vol}(G)_\mathrm{c} = 16\pi^2$, because $SU(2)$ in this metric is the round S^3 with circumference 4π, hence radius 2.

The relation (C.208) can be viewed as defining the coupling constant g given our normalization conventions for the kinetic terms and linearized gauge transformations. Of course the final result is independent of these conventions. Conventions without explicit factors of g in the curvature and gauge transformations are obtained by rescaling $\phi \to \phi/g$, $\xi \to \xi/g$. Then there won't be a factor g in (C.208), but instead g is read off from the action $S = \frac{1}{4g^2}\int (F^a)^2$. We could also write this without explicit reference to a basis as $S = \frac{1}{4g^2}\int \mathrm{Tr}\, F^2$, where the trace "Tr" is normalized such that $\mathrm{Tr}(L^a L^b) \equiv \delta^{ab}$. Then we can say the canonical bilinear/metric/volume is defined by the trace norm appearing in the YM action. We could choose a differently normalized trace $\mathrm{Tr}' = \lambda^2 \mathrm{Tr}$. The physics remains unchanged provided $g' = \lambda g$. Then $\mathrm{vol}(G)'_\mathrm{c} = \lambda^{\dim G}\mathrm{vol}(G)_\mathrm{c}$, hence, consistently, $Z'_G = Z_G$.

As a final example, for $SU(N)$ Yang-Mills with su(N) viewed as anti-hermitian $N \times N$ matrices, $S = -\frac{1}{4g^2}\int \mathrm{Tr}_N F^2$ in conventions without a factor g in the gauge algebra, and Tr_N the ordinary $N \times N$ matrix trace, $\mathrm{vol}(SU(N))_\mathrm{c} = (C.94)$.

Einstein gravity

The Einstein gravity case proceeds analogously. Now we have single massless spin-2 field $\phi_{\mu\nu}$. The gauge transformations are diffeomorphisms generated by vector fields ξ_μ. The subgroup G of diffeomorphisms leaving the background S^{d+1} invariant is $SO(d+2)$, generated by Killing vectors $\bar{\xi}_\mu$. The usual standard basis $M_{IJ} = -M_{JI}$, $I = 1, \ldots, d+2$ of the so$(d+2)$ Lie algebra satisfies $[M_{12}, M_{23}] = M_{13}$ etc. We define the canonical bilinear $\langle\cdot|\cdot\rangle_c$ to be the unique invariant form normalized such that the M_{IJ} have unit norm:

$$\langle M_{12}|M_{12}\rangle_c = 1\,. \qquad \text{(C.214)}$$

With respect to the corresponding metric ds_c^2, orbits $g(\varphi) = e^{\varphi M_{12}}$ with φ ranging from 0 to 2π have length 2π. The canonical volume is then given by (C.93).

To identify the standard generators M_{IJ} more precisely in our normalization conventions for $\bar{\xi}$, we need to look at the field theory realization in more detail. The so$(d+2)$ algebra generated by the Killing vectors $\bar{\xi}$ is realized in the interacting Einstein gravity theory as a subalgebra of the gauge (diffeomorphism) algebra. As in the Yang-Mills case (C.208), the bracket $[\![\cdot,\cdot]\!]$ of this subalgebra is inherited from the gauge algebra. Writing the Killing vectors as $\bar{\xi} = \bar{\xi}^\mu \partial_\mu$, the standard Lie bracket is $[\bar{\xi},\bar{\xi}']_{\rm L} = \big(\bar{\xi}^\mu \partial_\mu \bar{\xi}'^\nu - \bar{\xi}'^\mu \partial_\mu \bar{\xi}^\nu\big)\partial_\nu$. If we had normalized $\phi_{\mu\nu}$ as $\phi_{\mu\nu} \equiv g_{\mu\nu} - g^0_{\mu\nu}$ with $g^0_{\mu\nu}$ the background sphere metric, and if we had normalized ξ_μ by putting $\alpha_2 \equiv 1$ in (3.94), the bracket $[\![\cdot,\cdot]\!]$ would have coincided with the Lie bracket $[\cdot,\cdot]_{\rm L}$. However, we are working in different normalization conventions, in which $\phi_{\mu\nu}$ is canonically normalized and $\alpha_2 = \sqrt{2}$ according to (C.204). In these conventions we have instead

$$[\![\bar{\xi},\bar{\xi}']\!] = \sqrt{16\pi G_{\rm N}}\,[\bar{\xi}',\bar{\xi}]_{\rm L}\,, \tag{C.215}$$

where $G_{\rm N}$ is the Newton constant. This can be checked by starting from the Einstein-Hilbert action, expanding to quadratic order (see e.g. [204] for convenient and reliable explicit expressions in dS$_{d+1}$), and making the appropriate convention rescalings. This is the Einstein gravity analog of (C.208). To be more concrete, let us consider the ambient space description of the sphere S^{d+1}, i.e. $X^I X_I = 1$ with $X \in \mathbb{R}^{d+2}$. Then the basis of Killing vectors $M_{IJ} \equiv -(X_I\partial_J - X_J\partial_I)/\sqrt{16\pi G_{\rm N}}$ satisfy our standard so$(d+2)$ commutation relations $[\![M_{12},M_{23}]\!] = M_{13}$ etc, hence by (C.214), $\langle M_{12}|M_{12}\rangle_c = 1$. The path integral metric (C.203) on the other hand corresponds to the invariant bilinear $\langle\bar{\xi}|\bar{\xi}'\rangle_{\rm PI} = \frac{M^4}{2\pi}\int \bar{\xi}\cdot\bar{\xi}'$, so $\langle M_{12}|M_{12}\rangle_{\rm PI} = \frac{M^4}{2\pi}\frac{1}{16\pi G_{\rm N}}\int_{S^{d+1}}(X_1^2+X_2^2) = \frac{M^4}{2\pi}\frac{1}{16\pi G_{\rm N}}\frac{2}{d+2}{\rm vol}(S^{d+1})$. Thus we obtain the following relation between PI and canonical metrics and volumes for $G = SO(d+2)$:

$$ds^2_{\rm PI} = \frac{A_{d-1}}{4G_{\rm N}}\frac{1}{d(d+2)}\frac{M^4}{2\pi}\,ds^2_{\rm c} \qquad \Rightarrow \qquad \frac{{\rm vol}(G)_{\rm PI}}{{\rm vol}(G)_{\rm c}} = \left(\frac{A_{d-1}}{4G_{\rm N}}\frac{1}{d(d+2)}\frac{M^4}{2\pi}\right)^{\frac{1}{2}\dim G} \tag{C.216}$$

where $\dim G = \frac{1}{2}(d+2)(d+1)$, $A_{d-1} = \mathrm{vol}(S^{d-1})$ as in (C.213), and we again used $\mathrm{vol}(S^{d+1}) = \frac{2\pi}{d}\mathrm{vol}(S^{d-1})$. Combining this with (C.201), we get our desired result:

$$Z_G = \frac{\gamma^{\dim G}}{\mathrm{vol}(G)_{\mathrm{c}}}, \qquad \gamma \equiv \sqrt{\frac{8\pi G_{\mathrm{N}}}{A_{d-1}}}\,. \tag{C.217}$$

Higher-spin gravity

We follow the same template for the higher-spin case. In the interacting higher-spin theory, the Killing tensors generate a subalgebra of the nonlinear gauge algebra, with bracket $[\![\cdot,\cdot]\!]$ inherited from the gauge algebra, just like in the Yang-Mills and Einstein examples, except the gauge algebra is much more complicated in the higher-spin case. Fortunately it is not necessary to construct the exact gauge algebra to determine the Killing tensor algebra: it suffices to determine the lowest order deformation of the linearized gauge transformation (3.94) fixed by the transverse-traceless cubic couplings of the theory [68]. The Killing tensor algebra includes in particular an $\mathrm{so}(d+2)$ subalgebra, that is to say an algebra of the same general form (C.215) as in Einstein gravity, with some constant appearing on the right-hand side determined by the spin-2 cubic coupling in the TT action. We *define* the "Newton constant" G_{N} of the higher-spin theory to be this constant, that is to say we read off G_{N} from the $\mathrm{so}(d+2)$ Killing vector subalgebra by writing it as

$$[\![\bar{\xi},\bar{\xi}']\!] = \sqrt{16\pi G_{\mathrm{N}}}\,[\bar{\xi}',\bar{\xi}]_{\mathrm{L}}\,. \tag{C.218}$$

The standard Killing vector basis is then again given by $M_{IJ} \equiv -(X_I\partial_J - X_J\partial_I)/\sqrt{16\pi G_{\mathrm{N}}}$, satisfying $[\![M_{12}, M_{23}]\!] = M_{13}$ etc.

It was argued in [68] that for the most general set of consistent parity-preserving cubic interactions, assuming the algebra does not split as a direct sum of subalgebras, i.e. assuming the algebra is simple, there exists an up to normalization unique invariant bilinear form $\langle\cdot|\cdot\rangle_c$ on the Killing tensor algebra. We fix its normalization again by requiring the standard $\mathrm{so}(d+2)$ Killing vectors

M_{IJ} have unit norm,

$$\langle M_{12}|M_{12}\rangle_c \equiv 1\,. \tag{C.219}$$

Expressed in terms of the bilinears $\langle\bar{\xi}_{s-1}|\bar{\xi}_{s-1}\rangle_{\rm PI} = \frac{M^4}{2\pi}\int \bar{\xi}_{s-1}\cdot\bar{\xi}_{(s-1)}$ corresponding to (C.203), the invariant bilinear on the Killing tensor algebra takes the general form

$$\langle\bar{\xi}|\bar{\xi}'\rangle_c = \sum_s B_s\,\langle\bar{\xi}_{s-1}|\bar{\xi}'_{s-1}\rangle_{\rm PI}\,, \tag{C.220}$$

where B_s are certain constants fixed in principle by the algebra. The arguments given in [68] moreover imply that up to overall normalization, the coefficients B_s are independent of the coupling constants in the theory. More specifically, adapted (with some work, as described below) to our setting and conventions, and correcting for what we believe is a typo in [68], the coefficients are $B_s \propto (2s+d-4)(2s+d-2)$. We confirmed this by comparison to [71], where the invariant bilinear form for minimal Vasiliev gravity in AdS_{d+1}, dual to the free $O(N)$ model, was spelled out in detail, building on [68–70]. Analytically continuing to positive cosmological constant, implementing their ambient space X-contractions by a Gaussian integral, and reducing this integral to the sphere by switching to spherical coordinates, the expression in [71] can be brought to the form (C.220). This transformation almost completely cancels the factorials in the analogous coefficients b_s in [71], reducing to the simple $B_s \propto (2s+d-4)(2s+d-2)$. (The alternating signs of [71] are absent here due to the analytic continuation to positive cc.) Taking into account our normalization prescription (C.219) (which is different from the normalization chosen in [71]), we thus get

$$\left\langle\bar{\xi}|\bar{\xi}\right\rangle_{\rm c} = \frac{2\pi}{M^4}\cdot\frac{4G_{\rm N}}{A_{d-1}}\sum_s(2s+d-4)(2s+d-2)\left\langle\bar{\xi}^{(s-1)}\middle|\bar{\xi}^{(s-1)}\right\rangle_{\rm PI}, \tag{C.221}$$

with $A_{d-1} = \mathrm{vol}(S^{d-1})$ as before. In view of the independence of the coefficients B_s of the couplings within the class of theories considered in [68], i.e. all parity-invariant massless higher-spin gravity theories consistent to cubic order, this result is universal, valid for this entire class.

As before for Einstein gravity and Yang Mills, from (C.221) we get the ratio

$$\frac{\mathrm{vol}(G)_{\mathrm{PI}}}{\mathrm{vol}(G)_{\mathrm{c}}} = \prod_s \left(\frac{M^4}{2\pi} \cdot \frac{A_{d-1}}{4G_{\mathrm{N}}} \cdot \frac{1}{\big(2s+d-4\big)\big(2s+d-2\big)} \right)^{\frac{1}{2} N^{\mathrm{KT}}_{s-1}} . \tag{C.222}$$

Combining this with (C.201) we see that, rather delightfully, all the unpleasant-looking factors cancel, leaving us with

$$\boxed{ Z_G = \frac{\gamma^{\dim G}}{\mathrm{vol}(G)_{\mathrm{c}}} \, , \qquad \gamma \equiv \sqrt{\frac{8\pi G_{\mathrm{N}}}{A_{d-1}}} } \tag{C.223}$$

This takes exactly the same form as the Einstein gravity result (C.217) except G is now the higher-spin symmetry group rather than the $SO(d+2)$ spin-2 symmetry group.

The cancelation of the UV divergent factors M is as expected from consistency with locality. The cancelation of the s-dependent factors on the other hand seems surprising, in view of the different origin of the numerator (spectrum of quadratic action) and the denominator (invariant bilinear form on higher spin algebra of interactions). Apparently the former somehow knows about the latter. We do not see an obvious reason why this is the case, although the simplicity and universality of the result suggests we should, and that this entire section should be replaceable by a one-line argument. Perhaps it is obvious in a frame-like formalism.

Newton constant from central charge

Recall that the Newton constant G_{N} appearing in (C.223) was defined by the so$(d+2)$ algebra (C.218) in our normalization conventions. An analogous definition can be given in dS$_{d+1}$ or AdS$_{d+1}$ where the algebra becomes so$(1, d+1)$ resp. so$(2, d)$. Starting from this definition, G_{N} can also be formally related to the Cardy central charge C of a putative[21] boundary CFT for AdS or dS, defined as the coefficient of the CFT 2-point function of the putative energy-momentum tensor.

[21] There is no assumption whatsoever this CFT actually exists. One just imagines it exists and uses the formal holographic dictionary to infer the two-point function of this imaginary CFT's stress tensor. In dS, this "dual CFT" can be thought of as computing the Hartle-Hawking wave function of the universe [170].

With our definition of G_N, the computation of [205] remains unchanged, so we can just copy the result obtained there:

$$\boxed{C = \frac{(\pm 1)^{\frac{d-1}{2}}\,\Gamma(d+2)}{(d-1)\,\Gamma\big(\frac{d}{2}\big)^2} \cdot \frac{A_{d-1}}{8\pi G_\mathrm{N}}} \tag{C.224}$$

where as before $A_{d-1} = 2\pi^{d/2}\ell^{d-1}/\Gamma(\frac{d}{2})$, and $\pm 1 = +1$ for AdS and -1 for dS. The central charge of N free real scalars equals $C = \frac{d}{2(d-1)}\,N$ in the conventions used here. Note that (C.224) reduces to the Brown-Henneaux formula $C = 3\ell/2G_\mathrm{N}$ for $d = 2$. In [181] it was argued that the Hartle-Hawking wave function of minimal Vasiliev gravity in dS_4 is perturbatively computed by a $d = 3$ CFT of N free Grassmann scalars. This CFT has central charge $C = -\frac{3}{4}N$, hence according to (C.224), $G_\mathrm{N} = 2^5/\pi N$ and $\gamma = \sqrt{2G_\mathrm{N}} = 8/\sqrt{\pi N}$.

The final result of this appendix, putting everything together, is stated in (3.112).

C.8 One-loop and exact results for 3D theories

C.8.1 Character formula for $Z_\mathrm{PI}^{(1)}$

For $d = 2$, i.e. dS_3 / S^3, some of the generic-d formulae in sections 3.4 and 3.5 become a bit degenerate, requiring separate discussion. One reason $d = 2$ is a bit more subtle is that the spin-s irreducible representation of $SO(2)$ actually comes in two distinct chiral versions $\pm s$, as do the corresponding $SO(1,3)$ irreducible representations $(\Delta, \pm s)$. Likewise the field modes of a spin s field in the path integral on S^3 split into chiral irreps $(n, \pm s)$ of $SO(4)$. The dimensions $D^2_s = D^2_{-s} = 1$ and $D^4_{n,s} = D^4_{n,-s} = (1+n-s)(1+n+s)$ of the $SO(2)$ and $SO(4)$ irreps are correctly reproduced by the Weyl dimension formula (C.87), rather than (C.15). It should however be kept in mind that the single-particle Hilbert space of for instance a massive spin-$s \geq 1$ Pauli-Fierz field on dS_3 carries *both* helicity versions $(\Delta, \pm s)$ of the massive spin-s SO(1,3) irrep, hence the character χ to be used in expressions for Z_PI in this case is $\chi = \chi_{+s} + \chi_{-s} = 2\chi_{+s} = 2(q^\Delta + q^{2-\Delta})/(1-q)^2$. On the other hand for a real scalar field, we just have $\chi = \chi_0 = (q^\Delta + q^{2-\Delta})/(1-q)^2$.

For massless higher-spin gauge fields of spin $s \geq 2$, a similar reasoning implies we should include an overall factor of 2 in (C.180)-(C.181). For an $s = 1$ Maxwell field on the other hand, we get a factor of 2 in the first term but not in the second term (since the gauge parameter/ghost field is a scalar). The proper massless spin-s bulk and edge characters are then obtained from these by the polar term flip (C.183) as usual. This results in

$$\chi_{\text{bulk,s}} = 0 \quad (s \geq 2) , \qquad \chi_{\text{bulk}}^{(s=1)} = \frac{2q}{(1-q)^2} , \qquad \chi_{\text{edge,s}} = 0 \quad (\text{all } s) , \tag{C.225}$$

expressing the absence of propagating degrees of freedom (i.e. particles) for massless spin-$s \geq 2$ fields on dS_3.

This can also be derived more directly from the general path integral formula (C.161), taking into account the $\pm s$ doubling. In particular for massless $s \geq 2$, (C.171) gets replaced by

$$\hat{F}_s = \sum_{n \geq -1} \Theta(1+n)\, 2D^4_{n,s}\big(q^{s+n} + q^{2-s+n}\big) - \sum_{n \geq -1} \Theta(1+n)\, 2D^4_{n,s-1}\big(q^{s+1+n} + q^{1-s+n}\big), \tag{C.226}$$

which matters for the $n = -1$ term because $\Theta(0) \equiv \frac{1}{2}$. For $s = 1$, we get instead

$$\hat{F}_1 = \sum_{n \geq -1} \Theta(1+n)\, 2D^4_{n,1}\big(q^{1+n} + q^{1+n}\big) - \sum_{n \geq 0} D^4_{n,0}\big(q^{2+n} + q^n\big) . \tag{C.227}$$

For $s \geq 2$, the computation of Z_{char} and Z_G remains essentially unchanged. For $s = 1$ there are some minor changes. The edge character in (C.181) acquires an extra q^0 term in $d = 2$ because $q^{s+d-3} = q^0$, so the map $\hat{\chi}_{\text{edge}} \to [\hat{\chi}_{\text{edge}}]_+$ gets an extra -1 subtraction, as a result of which the factor -2 in (C.182) becomes a -3. Relatedly we get an extra q^0 term in $q^{2s+d-4} + q^{2s+d-2} + 2 = q^2 + 3$ in (C.200), and we end up with $Z_G = \tilde{\gamma}^{\dim G}/\mathrm{vol}(G)_{\text{c}}$ with $\tilde{\gamma} = g\ell/\sqrt{A_1} = g\sqrt{\ell}/\sqrt{2\pi}$ instead of (C.213). Everything else remains the same.

Finally, the phase i^{-P_s} (C.173) is somewhat modified. For $s \geq 2$, from (C.226),

$$P_s = -\sum_{n=-1}^{s-3} \Theta(1+n)\, 2D^4_{n,s} - \sum_{n=-1}^{s-2} \Theta(1+n)\, 2D^4_{n,s-1} = \frac{1}{3}(2s-3)(2s-1)(2s+1) \tag{C.228}$$

Note that $P_2 = 5$, in agreement with [59]. $P_1 = 0$ as before, since there are no negative modes.

Conclusion

The final result for $Z^{(1)}_{\rm PI} = Z_G Z_{\rm char}$ in dS_3 replacing (3.112)-(3.113) is:

- For Einstein and HS gravity theories with $s \geq 2$,

$$Z^{(1)}_{\rm PI} = i^{-P}\, \frac{\gamma^{\dim G}}{{\rm vol}(G)_{\rm c}} \cdot Z_{\rm char}\,, \qquad Z_{\rm char} = e^{-2\dim G \int^{\times} \frac{dt}{2t}\, \frac{1+q}{1-q}} = (2\pi)^{\dim G} e^{-\dim G \cdot c\, \ell\, \epsilon^{-1}}\,, \tag{C.229}$$

where as before $\gamma = \sqrt{8\pi G_{\rm N}/A_1} = \sqrt{4G_{\rm N}/\ell}$, $P = \sum_s P_s$, and ${\rm vol}(G)_{\rm c}$ is the volume with respect to the metric for which the standard so(4) generators M_{IJ} have norm 1. We used (C.75) to evaluate $Z_{\rm char}$. The coefficient c of the linearly divergent term is an order 1 constant depending on the regularization scheme. (For the heat kernel regularization of appendix C.3, following section C.3.3, $c = \frac{3\pi}{4}$. For a simple cutoff at $t = \epsilon$ as in section C.3.4, $c = 2$.) The finite part is

$$\boxed{Z^{(1)}_{\rm PI,fin} = i^{-P}\, \frac{(2\pi\gamma)^{\dim G}}{{\rm vol}(G)_{\rm c}}\,, \qquad \gamma = \sqrt{\frac{8\pi G_{\rm N}}{2\pi\ell}}} \tag{C.230}$$

For example for Einstein gravity with $G = SO(4)$, we get

$$Z^{(1)}_{\rm PI,fin} = i^{-5}\, \frac{(2\pi\gamma)^6}{(2\pi)^4} = -i\, 4\pi^2\gamma^6\,. \tag{C.231}$$

- For Yang-Mills theories with gauge group G and coupling constant g, we get

$$Z^{(1)}_{\rm PI} = \frac{\tilde{\gamma}^{\dim G}}{{\rm vol}(G)_{\rm c}} \cdot e^{\dim G \int^{\times} \frac{dt}{2t}\, \frac{1+q}{1-q}\left(\frac{2q}{(1-q)^2} - 3\right)}\,, \qquad \tilde{\gamma} = \frac{g\sqrt{\ell}}{\sqrt{2\pi}}\,. \tag{C.232}$$

Using (C.57), (C.75), the finite part evaluates to

$$Z^{(1)}_{\text{PI,fin}} = \frac{(2\pi g\sqrt{\ell}\, Z_1)^{\dim G}}{\text{vol}(G)_{\text{c}}}, \qquad Z_1 = e^{-\frac{\zeta(3)}{4\pi^2}}\,. \tag{C.233}$$

As in (3.112), $\text{vol}(G)_{\text{c}}$ is the volume of G with respect to the metric defined by the trace appearing in the Yang-Mills action. As a check, for $G = U(1)$ we have $\text{vol}(G)_{\text{c}} = 2\pi$, so $Z = g\, e^{-\zeta(3)/4\pi^2}\sqrt{\ell}$ in agreement with [206] eq. (3.25).

- We could also consider the Chern-Simons partition function on S^3,

$$Z_k = \int \mathcal{D}A\, e^{i\,k\,S_{\text{CS}}[A]}, \qquad S_{\text{CS}}[A] \equiv \frac{1}{4\pi}\int \text{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A), \tag{C.234}$$

with $k > 0$ suitably quantized ($k \in \mathbb{Z}$ for $G = SU(N)$ with Tr the trace in the N-dimensional representation). Because in this case the action is first order in the derivatives and not parity-invariant, it falls outside the class of theories we have focused on in this paper. It is not too hard though to generalize the analysis to this case. The main difference with Yang-Mills is that $\chi_{\text{bulk}} = 0 = \chi_{\text{edge}}$: like in the $s \geq 2$ case, the $s = 1$ Chern-Simons theory has no particles. The function $\hat{F}_1$ is no longer given by the Maxwell version (C.227), but rather by (C.226), except without the factors of 2, related to the fact that the CS action is first order in the derivatives. This immediately gives $F_1 = \hat{F}_1 = -2\,\frac{1+q}{1-q}$. The computation of the volume factor is analogous to our earlier discussions. The result (in canonical framing [207]) is

$$Z^{(1)}_k = \frac{\tilde{\gamma}^{\dim G}}{\text{vol}(G)_{\text{Tr}}}\, e^{-2\dim G\int^{\times}\frac{dt}{2t}\frac{1+q}{1-q}}, \qquad Z^{(1)}_{k,\text{fin}} = \frac{(2\pi\tilde{\gamma})^{\dim G}}{\text{vol}(G)_{\text{Tr}}}, \qquad \tilde{\gamma} = \frac{1}{\sqrt{k}}, \tag{C.235}$$

where $\text{vol}(G)_{\text{Tr}}$ is the volume with respect to the metric defined by the trace appearing in the Chern-Simons action (C.234). This agrees with the standard results in the literature, nicely reviewed in section 4 of [208].

C.8.2 Chern-Simons formulation of Einstein gravity

3D Einstein gravity can be reformulated as a Chern-Simons theory [143, 209]. Although well-known, we briefly review some of the basic ingredients and conceptual points here to facilitate the discussion of the higher-spin generalization in section C.8.3. A more detailed review of certain aspects, including more explicit solutions, can be found in section 4 of [210]. Explicit computations using the Chern-Simons formulation of $\Lambda > 0$ Euclidean quantum gravity with emphasis on topologies more sophisticated than the sphere can be found in [84, 211, 212].

Lorentzian gravity

For the Lorentzian theory with positive cosmological constant, amplitudes are computed by path integrals $\int \mathcal{D}A\, e^{iS_L}$ with real Lorentzian $SL(2,\mathbb{C})$ Chern-Simons action [144]

$$S_L = (l + i\kappa)S_{\rm CS}[A_+] + (l - i\kappa)S_{\rm CS}[A_-] \,, \qquad A_+^* = A_-, \tag{C.236}$$

where $S_{\rm CS}$ is as in (C.234) with $A_\pm$ an sl$(2,\mathbb{C})$-valued connection and $\mathrm{Tr} = \mathrm{Tr}_2$. The vielbein e and spin connection ω are the real and imaginary parts of the connection:

$$A_\pm = \omega \pm ie/\ell \,, \qquad ds^2 = 2\,\mathrm{Tr}_2\, e^2 = \eta_{ij} e^i e^j \,. \tag{C.237}$$

For the last equality we decomposed $e = e^i L_i$ in a basis L_i of sl$(2,\mathbb{R})$, say

$$(L_1, L_2, L_3) \equiv (\tfrac{1}{2}\sigma_1, \tfrac{1}{2}i\sigma_2, \tfrac{1}{2}\sigma_3) \qquad \Rightarrow \qquad \eta_{ij} \equiv 2\,\mathrm{Tr}_2(L_i L_j) = \mathrm{diag}(1,-1,1) \,. \tag{C.238}$$

Note that $[L_i, L_j] = -\epsilon_{ijk} L^k$ with $L^k \equiv \eta^{kk'} L_{k'}$. When $l = 0$, the action reduces to the firs-order form of the Einstein action with Newton constant $G_{\rm N} = \ell/4\kappa$ and cosmological constant $\Lambda = 1/\ell^2$. The equations of motion stipulate $A_\pm$ must be flat connections:

$$dA_\pm + A_\pm \wedge A_\pm = 0 \,, \tag{C.239}$$

equivalent with the Einstein gravity torsion constraint (with $\omega^i{}_j \equiv \eta^{il}\epsilon_{ljk}\omega^k$) and the Einstein equations of motion [143]. Turning on l deforms the action by parity-odd terms of gravitational Chern-Simons type. This does not affect the equations of motion (C.239). We can take $l \geq 0$ without loss of generality. The part of the action multiplied by l has a discrete ambiguity forcing l to be integrally quantized, like k in (C.234). Summarizing,

$$0 \leq l \in \mathbb{Z}\,, \qquad 0 < \kappa = \frac{2\pi\ell}{8\pi G_{\rm N}} \in \mathbb{R}, \tag{C.240}$$

dS$_3$ vacuum solution

A flat connection corresponding to the de Sitter metric can be obtained as follows. (We will be brief because the analog for the sphere below will be simpler and make this more clear.) Define $Q(X) \equiv 2\,(X^4 L_4 + iX^i L_i)$ with $L_4 \equiv \frac{1}{2}\mathbf{1}$ and note that $\det Q = X_4^2 + \eta_{ij}X^iX^j =: \eta_{IJ}X^IX^J$, so $\mathcal{M} \equiv \{X|\det Q(X) = 1\}$ is the dS$_3$ hyperboloid, and Q is a map from $\mathcal{M}$ into $SL(2,\mathbb{C})$. Its square root $h \equiv Q^{1/2}$ is then a map from $\mathcal{M}$ into $SL(2,\mathbb{C})/\mathbb{Z}_2 \simeq SO(1,3)$, so $A_+ \equiv h^{-1}dh$ is a flat sl$(2,\mathbb{C})$-valued connection on $\mathcal{M}$. Moreover on $\mathcal{M}$ we have $Q^* = Q^{-1}$, so $h^* = h^{-1}$, $A_- = A_+^* = -(dh)h^{-1}$, and $ds^2 = -\frac{1}{2}\ell^2\,\mathrm{Tr}(A_+ - A_-)^2 = -\frac{1}{2}\ell^2\,\mathrm{Tr}\,(Q^{-1}dQ)^2 = \ell^2\eta_{IJ}dX^IdX^J$, which is the de Sitter metric of radius ℓ on $\mathcal{M}$.

Euclidean gravity

Like the Einstein-Hilbert action — or any other action for that matter — (C.236) may have complex saddle points, that is to say flat connections $A_\pm$ which do not satisfy the reality constraint (C.236), or equivalently solutions for which some components of the vielbein and spin connection are not real. Of particular interest for our purposes is the solution corresponding to the round metric on S^3. This can be obtained from the dS$_3$ solution as usual by a Wick rotation of the time coordinate. Given our choice of sl$(2,\mathbb{R})$ basis (C.238), this means $X^2 \to -iX^2$. At the level of the vielbein $e = e^iL_i$ such a Wick rotation is implemented as $e^2 \to -ie^2$. Similarly, recalling $\omega_{ij} = \epsilon_{ijk}\omega^k$, the spin connection $\omega = \omega^iL_i$ rotates as $\omega^1 \to i\omega^1$, $\omega^3 \to i\omega^3$. Equivalently, $A_\pm \to$

$(\omega^i \pm e^i/\ell)S_i$ where $S_i \equiv \frac{1}{2}i\sigma_i$. Notice the S_i are the generators of su(2), satisfying $[S_i, S_j] = -\epsilon_{ijk}S_k$, $-2\,\mathrm{Tr}_2(S_iS_j) = \delta_{ij}$ and $S_i^\dagger = -S_i$. Thus the Lorentzian metric η_{ij} gets replaced by the Euclidean metric δ_{ij}, the Lorentzian sl(2, $\mathbb{C}$) = so(1, 3) reality condition gets replaced by the Euclidean su(2)$\oplus$ su(2) = so(4) reality condition, and the Lorentzian path integral $\int \mathcal{D}A\, e^{iS_L}$ becomes a Euclidean path integral $\int \mathcal{D}A\, e^{-S_E}$, where $S_E \equiv -iS_L$ is the Euclidean action:

$$S_E = (\kappa - il)\, S_{\mathrm{CS}}[A_+] - (\kappa + il)\, S_{\mathrm{CS}}[A_-], \qquad A_\pm^\dagger = -A_\pm\,. \tag{C.241}$$

This can be interpreted as the Chern-Simons formulation of Euclidean Einstein gravity with positive cosmological constant. The su(2) $\oplus$ su(2)-valued connection (A_+, A_-) encodes the Euclidean vielbein, spin connection and metric as

$$A_\pm = \omega \pm e/\ell = (\omega^i \pm e^i/\ell)S_i\,, \qquad S_i \equiv \tfrac{1}{2}i\sigma_i\,, \qquad ds^2 = -2\,\mathrm{Tr}_2\, e^2 = \delta_{ij}e^ie^j\,. \tag{C.242}$$

The Euclidean counterpart of the reality condition of the Lorentzian action is that S_E gets mapped to S_E^* under reversal of orientation. Reversal of orientation maps $S_{\mathrm{CS}}[A] \to -S_{\mathrm{CS}}[A]$, and in addition here it also exchanges the $\pm$ parts of the decomposition so(4) = su(2)$_+$ $\oplus$ su(2)$_-$ into self-dual and anti-self-dual parts, that is to say it exchanges $A_+ \leftrightarrow A_-$. Thus orientation reversal maps $S_E \to -(\kappa - il)\, S_{\mathrm{CS}}[A_-] + (\kappa + il)\, S_{\mathrm{CS}}[A_+] = S_E^*$, as required.

Round sphere solutions

Parametrizing S^3 by $g \in SU(2) \simeq S^3$, it is easy to write down a flat su(2) $\oplus$ su(2) connection yielding the round metric of radius ℓ:

$$(A_+, A_-) = (g^{-1}dg, 0) \qquad \Rightarrow \qquad e/\ell = \tfrac{1}{2}g^{-1}dg = \omega\,, \quad ds^2 = -\tfrac{1}{2}\ell^2\,\mathrm{Tr}(g^{-1}dg)^2\,. \tag{C.243}$$

The radius can be checked by observing that along an orbit $g(\varphi) = e^{\varphi S_3}$, we get $g^{-1}dg = d\varphi\, S_3$ so $ds = \frac{1}{2}\ell d\varphi$ and the orbit length is $\int_0^{4\pi} ds = 2\pi\ell$. The on-shell action is $S_E = -\frac{\kappa - il}{12\pi}\int_{S^3}\mathrm{Tr}_2(g^{-1}dg)^3 =$

$-\frac{2(\kappa-il)}{3\pi\ell^3}\int e^i e^j e^k \, \mathrm{Tr}_2(S_i S_j S_k) = -\frac{\kappa-il}{6\pi\ell^3}\int e^i e^j e^k \, \epsilon_{ijk} = -2\pi(\kappa - il)$, so

$$\exp(-S_E) = \exp\big(2\pi\kappa + 2\pi i l\big) = \exp\Big(\frac{2\pi\ell}{4G_{\mathrm{N}}}\Big), \tag{C.244}$$

where we used (C.240). This reproduces the standard Gibbons-Hawking result [10] for dS_3.

More generally we can consider flat connections of the form $(A_+, A_-) = (h_+^{-1}dh_+, h_-^{-1}dh_-)$ with $h_\pm = g^{n_\pm}$, where $n_\pm \in \mathbb{Z}$ if we take the gauge group to be $G = SU(2) \times SU(2)$ and $n_\pm \in \mathbb{Z}$ or $n_\pm \in \frac{1}{2} + \mathbb{Z}$ if we take $G = \big(SU(2) \times SU(2)\big)/\mathbb{Z}_2 \simeq SO(4)$. These are all related to the trivial connection $(0, 0)$ by a *large* gauge transformation $g \in S^3 \to (h_+, h_-) \in G$. All other flat connections on S^3 are obtained from these by gauge transformations continuously connected to the identity, which are equivalent to diffeomorphisms and vielbein rotations continuously connected to the identity in the metric description [143]. Large gauge transformations on the other hand are in general *not* equivalent to large diffeomorphisms. Indeed,

$$e^{-S_E} = e^{2\pi n\kappa + 2\pi i\tilde{n}l} = e^{2\pi n\kappa}, \qquad n \equiv n_+ - n_-, \quad \tilde{n} \equiv n_+ + n_-, \tag{C.245}$$

so evidently different values of n are physically inequivalent. Conversely, for a fixed value of n but different values of $\tilde{n}$, we get the same metric, so these solutions are geometrically equivalent. In particular the $n = 1$ solutions all produce the same round metric (C.243). For $n = 0$, the metric vanishes. For $n < 0$, we get a vielbein with negative determinant. Only vielbeins with positive determinant reproduce the Einstein-Hilbert action with the correct sign, so from the point of view of gravity we should discard the $n < 0$ solutions. Finally the cases $n > 1$ correspond to a metric describing a chain of n spheres connected by throats of zero size, presumably more appropriately thought of as n disconnected spheres. The Wick rotation $X^2 \to -iX^2$ of our earlier constructed Lorentzian dS_3 equals the $(n+, n_-) = (\frac{1}{2}, -\frac{1}{2})$ solution constructed here.

Euclidean path integral

The object of interest to us is the Euclidean path integral $Z = \int \mathcal{D}A\, e^{-S_E[A]}$, defined perturbatively around an $n = n_+ - n_- = 1$ round sphere solution $(\bar{A}_+, \bar{A}_-) = (g^{-n_+} dg^{n_+}, g^{-n_-} dg^{n_-})$, such as the $(1,0)$ solution (C.243). Physically, this can be interpreted as the all-loop quantum-corrected Euclidean partition function of the dS_3 static patch. For simplicity we take $G = SU(2) \times SU(2)$, so $n_\pm \in \mathbb{Z}$ and we can formally factorize Z as an $SU(2)_{k_+} \times SU(2)_{k_-}$ CS partition function where $k_\pm = l \pm i\kappa$, with $l \in \mathbb{Z}^+$ and $\kappa \in \mathbb{R}^+$:

$$Z = \int_{(n_+, n_-)} \mathcal{D}A\, e^{ik_+ S_{\mathrm{CS}}[A_+] + ik_- S_{\mathrm{CS}}[A_-]} = Z_{\mathrm{CS}}\big(SU(2)_{k_+} | \bar{A}_{n_+}\big)\, Z_{\mathrm{CS}}\big(SU(2)_{k_-} | \bar{A}_{n_-}\big) , \tag{C.246}$$

Here the complex-k CS partition function $Z_{\mathrm{CS}}(SU(2)_k | \bar{A}_m) \equiv \int_m \mathcal{D}A\, e^{ikS_{\mathrm{CS}}[A]}$ is defined perturbatively around the critical point $\bar{A} = g^{-m} dg^m$. It is possible, though quite nontrivial in general, to define Chern-Simons theories at complex level k on general 3-manifolds M_3 [213, 214]. Our goal is less ambitious, since we only require a perturbative expansion of Z around a given saddle, and moreover we restrict to $M_3 = S^3$. In contrast to generic M_3, at least for integer k, the CS action on S^3 has a unique critical point modulo gauge transformations, and its associated perturbative large-k expansion is not just asymptotic, but actually converges to a simple, explicitly known function: in canonical framing [207],

$$Z_{\mathrm{CS}}\big(SU(2)_k | \bar{A}\big)_0 = \sqrt{\tfrac{2}{2+k}}\, \sin\!\big(\tfrac{\pi}{2+k}\big)\, e^{i(2+k)S_{\mathrm{CS}}[\bar{A}]} \qquad (k \in \mathbb{Z}^+) . \tag{C.247}$$

The dependence on the choice of critical point $\bar{A} = g^{-m} dg^m$ actually drops out for integer k, as $S_{\mathrm{CS}}[\bar{A}] = -2\pi m \in 2\pi\mathbb{Z}$. We have kept it in the above expression to because this is no longer the case for complex k. Analytic continuation to $k_\pm = l \pm i\kappa$ with $l \in \mathbb{Z}^+$ and $\kappa \in \mathbb{R}^+$ in (C.246) then gives:

$$Z_0 = \left| \sqrt{\tfrac{2}{2+l+i\kappa}}\, \sin\!\big(\tfrac{\pi}{2+l+i\kappa}\big) \right|^2 e^{2\pi n\kappa - 2\pi i\, \tilde{n}(2+l)} = \left| \sqrt{\tfrac{2}{2+l+i\kappa}}\, \sin\!\big(\tfrac{\pi}{2+l+i\kappa}\big) \cdot e^{\pi\kappa} \right|^2 . \tag{C.248}$$

Framing dependence of phase and one-loop check

For a general choice of S^3 framing with $SO(3)$ spin connection $\hat{\omega}$, (C.247) gets replaced by [207]

$$Z_{\rm CS}\big(SU(2)_k|\bar{A}\big) = \exp\big(\tfrac{i}{24}c(k)I(\hat{\omega})\big)\, Z_{\rm CS}\big(SU(2)_k|\bar{A}\big)_0\,, \qquad c(k) = 3\big(1 - \tfrac{2}{2+k}\big)\,, \tag{C.249}$$

where $I(\hat{\omega}) = \frac{1}{4\pi}\int {\rm Tr}_3(\hat{\omega}\wedge d\hat{\omega} + \frac{2}{3}\hat{\omega}\wedge\hat{\omega}\wedge\hat{\omega})$ is the gravitational Chern-Simons action. The action $I(\hat{\omega})$ can be defined more precisely as explained under (2.22) of [215], by picking a 4-manifold M with boundary $\partial M = S^3$ and putting

$$I(\hat{\omega}) \equiv I_M \equiv \frac{1}{4\pi}\int_M {\rm Tr}(R\wedge R)\,, \tag{C.250}$$

where R is the curvature form of M, $R^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\rho\sigma}dx^\rho \wedge dx^\sigma$. Taking M to be a flat 4-ball B, the curvature vanishes so $I_B = 0$, corresponding to canonical framing. Viewing B as a 4-hemisphere with round metric has ${\rm Tr}(R\wedge R) = 0$ pointwise so again $I_B = 0$. Gluing any other 4-manifold M with boundary S^3 to B, we get a closed 4-manifold $X = M - B$, with $I_M - I_B = \frac{1}{4\pi}\int_X {\rm Tr}(R\wedge R) = 2\pi p_1(X)$, where $p_1(X)$ is the Pontryagin number of X. According to the Hirzebruch signature theorem, the signature $\sigma(X) = b_2^+ - b_2^-$ of the intersection form of the middle cohomology of X equals $\frac{1}{3}p_1(X)$. Therefore, for any choice of M,

$$I_M = 6\pi r\,, \qquad r = \sigma(X) \in \mathbb{Z}\,. \tag{C.251}$$

For example $r = 1$ for $X = \mathbb{CP}^2$ and $r = p - q$ for $X = p\mathbb{CP}^2\#q\overline{\mathbb{CP}}^2$. Thus for general framing, (C.249) becomes $Z_{\rm CS}(k|m) = Z_{\rm CS}(k|m)_0\, \exp\big(r\,c(k)\,\frac{i\pi}{4}\big)$ and (C.248) becomes

$$Z_r = e^{ir\phi}\left|\sqrt{\tfrac{2}{2+l+i\kappa}}\,\sin\big(\tfrac{\pi}{2+l+i\kappa}\big)e^{\pi\kappa}\right|^2, \qquad r \in \mathbb{Z}, \tag{C.252}$$

where, using $c(k) = 3\big(1 - \frac{2}{2+k}\big)$, the phase is given by

$$r\phi = r\big(c(l + i\kappa) + c(l - i\kappa)\big)\frac{\pi}{4} = r\big(1 - \tfrac{2(2+l)}{(2+l)^2+\kappa^2}\big)\frac{3\pi}{2}\,. \tag{C.253}$$

In the weak-coupling limit $\kappa \to \infty$,

$$Z_r \to (-i)^r\,\frac{2\pi^2}{\kappa^3}\cdot e^{2\pi\kappa}\,. \tag{C.254}$$

Using (C.240) and taking into account that we took $G = SU(2)\times SU(2)$ here, the absolute value agrees with our general one-loop result (C.230) in the metric formulation, with the phase $(-i)^r$ matching Polchinski's phase $i^{-P} = i^{-5} = -i$ in (3.112) for odd framing r.[22] We do not have any useful insights into why (or whether) CS framing and the phase i^{-P} might have anything to do with each other, let alone why odd but not even framing should reproduce the phase of [59]. Perhaps different contour rotation prescriptions as those assumed in [59] might reproduce the canonically framed ($r = 0$) result in the metric formulation of Euclidean gravity. We leave these questions open.

Comparison to previous results: The Chern-Simons formulation of gravity was applied to calculate Euclidean $\Lambda > 0$ partition functions in [84, 211, 212]. The focus of these works was on summing different topologies. Our one-loop (C.254) in canonical framing agrees with [211] up to an unspecified overall normalization constant in the latter, agrees with $Z(S^3)/Z(S^1\times S^2)$ in [212] combining their eqs. (13),(32), and disagrees with eq. (4.39) in [84], $Z^{(1)}(S^3) = \pi^3/(2^5\kappa)$.

C.8.3 Chern-Simons formulation of higher-spin gravity

The $SL(2,\mathbb{C})$ Chern-Simons formulation of Einstein gravity (C.236) has a natural extension to an $SL(n,\mathbb{C})$ Chern-Simons formulation of higher-spin gravity — the positive cosmological constant

[22] Strictly speaking for $r = 1 \bmod 4$, but i^P vs i^{-P} in (3.112) is a matter of conventions, so there is no meaningful distinction we can make here.

analog of the theories studied e.g. in [145–148, 153, 154]. The Lorentzian action is

$$S_L = (l + i\kappa)\, S_{\rm CS}[\mathcal{A}_+] + (l - i\kappa)\, S_{\rm CS}[\mathcal{A}_-]\,, \qquad \mathcal{A}_+^* = \mathcal{A}_-\,, \tag{C.255}$$

where $S_{\rm CS}[\mathcal{A}] = \frac{1}{4\pi}\int \mathrm{Tr}_n\big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)$, an $\mathcal{A}$ is an sl$(n,\mathbb{C})$-valued connection, $\kappa \in \mathbb{R}^+$ and $l \in \mathbb{Z}^+$. The corresponding Euclidean action $S_E = -iS_L$ extending (C.241) is given by

$$S_E = (\kappa - il)\, S_{\rm CS}[\mathcal{A}_+] - (\kappa + il)\, S_{\rm CS}[\mathcal{A}_-]\,, \qquad \mathcal{A}_\pm^\dagger = -\mathcal{A}_\pm\,, \tag{C.256}$$

where $\mathcal{A}_\pm$ are now independent su(n)-valued connections.

Landscape of dS$_3$ vacua

The solutions A of the original ($n = 2$) Einstein gravity theory can be lifted to solutions $\mathcal{A} = R(A)$ of the extended ($n > 2$) theory by choosing an embedding R of sl(2) into sl(n). More concretely, such lifts are specified by picking an n-dimensional representation R of su(2),

$$R = \oplus_a \mathbf{m}_a\,, \qquad \sum_a m_a = n\,, \qquad \mathcal{S}_i = R(S_i) = \oplus_a J_i^{(m_a)}\,. \tag{C.257}$$

Here $J_i^{(m)}$ are the standard anti-hermitian spin $j = \frac{m-1}{2}$ representation matrices of su(2), satisfying the same commutation relations and reality properties as the spin-$\frac{1}{2}$ generators S_i in (C.242). Then the matrices $\mathcal{L}_i \equiv R(L_i)$ with the L_i as in (C.238) are real, generating the corresponding n-dimensional representation of sl$(2,\mathbb{R})$. The Casimir eigenvalue of the spin $j = \frac{m-1}{2}$ irrep is $j(j+1) = \frac{1}{4}(m^2 - 1)$, so

$$\mathrm{Tr}_n(\mathcal{S}_i\mathcal{S}_j) = -\tfrac{1}{2}T_R\,\delta_{ij}\,, \qquad \mathrm{Tr}_n(\mathcal{L}_i\mathcal{L}_j) = \tfrac{1}{2}T_R\,\eta_{ij}\,, \qquad T_R \equiv \frac{1}{6}\sum_a m_a(m_a^2 - 1)\,; \tag{C.258}$$

A general $SL(2,\mathbb{C})$ connection $A = A^i L_i$ has curvature $dA + A\wedge A = \big(dA^i - \frac{1}{2}\epsilon^i{}_{jk}A^j A^k\big)L_i$, and an $SL(n,\mathbb{C})$ connection of the form $\mathcal{A} = R(A) = A^i\mathcal{L}_i$ has curvature $d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = \big(dA^i -$

$\frac{1}{2}\epsilon^i{}_{jk}A^jA^k\big)\mathcal{L}_i$, hence $\mathcal{A} = R(A)$ solves the equations of motion of the extended $SL(n,\mathbb{C})$ theory iff A solves the equations of motion of the original Einstein $SL(2,\mathbb{C})$ theory. In other words, restricting to connections $\mathcal{A} = A^i\mathcal{L}_i$ amounts to a consistent truncation, which may be interpreted as the gravitational subsector of the $n > 2$ theory. Substituting $\mathcal{A} = R(A)$ into the action (C.255) gives the consistently truncated action

$$S_L = (l + i\kappa)T_R\, S_{\rm CS}[A_+] + (l - i\kappa)T_R\, S_{\rm CS}[A_-]\,, \qquad A_+^* = A_-\,, \tag{C.259}$$

which is of the exact same form as the original Einstein CS gravity theory (C.236), except $l + i\kappa$ is replaced by $(l + i\kappa)T_R$. Thus we can naturally interpret the components $A^i_\pm$ again as metric/vielbein/spin connection degrees of freedom, just like in (C.242), i.e. $A^i_\pm = \omega^i \pm ie^i/\ell$, $ds^2 = \eta_{ij}e^ie^j$, and the lift $\mathcal{A} = R(A)$ of the original solution A corresponding to the dS_3 metric again as a solution corresponding to the dS_3 metric. The difference is that the original relation (C.240) between κ and $\ell/G_{\rm N}$ gets modified to

$$\kappa\, T_R = \frac{2\pi\ell}{8\pi G_{\rm N}}\,. \tag{C.260}$$

Since κ is fixed, this means the dimensionless ratio $\ell/G_{\rm N}$ depends on the choice of R. Thus the different solutions $\mathcal{A} = R(A)$ of the $SL(n,\mathbb{C})$ theory can be thought of as different de Sitter vacua of the theory, labeled by R, with different values of the curvature radius in Planck units $\ell/G_{\rm N}$. These are the dS analog of the AdS vacua discussed in [154]. The total number of vacua labeled by $R = \oplus_a \mathbf{m}_a$ equals the number of partitions of $n = \sum_a m_a$,

$$\mathcal{N}_{\rm vac} \sim e^{2\pi\sqrt{n/6}} \qquad (n \gg 1)\,. \tag{C.261}$$

For, say, $n \sim 2\times 10^5$, this gives $\mathcal{N}_{\rm vac} \sim 10^{500}$.

Analogous considerations hold for the Euclidean version of the theory. For example the round sphere solution (C.243) is lifted to

$$(\mathcal{A}_+, \mathcal{A}_-) = \big(R(A_+), 0\big) = \big(R(g)^{-1} dR(g), 0\big), \qquad R(e^{\alpha^i S_i}) \equiv e^{\alpha^i \mathcal{S}_i}, \tag{C.262}$$

with the sphere radius ℓ in Planck units given again by (C.260). The tree-level contribution of the solution (C.262) to the Euclidean path integral is

$$\exp(-S_E) = \exp\big(2\pi\kappa\, T_R\big) = \exp\Big(\frac{2\pi\ell}{4\, G_{\rm N}}\Big). \tag{C.263}$$

Note that $\mathcal{S}^{(0)} \equiv -S_E = \frac{2\pi\ell}{4G_{\rm N}}$ is the usual dS_3 Gibbons-Hawking horizon entropy [10]. Its value $\mathcal{S}^{(0)} = 2\pi\kappa\, T_R$ depends on the vacuum $R = \oplus_a \mathbf{m}_a$ through T_R as given by (C.258). The vacuum R maximizing e^{-S_E} corresponds to the partition of $n = \sum_a m_a$ maximizing T_R. Clearly the maximum is achieved for $R = \mathbf{n}$:

$$\max_R \, T_R = T_{\mathbf{n}} = \frac{n(n^2-1)}{6}. \tag{C.264}$$

The corresponding embedding of su(2) into su(n) is called the "principal embedding". Thus the "principal vacuum" maximizes the entropy at $S_{\mathrm{GH},\mathbf{n}} = \frac{1}{6} n(n^2-1)\, 2\pi\kappa$, exponentially dominating the Euclidean path integral in the semiclassical (large-κ) regime. In the remainder we focus on the Euclidean version of the theory.

Higher-spin field spectrum and algebra

Of course for $n > 2$, there are more degrees of freedom in the $2(n^2-1)$ independent components of $\mathcal{A}_\pm$ than just the 3+3 vielbein and spin connection degrees of freedom $\mathcal{A}^i_\pm \mathcal{S}_i$. The full set of fluctuations around the vacuum solution can be interpreted in a metric-like formalism as higher-spin field degrees of freedom. The precise spectrum depends on the vacuum R. For the principal vacuum $R = \mathbf{n}$, we get the higher-spin vielbein and spin connections of a set of massless spin-s fields of $s = 2, 3, \ldots, n$, as was worked out in detail for the AdS analog in [153]. Indeed su(n)

decomposes under the principally embedded su(2) subalgebra into spin-r irreps, $r = 1, 2, \ldots, n-1$, generated by the traceless symmetric products $\mathcal{S}_{i_1 \cdots i_r}$ of the generators $\mathcal{S}_i$. As reviewed in [69], this means we can identify the $\mathrm{su}(n)_+ \oplus \mathrm{su}(n)_-$ Lie algebra of the theory (C.256) with the higher-spin algebra $\mathrm{hs}_n(\mathrm{su}(2))_+ \oplus \mathrm{hs}_n(\mathrm{su}(2))_-$, where $\mathrm{su}(2)_+ \oplus \mathrm{su}(2)_- = \mathrm{so}(4)$ is the principally embedded gravitational subalgebra. In the metric-like formalism the spin-r generators correspond to (anti-)self-dual Killing tensors of rank r. These are the Killing tensors of massless symmetric spin-s fields with $s = r + 1$. As a check, recall the number of (anti-)self-dual rank r Killing tensors on S^3 equals $D^4_{r,\pm r} = 2r + 1$, correctly adding up to

$$\sum_{\pm} \sum_{r=1}^{n-1} D^4_{r,\pm r} = 2 \sum_{r=1}^{n-1} (2r + 1) = 2(n^2 - 1)\,. \tag{C.265}$$

For different choices of embedding R, we get different su(2) decompositions of su(n). For example for $n = 12$, while the principal embedding $R = \mathbf{12}$ considered above gives the su(2) decomposition $\mathbf{143}_{\mathrm{su}(12)} = \mathbf{3} + \mathbf{5} + \mathbf{7} + \mathbf{9} + \mathbf{11} + \mathbf{13} + \mathbf{15} + \mathbf{17} + \mathbf{19} + \mathbf{21} + \mathbf{23}$, taking $R = \mathbf{6} \oplus \mathbf{4} \oplus \mathbf{2}$ gives $\mathbf{143}_{\mathrm{su}(12)} = 2 \cdot \mathbf{1} + 7 \cdot \mathbf{3} + 8 \cdot \mathbf{5} + 6 \cdot \mathbf{7} + 3 \cdot \mathbf{9} + \mathbf{11}$. Interpreting these as Killing tensors for n_s massless spin-s fields, we get for the former $n_2 = 1, n_3 = 1, \ldots, n_{12} = 1$, and for the latter $n_1 = 2, n_2 = 7, n_3 = 8, n_4 = 6, n_5 = 3, n_6 = 1$. The tree-level entropy $\mathcal{S}^{(0)} = 2\pi\ell/4G_{\mathrm{N}}$ for $R = \mathbf{12}$ is $\mathcal{S}^{(0)} = 286 \cdot 2\pi\kappa$, and for $R = \mathbf{6} \oplus \mathbf{4} \oplus \mathbf{2}$ it is $\mathcal{S}^{(0)} = 46 \cdot 2\pi\kappa$.

One-loop Euclidean path integral from metric-like formulation

In view of the above higher-spin interpretation of the theory, we can apply our general massless HS formula (C.230) with $G = SU(n) \times SU(n)$ to obtain the one-loop contribution to the Euclidean path integral (for $l = 0$). In combination with (C.260) this takes the form

$$Z^{(1)}_{\mathrm{PI}} = i^{-P}\, \frac{(2\pi\gamma)^{\dim G}}{\mathrm{vol}(G)_{\mathrm{c}}}\,, \qquad \gamma = \sqrt{\frac{8\pi G_{\mathrm{N}}}{2\pi\ell}} = \frac{1}{\sqrt{\kappa\, T_R}}\,. \tag{C.266}$$

Recall that $\mathrm{vol}(G)_{\mathrm{c}}$ is the volume of G with respect to the metric normalized such that $\langle M|M\rangle_{\mathrm{c}} = 1$, where M is one of the standard $\mathrm{so}(4) = \mathrm{su}(2) \oplus \mathrm{su}(2)$ generators, which we can for instance take to be the rotation generator $M = \mathcal{S}_3 \oplus \mathcal{S}_3$. In the context of Chern-Simons theory, it is more natural to consider the volume $\mathrm{vol}(G)_{\mathrm{Tr}_n}$ with respect to the metric defined by the trace appearing in the Chern-Simons action (C.256). Using the definition of T_R in (C.259), we see the trace norm of M is $\langle M|M\rangle_{\mathrm{Tr}_n} = -2\,\mathrm{Tr}_n(\mathcal{S}_3\mathcal{S}_3) = T_R = T_R\langle M|M\rangle_{\mathrm{c}}$, hence $\mathrm{vol}(G)_{\mathrm{Tr}_n} = (\sqrt{T_R})^{\dim G}\,\mathrm{vol}(G)_{\mathrm{c}}$. Note that upon substituting this in (C.266), the T_R-dependent factors cancel out. Finally, using (C.228), we get $P = \sum_{s=2}^{n} P_s = \frac{1}{3}(2s-3)(2s-1)(2s+1) = \frac{2}{3}n^2(n-1)(n+1) - (n^2-1)$. Because $(n-1)\cdot n\cdot(n+1)$ is divisible by 3, the first term is an integer, and moreover a multiple of 8 because either n^2 or $(n+1)(n-1)$ is a multiple of 4. Hence $i^{-P} = i^{(n^2-1)}$, which equals $-i$ for even n and $+1$ for odd n. Thus we get

$$Z_{\mathrm{PI}}^{(1)} = i^{n^2-1}\,\frac{(2\pi\tilde{\gamma})^{\dim G}}{\mathrm{vol}(G)_{\mathrm{Tr}_n}}\,, \qquad \tilde{\gamma} \equiv \frac{1}{\sqrt{\kappa}}\,. \tag{C.267}$$

Euclidean path integral from CS formulation

As in the $SU(2)\times SU(2)$ Einstein gravity case, we can derive an all-loop expression for the Euclidean partition function $Z(R)$ of the $SU(n)\times SU(n)$ higher-spin gravity theory (C.256) expanded around a lifted round sphere solution $\bar{\mathcal{A}} = R(\bar{A})$ such as (C.262), by naive analytic continuation of the exact $SU(n)_{k_+}\times SU(n)_{k_-}$ partition function on S^3 to $k_\pm = l \pm i\kappa$, paralleling (C.246) and the subsequent discussion there. The $SU(n)_k$ generalization of the canonically framed $SU(2)_k$ result (C.247) as spelled out e.g. in [80, 216] is

$$Z_{\mathrm{CS}}(SU(n)_k|\bar{\mathcal{A}})_0 = \frac{1}{\sqrt{n}}\frac{1}{(n+k)^{\frac{n-1}{2}}}\prod_{p=1}^{n-1}\Big(2\sin\frac{\pi p}{n+k}\Big)^{(n-p)}\cdot e^{i(n+k)S_{\mathrm{CS}}[\bar{\mathcal{A}}]}\,. \tag{C.268}$$

The corresponding higher-spin generalization of (C.248) is therefore

$$Z(R)_0 = \left|\frac{1}{\sqrt{n}}\frac{1}{(n+l+i\kappa)^{\frac{n-1}{2}}}\prod_{p=1}^{n-1}\Big(2\sin\frac{\pi p}{n+l+i\kappa}\Big)^{(n-p)}\right|^2\cdot e^{2\pi\kappa T_R}\,. \tag{C.269}$$

Physically this can be interpreted as the all-loop quantum-corrected Euclidean partition function of the dS_3 static patch in the vacuum labeled by R. The analog of the result (C.249) for more general framing I_M is

$$Z_{\rm CS}(SU(n)_k|\bar{\mathcal{A}}) = \exp\left(\tfrac{i}{24}c(k)I_M\right) Z_{\rm CS}(SU(n)_k|\bar{\mathcal{A}})_0\,, \qquad c(k) = (n^2-1)\left(1-\tfrac{n}{n+k}\right), \tag{C.270}$$

hence the generalization of (C.252) for arbitrary framing $I_M = 6\pi r$, $r \in \mathbb{Z}$, is

$$Z(R)_r = e^{ir\phi}\, Z(R)_0\,, \tag{C.271}$$

where $\phi = \left(c(l+i\kappa)+c(l-i\kappa)\right)\frac{\pi}{4} = \left(1-\frac{2(n+l)}{(n+l)^2+\kappa^2}\right)(n^2-1)\frac{\pi}{2}$. In the limit $\kappa \to \infty$,

$$Z(R)_r \to i^{r(n^2-1)}\,\frac{1}{n}\frac{1}{\kappa^{n-1}}\prod_{r=1}^{n-1}\left(\frac{2\pi r}{\kappa}\right)^{2(n-r)} = i^{r(n^2-1)}\left(\frac{2\pi}{\sqrt{\kappa}}\right)^{2(n^2-1)}\left(\frac{1}{\sqrt{n}}\prod_{s=2}^{n}\frac{\Gamma(s)}{(2\pi)^s}\right)^2. \tag{C.272}$$

Recognizing $n^2-1 = \dim SU(n)$ and $\sqrt{n}\prod_{s=2}^{n}(2\pi)^s/\Gamma(s) = \mathrm{vol}(SU(n))_{\mathrm{Tr}_n}$ (C.94), we see this precisely reproduces the one-loop result (C.267). Like in the original $n=2$ case, the phase again matches for odd framing r. (The agreement at one loop can also be seen more directly by a slight variation of the computation leading to (C.235).) This provides a nontrivial check of our higher-spin gravity formula (C.230) and more generally (3.112).

Large-n limit and topological string description

In generic dS_{d+1} higher-spin theories, $\dim G = \infty$. To mimic this case, consider the $n \to \infty$ limit of $SU(n)\times SU(n)$ dS_3 higher-spin theory with $l = 0$. A basic observation is that the loop expansion is only reliable then if $n/\kappa \ll 1$. Using (C.260), this translates to $T_R\, n \ll \frac{\ell}{G_{\rm N}}$ For the exponentially dominant principal vacuum $R = \mathbf{n}$, this becomes $n^4 \ll \ell/G_{\rm N}$ while at the other extreme, for the nearly-trivial $R = \mathbf{2}\oplus\mathbf{1}\oplus\cdots\oplus\mathbf{1}$, this becomes $n \ll \ell/G_{\rm N}$. Either way, for fixed ℓ/G_N, the large-n limit is necessarily strongly coupled, and the one-loop formula (C.266), or equivalently (C.267) or (C.272), becomes unreliable. Indeed, according to this formula, $\log Z^{(1)} \sim$

$\log\left(\frac{n}{\kappa}\right) \cdot n^2$ in this limit, whereas the exact expression (C.269) actually implies $\log Z^{(\text{loops})} \to 0$.

In fact, the partition function *does* have a natural weak coupling expansion in the $n \to \infty$ limit — not as a 3D higher-spin gravity theory, but rather as a topological string theory. $U(n)_k$ Chern-Simons theory on S^3 has a description [217] as an open topological string theory on the deformed conifold T^*S^3 with n topological D-branes wrapped on the S^3, and a large-n 't Hooft dual description [79] as a *closed* string theory on the *resolved* conifold. Both descriptions are reviewed in [80], whose notation we follow here. The string coupling constant is $g_s = 2\pi/(n+k)$ and the Kähler modulus of the resolved conifold is $t = \int_{S^2} J + iB = i g_s n = 2\pi i n/(n+k)$. Under this identification,

$$Z_{\text{CS}}(SU(n)_k)_0 = \sqrt{\tfrac{n+k}{n}}\, Z_{\text{CS}}(U(n)_k)_0 = \sqrt{\tfrac{2\pi i}{t}}\, Z_{\text{top}}(g_s, t) \equiv \tilde{Z}_{\text{top}}(g_s, t)\,. \tag{C.273}$$

Thus we can write the $SU(n)_{l+i\kappa} \times SU(n)_{l-i\kappa}$ higher-spin Euclidean gravity partition function (C.269) expanded around the round S^3 solution $\bar{\mathcal{A}} = R(\bar{A})$ as

$$\boxed{Z(R)_0 = \left|\tilde{Z}_{\text{top}}(g_s, t)\, e^{-\pi T_R \cdot 2\pi i/g_s}\right|^2} \tag{C.274}$$

where T_R was defined in (C.258), maximized for $R = \mathbf{n}$ at $T_{\mathbf{n}} = \frac{1}{6} n(n^2-1)$, and

$$g_s = \frac{2\pi}{n+l+i\kappa}\,, \qquad t = i g_s n = \frac{2\pi i n}{n+l+i\kappa}\,. \tag{C.275}$$

Note that t takes values inside a half-disk of radius $\frac{1}{2}$ centered at $t = i\pi$, with $\operatorname{Re} t > 0$. The higher-spin gravity theory (or the open string theory description on the deformed conifold) is weakly coupled when $\kappa \gg n$, which implies $|t| \ll 1$. In the free field theory limit $\kappa \to \infty$, we get $g_s \sim -2\pi i/\kappa \to 0$ and $t \sim 2\pi n/\kappa \to 0$, which is singular from the closed string point of view. In the 't Hooft limit $n \to \infty$ with t kept finite, the closed string is weakly coupled and sees a smooth geometry. The earlier discussed Vasiliev-like limit $n \to \infty$ with $l = 0$ and $\ell/G_{\mathrm{N}} \sim T_{\mathbf{n}}\kappa \sim n^3\kappa$ fixed, infinitely strongly coupled from the 3D field theory point of view, maps to $g_s \sim 2\pi/n \to 0$

and $t \sim 2\pi i + 2\pi\kappa/n \to 2\pi i$, which is again singular from the closed string point of view, differing from the 3D free field theory singularity by a mere B-field monodromy, reflecting the more general $n \leftrightarrow l + i\kappa$, $t \leftrightarrow 2\pi i - t$ level-rank symmetry.

C.9 Quantum dS entropy: computations and examples

Here we provide the details for section 3.8.

C.9.1 Classical gravitational dS thermodynamics

3D Einstein gravity example

For concreteness we start with pure 3D Einstein gravity as a guiding example, but we will phrase the discussion so generalization will be clear. The Euclidean action in this case is

$$S_E[g] = \frac{1}{8\pi G}\int d^3x\sqrt{g}\Big(\Lambda - \frac{1}{2}R\Big), \tag{C.276}$$

with $\Lambda > 0$. The tree-level contribution to the entropy (3.153) is

$$\mathcal{S}^{(0)} = \log\mathcal{Z}^{(0)}, \qquad \mathcal{Z}^{(0)} = \int_{\text{tree}}\mathcal{D}g\, e^{-S_E[g]}. \tag{C.277}$$

The dominant saddle of (C.277) is a round S^3 metric g_ℓ of radius $\ell = \ell_0$ minimizing $S_E(\ell) \equiv S_E[g_\ell]$:

$$\mathcal{Z}^{(0)} = \int_{\text{tree}} d\ell\, e^{-S_E(\ell)}, \qquad S_E(\ell) = \frac{2\pi^2}{8\pi G}\big(\Lambda\ell^3 - 3\ell\big), \tag{C.278}$$

where $\int_{\text{tree}}$ means evaluation at the saddle point, here at the on-shell radius $\ell = \ell_0$:

$$\partial_\ell S_E(\ell_0) = 0 \qquad \Rightarrow \qquad \Lambda = \frac{1}{\ell_0^2}, \qquad \mathcal{S}^{(0)} = -S_E(\ell_0) = \frac{2\pi\ell_0}{4G}, \tag{C.279}$$

reproducing the familiar area law $\mathcal{S}^{(0)} = A/4G$ for the horizon entropy.

We now recast the above in a way that will allow us to make contact with the formulae of section 3.7.1 and will naturally generalize beyond tree level in a diffeomorphism-invariant way. To this end we define an "off-shell" tree-level partition function at fixed (off-shell) volume V:

$$Z^{(0)}(V) \equiv \int_{\text{tree}} d\sigma \int_{\text{tree}} \mathcal{D}g \, e^{-S_E[g]+\sigma(\int \sqrt{g}-V)} \,. \tag{C.280}$$

Evaluating the integral is equivalent to a constrained extremization problem with Lagrange multiplier σ enforcing the constraint $\int \sqrt{g} = V$. The dominant saddle is the round sphere $g = g_\ell$ of radius $\ell(V)$ fixed by the volume constraint:

$$Z^{(0)}(V) = e^{-S_E(\ell)} \,, \qquad 2\pi^2 \ell^3 = V \,. \tag{C.281}$$

Paralleling (3.135) and (3.136), we define from this an off-shell energy density and entropy,

$$\begin{aligned} \rho^{(0)} &\equiv -\partial_V \log Z^{(0)} = -\tfrac{1}{3}\ell\partial_\ell \log Z^{(0)} \,/\, V = \big(\Lambda - \ell^{-2}\big)/8\pi G \\ S^{(0)} &\equiv \big(1 - V\partial_V\big) \log Z^{(0)} = \big(1 - \tfrac{1}{3}\ell\partial_\ell\big) \log Z^{(0)} = \frac{2\pi\ell}{4G} \,. \end{aligned} \tag{C.282}$$

$\rho^{(0)}$ is the sum of the positive cosmological constant and negative curvature energy densities. $S^{(0)}$ is independent of Λ. It is the Legendre transform of $\log Z^{(0)}$:

$$S^{(0)} = \log Z^{(0)} + V\rho^{(0)} \,, \qquad d\log Z^{(0)} = -\rho^{(0)} dV \,, \qquad dS^{(0)} = V d\rho^{(0)} \,, \tag{C.283}$$

Note that evaluating $\int_{\text{tree}} d\sigma$ in (C.280) sets $\sigma = -\partial_V \log Z^{(0)} = \rho^{(0)}(V)$. On shell,

$$\rho^{(0)}(\ell_0) = 0 \,, \qquad \mathcal{S}^{(0)} = \log Z^{(0)}(\ell_0) = S^{(0)}(\ell_0) = \frac{2\pi\ell_0}{4G} \,. \tag{C.284}$$

Paralleling (3.137), the differential relations in (C.283) can be viewed as the first law of tree-level de Sitter thermodynamics. We can also consider variations of coupling constants such as Λ. Then $d\log Z^{(0)} = -\rho^{(0)} dV - \frac{1}{8\pi G} V d\Lambda$, $dS^{(0)} = V d\rho^{(0)} - \frac{1}{8\pi G} V d\Lambda$. On shell, $d\mathcal{S}^{(0)} = -\frac{V_0}{8\pi G}\, d\Lambda$.

General d and higher-order curvature corrections

The above formulae readily extend to general dimensions and to gravitational actions $S_E[g]$ with general higher-order curvature corrections. Using that $R_{\mu\nu\rho\sigma} = (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})/\ell^2$ for the round[23] S^{d+1}, $Z^{(0)}(V)$ (C.280) can be evaluated explicitly for any action. It takes the form

$$\log Z^{(0)}(V) = -S_E[g_\ell] = \frac{\Omega_{d+1}}{8\pi G}\left(-\Lambda\ell^{d+1} + \tfrac{d(d+1)}{2}\,\ell^{d-1} + \cdots\right), \qquad \Omega_{d+1}\ell^{d+1} = V\,, \tag{C.285}$$

where $+\cdots$ is a sum of R^n higher-order curvature corrections $\propto \ell^{-2n}$ and $\Omega_{d+1} = (C.92)$. The off-shell energy density and entropy are defined as in (C.282)

$$\begin{aligned} \rho^{(0)} &= -\tfrac{1}{d+1}\ell\partial_\ell \log Z^{(0)} \,/\, V = \left(\Lambda - \tfrac{d(d-1)}{2}\,\ell^{-2} + \cdots\right)/8\pi G \\ S^{(0)} &= \left(1 - \tfrac{1}{d+1}\ell\partial_\ell\right)\log Z^{(0)} = \frac{A}{4G}\left(1 + \cdots\right). \end{aligned} \tag{C.286}$$

where $A = \Omega_{d-1}\ell^{d-1}$ and $+\cdots$ are $1/\ell^{2n}$ curvature corrections. The on-shell radius ℓ_0 solves $\rho^{(0)}(\ell_0) = 0$, most conveniently viewed as giving a parametrization $\Lambda(\ell_0)$.

As an example, consider the general action up to order R^2 written as

$$S_E = \frac{1}{8\pi G}\int\!\sqrt{g}\Big(\Lambda - \tfrac{1}{2}R - l_s^2\big(\lambda_{C^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \lambda_{R^2}R^2 + \lambda_{E^2}E^{\mu\nu}E_{\mu\nu}\big)\Big), \tag{C.287}$$

where $E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{d+1}R\,g_{\mu\nu}$, $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, l_s is a length scale and the λ_i are dimensionless. The Weyl tensor vanishes on the round sphere and $R_{\mu\nu} = d\,g_{\mu\nu}/\ell^2$, hence

$$\log Z^{(0)} = \frac{\Omega_{d+1}}{8\pi G}\left(-\Lambda\,\ell^{d+1} + \tfrac{1}{2}d(d+1)\,\ell^{d-1} + \lambda_{R^2}\,d^2(d+1)^2\,l_s^2\,\ell^{d-3}\right), \tag{C.288}$$

[23] By virtue of its $SO(d+2)$ symmetry, the round sphere metric g_ℓ with $\Omega_{d+1}\ell^{d+1} = V$ is a saddle of (C.280). Spheres of dimension ≥ 5 admit a plentitude of Einstein metrics that are *not* round [218–221], but as explained e.g. in [222], by Bishop's theorem [223], these saddles are subdominant in Einstein gravity. In the large-size limit, higher-order curvature corrections are small, hence the round sphere dominates in this regime.

For example for $d = 2$,

$$\log Z^{(0)} = \frac{\pi}{4G}\Big(-\Lambda\,\ell^3 + 3\,\ell + \frac{36\, l_s^2 \lambda_{R^2}}{\ell}\Big)\,, \tag{C.289}$$

hence, using (C.286) and $\rho^{(0)}(\ell_0) = 0$,

$$\mathcal{S}^{(0)} = S^{(0)}(\ell_0) = \frac{2\pi\ell_0}{4G}\Big(1 + \frac{24\, l_s^2 \lambda_{R^2}}{\ell_0^2}\Big)\,, \qquad \Lambda = \frac{1}{\ell_0^2}\Big(1 - \frac{12\, l_s^2 \lambda_{R^2}}{\ell_0^2}\Big)\,. \tag{C.290}$$

Effective field theory expansion and field redefinitions

Curvature corrections such as those considered above naturally appear as terms in the derivative expansion of low-energy effective field theories of quantum gravity, with l_s the characteristic length scale of UV-completing physics and higher-order curvature corrections terms suppressed by higher powers of $l_s^2/\ell^2 \ll 1$. The action (C.287) is then viewed as a truncation at order l_s^2, and (C.290) can be solved perturbatively to obtain ℓ_0 and $\mathcal{S}^{(0)}$ as a function of Λ.

Suppose someone came up with some fundamental theory of de Sitter quantum gravity, producing both a precise microscopic computation of the entropy and a precise low-energy effective action, with the large-ℓ_0/l_s expansion reproduced as some large-N expansion. At least superficially, the higher-order curvature-corrected entropy obtained above looks like a Wald entropy [158]. In the spirit of for instance the nontrivial matching of R^2 corrections to the macroscopic BPS black hole entropy computed in [224] and the microscopic entropy computed from M-theory in [225], it might seem then that matching microscopic $1/N$-corrections and macroscopic l_s^2/ℓ_0^2-corrections to the entropy such as those in (C.290) could offer a nontrivial way of testing such a hypothetical theory.

However, this is not the case. Unlike the Wald entropy, there are no charges Q (such as energy, angular momentum or gauge charges) available here to give these corrections physical meaning as corrections in the large-Q expansion of a function $S(Q)$. Indeed, the detailed structure of the l_s/ℓ_0 expansion of $\mathcal{S}^{(0)} = S^{(0)}(\ell_0)$ has no intrinsic physical meaning at all, because all of it can be wiped out by a local metric field redefinition, order by order in l_s/ℓ_0, bringing the entropy to pure Einstein area law form, and leaving only the value of $\mathcal{S}^{(0)}$ itself as a physically meaningful, field-redefinition invariant, dimensionless quantity.

This is essentially a trivial consequence of the fact that in perturbation theory about the round sphere, the round sphere itself is the unique solution to the equations of motion. Let us however recall in more detail how this works at the level of local field redefinitions, and show how this is expressed at the level of $\log Z^{(0)}(\ell)$, as this will be useful later in interpreting quantum corrections. For concreteness, consider again (C.287) viewed as a gravitational effective field theory action expanded to order $l_s^2 R^2$. Under a local metric field redefinition

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} + O(l_s^4), \qquad \delta g_{\mu\nu} \equiv l_s^2 \big(u_0\,\Lambda\, g_{\mu\nu} + u_1\, R\, g_{\mu\nu} + u_2\, R_{\mu\nu}\big)\,, \tag{C.291}$$

where the u_i are dimensionless constants, the action transforms as

$$S_E \to S_E + \frac{1}{16\pi G}\int\!\sqrt{g}\,\big(R^{\mu\nu} - \tfrac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu}\big)\,\delta g_{\mu\nu} \;+\; O(l_s^4)\,, \tag{C.292}$$

shifting $\lambda_{R^2}, \lambda_{E^2}$ and rescaling G, Λ in (C.287). A suitable choice of u_i brings S_E to the form

$$S_E = \frac{1}{8\pi G}\int\!\sqrt{g}\Big(\Lambda' - \tfrac{1}{2}R - l_s^2\lambda_{C^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + O(l_s^4)\Big), \qquad \Lambda' = \Lambda\big(1 - \tfrac{4(d+1)^2}{(d-1)^2}\,\lambda_{R^2} l_s^2\Lambda\big). \tag{C.293}$$

Equivalently, this is obtained by using the $O(l_s^0)$ equations of motion $R_{\mu\nu} = \frac{2}{d-1}\Lambda g_{\mu\nu}$ in the $O(l_s^2)$ part of the action. Since $\lambda'_{R^2} = 0$, the entropy computed from this equivalent action takes a pure Einstein area law form $\mathcal{S}^{(0)} = \Omega_{d+1}{\ell'_0}^{d-1}/4G$, with $\ell'_0 = \sqrt{d(d-1)/2\Lambda'}$. The on-shell value $\mathcal{S}^{(0)}$ itself remains unchanged of course under this change of variables.

In the above we picked a field redefinition keeping $G' = G$. Further redefining $g_{\mu\nu} \to \alpha\, g_{\mu\nu}$ leads to another equivalent set of couplings $G'', \Lambda'', \ldots$ rescaled with powers of α according to their mass dimension. We could then pick α such that instead $\Lambda'' = \Lambda$, or such that $\ell''_0 = \ell_0$, now with $G'' \neq G$. If we keep $\ell_0'' = \ell_0$, we get

$$\mathcal{S}^{(0)} = \frac{\Omega_{d+1}\ell_0^{d-1}}{4G''}, \qquad \Lambda'' = \frac{d(d-1)}{2\ell_0^2}, \tag{C.294}$$

where for example in $d = 2$ starting from (C.290), $G'' = G\big(1 - 24\,\lambda_{R^2} l_s^2/\ell_0^2 + O(l_s^4)\big)$.

At the level of $\log Z^{(0)}(\ell)$ in (C.288) the metric redefinition (C.291) amounts to a radius redefininition $\ell \to \ell\, f(\ell)$ with $f(\ell) = 1 + l_s^2\big(v_{10}\Lambda + v_{11}\ell^{-2}\big) + O(l_s^4)$. For suitable v_i this brings $\log Z^{(0)}$ and therefore $\mathcal{S}^{(0)}$ to pure Einstein form. E.g. for the $d = 2$ example (C.289),

$$\ell = \big(1 - 12\,\lambda_{R^2} l_s^2\big(\Lambda + \ell'^{-2}\big)\big)\ell' \qquad \Rightarrow \qquad \log Z^{(0)} = \frac{\pi}{4G}\big(-\Lambda'\,\ell'^{3} + 3\,\ell' + O(l_s^4)\big)\,. \tag{C.295}$$

The above considerations generalize to all orders in the l_s expansion. R^n corrections to $\log Z^{(0)}$ are $\propto (l_s/\ell)^{2n}$ and can be removed order by order by a local metric/radius redefinition

$$\ell \to \alpha\, f(\ell)\,\ell\,, \qquad f(\ell) = 1 + l_s^2\big(v_{10}\Lambda + v_{11}\ell^{-2}\big) + l_s^4\big(v_{20}\Lambda^2 + v_{21}\Lambda\ell^{-2} + v_{22}\ell^{-4}\big) + \cdots \tag{C.296}$$

bringing $\log Z^{(0)}$ and thus $\mathcal{S}^{(0)}$ to Einstein form to any order in the l_s expansion.

In $d = 2$, the Weyl tensor vanishes identically. The remaining higher-order curvature invariants involve the Ricci tensor only, so can be removed by field redefinitions, reducing the action to Einstein form in general. Thus in $d = 2$, $\mathcal{S}^{(0)}$ is the only tree-level invariant in the theory, i.e. the only physical coupling constant. In the Chern-Simons formulation of C.8.2, $\mathcal{S}^{(0)} = 2\pi\kappa$. In $d \geq 3$, there are infinitely many independent coupling constants, such as the Weyl-squared λ_{C^2} in (C.287), which are not picked up by $\mathcal{S}^{(0)}$, but are analogously probed by invariants $\mathcal{S}_M^{(0)} = \log \mathcal{Z}^{(0)}[g_M] = -S_E[g_M]$ for saddle geometries g_M different from the round sphere. We comment on those and their role in the bigger picture in section C.9.5.

The point of considering quantum corrections to the entropy $\mathcal{S}$ is that these include nonlocal contributions, not removable by local redefinitions, and thus, unlike the tree-level entropy $\mathcal{S}^{(0)}$, offering actual data quantitatively constraining candidate microscopic models.

C.9.2 Quantum gravitational thermodynamics

The *quantum* off-shell partition function $Z(V)$ generalizing the tree-level $Z^{(0)}(V)$ (C.280) is defined by replacing $\int_{\text{tree}} \mathcal{D}g \to \int \mathcal{D}g$ in that expression:[24]

$$Z(V) \equiv \int_{\text{tree}} d\sigma \int \mathcal{D}g\, e^{-S_E[g]+\sigma(\int \sqrt{g}-V)}\,. \tag{C.297}$$

The quantum off-shell energy density and entropy generalizing (C.282) are

$$\rho(V) \equiv -\partial_V \log Z\,, \qquad S(V) \equiv \big(1 - V\partial_V\big)\log Z\,. \tag{C.298}$$

S is the Legendre transform of $\log Z$:

$$S = \log Z + V\rho\,, \qquad d\log Z = -\rho\, dV\,, \qquad dS = V d\rho\,. \tag{C.299}$$

Writing $e^{-\Gamma(V)} \equiv Z(V)$, the above definitions imply that as a function of ρ,

$$S(\rho) = \log \int_{\text{tree}} dV\, e^{-\Gamma(V)+\rho V} = \log \int \mathcal{D}g\, e^{-S_E[g]+\rho\int\sqrt{g}} \tag{C.300}$$

hence $S(\rho)$ is the generating function for moments of the volume. In particular

$$V = \left\langle \int \sqrt{g} \right\rangle_\rho = \partial_\rho S(\rho) \tag{C.301}$$

is the expectation value of the volume in the presence of a source ρ shifting the cosmological constant $\frac{\Lambda}{8\pi G} \to \frac{\Lambda}{8\pi G} - \rho$. $\Gamma(V)$ can be viewed as a quantum effective action for the volume, in the spirit of the QFT 1PI effective action [227–229] but taking only the volume off-shell. At tree level

[24] $Z(V)$ is reminiscent of but different from the fixed-volume partition function considered in the 2D quantum gravity literature, e.g. (2.20) in [226]. The latter would be defined as above but with $\frac{1}{2\pi i}\int_{i\mathbb{R}} d\sigma$ instead of $\int_{\text{tree}} d\sigma$, constraining the volume to V, whereas $Z(V)$ constrains the *expectation value* of the volume to V.

it reduces to $S_E[g_\ell]$ appearing in (C.285). At the quantum on-shell value $V = \bar{V} = \big\langle \int\sqrt{g}\big\rangle_0$,

$$\rho(\bar{V}) = 0\,, \qquad \mathcal{S} = \log Z(\bar{V}) = S(\bar{V})\,. \tag{C.302}$$

It will again be convenient to work with a linear scale variable ℓ instead of V, *defined* by

$$\Omega_{d+1}\ell^{d+1} \equiv V\,, \tag{C.303}$$

Since the mean volume $V = \big\langle \int\sqrt{g}\big\rangle_\rho$ is diffeomorphism invariant, (C.303) gives a manifestly diffeomorphism-invariant definition of the "mean radius" ℓ of the fluctuating geometry. Given $Z(\ell) \equiv Z(V(\ell))$, the off-shell energy density and entropy are then computed as

$$\rho(\ell) = -\frac{\frac{1}{d+1}\ell\partial_\ell \log Z}{V}\,, \qquad S(\ell) = \big(1 - \tfrac{1}{d+1}\ell\partial_\ell\big)\log Z\,. \tag{C.304}$$

The quantum on-shell value of ℓ is denoted by $\bar{\ell}$ and satisfies $\rho(\bar{\ell}) = 0$.

The magnitude of quantum fluctuations of the volume about its mean value is given by $\delta V^2 \equiv \big\langle \big(\int\sqrt{g} - V\big)^2\big\rangle_\rho = S''(\rho) = 1/\Gamma''(V) = 1/\rho'(V) = V/S'(V)$. At large V, $\delta V/V \propto 1/\sqrt{S}$.

C.9.3 One-loop corrected de Sitter entropy

The path integral (C.297) for $\log Z$ can be computed perturbatively about its round sphere saddle in a semiclassical expansion in powers of G. To leading order it reduces to $\log Z^{(0)}$ defined in (C.280). For 3D Einstein gravity,

$$\log Z(V) = \log Z^{(0)}(V) + O(G^0) = \frac{\Omega_3}{8\pi G}\big(-\Lambda\,\ell^3 + 3\,\ell\big) \,+\, O(G^0)\,, \tag{C.305}$$

To compute the one-loop $O(G^0)$ correction, recall that evaluation of $\int_{\text{tree}} d\sigma$ in (C.297) is equivalent to extremization with respect to σ, which sets $\sigma = -\partial_V \log Z(V) = \rho(V)$ and

$$\log Z(V) = \log \int \mathcal{D}g \, e^{-S_E[g]+\rho(V)(\int \sqrt{g}-V)} . \tag{C.306}$$

To one-loop order, we may replace ρ by its tree-level approximation $\rho^{(0)} = -\partial_V \log Z^{(0)}$. By construction this ensures the round sphere metric $g = g_\ell$ of radius $\ell(V)$ given by (C.303) is a saddle. Expanding the action to quadratic order in fluctuations about this saddle then gives a massless spin-2 Gaussian path integral of the type solved in general by (3.112), or more explicitly in (3.148)-(3.149). For 3D Einstein gravity, using (3.149),

$$\log Z = -\Big(\frac{\Lambda}{8\pi G} + c_0'\Big)\Omega_3 \ell^3 + \Big(\frac{1}{8\pi G} + c_2' - \frac{3}{4\pi\epsilon}\Big)3\Omega_3 \ell - 3\log\frac{2\pi\ell}{4G} + 5\log(2\pi) + O(G) \tag{C.307}$$

Here c_0' and c_2' arise from $O(G^0)$ local counterterms

$$S_{E,\text{ct}} = \int \sqrt{g}\Big(c_0' - \frac{c_2'}{2}R\Big), \tag{C.308}$$

split off from the bare action (C.276) to keep the tree-level couplngs Λ and G equal to their "physical" (renormalized) values to this order. We define these physical values as the coefficients of the local terms $\propto \ell^3, \ell$ in the $V \to \infty$ asymptotic expansion of the *quantum* $\log Z(V)$. That is to say, we fix c_0' and c_2' by imposing the renormalization condition

$$\log Z(V) = \frac{\Omega_3}{8\pi G}\big(-\Lambda\,\ell^3 + 3\,\ell\,\big) + \cdots \qquad (V \to \infty) . \tag{C.309}$$

This renormalization prescription is diffeomorphism invariant, since $Z(V)$, V and ℓ were all defined in a manifestly diffeomorphism-invariant way. In (C.307) it fixes $c_0' = 0$, $c_2' = \frac{3}{4\pi\epsilon}$, hence

$\log Z(\ell) = \log Z^{(0)} + \log Z^{(1)} + O(G)$, where

$$\log Z^{(1)} = -3 \log \frac{2\pi\ell}{4G} + 5 \log(2\pi) \, . \tag{C.310}$$

We can express the renormalization condition (C.309) equivalently as

$$\boxed{\log Z^{(1)} = \log Z_{\rm PI}^{(1)} + \log Z_{\rm ct} \, , \qquad \lim_{\ell \to \infty} \partial_\ell \log Z^{(1)} = 0} \tag{C.311}$$

where $\log Z_{\rm ct} = -S_{E,{\rm ct}}[g_\ell]$ with g_ℓ the round sphere metric of volume V. On S^3 we have $\log Z_{\rm ct} = c_0 \ell^3 + c_2 \ell$, and the $\ell \to \infty$ condition fixes c_0 and c_2. Recalling (3.135), we can physically interpret this as requiring the renormalized one-loop Euclidean energy $U^{(1)}$ of the static patch vanishes in the $\ell \to \infty$ limit.

For general d, the UV-divergent terms in $\log Z_{\rm PI}^{(1)}$ come with non-negative powers $\propto \ell^{d+1-2n}$, canceled by counterterms consisting of n-th order curvature invariants. For example on S^5, $\log Z_{\rm ct} = c_0 \ell^5 + c_2 \ell^3 + c_4 \ell$. In odd $d+1$, the renormalization prescription (C.311) then fixes the c_{2n}. In even $d+1$, $\log Z_{\rm ct}$ has a constant term c_{d+1}, which is not fixed by (C.311). As we will make explicit in examples later, it can be fixed by $\lim_{\ell \to \infty} Z^{(1)} = 0$ for massive field contributions, and for massless field contributions by minimal subtraction at scale L, $c_{d+1} = -\alpha_{d+1} \log(M_\epsilon L)$, $M_\epsilon = 2e^{-\gamma}/\epsilon$ (C.67), with $L \partial_L \log Z = 0$, i.e. $L \partial_L \log Z^{(0)} = \alpha_{d+1}$.

The renormalized off-shell ρ and S are obtained from $\log Z$ as in (C.304). For 3D Einstein,

$$\rho^{(1)} = \frac{1}{2\pi^2 \ell^3} \, , \qquad S^{(1)} = -3 \log \frac{2\pi\ell}{4G} + 5 \log(2\pi) + 1 \, . \tag{C.312}$$

The on-shell quantum dS entropy $\mathcal{S} = \log Z(\bar\ell) = S(\bar\ell)$ (C.302) is

$$\boxed{\mathcal{S} = S(\bar\ell) = S^{(0)}(\bar\ell) + S^{(1)}(\bar\ell) \; + \; O(G)} \tag{C.313}$$

where $\bar{\ell}$ is the quantum mean radius satisfying $\rho(\bar{\ell}) \propto \partial_\ell \log Z(\bar{\ell}) = 0$. For 3D Einstein,

$$\mathcal{S} = \frac{2\pi\bar{\ell}}{4G} - 3\log\frac{2\pi\bar{\ell}}{4G} + 5\log(2\pi) + 1 \;+\; O(G)\,, \qquad \Lambda = \frac{1}{\bar{\ell}^2} - \frac{4G}{\pi\bar{\ell}^3} + O(G^2)\,. \tag{C.314}$$

Alternatively, $\mathcal{S}$ can be expressed in terms of the tree-level ℓ_0, $\rho^{(0)}(\ell_0) \propto \partial_\ell \log Z^{(0)}(\ell_0) = 0$, using $\mathcal{S} = \log Z^{(0)}(\bar{\ell}) + \log Z^{(1)}(\bar{\ell}) + O(G)$, $\bar{\ell} = \ell_0 + O(G)$ and Taylor expanding in G:

$$\boxed{\mathcal{S} = \log Z(\bar{\ell}) = S^{(0)}(\ell_0) + \log Z^{(1)}(\ell_0) \;+\; O(G)} \tag{C.315}$$

This form would be obtained from (3.153) by a more standard computation. For 3D Einstein,

$$\mathcal{S} = \frac{2\pi\ell_0}{4G} - 3\log\frac{2\pi\ell_0}{4G} + 5\log(2\pi) \;+\; O(G)\,, \qquad \Lambda = \frac{1}{\ell_0^2}\,. \tag{C.316}$$

The equivalence of (C.313) and (C.315) can be checked directly here noting $\bar{\ell} = \ell_0 - \frac{2}{\pi}G + O(G^2)$, so $\frac{2\pi\bar{\ell}}{4G} = \frac{2\pi\ell_0}{4G} - 1 + O(G)$. The -1 cancels the $+1$ in (C.314), reproducing (C.316).

More generally and more physically, the relation between these two expressions can be understood as follows. At tree level, the entropy equals the geometric horizon entropy $S^{(0)}(\ell_0)$, with radius ℓ_0 such that the geometric energy density $\rho^{(0)}$ vanishes. At one loop, we get additional contributions from quantum field fluctuations. The UV contributions are absorbed into the gravitational coupling constants. The remaining IR contributions shift the entropy by $S^{(1)}$ and the energy density by $\rho^{(1)}$. The added energy backreacts on the fluctuating geometry: its mean radius changes from ℓ_0 to $\bar{\ell}$ such that the geometric energy density changes by $\delta\rho^{(0)} = -\rho^{(1)}$, ensuring the total energy density vanishes. This in turn changes the geometric horizon entropy by an amount dictated by the first law (C.283),

$$\delta S^{(0)} = V_0\,\delta\rho^{(0)} = -V_0\,\rho^{(1)}\,. \tag{C.317}$$

We end up with a total entropy $\mathcal{S} = S^{(0)}(\bar{\ell}) + S^{(1)} = S^{(0)}(\ell_0) - V_0\,\rho^{(1)} + S^{(1)} = S^{(0)}(\ell_0) + \log Z^{(1)}$, up

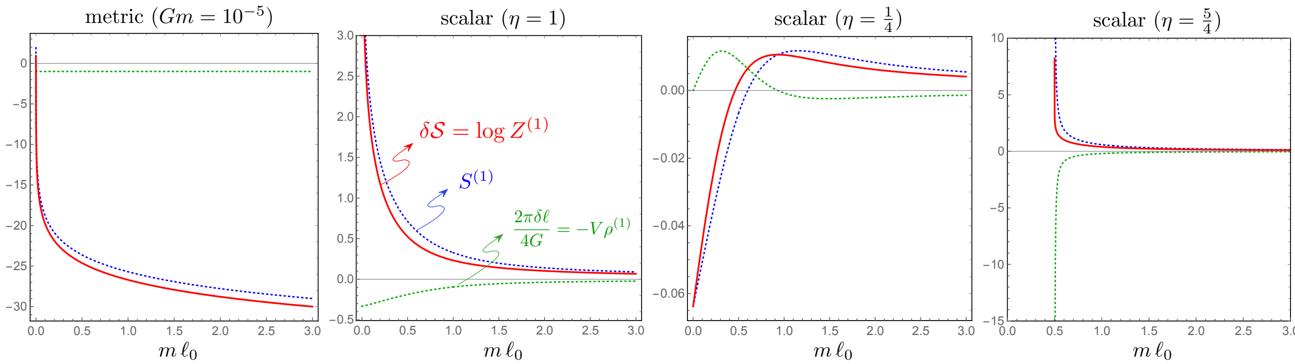

Figure C.12: One-loop contributions to the dS$_3$ entropy from metric and scalars with $\eta = 1, \frac{1}{4}, \frac{5}{4}$, i.e. $\xi = 0, \frac{1}{8}, -\frac{1}{24}$. Blue dotted line = renormalized entropy $S^{(1)}$. Green dotted line = horizon entropy change $\delta S^{(0)} = 2\pi\delta\ell/4G = -V\rho^{(1)}$ due to quantum backreaction $\ell_0 \to \bar{\ell} = \ell_0 + \delta\ell$, as dictated by first law. Solid red line = total $\delta\mathcal{S} = S^{(1)} - V\rho^{(1)} = \log Z^{(1)}$. The metric contribution is negative within the semiclassical regime of validity $\ell \gg G$. The renormalized scalar entropy and energy density are positive for $m\ell \gg 1$, and for all $m\ell$ if $\eta = 1$. If $\eta > 1$ and $\ell_0 \to \ell_* \equiv \frac{\sqrt{\eta-1}}{m}$, the correction $\delta\ell \sim -\frac{G}{3\pi}\frac{\ell_*}{\ell_0-\ell_*} \to -\infty$, meaning the one-loop approximation breaks down. The scalar becomes tachyonic beyond this point. If a ϕ^4 term is included in the action, two new dominant saddles emerge with $\phi \neq 0$.

to $O(G)$ corrections, relating (C.313) to (C.315). (See also fig. C.12.)

More succinctly, obtaining (C.315) from (C.313) is akin to obtaining the canonical description of a thermodynamic system from the microcanonical description of system + reservoir. The analog of the canonical partition function is $Z^{(1)} = e^{S^{(1)} - V_0\,\rho^{(1)}}$, with $-V_0\,\rho^{(1)}$ capturing the reservoir (horizon) entropy change due to energy transfer to the system.

C.9.4 Examples

3D scalar

An example with matter is 3D Einstein gravity + scalar ϕ as in (3.140). Putting $\xi \equiv \frac{1-\eta}{6}$,

$$S_E[g,\phi] = \frac{1}{8\pi G}\int\sqrt{g}\,\left(\Lambda - \tfrac{1}{2}R\right) + \frac{1}{2}\int\sqrt{g}\,\phi\left(-\nabla^2 + m^2 + \tfrac{1-\eta}{6}R\right)\phi\,, \qquad \text{(C.318)}$$

The metric contribution to $\log Z^{(1)}$ remains $\log Z^{(1)}_{\rm metric} = -3 \log \frac{2\pi\ell}{4G} + 5\log(2\pi)$ as in (C.310). The scalar $Z^{(1)}_{\rm PI}$ was given in (3.143). Its finite part is

$$\log Z^{(1)}_{\rm PI,fin,scalar} = \frac{\pi\nu^3}{6} - \sum_{k=0}^{2} \frac{\nu^k}{k!} \frac{\mathrm{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}} , \qquad \nu \equiv \sqrt{m^2\ell^2 - \eta} . \tag{C.319}$$

The polynomial $\log Z_{\rm ct}(\ell) = c_0\ell^3 + c_2\ell$ corresponding to the counterterm action (C.308) is fixed by the renormalization condition (C.311), resulting in

$$\log Z^{(1)}_{\rm scalar} = \log Z^{(1)}_{\rm PI,fin,scalar} - \frac{\pi}{6} m^3\ell^3 + \frac{\pi\eta}{4} m\ell . \tag{C.320}$$

The finite polynomial cancels the local terms $\propto \ell^3, \ell$ in the large-ℓ asymptotic expansion of the finite part: $\log Z^{(1)}_{\rm scalar} = \frac{\pi\eta^2}{16}(m\ell)^{-1} + \frac{\pi\eta^3}{96}(m\ell)^{-3} + \cdots$ when $m\ell \to \infty$. The $(m\ell)^{-2n-1}$ terms have the ℓ-dependence of R^n terms in the action and can effectively be thought of as finite shifts of higher-order curvature couplings in the $m\ell \gg 1$ regime. In the opposite regime $m\ell \ll 1$, IR bulk modes of the scalar becomes thermally activated and $\log Z^{(1)}_{\rm scalar}$ ceases to have a local expansion. In particular in the minimally-coupled case $\eta = 1$,

$$\log Z^{(1)}_{\rm scalar} \simeq -\log(m\ell) \qquad (m\ell \to 0) . \tag{C.321}$$

The total energy density is $\rho = -\frac{1}{3}\ell\partial_\ell \log Z/V = \frac{1}{8\pi G}\left(\Lambda - \ell^{-2}\right) + 1 + \rho^{(1)}_{\rm scalar}$ where

$$V\rho^{(1)}_{\rm scalar} = -\frac{\pi}{6}(m\ell)^2 \nu \coth(\pi\nu) + \frac{\pi}{6}(m\ell)^3 - \frac{\pi\eta}{12} m\ell . \tag{C.322}$$

The on-shell quantum dS entropy is given to this order by (C.315) or by (C.313) as

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} = S^{(0)}(\ell_0) + \log Z^{(1)} = S^{(0)}(\ell_0) - V\rho^{(1)} + S^{(1)} = S^{(0)}(\bar\ell) + S^{(1)} , \tag{C.323}$$

where $\ell_0^{-2} = \Lambda = \bar\ell^{-2} - 8\pi G\, \rho^{(1)}(\bar\ell)$ and $S^{(1)} = S^{(1)}_{\rm PI,fin} + \frac{1}{6}\pi\eta\, m\ell$, with the scalar contribution to $S^{(1)}_{\rm PI,fin}$ given by the finite part of (3.145). Some examples are shown in fig. C.12.

For a massless scalar, $m = 0$, the renormalized scalar one-loop correction to $\mathcal{S}$ is a constant independent of ℓ_0 given by (C.319) evaluated at $\nu = \sqrt{-\eta}$, and $\rho^{(1)}_{\text{scalar}} = 0$. For example for a massless conformally coupled scalar, $\eta = \frac{1}{4}$, $Z^{(1)}_{\text{scalar}} = \frac{3\,\zeta(3)}{16\pi^2} - \frac{\log(2)}{8}$.

3D massive spin s

The renormalized one-loop correction $\mathcal{S}^{(1)}_s = \log Z^{(1)}_s$ to the dS_3 entropy from a massive spin-s field is obtained similarly from (3.90):

$$\mathcal{S}^{(1)}_s = \log Z^{(1)}_{s,\text{bulk}} - \log Z^{(1)}_{s,\text{edge}}\,, \tag{C.324}$$

where $\log Z^{(1)}_{s,\text{bulk}}$ equals twice the contribution of an $\eta = (s-1)^2$ scalar as given in (C.320), while the edge contribution is, putting $\nu \equiv \sqrt{m^2\ell^2 - (s-1)^2}$,

$$\log Z^{(1)}_{s,\text{edge}} = s^2\big(\pi(m\ell - \nu) - \log\!\big(1 - e^{-2\pi\nu}\big)\big)\,. \tag{C.325}$$

The edge contribution to (C.324) is manifestly negative. It dominates the bulk part, and increasingly so as s grows. Examples are shown in fig. C.13.

2D scalar

As mentioned below (C.311), the counterterm polynomial $\log Z^{(0)}_{\text{ct}}$ has a constant term in even spacetime dimensions $d+1$, which is not fixed yet by the renormalization prescription given there. Let us consider the simplest example: a $d = 1$ scalar with action (3.140). Denoting

$$\mathtt{f}(\nu) \equiv \sum_{\pm} \zeta'(-1, \tfrac{1}{2} \pm i\nu) \mp i\nu\zeta'(0, \tfrac{1}{2} \pm i\nu)\,,\,, \tag{C.326}$$

and $M_\epsilon = 2e^{-\gamma}/\epsilon$ as in (C.67), we get from (C.59) with $\nu \equiv \sqrt{m^2\ell^2 - \eta}$, $\eta \equiv \frac{1}{4} - 2\xi$,

$$\log Z^{(1)}_{\text{PI}} = \big(2\epsilon^{-2} - m^2\log(M_\epsilon\ell) + m^2\big)\,\ell^2 + \big(\eta + \tfrac{1}{12}\big)\log\!\big(M_\epsilon\ell\big) - \eta + \mathtt{f}(\nu)\,, \tag{C.327}$$

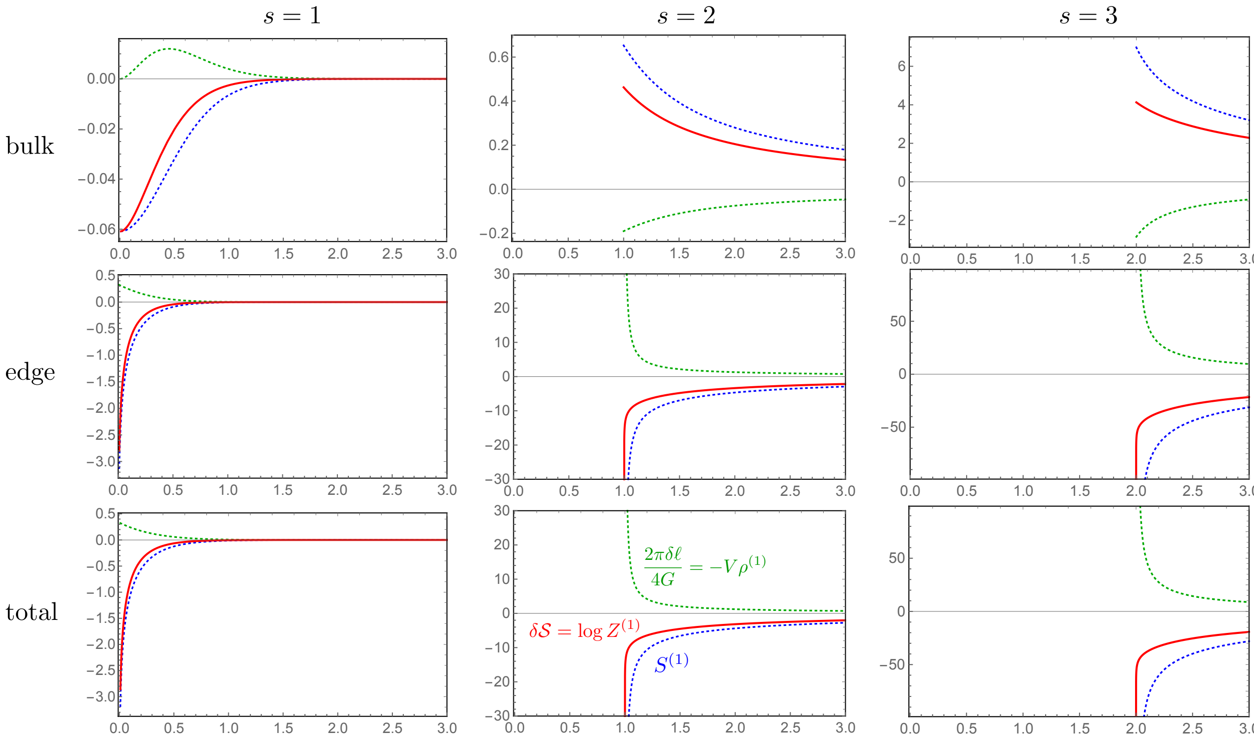

Figure C.13: Contributions to the dS_3 entropy from massive spin $s = 1, 2, 3$ fields, as a function of $m\ell_0$, with coloring as in fig. C.12. Singularities = Higuchi bound, as discussed under (3.145).

In the limit $m\ell \to \infty$, using the asymptotic expansion of the Hurwitz zeta function[25],

$$\log Z_{\rm PI}^{(1)} = \left(2\epsilon^{-2} - m^2 \log(M_\epsilon/m) - \tfrac{1}{2}m^2\right)\ell^2 + (\eta + \tfrac{1}{12})\log(M_\epsilon/m) + O((m\ell)^{-2})\,. \qquad \text{(C.328)}$$

Notice the $\log\ell$ dependence apparent in (C.327) has canceled out. The counterterm action to this order is again of the form (C.308), corresponding to $\log Z_{\rm ct} = 4\pi(-c_0'\ell^2 + c_2')$. The renormalization condition (C.311) fixes c_0' but leaves c_2' undetermined. Its natural extension here is to pick $c_2 = 4\pi c_2'$ to cancel off the constant term as well, that is

$$c_2 = -(\eta + \tfrac{1}{12})\log(M_\epsilon/m) \qquad \Rightarrow \qquad \lim_{\ell\to\infty} \log Z^{(1)} = 0\,, \qquad \text{(C.329)}$$

[25]E.g. appendix A of [230]

ensuring the tree-level G equals the renormalized Newton constant to this order, as in (C.309). The renormalized scalar one-loop contribution to the off-shell partition function is then

$$\log Z^{(1)} = \big(\tfrac{3}{2} - \log(m\ell)\big)(m\ell)^2 + (\eta + \tfrac{1}{12})\log(m\ell) - \eta + \mathtt{f}(\nu)\,. \tag{C.330}$$

In the large-$m\ell$ limit, $\log Z^{(1)} = \frac{240\,\eta^2+40\,\eta+7}{960}(m\ell)^{-2} + \cdots > 0$, while in the small-$m\ell$ limit

$$\log Z^{(1)} \simeq \big(\eta + \tfrac{1}{12}\big)\log(m\ell) \quad (\eta < \tfrac{1}{4}), \qquad \log Z^{(1)} \simeq \big(\tfrac{1}{4} + \tfrac{1}{12} - 1\big)\log(m\ell) \quad (\eta = \tfrac{1}{4}). \tag{C.331}$$

The extra $-\log(m\ell)$ in the minimally-coupled case $\eta = \frac{1}{4}$ is the same as in (C.321) and has the same thermal interpretation. The energy density is $\rho^{(1)} = -\frac{1}{2}\ell\partial_\ell \log Z^{(1)}/V$ with $V = 4\pi\ell^2$:

$$V\rho^{(1)} = -\tfrac{1}{2}(\eta + \tfrac{1}{12}) + \tfrac{1}{2}(m\ell)^2\big(2\log(m\ell) - \textstyle\sum_\pm \psi^{(0)}\big(\tfrac{1}{2} \pm i\nu\big)\big) \tag{C.332}$$

In the massless case $m = 0$, $\nu = \sqrt{-\eta}$ is ℓ-independent, and we cannot use the asymptotic expansion (C.328), nor the renormalization prescription (C.329). Instead we fix c_2 by minimal subtraction, picking a reference length scale L and putting (with $M_\epsilon = \frac{2e^{-\gamma}}{\epsilon}$ (C.67) as before)

$$c_2(L) \equiv -(\eta + \tfrac{1}{12})\log(M_\epsilon L)\,, \tag{C.333}$$

The renormalized G then satisfies $\partial_L(\frac{4\pi}{8\pi G} + c_2) = 0$, i.e. $L\partial_L \frac{1}{2G} = \eta + \frac{1}{12}$, and

$$\log Z^{(1)} = (\eta + \tfrac{1}{12})\log(\ell/L) - \eta + \mathtt{f}\big(\sqrt{-\eta}\big)\,, \qquad V\rho^{(1)} = -\tfrac{1}{2}(\eta + \tfrac{1}{12})\,. \tag{C.334}$$

The total $\log Z = \frac{1}{2G}(-\Lambda\ell^2 + 1) + \log Z^{(1)}$ is of course independent of the choice of L.

4D massive spin s

4D massive spin-s fields can be treated similarly, starting from (C.65). In particular the edge contribution $\log Z^{(1)}_{\rm edge}$ equals *minus* the $\log Z^{(1)}$ of $D^5_{s-1} = \frac{1}{6}s(s+1)(2s+1)$ scalars on S^2, computed

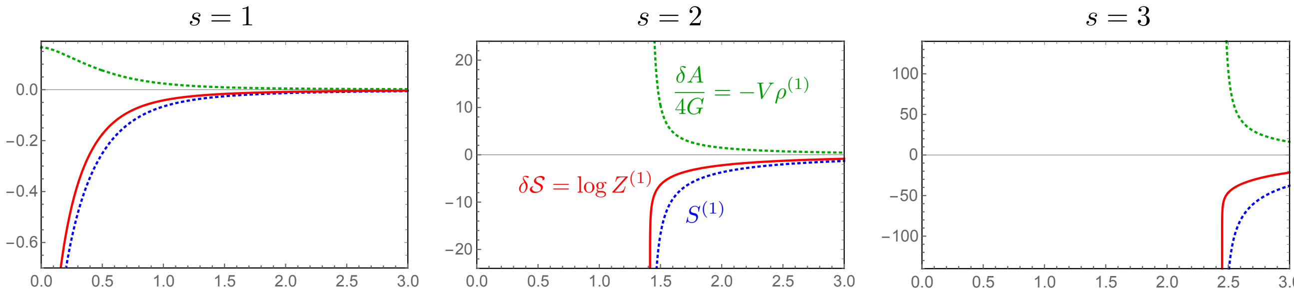

Figure C.14: Edge contributions to the dS_4 entropy from massive spin $s = 1, 2, 3$ fields, as a function of $m\ell_0$, with coloring as in fig. C.12. The Higuchi/unitarity bound in this case is $(m\ell_0)^2 - (s - \frac{1}{2})^2 > -\frac{1}{4}$.

earlier in (C.330), with the same $\nu = \sqrt{m^2\ell^2 - \eta_s}$ as the bulk spin-s field, which according to (3.79) means $\eta_s = (s - \frac{1}{2})^2$. The corresponding contribution to the renormalized energy density is $\rho^{(1)}_{\rm edge} = -\frac{1}{4}\ell\partial_\ell \log Z^{(1)}_{\rm edge}/V$ with $V = \Omega_4\ell^4$, so $V\rho^{(1)}_{\rm edge}$ equals $-\frac{1}{2}D^5_{s-1}$ times the scalar result (C.332).

As in the $d = 2$ case, the renormalized one-loop edge contribution $\mathcal{S}^{(1)}_{\rm edge}$ to the entropy is negative and dominant. Some examples are shown in fig. C.14.

Graviton contribution for general d

For $d \geq 3$, UV-sensitive terms in the loop expansion renormalize higher-order curvature couplings in the gravitational action, prompting the inclusion of such terms in $S_E[g]$. Some caution is in order then if we wish to apply (3.112) or (3.148)-(3.149) to compute $\log Z^{(1)}$. The formula (3.112) for $Z^{(1)}_{\rm PI}$ depends on $\gamma = \sqrt{8\pi G_{\rm N}/A_{d-1}}$, gauge-algebraically defined by (3.109) and various normalization conventions. We picked these such that in pure Einstein gravity, $\gamma = \sqrt{8\pi G/A(\ell_0)}$, $\ell_0 = \sqrt{d(d-1)/2\Lambda}$, with G and Λ read off from the gravitational Lagrangian. However this expression of γ in terms of Lagrangian parameters will in general be modified in the presence of higher-order curvature terms. This is clear from the discussion in C.9.1, and (C.294) in particular. Since γ_0 is field-redefinition invariant, and since after transforming to a pure Einstein frame we have $\gamma_0 = \sqrt{2\pi/\mathcal{S}^{(0)}}$, with the right hand side also invariant, we have in general (for Einstein + perturbative higher-order curvature corrections)

$$\gamma = \sqrt{2\pi/\mathcal{S}^{(0)}}\,. \tag{C.335}$$

From (3.148) we thus get (ignoring the phase)

$$\mathcal{S} = \mathcal{S}^{(0)} - \frac{D_d}{2} \log \mathcal{S}^{(0)} + \alpha^{(2)}_{d+1} \log \frac{\ell_0}{L} + K_{d+1} \; + \; O\big(1/\mathcal{S}^{(0)}\big) \tag{C.336}$$

where $D_d = \frac{(d+2)(d+1)}{2}$, $\alpha^{(2)}_{d+1} = 0$ for even d and given by (3.116) for odd d, and K_{d+1} a numerical constant obtained by evaluating (3.115). For odd d the constant in the counterterm $\log Z_{\rm ct}(\ell)$ is fixed by minimal subtraction at a scale L, $c_{d+1}(L) \equiv -\alpha_{d+1} \log(M_\epsilon L)$, with $M_\epsilon = 2e^{-\gamma}/\epsilon$ determined by the heat kernel regulator as in (C.67), and $L\partial_L \mathcal{S} = 0$, i.e. $L\partial_L \mathcal{S}^{(0)} = \alpha^{(2)}_{d+1}$. Explicitly for $d = 2, 3, 4$, using (3.149), (3.12)

$$\begin{array}{c|l} d & \mathcal{S} \\ \hline 2 & \mathcal{S}^{(0)} - 3 \log \mathcal{S}^{(0)} + 5 \log(2\pi) \\ 3 & \mathcal{S}^{(0)} - 5 \log \mathcal{S}^{(0)} - \frac{571}{90} \log\Big(\frac{\ell_0}{L}\Big) - \log\Big(\frac{8\pi}{3}\Big) + \frac{715}{48} - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3) \\ 4 & \mathcal{S}^{(0)} - \frac{15}{2} \log \mathcal{S}^{(0)} + \log(12) + \frac{27}{2} \log(2\pi) + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4} \end{array} \tag{C.337}$$

For a $d = 3$ action (C.287) up to $O(l_s^2 R^2)$, with dots denoting $O(l_s^4)$ terms,

$$\mathcal{S}^{(0)} = \frac{\pi}{G} \left(\ell_0^2 + 48\, \lambda_{R^2}\, l_s^2 + \cdots \right), \qquad \Lambda = \frac{3}{\ell_0^2} + \cdots . \tag{C.338}$$

where $L\partial_L \lambda_{R^2} = -\frac{G}{48\,\pi\, l_s^2} \cdot \frac{571}{45}$. Putting $L = \ell_0$, and defining the scale ℓ_{R^2} by $\lambda_{R^2}(\ell_{R^2}) = 0$,

$$\mathcal{S} = \mathcal{S}^{(0)} - 5 \log \mathcal{S}^{(0)} + K_4 \; + \; O\big(1/\mathcal{S}^{(0)}\big), \qquad \mathcal{S}^{(0)} = \frac{\pi \ell_0^2}{G} - \frac{571}{45} \log \frac{\ell_0}{\ell_{R^2}} + \cdots . \tag{C.339}$$

The constant K_4 could be absorbed into λ_{R^2} at this level. Below, in (C.345), we will give it relative meaning however, by considering saddles different from the round S^4.

C.9.5 Classical and quantum observables

Here we address question 3 in our list below (3.153). To answer this, we need "observables" of the $\Lambda > 0$ Euclidean low-energy effective field theory probing independent gravitational couplings (for simplicity we restrict ourselves to purely gravitational theories here), i.e. diffeomorphism and field-redefinition invariant quantities, analogous to scattering amplitudes in asymptotically flat space. For this to be similarly useful, an infinite amount of unambiguous data should be extractable, at least in principle, from these observables.

As discussed above, $\mathcal{S}^{(0)} = \log \mathcal{Z}^{(0)} = -S_E[g_{\ell_0}]$ invariantly probes the dimensionless coupling given by $\ell_0^{d-1}/G \propto 1/G\Lambda^{(d-1)/2}$ in Einstein frame. The obvious tree-level invariants probing different couplings in the gravitational low-energy effective field theory are then the analogous $\mathcal{S}_M^{(0)} \equiv \log \mathcal{Z}_M^{(0)} = -S_E[g_M]$ evaluated on saddles g_M different from the round sphere, in the parametric $\ell_0 \gg l_s$ regime of validity of the effective field theory, with g_M asymptotically Einstein in the $\ell_0 \to \infty$ limit. These are the analogs of tree-level scattering amplitudes. The obvious quantum counterparts are the corresponding generalizations of $\mathcal{S}$, i.e. $\mathcal{S}_M \equiv \log \mathcal{Z}_M$ evaluated in large-ℓ_0 perturbation theory about the saddle g_M. These are the analogs of quantum scattering amplitudes. Below we make this a bit more concrete in examples.

3D

In $d = 2$, the Weyl tensor vanishes identically, so higher-order curvature invariants involve $R_{\mu\nu}$ only and can be removed from the action by a field redefinition in large-ℓ_0 perturbation theory, reducing it to pure Einstein form in general. As a result, $\mathcal{S}^{(0)}$ is the only independent invariant in pure 3D gravity, all g_M are Einstein, and the $\mathcal{S}_M^{(0)}$ are all proportional to $\mathcal{S}_0 \equiv \mathcal{S}_{S^3}^{(0)}$.

As discussed under (3.168), the quantum $\mathcal{S} = \mathcal{S}_{S^3}$ takes the form

$$\mathcal{S} = \mathcal{S}_0 = \mathcal{S}_0 - 3 \log \mathcal{S}_0 + 5 \log(2\pi) + \textstyle\sum_n c_n \, \mathcal{S}_0^{-2n} \tag{C.340}$$

The corrections terms in the expansion are all nonlocal (no odd powers of ℓ_0), and the coefficients provide an unambiguous, infinite data set.

odd D ≥ 5

In 5D gravity, there are infinitely many independent coupling constants. There are also infinitely many different $\Lambda > 0$ Einstein metrics on S^5, including a discrete infinity of Böhm metrics with $SO(3)\times SO(3)$ symmetry [219] amenable to detailed numerical analysis [220], and 68 Sasaki-Einstein families with moduli spaces up to real dimension 10 [221]. Unlike the round S^5, these are not conformally flat, and thus, unlike $\mathcal{S}^{(0)}$, the corresponding $\mathcal{S}_M^{(0)}$ will pick up couplings such as the Weyl-squared coupling λ_{C^2} in (C.287). It is plausible that this set of known Einstein metrics (perturbed by small higher-order corrections to the Einstein equations of motion at finite ℓ_0) more than suffices to invariantly probe all independent couplings of the gravitational action, delivering moreover infinitely many quantum observables $\mathcal{S}_M$, providing an infinity of unambiguous low-energy effective field theory data to any order in perturbation theory, without ever leaving the sphere — at least in principle.

The landscape of known $\Lambda > 0$ Einstein metrics on odd-dimensional spheres becomes increasingly vast as the dimension grows, with double-exponentially growing numbers [221]. For example there are at least 8610 families of Sasaki-Einstein manifolds on S^7, spanning all 28 diffeomorphism classes, with the standard class admitting a 82-dimensional family, and there are at least 10^{828} distinct families of Einstein metrics on S^{25}, featuring moduli spaces of dimension greater than 10^{833}.

4D

4D gravity likewise has infinitely many independent coupling constants. It is not known if S^4 has another Einstein metric besides the round sphere. In fact the list of 4D topologies known to admit $\Lambda > 0$ Einstein metrics is rather limited [231]: S^4, $S^2 \times S^2$, $\mathbb{CP}^2$, and the connected sums $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $1 \leq k \leq 8$. However for $k \geq 5$ these have a moduli space of nonzero dimension [232, 233], which might suffice to probe all couplings. (The moduli space would presumably be lifted at sufficiently high order in the l_s expansion upon turning on higher-order curvature perturbations.)

Below we illustrate in explicit detail how the Weyl-squared coupling can be extracted from suitable linear combinations of pairs of $\mathcal{S}_M^{(0)}$ with $M \in \{S^4, S^2 \times S^2, \mathbb{CP}^2\}$, and how a suitable linear combination of all three can be used to extract an unambiguous linear combination of the constant terms arising at one loop.

The Weyl-squared coupling λ_{C^2} in $S_E[g] = \text{(C.287)} + \cdots$ is invisible to $\mathcal{S}^{(0)}$ (C.338) but it is picked up by $\mathcal{S}_M^{(0)}$ by $M = S^2 \times S^2$:

$$\mathcal{S}^{(0)}_{S^2\times S^2} = \frac{2}{3} \cdot \frac{\pi}{G}\big(\ell_0^2 + 48\,\lambda_{R^2}\, l_s^2 + 16\,\lambda_{C^2}\, l_s^2\; + \;\cdots\big)\,, \tag{C.341}$$

with the dots denoting $O(l_s^4)$ terms and $\ell_0 = \sqrt{3/\Lambda} + \cdots$ as in (C.338). Physically, $\mathcal{S}^{(0)}_{S^2\times S^2}$ is the horizon entropy of the $\mathrm{dS}_2 \times S^2$ static patch, i.e. the Nariai spacetime between the cosmological and maximal Schwarzschild-de Sitter black hole horizons, both of area $A = \frac{1}{3} \cdot 4\pi\ell_0^2$. Comparing to (C.338), the linear combination

$$\mathcal{S}^{(0)}_{C^2} \equiv 3\,\mathcal{S}^{(0)}_{S^2\times S^2} - 2\,\mathcal{S}^{(0)} = \frac{32\pi l_s^2}{G}\big(\,\lambda_{C^2}\; + \;\cdots\big) \tag{C.342}$$

extracts the Weyl-squared coupling of $S_E[g]$. Analogously, for the Einstein metric on $\mathbb{CP}^2$, we get $\tilde{\mathcal{S}}^{(0)}_{C^2} \equiv 8\,\mathcal{S}^{(0)}_{\mathbb{CP}^2} - 6\,\mathcal{S}^{(0)} = \frac{48\pi l_s^2}{G}(\lambda_{C^2} + \cdots)$. Then

$$\mathcal{S}^{(0)}_{\text{cub}} \equiv 2\,\tilde{\mathcal{S}}^{(0)}_{C^2} - 3\,\mathcal{S}^{(0)}_{C^2} = 16\,\mathcal{S}^{(0)}_{\mathbb{CP}^2} - 9\,\mathcal{S}^{(0)}_{S^2\times S^2} - 6\,\mathcal{S}^{(0)} \;= 0\; + \;\cdots\,, \tag{C.343}$$

which extracts some curvature-cubed coupling in the effective action.

To one loop, the quantum $\mathcal{S}_M = \log \mathcal{Z}_M$ can be expressed in a form paralleling (C.336):

$$\mathcal{S}_M = \mathcal{S}^{(0)}_M - \frac{D_M}{2}\log \mathcal{S}^{(0)}_M + \alpha_M \log\frac{\ell_0}{L} + K_M\; + \;\cdots\,, \tag{C.344}$$

where D_M is the number of Killing vectors of M: $D_{S^4} = 10$, $D_{S^2\times S^2} = 6$, $D_{\mathbb{CP}^2} = 8$, and α_M can be obtained from the local expressions in [51]: $\alpha_{S^4} = -\frac{571}{45}$, $\alpha_{S^2\times S^2} = -\frac{98}{45}$, $\alpha_{\mathbb{CP}^2} = -\frac{359}{60}$. Computing the constants K_M generalizing K_{S^4} given in (C.337) would require more work. Moreover, computing them for one or two saddles would provide no unambiguous information because they may be absorbed into λ_{C^2} and λ_{R^2}. However, since there only two undetermined coupling constants at this order, computing them for all *three* does provide unambiguous information, extracted by the quantum counterpart of (C.343):

$$\mathcal{S}_{\text{cub}} \equiv 16\,\mathcal{S}_{\mathbb{CP}^2} - 9\,\mathcal{S}_{S^2\times S^2} - 6\,\mathcal{S}_{S^4} = -7\,\log\mathcal{S}^{(0)} + 16\,K_{\mathbb{CP}^2} - 9\,K_{S^2\times S^2} - 6\,K_{S^4} + \cdots \tag{C.345}$$

The $\log(\ell_0/L)$ terms had to cancel in this linear combination because the tree-level parts at this order cancel by design and $L\partial_L\mathcal{S}_{\text{cub}} = 0$.