Me, Myself and I: Time-Inconsistent Stochastic Control, Contract Theory and Backward Stochastic Volterra Integral Equations

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Abstract

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This thesis studies the decision-making of agents exhibiting time-inconsistent preferences and its implications in the context of contract theory. We take a probabilistic approach to continuous-time non-Markovian time-inconsistent stochastic control problems for sophisticated agents. By introducing a refinement of the notion of equilibrium, an extended dynamic programming principle is established. In turn, this leads to consider an infinite family of BSDEs analogous to the classical Hamilton–Jacobi–Bellman equation. This system is fundamental in the sense that its well-posedness is both necessary and sufficient to characterise equilibria and its associated value function. In addition, under modest assumptions, the existence and uniqueness of a solution is established.

With the previous results in mind, we then study a new general class of multidimensional type-I backward stochastic Volterra integral equations. Towards this goal, the well-posedness of a system of infinite family of standard backward stochastic differential equations is established. Interestingly, its well-posedness is equivalent to that of the type-I backward stochastic Volterra integral equation. This result yields a representation formula in terms of semilinear partial differential equation of Hamilton–Jacobi–Bellman type. In perfect analogy to the theory of backward stochastic differential equations, the case of Lipschitz continuous generators is addressed first and subsequently the quadratic case. In particular, our results show the equivalence of the probabilistic and analytic approaches to time-inconsistent stochastic control problems.

Finally, this thesis studies the contracting problem between a standard utility maximiser principal and a sophisticated time-inconsistent agent. We show that the contracting problem faced by the principal can be reformulated as a novel class of control problems exposing the complications of the agent's preferences. This corresponds to the control of a forward Volterra equation via constrained Volterra type controls. The structure of this problem is inherently related to the representation of the agent's value function via extended type-I backward stochastic differential equations. Despite the inherent challenges of this class of problems, our reformulation allows us to study the solution for different specifications of preferences for the principal and the agent. This allows us to discuss the qualitative and methodological implications of our results in the context of contract theory: (i) from a methodological point of view, unlike in the time-consistent case, the solution to the moral hazard problem does not reduce, in general, to a standard stochastic control problem; (ii) our analysis shows that slight deviations of seminal models in contracting theory seem to challenge the virtues attributed to linear contracts and suggests that such contracts would typically cease to be optimal in general for time-inconsistent agents; (iii) in line with some recent developments in the time-consistent literature, we find that the optimal contract in the time-inconsistent scenario is, in general, non-Markovian in the state process X.

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Preface

In the typical situation of interest, a principal, who is offering a contract, is imperfectly informed about the actions of a time-inconsistent agent, who can accept or reject the contract. The goal is to design a contract, compatible with the agent's preferences, that maximises the utility of the principal while that of the agent is held to a given level.

Being able to provide a satisfactory solution requires a complete understanding of the agent's preferences. As such, a large part of this thesis is devoted to tackling several fundamental questions in the study of time-inconsistent stochastic control problems for sophisticated agents, also known as the game-theoretic approach. First, the results in the existing literature depend critically on the Markovian structure of the formulation: can these results be extended once we abandon this framework? Second, it is known that a solution to a system of equations provide an equilibrium strategy and its associated value function: in what sense is the converse true? Do all equilibria arise as solutions to such system? Third, what can be said about the existence and uniqueness of equilibria? Answering these questions is necessary to proceed to study the challenges of the problem faced by the principal. In the time-consistent case, the contracting problem can, without loss of generality, be reformulated as an optimal control problem: is there an analogue in the time-inconsistent case? To what extend does said reformulation help in finding a solution? Finally, how do the agent's preferences change the form of the contract?

In the following, we put forward the tools and the pieces needed to provide answers to these questions.

Notation

Throughout this document we take the convention $\infty - \infty := -\infty$, and we fix a time horizon $T > 0$. $\mathbb{R}_+$ and $\mathbb{R}_+^\star$ denote the sets of non-negative and positive real numbers, respectively. Given $(E, \|\cdot\|)$ a Banach space, a positive integer p, and a non-negative integer q, $\mathcal{C}_q^p(E)$ (resp. $\mathcal{C}_{q,b}^p(E)$) will denote the space of functions from E to $\mathbb{R}^p$ which are at least q times continuously differentiable (resp. and bounded with bounded derivatives). We set $\mathcal{C}_{q,b}(E) := \mathcal{C}_{q,b}^1(E)$, i.e. the space of q times continuously differentiable bounded functions with bounded derivatives from E to $\mathbb{R}$. Whenever $E = [0, T]$ (resp. $q = 0$ or b is not specified), we suppress the dependence on E (resp. on q or b), e.g. $\mathcal{C}^p$ denotes the space of continuous functions from $[0, T]$ to $\mathbb{R}^p$. Given $x \in \mathcal{C}^p$, we denote by $x_{\cdot\wedge t}$ the path of x stopped at time t, i.e. $x_{\cdot\wedge t} := (x(r \wedge t), r \geq 0)$. Given $(x, \tilde{x}) \in \mathcal{C}^p \times \mathcal{C}^p$ and $t \in [0, T]$, we define their concatenation $x \otimes_t \tilde{x} \in \mathcal{C}^p$ by $(x \otimes_t \tilde{x})(r) := x(r)1_{\{r\leq t\}} + (x(t) + \tilde{x}(r) - \tilde{x}(t))1_{\{r\geq t\}}$, $r \in [0, T]$.

For $\varphi \in \mathcal{C}_q^p(E)$ with $q \geq 2$, $\partial_{xx}^2\varphi$ will denote its Hessian. For a function $\phi : [0, T] \times E$ with $s \longmapsto \phi(s, \alpha)$ uniformly continuous uniformly in α, we denote by $\rho_\phi : [0, T] \longrightarrow \mathbb{R}$ its modulus of continuity, which we recall satisfies $\rho_\phi(\ell) \longrightarrow 0$ as $\ell \longrightarrow 0$. For $(u, v) \in \mathbb{R}^p \times \mathbb{R}^p$, $u \cdot b$ will denote their usual inner product, and $|u|$ the corresponding norm. For positive integers m and n, we denote by $\mathcal{M}_{m,n}(\mathbb{R})$ the space of $m \times n$ matrices with real entries. For $M \in M_{m,n}(\mathbb{R})$, $M_{:i}$ and $M_{i:}$ denote the i-th column and row. $\mathbb{S}_n^+(\mathbb{R})$ denotes the set of $n \times n$ symmetric positive semi-definite matrices, while $\mathrm{Tr}[M]$ denotes the trace of $M \in M_m(\mathbb{R})$, and $|M| := \sqrt{\mathrm{Tr}[M^\top M]}$ for $M \in M_{m,n}(\mathbb{R})$. By $0_{m,n}$ and I_n we denote the $m \times n$ matrix of zeros and the identity matrix of $\mathcal{M}_n(\mathbb{R}) := \mathcal{M}_{n,n}(\mathbb{R})$, respectively. $\mathbb{S}_n^+(\mathbb{R})$ denotes the set of $n \times n$ symmetric positive semi-definite matrices. $\mathrm{Tr}[M]$ denotes the trace of a matrix $M \in \mathcal{M}_n(\mathbb{R})$.

For $(\Omega, \mathcal{F})$ a measurable space, $\mathrm{Prob}(\Omega)$ denotes the collection of all probability measures on $(\Omega, \mathcal{F})$. For a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F})$, $\mathcal{P}_{\mathrm{prog}}(E, \mathbb{F})$ (resp. $\mathcal{P}_{\mathrm{pred}}(E, \mathbb{F})$, $\mathcal{P}_{\mathrm{opt}}(E, \mathbb{F})$) will denote the set of E-valued, $\mathbb{F}$-progressively measurable processes (resp. $\mathbb{F}$-predictable processes, $\mathbb{F}$-optional processes). For $\mathbb{P} \in \mathrm{Prob}(\Omega)$ and a filtration $\mathbb{F}$, $\mathbb{F}^{\mathbb{P}} := (\mathcal{F}_t^{\mathbb{P}})_{t\in[0,T]}$, denotes the $\mathbb{P}$-

augmentation of $\mathbb{F}$. We recall that for any $t \in [0,T]$, $\mathcal{F}_t^{\mathbb{P}} := \mathcal{F}_t \vee \sigma(\mathcal{N}^{\mathbb{P}})$, where $\mathcal{N}^{\mathbb{P}} := \{N \subseteq \Omega : \exists B \in \mathcal{F}, N \subseteq B \text{ and } \mathbb{P}[B] = 0\}$. With this, the probability measure $\mathbb{P}$ can be extended so that $(\Omega, \mathcal{F}, \mathbb{F}^{\mathbb{P}}, \mathbb{P})$ becomes a complete probability space, see Karatzas and Shreve [150, Chapter II.7]. $\mathbb{F}_+^{\mathbb{P}}$ denotes the right limit of $\mathbb{F}^{\mathbb{P}}$, i.e. $\mathcal{F}_{t+}^{\mathbb{P}} := \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^{\mathbb{P}}$, $t \in [0,T)$, and $\mathcal{F}_{T+}^{\mathbb{P}} := \mathcal{F}_T^{\mathbb{P}}$, so that $\mathbb{F}_+^{\mathbb{P}}$ is the smallest filtration that contains $\mathbb{F}$ and satisfies the usual conditions. Moreover, given $\mathcal{P} \subseteq \text{Prob}(\Omega)$ we introduce the set of $\mathcal{P}$-polar sets $\mathcal{N}^{\mathcal{P}} := \{N \subseteq \Omega : N \subseteq B, \text{ for some } B \in \mathcal{F} \text{ with } \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[B] = 0\}$, as well as the $\mathcal{P}$-completion of $\mathbb{F}$, $\mathbb{F}^{\mathcal{P}} := (\mathcal{F}_t^{\mathcal{P}})_{t \in [0,T]}$, with $\mathcal{F}_t^{\mathcal{P}} := \mathcal{F}_t \vee \sigma(\mathcal{N}^{\mathcal{P}})$, $t \in [0,T]$ together with the corresponding right-continuous limit $\mathbb{F}_+^{\mathcal{P}} := (\mathcal{F}_{t+}^{\mathcal{P}})_{t \in [0,T]}$, with $\mathcal{F}_{t+}^{\mathcal{P}} := \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^{\mathcal{P}}$, $t \in [0,T)$, and $\mathcal{F}_{T+}^{\mathcal{P}} := \mathcal{F}_T^{\mathcal{P}}$. For $\{s,t\} \subseteq [0,T]$, with $s \leq t$, $\mathcal{T}_{s,t}(\mathbb{F})$ denotes the collection of $[t,T]$-valued $\mathbb{F}$–stopping times.

Additionally, given $A \subseteq \mathbb{R}^k$, $\mathbb{A}$ denotes the collection of finite and positive Borel measures on $[0,T] \times A$ whose projection on $[0,T]$ is the Lebesgue measure. In other words, any $q \in \mathbb{A}$ can be disintegrated as $q(\mathrm{d}t, \mathrm{d}a) = q_t(\mathrm{d}a)\mathrm{d}t$, for an appropriate Borel measurable kernel q_t which is unique up to (Lebesgue–) almost everywhere equality. We are particularly interested in the set $\mathbb{A}_0$, of $q \in \mathbb{A}$ of the form $q = \delta_{\phi_t}(\mathrm{d}a)\mathrm{d}t$, where δ_ϕ the Dirac mass at a Borel function $\phi : [0,T] \longrightarrow A$.

We also recall the elementary inequalities

$$\left(\sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2, \tag{I.1}$$

valid for any positive integer n and any collection $(a_i)_{1 \leq i \leq n}$ of non-negative numbers, as well as, Young's inequality which guarantees that for any $\varepsilon > 0$, $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$.

Part I

Introduction

Chapter 1

Introduction

An extremely important point that has redrawn the attention of many academic disciplines in recent years is that human beings do not necessarily behave as perfectly rational economic agents. Hence, their criteria for evaluating their well-being are most of the time a lot more involved than the ones used in the classic economic literature. This has been amplified due to the rapid advent and flourishment of new economies, such as e-commerce and online advertisement. These activities are characterised by, notably, a higher number of agents from both the supply and demand sides, and an unprecedentedly large menu of tailor-made and personalised services. Naturally, this has led to more intricate behaviours and interactions at all levels of the economy. Though the development of large scale algorithms provides practical tools to circumvent the intricacies of these kind of behaviours, there is also a growing demand for models that shed some light into how these agents make their decisions and facilitate a deeper analysis. Notably of interest are models that study how to incentivise such agents so that they perform a particular task. This is how time-inconsistency and contract theory intersect.

In the moral hazard contracting problem between a principal (she) and an agent (he), the principal's objective is to use the information available to create the appropriate incentives, in the form of a contract, to: (i) encourage the agent, whose actions influence an output process, to accept the contract; (ii) maximise her utility. Though this is quite a simple scenario it has the virtue of being able to accommodate a wide spectrum of possibilities: the agent performing a task on the principal's behalf, the agent consuming a good produced by the principal, the agent buying something from the principal... The main problem is then to design contracts (that is wages, prices,...) such that the agent accepts them, and is given proper incentives to behave in a way that allows the principal to get the most out of the contract.

This thesis studies the previous situation in the case were the agent's preferences are time-inconsistent. Fortunately, there is a well-established blueprint for how to solve principal–agent problems in continuous-time in the case of a classical time-consistent utility maximiser agent. The following section serves as an introduction to contract theory models in continuous-time and, in particular, the dynamic programming approach will be presented. By virtue of the clarity of this approach, the steps necessary to successfully extend these ideas to the time-inconsistent case will become apparent. Developing and putting these steps together will guide much of the work of this thesis.

1.1 Contract theory in continuous-time models

In this thesis, we are interested in the moral hazard contracting problem between a principal and an agent with time-inconsistent preferences. A principal–agent problem is a problem of optimal contracting between two parties. The principal, who is interested in hiring the agent, offers a contract. Provided the agent accepts, he can influence a random process, the outcome, via his actions. A key feature in these models is the amount of information available to the principal when designing the contract. There are three classical cases studied in the literature: risk-sharing with symmetric information, hidden action, and hidden type. We are only concerned with the first two in this work.

In the risk-sharing scenario, also referred to as the first-best, both parties have the same information and have to agree on how to share the underlying risk. In the first-best problem, the principal has all the bargaining power, *i.e.* she offers the contract and dictates the agent's actions (the agent is compelled to follow or else he would be severely penalised). In the case of hidden actions, the principal is imperfectly informed about the agent's actions. Either they are too costly to be monitored or simply unobservable. Consequently, the principal expects to receive a second-best utility compared to the risk-sharing scenario. As the agent is allowed to take actions that are not in the principal's best interest, this situation is also referred to as moral hazard, and incentives play a crucial role. Indeed, the principal hopes to influence the agent's actions by offering an appropriate contract.

In the case of a traditional (time-consistent) agent, a common feature of these models is that their resolution boils down to standard stochastic control models. Indeed, in light of the principal's bargaining power, the first-best case is always cast as a stochastic control problem for a single individual, the principal, who chooses both the contract and the actions under the participation constraint. On the other hand, in the second-best problem, it being a two-stage Stackelberg game, one has to solve the agent's problem for any given fixed contract before moving to study the principal's problem. In principle, this renders a much more complicated structure on the problem. Since the introduction of the continuous-time framework, it took time for the literature to present a general approach that arrived at the same conclusion for the second-best problem.

The study of the moral hazard problem in continuous time has its roots in the seminal paper of Holmström and Milgrom [130]. In this model, the principal and the agent have CARA utility functions, and the agent's effort influences the drift of the output process, the solution to a controlled diffusion, but not the volatility. The resulting optimal contract is an affine function of the aggregate output. The model in [130] drew great attention as the resolution of the, seemingly more complicated, continuous-time formulation was much more amenable, could be rigorously justified and provided useful explicit solutions for the economic analysis. This was not the case for most of the discrete-time models that dominated the existing literature., see Laffont and Martimort [162]. For instance, Schättler and Sung [222, 223] study the validity of the so-called first order approach. Sung [234, 235] provides extensions to the case of diffusion control and hierarchical structures. The linearity of the optimal contract, a feature also present in [130] and [234], is further studied in Müller [187, 188], Hellwig [126] and Hellwig and Schmidt [127] for the first-best problem and in the interplay between the discrete- and continuous-time models, respectively.[1] Notably, Williams [262, 263, 264] and Cvitanić, Wan, and Zhang [60, 61, 62] characterise the optimal contract for general utilities by means of the so-called stochastic maximum principle and forward–backward stochastic differential equations (FBSDEs).[2]

[1]The previous list is my no means exhaustive. Other early continuous-time contract theory models were introduced in Adrian and Westerfield [1], Biais, Mariotti, Plantin, and Rochet [26], Biais, Mariotti, Rochet, and Villeneuve [27], Biais, Mariotti, and Rochet [28], Capponi and Frei [44], DeMarzo and Sannikov [72], DeMarzo, Fishman, He, and Wang [73], Fong [100], He [124], Hoffmann and Pfeil [129], Ju and Wan [145], Keiber [154], Leung [166], Mirrlees and Raimondo [186], Myerson [189], Ou-Yang [195], Pagès [198], Pagès and Possamaï [199], Piskorski and Tchistyi [204], Piskorski and Westerfield [205], Sannikov [218], Schroder, Sinha, and Levental [224], Van Long and Sorger [239], Westerfield [261], Zhang [274], Zhou [275], and Zhu [276]

[2]We refer to the monograph Cvitanić and Zhang [59] for a general framework that systematically surveys a great portion of the literature exploiting the maximum principle, in models driven by Brownian Motion.

Nevertheless, it was not until the approach in Sannikov [219], see also Sannikov [220], was available that the study of the moral hazard problem was, once again, reinvigorated and arrived finally at the methodical programme presented in Cvitanić, Possamaï, and Touzi [63, 64]. The idea of this approach is to focus on the dynamic continuation value of the agent as a state variable for the principal's problem, an idea already acknowledged in the discrete-time literature, see for instance Spear and Srivastava [231]. In a nutshell, this method leverages the dynamic programming principle and the theory of backward stochastic differential equations (BSDEs) to reformulate the principal's problem as a standard optimal stochastic control problem with an additional state variable, namely, the agent's continuation utility.

1.1.1 The dynamic programming approach

Let us describe, informally, the dynamic programming approach as derived in [64], for the case of a time-consistent agent, *i.e.* an agent that discounts according to an exponential parameter $\rho > 0$. To ease the presentation, we will exclude the case of volatility control here, and will take both the principal and agent risk-neutral. $\mathcal{A}$ denotes the set of A-valued[3] admissible actions used by the agent to control the distribution of the state process X, the canonical process on the space of $\mathbb{R}^d$-valued continuous functions on $[0,T]$, as follows. For $\nu \in \mathcal{A}$ and a $\mathbb{P}^\nu$–Brownian motion W^ν (depending on ν), X satisfies the dynamics

$$X_t = x_0 + \int_0^t \sigma_r(X_{\cdot\wedge r})\big(b_r(X_{\cdot\wedge r},\nu_r)\mathrm{d}r + \mathrm{d}W_r^\nu\big),\ t\in[0,T],\ \mathbb{P}^\nu\text{–a.s.},$$

where $X_{\cdot\wedge t}$ denotes the path up to time t of the state process X and x denotes its past trajectory. Let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ denote the augmented filtration generated by X, and $R_0 \in \mathbb{R}$ the agent's reservation utility below which he refuses the contract. Given an arbitrary fixed contract $\xi \in \Xi$, the utility drawn by the agent is given by

$$\mathrm{V}^{\mathrm{A}}(\xi) = \sup_{\nu\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[\mathrm{e}^{-\rho T}\xi - \int_0^T \mathrm{e}^{-\rho r} c_r(X_{r\wedge\cdot},\nu_r)\mathrm{d}r\bigg].$$

Let $\mathcal{A}^\star(\xi)$ denote the optimal responses to contract ξ, $\nu^\star \in \mathcal{A}$. The problem of the principal is

[3] A is, for instance, a subset of $\mathbb{R}^k$ for some non-negative integer k

given by

$$\mathrm{V}^{\mathrm{P}} = \sup_{\xi\in\Xi} \sup_{\nu^\star\in\mathcal{A}^\star(\xi)} \mathbb{E}^{\mathbb{P}^{\nu^\star}}\big[X_T - \xi\big].$$

Let us note that both the agent's and the principal's problems are non-standard stochastic control problems. Indeed, the agent's problem is non-Markovian and the principal's involves the optimisation over the set Ξ. Recall ξ, and all the data, is allowed to be of non-Markovian nature. Moreover, the principal's optimisation is, *a priori*, a control problem that can not be approached by dynamic programming. Let us expand on this last comment regarding V^{P}. As a typical two-stage Stackelberg game, for a long time, the predominant approach taken in the literature consisted of characterising the agent's value process, or continuation/promised utility, and his optimal actions given an arbitrary contract payoff. This, in turn, enabled the analysis of the principal's maximisation problem over all possible payoffs.[4] Yet, this approach may be challenging for several reasons: (i) it may be difficult to solve the agent's stochastic control problem given an arbitrary, possibly non-Markovian, payoff; (ii) it may be hard for the principal to maximise over all such contracts.; (iii) the agent's optimal control may depend on the given contract in a highly nonlinear manner, rendering the principal's optimisation problem even more complicated.

The following three steps summarise the dynamic programming approach.

Step 1: Establish a dynamic programming principle (DPP) for $\mathrm{V}^{\mathrm{A}}_0(\xi)$, *i.e.*

$$\mathrm{V}^{\mathrm{A}}(\xi) = \sup_{\nu\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[\mathrm{V}^{\mathrm{A}}_\tau(\xi) - \int_\sigma^\tau \mathrm{e}^{-\rho(r-\sigma)} c_r(X_{r\wedge\cdot},\nu_r)\mathrm{d}r \bigg| \mathcal{F}_\sigma\bigg],\ \mathrm{V}^{\mathrm{A}}_T(\xi) = \xi.$$

Note that this implies that the value process $\mathrm{V}^{\mathrm{A}}_\cdot(\xi)$ admits the representation via the standard backward stochastic differential equation (BSDE)

$$Y_t = \xi + \int_t^T H_r(X_{r\wedge\cdot}, Y_r, Z_r)\mathrm{d}r - \int_t^T Z_r \mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}^\nu\text{–a.s.},$$

where H denotes the Hamiltonian, given by

$$H_t(x,y,z) := \sup_{a\in A}\big\{\sigma_t(x)b_t(x,a)\cdot z - c_t(x,a)\big\} - \rho y.$$

[4] For completeness we remark the different approach in Evans, Miller, and Yang [97], where for each possible action process of the agent, they characterise contracts that are incentive compatible for it.

Moreover, the principal identifies all the agent's optimal actions as the maximisers of H, $a^\star(r,x,z)$, which, to ease the presentation, we assume to be unique.

Step 2: For an appropriate admissibility class $\mathcal{H}$, introduce the family of contracts

$$\widetilde{\Xi} := \bigg\{ Y_T^{Y_0,Z} : Y_t^{Y_0,Z} := Y_0 - \int_0^t H_r(X_{r\wedge\cdot}, Y_r^{Y_0,Z}, Z_r)\mathrm{d}r + \int_0^t Z_r \mathrm{d}X_r, t \in [0,T], (Y_0,Z) \in [R_0,\infty)\times\mathcal{H} \bigg\},$$

and establish there is no lost of generality in offering such contracts, *i.e.* $\Xi = \widetilde{\Xi}$.

Step 3: Conclude that the problem of the principal equals the standard stochastic control problem

$$\mathrm{V}^{\mathrm{P}} = \sup_{Y_0 \geq R_0} \sup_{Z\in\mathcal{H}} \mathbb{E}^{\mathbb{P}^\star(Z)}\Big[X_T - Y_T^{Y_0,Z}\Big]$$

where $\mathbb{P}^\star(Z) := \mathbb{P}^{a^\star(\cdot,X_\cdot,Z_\cdot)}$ denotes the probability induced by the agent's optimal response. In the above problem, Z is the control variable and $(X, Y^{Y_0,Z})$ the state variables. The control on X is via the probability $\mathbb{P}^\star(Z)$.

This methodology has been extended to several scenarii including random horizon contracting Possamaï and Touzi [209] and Lin, Ren, Touzi, and Yang [172], ambiguity features from the point of view of the principal, as in Mastrolia and Possamaï [180], Hernández Santibáñez and Mastrolia [128], Chen and Sung [48], and Sung [236], a principal contracting a finite number of agents Élie and Possamaï [90], several principals contracting a common agent Mastrolia and Ren [181], hierarchical contracting problems Hubert [140], a principal contracting a mean-field of agents Élie, Mastrolia, and Possamaï [91], and applications in, optimal electricity demand response contracting, as in Aïd, Possamaï, and Touzi [6], Alasseur, Chaton, and Hubert [7], and Élie, Hubert, Mastrolia, and Possamaï [92], market microstructure, as in El Euch, Mastrolia, Rosenbaum, and Touzi [83] and Baldacci, Possamaï, and Rosenbaum [16], green bond markets Baldacci and Possamaï [15], pandemic control Hubert, Mastrolia, Possamaï, and Warin [141]. The road map suggested by this approach is quite clear: (i) identify the generic dynamic programming representation of the agent's value process, (ii) express the contract payment in terms of the value process, (iii) optimise the principal's objective over such payments.

Not surprisingly, each of these steps would bring its own challenges in the case of a time-inconsistent agent. Notably, as we will see next, the distinctive feature of time-inconsistent preferences is that they do not satisfy a dynamic programming principle. This casts doubt upon the very starting point of our strategy.

1.2 Part II: non-Markovian time-inconsistent control

Let us begin by considering the following illustrative example in discrete time. When choosing his working routine an agent chooses an action $\nu \in \{\text{nap}, \text{work}\}$. The utility at time t corresponds to the discounted flow of utilities, given by

$$J(t,\nu) := u_t(x_t^\nu) + \beta\big(\rho u_{t+1}(x_{t+1}^\nu) + \rho^2 u_{t+2}(x_{t+2}^\nu) + \dots\big),$$

where x_t^ν denotes the value of the Agent's state variable induced by ν, and $u_t(x_t^\nu)$ the corresponding utility at time t. (ρ, β) are the parameters of the so-called quasi-hyperbolic discounting for which its distinguishing parameter β acts as an intra-personal weight which may bias towards present $(\beta < 1)$ or future (resp. $\beta > 1$) utilities and δ is a classic discount factor. Assume that working today generates an immediate disutility of $-2/3$ and a postponed benefit of 1, and normalise the utility of napping to 0. Then, for $\rho = 1$ and $\beta = 1/2$, it is not hard to see that at time t the agent prefers napping today $(0 > -2/3 + 1/2)$ and defers working for tomorrow $(-1/3 + 1/2 > 0)$. All together, the discounting structure induces dynamically inconsistent preferences, *i.e.* for any time reference t, at time t the agent prefers to work at time $t+1$, but at time $t+1$ the agent prefers to nap.

Time-inconsistency has recently redrawn the attention of many academic disciplines, ranging from mathematics to economics, due to both the mathematical challenges that rigorously understanding this phenomenon carries, as well as the need for the development of economic theories that are able to explain the behaviour of agents that fail to comply with the usual rationality assumptions. Indeed, one can find clear evidence of such attitudes in a number of applications, from consumption problems to finance, from crime to voting, from charitable giving to labour supply, see Rabin [213] and Dellavigna [71] for detailed reviews. In recent years, the need for thorough studies of this phenomenon has become more urgent, due notably to the rapid advent of new economies such as e-commerce and online advertisement.

The distinctive feature in these situations is that human beings do not necessarily behave as what neoclassical economists refer to *perfectly rational decision-makers*. Such idealised individuals are aware of their alternatives, form expectations about any unknowns, have clear preferences, and choose their actions deliberately after some process of optimisation, see Osborne and Rubinstein [194, Chapter 1]. In reality, their criteria for evaluating their well-being are in many cases a lot more involved than the ones considered in the classic literature. For instance, empirical studies suggest that relative preferences of agents do seem to change with time, see Frederick, Loewenstein, and O'Donoghue [101] and Fang and Silverman [98]. Similarly, there is robust evidence of an inclination for imminent gratification even if accompanied by harmful delayed consequences, see Mehra and Prescoot [183], Friedman and Savage [104], Ellsberg [93] and Allais [8]. In mathematical terms, this translates into stochastic control problems in which the classic dynamic programming principle, or in other words the Bellman optimality principle, is not satisfied.

Let us consider the form of pay-off functionals at the core of the continuous-time optimal stochastic control literature in a non-Markovian framework. Given a time reference $t \in [0,T]$, where $T > 0$ is a fixed time horizon, a past trajectory x for the state process X, whose path up to t we denote by $X_{\cdot\wedge t}$, and an action plan ν, that is to say, a probability distribution for X and an action process, the reward derived by an agent is

$$J(t,x,\nu) = \mathbb{E}^{\mathbb{P}}\bigg[\int_t^T f_r(X_{\cdot\wedge r},\nu_r)\mathrm{d}r + \xi(X_{\cdot\wedge T})\bigg].$$

However, as pointed out by Samuelson [217, pp. 159], 'it is completely arbitrary to assume that the individual behaves so as to maximise an integral of the form envisaged above. This involves the assumption that at every instant of time the individual's satisfaction depends only upon the action at that time, and that, furthermore, the individual tries to maximise the sum of instantaneous satisfactions reduced to some comparable base by time discount.' As a consequence, adds [217], 'the solution to the problem of maximising these type of rewards holds only for an agent deciding her actions throughout the period at the beginning of it, and, as she moves along in time, there is a *perspective phenomenon* in that her view of the future in relation to her instantaneous time position remains invariant, rather than her evaluation of any particular year.[...] Moreover, these results will remain unchanged even if she were to discount from the existing point of time rather than from the beginning of the period. Therefore, the fact that this is so is in itself a presumption that individuals do behave in terms of these functionals.' Consequently, understanding the rationale behind the actions of a broader class of economic individuals calls for the incorporation of functionals able to include the previous one as a particular case. This is the motivation behind any theory of time-inconsistency.

Time-inconsistency is generally the fact that marginal rates of substitution between goods consumed at different dates change over time, see Strotz [233], Laibson [163], Gul and Pesendorfer [110], Fudenberg and Levine [105], O'Donoghue and Rabin [192, 193]. For example, the marginal rate of substitution between immediate consumption and some later consumption is different from what it was when these two dates were seen from a remote prior date. In many applications, these time-inconsistent preferences introduce a conflict between 'an impatient *present self* and a patient *future self*', see Brutscher [43]. In [233], where this phenomenon was first treated, three different types of agents are described: the pre-committed agent does not revise her initially decided strategy even if that makes her strategy time-inconsistent; the naive agent revises his strategy without taking future revisions into account even if that makes her strategy time-inconsistent; the sophisticated agent revises her strategy taking possible future revisions into account, and by avoiding such makes her strategy time-consistent.

Which type is more relevant depends on the entire framework of the decision in question. Marín-Solano and Navas [179] comment on which strategies the three different types of agents should use, see also Harris and Laibson [117] and Vieille and Weibull [242] for explanations of the deep mathematical problems arising in seemingly benign situations, such as non uniqueness. Indeed, in some applications one is interested in the rational decision-maker who *pre-commits* his future behaviour by precluding future options and conforming to his present desire, for instance individuals who make irrevocable trusts or buy life insurance. This is in stark contrast to the one who, aware of his inconsistency, searches for strategies where the inconsistency is anticipated and embedded in her decision plan. In this thesis we are interested in the latter type.

Let us illustrate these ideas in the context of contract theory, in line with the example at the beginning of this section. Suppose an online ride-sharing platform wants to revise its contracts scheme as it has noticed a decline in the frequency drivers are available in the platform. The company is considering to make available different bonus packs whose access depends on the number of rides fulfilled. In such scenario one could find, due to time-inconsistency, how bonus plans can be designed so as to motivate drivers to fulfil rides in a short amount of time. This is similar in spirit as to how, with credit cards and non-traditional mortgages, consumers are motivated to repay their loan fast, and why delaying repayment carries large penalties, see Heidhues [125] and Eliaz and Splieger [89]. Now, provided empirical evidence confirms drivers indeed have time-inconsistent preferences, in light of the dynamic programming approach presented above one must develop the tools in order to thoroughly understand the decision-making behind time-inconsistent sophisticated drivers before addressing the contracting situation mentioned above.

The study of time-inconsistency has a long history. The game-theoretic approach started with [233] where the phenomenon was introduced in a continuous-time setting, and it was proved that preferences are time-consistent if, and only if, the discount factor representing time preferences is exponential with a constant discount rate. Pollak [206] gave the right solution to the problem for both naive and sophisticated agents under a logarithmic utility function. For a long period of time, most of the attention was given to the discrete-time setting introduced by Phelps and Pollak [203]. This was, presumably, due to the unavailability of a well-stated system of equations providing a general method for solving the problem, at least for sophisticated agents. Nonetheless, the theory for time-inconsistent problems for sophisticated agents progressed, and results were extended to new frameworks, although this was mostly on a case-by-case basis. For example, Barro [18] studied a modified version of the neoclassical growth model by including a variable rate of time preference, and [163] considered the case of quasi-hyperbolic time preferences. Notably, Basak and Chabakauri [19] treated the mean–variance portfolio problem and derived its time-consistent solution by cleverly decomposing the nonlinear term and then applying dynamic programming. In addition, Goldman [107] presented one of the first proofs of existence of an equilibrium under quite general conditions. More recently, [242] showed how for infinite horizon dynamic optimisation problems with non-exponential discounting, the multiplicity of solutions (with different pay-offs) was the rule rather than the exception.

To treat these problems in a systematic way, the series of works carried out by Ekeland and Lazrak [80, 81], and Ekeland and Pirvu [82] introduced and characterised the first notion of sub-game perfect equilibria in continuous-time, where the source of inconsistency is non-exponential discounting. [80] consider a deterministic setting, whereas [81] extend these ideas to Markovian diffusion dynamics. In [82], the authors provide the first existence result in a Markovian context encompassing the one in their previous works. This was the basis for a general Markovian theory developed by Björk and Murgoci [35] in discrete-time and Björk, Khapko, and Murgoci [38] in continuous-time. Inspired by the notion of equilibrium in [81] and their study in the discrete-time scenario in [35], in [38] the authors consider a general Markovian framework with diffusion dynamics for the controlled state process X, and provide a system of PDEs whose solution allows to construct an equilibrium for the problem. Recently, He and Jiang [120] fills in a missing step in [38] by deriving rigorously the PDE system and refining the definition of equilibrium while Lindensjö [173] shows that solving the PDE system is a necessary condition for a refinement of the notion of equilibrium which enforces additional regularity.

Simultaneously, extensions have been considered, and unsatisfactory seemingly simple scenarii have been identified. Björk, Murgoci, and Zhou [36] study the time-inconsistent version of the portfolio selection problem for diffusion dynamics and a mean–variance criterion. Czichowsky [65] considers an extension of this problem for general semi-martingale dynamics. Hu, Jin, and Zhou [135, 136] provide a rigorous characterisation of the linear-quadratic model, and Huang and Zhou [137] perform a careful study in a Markov chain environment. Regarding the expected utility paradigm, Karnam, Ma, and Zhang [151] introduce the idea of the *dynamic utility* under which an original time-inconsistent problem (under the originally fixed utility) becomes a time-consistent one. He, Strub, and Zariphopoulou [123] propose the concept of forward rank-dependent performance processes, by means of the notion of conditional nonlinear expectation introduced by Ma, Wong, and Zhang [178], to incorporate probability distortions without assuming that the model is fully known at the initial time. One of the first negative results was introduced by Landriault, Li, Li, and Young [164], here the authors present an example, stemming from a mean–variance investment problem, in which uniqueness of the equilibrium via the PDE characterisation of [35] fails.

A different approach is presented in Yong [270] and Wei, Yong, and Yu [260], where, in the framework of recursive utilities, an equilibrium is defined as a limit of discrete-time games leading to a system of FBSDEs. Building upon the analysis in [260], Wang and Yong [245] consider the case where the cost functional is determined by a so-called backward stochastic Volterra integral equation (BSVIE) which covers the general discounting situation with a recursive feature. An HJB equation is associated in order to obtain a verification result. Moreover, Wang and Yong [245] establish the well-posedness of the HJB equation and derive a probabilistic representation in terms of a novel type of BSVIEs. Han and Wong [116] study the case where the state variable follows a Volterra process and, by associating an extended path-dependent Hamilton–Jacobi–Bellman equation system, obtains a verification theorem. Hamaguchi [115] provides a necessary condition for an open-loop equilibria in a Markovian time-inconsistent consumption–investment problem. Finally, Mei and Zhu [184] deals with a class of time-inconsistent control problems for McKean–Vlasov dynamics which are, for example, a natural framework to study mean–variance problems.

When it comes to time-inconsistent stopping problems, recent works have progressed significantly in understanding this setting, yet many peculiarities and questions remain open. A novel treatment of optimal stopping for a Markovian diffusion process with a payoff functional involving probability distortions, for both naïve and sophisticated agents, is carried out by Huang, Nguyen-Huu, and Zhou [139]. Huang and Zhou [138] consider a stopping problem under non-exponential discounting, and looks for an *optimal equilibrium*, one which generates larger values than any other equilibrium does on the entire state space. He, Hu, Obłój, and Zhou [122] study the problem of a pre-committed gambler and compare his behaviour to that of a naïve one. Another series of works is that of Christensen and Lindensjö [55]; Christensen and Lindensjö [54, 53] and Bayraktar, Zhang, and Zhou [20]. [55] study a discrete-time Markov chain stopping problem and propose a definition of sub-game perfect Nash equilibria for which necessary and sufficient equilibrium conditions are derived, and an existence result is obtained. They extend their study to the continuous-time setting by considering diffusion dynamics in [54], and study in [53] a moment constrained version of the optimal dividend problem for which both the pre-committed and sub-game perfect solutions are studied. Independently, [20] studied a continuous Markov chain process and proposed another notion of equilibrium. The authors thoroughly obtain the relation between the notions of optimal-mild, weak and strong equilibrium introduced in [139], [54] and [20], respectively, and provide a novel iteration method which directly constructs an optimal-mild equilibrium bypassing the need to find first all mild-equilibria. Notably, Tan, Wei, and Zhou [237] gives an example of nonexistence of an equilibrium stopping plan. On the other hand, Nutz and Zhang [191] provide a first approach to the recently introduced conditional optimal stopping problems which are inherently time-inconsistent.

In light of the limitations of the existing literature and keeping in mind the dynamic programming approach, presented in Section 1.1.1, it is clear that our analysis needs to start at the level of the problem faced by a generic time-inconsistent agent in a non-Markovian framework. The purpose of the next section is to review some of the main results in the literature which remained focused on the Markovian case.

1.2.1 Existing results for Markovian time-inconsistent problems

In this section, we present some of the results available in the literature in the Markovian case which are relevant to our goal. As a note to the reader, we will do our best to balance the main ideas, and the notation and specifics of the statements. That being said, they should nevertheless be considered as informal presentations. We will refer to the original sources accordingly. We also mention that a review encompassing most of the classical results, as well as some recent development in the time-inconsistent literature is available in He and Zhou [121].

The existence literature on time-inconsistent control has focused on the strong formulation of the problem in a Markovian framework. Given $T > 0$, on the time interval $[0, T]$ a fixed filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}_T, \mathbb{P})$ supporting a Brownian motion W is given. Here, $\mathbb{F}$ denotes the $\mathbb{P}$-augmented Brownian filtration. Let $\mathcal{A}$ denote the set of admissible actions. For a process $\nu \in \mathcal{A}$, representing an action process, the state process X is given by the unique strong solution to the $\mathbb{R}^n$-valued SDE

$$X_t^{r,\mathrm{x},\nu} = \mathrm{x} + \int_r^t b_s(X_s^{0,\mathrm{x},\nu}, \nu_s)\mathrm{d}s + \int_r^t \sigma_s(X_s^{0,\mathrm{x},\nu}, \nu_s)\mathrm{d}W_s, \text{ for } t \in [r, T].$$

The agent's reward is given, for $(t, \mathrm{x}, \nu) \in [0, T] \times \mathbb{R}^n \times \mathcal{A}$, by

$$J(t, \mathrm{x}, \nu) = \mathbb{E}\bigg[\int_t^T f_r(t, \mathrm{x}, X_r^{t,\mathrm{x},\nu}, \nu_r)\mathrm{d}r + \xi(t, \mathrm{x}, X_T^{t,\mathrm{x},\nu})\bigg|\mathcal{F}_t\bigg] + G\big(t, \mathrm{x}, \mathbb{E}[X_T^{t,\mathrm{x},\nu}|\mathcal{F}_t]\big),$$

where $\mathbb{E}[\cdot|\mathcal{F}_t]$ denotes the expectation operator conditional on $X_t^{t,\mathrm{x},\nu} = \mathrm{x}$. Let us remark that the dependence of f, ξ (resp. of G) on (t, x) (resp. on $(t, \mathrm{x}, \mathbb{E}[X_T^{t,\mathrm{x},\nu}|\mathcal{F}_t]))$ are the sources of time-inconsistency. In the game-theoretic approach, the individual reconciles his current and future preferences by seeking for an equilibrium as defined next.

Definition (Equilibrium [82])**.** *$\nu^\star \in \mathcal{A}$ is said to be an equilibrium if for all $(t, \ell, \mathrm{x}, \nu) \in [0, T] \times (0, T-\ell] \times \mathbb{R}^n \times \mathcal{A}$, and $\nu \otimes_{t+\ell} \nu^\star := \nu \mathbf{1}_{[t,t+\ell)} + \nu^\star \mathbf{1}_{[t+\ell,T]}$,*

$$\liminf_{\ell \searrow 0} \frac{J(t, \mathrm{x}, \nu^\star) - J(t, \mathrm{x}, \nu \otimes_{t+\ell} \nu^\star)}{\ell} \geq 0.$$

For an equilibrium $\nu^\star$ we write $\nu^\star \in \mathcal{E}$, and define the value function $v(t, \mathrm{x}) := J(t, \mathrm{x}, \nu^\star)$.

Analogue to the HJB equation for classic time-consistent stochastic control problem, an extended PDE system is available to study the time-inconsistent analogue in a systematic way. We follow [38]. Let ν be a Markovian feedback control, *i.e.* $\nu_t = u(t, X_t^{0,\mathrm{x},\nu})$ for an function $u : [0, T] \times \mathbb{R}^n \longrightarrow A$, $a \in A$, and a smooth function $\varphi : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$. We define $b_t^a(\mathrm{x}) := b_t(\mathrm{x}, a)$, $\sigma_t^a(\mathrm{x}) := \sigma_t(\mathrm{x}, a)$, $b_t^\nu(\mathrm{x}) := b_t(\mathrm{x}, u(t, \mathrm{x}))$, $\sigma_t^\nu(\mathrm{x}) := \sigma_t(\mathrm{x}, u(t, \mathrm{x}))$, and the operators

$$(\mathrm{A}^\nu \varphi)(t, \mathrm{x}) := \partial_t \varphi(t, \mathrm{x}) + b_t^\nu(\mathrm{x}) \cdot \partial_\mathrm{x} \varphi(t, \mathrm{x}) + \frac{1}{2} \mathrm{Tr}\big[\sigma^\nu(\mathrm{x}) \sigma_t^{\nu\top}(\mathrm{x}) \partial_{\mathrm{xx}} \varphi(t, \mathrm{x})\big],$$
$$(\mathrm{A}^a \varphi)(t, \mathrm{x}) := \partial_t \varphi(t, \mathrm{x}) + b_t^a(\mathrm{x}) \cdot \partial_\mathrm{x} \varphi(t, \mathrm{x}) + \frac{1}{2} \mathrm{Tr}\big[\sigma^a(\mathrm{x}) \sigma_t^{a\top}(\mathrm{x}) \partial_{\mathrm{xx}} \varphi(t, \mathrm{x})\big].$$

Moreover, for functions $G(t, \mathrm{x}, \mathrm{y})$ and $g(t, \mathrm{x})$, we introduce $(G \diamond g)(t, \mathrm{x}) := G(t, \mathrm{x}, g(t, x))$, and the operator $\mathrm{H}^a g(t, \mathrm{x}) := \partial_y G(t, \mathrm{x}, g(t, \mathrm{x}))(\mathrm{A}^a g)(t, \mathrm{x})$.

Definition (Extended HJB [38])**.** *We define smooth function v, J, and g via the equations*

(i) the v function is determined by, $v(T, \mathrm{x}) = \xi(T, \mathrm{x}, \mathrm{x}) + G(T, \mathrm{x}, \mathrm{x})$, and

$$\sup_{a \in A} \Big\{ (\mathrm{A}v)(t, \mathrm{x}) + f(t, \mathrm{x}, t, \mathrm{x}, a) - (\mathrm{A}^a J)(t, \mathrm{x}, t, \mathrm{x}) + (\mathrm{A}^a J^{t,\mathrm{x}})(t, \mathrm{x}) - (\mathrm{A}^a G \diamond g)(t, \mathrm{x}) + (\mathrm{H}^a g)(t, \mathrm{x}) \Big\} = 0;$$

(ii) for $(s, \mathrm{y}) \in [0, T] \times \mathbb{R}^n$, the function $J^{s,\mathrm{y}}(t, x)$ is determined by, $J^{s,\mathrm{y}}(T, \mathrm{x}) = F(s, \mathrm{y}, \mathrm{x})$, and

$$(\mathrm{A}^{\nu^\star} J^{t,\mathrm{y}})(t, \mathrm{x}) + f(s, \mathrm{y}, t, \mathrm{x}, u^\star(t, \mathrm{x})) = 0;$$

(iii) the function $g(t,\mathrm{x})$ *is determined by,* $g(T,\mathrm{x})=x$, *and*

$$(\mathrm{A}^{\nu^\star} g)(t,\mathrm{x})=0,$$

where $\nu^\star$ *denotes the Markovian feedback control induced by the mapping* $u^\star(t,\mathrm{x})$ *that realises the supremum in* (*i*).

Several comments are in order: it is immediate to the eye that instead of a single equation for the value function, as it is the case for time-consistent problems, in the game-theoretic approach the so-called extended HJB equation consists of a system equations. We remark that the second equation defines a infinite family of functions $(J^{s,\mathrm{y}})_{(s,\mathrm{y})\in[0,T]\times\mathbb{R}^n}$. Second, the system is fully coupled in the sense that the entire family $(J^{s,\mathrm{y}})_{(s,\mathrm{y})\in[0,T]\times\mathbb{R}^n}$ as well as the function $g(t,x)$ appear in the first equation and determine the action $\nu^\star$ which in turn appears in the definition of the second and third equation. Lastly, we notice that in the absence of time-inconsistency the previous system reduces to the classic HJB equation. Indeed, if f does not depend on $(s,\mathrm{y})\in[0,T]\times\mathbb{R}^n$ and G is linear in y then the G term can be added into the conditional expectation of the reward J. It then follows that J coincides with the value function. This is, the second and third equation are redundant and we are left with the classical HJB equation.

All things considered, the previous system yields a verification result analogous to the one for classic time-consistent control problems.

Theorem (Verification [38])**.** *Assume that for all* $(s,\mathrm{y})\in[0,T]\times\mathbb{R}^n$, *the functions* $v(t,\mathrm{x})$, $J^{s,\mathrm{y}}(t,\mathrm{x})$, $g(t,\mathrm{x})$ *and* $u^\star(t,\mathrm{x})$ *have the following properties:*

(i) $v, J^{s,\mathrm{y}}$, *and* g, *are smooth solutions to the extended* HJB *system*;

(ii) the function $u^\star(t,x)$ *realizes the supremum in the extended* HJB *and* $\nu^\star=u^\star(\cdot,X_\cdot)\in\mathcal{A}$;

(iii) $v, J, J^{s,\mathrm{y}}, g$, *and* $G\diamond g$, *have appropriate integrability.*

Then $\nu^\star$ *is an equilibrium law, and* v *is the corresponding equilibrium value function. Furthermore,* $J^{s,\mathrm{y}}$ *and* g *have the probabilistic representations*

$$J^{s,\mathrm{y}}(t,\mathrm{x})=\mathbb{E}\bigg[\int_t^T f_r(s,\mathrm{y},X_r^{t,\mathrm{x},\nu^\star},\nu_r^\star)\mathrm{d}r+\xi(s,\mathrm{y},X_T^{t,\mathrm{x},\nu^\star})\bigg|\mathcal{F}_t\bigg],\ g(t,\mathrm{x})=\mathbb{E}[X_T^{t,\mathrm{x},\nu^\star}|\mathcal{F}_t],\ t\in[0,T].$$

In simple terms, the previous theorem provides a sufficient set of conditions that certifies whenever an action is an equilibrium and identifies its corresponding value function. Regularity assumptions aside, we remark that for the purposes of solving a time-inconsistent control problem, *i.e.* finding an equilibrium and its value function, the previous result does the job.

Another matter of interest bring us back to the very own definition of the sophisticated agent. Let us recall, this agent revises his strategy taking possible future revisions into account, and by avoiding such makes his strategy time-consistent. Therefore, one expects some form of dynamic programming principle to hold for this kind of problems. This is the purpose of [37, Proposition 8.1], which provides a link between time-inconsistent and time-consistent Markovian control problems. Recall this is a crucial, actually the initial, step in the dynamic programming approach of [64], see Section 1.1.1. Indeed, it allows to establish the reverse direction of the standard verification result, which we will refer to as a necessity result.

Proposition (A primer DPP [37])**.** *For every time inconsistent problem in the present framework there exists a standard, time consistent optimal control problem with the following properties:*

(i) *the optimal value function for the standard problem coincides with the equilibrium value function for the time inconsistent problem*;

(ii) *the optimal control for the standard problem coincides with the equilibrium control for the time inconsistent problem.*

[37, Proposition 8.1] states that (in the framework of strong formulation with Markovian feedback actions) given an equilibrium, as defined above, for any time-inconsistent stochastic control problem it is possible to associate a classical time-consistent optimal stochastic control problem which attains the same value and satisfies a dynamic programming principle. However, the argument layed down in [37, Proposition 8.1], which we have ommitted, assumes *a priori* a smooth solution to the above PDE system. This is, it presupposes that the value function associated to the equilibrium action and the decoupled pay-off functionals belong to $C^{1,2}([0,T]\times\mathbb{R}^d)$, and is in the spirit of the (easy direction) of the Feynman–Kac representation formula. This means that the class of equilibria for which the DPP used in [37] holds is actually a sub-class of the ones given by classical definition via the liminf presented above. Moreover, it was recently shown that these would correspond to *regular equilibria* as defined in in [173]. In fact, as explained in [173, Remark 3.9], regular equilibria are *a priori* required to be continuous in time and space. We remark that even for time-consistent problems the previous regularity assumptions on the value function and the control it is known to often not hold.

Even more critical in our view, in the context of classic time-consistent control, the argument in [37, Proposition 8.1] would be equivalent to assuming that the HJB equation has a smooth classical solution to prove the DPP. We find this line of argument to be quite atypical in the sense that: (*i*) this is usually done the other way around: the DPP allows one to show that the value function is related to the HJB PDE, usually in the viscosity sense; (*ii*) *quid* of the cases where the value function fails to be smooth, which are ubiquitous in the literature? In classical control problems, the DPP holds under mere Borel-measurability of the value function, see El Karoui and Tan [85, 86], and we do not feel that it is reasonable to assume smoothness *a priori* to prove it.

All in all, in order to obtain a useful necessity result one should obtain the underlying DPP circumventing these *a priori* assumptions on the value function of the associated equilibria. More generally, one should aim to obtain a proof of a DPP from first principles, by which we mean that by introducing a refinement on the notion of equilibrium a DPP can be establish as a direct consequence of it. Such approach, will open the door to a necessity result which remained mostly absent in the literature.

Indeed, the community quickly started to investigate such necessity type results hoping to arrive to a complete characterisation of equilibria and their value function. This is, both a necessary and sufficient condition for an admissible action to be an equilibrium. For this we follow [120] and introduce some additional notation.

For $(s,t,r,\mathrm{y},\mathrm{x},\nu) \in [0,T]^3 \times \mathbb{R}^n \times \mathcal{A}$, let $\hat f^{s,\mathrm{y},\nu}(t,\mathrm{x}) := \mathbb{E}_{t,\mathrm{x}}\big[f_t(s,\mathrm{y},\mathrm{x},u(t,\mathrm{x}))\big]$, $\hat f^{s,\mathrm{y},r}(t,\mathrm{x}) := \mathbb{E}_{t,x}\big[f^{s,\mathrm{y},\nu^\star}(r,X_r^{t,\mathrm{x},\nu^\star})\big]$, $\hat\xi^{s,\mathrm{y}}(t,x) := \mathbb{E}_{t,x}\big[\xi(s,\mathrm{y},X_T^{t,\mathrm{x},\nu^\star})\big]$, and

$$\Gamma^{s,\mathrm{y},\nu^\star}(t,\mathrm{x},\nu) := f^{s,\mathrm{y},\nu}(t,\mathrm{x}) - f^{s,\mathrm{y},\nu^\star}(t,\mathrm{x}) + \int_t^T (\mathrm{A}^\nu \hat f^{s,\mathrm{y},r})(t,\mathrm{x})\mathrm{d}r$$
$$+ (\mathrm{A}^\nu \xi^{s,\mathrm{y}})(t,x) + \partial_\mathrm{y} G(s,\mathrm{y},g(t,\mathrm{x}))(\mathrm{A}^\nu g)(t,x).$$

Theorem (Characterisation of Markovian equilibria [120]). *Under appropriate assumptions*

$$\liminf_{\ell \searrow 0} \frac{J(t,\mathrm{x},\nu^\star) - J(t,\mathrm{x},\nu \otimes_{t+\ell} \nu^\star)}{\ell} = -\Gamma^{t,\mathrm{x},\nu^\star}(t,\mathrm{x},\nu),\ \forall (t,\mathrm{x},\nu) \in [0,T] \times \mathbb{R}^n \times \mathcal{A}.$$

Moreover, $\nu^\star$ is an equilibrium if and only if

$$\Gamma^{t,\mathrm{x},\nu^\star}(t,\mathrm{x},a) \geq 0,\ \forall (t,\mathrm{x},a) \in [0,T) \times \mathbb{R}^n \times A.$$

The previous result is quite remarkable as [120] succeeds in characterising equilibria via the mapping Γ. Yet, we highlight that we have remained vague regarding the assumptions of the previous theorem. Indeed, the conditions under which such result holds are quite stringent, requiring for instance that the optimal action and the mappings involve in the definition of Γ to be differentiable in time and with spacial derivatives of polynomial growth. This relates to the fact that the argument in [120] makes a strong use of the infinitesimal generator operator and may, in general, limit the class of equilibria that are able to be identified by the previous result. Moreover, the previous characterisation bypasses the extended HJB system and makes no direct link between it and Γ.

For the purposes of the discussion in this chapter, the previous results summarise the state of the understanding of Markovian time-inconsistent control problems. According to [38, Section 10], the following remained as open research problems at the time of its writing:

- '[t]he present theory depends critically on the Markovian structure. It would be interesting to see what can be done without this assumption';
- '[a] related (hard) open problem is to prove existence and/or uniqueness for solutions of the extended HJB system';
- '[a]n open and difficult problem is to provide conditions on primitives which guarantee that the functions V and f are regular enough to satisfy the extended HJB system';
- '[a]nother problem is to give conditions on primitives which guarantee that the assumptions of the verification theorem are satisfied'.

To not get ahead of ourselves, we will just mention that this thesis makes contributions to each of these problems. We highlight that the third point above is making reference to a type of necessity result, which we stress again, is crucial in the dynamic programming approach of [64], see Section 1.1.1. As we mentioned earlier, one avenue to arrive at this result is to provide a DPP circumventing any kind of *a priori* assumptions on the value function of the associated equilibria. In this way, we can conduct the analysis in a non-Markovian framework, and, at the same time, we will remain at the greatest level of generality possible, *i.e.* without implicit assumptions on the class of equilibria into consideration.

In the following section, we present an informal description of our contributions to the time-inconsistent literature. In synthesis, inspired by the results of [64] in the context of contract theory, we take a probabilistic approach that allow us to, among other things, extends the results for Markovian models with time-inconsistency to the non-Markovian framework and provide answers to each of the previous items.

1.2.2 Contributions

In the second part of this thesis, we develop a probabilistic theory for continuous-time non-Markovian stochastic control problems which are inherently time-inconsistent. Our formulation is cast within the framework of a controlled non-Markovian forward stochastic differential equation, and a general objective functional. To illustrate our results, suppose the dynamics of the controlled state process are given as in Section 1.1.1. Suppose the utility drawn by the Agent, from an effort action ν at time $t \in [0,T]$ and past trajectory $x \in \mathcal{X}$[5], is given by

$$\mathrm{J}^{\mathrm{A}}(t,x,\nu) = \mathbb{E}^{\mathbb{P}^\nu}\bigg[f(T-t)\mathrm{U}_{\mathrm{A}}(\xi) - \int_t^T f(r-t)c_r(X_{\cdot\wedge r},\nu_r)\mathrm{d}r\bigg|\mathcal{F}_t\bigg],$$

where $f:[0,T]\longrightarrow\mathbb{R}$, $f(0)=1$, denotes a generic discount function. We remark that its presence in the terminal utility and the running cost, are the sources of inconsistency. We adopt a game-theoretic approach to study such problems, meaning that we seek for *sub-game perfect Nash equilibrium* points. As a first novelty of this work, we introduce and motivate a refinement on the

[5]To alleviate the notation we set $\mathcal{X} := \mathcal{C}^d$, which we recall denotes the space of $\mathbb{R}^d$-valued continuous functions on $[0,T]$ endowed with the sup topology

notion of equilibrium from which one can directly and rigorously establish an *extended dynamic programming principle*, in the same spirit as in the classical theory, which takes the following form.

Theorem (Extended DPP Chapter 2). *Let $\nu^\star \in \mathcal{E}$. For any stopping times $\sigma \leq \tau$,* $\mathbb{P}$–a.s.

$$\mathrm{V}^{\mathrm{A}}_\sigma = \sup_{\nu\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[\mathrm{V}^{\mathrm{A}}_\tau - \int_\sigma^\tau \bigg(c_r(X_{\cdot\wedge r},\nu_r) + \mathbb{E}^{\mathbb{P}^{\nu^\star}}\bigg[f'(T-r)\mathrm{U}_{\mathrm{A}}(\xi) - \int_r^T f'(u-r)c_u(X_{\cdot\wedge r},\nu^\star_u)\mathrm{d}u\bigg|\mathcal{F}_r\bigg]\bigg)\mathrm{d}r\bigg|\mathcal{F}_\sigma\bigg].$$

As an immediate consequence of this result, we can associate a system of BSDEs to study equilibria in the case of uncontrolled volatility, *i.e.* $\sigma_t(x,a) = \sigma_t(x)$. Let us introduce the system which for any $s \in [0,T]$ satisfies

$$\begin{aligned} \mathcal{Y}_t &= \mathrm{U}_{\mathrm{A}}(\xi) + \int_t^T \big(H_r(X_{\cdot\wedge r},\mathcal{Z}_r) - \partial Y^r_r\big)\mathrm{d}r - \int_t^T \mathcal{Z}_r\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.},\\ \partial Y^s_t &= -f'(T-s)\mathrm{U}_{\mathrm{A}}(\xi) + \int_t^T \nabla h^\star_r(s,X_{\cdot\wedge r},\partial Z^s_r,\mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial Z^s_r\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.,} \end{aligned} \tag{$\mathcal{H}$}$$

where

$$\begin{aligned} H_t(x,z) &:= \sup_{a\in A}\big\{\sigma_t(x)b_t(x,a)\cdot z - c_t(x,a)\big\},\\ \nabla h^\star_t(s,x,z,\mathrm{z}) &:= \sigma_t(x)b_t(x,a^\star(t,x,\mathrm{z}))\cdot z + f'(t-s)c_t(x,a^\star(t,x,\mathrm{z})), \end{aligned}$$

and $a^\star(t,x,z)$ denotes the unique, by assumption, maximiser in H. The previous system is of an infinite-dimensional nature as the second equation induces a family of BSDEs, one for every $s\in[0,T]$. Moreover, it is fully-coupled as the diagonal term of the family $(\partial Y^s)_{s\in[0,T]}$ appears in the generator of the first equation and $\mathcal{Z}$ appears, via $a^\star$, in the generator of the family of BSDEs. Be it as it may, we are able to show that: (*i*) ($\mathcal{H}$) is of both sufficient and necessary to characterise equilibria $\alpha^\star\in\mathcal{E}$, in particular our results establish that any equilibria must arise from a solution to the system, and thus, the existence of an equilibria is equivalent to the existence of a solution to ($\mathcal{H}$); (*ii*) $\mathcal{Y}$ coincides with the value function V^{A}, and $\alpha^\star$ always arises as a maximiser of the Hamiltonian; and (*iii*) ($\mathcal{H}$) is well-defined, and consequently, in the case of drift control, there exists a unique equilibria.

Let us mention that ($\mathcal{H}$) extends naturally to a system incorporating second-order BSDEs (2BSDEs) in the case the agent is allowed to control the volatility. Indeed, the extended DPP holds true in the case the Agent controls the volatility as well. For such system we are able to show (i) and (ii) still hold. Nevertheless, (iii) becomes a much more delicate matter as the existence of a solution requires the existence of an optimal measure $\mathbb{P}^\star$. This problem is worth of analysis in the future, as we discuss later in this chapter, see Section 1.5. As a final comment, we also address the extensions of $(\mathcal{H})$ to the case of more general rewards. In particular, to those with mean–variance type of criteria which are indispensable, for example, in applications in portfolio selection, or energy consumption management.

Lastly, we remark that a central part of our analysis is based on establishing that letting $h_t^\star(s,x,z,\mathrm{z}) := \sigma_t(x)b_t(x,a^\star(t,x,\mathrm{x}))\cdot z - f(t-s)c_t(x,a^\star(t,x,\mathrm{x}))$, given a solution to $(\mathcal{H})$ we can introduce the family of BSDEs

$$Y_t^s = f(T-s)\mathrm{U_A}(\xi) + \int_t^T h_r^\star(s, X_{\cdot\wedge r}, Z_r^s, \mathcal{Z}_r)\mathrm{d}r - \int_t^T Z_r^s \cdot \mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

and verify that

$$\mathrm{V}_t^{\mathrm{A}} = \mathcal{Y}_t = Y_t^t,\ t\in[0,T],\ \mathbb{P}\text{–a.s., and, } \mathcal{Z}_t = Z_t^t,\ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e.}$$

This is, (Y,Z) prescribe a solution to a so called type-I backward stochastic Volterra integral equation (BSVIE) of the form

$$Y_t^s = f(T-s)\mathrm{U_A}(\xi) + \int_t^T h_r^\star(s, X_{\cdot\wedge r}, Z_r^s, Z_r^r)\mathrm{d}r - \int_t^T Z_r^s \cdot \mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

In the context of Section 1.1.1, the extended DPP and the previous observation are the analogous version of **Step 1** in the time-inconsistent case. This is, we have successfully provided an extended DPP and are able to identify the natural probabilistic objects, analogous to the BSDE, that are associated with it. Moreover, in Chapter 2 we show that in the absence of time-inconsistency, *i.e.* when $f(t)=\mathrm{e}^{-\rho t}$, the extended DPP (resp. the system $(\mathcal{H})$ or, equivalently, the previous BSVIE) reduces to the classic DPP (resp. the BSDE in Section 1.1.1).

The link between the previous time-inconsistent control problem and type-I BSVIEs was the motivation of our analysis in the following part of the thesis were we explore the well-posedness of a general class of type-I BSVIEs in both the case of Lipschitz and quadratic generators.

1.3 Part III: backward stochastic Volterra integral equations

Seeking to understand the parallel stream of works that study time-inconsistency via BSVIEs, in Part III we questioned the extent to which our approach via system $(\mathcal{H})$ relates to those via BSVIEs of the form described at the end of the previous section. Building upon the strategy devised in Chapter 2, we addressed the well-posedness of a general and novel class of multidimensional type-I BSVIEs, that we coined extended BSVIEs.

BSVIEs are regarded as natural extensions of backward stochastic differential equations, BSDEs for short. On a complete filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$, supporting an n-dimensional Brownian motion B, and denoting by $\mathbb{G}$ the $\mathbb{P}$-augmented natural filtration generated by B, one is given data, that is to say a $\mathcal{G}_T$-measurable random variable ξ, and a mapping g, referred to respectively as the *terminal condition* and the *generator*. A solution to a BSDE is a pair of $\mathbb{G}$-adapted processes $(Y_\cdot, Z_\cdot)$ such that

$$Y_t = \xi + \int_t^T g_r(Y_r, Z_r)\mathrm{d}r - \int_t^T Z_r \mathrm{d}B_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.} \tag{1.3.1}$$

BSDEs of linear type were first introduced by Bismut [31, 32] as an adjoint equation in the Pontryagin stochastic maximum principle. Actually, the contemporary work of Davis and Varaiya [67][6] studied a precursor of a linear BSDE for characterising the value function and the optimal controls of stochastic control problems with drift control only. In the same context of the stochastic maximum principle, BSDEs of linear type are present in Arkin and Saksonov [12], Bensoussan [22] and Kabanov [146]. Remarkably, the extension to the non-linear case is due to Bismut [33], as a type of Riccati equation, as well as Chitashvili [50], and Chitashvili and Mania [51, 52]. Later, the seminal work of Pardoux and Peng [201] presented the first systematic treatment of BSDEs in the general nonlinear case, while the celebrated survey paper of El Karoui, Peng, and Quenez [88] collected a wide range of properties of BSDEs and their applications to finance. Among such

[6] Indeed, [67] was received for publication on October 27, 1971 and it is part of the bibliography of [32].

properties we recall the so-called *flow property*, that is to say, for any $0 \leq r \leq T$,

$$Y_t(T,\xi) = Y_t(r, Y_r(T,\xi)),\ t \in [0,r],\ \mathbb{P}\text{–a.s., and } Z_t(T,\xi) = Z_t(r, Y_r(T,\xi)),\ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e. on } [0,r]\times\Omega,$$

where $(Y(T,\xi), Z(T,\xi))$ denotes the solution to the BSDE with terminal condition ξ and final time horizon T.

A natural extension of (1.3.1) arises by considering a collection of $\mathcal{G}_T$-measurable random variables $(\xi(t))_{t\in[0,T]}$, referred in the literature of BSVIEs as the *free term*, as well as a generator g. In such a setting, a solution to a BSVIE is a pair $(Y_\cdot, Z_\cdot^\cdot)$ of processes such that

$$Y_t = \xi(t) + \int_t^T g_r(t, Y_r, Z_r^t, Z_t^r)\mathrm{d}r - \int_t^T Z_r^t \mathrm{d}B_r,\ \mathbb{P}\text{–a.s.},\ t \in [0,T]. \tag{1.3.2}$$

Of noticeable interest is the case in which the term Z_t^r is absent in the generator, *i.e.*

$$Y_t = \xi(t) + \int_t^T g_r(t, Y_r, Z_r^t)\mathrm{d}r - \int_t^T Z_r^t \mathrm{d}B_r,\ \mathbb{P}\text{–a.s.},\ t \in [0,T]. \tag{1.3.3}$$

Nowadays (1.3.3) and (1.3.2) are referred in the literature as type-I and type-II BSVIEs, respectively. The first mention of such equations is, to the best of our knowledge, due to Hu and Peng [132]. Indeed, in the context of well-posedness of BSDEs valued in a Hilbert space, a prototype of type-I BSVIEs (1.3.3) is considered, see the comments following [132, Remark 1.1]. Two decades passed before a direct consideration of BSVIEs of the form given by (1.3.3) was made by Lin [171], where the author studied the case $\xi(t) = \xi$, $t \in [0,T]$, for a $\mathcal{G}_T$-measurable ξ. The general form of (1.3.2) was first addressed in Yong [267, 269] in the context of optimal control of (forward) stochastic Volterra integral equations (FSVIEs, for short).

There are significant distinctions between BSDEs and BSVIEs. Nevertheless, a satisfactory concept of solution for such equations can be defined by extrapolating from the theory of BSDEs. In broad terms, a pair $(Y_\cdot, Z_\cdot^\cdot)$ is said to be a solution to a BSVIE, see [269], if for each $s \in [0,T)$, the mapping $t \longmapsto (Y_t, Z_t^s)$ is $\mathbb{G}$-adapted on $[s,T]$, (Y,Z) is appropriately integrable and satisfies (1.3.2). It is also worth pointing out the distinctions between type-I and type-II BSVIEs. As a consequence of the presence of Z_t^s in the generator, to obtain a solution to a type-II BSVIE one has to determine Z_t^s for $(t,s) \in [0,T]^2$, and (1.3.2) alone does not give enough restrictions. Indeed, [269] showed that an adapted solution to the type-II BSVIE (1.3.2) may, in general, not be unique. This is in contrasts with type-I BSVIEs, where it suffices to determine Z_t^s for $(t,s) \in [0,T]^2$, $0 \leq s \leq t \leq T$. Moreover, without additional assumptions, a solution to a general type-II BSVIE does not satisfy the flow property.

Since their introduction, BSVIEs have been extended to much more general frameworks than the one presented above. Hence, Wang [248] studies the case of random Lipschitz data; Wang and Zhang [258] and Shi and Wang [225] deal with general non-Lipschitz data; Coulibaly and Aman [58] study time-delayed generators; *mean-field* BSVIEs are considered in Shi, Wang, and Yong [226]; Lu [176], Hu and Øksendal [131], Overbeck and Röder [196] and Popier [207] studied extensions to general filtrations and the case where B is replaced by more general processes; infinite horizon BSVIEs are investigated in Hamaguchi [113]; Djordjević and Janković [79, 78] were interested in perturbed BSVIEs, *i.e.* when the coefficients depend additively on small perturbations; BSVIEs in Hilbert spaces have been investigated in Anh and Yong [10], Anh, Grecksch, and Yong [11], and Ren [214]; and an analysis of numerical schemes for BSVIEs has been proposed in Bender and Pokalyuk [21]. There is also a wide spectrum of applications of BSVIEs. Hence, dynamic risk measures have been considered in Yong [268], Wang and Shi [252, 253], Wang, Sun, and Yong [246] and Agram [2]. Kromer and Overbeck [159] also studied the question of dynamic capital allocations via BSVIEs. Wang and Shi [251] dealt with a risk minimisation problem by means of the maximum principle for FBSVIEs, while the optimal control of SVIEs and BSVIEs via the maximum principle has been studied in Chen and Yong [47], Wang [250], Agram, Øksendal, and Yakhlef [4, 5], Shi, Wang, and Yong [227], Shi, Wen, and Xiong [228], see also Wei and Xiao [259] for the case with state constraints.

Since their first appearance, a natural and non-trivial question for BSVIEs has been that of the regularity in time of their solutions. The best known probabilistic results for general type-II BSVIEs guarantee the regularity of the solutions in an $\mathbb{L}^p$ sense only, see Wang [249] and Li, Wu, and Wang [169]. Nevertheless, analytic results via a representation formula, guarantee the pathwise regularity of a solution to type-I BSVIEs, see Wang and Yong [255] and Wang, Yong, and Zhang [247] for results regarding the representation of BSVIEs in the Markovian and non-Markovian framework, respectively. Regarding BSVIEs driven by discontinuous processes, it is known that type-I BSVIEs are known to be much more amenable to the analysis, for example [207; 258] are able to study the regularity of type-I BSVIEs with jumps by probabilistic methods.These results stress the fact that extending these ideas to type-II BSVIEs is a more challenging task. General discontinuous BSVIEs are out of the scope of the current document.

Out of the class of processes described by BSVIEs, a broader family than that of standard type-I BSVIEs (1.3.3) is known to arise in the study of time-inconsistent control problems. Recently, Agram and Djehiche [3] studied reflected backward stochastic Volterra integral equations and their relations to a time-inconsistent optimal stopping problem. Earlier connections were suggested in the concluding remarks of Wang and Yong [255]. Indeed, BSVIEs provide a probabilistic representations of the system of partial differential equation (PDE, for short) appearing in the study of time-inconsistent optimal control problems, e.g. see Yong [270] and Wei, Yong, and Yu [260] for PDEs obtained via Pontryagin's and Bellman's principle, respectively. A natural link was then made rigorous independently by Wang and Yong [245, Section 5] and Lemma 2.10.3.1 in Chapter 2. Although following different approaches, their analyses lead to introduce type-I BSVIEs of the form

$$Y_t = \xi(t) + \int_t^T g_r(t, Y_r, Z_r^t, Z_r^r)\mathrm{d}r - \int_t^T Z_r^t \mathrm{d}B_r,\ \mathbb{P}\text{–a.s.},\ t \in [0,T]. \tag{1.3.4}$$

These are BSVIEs in which the diagonal of Z appears in the generator. We highlight that, until the present work, the only well-posedness results in the literature for type-I BSVIEs (1.3.4) are available in [245] and Chapter 2. Both results hold for the particular case in which the driver g is linear in Z_r^t. Indeed, the argument in [245] follows as a consequence of the representation formula, *i.e.* an analytic argument via PDEs, and holds in a Markovian setting. On the other hand, the probabilistic argument in Chapter 2 holds in the non-Markovian case.

Likewise, Hamaguchi [112, 114] studied a time-inconsistent control problem where the cost functional is defined by the Y component of the solution of a type-I BSVIE (1.3.3), in which g depends on a control. Via Pontryagin's optimal principle, the author noticed that the adjoint equations correspond to an *extended* type-I BSVIE, as first introduced in Wang [244] in the context of generalising the celebrated Feynman–Kac formula. An extended type-I BSVIE consists of a pair

$(Y^{\cdot}_{\cdot}, Z^{\cdot}_{\cdot})$, with appropriate integrability, such that $s \longmapsto Y^s$ is continuous in an appropriate sense for $s \in [0,T]$, $Y^s_{\cdot}$ is pathwise continuous, $Z^s_{\cdot}$ is predictable, and

$$Y_t^s = \xi(s) + \int_t^T g_r(s, Y_r^s, Z_r^s, Y_r^r)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}B_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\ s \in [0,T]. \tag{1.3.5}$$

We highlight that the noticeable feature of (1.3.4) and (1.3.5) is the appearance of the 'diagonal' processes $(Y_t^t)_{t\in[0,T]}$ and $(Z_t^t)_{t\in[0,T]}$, respectively. A prerequisite for rigorously introducing these processes is some regularity of the solution. Indeed, the regularity of $s \longmapsto (Y^s, Z^s)$ in combination with the pathwise continuity of Y and the introduction of a derivative of Z^s with respect to s, as first discussed in Chapter 2, make the analysis possible.

Put succinctly, type-I BSVIEs, understood in a broader sense than that of (1.3.3), provide a rich framework to address new classes of problems in mathematical finance and control. In the case of time-inconsistent control problems, (1.3.4) and (1.3.5) appear as a consequence of the study of such problems via Bellman's and Pontryagin's principles, respectively.

We also remark that, to the best of our knowledge, there are no well-posedness results for multidimensional quadratic type-I BSVIEs as general as (1.3.4) or (1.3.5). In fact, the study of non-Lipschitz BSVIEs remains limited to Ren [214], Shi and Wang [225], Wang, Sun, and Yong [246], and Wang and Zhang [258]. In [258] and [214], the authors study solutions to general multidimensional type-I (1.3.3) and type-II (1.3.2) BSVIEs, respectively, where the generator is increasing and concave in y and Lipschitz in z. [225] continued the study of type-II BSVIEs (1.3.2) and settled some flaws in the analysis of [214]. On the other hand, [246] presents the first analysis of BSVIEs whose generator have quadratic growth on z. Indeed, the authors consider a standard one dimensional type-I BSVIE (1.3.3) in which the generator is Lipschitz in y and quadratic in z, which we will refer as the Lipschitz–quadratic case, provided the data of the BSVIE is bounded. The need to introduce this additional assumption is due to underlying results for quadratic BSDEs that were employed in [246].

Even in the case of one dimensional BSDEs, establishing their well-posedness in the case of generators with quadratic growth in z is known to be more delicate that in the Lipschitz case and requires to impose extra assumptions on the data of the problem. The first result in this setting was provided by Kobylanski [155] in the case of bounded and Lipschitz–quadratic data. Building upon the ideas in [155], Briand and Hu [41, 42] showed that imposing sufficiently large exponential moments on the terminal condition ξ are actually enough. We stress that this approach allows for unbounded terminal conditions but is limited to the one dimensional Lipschitz–quadratic case. BSDEs with superquadratic growth were studied in Lepeltier and San Martín [165] and Delbaen, Hu, and Bao [68]. On the other hand, the original method introduced in Tevzadze [238] takes a different view of this problem and is able to cover quadratic BSDEs, in both y and z, but requires, once again, the data to be bounded.

In the multidimensional case, new approaches become necessary as the tools that are usually used in the one dimensional case, like monotone convergence or Girsanov transformation, are no longer available. In fact, Frei and Dos Reis [103] provide a simple example of a multidimensional quadratic BSDE with a bounded terminal condition for which there is no solution. This counter-example shows that a direct generalisation of the approaches in [155; 41; 42] or the classic approach in [201] would be unsuccessful in the case of BSDEs and, naturally, of BSVIEs. Nevertheless, it is known that well-posedness results for BSDEs can be obtained exploiting the theory of BMO martingales and imposing further structural conditions on the generator. See, for instance, Cheridito and Nam [49] for specific choices of generators, Hu and Tang [133] for the case of diagonally quadratic generators, the generalisations in Jamneshan, Kupper, and Luo [143] and Kupper, Luo, and Tangpi [161], as well as Rouge and El Karoui [215] Hu, Imkeller, and Müller [134] for some applications. To unify these approaches, Harter and Richou [118] provides a well-posedness result by approximating the solution of a Lipschitz–quadratic BSDE, assuming the *a priori* existence of uniform estimates on the BMO norm of the local martingale $\int_0^\cdot Z_r^n \mathrm{d}W_r$ and exploiting the Malliavin calculus. In providing scenarri in which such estimates can be obtained [118] recover several results available in the literature. On the other hand, the methodology devised in [238] reveals that a positive answer can be stated if the data of the problem is bounded and sufficiently small. We also mention Frei [102] and Kramkov and Pulido [158] which provide generalisations to slightly more general terminal condition, as well as, the remarkable global existence result in Xing and Žitković [265] for Markovian quadratic BSDEs.

Extending the ideas in [41; 42] to the case of BSVIEs was the motivation behind [246] and it explains the framework of their result, *i.e.* scalar BSVIEs with Lipschitz quadratic generator a bounded data. Moreover, as [246, pp. 4] states: '[t]he case [of] Y being higher dimensional will be significantly different in general.' Therefore, it is not expected that these ideas would be able to cover the general multidimensional quadratic BSVIE. Fortunately, a close look at the argument provided in [238] reveals that a positive answer can be stated as soon as the data of the problem is bounded and sufficiently small. This will be the inspiration of our approach to address the quadratic case.

1.3.1 Contributions

In this part of the thesis we built upon the strategy devised in Chapter 2 and address the well-posedness of a general and novel class of type-I BSVIEs. We let X be the solution to a drift-less stochastic differential equation (SDE, for short) under a probability measure $\mathbb{P}$, and $\mathbb{F}$ be the $\mathbb{P}$-augmentation of the filtration generated by X, and consider a tuple $(Y^{\cdot}_{\cdot}, Z^{\cdot}_{\cdot}, N^{\cdot}_{\cdot})$, of $\mathbb{F}$-adapted processes, satisfying for every $s \in [0,T]$

$$Y_t^s = \xi(s) + \int_t^T g_r(s, X, Y_r^s, Z_r^s, Y_r^r, Z_r^r)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r - \int_t^T \mathrm{d}N_r^s,\ t \in [0,T],\ \mathbb{P}\text{–a.s.} \qquad (1.3.6)$$

We remark that the additional process N corresponds to a martingale process which is $\mathbb{P}$-orthogonal to X. This is a consequence of the fact that we work with a general filtration $\mathbb{F}$. To the best of our knowledge, a theory for type-I BSVIEs, as general as the ones introduced above, remains absent in the literature. Moreover, such class of type-I BSVIEs has only been mentioned in [114, Remark 3.8] as an interesting generalisation of (1.3.5).

Our approach is based on the following class of infinite families of BSDEs, which for every $s \in [0,T]$ satisfy

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T, X_{\cdot\wedge T}) + \int_t^T h_r(X_{\cdot\wedge r}, \mathcal{Y}_r, \mathcal{Z}_r, Y_r^r, Z_r^r, \partial Y_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\\
Y_t^s &= \eta(s, X_{\cdot\wedge T}) + \int_t^T g_r(s, X_{\cdot\wedge r}, Y_r^s, Z_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T V_r^{s\top} \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\\
\partial Y_t^s &= \partial_s \eta(s, X_{\cdot\wedge T}) + \int_t^T \nabla g_r(s, X_{\cdot\wedge r}, \partial Y_r^s, \partial Z_r^s, Y_r^s, Y_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial Z_r^{s\top} \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},
\end{aligned}$$

where $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, Y, Z, N, \partial Y, \partial Z, \partial N)$ are unknown, and required to have appropriate integrability.

As a first step, in Chapter 3 we worked under classical Lipschitz growth assumptions and showed that for an appropriate choice of η and h its well-posedness is equivalent to that of the type-I BSVIE (1.3.6). In addition, we recovered classical results for this general class of BSVIEs: (i) we provided *a priori* estimates; (ii) established the stability of solutions; and (iii) provided a representation formula in terms of a semilinear PDEs. Our approach can naturally be specialised to recover the version of these results for BSVIEs previously studied in the literature. Lastly, as it was the motivation for this project, we considered the game-theoretic approach to time-inconsistent stochastic control problems and showed that as a consequence of our results, one can reconcile the two current probabilistic approaches to this problem. Moreover, we provided an equivalent result for two earlier analytic approaches, based on semi-linear PDEs. We believe this helps to elucidate connections between the different takes on the problem available in the literature.

Our work in this part of the thesis does not end there. Some of the cornerstone models in the literature in stochastic control are based on either mean–variance or linear–quadratic models. In fact, we recall the results in [164], where the authors present a mean–variance investment problem in which uniqueness of the equilibrium via the PDE characterisation of [35] fails. Consequently, a natural extension of the results in Chapter 3 is the case of generator with quadratic growth. For example, such extension becomes necessary when dealing with applications to contract theory models were the agent has a cost which is quadratic in his effort. As such, this is the objective of Chapter 4, the second chapter of Part III.

Recall that in Chapter 3 and Chapter 2 we studied a system consisting of an infinite family of BSDEs in a Lipschitz framework. In general terms, by introducing the equivalent weighted versions of the classic integrability spaces in the literature on BSDEs and choosing a weight large enough we were able to obtain its well-posedness. In the setting of this paper, the Lipschitz assumption for the generators is abandoned. As a consequence, one cannot recover a contraction by simply choosing a weight large enough. In fact, given our growth assumptions, the usual candidate for providing a contractive map is no longer Lipschitz–continuous, but only locally Lipschitz-continuous. Fortunately, by exploiting the theory of bounded mean oscillating (BMO) martingales we can obtain well-posedness results in the quadratic case in two scenarii: (i) Lipschitz–quadratic generators in (Y, Z), respectively and data with sufficiently small norm; (ii) fully quadratic generators in (Y, Z)

and data with sufficiently small norm.

Our results in Chapter 4 follows the idea initially proposed in Tevzadze [238] for multidimensional quadratic BSDEs, namely, that we can localise the usual procedure to a ball of radius R, thus making the application Lipschitz-continuous again, and then to choose the radius of such a ball so as to recover a contraction. The crucial contribution of [238] was to show that such controls can be obtained by taking the norm of the data of the system to be small enough. The procedure we followed is inspired by this idea and incorporates it into the strategy devised in Chapter 3 to address the well-posedness of these kind of systems and of extended type-I BSVIEs. We decided to work on weighted spaces as, in our opinion, it does significantly simplify the arguments in the proof. We also mention that we estimated the greatest ball, *i.e.* the largest radius R, for which such a localisation procedure leads to a contraction.

Bringing ourselves back to the context of contract theory, we echo that the analysis in this part of the thesis does elucidate the connections between the different approaches to time-inconsistent control problem at the analytic level, between PDEs in the Markovian setting, and the probabilistic level, between $(\mathcal{H})$ and extended type-I BSVIEs in the non-Markovian counterpart. This in turn, leaves us educated enough to bring our attention back to the contracting problem.

1.4 Part IV: time-inconsistent contract theory

When it comes to incorporating time-inconsistent features into contract theory models, the economic literature is abundant in discrete-time models with two, and up to three periods. A common feature in this literature is adopting quasi-hyperbolic discounting structures to draw conclusions in different mechanism design problems. Yet, the method of resolution in each problem remained limited to a case-by-case analysis. For instance, Amador, Werning, and Angeletos [9] and Bond and Sigurdsson [39] study the feasibility of commitment in models of consumption and savings, whereas Galperti [106] considers the optimal provision of commitment devices to people who value both commitment and flexibility. Bisin, Lizzeri, and Yariv [30] examines policymakers' responses to the political demands of agents with self-control problems, Halac and Yared [111] looks into a fiscal policy model in which the government has time-inconsistent preferences, while Lim and Yurukoglu [170] assesses the effects of time-inconsistency on monopoly regulation of electricity distribution.

Heidhues [125] and Karaivanov and Martin [148] integrate time-inconsistent preferences into credit, mortgage and insurance contract design problems, respectively. Englmaier, Fahn, and Schwarz [94], Gottlieb [108] and Gottlieb and Zhang [109] study contracting problems between firms and sophisticated, partially naive and naive present-biased consumers. Yılmaz [271, 272] considers a repeated moral hazard problem involving a sophisticated and naive agent, respectively. Ma [177] studies a multi-period model in which contracts are subject to renegotiations, and the agent's action has a long-term effect. Balbus, Reffett, and Wozny [14] shows the existence of time-consistent equilibria for dynamic models with generalised discounting. A survey of some of the state of behavioural economics research in contract theory was provided in Kószegi [156].

In continuous-time, where the dynamic models are sometimes more tractable and the solutions enjoy better interpretability, the literature becomes rather scarce. Models dealing with a pre-committed agent have been considered in Li and Qiu [167], in which a non-constant exponential discount factor is the source of time-inconsistency, and Djehiche and Helgesson [75], where the agent is allowed to have mean–variance utility functions. The case of a sophisticated agent was considered in Li, Mu, and Yang [168], Liu, Mu, and Yang [174], Liu, Huang, Liu, and Mu [175] and Wang, Huang, Liu, and Zhang [257] in the case of hyperbolic discounting. However, the time-inconsistency is restricted in the sense that it manifests only at discrete random times that are exponentially distributed. Lastly, Cetemen, Feng, and Urgun [46] considers a Markovian continuous-time contracting problem and dynamic inconsistency arising from non-exponential discounting. The authors' examples are limited to the case of a principal having time-inconsistent preferences and the agent has standard time-consistent preferences. Altogether, a thorough analysis of the general non-Markovian continuous-time contracting problem between a standard utility maximiser principal and a sophisticated time-inconsistent agent is still missing in the literature. This is because, in our opinion, the crux of the problem lies in identifying a proper description of the problem of the principal. In the case of a classic time-consistent agent and a time-inconsistent principal, following [64], one expects the problem of the principal to boil down to a non-Markovian time-inconsistent control problem with an additional state variable. As studied in [38], these problems are characterised by the so called extended HJB equation. Therefore, we expect that the problem considered in this document will open the door to a complete analysis of the problem in which both the principal and the agent are time-inconsistent.

1.4.1 Contributions

The set-up is as follows: given a contract ξ, *i.e.* an $\mathcal{F}_T$-measurable random variable, the agent, allowed to control only the drift b, as in Section 1.1.1, receives a reward given by $\mathrm{J}^{\mathrm{A}}(\,\cdot\,;\xi)$, as in Section 1.2.2 and where we now emphasise its dependence on the contract ξ. We call $\mathcal{E}(\xi)$ the set of all equilibria associated to ξ. Moreover, the agent has a reservation utility $R_0 \in \mathbb{R}$ below which he refuses the contract ξ. We therefore let the set of admissible contracts be given by $\Xi := \{\xi : \mathrm{V}_0^{\mathrm{A}}(\xi,\alpha) \geq R_0, \text{ for } \alpha \in \mathcal{E}(\xi)\}$. The principal has her own utility function $\mathrm{U_P} : \mathcal{X} \times \mathbb{R} \longrightarrow \mathbb{R}$, and solves the infinite dimensional problem

$$\mathrm{V^P} := \sup_{\xi \in \Xi} \sup_{\alpha \in \mathcal{E}(\xi)} \mathbb{E}^{\mathbb{P}^\alpha}\big[\mathrm{U_P}\big(X_{\cdot\wedge T},\xi\big)\big].$$

In light of our results in Chapter 2, for $\xi \in \Xi$ the problem solved by the Agent is characterised by the solution to $(\mathcal{H})$, which we denote $(\mathcal{Y}(\xi), \mathcal{Z}(\xi), \partial Y(\xi), \partial Z(\xi))$. Nevertheless, the infinite dimensional nature of $(\mathcal{H})$ posed an obstacle for a direct application of the results in [64]. Fortunately, in Chapter 2 we identified a link between time-inconsistent control problems and BSVIEs. Such type of links appeared first in the concluding remarks of Wang and Yong [255] and then later in [245]. In our setting, such an equation is satisfied by the value of the agent along the equilibrium, namely $\mathcal{Y}(\xi)$, which coincides with the diagonal process $(Y_t^t(\xi))_{t\in[0,T]}$ of the family $(Y^s(\xi))_{s\in[0,T]}$ defined, for $s \in [0,T]$, by

$$Y_t^s(\xi) = f(T-s)\mathrm{U_A}(\xi) + \int_t^T h_r^\star(s, X_{\cdot\wedge r}, Z_r^s(\xi), Z_r^r(\xi))\mathrm{d}r - \int_t^T Z_r^s(\xi)\cdot \mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

The previous equation provides, in principle, infinitely many representations of ξ. Nevertheless, provided $\mathrm{U^A}$ has an inverse, it holds that

$$\frac{\mathrm{U_A^{(-1)}}(Y_T^s(\xi))}{f(T-s)} = \xi = \frac{\mathrm{U_A^{(-1)}}(Y_T^u(\xi))}{f(T-u)},\ \mathbb{P}\text{–a.s.},\ (s,u)\in[0,T]^2.$$

This is, understanding the properties of solutions to the BSVIE, we are able to identify the structure of the contracts that induce equilibria for the Agent. In light of our previous observation, we introduce a family of restricted contract payments $\overline{\overline{\Xi}}$. For any contract in this family, we can solve the associated time-inconsistent control problem faced by the agent. Moreover, we show that any admissible contract available to the principal admits a representation as a contract in $\overline{\overline{\Xi}}$. Consequently, the principal's optimal expected utility is not reduced if she restricts herself to offer contracts in this family and optimises. To present these results, we need to introduce the set $\mathcal{I} := \{y_0 \in \mathcal{C}_1^1 : y_0^0 \geq R_0\}$

Definition (Reduced family of contracts Chapter 5)**.** (i) *We denote by $\mathcal{H}^{2,2}$ the collection of processes Z, with appropriate integrability satisfying:*

for $y_0 \in \mathcal{I}$, there exists a process $Y^{y_0,Z}$, which satisfies for every $s \in [0,T]$,

$$Y_t^{s,y_0,Z} = y_0^s - \int_0^t h_r^\star(s, X_{\cdot\wedge r}, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r + \int_0^t Z_r^s \cdot \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.}$$

and

$$\frac{\mathrm{U}_\mathrm{A}^{(-1)}(Y_T^{s,y_0,Z})}{f(T-s)} = \xi = \frac{\mathrm{U}_\mathrm{A}^{(-1)}(Y_T^{u,y_0,Z})}{f(T-u)},\ \mathbb{P}\text{–a.s.},\ (s,u) \in [0,T]^2.$$

(ii) *We denote $\overline{\overline{\Xi}}$ the set of contracts of the form*

$$\mathrm{U}_\mathrm{A}^{(-1)}\Big(T, Y_T^{T,y_0,Z}\Big),\ (y_0, Z) \in \mathcal{I} \times \mathcal{H}^{2,2}.$$

The main novelties of our argument, compared to that in the time-consistent case are: (i) the agent's time-inconsistent preferences yield that the so-called continuation utility $Y^{y_0,Z}$ is now described by a forward Volterra equation; (ii) as such, the set $\mathcal{H}^{2,2}$ now consist of family of processes Z; (iii) lastly $\mathcal{H}^{2,2}$ imposes a constraint on its elements, and therefore, it is essential to verify that $\mathcal{H}^{2,2} \neq \emptyset$.

We are now ready to state our main result, in words it guarantees that there is no loss of generality for the principal in offering contracts of the form given by $\overline{\overline{\Xi}}$.

Theorem (Characterisation of principal's problem Chapter 5)**.** (i) *We have $\overline{\overline{\Xi}} = \Xi$. Moreover,* *for any contract* $\xi \in \overline{\Xi}$, *associated to* $(y_0, Z) \in \mathcal{I} \times \mathcal{H}^{2,2}$, *we have*

$$\mathcal{E}(\xi) = \big\{a^\star(t, X_{\cdot\wedge t}, Y_t^{t,y_0,Z}, Z_t^t)_{t\in[0,T]}\big\},\ \mathrm{V}_0^{\mathrm{A}}(\xi) = y_0^0.$$

(ii) Let $\mathbb{P}^\star(Z) := \mathbb{P}^{a^\star(\cdot, X_\cdot, Y_\cdot^{\cdot,y_0,Z}, Z_\cdot^\cdot)}$. *The problem of the principal admits the following representation*

$$\mathrm{V}^{\mathrm{P}} = \sup_{y_0\in\mathcal{I}} \underline{V}(y_0),\ \text{where}\ \underline{V}(y_0) := \sup_{Z\in\mathcal{H}^{2,2}} \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\mathrm{U}_{\mathrm{P}}\Big(X_{\cdot\wedge T}, \mathrm{U}_{\mathrm{A}}^{(-1)}\Big(T, Y_T^{T,y_0,Z}\Big)\Big)\bigg].$$

Let us emphasise that contrary to the classic time-consistent scenario studied in [64], the previous problem $\underline{V}(y_0)$ is not a standard control problem in the sense that the Principal optimises over a family of Volterra controls in $\mathcal{H}^{2,2}$ which satisfy a novel type of Volterra constraint. This motivates the study of $\mathcal{H}^{2,2}$ under particular specifications of utility functions for both the agent and the principal, hoping to be able to: (i) reduce the complexity of the set $\mathcal{H}^{2,2}$; (ii) exploit its particular structure to formulate an ansatz to the problem of the principal. This is exactly what we do in the last part of Chapter 5. The general case remains subject of further research.

To give a hint of our results let us present the solution to the example considered in this section. Under additional assumptions on the model, including coefficients independent of X, one can proceed with the resolution. For instance, in the case of risk-neutral principal and agent, *i.e.* $\mathrm{U}_{\mathrm{A}}(x) = \mathrm{U}_{\mathrm{P}}(x) = x$, we found that an optimal contract is given by

$$\xi^\star := \frac{R_0}{f(T)} - f(T)^{-1}\int_0^T \big(\sigma_t b_t^\star(z^\star(t)) - f(t)c_t^\star(z^\star(t))\big)\mathrm{d}t + \int_0^T \frac{z^\star(t)}{f(T-t)}\mathrm{d}X_t,$$

for some deterministic function $z^\star(t)$ which depends on the data of the problem, and in particular, on the discounting function f. Moreover, the family $Z^\star \in \mathcal{H}^{2,2}$, solution to $\underline{V}(y_0)$, is given, for any $s \in [0, T]$, by

$$Z_t^s := \frac{f(T-s)}{f(T-t)} z^\star(t),\ t \in [0, T].$$

Two noticeable features are worth commenting: (i) the optimal contract takes the form of a non-Markovian linear function of the output process X, which is a reflection of the intrinsic non-Markovian nature of contracting theory in continuous time; (ii) the optimal family $Z^\star$ is, surprisingly, deterministic.

Nevertheless, the extent to which the previous structure of the family Z and the optimal contral $\xi^\star$ is true in greater generality is a challenging problem. Let us just mention that: (i) we succeed in extending the previous result to the case of random non-Markovian coefficients; (ii) we also address the solution under another two specfications of preferences for both the prinpical and the agent. In particular, we consider the case of an agent with time-inconsistent CARA preferences analogue to the seminal work of [130].

Regarding the implications of our results we can mention the following:

(i) from a methodological point of view, unlike in the time-consistent case, the solution to the moral hazard problem does not reduce, in general, to a standard stochastic control problem. Nevertheless, the solution to the risk-sharing problem between a utility maximiser principal and a time-inconsistent sophisticated agent does, see Section 2. This suggest a dire difference between the first-best and second-best problems as soon as the agent is allowed to have time-inconsistent preferences;

(ii) a second takeaway from our analysis is associated with the so-called optimality of linear contracts. These are contracts consisting of a constant part and a term proportional to the terminal value of the state process as in the seminal work of [130]. We study two examples that can be regarded as (time-inconsistent) variations of [130], which we refer to as discounted utility, and utility of discounted income. In the former case, by virtue of the simplicity of the source of time-inconsistency, we find that optimal contract is linear. In the latter case, we find that the optimal contract is no longer linear unless there is no discounting (as in [130]). Our point here is that slight deviations of the model in [130] seem to challenge the virtues attributed to linear contracts and this suggests that would typically cease to be optimal in general for time-inconsistent agents;

(iii) lastly, we comment on the non-Markovian nature of the optimal contract. It is known that, beyond the realm of the model in [130], the optimal contract in the time-consistent scenario is, in general, non-Markovian in the state process X, see [64]. Indeed, we find the same result in the case of an agent with separable time-inconsistent preferences. Moreover, in our context the non-Markovian structure is also manifestation of the agent's time-inconsistent preferences.

1.5 Perspectives and future research

The topic of time-inconsistency has proven to be a sufficiently rich area of study, yet, the research conducted for this thesis has revealed numerous questions that still remained unanswered and cleared the way for a whole new set of applications incorporating the behaviour of non-traditional economic agents. Let us now comment on some of them.

Inconsistent mixed optimal control/stopping

Arguably, one of the limitations of the framework presented in the previous section is the fact that the contracts are strictly binding and do not allow the Agent to terminate the contract, *i.e.* to quit, before the terminal time T. Therefore, extensions to mixed optimal control/stopping problems could also prove interesting. This is the scenario in which the Agent chooses a pair (α, τ) consisting of an action and a stopping time. In this case, the first equation in $(\mathcal{H})$ should become a reflected BSDE, giving us a candidate equilibrium stopping time in addition to a candidate equilibrium action. Such stopping time will in turn play the role of the terminal horizon for the second equation in $(\mathcal{H})$. We conjecture the system to be of the following form, where $(\xi_t)_{t\in[0,T]}$ is the salary when contract ends at t, and $\mathcal{K}$ is a non-decreasing process starting at 0,

$$\begin{aligned}
\mathcal{Y}_t &= \mathrm{U_A}(\xi_T) + \int_t^T \big(H_r(X_{\cdot\wedge r}, \mathcal{Z}_r) - \partial Y_r^r\big)\mathrm{d}r - \int_t^T \mathcal{Z}_r \cdot \mathrm{d}X_r + \int_t^T \mathrm{d}\mathcal{K}_r \\
\partial Y_t^s &= f'(T-s)\mathrm{U_A}(\xi_{\tau^\star}) + \int_t^{\tau^\star} \nabla h_r^\star(s, X_{\cdot\wedge r}, \partial Z_r^s, \mathcal{Z}_r)\mathrm{d}r - \int_t^{\tau^\star} \partial Z_r^s \cdot \mathrm{d}X_r, \\
\mathcal{Y}_t &\geq \mathrm{U_A}(\xi_t),\ t\in[0,T],\ \int_0^T \big(\mathcal{Y}_r - \mathrm{U_A}(\xi_r)\big)\mathrm{d}\mathcal{K}_r = 0,\ \tau^\star := \inf\{t \geq 0:\ \mathcal{K}_t > 0\}.
\end{aligned}$$

Naturally, these results would lead to contracting models in which the Agent is given the possibility to quit his job and in this way to draw economic intuitions on how the principal may deter the agent to do so.

Relaxed BSDEs and existence of equilibria

One central question that has been addressed for discrete-time principal–agent models, see Balder [17], Page Jr. [197], is that of existence of a contract. Yet, there are no results available in the continuous-time literature, not even in the time-consistent setting. As can be inferred from the results [64], existence of an optimal contract boils down to the existence of optimal controls in a general stochastic control problem. It is known, see Haussmann and Lepeltier [119], El Karoui, Nguyen, and Jeanblanc-Picqué [87], that randomisation (that is considering actions as probability measure-valued processes) is in general necessary to compactify the set of controls and obtain existence. Moreover, randomisation has been employed in one-period principal–agent models to argue existence of optimal contracts, see Kadan, Reny, and Swinkels [147]. As such, we plan to study this question using these techniques and prove general existence results for randomised contracts. Recall that in this setting the agent solves

$$\mathrm{V}_t^{\mathrm{A}} := \sup_{\alpha} \mathbb{E}^{\mathbb{P}^\alpha}\bigg[\mathrm{U_A}(\xi) - \int_t^T c_r(X_{\cdot\wedge r},\alpha)\mathrm{d}r\bigg|\mathcal{F}_t\bigg],\ t\in[0,T],$$

Let $\mathbb{A}$ be the collection of all finite and positive Borel measures on $[0,T]\times A$, whose projection on $[0,T]$ is the Lebesgue measure. Consider the canonical space $\Omega := \mathcal{C}^d\times\mathbb{A}$, with canonical process (X,Λ) and Borel σ-algebra $\mathcal{F}$. Let $\mathcal{P}_t := \{\mathbb{P}\in\mathrm{Prob}(\Omega,\mathcal{F}) : M(\varphi)$ is a $\mathbb{P}$-local martingale on $[t,T]$ for all $\varphi\in\mathcal{C}_{2,b}(\mathbb{R}^d)\}$,[7] and for any $(t,\varphi)\in[0,T]\times\mathcal{C}_{2,b}(\mathbb{R}^d)$

$$M_t(\varphi) := \varphi(X_t) - \iint_{[0,t]\times A}\Big(\sigma_r(X,a)b_r(X,a)\cdot\partial_x\varphi(X_r) + \frac{1}{2}\mathrm{Tr}\big[(\sigma\sigma^\top)_r(X,a)\partial^2_{xx}\varphi(X_r)\big]\Big)\Lambda(\mathrm{d}r,\mathrm{d}a).$$

We then consider the so-called relaxed formulation to the Agent's problem defined by

$$\mathrm{V}_t^{\mathrm{A,r}} := \sup_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}^{\mathbb{P}^\alpha}\bigg[\mathrm{U_A}(\xi) - \int\int_{[t,T]\times A} c_r(X,a)\Lambda(\mathrm{d}r,\mathrm{d}a)\bigg],\ t\in[0,T].$$

Our goal is then to obtain a probabilistic representation of the relaxed formulation. We recall the theory of 2BSDEs provides such representation for both weak and strong formulations, *i.e.* without

[7] $\mathrm{Prob}(\Omega,\mathcal{F})$ denotes the set of all probability measures on $(\Omega,\mathcal{F})$ and $\mathcal{C}_{2,b}(\mathbb{R}^d)$ is the set of bounded twice continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$, with bounded first and second derivatives.

randomisation, of this control problem, see Possamaï, Tan, and Zhou [210]. To extend these results to the relaxed formulation we would develop a new theory of relaxed 2BSDEs. Preliminary results suggest the following.

Conjecture. *We have* $\mathrm{V}_0^{\mathrm{A,r}} = \sup_{\mathbb{P}\in\overline{\mathcal{P}}_0} \mathbb{E}^{\mathbb{P}}[Y_0]$ *where* (Y, Z, K) *solves the following relaxed* 2BSDE

$$Y_t = \xi - \iint_{[t,T]\times A} F_r(X_{\cdot\wedge r}, Z_r(a), \Sigma(a))\Lambda(\mathrm{d}r, \mathrm{d}a) - \iint_{[t,T]\times A} Z_r(a)\Lambda^{\mathbb{P}}(\mathrm{d}r, \mathrm{d}a) + \int_t^T \mathrm{d}K_r, \mathbb{P}\text{–a.s.},$$

where,

- $\overline{\mathcal{P}}_t$ *is defined as* $\mathcal{P}_t$ *with* $b = 0$;
- $\Lambda^{\mathbb{P}}$ *is a martingale measure, in the sense of* [87], *with* $\mathbb{P}$*-quadratic variation* Λ*, and*

$$F_t(x, z, \Sigma) := \sup_{\{a\in A:(\sigma\sigma^\top)_t(x,a)=\Sigma\}} \Big\{\Sigma^{\frac{1}{2}} b_t(x, a)\cdot z - c_t(x, a)\Big\};$$

- K *is a non-decreasing process satisfying* $\inf_{\mathbb{P}\in\overline{\mathcal{P}}} \mathbb{E}^{\mathbb{P}}[K_T] = 0$;
- *moreover, there exists* $\mathbb{P}^\star \in \overline{\mathcal{P}}$ *such that* $\mathbb{E}^{\mathbb{P}^\star}[K_T] = 0$.

Provided this holds, one expects to be able to characterise optimal relaxed controls as maximisers in F and the probability measure $\mathbb{P}^\star$. This is in itself a result with implications that go beyond the applications in contract theory. Indeed, it encompasses any stochastic control problem and will also open the way to numerical considerations for relaxed optimal control, through the numerical simulation of these new 2BSDEs, see Possamaï and Tan [208]. However, these existence results are abstract and non-constructive in general, which makes the study of properties of optimal controls (and thus of optimal contracts) delicate. In a second step, one should push the analysis further, by finding characterisations of optimal controls in relaxed stochastic control problems.

Volterra control of forward Volterra processes

Another avenue of research that emerges from our study of principal–agent problems involving a time-inconsistent agent pertains the thorough understanding of the novel class of control problems that characterise the problem of the principal. In the easiest set up we could consider the following.

Let $\mathcal{A}$ be an appropriate class of A-valued doubly indexed controls of the form $(\alpha_t^s)_{(s,t)\in[0,T]^2}$. We would then let the dynamics of the controlled state process X be given by

$$X_t^{s,x_0,\alpha} = x_0^s + \int_0^t b_r(s, X_r^{s,x_0,\alpha}, X_r^{r,x_0,\alpha}, \alpha_r^s, \alpha_r^r)\mathrm{d}r + \int_0^t \sigma_r(s, X_r^{s,x_0,\alpha}, X_r^{r,x_0,\alpha}, \alpha_r^s, \alpha_r^r)\mathrm{d}B_r,$$

where B is a $\mathbb{P}$–Brownian motion. This is, the dynamics of X correspond to a forward Volterra process. Notably the control α is allowed to impact the dynamics of X via both processes α^s and its *diagonal* $(\alpha_t^t)_{t\in[0,T]}$. With this a control problem is then defined by

$$v(t,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\Big[\xi(X_T^{T,x_0,\alpha})\Big|\mathcal{F}_t\Big]$$

In the general situation, it is unclear—at least to us—how to address this class of problems. In fact, our discussion in Chapter 5 shows that under mild regularity assumptions on the data and the class of controls there are two possible representations for the term $X_T^{T,x_0,\alpha}$ appearing in the reward of the principal, each coming with its own challenges. For simplicity, let us assume $\sigma_t(s,x,\mathrm{x},a,\mathrm{a}) = \sigma_t(s)$ for some deterministic mapping $\sigma_t(s)$.

Indeed, it is possible to show that $X_T^{T,x_0,\alpha}$ corresponds to the final value of the process $X_t^{t,x_0,\alpha}$ given by the system

$$\begin{aligned} X_t^{t,x_0,Z} &= x_0^0 - \int_0^t \Big(b_r(r, X_r^{r,x_0,\alpha}, X_r^{r,x_0,\alpha}, \alpha_r^r, \alpha_r^r) - \partial X_r^{r,x_0,\alpha}\Big)\mathrm{d}r + \int_0^t \sigma_r(r)\cdot\mathrm{d}B_r,\\ \partial X_t^{s,x_0,Z} &= \partial x_0^s - \int_0^t \nabla b_r(s, \partial X_r^{s,x_0,\alpha}, X_r^{s,x_0,\alpha}, X_r^{r,x_0,\alpha}, \alpha_r^s, \partial\alpha_r^s, \alpha_r^r)\mathrm{d}r + \int_0^t \nabla\sigma_r(s)\cdot\mathrm{d}B_r. \end{aligned}$$

(i) If we choose to focus on the dynamics of $X_t^{t,x_0,Z}$ as given by the first equation, one can exploit the fact that the action of the control is only through the diagonal $(\alpha^t)_{t\in[0,T]}$. Moreover, given $(\partial X_t^{t,x_0,Z})_{t\in[0,T]}$, the first equation implies $(X_t^{t,x_0,\alpha})_{t\in[0,T]}$ is an Itô process. Nevertheless, as $(\partial X_t^{t,x_0,\alpha})_{t\in[0,T]}$ is not given, there is no direct access to the dynamics of $(\partial X_t^{t,x_0,\alpha})_{t\in[0,T]}$ that is amenable to the analysis, *i.e.* that would allow us to use Itô calculus. Moreover, the process ∂X process is neither Markov nor a semi-martingale and does not satisfy a flow property

$$\partial X_s^{s,x_0,Z} \neq \partial X_t^{t,x_0,Z} - \int_t^s \nabla b_r(s, \partial X_r^{s,x_0,\alpha}, X_r^{s,x_0,\alpha}, X_r^{r,x_0,\alpha}, \alpha_r^s, \partial\alpha_r^s, \alpha_r^r)\mathrm{d}r + \int_0^t \nabla\sigma_r(s)\cdot\mathrm{d}B_r,$$

(ii) Alternatively, suppose we choose the representation as in the statement of the problem. In that case, the problem of the principal involves the control of the family of FSVIEs $X_t^{s,x_0,Z}$, where the controls consist of a Volterra type processes $\alpha \in \mathcal{A}$ that impacts the dynamics via $((\alpha^t)_{t\in[0,T]}, (\alpha^s)_{t\in[0,T]})$. The fact that the dependence is through both arguments makes the recent approach in Viens and Zhang [243] for controlled FSVIEs inoperable.

In general, this seems to be quite a challenging problem. We shall leave the analysis of these class of problems for future research.

1.6 Outline of the thesis

Overall, the contributions of this thesis can be encompassed into the area of stochastic control and can be classified into two categories:

(i) infinite families backward stochastic differential equations (BSDEs), and backward stochastic Volterra integral equations (BSVIEs);

(ii) time-inconsistency and contract theory for sophisticated agents.

These categories correspond to the tools and their applications. That being said, this thesis is structure into four parts, the third of which being divided into two chapters.

Part I, which consists of this chapter, was dedicated to introduce the main ideas in contract theory, including a description of the dynamic programming approach; to describe the phenomenon known as time-inconsistency; to comment on a selection of the existing results in the literature on Markovian time-inconsistent control problem; to present our contributions; and to communicate some perspectives and future directions of research.

Part II, which consist of Chapter 2, we develop a probabilistic theory for continuous-time non-Markovian stochastic control problems which are inherently time-inconsistent. Our formulation is cast within the framework of a controlled non-Markovian forward stochastic differential equation, and a general objective functional setting. We adopt a game-theoretic approach to study such problems, meaning that we seek for *sub-game perfect Nash equilibrium* points. As a first novelty of this work, we introduce and motivate a refinement on the definition of equilibrium that allows us to establish rigorously an extended dynamic programming principle, in the same spirit as in the classical theory. This in turn allows us to introduce a system of backward stochastic differential equations analogous to the classical HJB equation. We prove that this system is fundamental, in the sense that its well-posedness is both necessary and sufficient to characterise the value function and equilibria. As a final step we provide an existence and uniqueness result. Some examples and extensions of our results are also presented.

Part III, which consist of Chapter 3 and Chapter 4, we take a temporary detour from the world of contract theory and dedicate ourselves to understanding the tools the appeared in the study of the time-inconsistent problem faced by a sophisticated agent. This is we study a novel general class of multidimensional type-I backward stochastic Volterra integral equations. Toward this goal, we introduce an infinite family of standard backward SDEs and establish its well-posedness, and we show that it is equivalent to that of a type-I backward stochastic Volterra integral equation. In light of the inherent nature of the cornerstone models in the literature, we study the cases of Lipschitz, Chapter 3, and quadratic generators, Chapter 4.

In Part IV, which consist of Chapter 5, we come back to the moral hazard problem equipped with all the knowledge gained in Part II and Part III. This is, we study the contracting problem between a time-consistent principal and sophisticated time-inconsistent agent. Our main contribution consists of a characterisation of the moral hazard problem faced by the principal. Nevertheless, this characterisation yields, as far as we know, a novel class of control problems that involve the control of a forward Volterra equation via constrained Volterra type controls. Despite the inherent challenges of these class of problems, we study the solution to this problem under three different specifications of utility functions for both the agent and the principal and draw implications from the form of the optimal contract. The general case remains the subject of future research.

Part II

Time-inconsistent control

Chapter 2

Non-Markovian time-inconsistent control for sophisticated agents

This chapter is devoted to develop a theory for continuous-time non-Markovian stochastic control problems which are inherently time-inconsistent. Their distinguishing feature is that the classical Bellman optimality principle no longer holds. Our formulation is cast within the framework of a controlled non-Markovian forward stochastic differential equation, and a general objective functional setting. We adopt a game-theoretic approach to study such problems, meaning that we seek for *sub-game perfect Nash equilibrium* points. As a first novelty of this work, we introduce and motivate a new definition of equilibrium that allows us to establish rigorously an *extended dynamic programming principle*, in the same spirit as in the classical theory. This in turn allows us to introduce a system of backward stochastic differential equations analogous to the classical HJB equation. We prove that this system is fundamental, in the sense that its well-posedness is both necessary and sufficient to characterise the value function and equilibria. As a final step we provide an existence and uniqueness result. Some examples and extensions of our results are also presented.

2.1 Problem formulation

2.1.1 Probabilistic framework

Let d and n be two positive integers. For consistency and in order to alleviate the notation we set $\mathcal{X} := \mathcal{C}^d$. We will work on the canonical space $\Omega := \mathcal{X} \times \mathcal{C}^n \times \mathbb{A}$, whose elements we will denote generically by $\omega := (x, \mathrm{w}, q)$, and with canonical process (X, W, Λ), where

$$X_t(\omega) := x(t),\ W_t(\omega) := \mathrm{w}(t),\ \Lambda(\omega) := q,\ (t, \omega) \in [0, T] \times \Omega.$$

$\mathcal{X}$ and $\mathcal{C}^d$ are endowed with the topology $\mathfrak{T}_\infty$, induced by the norm $\|x\|_\infty := \sup_{0 \leq t \leq T} |x(t)|$, $x \in \mathcal{X}$, while $\mathbb{A}$ is endowed with the topology $\mathfrak{T}_{\mathrm{w}}$ induced by weak convergence, which we recall is metrisable, for instance, by the Prohorov metric, see Stroock and Varadhan [232, Theorem 1.1.2]. With these norms, both spaces are Polish.

For $(t, \varphi) \in [0, T] \times \mathcal{C}_b([0, T] \times A)$, we define

$$\Delta_t[\varphi] := \iint_{[0,t] \times A} \varphi(r, a) \Lambda(\mathrm{d}r, \mathrm{d}a),\ \text{so that}\ \Delta_t[\varphi](\omega) = \iint_{[0,t] \times A} \varphi(r, a) q_r(\mathrm{d}a)\mathrm{d}r,\ \text{for any}\ \omega \in \Omega.$$

We denote by $\mathcal{F}$ the Borel σ-field on Ω. We will work with the filtrations $\mathbb{F} := (\mathcal{F}_t)_{t \in [0,T]}$ and $\mathbb{F}^X := (\mathcal{F}_t^X)_{t \in [0,T]}$ defined for $t \in [0, T]$ by

$$\mathcal{F}_t := \sigma\Big((X_r, W_r, \Delta_r[\varphi]) : (r, \varphi) \in [0, t] \times \mathcal{C}_b([0, T] \times A) \Big),$$
$$\mathcal{F}_t^X := \sigma\Big((X_r, \Delta_r[\varphi]) : (r, \varphi) \in [0, t] \times \mathcal{C}_b([0, T] \times A) \Big).$$

Additionally, we will work with processes $\psi : [0, T] \times \mathcal{X} \longrightarrow E, (t, x) \longmapsto \psi(t, x)$, for E some Polish space, which are $\mathbb{G}$-optional, with $\mathbb{G}$ an arbitrary filtration, *i.e.* $\mathcal{P}_{\mathrm{opt}}(E, \mathbb{G})$-measurable. In particular, these processes are automatically non-anticipative, that is to say, $\psi_r(X) := \psi_r(X_{\cdot \wedge r})$ for any $r \in [0, T]$. We denote by $\pi^{\mathcal{X}}$ the canonical projection from Ω to $\mathcal{X}$ and let $\pi^{\mathcal{X}}_{\#}\mathbb{P} := \mathbb{P} \circ (\pi^{\mathcal{X}})^{-1}$ denote the push-forward of $\mathbb{P}$. As the previous processes are defined on $[0, T] \times \mathcal{X} \subsetneq [0, T] \times \Omega$, we emphasise that throughout this chapter, the assertion

$$\text{`}\mathbb{P}\text{–a.e.} x \in \mathcal{X}\text{'},\ \text{will always mean that}\ (\pi^{\mathcal{X}}_{\#}\mathbb{P})[\mathcal{X}] = 1. \tag{2.1.1}$$

$\mathbb{P} \in \mathrm{Prob}(\Omega)$ will be called a semi-martingale measure if X is an $(\mathbb{F}, \mathbb{P})$–semi-martingale. By Karandikar [149], there then exists an $\mathbb{F}$-predictable process, denoted by $\langle X \rangle = (\langle X \rangle_t)_{t \in [0,T]}$, which coincides with the quadratic variation of X, $\mathbb{P}$–a.s., for every semi-martingale measure $\mathbb{P}$. Thus, we can introduce the $d \times d$ symmetric positive semi-definite matrix $\widehat{\sigma}$ as the square root of $\widehat{\sigma}^2$ given by

$$\widehat{\sigma}_t^2 := \limsup_{\varepsilon \searrow 0} \frac{\langle X \rangle_t - \langle X \rangle_{t-\varepsilon}}{\varepsilon},\ t \in [0,T]. \tag{2.1.2}$$

We also recall the celebrated result on the existence of a well-behaved ω-by-ω version of the conditional expectation. We also introduce the concatenation of a measure and a stochastic kernel. These objects are key for the statement of our results in the level of generality we are working with.

Recall Ω is a Polish space and $\mathcal{F}$ is a countably generated σ-algebra. For $\mathbb{P} \in \mathrm{Prob}(\Omega)$ and $\tau \in \mathcal{T}_{0,T}(\mathbb{F})$, $\mathcal{F}_\tau$ is also countably generated, so there exists an associated regular conditional probability distribution (r.c.p.d. for short) $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$, see [232, Theorem 1.3.4], satisfying

(i) for every $\omega \in \Omega$, $\mathbb{P}^\tau_\omega$ is a probability measure on $(\Omega, \mathcal{F})$;

(ii) for every $E \in \mathcal{F}$, the mapping $\omega \longmapsto \mathbb{P}^\tau_\omega[E]$ is $\mathcal{F}_\tau$-measurable;

(iii) the family $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ is a version of the conditional probability measure of $\mathbb{P}$ given $\mathcal{F}_\tau$, that is to say for every $\mathbb{P}$-integrable, $\mathcal{F}$-measurable random variable ξ, we have $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}^\tau_\omega}[\xi]$, for $\mathbb{P}$–a.e. $\omega \in \Omega$;

(iv) for every $\omega \in \Omega$, $\mathbb{P}^\tau_\omega[\Omega^\omega_\tau] = 1$, where $\Omega^\omega_\tau := \{\omega' \in \Omega : \omega'(r) = \omega(r), 0 \le r \le \tau(\omega)\}$.

Moreover, for $\mathbb{P} \in \mathrm{Prob}(\Omega)$ and an $\mathcal{F}_\tau$-measurable stochastic kernel $(\mathbb{Q}^\tau_\omega)_{\omega\in\Omega}$ such that $\mathbb{Q}^\tau_\omega[\Omega^\omega_\tau] = 1$ for every $\omega \in \Omega$, the concatenated probability measure is defined by

$$\mathbb{P} \otimes_\tau \mathbb{Q}.[A] := \int_\Omega \mathbb{P}(\mathrm{d}\omega) \int_\Omega \mathbf{1}_A(\omega \otimes_{\tau(\omega)} \tilde{\omega}) \mathbb{Q}_\omega(\mathrm{d}\tilde{\omega}),\ \forall A \in \mathcal{F}. \tag{2.1.3}$$

The following result, see [232, Theorem 6.1.2], gives a rigorous characterisation of the concatenation procedure.

Theorem 2.1.1 (Concatenated measure)**.** *Consider a stochastic kernel $(\mathbb{Q}_\omega)_{\omega\in\Omega}$, and let $\tau \in \mathcal{T}_{0,T}(\mathbb{F})$. Suppose the map $\omega \longmapsto \mathbb{Q}_\omega$ is $\mathcal{F}_\tau$-measurable and $\mathbb{Q}_\omega[\Omega^\omega_\tau] = 1$ for all $\omega \in \Omega$. Given $\mathbb{P} \in \mathrm{Prob}(\Omega)$, there is a unique probability measure $\mathbb{P} \otimes_{\tau(\cdot)} \mathbb{Q}.$ on $(\Omega, \mathcal{F})$ such that $\mathbb{P} \otimes_{\tau(\cdot)} \mathbb{Q}.$ equals $\mathbb{P}$ on $(\Omega, \mathcal{F}_\tau)$ and $(\delta_\omega \otimes_{\tau(\omega)} \mathbb{Q}_\omega)_{\omega\in\Omega}$ is an* r.c.p.d. *of $\mathbb{P} \otimes_{\tau(\cdot)} \mathbb{Q}.|\mathcal{F}_\tau$. In particular, for some $t \in [0,T]$, suppose that $\tau \ge t$, that $M : [t,T] \times \Omega \longrightarrow \mathbb{R}$ is a right-continuous, $\mathbb{F}$–progressively measurable* *function after t, such that M_t is $\mathbb{P}\otimes_{\tau(\cdot)}\mathbb{Q}_\cdot$-integrable, that for all $r\in[t,T]$, $(M_{r\wedge\tau})_{r\in[t,T]}$ is an $(\mathbb{F},\mathbb{P})$-martingale, and that $(M_r-M_{r\wedge\tau(\omega)})_{r\in[t,T]}$ is an $(\mathbb{F},\mathbb{Q}_\omega)$-martingale, for all $\omega\in\Omega$. Then $(M_r)_{r\in[t,T]}$ is an $(\mathbb{F},\mathbb{P}\otimes_{\tau(\cdot)}\mathbb{Q}_\cdot)$-martingale.*

In particular, if ξ is an $\mathcal{F}$-measurable function, we have $\mathbb{E}^{\mathbb{P}\otimes_\tau\mathbb{P}^\tau_\cdot}[\xi]=\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_\tau]]=\mathbb{E}^{\mathbb{P}}[\xi]$. This is nothing else than the classical tower property. We also mention that the reverse implication in the last statement in Theorem 2.1.1 holds as a direct consequence of [232, Theorem 1.2.10].

2.1.2 Controlled state dynamics

Let k be a positive integer, and let $A\subseteq\mathbb{R}^k$. An action process ν is an $\mathbb{F}^X$-predictable process taking values in A. Given an action process ν, the *controlled state equation* is given by the stochastic differential equation (SDE for short)

$$X_t=x_0+\int_0^t\sigma_r(X,\nu_r)\big(b_r(X,\nu_r)\mathrm{d}r+\mathrm{d}W_r\big),\ t\in[0,T]. \tag{2.1.4}$$

where W is an n-dimensional Brownian motion, $x_0\in\mathbb{R}^d$, and

$$\sigma:[0,T]\times\mathcal{X}\times A\longrightarrow\mathcal{M}_{d,n}(\mathbb{R}),\ b:[0,T]\times\mathcal{X}\times A\longrightarrow\mathbb{R}^n,\text{bounded}, \tag{2.1.5}$$

with $(t,x)\longmapsto(\sigma,b)(t,x,a)$, $\mathbb{F}^X$-optional for any $a\in A$.

In this work we characterise the controlled state equation in terms of *weak solutions* to (2.1.4). These are elegantly introduced via the so-called martingale problems, see [232, Chapter 6]. Let $\overline{X}:=(X,W)$ and $\overline{\sigma}:[0,T]\times\mathcal{X}\longrightarrow\mathcal{M}_{n+d}(\mathbb{R})$ given by

$$\overline{\sigma}:=\begin{pmatrix}\sigma & 0_{d,n}\\ \mathrm{I}_n & 0_{n,d}\end{pmatrix}.$$

For any $(t,x)\in[0,T]\times\mathcal{X}$, we define $\mathcal{P}(t,x)$ as the collection of $\mathbb{P}\in\mathrm{Prob}(\Omega)$ such that

(i) there exists $\mathrm{w}\in\mathcal{C}^n$ such that $\mathbb{P}\circ(X_{\cdot\wedge t},W_{\cdot\wedge t})^{-1}=\delta_{(x_{\cdot\wedge t},\mathrm{w}_{\cdot\wedge t})}$;

(ii) for all $\varphi \in \mathcal{C}_{2,b}(\mathbb{R}^{d+n})$, the process $M^\varphi : [t,T] \times \Omega \longrightarrow \mathbb{R}$ defined by

$$M_r^\varphi := \varphi(\overline{X}_r) - \iint_{[t,r]\times A} \frac{1}{2}\mathrm{Tr}\big[(\overline{\sigma\sigma}^\top)_u(X,a)(\partial^2_{\overline{xx}}\varphi)(\overline{X}_u)\big]\Lambda(\mathrm{d}u,\mathrm{d}a),\ r \in [t,T], \tag{2.1.6}$$

is an $(\mathbb{F},\mathbb{P})$–local martingale;

(iii) $\mathbb{P}[\Lambda \in \mathbb{A}_0] = 1$.

There are classical conditions ensuring that the set $\mathcal{P}(t,x)$ is non-empty. For instance, it is enough that the mapping $x \longmapsto \overline{\sigma}_t(x,a)$ is continuous for some constant control a, see [232, Theorem 6.1.6]. We also recall that uniqueness of a solution, *i.e.* there is a unique element in $\mathcal{P}(t,x)$, holds when in addition $\overline{\sigma\sigma}_t^\top(x,a)$ is uniformly positive away from zero, *i.e.* there is $\lambda > 0$ s.t. $\theta^\top\overline{\sigma\sigma}_t^\top(x,a)\theta \geq \lambda|\theta|^2,\ (t,x,\theta) \in [0,T] \times \mathcal{X} \times \mathbb{R}^d$, see [232, Theorem 7.1.6].

Given $\mathbb{P} \in \mathcal{P}(x) := \mathcal{P}(0,x)$, W is an n-dimensional $\mathbb{P}$–Brownian motion and there is an A-valued process ν such that

$$X_t = x_0 + \int_0^t \sigma_r(X,\nu_r)\mathrm{d}W_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.} \tag{2.1.7}$$

Remark 2.1.2. *We remark some properties of the previous martingale problem which, in particular, justify* (2.1.7)

(i) *for any* $\mathbb{P} \in \mathcal{P}(t,x)$ *and* ν *verifying* $\mathbb{P}[\Lambda \in \mathbb{A}_0] = 1$, (2.1.6) *implies*

$$\widehat{\sigma}_r^2 = (\sigma\sigma^\top)_r(X,\nu_r),\ \mathrm{d}r \otimes \mathrm{d}\mathbb{P}\text{–a.e.},\ \textit{on } [t,T] \times \Omega; \tag{2.1.8}$$

(ii) *we highlight the fact that our approach is to enlarge the canonical space right from the beginning of the formulation. This is in contrast to, for instance,* El Karoui and Tan [86, Remark 1.6]*, where the canonical space is taken as* $\mathcal{X} \times \mathbb{A}$ *and enlargements are considered as properly needed. As* $\mathcal{P}(t,x)$ *ought to describe the law of* X *as in* (2.1.7)*, this cannot be done unless the canonical space is extended. We feel our approach simplifies the readability and understanding of the analysis at no extra cost. Indeed, the extra canonical process* W *allows us to get explicitly the existence of a* $\mathbb{P}$*–Brownian motion by virtue of Lévy's characterisation. By* [232, Theorem 4.5.2] *and* (2.1.8), *since*

$$\begin{pmatrix} \sigma & 0 \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \sigma^\top & I_n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\sigma}^2 & \sigma \\ \sigma^\top & I_n \end{pmatrix},$$

it follows

$$\begin{pmatrix} X_t \\ W_t \end{pmatrix} = \begin{pmatrix} x_0 \\ \mathrm{w}_0 \end{pmatrix} + \int_0^t \begin{pmatrix} \sigma & 0 \\ I_n & 0 \end{pmatrix} \mathrm{d}W_s,\ t \in [0,T]; \tag{2.1.9}$$

(iii) *the reader might notice that the notation* $\mathcal{P}(t,x)$ *does not specify an initial condition for neither the process* W*, nor for the measure valued process* Λ*. Arguably, given our choice of* Ω*, one is naturally led to introduce* $\mathcal{P}(t,\omega)$*, with initial condition* $\omega = (x, \mathrm{w}, q) \in \Omega$*. Nevertheless, by* (2.1.6) *and* (2.1.9)*, we see that the dynamics of* X *depends on the increments of the application* $[0,T] \ni t \longmapsto \Delta_t[\sigma\sigma^\top](\omega) \in \mathcal{M}_d(\mathbb{R})$ *for* $\omega \in \Omega$*. It is clear from this that the initial condition on* W *and* Λ *are irrelevant. This yields* $\mathcal{P}(t,\omega) = \mathcal{P}(t,\tilde{\omega})$ *for all* $\tilde{\omega} = (x, \tilde{\mathrm{w}}, \tilde{q}) \in \Omega$*.*

We now introduce the class of admissible actions. We let $\mathfrak{A}$ denote the set of A-valued and $\mathbb{F}^X$-predictable processes. At the formal level, we will say $\nu \in \mathfrak{A}$ is admissible whenever (2.1.7) has a unique weak solution. A proper definition requires first to introduce some additional notations. We will denote by $(\mathbb{P}^\nu_{t,x})_{(t,x)\in[0,T]\times\mathcal{X}}$ the corresponding family of solutions associated to ν. Moreover, we recall that uniqueness guarantees the measurability of the application $(t,x) \longmapsto \mathbb{P}^\nu_{t,x}$, see [232, Exercise 6.7.4]. For $(t,x,\mathbb{P}) \in [0,T] \times \mathcal{X} \times \mathcal{P}(t,x)$, we define

$$\begin{aligned} \mathcal{A}^0(t,x,\mathbb{P}) &:= \{\nu \in \mathfrak{A} : \Lambda(\mathrm{d}r, \mathrm{d}a) = \delta_{\nu_r}(\mathrm{d}a)\mathrm{d}r,\ \mathrm{d}r \otimes \mathrm{d}\mathbb{P}\text{–a.e. on } [t,T]\times\Omega\}, \\ \mathfrak{M}^0(t,x) &:= \{(\mathbb{P},\nu) \in \mathcal{P}(t,x) \times \mathcal{A}^0(t,x,\mathbb{P})\}, \\ \mathcal{P}^0(t,x,\nu) &:= \{\mathbb{P} \in \mathrm{Prob}(\Omega) : (\mathbb{P},\nu) \in \mathfrak{M}^0(t,x)\}. \end{aligned}$$

Letting $\mathcal{A}^0(t,x) := \bigcup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathcal{A}^0(t,x,\mathbb{P})$, we define rigorously the class of admissible actions

$$\mathcal{A}(t,x) := \{\nu \in \mathcal{A}^0(t,x) : \mathcal{P}^0(t,x,\nu) = \{\mathbb{P}^\nu_{t,x}\}\}. \tag{2.1.10}$$

We set $\mathcal{A}(x) := \mathcal{A}(0,x)$ and define similarly, $\mathcal{A}(t,x,\mathbb{P})$, $\mathcal{P}(t,x,\nu)$, $\mathfrak{M}(t,x)$, $\mathcal{A}(x,\mathbb{P})$, $\mathfrak{M}(x)$ and $\mathcal{P}(x,\nu)$.

Remark 2.1.3. (i) *We remark that the sets* $\mathfrak{M}(t,x)$ *and* $\mathcal{A}(t,x)$ *are equivalent parametrisations of* *the admissible solutions to* (2.1.7)*. This follows since uniqueness of weak solutions for fixed actions implies that the sets* $\mathcal{A}(t,x,\mathbb{P})$ *are disjoint,* i.e.

$$\mathcal{A}(t,x) = \dot{\bigcup_{\mathbb{P}\in\mathcal{P}(t,x)}} \mathcal{A}(t,x,\mathbb{P}).$$

(ii) *The convenience of introducing the sets* $\mathcal{A}(t,x,\mathbb{P})$ *is that it allows us to better handle action processes for which the quadratic variation of* X *is the same. For different* $\mathbb{P}\in\mathcal{P}(t,x)$ *the discrepancy among such probability measures can be read from* (2.1.8)*,* i.e. *in the* (*support of the*) *quadratic variation of* X*. This reflects the fact that different diffusion coefficients of* (2.1.7) *might induce mutually singular probability measures in* $\mathcal{P}(t,x)$*. We also recall that in general* $\mathcal{P}(t,x)$ *is not finite since it is a convex set, see* Jacod and Shiryaev [142, Proposition III.2.8]*.*

Remark 2.1.4. (i) *Since* b *is bounded, it follows that given* $(\mathbb{P},\nu)\in\mathfrak{M}(x)$*, if we define* $\mathbb{M} := (\overline{\mathbb{P}}^\nu,\nu)$ *with*

$$\frac{\mathrm{d}\overline{\mathbb{P}}^\nu}{\mathrm{d}\mathbb{P}} := \exp\bigg(\int_0^T b_r(X,\nu_r)\cdot \mathrm{d}W_r - \int_0^T |b_r(X,\nu_r)|^2\mathrm{d}r\bigg),$$
$$W_t^{\mathbb{M}} := W_t - \int_0^t b_r(X,\nu_r)\mathrm{d}r,\ t\in[0,T],$$

we have that $W^{\mathbb{M}}$ *is a* $\overline{\mathbb{P}}^\nu$*–Brownian motion, and*

$$X_t = x_0 + \int_0^t \sigma_r(X,\nu_r)\big(b_r(X,\nu_r)\mathrm{d}r + \mathrm{d}W_r^{\mathbb{M}}\big),\ \overline{\mathbb{P}}^\nu\text{–a.s.}$$

(ii) *We will exploit the previous fact and often work under the drift-less dynamics* (2.1.7)*. We stress that in contrast to the strong formulation setting, in the weak formulation the state process* X *is fixed and the action process* ν *allows to control the distribution of* X *via* $\overline{\mathbb{P}}^\nu$*.*

In light of the previous discussion, we define the collection of *admissible models* with initial conditions $(t,x)\in[0,T]\times\mathcal{X}$

$$\mathcal{M}(t,x) := \big\{(\overline{\mathbb{P}}^\nu,\nu) : (\mathbb{P},\nu)\in\mathfrak{M}(t,x)\big\}.$$

and we set $\mathcal{M}(x) := \mathcal{M}(0,x)$. To ease notations we set $\mathcal{T}_{t,T} := \mathcal{T}_{t,T}(\mathbb{F})$, for any $\tau\in\mathcal{T}_{0,T}$,

$\mathbb{P}^{\tau}_{\cdot} := (\mathbb{P}^{\tau}_{\omega})_{\omega\in\Omega}$, and for any $(\tau,\omega) \in \mathcal{T}_{0,T} \times \Omega$, $\mathcal{P}(\tau,x) := \mathcal{P}(\tau(\omega),x)$, $\mathfrak{M}(\tau,x) := \mathfrak{M}(\tau(\omega),x)$, and $\mathcal{M}(\tau,x) := \mathcal{M}(\tau(\omega),x)$.

We will take advantage in the rest of this chapter of the fact that we can move freely from objects in $\mathcal{M}$ to their counterparts in $\mathfrak{M}$, see Lemma 2.10.3.2. Also, we mention that we will make a slight abuse of notation, and denote by $\mathbb{M}$ elements in both $\mathfrak{M}(t,x)$ and $\mathcal{M}(t,x)$. It will be clear from the context whether $\mathbb{M}$ refers to a model for (2.1.4), or the drift-less dynamics (2.1.7).

2.1.3 Objective functional

Let us introduce the running and terminal cost functionals

$$f : [0,T] \times [0,T] \times \mathcal{X} \times A \longrightarrow \mathbb{R},\ \xi : [0,T] \times \mathcal{X} \longrightarrow \mathbb{R},\ \text{Borel-measurable},$$

with $f_\cdot(s,\cdot,a)$, $\mathbb{F}^X$-optional, for any $(s,a) \in [0,T] \times A$.

We are interested in a generic pay-off functional of the form

$$J(t,x,\nu) := \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[\int_t^T f_r(t,X,\nu_r)\mathrm{d}r + \xi(t,X_{\cdot\wedge T})\bigg],\ (t,x,\nu) \in [0,T] \times \mathcal{X} \times \mathcal{A}(t,x).$$

Remark 2.1.5. (i) *As we work on an enlarged probability space, one might wonder whether the value of the reward under both formulations coincide or not. We recall that we chose to enlarge the canonical space, see* Section 2.1.2*, to explicitly account for the randomness driving* (2.1.4)*,* i.e. *the process* W*. Nevertheless, as for any* $\omega \in \Omega$ *and* $\mathbb{M} \in \mathfrak{M}(t,x)$*, the latter depends only on* x*, see* Remark 2.1.2*, we see that given* $\mathbb{M}$*,* J *is completely specified by* (t,x)*.*

(ii) *Given the form of the pay-off functional* J*, the problem of maximising* $\mathcal{A}(x) \ni \nu \longmapsto J(0,x,\nu)$ *has a time-inconsistent nature. More precisely, the dependence of both* f *and* ξ *on the current time* t *is here the source of inconsistency.*

Due to the nature of the pay-off functional, the classic dynamic programming principle falls apart, and the action prescribed by a maximiser is in general not optimal at future times. In this work, out of the three approaches to time-inconsistency, namely naive, pre-committed and sophisticated planning, we adopt the latter. Thus, we study this problem from a game-theoretic perspective and look for *equilibrium laws*. The next section is dedicated to explaining these concepts more thoroughly.

2.1.4 Game formulation

In order to define a game, say $\mathcal{G}$, one has to specify a certain number of elements, see Osborne and Rubinstein [194]: a set of I players; a set of decision nodes $\mathfrak{D}$; a specification of the unique immediate predecessor Υ; a set of actions Θ; a set $\mathfrak{H}$ forming a partition of $\mathfrak{D}$ referred to as information sets; a function $\mathcal{H} : \mathfrak{D} \longmapsto \mathfrak{H}$ that assigns each decision node into an information set; a probability distribution $\mathbb{P}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ that dictates nature's moves and the rewards $\mathfrak{J}$.

Let $\mathbf{x} \in \mathcal{X}$, $\mathfrak{D} := \{(s,t,x) \in [0,T] \times [0,T] \times \mathcal{X} : s = t \text{ and } x(0) = \mathbf{x}(0)\}$, $\mathfrak{H} := \{\{z\} : z \in \mathfrak{D}\}$ and $z \longmapsto \mathcal{H}(z) := \{z\} \in \mathfrak{H}$. Defining Υ is one of the bottlenecks in a continuum setting which we address via our definition of equilibrium. We define $\mathcal{G} := \{[0,T], [0,T], \mathbb{A}, \mathfrak{H}, \mathcal{H}, (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), J\}$ as a non-cooperative game in which

(i) we have one player for each point in time $s \in [0,T]$, we refer to this player as 'Player s';

(ii) at time $t = s$, Player s can control the process X exactly at time t. He does that by choosing an admissible action model, $(\overline{\mathbb{P}}^{\nu}, \nu) \in \mathcal{M}(t, X_{\cdot\wedge t})$, after negotiating with future versions of himself, *i.e.* $s \in [t,T]$;

(iii) when all players reach a 'successful' negotiation, we end up with an action model $(\overline{\mathbb{P}}^{\nu^\star}, \nu^\star) \in \mathcal{M}(\mathbf{x})$;

(iv) the reward to Player s is given by $J(s, s, X_{\cdot\wedge s}, \nu^\star)$, where following [35], for $(s,t,x,\nu) \in [0,T] \times [s,T] \times \mathcal{X} \times \mathcal{A}(s,x)$ we introduce

$$J(s,t,x,\nu) := \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[\int_t^T f_r(s, X, \nu_r)\mathrm{d}r + \xi(s, X_{\cdot\wedge T})\bigg]. \tag{2.1.11}$$

To motivate the concept of equilibrium law we will introduce in the sequel, we recall the idea behind the concept of a *Nash equilibrium*. An action profile is a Nash equilibrium if no player can do better by unilaterally deviating from his strategy while the other players still play the equilibrium. In other words, imagine that each player asks himself: 'knowing the strategies of the others, and treating them as fixed, can I benefit by changing my strategy?'. If the answer is no, then the current combination of strategies is a Nash equilibrium. In the context of dynamic games, the concept of sub-game perfection has been commonly adopted, see [194]. To introduce this concept we recall that a *sub-game* is any part, *i.e.* subset, of a game that satisfies

- the initial decision node is in a singleton information set, which means that only one possible sequence of actions could have led to the sub-game's initial decision node;
- if a decision node is contained in the sub-game, then so are all of its subsequent decision nodes;
- if a decision node in a particular information set is in the sub-game, then all members of that information set belong to the sub-game.

In our framework, the first condition translates into the fact that every player, together with a past trajectory, define a new sub-game. In the particular case of the first player, *i.e.* $s = 0$, we will write, now and in the following, $\mathbf{x} \in \mathcal{X}$ so that the initial decision node is well defined. Note this corresponds to fixing an initial value for the trajectories of X. The last two conditions imply that, given a player $s \in [0, T]$, every sub-game of which a subsequent player $s \leq t \leq T$ is part of, must be part of the sub-game defined by s. These three conditions together yield sub-games of the form $[s, T]$, together with a past trajectory. A strategy profile is therefore *sub-game perfect* if it prescribes a Nash equilibrium in any sub-game. This motivates the idea behind the definition of an equilibrium model, see among others [233], [80] and [34].

Let $\mathbf{x} \in \mathcal{X}$, $\nu^\star \in \mathcal{A}(\mathbf{x})$ be an action, which is a candidate for an equilibrium, $(t, \omega) \in [0, T] \times \Omega$ an arbitrary initial condition, $\ell \in (0, T - t]$, $\tau \in \mathcal{T}_{t,t+\ell}$ and $\nu \in \mathcal{A}(\tau, x)$. We define

$$\nu \otimes_\tau \nu^\star := \nu \mathbf{1}_{[t,\tau)} + \nu^\star \mathbf{1}_{[\tau,T]}.$$

Definition 2.1.6 (*Equilibrium*)**.** *Let $\mathbf{x} \in \mathcal{X}$, $\nu^\star \in \mathcal{A}(\mathbf{x})$. For $\varepsilon > 0$ let*

$$\ell_\varepsilon := \inf \Big\{ \ell > 0 : \exists \mathbb{P} \in \mathcal{P}(\mathbf{x}), \\ \mathbb{P}\big[\big\{ x \in \mathcal{X} : \exists (t, \nu) \in [0, T] \times \mathcal{A}(t, x),\ J\big(t, t, x, \nu^\star\big) < J\big(t, t, x, \nu \otimes_{t+\ell} \nu^\star\big) - \varepsilon \ell \big\}\big] > 0 \Big\}.$$

If for any $\varepsilon > 0$, $\ell_\varepsilon > 0$ then $\nu^\star$ is an ***equilibrium model****, and we write $\nu^\star \in \mathcal{E}(\mathbf{x})$.*

Remark 2.1.7. *We now make a few remarks regarding our definition.*

(i) The first advantage of Definition 2.1.6 *is that ℓ_ε is monotone in ε. Indeed, let $0 < \varepsilon' \leq \varepsilon$, then by definition, ℓ_ε satisfies the condition in the definition of $\ell_{\varepsilon'}$, thus $\ell_{\varepsilon'} \leq \ell_\varepsilon$.*

(ii) We chose $\mathcal{A}(t,x)$ for the class of actions to be compared to in the equilibrium condition. One could have chosen instead

$$\mathcal{A}(t,x,t+\ell) := \{\nu \in \mathcal{A}(t,x) : \nu \otimes_{t+\ell} \nu^\star \in \mathcal{A}(t,x)\}, \tag{2.1.12}$$

as it was actually considered in [82]. Clearly, $\mathcal{A}(t,x,t+\ell) \subseteq \mathcal{A}(t,x)$ for $\ell > 0$ and, a priori, *$\mathcal{A}(t,x,t+\ell)$ could be considered as a more sensible choice, since there is no guarantee that $\nu \otimes_{t+\ell} \nu^\star \in \mathcal{A}(t,x)$ for any $\nu \in \mathcal{A}(t,x)$. It turns out that under the assumption of weak uniqueness, this is indeed the case, see* Lemma 2.10.3.1.

*(iii) From the definition, given $(\varepsilon,\ell) \in \mathbb{R}_+^\star \times (0,\ell_\varepsilon)$, for $\mathcal{P}(\mathbf{x})$–*q.e. *$x \in \mathcal{X}$ and any $(t,\nu) \in [0,T] \times \mathcal{A}(t,x)$*

$$J(t,t,x,\nu^\star) - J(t,t,x,\nu \otimes_{t+\ell} \nu^\star) \geq -\varepsilon\ell. \tag{2.1.13}$$

From this we can better appreciate two distinguishing features of our definition. The fist one is that (2.1.13) *holds for $\mathcal{P}(\mathbf{x})$–*q.e. *$x \in \mathcal{X}$. This is in contrast to [35] and [120], where the rewards are respectively compared for all $x \in \mathcal{X}$ and x in the support of $\mathbb{P}^{\nu^\star} \in \mathcal{P}(\mathbf{x})$. The approach in [35] is too stringent as it might impose a condition on trajectories that are not reachable by any admissible action. On the other hand, requiring it to hold only in the support of the probability measure associated with the equilibrium might lead to situations which fail to be sub-game perfect. Our choice is motivated by the rationale behind sub-game perfection,* i.e. *a sub-game perfect equilibrium should prescribed a (ε–)Nash equilibrium in any possible sub-game,* i.e. *for $\mathcal{P}(\mathbf{x})$–*q.e. *$x \in \mathcal{X}$. The second one is that* Definition 2.1.6 *imposes* (2.1.13) *for all $\ell < \ell_\varepsilon$. This local feature had not been considered in the literature before, and it will be key to rigorously prove that these equilibria lead to an extended* DPP*, see* Section 2.2.1 *for more details.*

(iv) *Alternatively, we could have introduced* $\ell_{\varepsilon,t,\nu} > 0$, i.e. *let the choice of* ℓ *depend on* ε, t *and* ν, *via* (2.1.13), *which we will refer here as* ***lax equilibria***. *These equilibria are in close connection with the current standard definition of equilibrium in the Markovian context. Furthermore, we could have also considered* ***strict equilibria*** *as those lax equilibria for which* (2.1.13) *holds with* $\varepsilon = 0$. *While this document was being finalised, we were made aware of the works* [137] *and* [120] *where this last notion of equilibrium, referred there as regular equilibrium, was studied. See once more* Section 2.2.1 *for a discussion on how we compare our definition to this and other notions of equilibria in the literature.*

(v) *In addition, one could analyse something that we would coin* f***-equilibria***. *This corresponds to considering, for a given increasing right-continuous function* $f : \mathbb{R}_+^\star \longrightarrow \mathbb{R}_+$ *with* $f(\ell) \longrightarrow 0$ *as* $\ell \longrightarrow 0$, *the criterion*

$$J(t,t,x,\nu^\star) - J(t,t,x,\nu \otimes_{t+\ell} \nu^\star) \geq -\varepsilon f(\ell), \text{ for any } 0 < \ell < \ell_\varepsilon.$$

With this, one could study the optimal speed at which the term on the left-hand side converges to zero. It is known that provided extra regularity of the action process, f *would be in general linear in* ℓ. *Nevertheless, as in this work the action processes are only measurable, in general,* f *needs to be appropriately chosen, and it need not be linear in* ℓ. *In the case of strict equilibria,* [120, Theorems 1 and 2] *state sufficient conditions on the data of the problem for the term on the left hand side in the above inequality to be a linear (or quadratic) decreasing function of* ℓ, *respectively. Insights on how* f *is related to the dynamics of* X *can be drawn from* [65], *where* X *is a general semi-martingale. Motivated by the Dambis–Dubins–Schwarz theorem on time-change for continuous martingales, see* [150, Theorem III.4.6], *the quadratic variation of the controlled process is exploited to define an equilibria. See again* Section 2.2.1 *for more details.*

In the rest of the this chapter we fix $\mathbf{x} \in \mathcal{X}$ and study the problem

$$v(t,x) := J(t,t,x,\nu^\star), \ (t,x) \in [0,T] \times \mathcal{X}, \ \nu^\star \in \mathcal{E}(\mathbf{x}). \tag{P}$$

The previous functional is well-defined for all $(t,x) \in [0,T] \times \mathcal{X}$ and measurable. This follows from the weak uniqueness assumption.

Remark 2.1.8. *We emphasise that* (P) *is fundamentally different from the problem of maximising* $\mathcal{A}(t,x) \ni \nu \longmapsto J(t,t,x,\nu)$*. First,* (P) *is solved by finding* $\nu^\star \in \mathcal{A}(\mathbf{x})$ *and then defining the value function. This contrasts with the classical formulation of optimal control problems. Second, the previous maximisation problem will correspond to finding the pre-committed strategy for player* t*,* i.e. *solving an optimal control problem for which dynamic programming does not hold.*

In fact, given the sequential nature of $\mathcal{G}$*, our definition of equilibrium which compares rewards on sub-games of the form* $[t,T]$ *via coalitions among players* $[t,t+\ell]$*, is compatible with sub-game perfection. Indeed, to find his best response, Player* t *reasons as follows: for any* $\tau \in [t,t+\ell_\varepsilon)$ *knowing that Players* $[\tau,T]$ *are following the equilibrium* $\nu^\star$*,* i.e. *the sub-game* $[\tau,T]$ *is following the equilibrium, and treating the actions of those players as fixed, the action prescribed by the equilibrium to Player* t *is such that he is doing at least as good as in any* coalition/deviation ν *from the equilibrium with players* $t+\ell \in [t,\tau)$*, give or take* $\varepsilon\ell$*. This allows us to interpret the equilibrium as Player* t*'s* ε*–best response for all sub-games containing* $[t,T]$ *via coalitions. Consequently the equilibrium agrees with an* ε*–sub-game perfect Nash equilibrium.*

Additionally, note that according to our definition, if $\nu^\star$ *is an equilibrium action model for the game* $[0,T]$*, by sub-game perfection it is also an equilibrium for* $[t,T]$*. Indeed, this follows immediately from the fact* ℓ_ε *in* Definition 2.1.6 *is uniform in* $t \in [0,T]$*. Therefore, we expect the value of the game to be time-consistent. This feature motivates why these type of solutions are referred in the literature as consistent plans, see* [233]*.*

2.2 Related work and our results

As a preliminary to the presentation of our contributions, this section starts by comparing our setting with the ones considered in the extant literature.

2.2.1 On the different notions of equilibrium

We now make a few comments on our definition of equilibria, its relevance and compare it with the ones previously proposed. Let us begin by recalling the set-up adopted by most of the existing literature in time-inconsistent stochastic control models in continuous time.

Given $T > 0$, on the time interval $[0, T]$ a fixed filtered probability space $(\Omega, \mathbb{F}^W, \mathcal{F}_T^W, \mathbb{P})$ supporting a Brownian motion W is given. Here, $\mathbb{F}^W$ denotes the $\mathbb{P}$-augmented Brownian filtration. Let $\mathcal{A}$ denote the set of admissible actions and $\mathbb{G}$ a (possibly) smaller filtration than $\mathbb{F}^W$. For a $\mathbb{G}$-adapted process $\nu \in \mathcal{A}$, representing an action process, the state process X is given by the unique strong solution to the SDE

$$X_t^{0,x_0,\nu} = x_0 + \int_0^t b_s(X_s^{0,x_0,\nu}, \nu_s)\mathrm{d}s + \int_0^t \sigma_s(X_s^{0,x_0,\nu}, \nu_s)\mathrm{d}W_s, \text{ for } t \in [0, T]. \tag{2.2.1}$$

As introduced in [82], $\nu^\star$ is then said to be an equilibrium if for all $(t, x, \nu) \in [0, T] \times \mathcal{X} \times \mathcal{A}$

$$\liminf_{\ell \searrow 0} \frac{J(t, t, x, \nu^\star) - J(t, t, x, \nu \otimes_{t+\ell} \nu^\star)}{\ell} \geq 0.$$

If we examine closely the above condition, we obtain that for any $\kappa := (t, x, \nu) \in [0, T] \times \mathcal{X} \times \mathcal{A}$, $\varepsilon > 0$ and sequence $(\ell_n)_{n \in \mathbb{N}} \subseteq [0, T]$, $\ell_n \longrightarrow 0$, there is a positive integer N_ε^κ such that

$$\forall n \geq N_\varepsilon^\kappa,\ J(t, t, x, \nu^\star) - J(t, t, x, \nu \otimes_{t+\ell_n} \nu^\star) \geq -\varepsilon \ell_n. \tag{2.2.2}$$

From (2.2.2) it is clear that the classical definition of equilibrium is an ε-like notion of equilibrium. Now, ever since its introduction, the distinctive case in which the $\liminf$ is 0 has been noticed. In fact, a situation in which the agent is worse-off in a *sequence* of coalitions with future versions of himself but in the limit is indifferent, conforms with this definition. This case is excluded when $\varepsilon = 0$, *i.e.* the case of regular equilibria in [120]. We also remark that [120, Section 4] presents several examples from the existing literature on time-inconsistent control in which a classical equilibria fails to be a strict equilibria, recall Remark 2.1.7. This is a consequence of the fact that (2.1.13) does in general depend on (t, x, ν). From these examples one can further show that classical equilibria fail to be an equilibrium as in Definition 2.1.6.

At this point we want to emphasise the main idea in our approach to time-inconsistent control for sophisticated agents: *any useful definition of equilibrium for a sophisticated agent ought to lead to a dynamic programming principle.* This idea has substantial justification from both the economical as well as the mathematical perspectives to the problem. Firstly, the idea behind the reasoning of the sophisticated agent is that he 'internalises' his time-inconsistency, so as to design a plan that reconciles present and future preferences, *i.e.* versions of himself. Consequently, this leads to time-consistent game values under equilibria from which a dynamic programming principle is expected. Secondly, one expects, and we chose, this to be a direct consequence of the notion of equilibrium and as such a necessary condition to achieve this is a nice local behaviour. Thus, a definition with no local features in the windows for 'negotiation' prescribed by ℓ is not amenable to our analysis of the problem. As we will see in the next section, once a DPP is available, all the pieces necessary for a complete theory of time-inconsistent non-Markovian stochastic control will become apparent.

We also mention that the definition proposed in this work is by no means the only definition complying with the features just described. The notion of strict equilibrium, which is point-wise in (t,ν) and imposes a sign condition on (2.1.13) leads, under extra assumptions, to the same dynamic programming principle we obtain for equilibria, see Remark 2.2.5.

We use the rest of this section to address different works in the area.

(i) The inaugural papers on the sophisticated, *i.e.* game theoretic, approach to inconsistent control problems are the sequence of papers by Ekeland and Lazrak [80, 81], and Ekeland and Pirvu [82]. In their initial work [80], the authors consider a strong formulation framework, *i.e.* (2.2.1) and $\mathbb{G} = \mathbb{G}^X$ denotes the augmented natural filtration of $X^{0,x_0,\nu}$, and seek for closed loop, *i.e.* $\mathbb{G}^X$-measurable, action processes defined via spike perturbations of the form

$$(a \otimes_{t+\ell} \nu)_r := a 1_{[t,t+\ell)}(r) + \nu(X_r^{0,x_0,\nu}) 1_{[t+\ell,T]}(r), \tag{2.2.3}$$

that maximise the corresponding Hamiltonian. However as already pointed out by Wei, Yong, and Yu [260], the local comparison is made between a Markovian feedback control, *i.e.* $\mathbb{G} = \mathbb{G}^X$ and $\nu_r = \nu(X_r^{0,x_0,\nu})$, and an open-loop control value a, *i.e.* $\mathbb{G} = \mathbb{F}^W$, and there is no argument as to whether admissibility is preserved. The later two works [82] and [81] introduce the definition of equilibrium via (2.2.2), in which admissibility is defined by progressively measurable open loop processes with moments of all orders. This last condition is imposed to guarantee uniqueness of a strong solution to the dynamics of the state process. In our framework we do not need such condition as the state dynamics hold in the sense of weak solutions.

(ii) The study in the linear quadratic set-up is carried out by Hu, Jin, and Zhou [135, 136]. There, the dynamics are stated in strong formulation. To bypass the admissibility issues in [80], [82] and [81], the class of admissible actions is open loop, *i.e.* $\mathbb{G} = \mathbb{F}^W$. Equilibria are defined via (2.2.2) but contrasting against spike perturbation as in (2.2.3). In [135], the authors obtain a condition which ensures that an action process is an equilibrium, and in [136] the authors are able to complete their analysis and provide a sufficient and necessary condition, via a flow of forward-backward stochastic differential equations. The approach taken to characterise equilibria in both [135] and [136] leverages on the particular structure of the linear quadratic setting and the admissibility class. Moreover, the authors are able to prove uniqueness of the equilibrium in the setting of a mean–variance portfolio selection model in a complete financial market, where the state process is one-dimensional and the coefficients in the formulation are deterministic. Unlike theirs, our results require standard Lipschitz assumptions.

(iii) Another sequence of works that has received great attention is by Björk and Murgoci [35] and Björk, Khapko, and Murgoci [38]. There, the authors approach the problem of inconsistent control in strong formulation, *i.e.* (2.2.1), and admissibility is defined by Markovian feedback actions, *i.e.* $\mathbb{G} = \mathbb{G}^X$ and $\nu_r = \nu(X_r^{0,x_0,\nu})$. The first of these works deals with the problem in discrete-time, a scenario in which the classic backward induction algorithm is implemented to obtain an equilibrium by seeking for a sign on the difference of the pay-offs corresponding to $\nu^\star$ and $\nu \otimes_{t+\ell} \nu^\star$. In their subsequent paper, the results are extended to a continuous-time setting, implementing the definition of equilibrium (2.2.2). Their main contribution is to provide a system of PDEs associated to the problem and provide a verification theorem. Nevertheless, the derivation of such system is completely formal and, in addition, there is no rigorous argument about the well-posedness of the system.

(iv) The study of more general dynamics for the state processes X began in Czichowsky [65] where the mean–variance portfolio selection problem was considered for general semi-martingales. There, the author developed a time-consistent formulation, based on a local notion of optimality called local mean–variance efficiency. Namely, given $\mathbb{G}^X$, the augmented natural filtration of the semi-martingales X, an increasing partition $\Pi = (t_i)_{1 \leq i \leq m} \subseteq [0, T]$ and two action processes ν and μ, the author introduces

$$u^{\Pi}[\nu,\mu]:=\sum_{i=1}^{m-1}\frac{J(t_i,t_i,x,\mathbb{P},\nu)-J(t_i,t_i,x,\mathbb{P},\nu+\mu 1_{(t_i,t_{i+1}]})}{\mathbb{E}[\langle X\rangle_{t_{i+1}}-\langle X\rangle_{t_i}|\mathcal{G}^X_{t_i}]}.$$

With this, given a sequence of increasing partitions $(\Pi^n)_{n\in\mathbb{N}}$, $\Pi^n\subset[0,T]$ with mesh, *i.e.* $\|\Pi^n\|:=\max_i|t^n_{i+1}-t^n_i|$, tending to zero, a strategy $\nu^\star$ if said to be *locally mean–variance efficient* (LMVE) if

$$\liminf_{n\to\infty}u^{\Pi^n}[\nu^\star,\nu]\geq 0.$$

We highlight how the notion of LMVE serves as a justification as to why the dependence in ℓ in (2.2.2) is linear. Clearly extensions of our results to more general state processes would call for such considerations to be taken into account, via the study of *f-equilibria*, as mentioned in Remark 2.1.7.

(v) A different setting is taken in Wei, Yong, and Yu [260] and Wang and Yong [245], see references therein too. Both works study a time-inconsistent recursive optimal control problem in strong formulation setting, *i.e.* (2.2.1), in which the class of admissible actions are Markovian feedback, *i.e.* $\mathbb{G}=\mathbb{G}^X$ and $\nu_r=\nu_r(X_r^{0,x_0,\nu})$. The motivation is the following: when $\tilde J(t;x_0,\nu):=\mathbb{E}^{\mathbb{P}}[\mathrm{e}^{-\lambda(T-t)}\xi(X_T^{0,x_0,\nu})|\mathcal{G}_t]$, it is known that $\tilde J(t;x_0,\nu)=Y_t$, $t\in[0,T]$, $\mathbb{P}$–a.s., where the pair (Y,Z) solves the BSDE $\mathrm{d}Y_t=-\lambda Y_t\mathrm{d}t+Z_t\mathrm{d}X_t$, $Y_T=\xi(X_T^{0,x_0,\nu})$. In other words, in the case of exponential discounting, the generator of the BSDE representing the reward functional depends (linearly) on the Y term of the solution (Y,Z). Therefore, [260] studies the natural extension to a general time-inconsistent recursive utility problem. An equilibrium is defined as the unique continuous solution to a forward–backward system that describes the controlled state process and the reward functionals of the players, and satisfy a *local approximate optimality property*. More precisely, u is the limit of a sequence of locally optimal controls at discrete times. Such property builds upon the known fact, see [35], that in discrete-time, equilibria are optimal given the best response of subsequent players. We highlight that in addition to a verification theorem, the authors are able to prove well-posedness of their system in the uncontrolled volatility scenario. However, we do not think their definition of an equilibrium is tractable in theory or practice. The definition is lengthy enough to fit a whole page, making the overall definition quite hindered and hard to check in practice. We believe this is due to the fact that their approach relies heavily on the approximation of the solution to the continuous game by discretised problems, and requires the equilibrium to be continuous in time, as opposed to mere measurability which is the case for us, see Equation (2.1.10). Overall, we feel it might send the wrong impression and make the sophisticated approach to time-inconsistent stochastic control problems look somehow excessively cumbersome and convoluted to deal with. We believe the approach taken in this work makes a good case to show the opposite. We also would like to mention [245] which, building upon the ideas in [260], modelled the reward functional by a BSVIE and look for a time-consistent locally near optimal equilibrium strategy. The authors are able to extend the results of the previous paper, but more importantly, they argue that a BSVIE is a more suitable way to represent a recursive reward functional with non-exponential discounting. On this matter, we highlight that in our approach, (H) allows us to identify as well a BSVIE, see Section 2.2.3 and Section 2.10.3.

(vi) Other recent works on this subject are Huang and Zhou [137] and He and Jiang [120]. The first article considers an infinite-horizon stochastic control problem in which the agent can control the generator of a time-homogeneous, continuous-time, finite-state Markov chain at each time. The authors begin by introducing two variations of the notion of lax equilibria, referred there as strong and weak equilibria. Exploiting the structure on their dynamics, and in particular by computing an expansion of the reward at a given equilibria, they derive necessary and sufficient conditions for both notions of equilibria. Moreover, under compactness of the set of admissible actions, existence of an equilibria is proved. In the second work, motivated by the missing steps in [38] and working in the same framework, *i.e.* strong formulation (2.2.1), and Markovian feedback actions, *i.e.* $\mathbb{G} = \mathbb{G}^X$ and $\nu_r = \nu(X_r^{0,x_0,\nu})$, the authors 'perform the analysis of the derivation of the system posed in [38], *i.e.* lay down sufficient conditions under which the value function satisfies the system'. In addition to their analysis, the authors introduce two new notions of equilibria, *regular* and *strong* equilibria. Regular equilibria compare rewards in (2.1.13) with feasible actions different from $\nu^\star$ at t, and strong equilibria allow comparisons with any feasible actions. These notions correspond to pointwise versions of our notion of equilibrium with $\varepsilon = 0$, see Remark 2.1.7. By requiring extra regularity on the actions, the authors provide necessary and sufficient conditions for a strategy to be a regular or a strong equilibria. It is our opinion that our definition of equilibrium is both natural and advantageous compared to their setting, as it allows to derive an extended DPP which in turn naturally links the value function to an appropriate system of equations. Moreover, even though their notion of equilibrium is a variation of ours, an extended DPP can be also be obtained, but extra requirements are necessary, see Remark 2.2.5. Finally, we mention that even though [120] succeeds in characterising regular equilibria, the conditions under which such results hold are, as expected, quite stringent, requiring for instance that the optimal action is differentiable in time and with derivatives of polynomial growth.

We emphasise that in our setting the dynamics of the state process are non-Markovian, and the class of admissible actions is 'closed loop'[1] and non-Markovian, *i.e.* $\mathbb{F}^X$-adapted see Section 2.1.1, both being novelties of this work. Additionally, another novelty of our treatment of time-inconsistent control problems is our weak formulation set-up. which is an extension on all the previous works in the subject. See Zhang [273, Section 4.5] for a presentation of weak formulation in optimal stochastic control and [64] for an application in contract theory in which both features are essential.

The remaining of this section is devoted to present the main results of this chapter.

2.2.2 Dynamic programming principle

Our first main result for the study of time-inconsistent stochastic control problems for sophisticated agents, Theorem 2.2.2, concerns the local behaviour of the value function v as defined in (P). This result is the first of its kind in a continuous-time setting and it is the major milestone for a complete theory. This result confirms the intuition drawn from the behaviour of a sophisticated agent, in the sense that v does indeed satisfy a dynamic programming principle. In fact, it can be regarded as an extended one, as it does reduce to the classic result known in optimal stochastic control in the case of exponential discounting, see Remark 2.5.5. This result requires the following main assumptions.

Assumption A. (i) *The map* $s \longmapsto \xi(s,x)$ *(resp.* $s \longmapsto f_t(s,x,a)$*) is continuously differentiable uniformly in* x *(resp. in* (t,x,a)*), and we denote its derivative by* $\partial_s \xi(s,x)$ *(resp.* $\partial_s f_t(s,x,a)$*).*

[1] This is not a closed-loop formulation *per se*, as our controls are allowed to be adapted as well to the information generated by the canonical process Λ.

(ii) $a \longmapsto f_t(s,x,a)$ *is uniformly Lipschitz-continuous,* i.e. $\exists C > 0,$ *s.t.* $\forall (s,t,x,a,a') \in [0,T]^2 \times \mathcal{X} \times A^2$,

$$\big|f_t(s,x,a) - f_t(s,x,a')\big| \leq C|a-a'|.$$

(iii) (a) $x \longmapsto \xi(t,x)$ *is lower-semicontinuous uniformly in* t, i.e. $\forall (\tilde{x}, \varepsilon) \in \mathcal{X} \times \mathbb{R}^\star_+,\ \exists U_{\tilde{x}} \in \mathfrak{T}_\infty,\ \tilde{x} \in U_{\tilde{x}},\ \forall (t,x) \in [0,T] \times U_{\tilde{x}}$,

$$\xi(t,x) \geq \xi(t,x_0) - \varepsilon, \text{ when } \xi(t,x_0) > -\infty.$$

(b) $x \longmapsto (b,\sigma)(t,x,a)$ *is uniformly Lipschitz-continuous,* i.e. $\exists C > 0,$ *s.t.* $\forall (t,x,x',a) \in [0,T] \times \mathcal{X}^2 \times A$,

$$|b_t(x,a) - b_t(x',a)| + |\sigma_t(x,a) - \sigma_t(x',a)| \leq C\|x_{\cdot\wedge t} - x'_{\cdot\wedge t}\|_\infty.$$

Remark 2.2.1. *Let us comment on the above assumptions. As it will be clear from our analysis in* Section 2.5 *and* Section 2.7, *to study time-inconsistent stochastic control problems for sophisticated agents under our notion of equilibrium, one needs to make sense of a system. The fact that we get such a system should be compared to the classical stochastic control framework, where only one* BSDE *suffices to characterise the value function and the optimal control, see* [273, Section 4.5].

Consequently, Assumption A.(i) *and* Assumption A.(ii) *are fairly mild requirements in order to understand the behaviour of the reward functionals on the agent's type, which is the source of inconsistency. We also remark that* Assumption A.(i) *guarantees that the map* $t \longmapsto f_t(t,x,a)$ *is continuous, uniformly in* (x,a), *which ensures extra regularity in type and time for the player's running rewards. Lastly, given our approach and our choice to not impose regularity on the action process, in order to get a rigorous dynamic programming principle, we cannot escape imposing extra assumptions. Namely, if we do not want to assume that* $t \longmapsto \nu_t(X)$ *is continuous, we have to impose regularity in* x, *which is exactly what* Assumption A.(iii) *does. This allows us to use the result in* [86] *regarding piece-wise constant approximations of stochastic control problems. We believe our choice is the least stringent and it is clearly much weaker than any regularity assumptions* *made in the existing literature, see* [35], [260], [120]*. For details see the discussion after* (2.5.3) *and* Remark 2.5.3*.*

Our dynamic programming principle takes the following form.

Theorem 2.2.2. *Let* Assumption A *hold, and* $\nu^\star \in \mathcal{E}(\mathbf{x})$*. For* $\{\sigma, \tau\} \subseteq \mathcal{T}_{t,T}$*,* $\sigma \leq \tau$ *and* $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$*, we have*

$$v(\sigma, x) = \sup_{\nu \in \mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[v(\tau, X) + \int_\sigma^\tau \bigg(f_r(r, X, \nu_r) - \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,\cdot}}\bigg[\partial_s \xi(r, X_{\cdot \wedge T}) + \int_r^T \partial_s f_u(r, X, \nu_u^\star)\mathrm{d}u\bigg]\bigg)\mathrm{d}r\bigg].$$

Moreover, $\nu^\star$ *attains the* sup*.*

Remark 2.2.3. (*i*) *In light of* Theorem 2.2.2 *we are led to consider a system consisting of a second order* BSDE (2BSDE *for short*) *and a infinite dimensional collection of processes in order to solve* Problem (P)*. This will be the object of the next section. As a by-product, we recover below a connection already mentioned in* [37, Proposition 8.1] *between time-inconsistent control problems and optimal stochastic control problems in a Markovian setting. In words, it says that given an equilibrium for a time-inconsistent stochastic control problem, it is possible to associate a classical time-consistent optimal stochastic control problem which attains the same value.*

(*ii*) *We emphasize that* Theorem 2.2.2 *is a direct consequence of* Definition 2.1.6*. Moreover, it differs from* [37, Proposition 8.1] *in that the latter argues via the* PDE (2.2.9)*. In fact, the result in* [37] *is obtained assuming a solution exists and it is in the spirit of the Feynman–Kac representation formula. This would be analogous to us assuming a solution to the* 2BSDE *in the proof of* Theorem 2.2.2 *which, among other things, would automatically rule out the possibility to prove the necessity of the* 2BSDE*,* i.e. *that any equilibria prescribes a solution to such system. In our probabilistic framework, the proof of* Theorem 2.2.2 *will ultimately allow us to bypass this and establish the necessity result.*

(*iii*) *We also point out that even for classic optimal control in discrete time proving a* DPP *in fairly general settings is a very difficult task because of crucial measurability issues, see for instance* Bertsekas and Shreve [25, Section 1.2].

(*iv*) *Lastly, we mention that the extend to which this result holds true goes beyond the type of* *reward functionals considered in this chapter. Indeed, in* Section 2.4 (*resp.* Chapter 5) *we consider a broader class of time-inconsistent rewards, namely those with mean–variance type of rewards* (*resp. those given by standard* type-I BSVIEs).

Corollary 2.2.4. *Let* Assumption A *hold, and* $\nu^\star \in \mathcal{E}(\mathbf{x})$. *There exists a time-consistent stochastic control problem with the same value* v, *for which* $\nu^\star$ *prescribes an optimal control. Namely let*

$$[0,T]\times\mathcal{X}\times A \longrightarrow \mathbb{R},\ (t,x,a)\longmapsto k_t(x,a) := f_t(t,x,a) - \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}\bigg[\partial_s\xi(t,X_{\cdot\wedge T}) + \int_t^T \partial_s f_u(t,X,\nu_u^\star)\mathrm{d}u\bigg],$$

Then, for $\mathcal{P}(\mathbf{x})$–q.e. $x\in\mathcal{X}$

$$v(t,x) = \sup_{\nu\in\mathcal{A}(t,x)} \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[\int_t^T k_r(X,\nu_r)\mathrm{d}r + \xi(T,X_{\cdot\wedge T})\bigg].$$

Remark 2.2.5. (i) *For the purposes of this work, this is as much as we can gain from this last result as it requires knowing the equilibrium strategy a priori, since the functional* k *does depend on* $\nu^\star$. *This is obviously of little practical use as it was already mentioned in* [38, Proposition 8.1]. *Nevertheless, as a direct consequence of* Theorem 2.2.2, *we are led to consider a system consisting of a second order* BSDE *and a infinite dimensional collection of processes in order to solve Problem* (P), *which will be the object of the next section.*

(ii) *We remark that all the results in this section are true for strict equilibria, provided there is more control of* $\ell_{\varepsilon,t,\nu}$, *see* Remark 2.1.7 *and* Remark 2.5.4. *Namely, it suffices, for instance, to have that for* $\mathcal{P}(\mathbf{x})$–q.e. $x\in\mathcal{X}$

$$\inf_{(t,\nu)\in[0,T]\times\mathcal{A}(t,x)} \ell_{0,t,\nu} > 0.$$

2.2.3 BSDE system associated to (P)

Let us introduce the functionals needed to state the rest of our results. As in the classical theory of optimal control, we introduce the Hamiltonian operator associated to this problem. For $(s,t,x,z,\gamma,\Sigma,u,v,a)\in[0,T]^2\times\mathcal{X}\times\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})\times\mathbb{S}_d(\mathbb{R})\times\mathbb{R}\times\mathbb{R}^d\times A$, let

$$\mathsf{h}_t(s,x,z,\gamma,a) := f_t(s,x,a) + b_t(x,a)\cdot\sigma_t(x,a)^\top z + \frac{1}{2}\mathrm{Tr}\big[(\sigma\sigma^\top)_t(x,a)\gamma\big],$$

$$H_t(x,z,\gamma,u) := \sup_{a\in A}\Big\{\mathsf{h}_t(t,x,z,\gamma,a)\Big\} - u.$$

Following the approach of Soner, Touzi, and Zhang [230], we introduce the range of our squared diffusions coefficient and the inverse map which assigns to every squared diffusion the corresponding set of generating actions

$$\begin{aligned}
\boldsymbol{\Sigma}_t(x) &:= \big\{\Sigma_t(x,a)\in\mathbb{S}_d(\mathbb{R}) : a\in A\big\},\ \text{where } \Sigma_t(x,a) := (\sigma\sigma^\top)_t(x,a),\\
A_t(x,\Sigma) &:= \big\{a\in A : (\sigma\sigma^\top)_t(x,a) = \Sigma\big\},\ \Sigma\in\boldsymbol{\Sigma}_t(x).
\end{aligned}$$

The previous definitions allow us to isolate the partial maximisation with respect to the squared diffusion. Let

$$\begin{aligned}
h_t(s,x,z,a) &:= f_t(s,x,a) + b_t(x,a)\cdot\sigma_t(x,a)^\top z,\\
\nabla h_t(s,x,v,a) &:= \partial_s f_t(s,x,a) + b_t(x,a)\cdot\sigma_t(x,a)^\top v,\\
F_t(x,z,\Sigma,u) &:= \sup_{a\in A_t(x,\Sigma)}\big\{h_t(t,x,z,a)\big\} - u.
\end{aligned}$$

With this, $2H = (-2F)^*$ is the covex conjugate of $-2F$, *i.e.*

$$H_t(x,z,\gamma,u) = \sup_{\Sigma\in\boldsymbol{\Sigma}_t(x)}\Big\{F_t(x,z,\Sigma,u) + \frac{1}{2}\mathrm{Tr}[\Sigma\gamma]\Big\}. \tag{2.2.4}$$

Moreover, we assume there exists a unique A-valued Borel-measurable map $\mathcal{V}^\star(t,x,z)$ satisfying[2]

$$[0,T]\times\mathcal{X}\times\mathbb{R}^d \ni (t,x,z) \longmapsto \mathcal{V}^\star(t,x,z)\in \operatorname*{arg\,max}_{a\in A_t(x,\hat\sigma_t^2(x))} h_t(t,x,z,a). \tag{2.2.5}$$

In the most general setting for (P) considered in this chapter, where control on both the drift and the volatility are allowed, to ξ, $\partial_s\xi$, $\partial_s f$, and F as above, we associate the system

$$\begin{aligned}
Y_t =&\ \xi(T,X_{\cdot\wedge T}) + \int_t^T F_r(X,Z_r,\widehat\sigma_r^2,\partial Y_r^r)\mathrm{d}r - \int_t^T Z_r\cdot\mathrm{d}X_r + \int_t^T \mathrm{d}K_r^{\mathbb{P}},\ t\in[0,T],\ \mathcal{P}(\mathbf{x})\text{–q.s.},\\
\partial Y_t^s(\omega) :=&\ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}\bigg[\partial_s\xi(s,X_{\cdot\wedge T}) + \int_t^T \partial_s f_r(s,X,\mathcal{V}^\star(r,X,Z_r))\mathrm{d}r\bigg],\ (s,t)\in[0,T]^2,\ \omega\in\Omega.
\end{aligned} \tag{H}$$

[2]The existence of such mapping is guaranteed by Schäl [221, Theorem 3] in the case of A bounded and $a\mapsto h_t(t,x,z,\gamma,a)$ Lipschitz for every $(t,x,z,\gamma)\in[0,T]\times\mathcal{X}\times\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})$

where $\mathbb{P}^{\nu^\star} \in \mathcal{P}(\mathbf{x})$ with $\nu_t^\star := \mathcal{V}^\star(t, X, Z_t) \in \mathcal{A}(\mathbf{x})$.

If only drift control is allowed, *i.e.* $\sigma_t(x) := \sigma_t(x,a)$ for all $a \in A$, the weak uniqueness assumption for (2.1.7) implies $\mathcal{P}(\mathbf{x}) = \{\mathbb{P}\}$. Letting

$$h_t^o(s,x,z,a) := f_t(s,x,a) + b_t(x,a)\cdot\sigma_t(x)^\top z,\ \nabla h_t^o(s,x,z,a) := \partial_s f_t(s,x,a) + b_t(x,a)\cdot\sigma_t(x)^\top z,$$

$$H_t^o(x,z,u) := \sup_{a\in A}\{h_t^o(t,x,z,a)\} - u,$$

we show, see Proposition 2.9.1, that (H) reduces to the system, which for any $s \in [0,T]$, holds $\mathbb{P}$–a.s., for any $t \in [0,T]$

$$\begin{aligned} Y_t &= \xi(T, X_{\cdot\wedge T}) + \int_t^T H_r^o(X, Z_r, \partial Y_r^r)\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r, \\ \partial Y_t^s &= \partial_s\xi(s, X_{\cdot\wedge T}) + \int_t^T \nabla h_r^o(s, X, \partial Z_r^s, \mathcal{V}^\star(r, X, Z_r))\mathrm{d}r - \int_t^T \partial Z_r^s\cdot\mathrm{d}X_r. \end{aligned} \tag{$\mathrm{H_o}$}$$

Though BSDEs have been previously incorporated in the formulation and analysis of control problems which are time-inconsistent, the kind of systems of BSDEs prescribed by (H) and ($\mathrm{H_o}$) are new in the literature. Indeed, in [82] an auxiliary family of BSDEs was introduced in order to argue, in combination with the stochastic maximum principle, that a given action process complies with the definition of equilibrium (2.2.2). This was a reasonable line of arguments as we recall at the time no verification theorem was available. More recently, [245] studied the case in which the reward functional is represented by a type-I BSVIEs, a generalisation of the concept of BSDEs. In fact, [245] argues that when the running cost rate and the terminal cost are time dependent, then the reward functional does satisfy a BSVIE. As it happens, in order to conduct our analysis of ($\mathrm{H_o}$), we also identify a link to such type of equations, namely we obtained that the process $(\partial Y_t^t)_{t\in[0,T]}$ satisfies a type-I BSVIE, see Lemmata 2.8.2 and 2.8.3.

In the general case, the first equation in (H) defines a 2BSDE, whereas the second defines an infinite dimensional family of processes, $(\partial Y^s)_{0\leq s\leq T}$, each of which admits a BSDE representation. We chose to introduce such processes via the family $(\mathbb{P}_{t,x}^{\nu^\star})_{(t,x)\in[0,T]\times\mathcal{X}}$, since in the first equation one needs an object defined on the support of every $\mathbb{P} \in \mathcal{P}(\mathbf{x})$. We recall that when the volatility is controlled, the supports of the measures in $\mathcal{P}(\mathbf{x})$ may be disjoint, whereas in the drift case, the support is always the same. Had we chosen to introduce the BSDE representation, we would have obtained an object defined only on the support of $\overline{\mathbb{P}}^{\nu^\star}$. Moreover, as we work with non-dominated probabilities measures, this last choice would not have been consistent with our extended DPP, nor would have sufficed to obtain the rest of our results.

The novel features of system (H) mentioned above raise several theoretical challenges. At the core of such system is the coupling arising from the appearance of ∂Y_t^t, the *diagonal* process prescribed by the infinite dimensional family $(\partial Y^s)_{0\leq s\leq T}$, in the first equation and of Z in the definition of the family. This makes any standard results available in the literature immediately inoperative. Therefore, the natural questions of stating sufficient conditions under which well-posedness holds, *i.e.* existence and uniqueness of a solution, needed to be addressed. Of course part of these duties consists of determining in what sense such a solution exists, see Definition 2.2.7 and Definition 2.2.13 for the general case and the drift control case, respectively. Answers to well-posedness of $(\mathrm{H_o})$, *i.e.* when only drift control is allowed are part of our, rather technical, Section 2.2.6.

We now introduce the concept of solution of a 2BSDE which we will use to state the definition of a solution to System (H). To do so, we introduce for $(t,r,x)\in[0,T]^2\times\mathcal{X}$, $\mathbb{P}\in\mathcal{P}(t,x)$ and a filtration $\mathbb{G}$

$$\mathcal{P}_{t,x}(r,\mathbb{P},\mathbb{G}) := \Big\{\mathbb{P}'\in\mathcal{P}(t,x) : \mathbb{P}'=\mathbb{P} \text{ on } \mathcal{G}_r\Big\}. \tag{2.2.6}$$

We will write $\mathcal{P}_x(r,\mathbb{P},\mathbb{G})$ for $\mathcal{P}_{0,x}(r,\mathbb{P},\mathbb{G})$. We postpone to Section 2.6 the definition of the spaces involved in the following definitions.

Definition 2.2.6. *For a given process $(\partial Y_t^t)_{t\in[0.T]}$, consider the equation*

$$Y_t = \xi(T,X_{\cdot\wedge T}) + \int_t^T F_r(X,Z_r,\widehat{\sigma}_r^2,\partial Y_r^r)\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ 0\leq t\leq T. \tag{2.2.7}$$

We say $(Y,Z,(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})})$ is a solution to 2BSDE (2.2.7) *under $\mathcal{P}(\mathbf{x})$ if for some $p>1$*

(i) Equation (2.2.7) *holds* $\mathcal{P}(\mathbf{x})$–q.s.;

(ii) $(Y,Z,(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})})\in\mathbb{S}_{\mathbf{x}}^p(\mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})})\times\mathbb{H}_{\mathbf{x}}^p(\mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})},X)\times\mathbb{I}_{\mathbf{x}}^p\big((\mathbb{F}_+^{X,\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}\big)$;

(*iii*) *the family* $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}$ *satisfies the minimality condition*

$$0 = \underset{\mathbb{P}'\in\mathcal{P}_{\mathbf{x}}(t,\mathbb{P},\mathbb{F}^X_+)}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\big[K^{\mathbb{P}'}_T - K^{\mathbb{P}'}_t \big| \mathcal{F}^{X,\mathbb{P}'}_{t+}\big],\ 0\leq t\leq T,\ \mathbb{P}\text{–a.s.},\ \forall \mathbb{P}\in\mathcal{P}(\mathbf{x}). \tag{2.2.8}$$

We now state our definition of a solution to (H).

Definition 2.2.7. *We say* $(Y, Z, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}, \partial Y)$ *is a solution to* (H), *if for some* $p > 1$

(*i*) $(Y, Z, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})})$ *is a solution to the* 2BSDE *in* (H) *under* $\mathcal{P}(\mathbf{x})$;

(*ii*) $\partial Y \in \mathbb{S}^{p,2}_{\mathbf{x}}(\mathbb{F}^{X,\mathcal{P}(\mathbf{x})}_+)$;

(*iii*) *there exists* $\nu^\star\in\mathcal{A}(\mathbf{x})$ *such that*

$$0 = \mathbb{E}^{\mathbb{P}^{\nu^\star}}\Big[K^{\mathbb{P}^{\nu^\star}}_T - K^{\mathbb{P}^{\nu^\star}}_t\Big|\mathcal{F}^{X,\mathbb{P}^{\nu^\star}}_{t+}\Big],\ 0\leq t\leq T,\ \mathbb{P}^{\nu^\star}\text{–a.s.}$$

An immediate result about system (H) is that it is indeed a generalisation of the system of PDEs given in [35] in the Markovian framework. This builds upon the fact second-order, parabolic, fully nonlinear PDEs of HJB type admit a non-linear Feynman–Kac representation formula.

Theorem 2.2.8. *Consider the Markovian setting,* i.e. $\psi_t(X,\cdot)=\psi_t(X_t,\cdot)$ *for* $\psi = b,\sigma,f,\partial_s f$, *and,* $\psi(s,X)=\psi(s,X_T)$ *for* $\psi=\partial_s\xi,\xi$. *Assume that*

(*i*) *there exists a unique A-valued Borel-measurable map* $\overline{\mathcal{V}}^\star(t,x,z,\gamma)$ *satisfying*

$$[0,T]\times\mathcal{X}\times\mathbb{R}^d\times\mathbb{S}_d(\mathbb{R})\ni(t,x,z,\gamma)\longmapsto\overline{\mathcal{V}}^\star(t,x,z,\gamma)\in\underset{a\in A}{\arg\max}\,\{\mathsf{h}_t(t,x,z,\gamma,a)\};$$

(*ii*) *for* $(s,t,x)\in[0,T)\times[0,T]\times\mathbb{R}^d:=\mathcal{O}$, *there exists* $(v(t,x),J(s,t,x))\in\mathcal{C}_{1,2}([0,T]\times\mathbb{R}^d)\times\mathcal{C}_{1,1,2}([0,T]^2\times\mathbb{R}^d)$ *classical solution to the system*[3]

$$\begin{cases}\partial_t V(t,x)+H(t,x,\partial_x V(t,x),\partial_{xx}V(t,x),\partial_s\mathcal{J}(t,t,x))=0,\ (s,t,x)\in\mathcal{O},\\[2ex] \partial_t\mathcal{J}^s(t,x)+\mathsf{h}^s(t,x,\partial_x\mathcal{J}^s(t,x),\partial_{xx}\mathcal{J}^s(t,x),\nu^\star(t,x))=0,\ (s,t,x)\in\mathcal{O},\\[2ex] V(T,x)=\xi(T,x),\ \mathcal{J}^s(T,x)=\xi(s,x),\ (s,x)\in[0,T]\times\mathbb{R}^d.\end{cases} \tag{2.2.9}$$

[3]Following [35], for a function $(s,\cdot)\longmapsto\varphi(s,\cdot)$, $\varphi^s(\cdot)$, stresses that the s coordinate is fixed.

where $\nu^\star(t,x) := \overline{\mathcal{V}}^\star(t,x,\partial_x V(t,x),\partial_{xx}V(t,x))$.

(iii) v, $\partial_x v$, J, $\partial_x J$ *and* $\overline{f}(s,t,x) := f(s,t,x,\nu^\star(t,x))$ *have uniform exponential growth in* x[4], i.e. $\exists C>0,\ \forall (s,t,x)\in[0,T]^2\times\mathbb{R}^d$,

$$|v(t,x)| + |\partial_x v(t,x)| + |J(s,t,x)| + |\partial_x J(s,t,x)| + |\overline{f}(s,t,x)| \leq C\exp(C|x|_1),$$

Then, a solution to the 2BSDE *in* (H) *is given by*

$$Y_t := v(t,X_t),\ Z_t := \partial_x v(t,X_t),\ K_t := \int_0^t k_r \mathrm{d}r,\ \partial Y_t^s := \partial_s J^s(t,X_t),$$

where $k_t := H(t,X_t,Z_t,\Gamma_t,\partial Y_t^t) - F_t(X_t,Z_t,\widehat{\sigma}_t^2,\partial Y_t^t) - \frac{1}{2}\mathrm{Tr}[\widehat{\sigma}_t^2\Gamma_t]$ *and* $\Gamma_t := \partial_{xx}v(t,X_t)$. *Moreover, if for* $\nu_t^\star := \overline{\mathcal{V}}^\star(t,X_t,Z_t,\Gamma_t)$ *there exists* $\mathbb{P}^\star \in \mathcal{P}(\mathbf{x},\nu^\star)$, *then* $(Y,Z,K,\partial Y)$ *and* $\nu^\star$ *are a solution to* (H).

Remark 2.2.9. *We highlight the assumption on* $\mathcal{P}(\mathbf{x},\nu^\star)$ *is satisfied, for instance, if the map* $x \longmapsto \sigma\sigma_t^\top(x,\nu^\star(t,x))$ *is continuous for every* t, *see* [232, Theorem 6.1.6]. *Note that the latter is a property of* (2.2.9) *itself. In addition, we also remark that under the assumptions in the verification theorem in* [34, Theorem 5.2], *it is immediate that* (H) *admits a solution.*

The next sub-sections present the remaining of our results. The first of them is about the necessity of our system. We show that given an equilibrium and the associated game value function, one can construct a solution to (H). The second result is about the sufficiency of our system, *i.e.* a verification result. In words, it says that from a solution to (H) one can recover an equilibrium. Our last result is about the well-posedness of the system (H) when volatility control is forbidden.

2.2.4 Necessity of (H)

The next result, familiar to those acquainted with the literature on optimal stochastic control, is new in the context of time-inconsistent control problems for sophisticated agents. Up until now, the study of such problems, regardless of the notion of equilibrium considered, remained limited to a verification argument, and the study of multiplicity of equilibria. Ever since the work of [38,

[4]For $x\in\mathbb{R}^d$, $|x|_1 = \sum_{i=1}^d |x_i|$.

Section 5], it had been conjectured that, in a Markovian setting, given an equilibrium $\nu^\star$ and its value function v, the latter would satisfy the associated system of PDEs (2.2.9), and $\nu^\star$ would attain the supremum in the associated Hamiltonian. Nevertheless, according to [38] this remained an open and difficult problem. Fortunately, capitalising on the DPP satisfied by any equilibrium and our probabilistic approach, we are able to present a proof of this claim in a general non-Markovian setting.

Assumption B. (i) *There exists* $p > 1$ *such that for every* $(s,t,x) \in [0,T]^2 \times \mathcal{X}$

$$\sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\bigg[|\xi(T,X)|^p + |\partial_s \xi(s,X)|^p + \int_t^T |F_r(X,0,\widehat{\sigma}_r^2,0)|^p + |\partial_s f_r(s,X,\nu_r^\star)|^p \mathrm{d}r\bigg] < \infty.$$

(ii) $\mathbb{R}^d \ni z \longmapsto F_t(x,z,\Sigma,u)$ *is Lipschitz-continuous, uniformly in* (t,x,Σ,u)*,* i.e. $\exists C > 0$*, s.t.* $\forall (t,x,\Sigma,u,z,z') \in [0,T]\times\mathcal{X}\times\Sigma_t(x)\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$,

$$|F_t(x,z,\Sigma,u) - F_t(x,z',\Sigma,u)| \leq C|\Sigma^{1/2}(z-z')|.$$

(iii) $\mathbb{R}^d \ni z \longmapsto \mathcal{V}^\star(t,x,z) \in A$ *is Lipschitz-continuous, uniformly in* $(t,x)\in[0,T]\times\mathcal{X}$*,* i.e. $\exists C > 0$ *s.t.* $\forall (t,x,z,z') \in [0,T]\times\mathcal{X}\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\big|\mathcal{V}^\star(t,x,z) - \mathcal{V}^\star(t,x,z')\big| \leq C|z-z'|.$$

Theorem 2.2.10 (Necessity)**.** *Let* Assumption A *and* Assumption B *hold. Given* $\nu^\star \in \mathcal{E}(\mathbf{x})$*, one can construct* $(Y,Z,(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})},\partial Y)$ *solution to* (H)*, such that for any* $t\in[0,T]$ *and* $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$v(t,x) = \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}[Y_t].$$

Moreover, $\nu^\star$ *satisfies* Definition 2.2.7.(iii)*,* i.e. $\nu^\star$ *is a maximiser of the Hamiltonian.*

Remark 2.2.11. (i) *We stress that in light of* Theorem 2.2.10*, every equilibrium must necessarily maximise the Hamiltonian associated to* (P)*. This is, to the best of our knowledge, the first time such a statement is rigorously justified in the framework of time-inconsistent control problems at the level of generality of this paper.*

(*ii*) *Even for Markovian time-consistent control problems, we recall that necessity results are quite technical and typically require the theory of viscosity solutions, see* Fleming and Soner [99]*. To appreciate the scope of* Theorem 2.2.10*, we recall that Markovian* BSDEs (*resp.* 2BSDEs) *coincide with viscosity* (*resp. Sobolev type*) *solutions of* PDEs (*resp. path dependent* PDEs)*, see* [273, Theorem 5.5.8] (*resp.* [273, Proposition 11.3.8]) .

(*iii*) *We also comment on* [173, Theorem 3.11] *which states that given a regular equilibria, see* [173, Definition 3.7]*, one can define a classic solution the* PDE *system in* [38], i.e. (2.2.9)*. In the Markovian setting of* [173]*, regular equilibria render, by definition, smooth classic solutions to the value function and the decoupled pay-off functionals. Not surprisingly, one can construct a classic solution to* (2.2.9) *by means of Itô's formula. However, even for time-consistent problems this assumption rarely holds. Moreover, regular equilibria, which are feedback Markovian, are* a priori *required to be continuous. This contrast with our non-Markovian framework and the fact that we take admissible actions and equilibria to be, in general, measurable.*

2.2.5 Verification

As is commonplace for control problems, we are able to prove the sufficiency of our system. Indeed, our notion of equilibrium is captured by solutions to System (H). Our result is not the first of its kind, though our framework allows us to state a fairly simple proof with clear arguments. For instance, [35, Theorem 6.3] is stated to be a direct consequence of the simpler case of a state dependant final reward functional plus a non-linear functional of a conditional expectation of the final state value, but no proof is presented. Also, the proof of [260, Theorem 6.2] requires laborious arguments, as a consequence of the notion of equilibrium considered, and relies heavily on PDE arguments. Our theorem requires the following set of assumptions.

Assumption C. (*i*) *There exists* $p>1$ *such that for every* $(s,t,x)\in[0,T]^2\times\mathcal{X}$

$$\sup_{\mathbb{P}\in\mathcal{P}(t,x)}\mathbb{E}^{\mathbb{P}}\bigg[|\xi(T,X)|^p+|\partial_s\xi(s,X)|^p+\int_t^T|F_r(X,0,\widehat{\sigma}_r^2,0)|^p+|\partial_s f_r(s,X,\nu_r^\star)|^p\mathrm{d}r\bigg]<\infty;$$

(*ii*) $s\longmapsto\xi(s,x)$ (*resp.* $s\longmapsto f_t(s,x,a)$) *is continuously differentiable uniformly in* x (*resp. in* (t,x,a));

(*iii*) $x \longmapsto \Phi(r,x) := \partial_s \xi(r,x) + \int_r^T \partial_s f_u(r,x,a)\mathrm{d}u$ *is continuous in* x.

Theorem 2.2.12 (Verification). *Let* Assumption C *hold. Let* $(Y, Z, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}, \partial Y)$ *be a solution to* (H) *as in* Definition 2.2.7 *with* $\nu_t^\star := \mathcal{V}^\star(t, X_t, Z_t)$. *Then,* $\nu^\star \in \mathcal{E}(\mathbf{x})$ *and for* $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$v(t,x) = \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}[Y_t].$$

We stress that together, Theorem 2.2.10 and Theorem 2.2.12 imply that System (H) is fundamental for the study of time-consistent stochastic control problems for sophisticated agents.

2.2.6 Well-posedness

Our analysis would not be complete without a well-posedness result. The well-posedness result we present is limited to the drift control case, see Section 2.9. In this framework, there is a unique weak solution to (2.1.7) which we will denote by $\mathbb{P}$. In the context of PDEs, under a different and stronger notion of equilibrium, a well-posedness result for the corresponding system of PDEs was given in [260]. Nonetheless, as we present a probabilistic argument as opposed to an analytic one, our proof makes substantial improvements in both weakening the assumptions as well as the presentation and readability of the arguments. We state next our definition of a solution to $(\mathrm{H_o})$.

Definition 2.2.13. *The pairs* (Y, Z), *and* $(\partial Y, \partial Z)$ *are a* solution *to the system* $(\mathrm{H_o})$ *if for some* $p > 1$

(*i*) $(Y,Z) \in \mathbb{S}^p_{\mathbf{x}}(\mathbb{F}^{X,\mathbb{P}}_+) \times \mathbb{H}^p_{\mathbf{x}}(\mathbb{F}^{X,\mathbb{P}}_+, X)$ *satisfy the first equation in* $(\mathrm{H_o})$ $\mathbb{P}$–a.s.;

(*ii*) $(\partial Y, \partial Z) \in \mathbb{S}^{p,2}_{\mathbf{x}}(\mathbb{F}^{X,\mathbb{P}}_+) \times \mathbb{H}^{p,2}_{\mathbf{x}}(\mathbb{F}^{X,\mathbb{P}}_+, X)$ *and for every* $s \in [0,T]$, $(\partial Y^s, \partial Z^s)$ *satisfies the second equation in* $(\mathrm{H_o})$ $\mathbb{P}$–a.s.;

(*iii*) *there exists* $\nu^\star \in \mathcal{A}(\mathbf{x})$ *such that*

$$H_t^o(X, Z_t, \partial Y_t^t) = h_t^o(t, X, Z_t, \partial Y_t^t, \nu_t^\star)\ \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e. on } [0,T]\times\mathcal{X}.$$

Our well-posedness result in the uncontrolled volatility case is subject to the following assumption.

Assumption D. *(i)* $\sigma_t(x) := \sigma_t(x,a)$ *for any* $a \in A$, i.e. *the volatility is not controlled*;

(ii) $s \longmapsto \xi(s,x)$ *(resp.* $s \longmapsto f_t(s,x,a)$*) is continuously differentiable uniformly in* x *(resp. in* (t,x,a)*)*;

(iii) $z \longmapsto H^o_t(x,z,u)$ *is Lipschitz-continuous, uniformly in* (t,x,u), i.e. $\exists C > 0$, *s.t.* $\forall (t,x,u,z,z') \in [0,T] \times \mathcal{X} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$|H^o_t(x,z,u) - H^o_t(x,z',u)| \leq C|\sigma_t(x)^{1/2}(z-z')|;$$

(iv) $z \longmapsto \mathcal{V}^\star(t,x,z) \in A$ *is Lipschitz-continuous, uniformly in* (t,x), i.e. $\exists C > 0$, *s.t.* $\forall (t,x,z,z') \in [0,T] \times \mathcal{X} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\big|\mathcal{V}^\star(t,x,z) - \mathcal{V}^\star(t,x,z')\big| \leq C|\sigma_t(x)^{1/2}(z-z')|;$$

(v) $(z,a) \longmapsto \partial h^o_t(s,x,a)$ *is Lipschitz-continuous, uniformly in* (s,t,x), i.e. $\exists C > 0$ *s.t.* $\forall (s,t,x,z,z',a,a') \in [0,T]^2 \times \mathcal{X} \times \mathbb{R}^d \times \mathbb{R}^d \times A^2$

$$|\nabla h^o_t(s,x,z,a) - \nabla h^o_t(s,x,z',a')| \leq C(|\sigma_t(x)^{1/2}(z-z')| + |a-a'|);$$

(vi) there exists $p > 1$ *such that for any* $(s,t,x) \in [0,T]^2 \times \mathcal{X}$

$$\mathbb{E}^{\mathbb{P}}\bigg[\big|\xi(T,X)\big|^p + \int_t^T \big|H^o_t(x,0,0)\big|^p \mathrm{d}r\bigg] + \mathbb{E}^{\mathbb{P}}\bigg[\big|\partial_s \xi(s,X)\big|^p + \int_t^T \big|\nabla h^o_t(s,x,0,0)\big|^p \mathrm{d}r\bigg] < \infty.$$

Remark 2.2.14. *We would like to comment on the previous set of assumptions. We will follow a classic fix point argument to get the well-posedness of* System $(\mathrm{H_o})$, *thus* Assumption D *consists of a tailor-made version of the classic requirements to get a contraction in a Lipschitz context. Conditions* (*iii*), (*iv*) *and* (*v*) *will guarantee the Lipschitz property of the drivers. Condition* (*ii*) *will be exploited to control the coupling between the two* BSDEs.

The technical but simple results regarding well-posedness are deferred to Chapter 3. In fact, we are able to establish a well-posedness result for a more general class of systems, see System $(\mathcal{S})$ for which we allow for orthogonal martingales. Moreover, we stress that coupled systems as the ones considered in this work, where the coupling is via an uncountable family of BSDEs, have not been considered before in the literature.

Theorem 2.2.15 (Wellposedness)**.** *Let* Assumption D *hold with* $p = 2$. *There exists a unique solution, in the sense of* Definition 2.2.13*, to* ($\mathrm{H_o}$) *with* $p = 2$*.*

We would like to mention here that the assumption $p = 2$ is by no means crucial in our analysis and our results hold in the general case $p > 1$, a fact that should be clear to our readers familiar with the theory of BSDEs. Nevertheless, hoping to keep our arguments simple and to not drown them in additional unnecessary technicalities, we have opted to present the case $p = 2$ only. In this case, it is easier to distinguish between the essential ideas behind our assumptions, and how they work into the probabilistic framework we propose for the study of (P) and ($\mathrm{H_o}$).

2.3 Example: optimal investment

We consider the following non-exponential discounting framework

$$\xi(s,x) = \varphi(T-s)\tilde{\xi}(x),\ f_t(s,x,a) = \varphi(t-s)\tilde{f}_t(x,a),\ (t,s,x,a) \in [0,T]^2 \times \mathcal{X} \times A,$$

where $\tilde{\xi}$ is a Borel-map, and

$$\tilde{f} : [0,T] \times \mathcal{X} \times A \longrightarrow \mathbb{R},\ \text{Borel, with } \tilde{f}_\cdot(\cdot,a)\ \mathbb{F}^X\text{-optional for any } a \in A,$$

and

$$\varphi : [0,T] \longrightarrow \mathbb{R},\ \text{non-negative, differentiable, with } \varphi(0) = 1.$$

Let us assume in addition $d = m = 1$, $A = \mathbb{R}_+^\star \times \mathbb{R}_+^\star$ and a Markovian framework. We define an action process ν as a 2-dimensional $\mathbb{F}^X$-adapted process $(\alpha_t(X_t), c_t(X_t))_{t\in[0,T]}$ with exponential moments of all orders bounded by some arbitrary large constant M. We let $\sigma_t(X_t,\nu_t) := \alpha_t(X_t)$, $b_t(X_t,\nu_t) := \beta + \alpha_t^{-1}(X_t)(rX_t - c_t(X_t))$. Consequently, for $\mathbb{M} = (\mathbb{P},\nu) \in \mathfrak{M}(s,x)$, with $\mathfrak{M}(s,x) =$

$\{(\mathbb{P},\nu)\in \text{Prob}(\Omega)\times\mathcal{A}(s,x,\mathbb{P})\}$

$$\mathrm{d}X_t^{s,x,\nu}=\alpha_t(\beta+\alpha_t^{-1}(rX_t-c_t)\mathrm{d}t+\mathrm{d}W_t^{\mathbb{M}}),\ \overline{\mathbb{P}}^{\nu}\text{–a.s.}$$

Here we will study the case where the utility function is given, for all $x\in\mathbb{R}$ by

$$U(x):=\begin{cases}\dfrac{x^{1-\eta}-1}{1-\eta}, & \eta\in(0,1),\\ \log(x), & \eta=1,\end{cases}$$

so that

$$J(s,t,x,\nu)=\mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[\varphi(T-s)U(X_T)+\int_t^T\varphi(r-s)U(c_r)\mathrm{d}r\bigg].$$

This model, studied for specific cases of U and φ in [82] and [37], represents an agent who is seeking to find investment and consumption plans, in cash value, α and c respectively, which determine the wealth process X. he derives utility only from consumption. At time s, the present utility from consumption at time r is discounted according to $\varphi(r-s)$. Here, we present a solution based on the verification theorem introduced in this chapter. The system (H) is defined by

$$Y_t=U(X_T)+\int_t^T F_r(X_r,Z_r,\widehat{\sigma}_r^2,\partial Y_r^r)\mathrm{d}r-\int_t^T Z_r\cdot\mathrm{d}X_r+K_T^{\mathbb{P}}-K_t^{\mathbb{P}},\ 0\leq t\leq T,\ \mathcal{P}(\mathbf{x})\text{–q.s.},$$
$$\partial Y_t^s(\omega):=\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}\bigg[\partial_s\varphi(T-s)U(X_T)+\int_t^T\partial_s\varphi(r-s)U(c^\star(r,X_r,Z_r))\mathrm{d}r\bigg],\ (s,t)\in[0,T]^2,\ \omega\in\Omega,$$

where for $(t,x,z,\gamma,\Sigma)\in[0,T]\times\mathcal{X}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+^\star$

$$F_t(x,z,\Sigma,u)=\sup_{(c,\alpha)\in\mathbb{R}_+^\star\times\{\alpha^2=\Sigma\}}\{(rx+\beta\alpha-c)z+U(c)\}-u,$$
$$H_t(x,z,\gamma)=\sup_{\Sigma\in\mathbb{R}_+^\star}\Big\{F_t(x,z,\Sigma,u)+\frac{1}{2}\Sigma\gamma\Big\}.$$

Proposition 2.3.1. *Let (Y,Z,K) and $\nu^\star$ be given by*

$$Y_t:=a(t)U(X_t)+b(t),\ Z_t:=a(t)X_t^{-\eta},\ K_t:=\int_0^t k_r\mathrm{d}r,\ \nu_t^\star:=(\beta\eta^{-1}X_t,a(t)^{-\frac{1}{\eta}}X_t),$$

where

$$k_t := H(t, X_t, Z_t, \Gamma_t, \partial Y_t^t) - F_t(X_t, Z_t, \widehat{\sigma}_t^2, \partial Y_t^t) - \frac{1}{2}\mathrm{Tr}[\widehat{\sigma}_t^2 \Gamma_t],\ \Gamma_t := -\eta a(t) X_t^{-(1+\eta)},\ t \in [0,T],$$

∂Y^s *is defined as above, and* $a(t)$ *and* $b(t)$ *as in* Section 2.10.1*, which we assume has a unique and continuous solution. Then, there exists* $\mathbb{P}^\star \in \mathcal{P}(\mathbf{x}, \nu^\star)$*,* $(Y, Z, K, \partial Y)$ *define a solution to* (H) *and* $(\mathbb{P}^\star, \nu^\star)$ *is an equilibrium model. Moreover*

$$\mathrm{d}X_t = X_t\Big(\big(r + \beta^2\eta^{-1} + a(r)^{-\frac{1}{\eta}}\big)\mathrm{d}t + \beta\eta^{-1}\mathrm{d}W_t\Big),\ 0 \leq t \leq T,\ \overline{\mathbb{P}}^{\nu^\star}\text{–a.s.}$$

Proof. By computing the Hamiltonian, we obtain $(c^\star(t,x,z), \alpha^\star(t,x,z,\gamma)) := (z^{-\frac{1}{\eta}}, |\beta z \gamma^{-1}|)$ define the maximisers in F and H respectively. Therefore, in this setting, the 2BDE in (H) can be rewritten for any $\mathbb{P} \in \mathcal{P}(\mathbf{x})$. Indeed, $\mathbb{P}$–a.s. for $t \in [0,T]$ we have that

$$Y_t = U(X_T) + \int_t^T \Big(Z_r(rX_r + \beta\widehat{\sigma}_r - Z_r^{-\frac{1}{\eta}}) + U(Z_r^{-\frac{1}{\eta}}) - \partial Y_r^r\Big)\mathrm{d}r - \int_t^T Z_r \cdot \mathrm{d}X_r + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}.$$

Moreover, taking Y, Z and K as in the statement we obtain $\nu_t^\star := (\alpha_t^\star, c_t^\star) = (\beta\eta^{-1}X_t, a(t)^{-\frac{1}{\eta}}X_t)$, $t \in [0,T]$. Note that the map $(t, x, z, u) \longmapsto F_t(x, z, \Sigma, u)$ is clearly continuous for fixed Σ and that the processes X, Y, Z and Γ are continuous in time for fixed ω. This yields, as in the proof of Theorem 2.2.8, that the process K satisfies the minimality condition under every $\mathbb{P} \in \mathcal{P}(\mathbf{x})$. Moreover, note that $x \longmapsto \alpha_t^\star(x)$ is continuous for all $t \in [0,T]$. Therefore, there exists $\mathbb{P}^{\nu^\star} \in \mathcal{A}(\mathbf{x}, \nu^\star)$ such that

$$\mathrm{d}X_t = X_t\Big(\big(r + \beta^2\eta^{-1} + a(r)^{-\frac{1}{\eta}}\big)\mathrm{d}t + \beta\eta^{-1}\mathrm{d}W_t\Big),\ 0 \leq t \leq T,\ \overline{\mathbb{P}}^{\nu^\star}\text{–a.s.}$$

Moreover, we may find $a(t)$ and $b(t)$ given as in Section 2.10.1 so that for any $\mathbb{P} \in \mathcal{P}(\mathbf{x})$

$$\begin{aligned} Y_t &= U(X_T) + \int_t^T h_r(r, X_r, a(r)X_r^{-\eta}, a(t)^{-\frac{1}{\eta}}X_t) - \partial Y_r^r \mathrm{d}r - \int_t^T a(r)X_r^{-\eta} \cdot \mathrm{d}X_r + K_T - K_t, \\ &= U(X_T) + \int_t^T F_r(X_r, X_r, a(r)X_r^{-\eta}, \widehat{\sigma}_r^2, \partial Y_r^r)\mathrm{d}r - \int_t^T a(r)X_r^{-\eta} \cdot \mathrm{d}X_r + K_T - K_t, \end{aligned}$$

where we exploited the fact ∂Y satisfies (2.8.3). Also note that given our choice of $\nu^\star$, $K^{\mathbb{P}^{\nu^\star}} = 0$.

We are left to argue the integrability of Y, Z, K. This follows as in the proof of Theorem 2.2.8 as the uniform exponential growth assumption is satisfied, since $a(t)$ and $b(t)$ are by assumption continuous on $[0,T]$. The integrability follows as the action processes are assumed to have exponential moments of all orders bounded by M. With this we obtained that (Y,Z,K) is a solution to the 2BSDE in (H). The integrability of ∂Y is argued as in the proof of (2.2.10). $\square$

2.4 Extensions of our results

We use this section to present some extensions of our results to more general classes of pay-off functionals. This is motivated by [37, Section 7.4]. We present the corresponding results and defer their proofs to Section 2.10.2. We emphasise that the formulation of the dynamics of the controlled process X remains as in Section 2.1.2. We will consider

$$\begin{aligned}
&\mathsf{f}:[0,T]\times\mathcal{X}\longrightarrow\mathbb{R},\ \text{Borel-measurable, with } \mathsf{f}_\cdot(\cdot)\ \mathbb{F}^X\text{-optional};\ g:\mathcal{X}\longrightarrow\mathbb{R},\ \text{Borel-measurable;}\\
&\xi:[0,T]\times\mathcal{X}\longrightarrow\mathbb{R},\ \text{Borel-measurable; } G:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R},\ \text{Borel-measurable,}\\
&f:[0,T]^2\times\mathcal{X}\times\mathbb{R}\times A\longrightarrow\mathbb{R},\ \text{Borel-measurable, with } f_\cdot(s,\mathrm{n},\cdot,a)\ \mathbb{F}^X\text{-optional,}
\end{aligned}$$

for any $(s,\mathrm{n},a)\in[0,T]\times\mathbb{R}\times A$, and define for $(s,t,x,\nu)\in[0,T]^2\times\mathcal{X}\times\mathcal{A}(t,x)$

$$J(s,t,x,\nu)=\mathbb{E}^{\overline{\mathbb{P}}^\nu_{t,x}}\bigg[\int_t^T f_r\Big(s,X,\mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}[\mathsf{f}_r(X)],\nu_r\Big)\mathrm{d}r+\xi(s,X_{\cdot\wedge T})\bigg]+G\Big(s,\mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}[g(X_{\cdot\wedge T})]\Big).$$

As a motivation for the consideration for this kind of pay-off functionals, notice that the presence of the term $G(s,\mathbb{E}^{\mathbb{P}^\nu_{t,x}}[g(x)])$ allows, for example, to include classic mean–variance models into the analysis. In order to present the corresponding DPP in this framework we need to adapt the notation and assumptions that led to it.

We will continue to use the convention $\partial^2_{\mathrm{nn}}f_t(s,x,\mathrm{n},a):=\frac{\partial^2}{\partial\mathrm{n}^2}f_t(s,x,\mathrm{n},a)$ to denote the respective derivatives. Let $\nu^\star\in\mathcal{E}(\mathbf{x})$ and define

$$M^\star_t(x):=\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}[\mathsf{g}(X_{\cdot\wedge T})],\ N^{s,\star}_t(x):=\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{s,x}}[\mathsf{f}_t(X_{\cdot\wedge t})],\ (s,t,x)\in[0,T]^2\times\mathcal{X}.$$

We emphasise that N defines an infinite family of processes, when considered as functions of s, one for every $t \in [0,T]$. We also recall that under the weak uniqueness assumption, both processes are well–defined. Moreover, we know that the application $(t,x) \longmapsto \mathbb{P}^{\nu^\star}_{t,x}$ is measurable and continuous for the weak topology, see [232, Corollary 6.3.3] and the preceding comments.

To be able to extend our results, we work under the following set of assumptions.

Assumption E. *(i)* $s \longmapsto \xi(s,x)$ *is continuously differentiable uniformly in* x.$(s,\mathsf{m}) \longmapsto G(s,\mathsf{m})$ *belongs to* $\mathcal{C}_{1,2}([0,T]\times\mathbb{R},\mathbb{R})$ *with spatial derivatives Lipschitz-continuous uniformly in* s. $(s,\mathrm{n}) \longmapsto f_t(s,x,\mathrm{n},a)$ *belongs to* $\mathcal{C}_{1,2}([0,T]\times\mathbb{R},\mathbb{R})$ *with spatial derivatives Lipschitz-continuous uniformly in* (s,t,x,a).

(ii) $f_t(s,x,\mathrm{n},a) = \tilde f_t(s,x,a) + \hat f_t(s,\mathrm{n})$ *for an* $\mathbb{F}^X$*-optional (resp. deterministic) mapping* $\tilde f$ *(resp.* $\hat f$*). The mappings* $x \longmapsto \mathsf{f}_t(x)$, $t \longmapsto \mathsf{f}_t(x)$, *and* $x \longmapsto g(x)$ *are continuous uniformly in the other variables.*

(iii) $\exists C > 0$, $\rho : (0,\infty) \longrightarrow [0,\infty)$, $\rho(|\ell|) \longrightarrow 0$, $\ell \longrightarrow 0$, *such that for* $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}, \nu \in \mathcal{A}(t,x)$, $t \leq t' \leq r \leq T$,

$$\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\Big|\mathbb{E}^{\overline{\mathbb{P}}^\nu_{t,\cdot}}[N^{t',\star}_r] - N^{t,\star}_r\Big|^2 + \Big|\mathbb{E}^{\overline{\mathbb{P}}^\nu_{t,\cdot}}[M^\star_{t'}] - M^\star_t\Big|^2\bigg] \leq C|t'-t|\rho(|t'-t|).$$

(iv) $a \longmapsto f_t(s,x,a)$ *is uniformly Lipschitz-continuous,* i.e. $\exists C>0, \forall (s,t,x,a,a') \in [0,T]^2\times\mathcal{X}\times A^2$,

$$\big|f_t(s,x,a) - f_t(s,x,a')\big| \leq C|a-a'|.$$

(v) *(a)* $x \longmapsto \xi(t,x)$ *is lower-semicontinuous uniformly in* t, i.e. $\forall (\tilde x,\varepsilon) \in \mathcal{X}\times\mathbb{R}^\star_+$, $\exists U_{\tilde x} \in \mathfrak{T}_\infty$, $\tilde x \in U_{\tilde x}$, $\forall (t,x) \in [0,T]\times U_{\tilde x}$,

$$\xi(t,x) \geq \xi(t,x_0) - \varepsilon, \ \textit{when}\ \xi(t,x_0) > -\infty.$$

(b) $x \longmapsto (b,\sigma)(t,x,a)$ *is uniformly Lipschitz continuous,* i.e. $\exists C > 0$, *s.t.* $\forall (t,x,x',a) \in [0,T]\times$

$\mathcal{X}^2 \times A$,

$$|b_t(x,a) - b_t(x',a)| + |\sigma_t(x,a) - \sigma_t(x',a)| \leq C\|x_{\cdot\wedge t} - x'_{\cdot\wedge t}\|_\infty.$$

Remark 2.4.1. *We would like to comment on the previous set of assumptions. The extensions of our previous set of assumptions correspond to* (i), (iv) *and* (v). *In addition, the reader might have noticed the assumptions imposed on* f, f *and* g *in* Assumption E.(ii) *and* Assumption E.(iii).

The condition of f *basically disentangles the randomness coming from* $X_{\cdot\wedge r}$ *and* $\mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{s,x}}[f_r(X_{\cdot\wedge r})]$. *This helps us bypass some measurability issues arising from the interaction between these two variables. Given the non-linear dependence of the reward, when passing to the limit in the proof of* Theorem 2.4.3 *below, one should expect that first order,* i.e. *linear, approximations, would not suffice to rigorously obtain the limit. Not surprisingly, it is necessary to have access to the quadratic variations of the previously introduced processes. This is usually carried out by having a pathwise construction of the stochastic integral. For this, a viable way is to follow the approach in* [149]. *It is thus necessary to guarantee that* $M^\star$ *and* $N^{\star,\cdot}_r$ *are left limits of càdlàg processes. In light of the continuity of the map* $(t,x) \longmapsto \mathbb{P}^{\nu^\star}_{t,x}$, Assumption E.$(ii)$ *ensures that these processes are continuous. Hence, there exists a process* $[M^\star]$ *(resp.* $[N^{\star,\cdot}_r]$ *for any* $r \in [0,T]$*) which coincides with the quadratic variation of* $M^\star$ *(resp.* $N^{\star,\cdot}_r$ *for any* $r \in [0,T]$*) under* $\mathbb{P}^{\nu^\star}$.

Moreover, f *and* g *can be understood as* changes of variables *from the canonical process* X. *As such, it is expected to require some control on the quadratic difference under the laws induced by an arbitrary action* ν *and the equilibrium* $\nu^\star$. *This is precisely the goal of* Assumption E.(iii). *We highlight that both processes appearing in* Assumption E.(iii) *are* $\mathcal{F}_t$*-measurable and differ only, from* ν *to* $\nu^\star$, *on the action performed over the interval* $[t,t']$. *In fact, when* $\nu = \nu^\star$ *this condition holds trivially as both expression are equal to zero. In fact, when* $\nu = \nu^\star$ *this condition holds trivially as both expression are equal to zero. It is also possible to verify this condition in the case of uncontrolled volatility if* f *and* g *are regular in the sense of* Cont and Fournié [57] *so that the functional Itô formula holds, see below.*

Lemma 2.4.2. Assumption E.(iii) *holds if either*

(i) *the volatility is uncontrolled,* A *is bounded,* f *has bounded horizontal, first and second order* *vertical derivatives,* $\mathcal{D}_t f$*,* $\nabla_x f$ *and* ∇_x^2*, respectively. Moreover, the process* $\mathfrak{A}(t, X_{\cdot\wedge t}, \nu_t)$ *is square integrable for any* $\nu \in \mathcal{A}$*, where* $\mathfrak{A}(t,x,a) := \mathcal{D}_t f_t(x) + \sigma_t(x) b_u(x,a) \nabla_x f_u(x) + \frac{1}{2}(\sigma\sigma^\top)_u(x) \nabla_x^2 f_u(x)$*;*

(ii) *the problem is in strong formulation with state dependent coefficients, unique strong solution. This is, there is a probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *and a* $\mathbb{P}$*–Brownian motion* W *such that for any* $\nu \in \mathcal{A}(\mathbf{x})$ *there is a unique process* $X^{x,\nu}$ *that satisfies*

$$X_t^{x,\nu} = x_0 + \int_0^t b_r(X_r, \nu_r)\mathrm{d}r + \int_0^t \sigma_r(X_r, \nu_r)\mathrm{d}W_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s..}$$

Moreover, $(\mathsf{f}, \mathsf{g}) \in \mathcal{C}_{1,2}([0,T] \times \mathbb{R}, \mathbb{R}) \times \mathcal{C}_2(\mathbb{R}, \mathbb{R})$ *and* $\mathcal{A}(t, X_t, \nu_t)$ *is square integrable for any* $\nu \in \mathcal{A}$*, where* $\mathcal{A}(t,x,a) := \partial_t \mathsf{f}_t(x) + \sigma_t(x,a) b_u(x,a) \partial_x \mathsf{f}_u(x) + \frac{1}{2}(\sigma\sigma^\top)_u(a,x) \partial_{xx}^2 \mathsf{f}_u(x)$*.*

In light of Remark 2.4.1 we define $\widehat{m}$ and $\widehat{n}$ square roots of the processes

$$\widehat{m}_t^2 := \limsup_{\varepsilon \searrow 0} \frac{[M^\star]_t - [M^\star]_{t-\varepsilon}}{\varepsilon},\ \widehat{n}_r^{t\,2} := \limsup_{\varepsilon \searrow 0} \frac{[N_r^{\cdot,\star}]_t - [N_r^{\cdot,\star}]_{t-\varepsilon}}{\varepsilon},\ (r,t) \in [0,T]^2.$$

Building upon our previous analysis, we can profit from the recent results available in Djete, Possamaï, and Tan [76] to obtain the next dynamic programming principle, Theorem 2.4.3. It concerns the local behaviour of the value function v. We also remark that time-inconsistent McKean–Vlasov problems have been recently studied by Mei and Zhu [184].

Theorem 2.4.3. *Let* Assumption E *hold, and* $\nu^\star \in \mathcal{E}(\mathbf{x})$*. For* $\{\sigma, \tau\} \subseteq \mathcal{T}_{t,T}$*,* $\sigma \leq \tau$ *and* $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$*, we have*

$$\begin{aligned} v(\sigma, x) = \sup_{\nu \in \mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[v(\tau, X) + \int_\sigma^\tau \bigg(f_r(r, X, \mathsf{f}_r(X), \nu_r) - \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,\cdot}}\Big[\partial_s G(r, M_r^\star) - \frac{1}{2}\partial_{\mathsf{mm}}^2 G(r, M_r^\star) \widehat{m}_r^2 \Big] \bigg) \mathrm{d}r \\ - \int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,\cdot}} \Big[\partial_s \xi(r, X) + \int_r^T \Big(\partial_s f_u(r, X, N_u^{r,\star}, \nu_u^\star) + \frac{1}{2} \partial_{\mathrm{nn}}^2 f_u(r, X, N_u^{r,\star}, \nu_u^\star) \widehat{n}_u^{r\,2} \Big) \mathrm{d}u \Big] \mathrm{d}r \bigg] \end{aligned}$$

Analogously, we can associate a system of BSDEs to the problem. Define for $(s,t,x,z,\gamma,\Sigma,u,v,\mathrm{n},\mathrm{z},\mathsf{m},\mathsf{z},a) \in [0,T)^2 \times \mathcal{X} \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R}) \times \mathbb{S}_d(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times A$

$$\begin{aligned} h_t(s,x,z,a) &:= f_t(s,x,\mathsf{f}_t(x),a) + b_t(x,a) \cdot \sigma_t(x,a)^\top z; \\ F_t(x,z,\Sigma,u,\mathsf{m},\mathsf{z}) &:= \sup_{a \in A_t(x,\Sigma)} \{h_t(t,x,z,a)\} - u - \frac{1}{2} \mathsf{z}^\top \Sigma\, \mathsf{z}\, \partial_{\mathsf{mm}}^2 G(t,\mathsf{m}), \end{aligned}$$

and $\mathcal{V}^\star(t,x,z)$ denotes the unique (for simplicity) A-valued Borel-measurable map satisfying

$$[0,T]\times\mathcal{X}\times\mathbb{R}^d \ni (t,x,z) \longmapsto \mathcal{V}^\star(t,x,z) \in \underset{a\in A}{\arg\max}\, h_t(t,x,z,a).$$

In light of Theorem 2.4.3, we consider for $(s,t)\in[0,T]^2$ and $\omega\in\Omega$ the system

$$\begin{aligned}
Y_t &= \xi(T,X) + G(T,g(X)) + \int_t^T F_r(X,Z_r,\widehat{\sigma}_r^2,\partial Y_r^r, M_r^\star,\widehat{m}_t^2)\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\\
\partial Y_t^s(\omega) &= \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}\bigg[\partial_s\xi(s,X) + \int_t^T \Big(\partial_s f_r^\star(s,X,N_r^{\star,s},Z_r) + \frac{1}{2}\partial^2_{\mathrm{nn}} f_r^\star(s,X,N_r^{s,\star},Z_r)\widehat{n}_r^{\star\,2}\Big)\mathrm{d}r\bigg],\\
M_t^\star(\omega) &= \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,\omega}}\big[g(X)\big],\ N_t^{\star,s}(\omega) = \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{s,\omega}}\big[\mathsf{f}_t(X)\big],\ \widehat{m}_t^2 := \frac{\mathrm{d}[M^\star]_t}{\mathrm{d}t},\ \widehat{n}_r^{t\,2} := \frac{\mathrm{d}[N_r^{\star,\cdot}]_t}{\mathrm{d}t},
\end{aligned} \tag{H$_\mathrm{e}$}$$

where $\partial_s f_t^\star(s,x,n,z) := \partial_s f_t^\star(s,x,n,\mathcal{V}^\star(t,x,z))$ and $\partial^2_{\mathrm{nn}} f_t^\star(s,x,n,z) := \partial^2_{\mathrm{nn}} f_t^\star(s,x,n,\mathcal{V}^\star(t,x,z))$.

In the case of drift control only, we define

$$\begin{aligned}
h_t^o(s,x,z,a) &:= f_t(s,x,n,a) + b_t(x,a)\cdot\sigma_t(x)^\top z,\\
\partial h_t^o(s,x,v,\mathrm{n},\mathrm{z},a) &:= \partial_s f_t(s,x,\mathrm{n},a) + b_t(x,a)\cdot\sigma_t(x)^\top v + \frac{1}{2}\mathrm{z}^\top\sigma_t(x)^\top\sigma_t(x)\,\mathrm{z}\,\partial^2_{\mathrm{nn}} f_t(s,x,\mathrm{n},a),\\
H_t^o(x,z,u,\mathsf{m},\mathsf{z}) &:= \sup_{a\in A}\big\{h_t^o(t,x,z,a)\big\} - u - \partial_s G(t,\mathsf{m}) - \frac{1}{2}\,\mathsf{z}^\top\sigma_t(x)^\top\sigma_t(x)\,\mathsf{z}\,\partial^2_{\mathsf{mm}} G(t,\mathsf{m}),
\end{aligned}$$

and (H$_\mathrm{e}$) reduces to the infinite family of BSDEs which holds $\mathbb{P}$–a.s. (recall the notations in Section 2.2.3) for $(s,t)\in[0,T]^2$

$$\begin{aligned}
Y_t &= \xi(T,X) + G(T,g(X)) + \int_t^T H_r^o(X,Z_r,\partial Y_r^r, M_r^\star, Z_r^\star)\mathrm{d}r - \int_t^T Z_r\cdot\mathrm{d}X_r,\\
\partial Y_t^s &= \partial_s\xi(s,X) + \int_t^T \partial h_r^o(s,X,\partial Z_r^s, N_r^{s,\star}, Z_r^{s,\star},\mathcal{V}^\star(r,X,Z_r))\mathrm{d}r - \int_t^T \partial Z_r^s\cdot\mathrm{d}X_r,\\
M_t^\star &= g(X) + \int_t^T b_r(X,\mathcal{V}^\star(r,X,Z_r))\cdot\sigma_r^\top(X)Z_r^\star\mathrm{d}r - \int_t^T Z_r^\star\cdot\mathrm{d}X_r,\\
N_t^{s,\star} &= \mathsf{f}_t(X) + \int_s^T b_r(X,\mathcal{V}^\star(r,X,Z_r))\cdot\sigma_r^\top(X)Z_t^{r,\star}\mathrm{d}r - \int_s^T Z_t^{r,\star}\cdot\mathrm{d}X_r.
\end{aligned} \tag{H$^\mathrm{o}_\mathrm{e}$}$$

In the same way, a necessity theorem holds. It does require us to introduce the following set of assumptions.

Assumption F. Assumption B.(ii) and B.(iii) *together with* *(i) there exists $p > 1$ such that for every $(s,t,x) \in [0,T]^2 \times \mathcal{X}$*

$$\sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\bigg[|\xi(T,X)|^p + |\partial_s\xi(s,X)|^p + |G(T,g(X))|^p + |g(X)|^p + |f_s(X)|^p + \int_t^T |F_r(X,0,\widehat{\sigma}_r^2,0,0,0)|^p + |\partial h_r^o(s,X,0,N_r^{s,\star},\widehat{n}_r^{s,\star},\nu_r^\star)|^p \mathrm{d}r\bigg] < \infty.$$

Theorem 2.4.4 (Necessity)**.** *Let* Assumption E *and* Assumption F *hold. Given $\nu^\star \in \mathcal{E}(\mathbf{x})$, one can construct $(Y, Z, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}, \partial Y)$ solution to* (H)*, such that for any $t \in [0,T]$ and $\mathcal{P}(\mathbf{x})$*–q.e. *$x \in \mathcal{X}$*

$$v(t,x) = \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}[Y_t].$$

Moreover, $\nu^\star$ satisfies Definition 2.2.7.*(iii),* i.e. *$\nu^\star$ is a maximiser of the Hamiltonian.*

Remark 2.4.5. *We would like to comment that the well-posedness of* ($\mathrm{H}_\mathrm{e}^\mathrm{o}$)*,* i.e. *the extended system when only drift control is allowed, remains a much harder task. In particular, it is known that the presence of a non-linear functionals of conditional expectations opens the door to scenarii with multiplicity of equilibria with different game values, see* [164] *for an example in a mean–variance investment problem. Consequently and in line with current results available for systems of* BSDEs *with quadratic growth, see* Frei and Dos Reis [103], Harter and Richou [118]*, and* Xing and Žitković [265]*, we expect to be able to obtain existence of a solution, but not necessarily uniquness.*

Finally, one could also include into the analysis on the current chapter rewards allowing for recursive utilities as studied in [245]. Recall these are models in which the utility at time t is not only a function of the state variable process and the action but also of the *future utility.* This is, for $(t,x,\nu) \in [0,T]^2 \times \mathcal{X} \times \mathcal{A}(t,x)$, the reward in given by

$$J(t,x,\nu) := \mathbb{E}^{\mathbb{P}^\nu}[\mathcal{Y}_t],$$

where $\mathcal{Y}$ is the first coordinate of the solution to the following BSVIE

$$\mathcal{Y}_t = \xi(t,X) + \int_t^T k_r(t,X,\mathcal{Y}_r,\mathcal{Z}_r^s)\mathrm{d}r - \int_t^T \mathcal{Z}_r^s \cdot \mathrm{d}X_r,\ 0 \le t \le T,\ \mathbb{P}\text{–a.s.},$$

where k is a non-anticipative functional that allows for, possibly, non-linear dependences on future utilities. We highlight we have taken expectations with respect to $\mathbb{P} \in \mathcal{A}(t,x)$ in the definition of $J(t,x,\nu)$. As we work under a general filtration, the solution to the previous BSVIE is $\mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})}$ and therefore the need to take expectation. Of course, the analysis would require to derive the corresponding DPP and from it deduce the system encoding equilibria. This will be part of our analysis in Chapter 5.

2.5 Proof of Theorem 2.2.2

This section is devoted to the proof of Theorem 2.2.2, namely we wish to obtain the corresponding *extended* version of the *dynamic programming principle*. We begin with a sequence of lemmata that will allow us to study the local behaviour of the value of the game. These results are true in great generality and require mere extra regularity of the running and terminal rewards in the type variable.

Throughout this section we assume there exists $\nu^\star \in \mathcal{E}(\mathbf{x})$. We stress that no assumption about uniqueness of the equilibrium will be imposed. Therefore, in the spirit of keeping track of the notation, for $(t,x) \in [0,T] \times \mathcal{X}$ we recall we set

$$v(t,x) = J(t,t,x,\nu^\star) = \mathbb{E}^{\mathbb{P}^{\nu^\star}_{t,x}}\bigg[\int_t^T f_r(t,X,\nu_r^\star)\mathrm{d}r + \xi(t,X_{\cdot\wedge T})\bigg],$$

and $(\mathbb{P}^{\nu^\star}_{t,x})_{(t,x)\in[0,T]\times\mathcal{X}}$ denotes the unique solution to the martingale problem for (2.1.7), with initial condition (t,x) and fixed action process $\nu^\star$. Similarly, for $\omega = (x,\mathrm{w},q) \in \Omega$, $\{\sigma,\tau\} \subset \mathcal{T}_{0,T}$, with $\sigma \leq \tau$, and $\nu \in \mathcal{A}(\sigma,x)$ we also set

$$v(\sigma,X)(\omega) := v(\sigma(\omega), x_{\cdot\wedge\sigma(\omega)}), \text{ and } J(\sigma,\tau,X,\nu)(\omega) := J(\sigma(\omega),\tau(\omega),x_{\cdot\wedge\sigma(\omega)},\nu).$$

Our first result consists of a one step iteration of our equilibrium definition. This is in spirit similar to the analysis performed in a discrete-time set up in [38].

Lemma 2.5.1. *Let $\nu^\star \in \mathcal{E}(\mathbf{x})$ and v the value associated to $\nu^\star$ as in* (P). *Then, for $(\varepsilon,\ell,t,\sigma,\tau) \in$*

$\mathbb{R}_+^\star \times (0,\ell_\varepsilon) \times [0,T] \times \mathcal{T}_{t,t+\ell} \times \mathcal{T}_{t,t+\ell}$ *with* $\sigma \leq \tau$, *and* $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$\begin{aligned} v(\sigma,x) &\leq \sup_{\nu\in\mathcal{A}(\sigma,x)} J(\sigma,\sigma,x,\nu\otimes_\tau \nu^\star), \\ v(\sigma,x) &\geq \sup_{\nu\in\mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[v(\tau,X) + \int_\sigma^\tau f_r(\sigma,X,\nu_r)\mathrm{d}r + J(\sigma,\tau,X,\nu^\star) - J(\tau,\tau,X,\nu^\star)\bigg] - \varepsilon\ell. \end{aligned} \tag{2.5.1}$$

Proof. The first inequality is clear. Indeed for $\mathcal{P}(\mathbf{x})$–q.e. $x\in\mathcal{X}$, $\overline{\mathbb{P}}^{\nu^\star}_{\sigma(\omega),x} \in \mathcal{P}(\sigma(\omega),x)$ and therefore $\nu^\star \in \mathcal{A}(\sigma,x)$.

To get the second inequality note that for $(\varepsilon,\ell,t,\sigma,\tau) \in \mathbb{R}_+^\star \times (0,\ell_\varepsilon) \times [0,T] \times \mathcal{T}_{t,t+\ell} \times \mathcal{T}_{t,t+\ell}$ with $\sigma\leq\tau$, $\mathcal{P}(\mathbf{x})$–q.e. $x\in\mathcal{X}$ and $\nu \in \mathcal{A}(\sigma,x)$

$$\begin{aligned} v(\sigma,x) &= J(\sigma,\sigma,x,\nu^\star) \geq J(\sigma,\sigma,x,\nu\otimes_\tau\nu^\star) - \varepsilon\ell \\ &= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}}\bigg[\int_\sigma^\tau f_r(\sigma,X,(\nu\otimes_\tau\nu^\star)_r)\mathrm{d}r + \int_\tau^T f_r(\sigma,X,(\nu\otimes_\tau\nu^\star)_r)\mathrm{d}r + \xi(\sigma,X_{\cdot\wedge T})\bigg] - \varepsilon\ell \\ &= \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[v(\tau,X) + \int_\sigma^\tau f_r(\sigma,X,\nu_r)\mathrm{d}r + J(\sigma,\tau,X,\nu^\star) - J(\tau,\tau,X,\nu^\star)\bigg] - \varepsilon\ell, \end{aligned}$$

where the last equality follows by conditioning and the $\mathcal{F}_\tau$-measurability of all the terms. Indeed, in light of Lemma 2.10.3.1, an r.c.p.d. of $\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}$ with respect to $\mathcal{F}_\tau$, evaluated at x, agrees with $\overline{\mathbb{P}}^{\nu^\star}_{\tau(x),x}$, the weak solution to (2.1.7) with initial condition (τ,x) and action $\nu^\star$, for $\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}$–a.e. $x\in\mathcal{X}$. As all the terms inside the expectation are $\mathcal{F}_\tau$-measurable, the previous holds for $\overline{\mathbb{P}}^\nu$–a.e. $x\in\mathcal{X}$. □

From the previous result, we know that equilibrium models satisfy a form of ε-optimality in a sufficiently small window of time. We now seek to gain more insight from iterating the previous result. This will allow us to move forward the time window into consideration.

In the following, given $(\sigma,\tau)\in\mathcal{T}_{t,T}\times\mathcal{T}_{t,t+\ell}$, with $\sigma\leq\tau$, we denote by $\Pi^\ell := (\tau_i^\ell)_{i\in\{1,\dots,n_\ell\}} \subseteq \mathcal{T}_{t,T}$ a generic partition of $[\sigma,\tau]$ with mesh smaller than ℓ, *i.e.* for $n_\ell := \lceil(\tau-\sigma)/\ell\rceil$, $\sigma =: \tau_0^\ell \leq \cdots \leq \tau_{n^\ell}^\ell := \tau$, $\forall\ell$, and $\sup_{1\leq i\leq n_\ell}|\tau_i^\ell - \tau_{i-1}^\ell| \leq \ell$. We also let $\Delta\tau_i^\ell := \tau_i^\ell - \tau_{i-1}^\ell$. The previous definitions hold ω-by-ω.

Proposition 2.5.2. *Let* $\nu^\star\in\mathcal{E}(\mathbf{x})$ *and* $\{\sigma,\tau\}\subset\mathcal{T}_{t,T}$, *with* $\sigma\leq\tau$. *Fix* $\varepsilon>0$ *and some partition*

Π^ℓ *with* $\ell < \ell_\varepsilon$. *Then for* $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$v(\sigma,x) \geq \sup_{\nu\in\mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[v(\tau,X) + \sum_{i=0}^{n_\ell-1}\int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell,X,\nu_r)\mathrm{d}r \\ + \sum_{i=0}^{n_\ell-1} J(\tau_i^\ell,\tau_{i+1}^\ell,X,\nu^\star) - J(\tau_{i+1}^\ell,\tau_{i+1}^\ell,X,\nu^\star) - n_\ell\varepsilon\ell\bigg].$$

Proof. A straightforward iteration of Lemma 2.5.1 yields that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$\begin{aligned} v(\sigma,x) &\geq \sup_{\nu\in\mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^{\tau_1^\ell} f_r(\sigma,X,\nu_r)\mathrm{d}r + J(\sigma,\tau_1^\ell,X,\nu^\star) - J(\tau_1^\ell,\tau_1^\ell,X,\nu^\star) + v(\tau_1^\ell,X) - \varepsilon\ell\bigg] \\ &\geq \sup_{\nu\in\mathcal{A}(\sigma,x)} \int_\Omega \bigg(\int_\sigma^{\tau_1^\ell} f_r(\sigma,X,\nu_r)\mathrm{d}r + J(\tau_0^\ell,\tau_0^\ell,X,\nu^\star) - J(\tau_1^\ell,\tau_1^\ell,X,\nu^\star) \\ &\qquad + \sup_{\tilde\nu\in\mathcal{A}(\tau_1^\ell,\tilde x)} \mathbb{E}^{\widetilde{\mathbb{P}}^{\tilde\nu}}\bigg[v(\tau_2^\ell,X) + \int_{\tau_1^\ell}^{\tau_2^\ell} f_r(\tau_1^\ell,X,\tilde\nu_r)\mathrm{d}r \\ &\qquad\qquad + J(\tau_1^\ell,\tau_2^\ell,X,\nu^\star) - J(\tau_2^\ell,\tau_2^\ell,X,\nu^\star) - 2\varepsilon\ell\bigg]\bigg)\overline{\mathbb{P}}^\nu(\mathrm{d}\tilde\omega) \\ &= \sup_{\nu\in\mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^{\tau_1^\ell} f_r(\sigma,X,\nu_r)\mathrm{d}r + J(\tau_0^\ell,\tau_1^\ell,X,\nu^\star) - J(\tau_1^\ell,\tau_1^\ell,X,\nu^\star) \\ &\qquad + v(\tau_2^\ell,X) + \int_{\tau_1^\ell}^{\tau_2^\ell} f_r(\tau_1^\ell,X,\nu_r)\mathrm{d}r + J(\tau_1^\ell,\tau_2^\ell,X,\nu^\star) - J(\tau_2^\ell,\tau_2^\ell,X,\nu^\star) - 2\varepsilon\ell\bigg], \end{aligned}$$

where the second inequality follows by applying the definition of an equilibrium at (τ_1^ℓ, X). Now, the last step follows from [85, Theorem 4.6.], see also [190, Theorem 2.3.], which holds thanks to [86]. Indeed, as $\mathbb{F}$ is countably generated and $\mathcal{P}(t,x) \neq \varnothing$, for all $(t,x) \in [0,T]\times\mathcal{X}$, [86, Lemmata 3.2 and 3.3] hold. The general result follows directly by iterating and the fact the iteration is finite. □

In the same spirit as in the classic theory of stochastic control, a natural question at this point is whether there is, if any, an infinitesimal limit of the previous iteration and what kind of insights on the value function we can draw from it. The next theorem shows than under a mild extra regularity assumption on the running cost, namely Assumption A.(i), we can indeed pass to the limit.

To ease the readability of our main theorem, for $x \in \mathcal{X}$, $\{\sigma,\zeta,\tau\} \subset \mathcal{T}_{0,T}$ with $\sigma \leq \zeta \leq \tau$,

$\nu^\star \in \mathcal{A}(\mathbf{x})$, $\nu \in \mathcal{A}(\sigma, x)$, and any $\mathcal{F}_T^X$-measurable random variable ξ, we introduce the notation

$$\mathbb{E}_\zeta^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}[\xi] := \mathbb{E}^{\overline{\mathbb{P}}^\nu \otimes_\zeta \overline{\mathbb{P}}^{\nu^\star}_{\zeta,X}}[\xi],$$

where $\mathbb{P}^{\nu^\star}_{\zeta,X}$ is given by $\omega \longmapsto \mathbb{P}^{\nu^\star}_{(\zeta(\omega), x_{\cdot\wedge\zeta(\omega)})}$ and denotes the $\mathcal{F}_\zeta$-kernel prescribed by the family of solutions to the martingale problem associated with $\nu^\star$, see [232, Theorem 6.2.2]. Note in particular $\mathbb{E}_\sigma^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}[\xi] = \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\sigma,X}}[\xi]$.

Proof of Theorem 2.2.2. Let $\varepsilon > 0$, $0 < \ell < \ell_\varepsilon$ and Π^ℓ be as in the statement of Proposition 2.5.2. From Proposition 2.5.2 we know that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$\begin{aligned}
v(\sigma, x) \geq \sup_{\nu \in \mathcal{A}(\sigma,x)} \bigg\{ \mathbb{E}^{\overline{\mathbb{P}}^\nu} \bigg[& v(\tau, X) + \sum_{i=0}^{n_\ell - 1} \int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X, \nu_r) \mathrm{d}r \\
& + \sum_{i=0}^{n_\ell-1} J(\tau_i^\ell, \tau_{i+1}^\ell, X, \nu^\star) - J(\tau_{i+1}^\ell, \tau_{i+1}^\ell, X, \nu^\star) - n_\ell \varepsilon \ell \bigg] \bigg\} \\
= \sup_{\nu \in \mathcal{A}(\sigma,x)} \bigg\{ \mathbb{E}^{\overline{\mathbb{P}}^\nu} \bigg[& v(\tau, X) + \sum_{i=0}^{n_\ell - 1} \int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X, \nu_r) \mathrm{d}r - n_\ell \varepsilon \ell \bigg] \\
+ \sum_{i=0}^{n_\ell-1} \int_\Omega \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1}^\ell(\tilde\omega), X(\tilde\omega)}} \bigg[& \int_{\tau_{i+1}^\ell}^T \big(f_r(\tau_i^\ell, X, \nu_r^\star) - f_r(\tau_{i+1}^\ell, X, \nu_r^\star) \big) \mathrm{d}r \\
& + \xi(\tau_i^\ell, X) - \xi(\tau_{i+1}^\ell, X) \bigg] \overline{\mathbb{P}}^\nu(\mathrm{d}\tilde\omega) \bigg\} \\
= \sup_{\nu \in \mathcal{A}(\sigma,x)} \bigg\{ \mathbb{E}^{\overline{\mathbb{P}}^\nu} \bigg[& v(\tau, X) + \sum_{i=0}^{n_\ell - 1} \int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X, \nu_r) \mathrm{d}r - n_\ell \varepsilon \ell \bigg] \\
+ \sum_{i=0}^{n_\ell-1} \mathbb{E}_{\tau_{i+1}^\ell}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}} \bigg[& \int_{\tau_{i+1}^\ell}^T \big(f_r(\tau_i^\ell, X, \nu_r^\star) - f_r(\tau_{i+1}^\ell, X, \nu_r^\star) \big) \mathrm{d}r \bigg] \\
+ \sum_{i=0}^{n_\ell-1} \mathbb{E}_{\tau_{i+1}^\ell}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}} \bigg[& \xi(\tau_i^\ell, X_{\cdot\wedge T}) - \xi(\tau_{i+1}^\ell, X_{\cdot\wedge T}) \bigg] \bigg\},
\end{aligned} \tag{2.5.2}$$

where we used the definition of J and conditioned. For $(t, x, \nu) \in [0,T] \times \mathcal{X} \times \mathcal{A}(t,x)$, let $G^t(s) := \int_t^s f_r(s, x, \nu_r) \mathrm{d}r$, $s \in [t, T]$. We set $G(s) := G^\sigma(s)$, so that

$$\begin{aligned}
& \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu} \bigg[\int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X, \nu_r) \mathrm{d}r \bigg] \\
& = \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu} \bigg[G(\tau_{i+1}^\ell) - G(\tau_i^\ell) + \int_\sigma^{\tau_{i+1}^\ell} \big(f_r(\tau_i^\ell, X, \nu_r) - f_r(\tau_{i+1}^\ell, X, \nu_r) \big) \mathrm{d}r \bigg]
\end{aligned}$$

$$= \sum_{i=0}^{n_\ell -1} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\Big[G(\tau^\ell_{i+1}) - G(\tau^\ell_i)\Big] + \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_\sigma^{\tau^\ell_{i+1}} \big(f_r(\tau^\ell_i, X, \nu_r) - f_r(\tau^\ell_{i+1}, X, \nu_r)\big)\mathrm{d}r\bigg], \tag{2.5.3}$$

where the last equality follows from the $\mathcal{F}_{\tau^\ell_{i+1}}$-measurability of the integral and Theorem 2.1.1. Now we observe that we can add the integral terms in (2.5.2) and (2.5.3), *i.e.*

$$\begin{aligned}
&\sum_{i=0}^{n_\ell -1} \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_{\tau^\ell_{i+1}}^T \big(f_r(\tau^\ell_i, X, \nu^\star_r) - f_r(\tau^\ell_{i+1}, X, \nu^\star_r)\big)\mathrm{d}r + \int_\sigma^{\tau^\ell_{i+1}} \big(f_r(\tau^\ell_i, X, \nu_r) - f_r(\tau^\ell_{i+1}, X, \nu_r)\big)\mathrm{d}r\bigg]\\
&= \sum_{i=0}^{n_\ell -1} \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_\sigma^T \big(f_r(\tau^\ell_i, X, (\nu \otimes_{\tau^\ell_{i+1}} \nu^\star)_r) - f_r(\tau^\ell_{i+1}, X, (\nu \otimes_{\tau^\ell_{i+1}} \nu^\star)_r)\big)\mathrm{d}r\bigg].
\end{aligned}$$

Consequently, for every $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$\begin{aligned}
v(\sigma, x) \geq \sup_{\nu \in \mathcal{A}(\sigma,x)} \bigg\{ &\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[v(\tau, X) + \sum_{i=0}^{n_\ell -1} G(\tau^\ell_{i+1}) - G(\tau^\ell_i) - n_\ell \varepsilon \ell\bigg]\\
&+ \sum_{i=0}^{n_\ell -1} \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_\sigma^T f_r(\tau^\ell_i, X, (\nu \otimes_{\tau^\ell_{i+1}} \nu^\star)_r) - f_r(\tau^\ell_{i+1}, X, (\nu \otimes_{\tau^\ell_{i+1}} \nu^\star)_r)\mathrm{d}r\bigg]\\
&+ \sum_{i=0}^{n_\ell -1} \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\Big[\xi(\tau^\ell_i, X_{\cdot \wedge T}) - \xi(\tau^\ell_{i+1}, X_{\cdot \wedge T})\Big]\bigg\}.
\end{aligned} \tag{2.5.4}$$

The idea in the rest of the proof is to take the limit $\ell \longrightarrow 0$ on both sides of (2.5.4). As v is finite we can exchange the limit with the sup and study the limit inside. The analysis of all the above terms, except the error term $\lceil \tau - \sigma/\ell \rceil \varepsilon \ell$, is carried out below. Regarding the error term, we would like to make the following remarks as it is clear that simply letting ε go to zero will not suffice for our purpose. As ℓ_ε is bounded and monotone in ε, see Remark 2.1.7, we consider ℓ_0 given by $\ell_\varepsilon \longrightarrow \ell_0$ as $\varepsilon \longrightarrow 0$. We must consider two cases for ℓ_0: when $\ell_0 = 0$ the analysis in the next paragraph suffices to obtain the result; in the case $\ell_0 > 0$, we can then take at the beginning of this proof $\ell < \ell_0 \leq \ell_\varepsilon$, in which case all the sums in (2.5.4) are independent of ε, we then first let ε go to zero so that $\lceil \tau - \sigma/\ell \rceil \varepsilon \ell \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, and then study the limit $\ell \to 0$ as in the following. In both scenarii the conclusion holds.

We now carry-out the analysis of the remaining terms. To this end, and in order to prevent enforcing unnecessary time regularity on the action process, we will restrict our class of actions to piece-wise constant actions, *i.e.* $\nu_t := \sum_{k\geq 1} \nu_k \mathbf{1}_{(\varrho_{k-1}, \varrho_k]}(t)$ for a sequence of non-decreasing $\mathbb{F}$–stopping times $(\varrho_k)_{k\geq 0}$, and random variables $(\nu_k)_{k\geq 1}$, such that for any $k \geq 1$, ν_k is $\mathcal{F}^X_{\varrho_{k-1}}$-

measurable. We will denote by $\mathcal{A}^{\mathrm{pw}}(t,x)$ the corresponding subclass of actions. By [86] the supremum over $\mathcal{A}(t,x)$ and $\mathcal{A}^{\mathrm{pw}}(t,x)$ coincide. Indeed, under Assumption A.(ii) and A.(iii), we can apply [86, Theorem 4.2]. Assumption A.(ii), *i.e.* the Lipschitz-continuity of $a \longmapsto f_t(t,x,a)$, ensures the continuity of the drift coefficient when the space is extended to include the running reward, see [86, Remark 3.8]. Without lost of generality we assume $(\varrho_k)_{k\geq 0} \subseteq \Pi^\ell$, this is certainly the case as we can always refine Π^ℓ so that $\nu_r = \nu_i$ for $\tau_i^\ell \leq r \leq \tau_{i+1}^\ell$.

In the following, we fix $\omega \in \Omega$. A first-order Taylor expansion of the first summation term in (2.5.4) guarantees the existence of $\gamma_i^\ell \in (\tau_i^\ell, \tau_{i+1}^\ell)$, $i \in \{0, \dots, n_\ell\}$ such that

$$
\begin{aligned}
&\bigg| \sum_{i=0}^{n_\ell - 1} G(\tau_{i+1}^\ell) - G(\tau_i^\ell) - \Delta\tau_{i+1}^\ell \bigg(f_{\tau_i^\ell}(\tau_i^\ell, X, \nu_i) + \int_\sigma^{\tau_{i+1}^\ell} \partial_s f_r(\tau_{i+1}^\ell, X, \nu_r)\mathrm{d}r \bigg) \bigg| \\
&= \bigg| \sum_{i=0}^{n_\ell - 1} \Delta\tau_{i+1}^\ell \bigg(f_{\gamma_i^\ell}(\gamma_i^\ell, X, \nu_i) - f_{\tau_i^\ell}(\tau_i^\ell, X, \nu_i) \\
&\qquad\qquad + \sum_{k=0}^{i} \int_{\tau_k^\ell}^{\tau_{k+1}^\ell \wedge \gamma_i^\ell} \partial_s f_r(\gamma_i^\ell, X, \nu_k)\mathrm{d}r - \int_{\tau_k^\ell}^{\tau_{k+1}^\ell} \partial_s f_r(\tau_{i+1}^\ell, X, \nu_k)\mathrm{d}r \bigg) \bigg| \\
&\leq \sum_{i=0}^{n_\ell - 1} |\Delta\tau_{i+1}^\ell| \bigg(\rho_f(|\Delta\tau_{i+1}^\ell|) + \sum_{k=0}^{i} \int_{\tau_k^\ell}^{\tau_{k+1}^\ell} \rho_{\partial_s f}(|\Delta\tau_{i+1}^\ell|)\mathrm{d}r \bigg) \leq 2T\big(\rho_f(\ell) + \rho_{\partial_s f}(\ell)\big) \xrightarrow{\ell\to 0} 0.
\end{aligned}
$$

The equality follows by replacing the expansion of the terms $G(\tau_{i+1}^\ell)$ and the fact ν is constant between any two terms of the partition. The first inequality follows from Assumption A.(i), where ρ and $\rho_{\partial_s f}$ are the modulus of continuity of the maps $t \longmapsto f_t(t,x,a)$ and $s \longmapsto \partial_s f_r(s,x,a)$, for a constant. The limits follows by bounded convergence as the last term is independent of ω. Thus, both expressions on the first line have the same limit for every $\omega \in \Omega$. We claim that for a well chosen sequence of partitions of the interval $[\sigma, \tau]$

$$
\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell - 1} G(\tau_{i+1}^\ell) - G(\tau_i^\ell) \bigg] \\
&\qquad \xrightarrow{\ell\to 0} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^\tau f_r(r, X, \nu_r)\mathrm{d}r + \int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}\bigg[\int_\sigma^r \partial_s f_u(r, X, \nu_u)\mathrm{d}u \bigg]\mathrm{d}r \bigg],
\end{aligned}
\tag{2.5.5}
$$

where the integrals on the right-hand side are w.r.t the Lebesgue measure on $[0,T]$, and we recall the term inside $\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}$ is $\mathcal{F}_r^X$-measurable. Indeed, following McShane [182], for $\ell > 0$ fixed there exists, ω-by-ω, $\widehat{\Pi}^\ell := (\hat{\tau}_i^\ell)_{i\in[n_\ell]}$ a partition of $[\sigma, \tau]$ such that the Riemann sum evaluated at $\widehat{\Pi}^\ell$ converges to the Lebesgue integral ω-by-ω. With this, we are left to argue (2.5.5). Recall that so far, our analysis was for $\omega \in \Omega$ fixed, therefore one has to be careful about, for instance, the measurability of the partition $\widehat{\Pi}^\ell$. An application of Galmarino's test, see Dellacherie and Meyer [69, Ch. IV. 99–101], guarantees that $\hat{\tau}_i^\ell \in \mathcal{T}_{0,T}$ for all i, *i.e.* the random times $\hat{\tau}_i^\ell$ are in fact stopping times. See Lemma 2.10.3.3 for details. Finally, (2.5.5) follows by the bounded convergence theorem.

Similarly, a first-order expansion of the second term in (2.5.4) yields $(\gamma_i^\ell)_{0\leq i\leq n_\ell}$ such that

$$\begin{aligned}
&\bigg| \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}} \bigg[\int_\sigma^T \Big(f_r(\tau_i^\ell, X, (\nu\otimes_{\tau^\ell_{i+1}} \nu^\star)_r) - f_r(\tau_{i+1}^\ell, X, (\nu\otimes_{\tau^\ell_{i+1}} \nu^\star)_r) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \Delta\tau^\ell_{i+1}\partial_s f_r(\tau^\ell_{i+1}, X, (\nu\otimes_{\tau^\ell_{i+1}} \nu^\star)_r)\Big)\mathrm{d}r\bigg]\bigg| \\
&= \bigg| \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}} \bigg[\Delta\tau^\ell_{i+1} \int_\sigma^T \Big(\partial_s f_r(\gamma_i^\ell, X, (\nu\otimes_{\tau^\ell_{i+1}} \nu^\star)_r) - \partial_s f_r(\tau^\ell_{i+1}, X, (\nu\otimes_{\tau^\ell_{i+1}} \nu^\star)_r)\Big)\mathrm{d}r\bigg]\bigg| \\
&\leq T\rho_{\partial_s f}(\ell) \xrightarrow{\ell\to 0} 0.
\end{aligned}$$

Since the limits agree, we obtain that for an appropriate choice of Π^ℓ this term converges to

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1} \Delta\tau^\ell_{i+1} \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau^\ell_{i+1},X}}\bigg[\int_\sigma^T \partial_s f_u(\tau^\ell_{i+1}, X, (\nu\otimes_{\tau^\ell_{i+1}} \nu^\star)_u)\mathrm{d}u\bigg]\bigg] \\
&\qquad\qquad \xrightarrow{\ell\to 0} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}\bigg[\int_\sigma^T \partial_s f_u(r, X, (\nu\otimes_r \nu^\star)_u)\mathrm{d}u\bigg]\mathrm{d}r\bigg].
\end{aligned}$$

Combining the double integrals in (2.5.5) and the previous expression we obtain back in (2.5.4) that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$v(\sigma,x) \geq \sup_{\nu\in\mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[v(\tau,X) + \int_\sigma^\tau \bigg[f_r(r,X,\nu_r) - \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}\bigg[\partial_s\xi(r,X_{\cdot\wedge T}) + \int_r^T \partial_s f_u(r,X,\nu_u^\star)\mathrm{d}u\bigg]\bigg]\mathrm{d}r\bigg].$$

Now for the reverse inequality, note that for $\mathcal{P}(\mathbf{x})$–q.e.$x \in \mathcal{X}$, $\overline{\mathbb{P}}^{\nu^\star}_{\sigma(\omega),x} \in \mathcal{P}(\sigma(\omega),x)$, *i.e.* $\nu^\star \in \mathcal{A}(\sigma,x)$. Second, by definition

$$v(\sigma,x) = \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\sigma,x}}\bigg[v(\tau,X) + \int_\sigma^T f_r(\sigma,X,\nu^\star)\mathrm{d}r - \int_\tau^T f_r(\tau,X,\nu^\star)\mathrm{d}r + \xi(\sigma,X_{\cdot\wedge T}) - \xi(\tau,X_{\cdot\wedge T})\bigg].$$

In light of the regularity of $s \longmapsto f_t(s,x,a)$ and the measurability of $\nu^\star$, Fubini's theorem yield

$$\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\sigma,x}}\bigg[\int_\sigma^T f_r(\sigma,X,\nu^\star_r)\mathrm{d}r-\int_\tau^T f_r(\tau,X,\nu^\star_r)\mathrm{d}r\bigg]$$
$$=\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\sigma,x}}\bigg[\int_\sigma^\tau \bigg(f_r(r,X,\nu^\star_r)-\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}\bigg[\int_u^T \partial_s f_u(r,X,\nu^\star_u)\mathrm{d}u\bigg]\bigg)\mathrm{d}r\bigg],$$

where we also use the tower property. Proceeding similarly for $s \longmapsto \xi(s,x)$, we conclude that for $\mathcal{P}(\mathbf{x})$–q.e. $x\in\mathcal{X}$

$$v(\sigma,x)=\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\sigma,x}}\bigg[v(\tau,X)+\int_\sigma^\tau\bigg(f_r(r,X,\nu^\star_r)-\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}\bigg[\partial_s\xi(r,X_{\cdot\wedge T})+\int_r^T\partial_s f_u(r,X,\nu^\star_u)\mathrm{d}u\bigg]\bigg)\mathrm{d}r\bigg],$$

which gives us the desired equality and the fact that $\nu^\star$ does attain the supremum. □

Remark 2.5.3. *Let us comment on the necessity of* Assumption A.(iii) *for our result to hold. As commented in the proof, a crucial step in our approach is that the* sup *in* (2.5.4) *attains the same value over* $\mathcal{A}(t,x)$ *and* $\mathcal{A}^{pw}(t,x)$*. For this we used* [86, Theorem 4.5] *which holds in light of* Assumption A.(iii)*. Indeed, after inspecting the proof of* [86, Theorem 4.5]*, one sees that* [86, Assumption 1.1] *guarantees pathwise uniqueness of the solution to an auxiliary* SDE*. From this, using classic arguments as in* Yamada and Watanabe [266]*, the uniqueness of the associated auxiliary martingale problem follows. However, as pointed out also in* Claisse, Talay, and Tan [56, Section 2.1]*, the previous condition can be relaxed to weaker conditions which imply weak uniqueness. As it is not our purpose in this work to extend their results in such a direction, we have not pursued it. Possible generalisations of this results are certainly interesting questions worthy of further consideration.*

Remark 2.5.4. *A close look at our arguments in the above proof, right after* Equation (2.5.4)*, brings to light how to obtain* Theorem 2.2.2 *for strict equilibria, or regular equilibria in* [120] *which are point-wise defined. Indeed, we need to control* $\ell_{0,t,\nu}$*, the limit* $\varepsilon\longrightarrow 0$ *of* $\ell_{\varepsilon,t,\nu}$*, see* Remark 2.1.7*. In the case of equilibria, no extra condition was necessary as* ℓ_ε *is uniform in* (t,ν)*. However, for strict equilibria this is not the case and we could add, for instance, the condition that for* $\mathcal{P}(\mathbf{x})$*–q.e.* $x\in\mathcal{X}$

$$\inf_{(t,\nu)\in[0,T]\times\mathcal{A}(t,x)}\ell_{0,t,\nu}>0.$$

Remark 2.5.5 (Reduction in the exponential case)**.** *As a sanity check at this point, we can see what* Theorem 2.2.2 *yields in the case of exponential discounting. Let, for any* $(t,s,x,a)\in[0,T]^2\times\mathcal{X}\times A$

$$
\begin{aligned}
f(s,t,x,a) &= \mathrm{e}^{-\theta(t-s)}\tilde{f}(t,x,a),\ \xi(s,x)=\mathrm{e}^{-\theta(T-s)}\tilde{\xi}(x),\\
J(t,x,\nu) &= \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[\int_t^T e^{-\theta(r-t)}\tilde{f}(r,X,\nu_r)\mathrm{d}r + e^{-\theta(T-t)}\tilde{\xi}(X_{\cdot\wedge T})\bigg].
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\int_t^\tau\bigg(\partial_s\xi(r,X_{\cdot\wedge T})+\int_r^T\partial_s f_u(r,X,\nu_u^\star)\mathrm{d}u\bigg)\mathrm{d}r\\
&=(\mathrm{e}^{-\theta(T-\tau)}-\mathrm{e}^{\theta(T-t)})\tilde{\xi}(X_{\cdot\wedge T})+\int_t^\tau(1-\mathrm{e}^{-\theta(r-t)})\tilde{f}(r,X,\nu_r^\star)\mathrm{d}r\\
&\quad+\int_\tau^T(\mathrm{e}^{-\theta(r-\tau)}-\mathrm{e}^{-\theta(r-t)})\tilde{f}(r,X,\nu_r^\star)\mathrm{d}r.
\end{aligned}
$$

Now, replacing on the right side of the expression in Theorem 2.2.2 *and cancelling terms we obtain that for* $\mathcal{P}(\mathbf{x})$–q.e. $x\in\mathcal{X}$

$$
v(\sigma,x)=\sup_{\nu\in\mathcal{A}(\sigma,x)}\mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[\int_\sigma^\tau e^{-\theta(r-\sigma)}\tilde{f}(r,X,\nu_r)\mathrm{d}r+v(\tau,X_{\cdot\wedge T})\bigg],
$$

which is the classic dynamic programming principle, see [86, Theorem 3.5]*.*

Finally, we mention that representations in the spirit of Theorem 2.2.2 have been previously obtained, see [82, Proposition 3.2]. Nevertheless, this is an *a posteriori* result from a verification type of result and follows from a direct application of Feynman–Kac's formula.

2.6 The BSDE system

We begin this section introducing the spaces necessary to carry out our analysis.

2.6.1 Functional spaces and norms

Let $(\mathcal{P}(t,x))_{(t,x)\in[0,T]\times\mathcal{X}}$ be given family of sets of probability measures on $(\Omega,\mathcal{F})$ solutions to the corresponding martingale problems with initial condition $(t,x)\in[0,T]\times\Omega$. Fix $(t,x)\in[0,T]\times\mathcal{X}$

and let $\mathcal{G}$ be an arbitrary σ-algebra on Ω, $\mathbb{G} := (\mathcal{G}_r)_{s\leq r\leq T}$ be an arbitrary filtration on Ω, X be an arbitrary $\mathbb{G}$-adapted process, $\mathbb{P}$ an arbitrary element in $\mathcal{P}(t,x)$. For any $(p,q)\in(1,\infty)^2$ we introduce the space

- $\mathcal{L}^p_{t,x}(\mathcal{G})$ (resp. $\mathcal{L}^p_{t,x}(\mathcal{G},\mathbb{P})$) of $\mathcal{G}$-measurable $\mathbb{R}$-valued random variables ξ with

$$\|\xi\|^p_{\mathcal{L}^p_{t,x}} := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}[|\xi|^p] < \infty, \ \Big(\text{resp. } \|\xi\|^p_{\mathcal{L}^p_{t,x}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}[|\xi|^p] < \infty\Big).$$

- $\mathbb{S}^p_{t,x}(\mathbb{G})$ (resp. $\mathbb{S}^p_{t,x}(\mathbb{G},\mathbb{P})$) of $Y\in\mathcal{P}_{\text{prog}}(\mathbb{R},\mathbb{G})$, with $\mathcal{P}(t,x)$–q.s. (resp. $\mathbb{P}$–a.s.) càdlàg paths on $[t,T]$, with

$$\|Y\|^p_{\mathbb{S}^p_{t,x}} := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\bigg[\sup_{r\in[t,T]}|Y_r|^p\bigg] < \infty, \ \bigg(\text{resp. } \|Y\|^p_{\mathbb{S}^p_{t,x}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\bigg[\sup_{r\in[t,T]}|Y_r|^p\bigg] < \infty\bigg).$$

- $\mathbb{L}^{q,p}_{t,x}(\mathbb{G})$ (resp. $\mathbb{L}^{q,p}_{t,x}(\mathbb{G},\mathbb{P})$) of $Y\in\mathcal{P}_{\text{prog}}(\mathbb{R},\mathbb{G})$, with

$$\|Y\|^p_{\mathbb{L}^{q,p}_{t,x}} := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_0^T |Y_r|^q\mathrm{d}r\bigg)^{\frac{p}{q}}\bigg] < \infty,$$
$$\bigg(\text{resp. } \|Y\|^p_{\mathbb{L}^{q,p}_{t,x}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_0^T |Y_r|^q\mathrm{d}r\bigg)^{\frac{p}{q}}\bigg] < \infty\bigg).$$

- $\mathbb{H}^p_{t,x}(\mathbb{G})$ (resp. $\mathbb{H}^p_{t,x}(\mathbb{G},\mathbb{P})$) of $Z\in\mathcal{P}_{\text{pred}}(\mathbb{R}^d,\mathbb{G})$, which are defined $\widehat{\sigma}^2_t\mathrm{d}t$–a.e., with

$$\|Z\|^p_{\mathbb{H}^p_{t,x}} := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_0^T |\widehat{\sigma}_r Z_r|^2\mathrm{d}r\bigg)^{\frac{p}{2}}\bigg] < \infty,$$
$$\bigg(\text{resp. } \|Z\|^p_{\mathbb{H}^p_{t,x}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_0^T |\widehat{\sigma}_r Z_r|^2\mathrm{d}r\bigg)^{\frac{p}{2}}\bigg] < \infty\bigg).$$

- $\mathbb{I}^p_{t,x}(\mathbb{G},\mathbb{P})$ of $K\in\mathcal{P}_{\text{pred}}(\mathbb{R},\mathbb{G})$, with $\mathbb{P}$–a.s. càdlàg, non-decreasing paths with $K_t=0$, $\mathbb{P}$–a.s., and such that

$$\|K\|^p_{\mathbb{I}^p_{t,x}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}[|K_T|^p] < \infty.$$

We will say a family $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,x)}$ belongs to $\mathbb{I}^p_{t,x}((\mathbb{G}_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,x)})$, if for any $\mathbb{P}\in\mathcal{P}(t,x)$, $K^{\mathbb{P}}\in\mathbb{I}^p_{t,x}(\mathbb{G}_{\mathbb{P}},\mathbb{P})$, and

$$\|K\|_{\mathbb{I}^p_{t,x}} := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \|K\|^p_{\mathbb{I}^p_{t,x}(\mathbb{P})} < \infty.$$

• $\mathbb{M}^p_{t,x}(\mathbb{G},\mathbb{P})$ of martingales $M \in \mathcal{P}_{\rm opt}(\mathbb{R},\mathbb{G})$ which are $\mathbb{P}$-orthogonal to X (that is the product XM is a $(\mathbb{G},\mathbb{P})$-martingale), with $\mathbb{P}$–a.s. càdlàg paths, $M_0=0$ and

$$\|M\|^p_{\mathbb{M}^p_{t,x}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\Big[[M]_T^{\frac{p}{2}}\Big] < \infty.$$

Due to the time-inconsistent nature of the problem, for a metric space E we let $\mathcal{P}^2_{\rm meas}(E,\mathcal{G})$ be the space of two parameter processes $(U_\tau)_{\tau\in[0,T]^2} : ([0,T]^2\times\Omega, \mathcal{B}([0,T]^2)\otimes\mathcal{G}) \longrightarrow (\mathcal{B}(E),E)$ measurable.

• $\mathcal{L}^{p,2}_{t,x}(\mathcal{G})$ (resp. $\mathcal{L}^{p,2}_{t,x}(\mathcal{G},\mathbb{P})$) denotes the space of collections $(\xi(s))_{s\in[0,T]}$ of $\mathcal{G}$-measurable $\mathbb{R}$-valued random variables such that the mapping $([0,T]\times\Omega,\mathcal{B}([0,T])\otimes\mathcal{F}^X_T) \longrightarrow (\mathcal{L}^p_{t,x}(\mathcal{G}),\|\cdot\|_{\mathcal{L}^p_{t,x}})$ (resp. $(\mathcal{L}^{p,2}_{t,x}(\mathcal{G},\mathbb{P})),\|\cdot\|_{\mathcal{L}^p_{t,x}(\mathbb{P})}))$: $s\longmapsto \xi(s)$ is continuous and

$$\|\xi\|^p_{\mathcal{L}^{p,2}_{t,x}} := \sup_{s\in[0,T]}\|\xi\|^p_{\mathcal{L}^p_{t,x}} < \infty, \;\Big(\text{resp. } \|\xi\|^p_{\mathcal{L}^{p,2}_{t,x}(\mathbb{P})} := \sup_{s\in[0,T]}\|\xi\|^p_{\mathcal{L}^p_{t,x}(\mathbb{P})} < \infty\Big).$$

• $\mathbb{S}^{p,2}_{t,x}(\mathbb{G})$ (resp. $\mathbb{S}^{p,2}_{t,x}(\mathbb{G},\mathbb{P})$) denotes the space of processes $(U_\tau)_{\tau\in[0,T]^2}\in\mathcal{P}^2_{\rm meas}(\mathbb{R},\mathcal{G}_T)$ such that the mapping $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{S}^p_{t,x}(\mathbb{G}),\|\cdot\|_{\mathbb{S}^p_{t,x}})$ (resp. $(\mathbb{S}^p_{t,x}(\mathbb{G},\mathbb{P})),\|\cdot\|_{\mathbb{S}^p_{t,x}(\mathbb{P})}))$: $s\longmapsto U^s$ is continuous and

$$\|U\|^p_{\mathbb{S}^{p,2}_{t,x}} := \sup_{s\in[0,T]}\|U^s\|^p_{\mathbb{S}^p_{t,x}} < \infty, \;\Big(\text{resp. } \|U\|^p_{\mathbb{S}^{p,2}_{t,x}(\mathbb{P})} := \sup_{s\in[0,T]}\|U^s\|^p_{\mathbb{S}^p_{t,x}(\mathbb{P})} < \infty\Big).$$

• $\mathbb{L}^{q,p,2}_{t,x}(\mathbb{G})$ (resp. $\mathbb{L}^{q,p,2}_{t,x}(\mathbb{G},\mathbb{P})$) denotes the space of processes $(U_\tau)_{\tau\in[0,T]^2}\in\mathcal{P}^2_{\rm meas}(\mathbb{R},\mathcal{G}_T)$ such that the mapping $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{L}^{q,p}_{t,x}(\mathbb{G}),\|\cdot\|_{\mathbb{L}^{q,p}_{t,x}})$ (resp. $(\mathbb{L}^{q,p}_{t,x}(\mathbb{G},\mathbb{P})),\|\cdot\|_{\mathbb{L}^{q,p}_{t,x}(\mathbb{P})}))$: $s\longmapsto U^s$ is continuous and

$$\|U\|^p_{\mathbb{L}^{q,p,2}_{t,x}} := \sup_{s\in[0,T]}\|U^s\|^p_{\mathbb{L}^{q,p}_{t,x}} < \infty, \;\Big(\text{resp. } \|U\|^p_{\mathbb{L}^{q,p,2}_{t,x}(\mathbb{P})} := \sup_{s\in[0,T]}\|U^s\|^p_{\mathbb{L}^{q,p}_{t,x}(\mathbb{P})} < \infty\Big).$$

• $\mathbb{H}^{p,2}_{t,x}(\mathbb{G})$ (resp. $\mathbb{H}^{p,2}_{t,x}(\mathbb{G},\mathbb{P})$) denotes the space of processes $(V_\tau)_{\tau\in[0,T]^2}\in\mathcal{P}^2_{\rm meas}(\mathbb{R}^d,\mathcal{G}_T)$ such that the mapping $([0,T],\mathcal{B}([0,T])) \to (\mathbb{H}^{p,2}_{t,x}(\mathbb{G},X),\|\cdot\|_{\mathbb{H}^p_{t,x}(X)})$ (resp. $(\mathbb{H}^p_{t,x}(\mathbb{G},X,\mathbb{P})),\|\cdot\|_{\mathbb{H}^{p,2}_{t,x}(\mathbb{P})}))$: $s\longmapsto$

V^s is continuous and

$$\|V\|^p_{\mathbb{H}^{p,2}_{t,x}} := \sup_{s\in[0,T]} \|V^s\|^p_{\mathbb{H}^p_{t,x}} < \infty, \ \Big(\text{resp. } \|V\|^p_{\mathbb{H}^{p,2}_{t,x}(\mathbb{P})} := \sup_{s\in[0,T]} \|V^s\|^p_{\mathbb{H}^p_{t,x}(\mathbb{P})} < \infty\Big).$$

- $\mathbb{M}^{p,2}_{t,x}(\mathbb{G},\mathbb{P})$ denotes the space of two parameter processes $(N_\tau)_{\tau\in[0,T]^2} \in \mathcal{P}^2_{\text{meas}}(\mathbb{R},\mathcal{G}_T)$ such that the mapping $([0,T],\mathcal{B}([0,T])) \to (\mathbb{M}^p_{t,x}(\mathbb{G},\mathbb{P}), \|\cdot\|_{\mathbb{M}^p_{t,x}(\mathbb{P})}) : s \longmapsto N^s$ is continuous and

$$\|N\|^p_{\mathbb{M}^{p,2}_{t,x}(\mathbb{P})} := \sup_{s\in[0,T]} \|N^s\|^p_{\mathbb{M}^p_{t,x}(\mathbb{P})} < \infty.$$

- $\mathbb{I}^{p,2}_{t,x}(\mathbb{G},\mathbb{P})$ denotes the space of two parameter processes $(L_\tau)_{\tau\in[0,T]^2} \in \mathcal{P}^2_{\text{meas}}(\mathbb{R},\mathcal{G}_T)$ such that the mapping $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{I}^p_{t,x}(\mathbb{G},\mathbb{P}), \|\cdot\|_{\mathbb{I}^{p,o}_{t,x}}) : s \longmapsto L^s$ is continuous and

$$\|L\|^p_{\mathbb{I}^{p,2}_{t,x}(\mathbb{P})} := \sup_{s\in[0,T]} \|N^s\|^p_{\mathbb{I}^p(\mathbb{P})} < \infty.$$

Remark 2.6.1. *To ease the notation, when $p = q$ we will write $\mathbb{L}^p_{t,x}(\mathbb{G})$ (resp. $\mathbb{L}^{p,2}_{t,x}(\mathbb{G})$) for $\mathbb{L}^{q,p}_{t,x}(\mathbb{G})$ (resp. $\mathbb{L}^{q,p,2}_{t,x}(\mathbb{G})$). With this convention, $\mathbb{L}^2_{t,x}(\mathbb{G})$ (resp. $\mathbb{L}^{2,2}_{t,x}(\mathbb{G})$) will always mean $\mathbb{L}^{2,2}_{t,x}(\mathbb{G})$ (resp. $\mathbb{L}^{2,2,2}_{t,x}(\mathbb{G})$). The spaces are $\mathbb{L}^{q,p,2}_{t,x}(\mathbb{G})$ and $\mathbb{H}^{p,2}_{t,x}(\mathbb{G})$ are Hilbert spaces. For $U \in \mathbb{S}^{p,2}_{t,x}(\mathbb{G})$ we highlight the diagonal process $(U^t_t)_{t\in[0,T]}$ is well defined. Indeed, the path continuity of U^s for all $s \in [0,T]$ together with the (uniform) continuity of $s \longmapsto \|U^s\|_{\mathbb{S}^2}$ allows us to define a $\mathcal{B}[0,T]\otimes\mathcal{F}$-measurable version. Finally, we will suppress the dependence on $(0,\mathbf{x})$ and write $\mathbb{S}_{\mathbf{x}}(\mathbb{G})$ for $\mathbb{S}_{0,\mathbf{x}}(\mathbb{G})$ and similarly for the other spaces.*

We now begin our study of the system

$$\begin{aligned} Y_t =&\ \xi(T,X_{\cdot\wedge T}) + \int_t^T F_r(X,Z_r,\widehat{\sigma}^2_r,\partial Y^r_r)\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r + \int_t^T \mathrm{d}K^{\mathbb{P}}_r,\ t\in[0,T],\ \mathcal{P}(\mathbf{x})\text{–q.s.},\\ \partial Y^s_t(\omega) :=&\ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}\Big[\partial_s\xi(s,X_{\cdot\wedge T}) + \int_t^T \partial_s f_r(s,X,\mathcal{V}^\star(r,X,Z_r))\mathrm{d}r\Big],\ (s,t)\in[0,T]^2,\ \omega\in\Omega. \end{aligned} \tag{H}$$

As a motivation of the notion of solution to (H), let us note that the first equation is a 2BSDE under the set $\mathcal{P}(\mathbf{x})$, *i.e.* the dynamics holds $\mathcal{P}(\mathbf{x})$–q.s. A closer examination of Definition 2.1.6 reveals that, unlike in the classical stochastic control scenario, one needs to be able to make sense of a solution under any $\mathcal{P}(s,x)$ for $s\in[0,T]$ and x outside a $\mathcal{P}(\mathbf{x})$-polar set. Fortunately, the results in [210] allow us to verify that constructing the initial solution suffices, see Lemma 2.8.1.

Definition 2.6.2. *Let $(s,x) \in [0,T] \times \mathcal{X}$, ∂Y_r^r be a given process and consider the equation*

$$Y_t^{s,x} = \xi(T, X_{\cdot\wedge T}) + \int_t^T F_r(X, Z_r^{s,x}, \widehat{\sigma}_r^2, \partial Y_r^r)\mathrm{d}r - \int_t^T Z_r^{s,x} \cdot \mathrm{d}X_r \ + \int_t^T \mathrm{d}K_r^{s,x,\mathbb{P}},\ t \in [s,T]. \quad (2.6.1)$$

We say $(Y^{s,x}, Z^{s,x}, (K^{s,x,\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(s,x)})$ is a solution to 2BSDE (2.6.1) *under $\mathcal{P}(s,x)$ if for some $p>1$,*

(i) Equation (2.6.1) *holds $\mathcal{P}(s,x)$–*q.s.

(ii) $(Y^{s,x}, Z^{s,x}, (K^{s,x,\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(s,x)}) \in \mathbb{S}^p_{s,x}(\mathbb{F}_+^{X,\mathcal{P}(s,x)}) \times \mathbb{H}^p_{s,x}(\mathbb{F}_+^{X,\mathcal{P}(s,x)}) \times (\mathbb{I}^p_{s,x}(\mathbb{F}_+^{X,\mathbb{P}}, \mathbb{P}))_{\mathbb{P}\in\mathcal{P}(s,x)}$.

(iii) *The family $(K^{s,x,\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(s,x)}$ satisfies the minimality condition*

$$0 = \underset{\mathbb{P}'\in\mathcal{P}_{s,x}(t,\mathbb{P},\mathbb{F}_+^X)}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\big[K_T^{s,x,\mathbb{P}'} - K_t^{s,x,\mathbb{P}'}\big|\mathcal{F}_{t+}^{X,\mathbb{P}'}\big],\ s \le t \le T,\ \mathbb{P}\text{–a.s.},\ \forall \mathbb{P}\in\mathcal{P}(s,x).$$

Consistent with Definition 2.2.6, we set $(Y, Z, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}) = (Y^{0,\mathbf{x}}, Z^{0,\mathbf{x}}, (K^{0,\mathbf{x},\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})})$. We use the rest of this section to prove Theorem 2.2.8, justifying that in the setting of this section our approach encompasses that of [35].

Proof of Theorem 2.2.8. Let $\mathbb{P}\in\mathcal{P}(\mathbf{x})$ and consider $(\Omega, \mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})}, \mathbb{P})$. We first verify that (Y,Z,K) satisfies first equation in System (H). A direct application of Itô's formula to $Y_t = v(t, X_t)$ with X given by the SDE (2.1.7) yields that $\mathbb{P}$–a.s.

$$\begin{aligned} Y_t &= Y_T - \int_t^T \Big(\partial_t v(r, X_r) + \frac{1}{2}\mathrm{Tr}[\mathrm{d}\langle X\rangle_r \partial_{xx} v(r,X_r)]\Big)\mathrm{d}r - \int_t^T \partial_x v(r, X_r)\cdot \mathrm{d}X_r \\ &= Y_T + \int_t^T \sup_{\Sigma\in\Sigma_r(X_r)} \Big\{F_r(X_r, Z_r, \Sigma, \partial\mathcal{Y}_r^r) + \frac{1}{2}\mathrm{Tr}[\Sigma\,\Gamma_r]\Big\} - \frac{1}{2}\mathrm{Tr}\big[\mathrm{d}\langle X\rangle_r \Gamma_r\big]\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r, \end{aligned}$$

where we used (2.2.9) and the definition of H in terms of F as in (2.2.4). Next, by definition of $\widehat{\sigma}_t^2$ and with K_t as in the statement we obtain

$$Y_t = Y_T + \int_t^T F_r(X_r, Z_r, \widehat{\sigma}_r^2, \partial\mathcal{Y}_r^r) - \int_t^T Z_r\cdot\mathrm{d}X_r + K_T - K_t,\ \mathbb{P}\text{–a.s.}$$

Next, we verify the integrability conditions in Definitions 2.2.7.(i) and 2.2.7.(ii). As σ is bounded, it follows that for any $\mathbb{P}\in\mathcal{P}(\mathbf{x})$, X_t has exponential moments of any order which are bounded on $[0,T]$, *i.e.* $\exists C$, $\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{P}}[\exp(c|X_t|_1)] \leq C < \infty, \forall \mathbb{P} \in \mathcal{P}(\mathbf{x}), \forall c > 0$, where C depends on T and the bound on σ.

The exponential grown assumption on v and de la Vallée-Poussin's theorem yield that for $p > 1$ $Y \in \mathbb{S}^p_{\mathbf{x}}(\mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})}, \mathcal{P}(\mathbf{x}))$. Similarly, we obtain $(Z, \partial Y) \in \mathbb{H}^p_{\mathbf{x}}(\mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})}, \mathcal{P}(\mathbf{x})) \times \mathbb{S}^{p,2}_{\mathbf{x}}(\mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})}, \mathcal{P}(\mathbf{x}))$. To derive the integrability of K, let $\mathbb{P} \in \mathcal{P}(\mathbf{x})$ and note that

$$\mathbb{E}^{\mathbb{P}}[K_T^p] \leq C_p\bigg(\|Y\|^p_{\mathbb{S}^p_{\mathbf{x}}} + \sup_{\mathbb{P}\in\mathcal{P}(\mathbf{x})} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_0^T |F_r(X_r, Z_r, \widehat{\sigma}_r^2, 0)|\mathrm{d}r\bigg)^p\bigg] + \|\partial Y\|^p_{\mathbb{S}^{p,2}_{\mathbf{x}}} + \|Z\|^p_{\mathbb{H}^p_{\mathbf{x}}}\bigg) < \infty,$$

where the inequality follows from the fact $(t,x,z,a) \longmapsto F(t,x,z,a,0)$ is Lipschitz in z which follows from the exponential growth assumption on f^0 and the boundedness of the coefficients b and σ. The constant C_p depends on the Lipschitz constant and the value of p as in Bouchard, Possamaï, Tan, and Zhou [40, Lemma 2.1]. As the term on the right does not depend on $\mathbb{P}$, we conclude $K \in \mathbb{I}^p_{\mathbf{x}}((\mathbb{F}_+^{X,\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})})$. Note that the previous estimate shows as a by-product that the 2BSDE in (H) is well-posed, see [210, Theorem 4.1]. Therefore, provided K satisfies the minimality condition by [210, Theorem 4.2], we conclude 2.2.7.(i), *i.e.* (Y, Z, K) is the solution to the 2BSDE in (H).

We now argue that K satisfies (2.2.8). Following [230, Theorem 5.2], we can exploit the fact the σ is bounded and the continuity in time of X, Z and Γ for fixed $x \in \mathcal{X}$, to show that for $\varepsilon > 0$, $(t,\nu) \in [0,T] \times \mathcal{A}(\mathbf{x},\mathbb{P})$ and $\tau^{\varepsilon,t} := T \wedge \inf\{r > t : K_r \geq K_t + \varepsilon\}$, there exists $\mathbb{P}^{\nu^\varepsilon} \in \mathcal{P}_{\mathbf{x}}(t, \mathbb{P}, \mathbb{F}_+^X)$ such that $k_t \leq \varepsilon$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}^{\nu^\varepsilon}$ on $[\tau^{\varepsilon,t}, T] \times \Omega$. From this the minimality condition follows. Moreover, by assumption, we know there exists $\mathbb{P}^{\nu^\star} \in \mathcal{P}^0(0, \mathbf{x}, \nu^\star)$ where $\nu^\star$ maximises the Hamiltonian, *i.e.* $\widehat{\sigma}_r^2 = (\sigma\sigma^\top)_t(X_t, \nu_t^\star), \mathrm{d}t \otimes \mathrm{d}\mathbb{P}^\star$–a.e. on $[0,T] \times \Omega$, and $k_t = 0$, $\mathbb{P}^{\nu^\star}$–a.s. for all $t \in [0,T]$. Thus the minimality condition is attained under $\mathbb{P}^{\nu^\star}$, *i.e.* 2.2.7.(iii) holds. Moreover, note that this implies

$$\overline{\mathcal{V}}^\star(t, X_t, Z_t, \Gamma_t) = \mathcal{V}^\star(t, X_t, Z_t),\ \mathbb{P}^{\nu^\star}\text{–a.s.} \tag{2.6.2}$$

We are left to argue ∂Y satisfies the second equation in (H). Given the regularity of $s \longmapsto J(s,t,x)$, we can differentiate the second equation in (2.2.9). Using this, for $s \in [0,T]$ fixed and $\omega = (x, \mathbf{w}, q) \in \Omega$ we can apply Itô's formula to $\partial Y_t^s = \partial_s J(s,t,X_t)$ under $\overline{\mathbb{P}}^{\nu^\star}_{t,x}$. This yields,

$$\partial Y_t^s(\omega) = \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}\bigg[\partial_s \xi(s, X_T) + \int_t^T \partial_s f_r(s, X_r, \mathcal{V}^\star(r, X, Z_r))\mathrm{d}r\bigg],$$

where the stochastic integral term vanished in light of the growth assumption on $\partial_x J(s,t,x)$ and we used (2.6.2). $\square$

2.7 Proof of Theorem 2.2.10

We recall that throughout this section, we let Assumptions A and B hold. To begin with, from the definition of the set $\mathcal{A}(t,x)$, we can re-state the result of our dynamic programming principle Theorem 2.2.2, by decomposing a control $\nu \in \mathcal{A}(t,x)$ into a pair $(\mathbb{P},\nu) \in \mathcal{P}(t,x) \times \mathcal{A}(t,x,\mathbb{P})$, where $\mathbb{P}$ is the unique weak solution to (2.1.7) and ν. We remark that for any given $\mathbb{P} \in \mathcal{P}(t,x)$, there could be in general several admissible controls ν. With this we state Theorem 2.2.2 as, for $\nu^\star \in \mathcal{E}(\mathbf{x})$, $\sigma, \tau \in \mathcal{T}_{t,T}$, $\sigma \leq \tau$ and $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$\begin{aligned} v(\sigma,x) = \sup_{\mathbb{P}\in\mathcal{P}(\sigma,x)} \sup_{\nu\in\mathcal{A}(\sigma,x,\mathbb{P})} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[& v(\tau,X) + \int_\sigma^\tau f_r(r,X,\nu_r)\mathrm{d}r \\ & - \int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\star,\nu^\star}_{r,\cdot}}\bigg[\partial_s \xi(r,X_{\cdot\wedge T}) + \int_r^T \partial_s f_u(r,X,\nu^\star_u)\mathrm{d}u\bigg]\mathrm{d}r\bigg]. \end{aligned} \tag{2.7.1}$$

The goal of this section is to show that given $\nu^\star$, $\mathbb{P}^\star$, unique solution to the martingale problem asociated with $\nu^\star$, and $v(t,x)$, one can construct a solution to (H). To do so, we recall that given a family of BSDEs $(\mathcal{Y}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ indexed by $\mathcal{P}(t,\omega) \subseteq \mathrm{Prob}(\Omega)$ with $(t,\omega) \in [0,T]\times\Omega$, a 2BSDE is the supremum over $\mathcal{P}(t,\omega)$ of the $\mathbb{P}$-expectation of the afore mentioned family, see [230], [210]. This together with equation (2.7.1) reveals the road map we should take.

Let us begin by fixing an equilibrium $\nu^\star \in \mathcal{E}(\mathbf{x})$. For $(s,t,\omega,\mathbb{P}) \in [0,T]\times[0,T]\times\Omega\times\mathcal{P}(t,x)$ we consider the $\mathbb{F}^X$-adapted processes

$$\begin{aligned} \widetilde{\mathcal{Y}}^{\mathbb{P}}_t(\omega) &:= \sup_{\nu\in\mathcal{A}(t,x,\mathbb{P})} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\xi(T,X) + \int_t^T \big(f_r(r,X,\nu_r) - \partial Y^r_r\big)\mathrm{d}r\bigg], \\ \partial Y^s_t(\omega) &:= \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}\bigg[\partial_s \xi(s,X_{\cdot\wedge T}) + \int_t^T \partial_s f_r(s,X,\nu^\star_r)\mathrm{d}r\bigg]. \end{aligned} \tag{2.7.2}$$

and on $(\Omega, \mathbb{F}^{X,\mathbb{P}}_+, \mathbb{P})$ the BSDE

$$\mathcal{Y}^{\mathbb{P}}_t = \xi(T,X_{\cdot\wedge T}) + \int_t^T F_r(X,\mathcal{Z}^{\mathbb{P}}_r,\widehat{\sigma}^2_r,\partial Y^r_r)\mathrm{d}r - \bigg(\int_t^T \mathcal{Z}^{\mathbb{P}}_r \cdot \mathrm{d}X_r\bigg)^{\mathbb{P}},\ 0 \leq s \leq t \leq T. \tag{2.7.3}$$

Note we specify the stochastic integral w.r.t X is under the probability $\mathbb{P}$. Our first step is to relate $\widetilde{\mathcal{Y}}^{\mathbb{P}}$ with the solution to the BSDE (2.7.3). Namely, Lemma 2.7.1 says that $\widetilde{\mathcal{Y}}^{\mathbb{P}}$ corresponds to the first component of the solution to (2.7.3).

Lemma 2.7.1. *Let* Assumption B *hold,* $(t,\omega,\mathbb{P})\in[0,T]\times\Omega\times\mathcal{P}(t,x)$ *and* $(\mathcal{Y}^{\mathbb{P}},\mathcal{Z}^{\mathbb{P}})$ *be the solution to the* BSDE (2.7.3)*, as in* Papapantoleon, Possamaï, and Saplaouras [200, Definition 3.2]*, and* $\tilde{\nu}^\star_t:=\mathcal{V}^\star(t,X,\mathcal{Z}^{\mathbb{P}}_t)$*. Then*

$$\widetilde{\mathcal{Y}}^{\mathbb{P}}_t(\omega)=\mathbb{E}^{\mathbb{P}^{\tilde{\nu}^\star}}[\mathcal{Y}^{\mathbb{P}}_t]. \tag{2.7.4}$$

Proof. Let us consider on $(\Omega,\mathbb{F}^{X,\mathbb{P}}_+,\mathbb{P})$, for $t\le u\le T$ and $\nu\in\mathcal{A}(t,x,\mathbb{P})$ the BSDE

$$\mathcal{Y}^{\mathbb{P},\nu}_u=\xi(T,X_{\cdot\wedge T})+\int_u^T\big(h_r(r,X,\mathcal{Z}^{\mathbb{P},\nu}_r,\nu_r)-\partial Y^r_r\big)\mathrm{d}r-\bigg(\int_u^T\mathcal{Z}^{\mathbb{P},\nu}_r\cdot\mathrm{d}X_r\bigg)^{\mathbb{P}},\ \mathbb{P}\text{–a.s.} \tag{2.7.5}$$

Under Assumptions B.(ii) and B.(i), we know that $z\longmapsto h_t(t,x,z,a)$ is Lipschitz-continuous, uniformly in (t,x,a), that there exists $p>1$ such that $([0,T]\times\Omega,\mathbb{F}^X)\ni(t,\omega)\longmapsto h_t(t,x,0,0)\in\mathbb{H}^{p,2}_{s,\omega}(\mathbb{F}^{X,\mathbb{P}}_+,\mathbb{P})$ is well defined, and that $\partial Y\in\mathbb{H}^p_{s,x}(\mathbb{F}^{X,\mathbb{P}}_+,\mathbb{P})$. Moreover, as for any ν, and $\tilde{\nu}$ in $\mathcal{A}(s,x,\mathbb{P})$, $(\sigma\sigma^\top)_t(X,\nu_t)=\widehat{\sigma}^2_t=(\sigma\sigma^\top)_t(X,\tilde{\nu}_t),\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$–a.e., $\mathbb{P}$ is the unique solution to an uncontrolled martingale problem where $(\sigma\sigma^\top)_t(X):=(\sigma\sigma^\top)_t(X,\nu_t)$. Consequently, the martingale representation property holds for any local martingale in $(\Omega,\mathbb{F}^{\mathbb{P}},\mathbb{P})$ relative to X, see Jacod and Shiryaev [142, Theorem 4.29]. Therefore, conditions (H1)–(H6) in [200, Theorem 3.23] hold and the above BSDE is well defined. Its solution consists of a tuple $(\mathcal{Y}^{\mathbb{P},\nu},\mathcal{Z}^{\mathbb{P},\nu})\in\mathbb{D}^p_{s,x}(\mathbb{F}^{X,\mathbb{P}}_+,\mathbb{P})\times\mathbb{H}^p_{s,x}(\mathbb{F}^{X,\mathbb{P}}_+,\mathbb{P})$ and for every $\nu\in\mathcal{A}(t,x,\mathbb{P})$ and $\overline{\mathbb{P}}^\nu$ is as in Remark 2.1.4 we have

$$\mathbb{E}^{\overline{\mathbb{P}}^\nu}[\mathcal{Y}^{\mathbb{P},\nu}_t]=\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\xi(T,X)+\int_t^T\big(f_r(r,X,\nu_r)-\partial Y^r_r\big)\mathrm{d}r\bigg],\ t\in[0,T].$$

In addition, the solution $(\mathcal{Y}^{\mathbb{P}},\mathcal{Z}^{\mathbb{P}})\in\mathbb{D}^p_{s,\omega}(\mathbb{F}^{X,\mathbb{P}}_+,\mathbb{P})\times\mathbb{H}^p_{s,\omega}(\mathbb{F}^{X,\mathbb{P}}_+,\mathbb{P})$ to BSDE (2.7.3) is well defined, by Assumption B.(iii) guarantees. Furthermore, conditions (Comp1)–(Comp3) in [200, Theorem 3.25] are fulfilled, ensuring a comparison theorem holds. Indeed, as X is continuous (Comp1) is immediate, while (Comp2) and (Comp3) correspond in our setting to B.(ii) and B.(i), respectively. By definition, $\tilde{\nu}^\star_t$ satisfies $(\sigma\sigma^\top)_t(X,\tilde{\nu}^\star_t)=\widehat{\sigma}^2_t,\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$–a.e. on $[t,T]\times\mathcal{X}$, and so, $\tilde{\nu}^\star\in\mathcal{A}(t,x,\mathbb{P})$.

By comparison, we obtain

$$\widetilde{\mathcal{Y}}_t^{\mathbb{P}}(x) = \sup_{\nu \in \mathcal{A}(t,x,\mathbb{P})} \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[\mathcal{Y}_t^{\mathbb{P},\nu}] \leq \mathbb{E}^{\overline{\mathbb{P}}^{\tilde{\nu}^\star}}[\mathcal{Y}_t^{\mathbb{P}}].$$

□

Remark 2.7.2. *(i) In the literature on* BSDEs *one might find the additional term* $\int_t^T \mathrm{d}\mathcal{N}_r^{\mathbb{P}}$ *in* (2.7.3)*, where* $\mathcal{N}^{\mathbb{P}}$ *is a* $\mathbb{P}$*-martingale* $\mathbb{P}$*-orthogonal to* X*. Yet, as noticed in the proof once* $(\sigma\sigma^\top)_t(X)$ *is fixed, uniqueness of the associated martingale problem guarantees the representation property relative to* X*.*

(ii) We remark that an alternative constructive approach to relate $\widetilde{\mathcal{Y}}$ *to a* BSDE *is to consider for any* $\nu \in \mathcal{A}(s,x,\mathbb{P})$ *and* $t \geq s$ *the process*

$$N_t^{\mathbb{P},\nu} := J(t,t,X,\nu^\star(\mathbb{P})) + \int_0^t \big(f_r(r,X,\nu_r) - \partial Y_r^r\big)\mathrm{d}r.$$

However, as the careful reader might have noticed, this requires that for a given $\mathbb{P} \in \mathcal{P}(s,x)$ *we introduce* $\nu^\star(\mathbb{P})$ *the action process attaining the* sup *in* (2.7.2)*. However, the existence of an action with such property is not necessarily guaranteed for all* $\mathbb{P} \in \mathcal{P}(s,x)$*, at least without further assumptions, which we do not want to impose here.*

Remark 2.7.3. *At this point we are halfway from our goal in this section. For* $(t,\omega) \in [0,T]\times\Omega$*, the previous lemma defines a family* $(\mathcal{Y}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ *of* $\mathbb{F}_+^{X,\mathbb{P}}$*-adapted processes. Recalling our discussion at the beginning of this section, all we are left to do is to take* sup *over* $(\mathcal{P}(t,x))_{(t,x)\in[0,T]\times\mathcal{X}}$*. In other words, putting together* (2.7.4) *and* (2.7.1)*, we now know*

$$v(t,x) = \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\overline{\mathbb{P}}^{\tilde{\nu}^\star}}[\mathcal{Y}_t^{\mathbb{P}}]. \tag{2.7.6}$$

In light of the previous remark and the characterisation in [210], we consider the following 2BSDE

$$Y_t = \xi(T, X_{\cdot\wedge T}) + \int_t^T F_r(X, Z_r, \widehat{\sigma}_r^2, \partial Y_r^r)\mathrm{d}r - \int_t^T Z_r \cdot \mathrm{d}X_r + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ t \in [0,T],\ \mathcal{P}(\mathbf{x})\text{–q.s.} \tag{2.7.7}$$

With this we are ready to prove the necessity of System (H).

Proof of Theorem 2.2.10. We begin by verifying the integrability of ∂Y, defined as in (2.7.2). From Assumption B.(ii) we have that for any $s \in [0,T]$

$$\|\partial Y^s\|_{\mathbb{S}^p_{\mathbf{x}}} \leq \sup_{\mathbb{P}\in\mathcal{P}(\mathbf{x})} \mathbb{E}^{\mathbb{P}}\bigg[\big|\partial_s \xi(s, X_{\cdot\wedge T})\big|^p + \int_0^T |\partial_s f_r(s, X, \nu^\star_r)|^p \mathrm{d}r\bigg] < \infty.$$

Therefore, as Assumption A.(i) guarantees the continuity of the map $s \longmapsto \|\partial Y^s\|_{\mathbb{S}^p_{0,\mathbf{x}}(\mathbb{F}^X,\mathcal{P}(\mathbf{x}))}$ the result follows.

Let us construct such a solution from $\nu^\star \in \mathcal{E}(\mathbf{x})$. Under Assumption B.$(ii)$, it follows from (2.7.6) and [210, Lemma 3.2] that v is làdlàg outside a $\mathcal{P}(\mathbf{x})$-polar set. Therefore the process v^+ given by

$$v^+_t(x) := \lim_{r\in\mathbb{Q}\cap(t,T], r\downarrow t} v(t,x),$$

is well defined in the $\mathcal{P}(\mathbf{x})$–q.s. sense. Clearly v^+ is càdlàg, $\mathbb{F}^{X,\mathcal{P}(\mathbf{x})}_+$-adapted, and in light of [210, Lemmata 2.2 and 3.6], which hold under Assumption B, we have that for any $\mathbb{P}\in\mathcal{P}(\mathbf{x})$, there exist $(\mathcal{Z}^{\mathbb{P}}, \mathcal{K}^{\mathbb{P}}) \in \mathbb{H}^p_{0,\mathbf{x}}(\mathbb{F}^{X,\mathbb{P}}_+, \mathbb{P}) \times \mathbb{I}^p_{0,\mathbf{x}}(\mathbb{F}^{X,\mathbb{P}}_+, \mathbb{P})$ such that

$$v^+_t = \xi(T, X_{\cdot\wedge T}) + \int_t^T F_r(X, \mathcal{Z}^{\mathbb{P}}_r, \widehat{\sigma}^2_r, \partial Y^r_r)\mathrm{d}r - \bigg(\int_t^T \mathcal{Z}^{\mathbb{P}}_r \cdot \mathrm{d}X_r\bigg)^{\mathbb{P}} + \mathcal{K}^{\mathbb{P}}_T - \mathcal{K}^{\mathbb{P}}_t,\ 0 \leq t \leq T,\ \mathbb{P}\text{–a.s.}$$

Moreover, the process $Z_t := (\widehat{\sigma}^2_t)^{\oplus}\frac{\mathrm{d}[v^+,X]_t}{\mathrm{d}t}$, where $(\widehat{\sigma}^2_t)^{\oplus}$ denotes the Moore–Penrose pseudo-inverse of $\widehat{\sigma}^2_t$, aggregates the family $(Z^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}$. The proof that $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}$ satisfies the minimality condition (2.2.8) is argued as in [210, Section 4.4]. The result follows by well-posedness of (2.7.7).

Arguing as in [210, Lemma 3.5], we may obtain that for any $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$ and $\mathbb{P}\in\mathcal{P}(t,x)$

$$v^+_t = \operatorname*{ess\,sup^{\mathbb{P}}}_{\tilde{\mathbb{P}}\in\mathcal{P}(t,\mathbb{P},\mathbb{F}^X_+)} \mathcal{Y}^{\tilde{\mathbb{P}}}_t,$$

with $\mathcal{Y}^{\tilde{\mathbb{P}}}$ as in (2.7.3). Consequently

$$v(t,x) = \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}[v^+_t].$$

Moreover, as for any $t \in [0,T]$ and $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$, $\overline{\mathbb{P}}^{\nu^\star}_{t,x}$ attains equality in (2.7.6), see Theorem 2.2.2, we deduce, in light of (2.1.4), $\mathbb{P}^{\nu^\star}$ attains the minimality condition. This is, under $\mathbb{P}^{\nu^\star}$, the process $K^{\mathbb{P}^{\nu^\star}}$ equals 0. With this, we obtain $(v^+, Z, (K)^{\mathbb{P}}_{\mathbb{P}\in\mathcal{P}(\mathbf{x})})$ and ∂Y are a solution to (H). Moreover, Lemma 2.7.1 implies

$$\nu^\star_t \in \underset{a\in A_t(x,\hat{\sigma}^2_t(x))}{\arg\max}\ h_t(t, X, Z_t, a),\ \mathrm{d}t \otimes \mathrm{d}\mathbb{P}^{\nu^\star}\text{–a.e.., on } [0,T]\times\mathcal{X}.$$

□

2.8 Proof of Theorem 2.2.12

This section is devoted to proof the verification Theorem 2.2.12. To do so we will need to obtain a rigorous statement of how the processes defined by (H) relate. This is carried in the following series of lemmata.

Lemma 2.8.1. *For* $(\mathbf{x}, x, s) \in \mathcal{X}\times\mathcal{X}\times(0,T]$ *consider the* 2BSDEs

$$Y_t = \xi(T,X) + \int_t^T F_r(X, Z_r, \widehat{\sigma}^2_r, \partial Y^r_r)\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r + \int_t^T \mathrm{d}K^{\mathbb{P}}_r,\ t\in[0,T],\ \mathcal{P}(\mathbf{x})\text{–q.s.},$$
$$Y^{s,x}_t = \xi(T,X) + \int_t^T F_r(X, Z^{s,x}_r, \widehat{\sigma}^2_r, \partial Y^r_r)\mathrm{d}r - \int_t^T Z^{s,x}_r\cdot \mathrm{d}X_r + \int_t^T \mathrm{d}K^{s,x,\mathbb{Q}}_r,\ t\in[s,T],\ \mathcal{P}(s,x)\text{–q.s.}$$

Suppose both 2BSDEs *are well-posed. Then, for any* $s\in(0,T]$,

$$\begin{aligned}
Y_t &= Y^{s,x}_t,\ s\le t\le T,\ \mathcal{P}(s,x)\text{–q.s., } \textit{for } \mathcal{P}(\mathbf{x})\text{–q.e. } x\in\mathcal{X},\\
Z_t &= Z^{s,x}_t,\ \widehat{\sigma}^2_t\mathrm{d}t\otimes\mathrm{d}\mathcal{P}(s,x)\text{–q.e. } \textit{on } [s,T]\times\mathcal{X},\ \textit{for } \mathcal{P}(\mathbf{x})\text{–q.e. } x\in\mathcal{X},\\
K^{\mathbb{P}}_t &= K^{s,x,\mathbb{P}_{s,x}}_t,\ s\le t\le T,\ \mathbb{P}_{s,x}\text{–a.s., } \textit{for } \mathbb{P}\text{–a.e. } x\in\mathcal{X}, \forall\mathbb{P}\in\mathcal{P}(\mathbf{x}).
\end{aligned}$$

Proof. Following [210], we consider for $(t,x)\in[0,T]\times\mathcal{X}$

$$\widehat{\mathcal{Y}}_t(x) := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}^{\mathbb{P}}_t\Big],$$

where for an arbitrary $\mathbb{P}\in\mathcal{P}(s,x)$, $\mathcal{Y}^{\mathbb{P}}$ corresponds to the first coordinate of the solution to the

BSDE

$$\mathcal{Y}_t^{\mathbb{P}} = \xi(T, X_{\cdot\wedge T}) + \int_t^T F_r(X, \mathcal{Z}_r^{\mathbb{P}}, \widehat{\sigma}_r^2, \partial Y_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^{\mathbb{P}} \cdot \mathrm{d}X_r, s \leq t \leq T,\ \mathbb{P}\text{–a.s.}$$

It then follows by [210, Lemmata 3.2 and 3.6] that $\widehat{\mathcal{Y}}^+$, the right limit of $\widehat{\mathcal{Y}}$, is $\mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})}$-measurable, $\mathcal{P}(\mathbf{x})$–q.s. càdlàg, and for every $\mathbb{P} \in \mathcal{P}(\mathbf{x})$, there is $(\widehat{\mathcal{Z}}^{\mathbb{P}}, \widehat{\mathcal{K}}^{\mathbb{P}}) \in \mathbb{H}_{\mathbf{x}}^p(\mathbb{F}_+^{X,\mathbb{P}}, \mathbb{P}) \times \mathbb{I}_{\mathbf{x}}^p(\mathbb{F}_+^{X,\mathbb{P}}, \mathbb{P})$ such that for every $\mathbb{P} \in \mathcal{P}(\mathbf{x})$

$$\widehat{\mathcal{Y}}_t^+ = \xi(T, X_{\cdot\wedge T}) + \int_t^T F_r(X, \widehat{\mathcal{Z}}_r^{\mathbb{P}}, \widehat{\sigma}_r^2, \partial Y_r^r)\mathrm{d}r - \int_t^T \widehat{\mathcal{Z}}_r^{\mathbb{P}} \cdot \mathrm{d}X_r + \int_t^T \mathrm{d}\widehat{\mathcal{K}}_r^{\mathbb{P}},\ 0 \leq t \leq T,\ \mathbb{P}\text{–a.s.}$$

By [149], there exists a universal process $[\widehat{\mathcal{Y}}^+, X]$ which coincides with the quadratic co-variation of $\widehat{\mathcal{Y}}^+$ and X under each probability measure $\mathbb{P} \in \mathcal{P}(\mathbf{x})$. Thus, one can define a universal $\mathbb{F}_+^{X,\mathcal{P}(\mathbf{x})}$-predictable process Z by

$$\widehat{Z}_t := (\widehat{\sigma}_t^2)^{\oplus} \frac{\mathrm{d}[\widehat{\mathcal{Y}}^+, X]_t}{\mathrm{d}t}, \tag{2.8.1}$$

and obtain,

$$\widehat{\mathcal{Y}}_t^+ = \xi(T, X_{\cdot\wedge T}) + \int_t^T F_r(X, \widehat{Z}_r, \widehat{\sigma}_r^2, \partial Y_r^r)\mathrm{d}r - \int_t^T \widehat{Z}_r \cdot \mathrm{d}X_r + \widehat{\mathcal{K}}_T^{\mathbb{P}} - \widehat{\mathcal{K}}_t^{\mathbb{P}},\ 0 \leq t \leq T,\ \mathcal{P}(\mathbf{x})\text{–q.s.}$$

By well-posedness, we have that

$$\begin{aligned} &\widehat{\mathcal{Y}}_t^+ = Y_t,\ 0 \leq t \leq T,\ \mathcal{P}(\mathbf{x})\text{–q.s.};\ \widehat{Z}_t = Z_t,\ \widehat{\sigma}^2\mathrm{d}t \otimes \mathrm{d}\mathcal{P}(\mathbf{x})\text{–q.e. on } [0,T] \times \mathcal{X}; \\ &\widehat{\mathcal{K}}_t^{\mathbb{P}} = K_t^{\mathbb{P}}, 0 \leq t \leq T,\ \mathbb{P}\text{–a.s.},\ \forall \mathbb{P} \in \mathcal{P}(\mathbf{x}), \end{aligned} \tag{2.8.2}$$

where the latter denotes the solution to the first 2BSDE in the statement of the lemma. Thus, as $\mathcal{Y}^+$ is computed ω-by-ω, we can repeat the previous argument on the time interval $[s,T]$ and $\Omega_s^\omega = \{\tilde{\omega} \in \Omega : \tilde{x}_r = x_r, 0 \leq r \leq s\}$, *i.e.* fixing an initial trajectory. Reasoning as before, we then find that on Ω_s^ω, $\widehat{\mathcal{Y}}^+$ is $\mathbb{F}_+^{\mathcal{P}(s,x)}$-measurable and $\mathcal{P}(s,x)$–q.s. càdlàg. By well-posedness of the second 2BSDE in the statement of the lemma, this yields the analogous version of (2.8.2) between

$(\widehat{\mathcal{Y}}^+, \widehat{Z}, (\widehat{\mathcal{K}}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(s,x)})$ and $(Y^{s,x}, Z^{s,x}, (K^{s,x})_{\mathbb{P}\in\mathcal{P}(s,x)})$. It is then clear that

$$Y_t = Y_t^{s,x}, s \leq t \leq T, \mathcal{P}(s,x)\text{–q.s., for } \mathcal{P}(\mathbf{x})\text{–q.e. } x \in \mathcal{X},\ s \in (0,T].$$

The corresponding result for Z follows from (2.8.1). The relation for the family $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(\mathbf{x})}$ holds $\mathbb{P}$-by-$\mathbb{P}$ for every $\mathbb{P} \in \mathcal{P}(\mathbf{x})$ in light of the weak uniqueness assumption for the drift-less dynamics (2.1.7) and [56, Lemma 4.1], which guarantees that for any $\mathbb{P} \in \mathcal{P}(\mathbf{x})$ and $\mathbb{P}$–a.e. $x \in \mathcal{X}$

$$\bigg(\int_t^T Z_r \cdot X_r\bigg)^{\mathbb{P}} = \bigg(\int_t^T Z_r \cdot X_r\bigg)^{\mathbb{P}_{s,x}},\ 0 \leq s \leq t,\ \mathbb{P}_{s,x}\text{–a.s.}$$

□

Lemma 2.8.2. *Let $(\mathbb{P}^\nu, \nu) \in \mathfrak{M}(\mathbf{x})$. For $(s,x) \in [0,T] \times \mathcal{X}$ consider the system, assumed to hold $\mathbb{P}^\nu_{s,x}$–a.s.*

$$\begin{aligned} \partial\mathcal{Y}_t^r &= \partial_s\xi(r, X_{\cdot\wedge T}) + \int_t^T \partial h_u(r, X, \partial\mathcal{Z}_u^r, \nu_u)\mathrm{d}u - \int_t^T \partial\mathcal{Z}_u^r \cdot \mathrm{d}X_u,\ s \leq t \leq T, \\ \mathcal{Y}_t^r &= \xi(r, X_{\cdot\wedge T}) + \int_t^T h_u(r, X, \mathcal{Z}_u^r, \nu_u)\mathrm{d}u - \int_t^T \mathcal{Z}_u^r \cdot \mathrm{d}X_u,\ s \leq t \leq T. \end{aligned} \tag{D}$$

Let $(\partial\mathcal{Y}, \partial\mathcal{Z}) \in \mathbb{S}^{p,2}_{s,x}(\mathbb{F}_+^{X,\mathbb{P}^\nu_{s,x}}, \mathbb{P}^\nu_{s,x}) \times \mathbb{H}^{p,2}_{s,x}(\mathbb{F}_+^{X,\mathbb{P}^\nu_{s,x}}, \mathbb{P}^\nu_{s,x}, X)$ be the solution to the first BSDE *in* (D). *Then, the mapping $[0,T] \ni s \longmapsto (\partial\mathcal{Y}^s, \partial\mathcal{Z}^s) \in \mathbb{S}^p_{s,x}(\mathbb{F}_+^{X,\mathbb{P}^\nu_{s,x}}, \mathbb{P}^\nu_{s,x}) \times \mathbb{H}^p_{s,x}(\mathbb{F}_+^{X,\mathbb{P}^\nu_{s,x}}, \mathbb{P}^\nu_{s,x}, X)$ is Lebesgue-integrable with antiderivative $(\mathcal{Y}, \mathcal{Z})$, that is to say*

$$\bigg(\int_s^T \partial\mathcal{Y}^r \mathrm{d}r, \int_s^T \partial\mathcal{Z}^r \mathrm{d}r\bigg) = (\mathcal{Y}^T - \mathcal{Y}^s, \mathcal{Z}^T - \mathcal{Z}^s),\ \mathbb{P}^\nu_{s,x}\text{–a.s.}$$

Furthermore, letting

$$\partial Y_t^r(\omega) := \mathbb{E}^{\overline{\mathbb{P}}^\nu_{t,x}}\bigg[\partial_s\xi(r, X_{\cdot\wedge T}) + \int_t^T \partial_s f_u(r, X, \nu_u)\mathrm{d}u\bigg],\ (s,t) \in [0,T]^2,\ \omega \in \Omega,$$

it holds that $\partial Y_s^s = \mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}[\partial\mathcal{Y}_s^s]$ and $J(s,s,x,\nu) = \mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}[\mathcal{Y}_s^s]$. If in addition $\partial Y \in \mathbb{S}^{p,2}_{\mathbf{x}}(\mathbb{F}_+^{X,\mathbb{P}^\nu_{s,x}})$, for any $t \in [s,T]$

$$\mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}\bigg[\int_t^T \partial Y_r^r \mathrm{d}r\bigg] = \mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}\bigg[\int_t^T \partial\mathcal{Y}_r^r \mathrm{d}r\bigg].$$

Proof. We first prove the second part of the statement. Note that

$$\begin{aligned}\mathcal{Y}_t^s &= \xi(s, X_{\cdot\wedge T}) + \int_t^T \big(f_r(s, X, \nu_r) + b_r(X, \nu_r) \cdot \widehat{\sigma}_r^\top Z_r^s\big)\mathrm{d}r - \int_t^T Z_r^s \cdot \mathrm{d}X_r,\ \mathbb{P}^\nu_{s,x}\text{–a.s.},\\ &= \xi(s, X_{\cdot\wedge T}) + \int_t^T f_r(s, X, \nu_r)\mathrm{d}r - \int_t^T Z_r^s \cdot \mathrm{d}X_r,\ \overline{\mathbb{P}}^\nu_{s,x}\text{–a.s.}\end{aligned}$$

This implies

$$\mathcal{Y}_t^r(x) = \mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}\bigg[\xi(r, X_{\cdot\wedge T}) + \int_t^T f_u(r, X, \nu_u)\mathrm{d}u\Big|\mathcal{F}_{t+}^{X,\mathbb{P}^\nu_{s,x}}\bigg].$$

Therefore, by taking expectation

$$\mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}\big[\mathcal{Y}_s^s(\omega)\big] = \mathbb{E}^{\overline{\mathbb{P}}^\nu_{s,x}}\bigg[\xi(s, X_{\cdot\wedge T}) + \int_s^T f_r(s, X, \nu_r)\mathrm{d}r\bigg] = J(s, s, x, \nu).$$

The equality $\partial Y_s^s = \partial \mathcal{Y}_s^s$, is argued identically. Now, to obtain the last equality we use the fact that $\partial\mathcal{Y} \in \mathbb{S}^{p,2}_{s,x}(\mathbb{F}_+^{X,\mathbb{P}^\nu_{s,x}}, \mathbb{P}^\nu_{s,x})$. Indeed, the continuity of the mapping $s \longmapsto (\partial\mathcal{Y}^s, \partial\mathcal{Z}^s)$ guarantees the integral is well-defined. The equality follows from the tower property.

We now argue the first part of the statement. Again, we know the mapping $[0, T] \ni s \longmapsto (\partial\mathcal{Y}^s, \partial\mathcal{Z}^s)$ is continuous, in particular integrable. A formal integration with respect to s to the first equation in (D) leads to

$$\begin{aligned}\int_s^T \partial\mathcal{Y}_t^s \mathrm{d}s = \int_t^T \int_s^T \frac{\partial f_r}{\partial s}(s, X, \nu_r)\mathrm{d}s + b_r(X, \nu_r) \cdot \widehat{\sigma}_r^\top \int_s^T \partial\mathcal{Z}_r^s \mathrm{d}s\mathrm{d}r\\ + \int_s^T \partial_s\xi(s, X_{\cdot\wedge T})\mathrm{d}s - \int_t^T \int_s^T \partial\mathcal{Z}_r^s\mathrm{d}s \cdot \mathrm{d}X_r.\end{aligned}$$

Therefore, a natural candidate for solution to the second BSDE in (D) is $(\mathcal{Y}^s, \mathcal{Z}^s, \mathcal{N}^s)$, solution of the BSDE

$$\mathcal{Y}_t^s = \xi(s, X_{\cdot\wedge T}) + \int_t^T \big(f_r(s, X, \nu_r) + b_r(X, \nu_r) \cdot \widehat{\sigma}_r^\top \mathcal{Z}_r^s\big)\mathrm{d}r - \int_t^T \mathcal{Z}_r^s \cdot \mathrm{d}X_r.$$

Let $(\Pi^\ell)_\ell$ be a properly chosen sequence of partitions of $[s, T]$, as in [240, Theorem 1], $\Pi^\ell = (s_i)_{i=1,\dots,n_\ell}$ with $\|\Pi^\ell\| \leq \ell$. Recall $\Delta s_i^\ell = s_i^\ell - s_{i-1}^\ell$. For a generic family process $X \in \mathbb{H}^{p,2}_{s,x}(\mathbb{G})$, and mappings $s \longmapsto \partial_s \xi(s,x)$, $s \longmapsto \partial_s f(s,x,a)$ for $(s,x,a) \in [0,T] \times \mathcal{X} \times A$ we define

$$I^\ell(X) := \sum_{i=0}^{n_\ell} \Delta s_i^\ell X^{s_i^\ell},\ \delta X := X^T - X^s,\ I^\ell(\partial_s \xi(\cdot,x)) := \sum_{i=0}^{n} \Delta s_i^\ell \partial_s \xi(s_i^\ell, x),$$
$$I^\ell(\partial_s f)_t(\cdot,x,a) := \sum_{i=0}^{n_\ell} \Delta s_i^\ell \partial_s f_t(s_i^\ell, x, a),$$

and notice that for any $t \in [0,T]$

$$\begin{aligned} & I^\ell(\partial Y)_t - (\delta Y)_t \\ &= I^\ell(\partial_s \xi(\cdot, X_{\cdot\wedge T})) - (\xi(T, X_{\cdot\wedge T}) - \xi(s, X_{\cdot\wedge T})) - \int_t^T (I^\ell(\partial Z)_r - (\delta Z)_r) \cdot \mathrm{d}X_r \\ &\quad + \int_t^T [I^\ell(\partial_s f)_r(\cdot, X, \nu_r)) - (f_r(T,X,\nu_r) - f_r(s,X,\nu_r)] + \widehat{\sigma}_r b_r(X,\nu)[I^\ell(\partial Z)_r - (\delta Z)_r]\mathrm{d}r. \end{aligned}$$

Thanks to the integrability of $(\partial Y, \partial Z)$ and (Y,Z), it follows that $I^\ell(\partial Y) - (\delta Y) \in \mathbb{H}^{p,2}_{s,x}$ and similarly for ∂Z and Z. Therefore, [40, Theorem 2.2] yields

$$\begin{aligned} & \|I^\ell(\partial Y) - (\delta Y)\|^p_{\mathbb{H}^{p,2}_{s,x}} + \|I^\ell(\partial Z) - (\delta Z)\|^p_{\mathbb{H}^{p,2}_{s,x}} \\ &\leq \mathbb{E}^{\mathbb{P}^\nu_{s,x}}\bigg[\Big|I^\ell\big(\partial_s \xi(\cdot, X_{\cdot\wedge T})\big) - \big(\xi(T, X_{\cdot\wedge T}) - \xi(s, X_{\cdot\wedge T})\big)\Big|^p \\ &\quad + \int_t^T \Big|I^\ell\big(\partial_s f)_r(\cdot, X, \nu_r)\big) - \big(f_r(T,X,\nu_r) - f_r(s,X,\nu_r)\big)\Big|^p \mathrm{d}r\bigg]. \end{aligned}$$

The uniform continuity of $s \longmapsto \partial_s \xi(s,x)$ and $s \longmapsto \partial_s f(s,x,a)$, see Assumption C, justifies, via bounded convergence, the convergence in $\mathbb{S}^p_{s,x}(\mathbb{F}_+^{X,\mathbb{P}^\nu_{s,x}}, \mathbb{P}^\nu_{s,x})$ (resp $\mathbb{H}^p_{s,x}$) of $I^\ell(\partial Y^s)$ to $Y^T - Y^s$ (resp $I^\ell(\partial Z^s)$ to $Z^T - Z^s$) as $\ell \longrightarrow 0$. □

Lemma 2.8.3. *Let* $(\mathbb{P}^\nu, \nu) \in \mathfrak{M}(\mathbf{x})$, $(s,x) \in [0,T] \times \mathcal{X}$ *and* $(\partial\mathcal{Y}, \partial\mathcal{Z})$ *and* $(\mathcal{Y}, \mathcal{Z})$ *as in* (D). *Then*

$$\mathcal{Y}^t_t = \mathcal{Y}^T_T + \int_t^T h_r(r, X, \mathcal{Z}^r_r, \nu_r) - \partial\mathcal{Y}^t_t \mathrm{d}r - \int_t^T \mathcal{Z}^r_r \cdot \mathrm{d}X_r,\ \tilde{s} \leq t \leq T,\ \mathbb{P}_{s,x}\text{–a.s.} \tag{2.8.3}$$

Proof. By evaluating $\partial\mathcal{Y}$ at $r = t$ in (D) we get

$$\partial\mathcal{Y}^t_t = \partial_s \xi(t, X_{\cdot\wedge T}) + \int_t^T \partial h_r(t, X, \partial\mathcal{Z}^t_r, \nu_r) - \int_t^T \partial\mathcal{Z}^t_r \cdot \mathrm{d}X_r,\ \mathbb{P}^s_x\text{–a.s.}$$

We will show that for $s \leq t \leq T$, $\mathbb{P}_{s,x}$–a.s.

$$\xi(T, X_{\cdot\wedge T}) - \xi(t, X_{\cdot\wedge T}) + \int_t^T h_r(r, X, \mathcal{Z}_r^r, \nu_r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^r \mathrm{d}X_r$$
$$= \int_t^T h_r(t, X, \mathcal{Z}_r^t, \nu_r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^t \cdot \mathrm{d}X_r + \int_t^T \partial \mathcal{Y}_r^r \mathrm{d}r.$$

Indeed

$$\int_t^T \partial \mathcal{Y}_r \mathrm{d}r = \int_t^T \partial_s \xi(r, X_{\cdot\wedge T})\mathrm{d}r + \int_t^T \int_r^T \partial h_u(r, X, \partial \mathcal{Z}_u^r, \nu_u)\mathrm{d}u\, \mathrm{d}r - \int_t^T \int_r^T \partial \mathcal{Z}_u^r \cdot \mathrm{d}X_u\, \mathrm{d}r.$$

Now, Assumption C.(ii) and $\partial \mathcal{Z} \in \mathbb{H}_{s,x}^{p,2}(\mathbb{F}_+^{X,\mathbb{P}_{s,x}^\nu}, \mathbb{P}_{s,x}^\nu, X)$ yield

$$\int_t^T \int_r^T \partial h_u(r, X, \partial Z_u^r, \nu_u)\mathrm{d}u\, \mathrm{d}r = \int_t^T \int_t^u \big(\partial_s f_u(r, X, \nu_u) + \widehat{\sigma}_u b_u(X, \nu_u) \cdot \partial \mathcal{Z}_u^r\big)\mathrm{d}r\, \mathrm{d}u$$
$$= \int_t^T \big(h_u(u, X, \mathcal{Z}_u^u, \nu_u) - h_u(t, X, \mathcal{Z}_u^t, \nu_u)\big)\mathrm{d}u.$$

Moreover, $\|\partial \mathcal{Z}\|_{\mathbb{H}_{\tilde{s},x}^{p,2}(X)} < \infty$ guarantees $\int_0^T \mathbb{E}^{\mathbb{P}_{s,x}^\nu}\Big[\int_0^T |\widehat{\sigma}_t Z_t^r|^2 \mathrm{d}t\Big]^{\frac{p}{2}} \mathrm{d}r < \infty$ so a stochastic Fubini's theorem, see Da Prato and Zabczyk [66, Section I.4.5], justifies

$$\int_t^T \int_r^T \partial Z_u^s \cdot \mathrm{d}X_u \mathrm{d}r = \int_t^T [Z_u^u - Z_u^t] \cdot \mathrm{d}X_u,\ \mathbb{P}_{s,x}^\nu\text{–a.s.}$$

□

Proof of Theorem 2.2.12. We will first show that with $\nu^\star$ as in the statement of the theorem $Y_t(x) = J(t, t, x, \nu^\star)$ for all $t \in [0, T]$ and $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$. To do so, let $(s, x) \in [0, T] \times \mathcal{X}$ and note that Assumption C.(i) guarantees that the corresponding 2BSDE under $\mathcal{P}(s, x)$ is well-posed. Indeed, it follows from Soner, Touzi, and Zhang [229, Lemma 6.2] that for any $p > p' > \kappa > 1$,

$$\sup_{\mathbb{P} \in \mathcal{P}(s,x)} \mathbb{E}^{\mathbb{P}}\bigg[\operatorname*{ess\,sup}_{s \leq t \leq T}^{\mathbb{P}} \bigg(\operatorname*{ess\,sup}_{P' \in \mathcal{P}_{s,x}(t, \mathbb{P}, \mathbb{F}^+)}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}\bigg[|\xi(T, X_{\cdot\wedge T})|^\kappa + \int_s^T |F_r(X, 0, \widehat{\sigma}_r^2, \partial Y_r^r)|^\kappa \mathrm{d}r \bigg| \mathcal{F}_t^+\bigg]\bigg)^{\frac{p'}{\kappa}}\bigg] < \infty.$$

The well-posedness follows by [210, Theorem 4.1]. Now, in light of lemma 2.8.1, for any $s \in [0,T]$

$$\begin{aligned} Y_t &= Y_t^{s,x},\ s \leq t \leq T,\ \mathcal{P}(s,x)\text{–q.s., for } \mathcal{P}(\mathbf{x})\text{–q.e. } x \in \mathcal{X}, \\ Z_t &= Z_t^{s,x},\ \widehat{\sigma}_t^2 \mathrm{d}t \otimes \mathrm{d}\mathcal{P}(s,x)\text{–q.e. on } [s,T]\times\mathcal{X},\ \text{for } \mathcal{P}(\mathbf{x})\text{–q.e. } x \in \mathcal{X}, \\ K_t^{\mathbb{P}} &= K_t^{s,x,\mathbb{P}_{s,x}},\ s \leq t \leq T,\ \mathbb{P}_{s,x}\text{–a.s., for } \mathbb{P}\text{–a.e. } x \in \mathcal{X},\ \forall \mathbb{P} \in \mathcal{P}(\mathbf{x}). \end{aligned} \tag{2.8.4}$$

We first claim that given a solution to (H) for any $(s,x) \in (0,T]\times\mathcal{X}$, $\mathbb{P}^{\nu^\star}_{s,x}$ attains the minimality condition for the 2BSDE under $\mathcal{P}(s,x)$, see Definition 2.6.2. Indeed, by Definition 2.2.7.(iii)

$$\mathbb{E}^{\mathbb{P}^{\nu^\star}_{0,\mathbf{x}}}\Big[K_T^{\mathbb{P}^{\nu^\star}_{0,\mathbf{x}}} - K_t^{\mathbb{P}^{\nu^\star}_{0,\mathbf{x}}}\Big] = 0,\ 0 \leq t \leq T.$$

As $K^{\mathbb{P}^{\nu^\star}_{0,\mathbf{x}}}$ is a non-decreasing process, this implies $K^{\mathbb{P}^{\nu^\star}_{0,\mathbf{x}}} = 0$ and therefore $\mathbb{P}^{\nu^\star}_{0,\mathbf{x}}$ attains the minimality condition for the 2BSDE in (H) under $\mathcal{P}(\mathbf{x})$. This implies, together with (2.8.4), that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$ and $s \in [0,T]$

$$\mathbb{E}^{\mathbb{P}^{\nu^\star}_{s,x}}\Big[\mathbb{E}^{\mathbb{P}^{\nu^\star}_{s,x}}\Big[K_T^{s,x,\mathbb{P}^{\nu^\star}_{s,x}} - K_s^{s,x,\mathbb{P}^{\nu^\star}_{s,x}}\Big|\mathcal{F}^X_{s+}\Big]\Big] = \mathbb{E}^{\mathbb{P}^{\nu^\star}_{s,x}}\Big[K_T^{s,x,\mathbb{P}^{\nu^\star}_{s,x}} - K_s^{s,x,\mathbb{P}^{\nu^\star}_{s,x}}\Big] = 0,\ \mathbb{P}^{\nu^\star}_{s,x}\text{-a.s.},$$

which proves the claim. Consequently, for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$ and $s \in [0,T]$

$$Y_t = Y_t^{s,x} = \xi(T, X_{\cdot\wedge T}) + \int_t^T F_r(X, Z_r^{s,x}, \widehat{\sigma}_r^2, \partial Y_r^r)\mathrm{d}r - \int_t^T Z_r^{s,x}\cdot \mathrm{d}X_r,\ s \leq t \leq T,\ \mathbb{P}^{\nu^\star}_{s,x}\text{–a.s.}$$

We note the equation on the right side prescribes a BSDE under $\mathbb{P}^{\nu^\star}_{s,x} \in \mathcal{P}(s,x)$. Moreover, given that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$ and $s \in [0,T]$, $\mathcal{V}_t^\star(X, Z_t^{s,x}) = \mathcal{V}_t^\star(X, Z_t), \mathrm{d}t \otimes \mathrm{d}\mathbb{P}^{\nu^\star}_{s,x}$–a.e. on $[s,T]\times\mathcal{X}$, we obtain

$$Y_t^{s,x} = \xi(T, X_{\cdot\wedge T}) + \int_t^T (h_r(r, X, Z_r, \nu_r^\star) - \partial Y_r^r)\mathrm{d}r - \int_t^T Z_r^{s,x}\cdot \mathrm{d}X_r,\ s \leq t \leq T,\ \mathbb{P}^{\nu^\star}_{s,x}\text{–a.s.}$$

In particular, at $t = s$ we have that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$ and $t \in [0,T]$

$$\mathbb{E}^{\mathbb{P}^{\nu^\star}_{t,x}}[Y_t^{t,x}] = \mathbb{E}^{\mathbb{P}^{\nu^\star}_{t,x}}\Big[\xi(T, X_{\cdot\wedge T}) + \int_t^T (h_r(r, X, Z_r, \nu_r^\star) - \partial Y_r^r)\mathrm{d}r\Big]. \tag{2.8.5}$$

Now, in light of Assumption D, there exists $(\partial\mathcal{Y}, \partial\mathcal{Z}) \in \mathbb{S}^{2,2}_{t,x}(\mathbb{F}_+^{X,\mathbb{P}^{\nu^\star}_{t,x}}) \times \mathbb{H}^{2,2}_{t,x}(\mathbb{F}_+^{X,\mathbb{P}^{\nu^\star}_{t,x}}, X)$ such that for every $s \in [0,T]$

$$\partial \mathcal{Y}_r^s = \partial_s \xi(s, X_{\cdot \wedge T}) + \int_r^T \partial h_u(s, X, \partial \mathcal{Z}_u^s, \nu_u^\star)\mathrm{d}u - \int_r^T \partial \mathcal{Z}_u^s \cdot \mathrm{d}X_u,\ t \leq s \leq r \leq T,\ \mathbb{P}_{t,x}^{\nu^\star}\text{–a.s.}$$

In addition, Lemma 2.8.2 yields

$$\mathbb{E}^{\mathbb{P}_{t,x}^{\nu^\star}}\bigg[\int_t^T \partial Y_r^r \mathrm{d}r\bigg] = \mathbb{E}^{\mathbb{P}_{t,x}^{\nu^\star}}\bigg[\int_t^T \mathbb{E}^{\mathbb{P}_{t,\cdot}^{\nu^\star}}\bigg[\partial_s \xi(r, X_{\cdot\wedge T}) + \int_r^T \partial_s f_u(r, X, \nu_u^\star)\mathrm{d}u\bigg]\mathrm{d}r\bigg] = \mathbb{E}^{\mathbb{P}_{t,x}^{\nu^\star}}\bigg[\int_t^T \partial \mathcal{Y}_r^r \mathrm{d}r\bigg].$$

Therefore, from Lemma 2.8.3 and (2.8.5) we have that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in X$ and $t \in [0,T]$

$$\mathbb{E}^{\mathbb{P}_{t,x}^{\nu^\star}}[Y_t^{t,x}] = \mathbb{E}^{\mathbb{P}_{t,x}^{\nu^\star}}\bigg[\xi(T, X_{\cdot\wedge T}) + \int_t^T (h_r(r, X, Z_r^{t,x}, \nu_r^\star) - \partial \mathcal{Y}_r^r)\mathrm{d}r\bigg] = J(t,t,x,\nu^\star). \tag{2.8.6}$$

Finally, arguing as in [64, Proposition 4.6], (2.8.6) yields that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$ and $t \in [0,T]$

$$v(t,x) = \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}[Y_t^{t,x}].$$

It remains to show $\nu^\star \in \mathcal{E}(\mathbf{x})$. Let $(\varepsilon, \ell, t, x, \nu) \in \mathbb{R}_+^\star \times (0, \ell_\varepsilon) \times [0,T] \times \Omega \times \mathcal{A}(t,x)$, ℓ_ε to be chosen, and $\nu \otimes_{t+\ell} \nu^\star$ as in (2.1.12). Recall we established that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$, $t \in [0,T]$

$$J(t,t,x,\nu^\star) = \xi(T,X) + \int_t^T F_r(X, Z_r^{t,x}, \widehat{\sigma}_r^2, \partial Y_r^r)\mathrm{d}r - \int_t^T Z_r^{t,x} \cdot \mathrm{d}X_r + K_T^{t,x,\mathbb{P}} - K_t^{t,x,\mathbb{P}},\ \mathcal{P}(t,x)\text{–q.s.}$$

By computing the expectation of the stochastic integral under $\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}$ we obtain

$$\begin{aligned}
&J(t,t,x,\nu^\star) - J(t,t,x,\nu\otimes_{t+\ell}\nu^\star)\\
&= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}}\bigg[\xi(T,X) + \int_t^T (h_r(r,X,Z_r^{t,x},\nu_r^\star) - b_r(X,(\nu\otimes_{t+\ell}\nu^\star)_r)\cdot\widehat{\sigma}_r^\top Z_r^{t,x})\mathrm{d}r\\
&\quad - \xi(t,X) - \int_t^T (f_r(t,X,(\nu\otimes_{t+\ell}\nu^\star)_r) + \partial Y_r^r)\mathrm{d}r\bigg]\\
&= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}}\bigg[\xi(T,X) - \xi(t,X) + \int_t^T h_r(r,X,Z_r^{t,x},\nu_r^\star) - h_r(t,X,Z_r^{t,x},\nu_r^\star) - \partial Y_r^r\mathrm{d}r\bigg]\\
&\quad + \int_t^{t+\ell} h_r(r,X,Z_r^{t,x},\nu_r^\star) - h_r(t,X,Z_r^{t,x},\nu_r) + h_r(t,X,Z_r^{t,x},\nu_r^\star) - h_r(r,X,Z_r^{t,x},\nu_r)\mathrm{d}r\bigg],
\end{aligned}$$

where the inequality follows from dropping the K term. Since for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$, $\mathcal{V}_t^\star(X, Z_t^{s,x}) =$

$\mathcal{V}_t^\star(X,Z_t)$, $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$–a.e. on $[s,T]\times\mathcal{X}$ for all $\mathbb{P}\in\mathcal{P}(s,x)$, we have the previous expression is greater or equal than the sum of

$$
\begin{aligned}
I_1 &:= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}}\bigg[\xi(T,X)-\xi(t+\ell,X)+\int_{t+\ell}^T\big(h_r(r,X,Z_r^{t,x},\nu_r^\star)-h_r(t+\ell,X,Z_r^{t,x},\nu_r^\star)-\partial Y_r^r\big)\mathrm{d}r\bigg],\\
I_2 &:= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}}\bigg[\xi(t+\ell,X)-\xi(t,X)-\int_t^T f_r(t,X,\nu_r^\star)\mathrm{d}r+\int_{t+\ell}^T f_r(t+\ell,X,\nu_r^\star)\mathrm{d}r\\
&\qquad +\int_t^{t+\ell}\big(f_r(r,X,\nu_r^\star)-\partial Y_r^r\big)\mathrm{d}r\bigg],\\
I_3 &:= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}}\bigg[\int_t^{t+\ell}\big(f_r(r,X,\nu_r)-f_r(t,X,\nu_r)+f_r(t,X,\nu_r^\star)-f_r(r,X,\nu_r^\star)\big)\mathrm{d}r\bigg].
\end{aligned}
$$

We now study each remaining terms separately. First, regarding I_1, by conditioning we can see this term equals 0. Indeed, this follows analogously to (2.8.5), by using the fact that $(\delta_\omega\otimes_{t+\ell}\overline{\mathbb{P}}^{\nu^\star,t+\ell,x})_{\omega\in\Omega}$ is an r.c.p.d. of $\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}|\mathcal{F}_{t+\ell}^X$, see Lemma 2.10.3.1, together with Lemma 2.8.3.

We can next use Fubini's theorem and Φ as in Assumption C to express the term I_2 as

$$
\begin{aligned}
I_2 &= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}}\bigg[\int_t^{t+\ell}\partial_s\xi(r,X)\mathrm{d}r+\int_t^T\int_r^T\partial_s f_u(r,X,\nu_u^\star)\mathrm{d}u\mathrm{d}r\\
&\qquad -\int_{t+\ell}^T\int_r^T\partial_s f_u(r,X,\nu_u^\star)\mathrm{d}u\mathrm{d}r-\int_t^{t+\ell}\partial Y_r^r\mathrm{d}r\bigg]\\
&= \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[\int_t^{t+\ell}\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t+\ell,\cdot}}[\Phi(r,X)]-\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,\cdot}}[\Phi(r,X)]\mathrm{d}r\bigg],
\end{aligned}
$$

where the second equality follows by conditioning, see Lemma 2.10.3.1. Now, arguing as in the proof of [232, Corollary 6.3.3], under the weak uniqueness assumption for fixed actions ,$\mathbb{P}^{\nu^\star}_{t_n,x_n}\longrightarrow\mathbb{P}^{\nu^\star}_{t,x}$ weakly, whenever $(t_n,x_n)\longrightarrow(t,x)$.

By Assumption C.(iii), for every $(r,t,x)\in[0,T]^2\times\mathcal{X}$, $\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t+\ell,x}}[\Phi(r,X)]\longrightarrow\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}[\Phi(r,X)]$, $\ell\longrightarrow0$. Moreover, as $t\longmapsto\Phi(t,x)$ is clearly continuous and $(r,t)\in[0,T]^2$, the previous convergence holds uniformly in (r,t). Together with bounded convergence we obtain that for $\mathcal{P}(\mathbf{x})$–q.e. $x\in\mathcal{X}$ and $\nu\in\mathcal{A}(t,x)$, $\mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_{t+\ell}\nu^\star}}[\Phi(r,X)]\longrightarrow\mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_t\nu^\star}}[\Phi(r,X)]$, $\ell\longrightarrow0$, uniformly in (t,r).

We now argue that the above convergence holds uniformly in ν. Indeed, Assumption C.(i) guarantees that for $\mathcal{P}(\mathbf{x})$–q.e. $x\in\mathcal{X}$ the family $\mathcal{M}^x_{t+\ell}(\tilde{x}):=\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t+\ell,\tilde{x}}}[\Phi(r,X)]-\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{t,x}}[\Phi(r,X)]$, is $\mathcal{P}(t,x)$–uniformly integrable. Thus, provided $\mathcal{P}(t,x)$ is weakly compact, an application of the non-dominated monotone convergence theorem, see [74, Theorem 31], yields the result. In order to bypass the compactness assumption on $\mathcal{P}(t,x)$, we consider the compact set $\mathcal{A}^{\text{rel}}(t,x)$, see [86, Theorem 4.1], of solutions to the martingale problem for which relaxed action processes are allowed, *i.e.* ignoring condition (iii) in the definition of $\mathcal{P}(t,x)$. By [86, Theorem 4.5], the supremum over the two families coincide. With this we can find ℓ_ε such that for $\ell < \ell_\varepsilon$

$$\int_t^{t+\ell} \sup_{\nu \in \mathcal{A}(t,x)} \Big| \mathbb{E}^{\overline{\mathbb{P}}^{\nu \otimes_{t+\ell} \nu^\star}} \big[\Phi(r,X)\big] - \mathbb{E}^{\overline{\mathbb{P}}^{\nu \otimes_{t} \nu^\star}} \big[\Phi(r,X)\big] \Big| \mathrm{d}r \leq \varepsilon \ell.$$

Finally, to control I_3, we see that Assumption C.(ii) guarantees there is ℓ_ε such that for all $(r,x,a) \in [0,T] \times \mathcal{X} \times A$, $|f_r(s,x,a) - f_r(t,x,a)| < \varepsilon/2$ whenever $|s-t| < \ell_\varepsilon$, so that

$$\mathbb{E}^{\overline{\mathbb{P}}^{\nu \otimes_{t+\ell} \nu^\star}} \bigg[\int_t^{t+\ell} |f_r(r,X,\nu_r)dr - f_r(t,X,\nu_r)| + |f_r(t,X,\nu_r^\star) - f_r(r,X,\nu_r^\star)|dr \bigg] \leq \varepsilon \ell.$$

Combining the previous arguments, we obtain that for $0 < \ell < \ell_\varepsilon$, $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$ and $(t,\nu) \in [0,T] \times \mathcal{A}(t,x)$

$$J(t,t,x,\nu^\star) - J(t,t,x,\nu \otimes_{t+\ell} \nu^\star) \geq -\varepsilon \ell.$$

□

2.9 Well-posedness: the uncontrolled volatility case

We start this section studying how System (H) reduces when no control on the volatility is allowed. Intuitively speaking the first equation should reduce to a standard BSDE and under our assumption of weak uniqueness for (2.1.7) we end up with only one probability measure which allows a probabilistic representation of the second element in the system. We first study the reduction in the next proposition.

Proposition 2.9.1. *Suppose* $\sigma_t(x,a) = \sigma_t(x,\tilde{a}) =: \sigma_t(x)$ *for all* $a \in A$, i.e. *the volatility is not controlled, then* System (H) *reduces to the following system which for any* $s \in [0,T]$, *holds* $\mathbb{P}$–a.s. *for any* $t \in [0,T]$

$$\begin{aligned} Y_t &= \xi(T, X_{\cdot \wedge T}) + \int_t^T H_r^o(X, Z_r, \partial Y_r^r)\mathrm{d}r - \int_t^T Z_r \cdot \mathrm{d}X_r, \\ \partial Y_t^s &= \partial_s \xi(s, X_{\cdot \wedge T}) + \int_t^T \partial h_r^o(s, X, \partial Z_r^s, \mathcal{V}^\star(r,X,Z_r))\mathrm{d}r - \int_t^T \partial Z_r^s \cdot \mathrm{d}X_r. \end{aligned} \tag{H$_\text{o}$}$$

Proof. As the volatility is not controlled there is a unique solution to the martingale problem (2.1.6), *i.e.* $\mathcal{P}(\mathbf{x}) = \{\mathbb{P}\}$. In addition, since $(\sigma\sigma^\top)_t(x,a) = (\sigma\sigma^\top)_t(x)$ for all $t \in [0,T]$, then

$$\Sigma_t(x) = \{(\sigma\sigma^\top)_t(x)\} \in \mathbb{S}_n^+(\mathbb{R}),\ A_t(x,\sigma_t(x)) = A.$$

Let (Y,Z,K) be a solution to the 2BSDE in (H). As $\mathcal{P}(\mathbf{x}) = \{\mathbb{P}\}$, the minimality condition implies that the process $K^{\mathbb{P}}$ vanishes in the dynamics, thus (Y,Z) is a solution to the first BSDE in $(\mathrm{H_o})$. Now as the family ∂Y is defined $\mathbb{P}$–a.s., Y is well-defined in the $\mathbb{P}$–a.s. sense too. Finally, $\mathcal{P}(\mathbf{x}) = \{\mathbb{P}\}$ guarantees that the predictable martingale representation holds for $(\mathbb{F}_+^{X,\mathbb{P}},\mathbb{P})$-martingales. With this, it follows that for $s \in [0,T]$, ∂Y^s in (H) admits the representation in $(\mathrm{H_o})$, which holds up to a $\mathbb{P}$-null set. $\square$

Remark 2.9.2 (HJB system exponential case)**.** *As a sanity check at this point we can check what the above system leads to in the case of exponential discounting in a non-Markovian framework. Defining $f(s,t,x,a)$ and $F(s,x)$ as in* Remark 2.5.5*, note that*

$$J(t,x,\nu) = \mathbb{E}^{\mathbb{P}^\nu}\bigg[\int_t^T \mathrm{e}^{-\theta(r-t)}\tilde f(r,X,\nu_r)\mathrm{d}r + \mathrm{e}^{-\theta(T-t)}\tilde F(X_{\cdot\wedge T})\bigg] = Y_t^t.$$

Notice that

$$\partial Y_t^s = \theta \mathrm{e}^{-\theta(T-s)} + \int_t^T \big(\theta\mathrm{e}^{-\theta(r-s)}\tilde f(r,X,\nu_r^\star) + b(r,X,\nu_r^\star)\cdot\sigma(r,X)^\top Z_r^s\big)\mathrm{d}r - \int_t^T \partial Z_r^s \cdot \mathrm{d}X_r,$$

and as it does turn out that $Z_r^r = Z_r$, see Lemma 2.8.2 *and* Theorem 2.2.12*, we get*

$$\begin{aligned}\partial Y_t^t &= \theta\mathrm{e}^{-\theta(T-t)} + \int_t^T \big(\theta\mathrm{e}^{-\theta(r-t)}\tilde f(r,X,\nu_r^\star) + b(r,X,\nu_r^\star)\cdot\sigma(r,X)^\top Z_r^t\big)\mathrm{d}r - \int_t^T \partial Z_r^t\cdot\mathrm{d}X_r \\ &= \theta\mathbb{E}^{\mathbb{P}^{\nu^\star}}\bigg[\int_t^T \mathrm{e}^{-\theta(r-t)}\tilde f(r,X,\nu_r^\star)\mathrm{d}r + e^{-\theta(T-t)}\tilde F(X_{\cdot\wedge T})\Big|\mathcal{F}_{t+}^{X,\mathbb{P}}\bigg] = \theta Y_t^t.\end{aligned}$$

Thus

$$Y_t = \xi(T) + \int_t^T H_r^o(X_r, Z_r^{\nu^\star}, \theta Y_r)\mathrm{d}r - \int_t^T Z_r^{\nu^\star}\cdot\mathrm{d}X_r,\ \nu_t^\star(x,z,u) := \underset{a\in A}{\arg\max}\,\{h_t^o(t,x,z,a)\}.$$

In the classic Brownian filtration set up, e.g. assuming σ is non-degenerate and $n=d$, the above

BSDE *corresponds to the well known solution to an optimal stochastic control problem with exponential discounting, see* [273].

We would like to stress the fact that the existence of a solution to System ($\mathrm{H_o}$) is in general not guaranteed. Indeed, for a fixed ν one could try to find the family of BSDEs $(\partial Y^s)_{s\in[0,T]}$, determined by the second equation and use this solution to obtain Y. Nevertheless, this approach becomes nonviable as a result of the coupling arising from the choice of ν in the first equation and its appearance in the second one. The general treatment of systems as ($\mathrm{H_o}$) is carried out in the Appendix.

Proof of Theorem 2.2.15. The result is immediate from Theorem 3.3.5. □

Remark 2.9.3. *The general well-posedness result,* i.e. *in which both the drift and the volatility are controlled remains open. In fact, as this requires to be able to guarantee the existence of a probability measure* $\mathbb{P}^\star$ *under which the minimality condition* (2.2.8) *is attained, we believe a feasible direction to attain this result is to go one level beyond the weak formulation, and consider and work in a relaxed framework, see for example* [86] *for a precise formulation of control problems in relaxed form.*

2.10 Auxiliary results

2.10.1 Optimal investment and consumption for log utility

We provide the necessary results for Section 2.3. We start with expressions to determine the functions $a(\cdot)$ and $b(\cdot)$. We introduce the quantities

$$p := \frac{1}{\eta},\ q := 1-\frac{1}{\eta},\ \alpha_1(t) := r+\frac{1}{2}\beta^2 p + a(t)^{-p},\ \alpha_2(t) := (1-\eta)(\alpha_1(t)+\frac{\beta^2}{2\eta^2}(1+\eta)),\ t\in[0,T].$$

Under the optimal policy $(c^\star,\gamma^\star)$ we have that $\mathbb{P}^{\nu^\star}$–a.s.

$$\begin{aligned}\mathrm{d}X_t &= X_t(r+\beta^2\eta^{-1}+a(r)^{-\frac{1}{\eta}})\mathrm{d}t+\beta\eta^{-1}X_t\mathrm{d}W_t,\\ \mathrm{d}{X_t}^{1-\eta} &= (1-\eta)X_t^{1-\eta}\big[\alpha_1(t)\mathrm{d}t+p\beta\mathrm{d}W_t\big],\end{aligned}$$

$$\mathrm{d}U(c^\star(t,X_t)) = a(t)^q X_t^{1-\eta}\big((\alpha_1(t) - pa'(t)a(t)^{-1})\mathrm{d}t + p\beta \mathrm{d}W_t\big),$$

which we can use to obtain that for $\mathbb{P} \in \mathcal{P}(t,x)$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[U(c^\star(r,X_r))] &= U(c^\star(t,x)) + x^{1-\eta}\int_t^r \exp\bigg(\int_t^u \alpha_2(v)\mathrm{d}v\bigg)a(u)^q(\alpha_1(u) - pa'(u)a(u)^{-1})\mathrm{d}u,\\
\mathbb{E}^{\mathbb{P}}[U(X_T)] &= U(x)\exp\bigg(\int_t^T \alpha_2(u)\mathrm{d}u\bigg).
\end{aligned}$$

By direct computation in (2.8.3) one finds that in general a must satisfy

$$\begin{aligned}
0 &= a'(t) + (1-\eta)a(t)\alpha_1(t) + a(t)^q\varphi(T-t) + \varphi'(T-t)\exp\bigg(\int_t^T \alpha_2(u)\mathrm{d}u\bigg)\\
&+ \int_t^T \varphi'(r-t)\int_t^r \exp\bigg(\int_t^r \alpha_2(v)\mathrm{d}v\bigg)a(u)^q(\alpha_1(t) - pa'(u)a(u)^{-1})\mathrm{d}u\mathrm{d}r,\ t\in[0,T),\ a(T) = 1,
\end{aligned}$$

where we recall α_1 and α_2 are actually functions of a. The boundary condition follows as $Y_T(x) = U(x)$ for all $x \in \mathcal{X}$. The previous equation is, of course, an implicit formula that reflects the non-linearities inherent to the general case. A general expression for b can be written down too. We have refrained to do so here.

Nevertheless, in the particular case $\eta = 1$, which corresponds to the log utility scenario, the expressiona involved simplify considerably, which reflects the fact that all non-linearities vanish. Indeed, in this case $p = 1$, $q = 0$, $\alpha_1 = r + \beta/2 - a^{-1}$, $\alpha_2 = 0$ and one obtains

$$\begin{aligned}
0 = b'(t) + a(t)\alpha_1(t) - \log a(t) - \int_t^T \varphi'(s-t)\big(A(s) - A(t) - \log(a(s))\big)\mathrm{d}s\\
+ \varphi'(T-t)(A(T) - A(t)) + \log(x)[a'(t) + \varphi(T-t) + \varphi'(T-t)],
\end{aligned}$$

where $A(t)$ denotes the antiderivative of α_1. To find a we set

$$a'(t) + \varphi(T-t) + \varphi'(T-t) = 0,\ t\in[0,T),\ a(T) = 1.$$

This determines both α and A. b is then given by setting the first line in the above expression equal to zero together with the boundary condition $b(T) = 0$.

2.10.2 Extensions

This section contain the proofs of the results in 2.4. We begin with the proof of Lemma 2.4.2 which verifies conditions under which the additional requirement in Assumption E is satisfied.

Proof of Lemma 2.4.2. In the case of uncontrolled volatility, the weak uniqueness assumption implies that there exists a unique probability measure $\mathbb{P}$ and a $\mathbb{P}$–Brownian motion W such that

$$X_t = x_0 + \int_0^t \sigma_r(X_{\cdot\wedge r}) \cdot \mathrm{d}W_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.}$$

In light of the regularity of f, we have

$$\mathsf{f}_r(X_{\cdot\wedge r}) - \mathsf{f}_t(X_{\cdot\wedge t}) = \int_t^r \mathfrak{A}(u, X_{\cdot\wedge u}, \nu_u)\mathrm{d}r + \int_t^r \sigma_u(X_{\cdot\wedge u})\nabla_x f_u(X_{\cdot\wedge u}) \cdot \mathrm{d}W_r,$$

from which we obtain

$$\mathbb{E}^{\mathbb{P}^{\nu\otimes_{t'}\nu^\star}}[\mathsf{f}_r(X_r)|\mathcal{F}_t] = \mathsf{f}_t(X_t) + \int_t^{t'} \mathbb{E}^{\mathbb{P}}[\mathfrak{A}(u, X_{\cdot\wedge u}, \nu_u)\mathcal{E}^{\nu}_{t,T}|\mathcal{F}_t]\mathrm{d}u + \int_{t'}^r \mathbb{E}^{\mathbb{P}}[\mathfrak{A}(u, X_{\cdot\wedge u}, \nu^\star_u)\mathcal{E}^{\nu\otimes_{t'}\nu^\star}_{t,T}|\mathcal{F}_t]\mathrm{d}u.$$

Thus

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{\nu\otimes_{t'}\nu^\star}}[\mathsf{f}_r(X_r)|\mathcal{F}_t] - \mathbb{E}^{\mathbb{P}^{\nu^\star}}[\mathsf{f}_r(X_r)|\mathcal{F}_t] &= \int_t^{t'} \mathbb{E}^{\mathbb{P}}[\mathfrak{A}(u, X_{\cdot\wedge u}, \nu_u)\mathcal{E}^{\nu}_{t,T}|\mathcal{F}_t] - \mathbb{E}^{\mathbb{P}}[\mathfrak{A}(u, X_{\cdot\wedge u}, \nu^\star_u)\mathcal{E}^{\nu^\star}_{t,T}|\mathcal{F}_t]\mathrm{d}u \\ &\quad + \mathbb{E}^{\mathbb{P}}\bigg[(\mathcal{E}^{\nu\otimes_{t'}\nu^\star}_{t,T} - \mathcal{E}^{\nu^\star}_{t,T}) \int_{t'}^r \mathfrak{A}(u, X_{\cdot\wedge u}, \nu^\star_u)\mathrm{d}u\bigg|\mathcal{F}_t\bigg].\end{aligned}$$

The result then follows in light of the integrability of the first term, applying Hölder intequality to the second term and the estimate

$$\mathbb{E}^{\mathbb{P}}\Big[|\mathcal{E}^{\nu\otimes_{t'}\nu^\star}_{t,T} - \mathcal{E}^{\nu^\star}_{t,T}|^2\Big|\mathcal{F}_t\Big] \leq \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_t^{t'} |\nu_r - \nu^\star_r|\mathrm{d}r\bigg)^2\bigg] \leq C|t'-t|^2.$$

To obtain the result in (ii), notice that by uniqueness of the strong solution $X^{x,\nu}$ we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[\mathsf{f}_r(X^{\nu\otimes_{t'}\nu^\star}_r)|\mathcal{F}_t] &= \mathsf{f}_t(X_t) + \int_t^{t'} \mathbb{E}^{\mathbb{P}}[\mathcal{A}(u, X^{\nu}_r, \nu_u)|\mathcal{F}_t]\mathrm{d}u + \int_{t'}^r \mathbb{E}^{\mathbb{P}}[\mathcal{A}(u, X^{\nu\otimes_{t'}\nu^\star}_r, \nu^\star_u)|\mathcal{F}_t]\mathrm{d}u \\ &= \mathsf{f}_t(X_t) + \int_t^{t'} \mathbb{E}^{\mathbb{P}}[\mathcal{A}(u, X^{\nu}_r, \nu_u)|\mathcal{F}_t]\mathrm{d}u + \int_{t'}^r \mathbb{E}^{\mathbb{P}}[\mathcal{A}(u, X^{\nu^\star}_r, \nu^\star_u)|\mathcal{F}_t]\mathrm{d}u.\end{aligned}$$

Thus, the result follows from

$$\mathbb{E}^{\mathbb{P}}[\mathsf{f}_r(X_r^{\nu\otimes_{t'}\nu^\star})|\mathcal{F}_t] - \mathbb{E}^{\mathbb{P}}[\mathsf{f}_r(X_r^{\nu^\star})|\mathcal{F}_t] = \int_t^{t'} \mathbb{E}^{\mathbb{P}}[\mathcal{A}(u, X_r^{\nu}, \nu_u)|\mathcal{F}_t] - \mathbb{E}^{\mathbb{P}}[\mathcal{A}(u, X_r^{\nu^\star}, \nu_u^\star)|\mathcal{F}_t]\mathrm{d}u.$$

$\square$

The next sequence of results build up to the proof of Theorem 2.4.3, namely we wish to obtain the corresponding *extended* version of the *dynamic programming principle*. We begin with a sequence of lemmata that will allow us to study the local behaviour of the value of the game. These results are true in great generality and require mere extra regularity of the running and terminal rewards in the type variables $(s, \mathrm{n}, \mathrm{m})$. Our first result consists of a one step iteration of our equilibrium definition.

Lemma 2.10.1. *Let $\nu^\star \in \mathcal{E}(\mathbf{x})$ and v the value associated to $\nu^\star$. Then, for any $(\varepsilon, \ell, t, \sigma, \tau) \in (0, \infty) \times (0, \ell_\varepsilon) \times [0, T] \times \mathcal{T}_{t,t+\ell} \times \mathcal{T}_{t,t+\ell}$ with $\sigma \leq \tau$, and $\mathcal{P}(\mathbf{x})$–*q.e.* $x \in \mathcal{X}$,*

$$\begin{aligned}
v(\sigma, x) \geq \sup_{\nu \in \mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^{\nu}} \bigg[& v(\tau, X) + \int_\sigma^\tau f_r(\sigma, X_{\cdot\wedge r}, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[\mathsf{f}_r(X)], \nu_r)\mathrm{d}r + \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau,\cdot}} \Big[G(\sigma, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[M_\tau^\star]) - G(\tau, M_\tau^\star) \Big] \\
& + \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau,\cdot}} \Big[\int_\tau^T f_r(\sigma, X_{\cdot\wedge r}, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[N_r^{\tau,\star}], \nu_r^\star) - f_r(\tau, X_{\cdot\wedge r}, N_r^{\tau,\star}, \nu_r^\star)\mathrm{d}r \Big] - \varepsilon\ell \\
& + \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau,\cdot}} \Big[\xi(\sigma, X_{\cdot\wedge T}) - \xi(\tau, X_{\cdot\wedge T}) \Big] - \varepsilon\ell.
\end{aligned}$$

Proof. To get the second inequality note that for $(\varepsilon, \ell, t, \sigma, \tau) \in \mathbb{R}_+^\star \times (0, \ell_\varepsilon) \times [0, T] \times \mathcal{T}_{t,t+\ell} \times \mathcal{T}_{t,t+\ell}$ with $\sigma \leq \tau$, $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$ and $\nu \in \mathcal{A}(\sigma, x)$

$$\begin{aligned}
v(\sigma, x) &= J(\sigma, \sigma, x, \nu^\star) \geq J(\sigma, \sigma, x, \nu \otimes_\tau \nu^\star) - \varepsilon\ell \\
&= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}} \bigg[\int_\sigma^\tau f_r(\sigma, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}}[\mathsf{f}_r(X)], (\nu \otimes_\tau \nu^\star)_r)\mathrm{d}r \\
&\qquad + \int_\tau^T f_r(\sigma, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}}[\mathsf{f}_r(X)], (\nu \otimes_\tau \nu^\star)_r)\mathrm{d}r \\
&\qquad + \xi(\sigma, X_{\cdot\wedge T}) + G(\sigma, \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}}[g(X_{\cdot\wedge T})]) \bigg] - \varepsilon\ell \\
&= \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}} \bigg[\int_\sigma^\tau f_r(\sigma, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[\mathsf{f}_r(X)], \nu_r)\mathrm{d}r + \int_\tau^T f_r(\sigma, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[N_r^{\tau,\star}], \nu_r^\star)\mathrm{d}r \bigg] \\
&\qquad + \mathbb{E}^{\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}} \Big[\xi(\sigma, X_{\cdot\wedge T}) + G(\sigma, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[M_\tau^\star]) \Big] - \varepsilon\ell
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[v(\tau, X) + \int_\sigma^\tau f_r(\sigma, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[\mathsf{f}_r(X)], \nu_r)\mathrm{d}r \\
&\qquad + \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau,\cdot}}\bigg[\int_\tau^T f_r(\sigma, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[N_r^{\tau,\star}], \nu_r^\star) - f_r(\tau, X, N_r^{\tau,\star}, \nu_r^\star)\mathrm{d}r \\
&\qquad + \xi(\sigma, X_{\cdot\wedge T}) - \xi(\tau, X_{\cdot\wedge T}) + G(\sigma, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}[M_\tau^\star]) - G(\tau, M_\tau^\star)\bigg]\bigg] - \varepsilon\ell,
\end{aligned}
$$

where the last equality follows by conditioning and the $\mathcal{F}_\tau$ measurability of all the terms. Indeed, in light of [232, Theorem 6.1.2], an r.c.p.d. of $\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}|\mathcal{F}_\tau$ evaluated at x agrees with $\overline{\mathbb{P}}^{\nu^\star}_{\tau(x),x}$, the weak solution to the martingale problem with initial condition (τ, x) and action $\nu^\star$, for $\overline{\mathbb{P}}^{\nu\otimes_\tau\nu^\star}$–a.e. $x \in \mathcal{X}$. As all the terms inside the second expectation are $\mathcal{F}_\tau$-measurable the previous holds for $\overline{\mathbb{P}}^{\nu}$–a.e. $x \in \mathcal{X}$. □

From the previous result, we know that equilibrium models satisfy a form of ε-optimality in a sufficiently small window of time. We now seek to gain more insight from iterating the previous result. This will allow us to move forward the time window into consideration.

In the following, given $(\sigma, \tau) \in \mathcal{T}_{t,T} \times \mathcal{T}_{t,t+\ell}$, with $\sigma \leq \tau$, we denote by $\Pi^\ell := (\tau_i^\ell)_{i=1,\dots,n_\ell} \subseteq \mathcal{T}_{t,T}$ a generic partition of $[\sigma, \tau]$ with mesh smaller than ℓ, *i.e.* for $n_\ell := \lceil(\tau - \sigma)/\ell\rceil$, $\sigma =: \tau_0^\ell \leq \cdots \leq \tau_{n^\ell}^\ell := \tau$, $\forall \ell$, and $\sup_{i\leq n_\ell}|\tau_i^\ell - \tau_{i-1}^\ell| \leq \ell$. We also let $\Delta\tau_i^\ell := \tau_i^\ell - \tau_{i-1}^\ell$. The previous definitions hold ω–by–ω.

Proposition 2.10.2. *Let $\nu^\star \in \mathcal{E}(\mathbf{x})$ and $\{\sigma, \tau\} \subset \mathcal{T}_{t,T}$, with $\sigma \leq \tau$. Fix $\varepsilon > 0$ and some partition Π^ℓ with $\ell < \ell_\varepsilon$. Then for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$*

$$
\begin{aligned}
v(\sigma, x) \geq \sup_{\nu\in\mathcal{A}(\sigma,x)} &\mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[v(\tau, X) + \sum_{i=0}^{n_\ell-1}\int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X_{\cdot\wedge r}, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_i^\ell,\cdot}}[\mathsf{f}_r(X)], \nu_r)\mathrm{d}r \\
&+ \sum_{i=0}^{n_\ell-1}\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1}^\ell,\cdot}}\bigg[\int_{\tau_{i+1}^\ell}^T f_r(\tau_i^\ell, X_{\cdot\wedge r}, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_i^\ell,\cdot}}[N_r^{\tau_{i+1}^\ell,\star}], \nu_r^\star) - f_r(\tau_{i+1}^\ell, X_{\cdot\wedge r}, N_r^{\tau_{i+1}^\ell,\star}, \nu_r^\star)\mathrm{d}r\bigg] \\
&+ \sum_{i=0}^{n_\ell-1}\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1}^\ell,\cdot}}\Big[G(\tau_i^\ell, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_i^\ell,\cdot}}[M^\star_{\tau_{i+1}^\ell}]) - G(\tau_{i+1}^\ell, M^\star_{\tau_{i+1}^\ell})\Big] \\
&+ \sum_{i=0}^{n_\ell-1}\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1}^\ell,\cdot}}\Big[\xi(\tau_i^\ell, X_{\cdot\wedge T}) - \xi(\tau_{i+1}^\ell, X_{\cdot\wedge T}) - n_\ell\varepsilon\ell\Big]\bigg].
\end{aligned}
$$

Proof. A straightforward iteration of Lemma 2.10.1 yields that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$
\begin{aligned}
v(\sigma,x) \geq \sup_{\nu\in\mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[\int_{\tau_0^\ell}^{\tau_1^\ell} f_r(\tau_0^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[\mathsf{f}_r(X)],\nu_r)\mathrm{d}r + \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_1^\ell,X}}\Big[G(\tau_0^\ell, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[M^\star_{\tau_1^\ell}]) - G(\tau_1^\ell, M^\star_{\tau_1^\ell})\Big] \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_1^\ell,X}}\bigg[\int_{\tau_1^\ell}^{T} f_r(\tau_0^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[N_r^{\tau_1^\ell,\star}],\nu_r^\star) - f_r(\tau_1^\ell, X, N_r^{\tau_1^\ell,\star},\nu_r^\star)\mathrm{d}\bigg] \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_1^\ell,X}}\Big[\xi(\tau_0^\ell,X) - \xi(\tau_1^\ell,X) + v(\tau_1,X)\Big] - \varepsilon\ell
\end{aligned}
$$

$$
\begin{aligned}
\geq \sup_{\nu\in\mathcal{A}(\sigma,x)} \int_\Omega \bigg\{ \int_{\tau_0^\ell}^{\tau_1^\ell} f_r(\tau_0^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[\mathsf{f}_r(X)],\nu_r)\mathrm{d}r + \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_1^\ell,X}}\Big[G(\tau_0^\ell, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[M^\star_{\tau_1^\ell}]) - G(\tau_1^\ell, M^\star_{\tau_1^\ell})\Big] \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_1^\ell,X}}\bigg[\int_{\tau_1^\ell}^{T} f_r(\tau_0^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[N_r^{\tau_1^\ell,\star}],\nu_r^\star) - f_r(\tau_1^\ell, X, N_r^{\tau_1^\ell,\star},\nu_r^\star)\mathrm{d}r + \xi(\tau_0^\ell,X) - \xi(\tau_1^\ell,X)\bigg] \\
+ \sup_{\tilde\nu\in\mathcal{A}(\tau_1^\ell,\tilde x)} \mathbb{E}^{\overline{\mathbb{P}}^{\tilde\nu}}\bigg[\int_{\tau_1^\ell}^{\tau_2^\ell} f_r(\tau_1^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\tilde\nu}_{\tau_1^\ell,\tilde x}}[\mathsf{f}_r(X)],\nu_r)\mathrm{d}r \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_2^\ell,X}}\bigg[\int_{\tau_2^\ell}^{T} f_r(\tau_1^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\tilde\nu}_{\tau_1^\ell,\tilde x}}[N_r^{\tau_2^\ell,\star}],\nu_r^\star) - f_r(\tau_2^\ell, X, N_r^{\tau_2^\ell,\star},\nu_r^\star)\mathrm{d}r + \xi(\tau_1^\ell,X) - \xi(\tau_2^\ell,X)\bigg] \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_2^\ell,X}}\Big[G(\tau_1^\ell, \mathbb{E}^{\overline{\mathbb{P}}^{\tilde\nu}_{\tau_1^\ell,\tilde x}}[M^\star_{\tau_2^\ell}]) - G(\tau_2^\ell, M^\star_{\tau_2^\ell})\Big] + v(\tau_2,X)\bigg]\bigg\}\mathbb{P}^\nu(\mathrm{d}\tilde\omega) - 2\varepsilon\ell
\end{aligned}
$$

$$
\begin{aligned}
= \sup_{\nu\in\mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^{\nu}}\bigg[v(\tau_2,X) + \int_{\tau_0^\ell}^{\tau_1^\ell} f_r(\tau_0^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[\mathsf{f}_r(X)],\nu_r)\mathrm{d}r + \int_{\tau_1^\ell}^{\tau_2^\ell} f_r(\tau_1^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_1^\ell,\cdot}}[\mathsf{f}_r(X)],\nu_r)\mathrm{d}r \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_1^\ell,\cdot}}\bigg[\int_{\tau_1^\ell}^{T} f_r(\tau_0^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[N_r^{\tau_1^\ell,\star}],\nu_r^\star) - f_r(\tau_1^\ell, X, N_r^{\tau_1^\ell,\star},\nu_r^\star)\mathrm{d}r\bigg] \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_1^\ell,\cdot}}\Big[\xi(\tau_0^\ell,X) - \xi(\tau_1^\ell,X) + G(\tau_0^\ell, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_0^\ell,x}}[M^\star_{\tau_1^\ell}]) - G(\tau_1^\ell, M^\star_{\tau_1^\ell})\Big] \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_2^\ell,\cdot}}\bigg[\int_{\tau_2^\ell}^{T} f_r(\tau_1^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_1^\ell,\cdot}}[N_r^{\tau_2^\ell,\star}],\nu_r^\star) - f_r(\tau_2^\ell, X, N_r^{\tau_2^\ell,\star},\nu_r^\star)\mathrm{d}r\bigg] \\
+ \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_2^\ell,\cdot}}\Big[\xi(\tau_1^\ell,X) - \xi(\tau_2^\ell,X) + G(\tau_1^\ell, \mathbb{E}^{\overline{\mathbb{P}}^{\nu}_{\tau_1^\ell,\cdot}}[M^\star_{\tau_2^\ell}]) - G(\tau_2^\ell, M^\star_{\tau_2^\ell})\Big]\bigg] - 2\varepsilon\ell,
\end{aligned}
$$

where the second inequality follows by applying the definition of an equilibrium at (τ_1^ℓ, X). Now, to justify the last step follows we first note that as we let the reward depend on $\mathbb{E}^{\mathbb{P}_{t,x}}[f_r(X_{\cdot\wedge r})]$ and $\mathbb{E}^{\mathbb{P}_{t,x}}[g(X_{\cdot\wedge T})]$ we need a dynamic programming result for McKean–Vlasov control. This result is recently available in [76, Theorem 3.1]. The statement of the lemma follows directly by iterating and the fact the iteration is countable. $\square$

To ease the readability of the next result, recall that for $x \in \mathcal{X}$, $\{\sigma,\zeta,\tau\} \subset \mathcal{T}_{0,T}$ with $\sigma \leq \zeta \leq \tau$,

$\nu^\star \in \mathcal{A}(\mathbf{x})$, $\nu \in \mathcal{A}(\sigma, x)$, and any $\mathcal{F}_T^X$-measurable random variable ξ, we introduce the notation

$$\mathbb{E}_\zeta^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}[\xi] := \mathbb{E}^{\overline{\mathbb{P}}^\nu \otimes_\zeta \overline{\mathbb{P}}^{\nu^\star}_{\zeta, X}}[\xi],$$

where $\mathbb{P}^{\nu^\star}_{\zeta,X}$ is given by $\omega \longmapsto \mathbb{P}^{\nu^\star}_{(\zeta(\omega), x_{\cdot\wedge\zeta(\omega)})}$ and denotes the $\mathcal{F}_\zeta$-kernel prescribed by the family of solutions to the martingale problem associated with $\nu^\star$, see [232, Theorem 6.2.2]. Note in particular $\mathbb{E}_\sigma^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}[\xi] = \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\sigma, X}}[\xi]$.

Proof of Theorem 2.4.3. Let $\varepsilon > 0$, $0 < \ell < \ell_\varepsilon$ and Π^ℓ be as in Proposition 2.10.2. It then follows that for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$\begin{aligned}
v(\sigma, x) \geq \sup_{\nu \in \mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[& v(\tau, X) + \sum_{i=0}^{n_\ell - 1} \int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[\mathsf{f}_r(X)], \nu_r) \mathrm{d}r \\
& + \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1}^\ell,\cdot}}\bigg[\int_{\tau_{i+1}^\ell}^T f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[N_r^{\tau_{i+1}^\ell,\star}], \nu_r^\star) - f_r(\tau_{i+1}^\ell, X, N_r^{\tau_{i+1}^\ell,\star}, \nu_r^\star) \mathrm{d}r \bigg] \\
& + \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1}^\ell,\cdot}}\Big[\xi(\tau_i^\ell, X) - \xi(\tau_{i+1}^\ell, X) + G(\tau_i^\ell, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[M^\star_{\tau_{i+1}^\ell}]) - G(\tau_{i+1}^\ell, M^\star_{\tau_{i+1}^\ell}) - n_\ell \varepsilon \ell \Big] \bigg] \\
= \sup_{\nu \in \mathcal{A}(\sigma,x)} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[& v(\tau, X) + \sum_{i=0}^{n_\ell - 1} \int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[\mathsf{f}_r(X)], \nu_r) \mathrm{d}r \bigg] \\
& + \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau_{i+1}^\ell}\bigg[\int_{\tau_{i+1}^\ell}^T f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[N_r^{\tau_{i+1}^\ell,\star}], \nu_r^\star) - f_r(\tau_{i+1}^\ell, X, N_r^{\tau_{i+1}^\ell,\star}, \nu_r^\star) \mathrm{d}r \bigg] \\
& + \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau_{i+1}^\ell}\Big[\xi(\tau_i^\ell, X) - \xi(\tau_{i+1}^\ell, X) + G(\tau_i^\ell, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[M^\star_{\tau_{i+1}^\ell}]) - G(\tau_{i+1}^\ell, M^\star_{\tau_{i+1}^\ell}) - n_\ell \varepsilon \ell \Big].
\end{aligned}$$

For $(t, x, \nu) \in [0, T] \times \mathcal{X} \times \mathcal{A}(t, x)$, let $\mathsf{R}(s) := \int_\sigma^s f_r(s, X, \mathsf{f}_r(X), \nu_r)\mathrm{d}r$, $s \in [\sigma, T]$, so that for $i = 0, \dots, n_\ell - 1$,

$$\begin{aligned}
\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_{\tau_i^\ell}^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[\mathsf{f}_r(X)], \nu_r) \mathrm{d}r \bigg] &= \mathbb{E}^{\overline{\mathbb{P}}^\nu}\Big[\mathsf{R}(\tau_{i+1}^\ell) - \mathsf{R}(\tau_i^\ell) \Big] \\
&\quad + \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^{\tau_{i+1}^\ell} (f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[\mathsf{f}_r(X)], \nu_r) - f_r(\tau_{i+1}^\ell, X, \mathsf{f}_r(X), \nu_r)) \mathrm{d}r \bigg],
\end{aligned}$$

where we used the fact that $\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\zeta,\cdot}}[\mathsf{f}_t(X)] = \mathsf{f}_t(X)$ for $t \in [\sigma,\zeta]$. Consequently, for $\mathcal{P}(\mathbf{x})$–q.e. $x \in \mathcal{X}$

$$
\begin{aligned}
v(\sigma,x) \geq \sup_{\nu\in\mathcal{A}(\sigma,x)} \bigg\{ & \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[v(\tau,X) + \sum_{i=0}^{n_\ell-1} \mathsf{R}(\tau_{i+1}^\ell) - \mathsf{R}(\tau_i^\ell) - n_\ell \varepsilon\ell \bigg] \\
& + \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^{\tau_{i+1}^\ell} \big(f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[\mathsf{f}_r(X)], \nu_r) - f_r(\tau_{i+1}^\ell, X, \mathsf{f}_r(X), \nu_r)\big)\mathrm{d}r \bigg] \\
& + \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau_{i+1}^\ell}\bigg[\int_{\tau_{i+1}^\ell}^T f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[N_r^{\tau_{i+1}^\ell,\star}], \nu_r^\star) - f_r(\tau_{i+1}^\ell, X, N_r^{\tau_{i+1}^\ell,\star}, \nu_r^\star)\mathrm{d}r \bigg] \\
& + \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau_{i+1}^\ell}\bigg[\xi(\tau_i^\ell, X) - \xi(\tau_{i+1}^\ell, X) + G(\tau_i^\ell, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[M^\star_{\tau_{i+1}^\ell}]) - G(\tau_{i+1}^\ell, M^\star_{\tau_{i+1}^\ell}) \bigg] \bigg\}.
\end{aligned} \tag{2.10.1}
$$

The idea in the rest of the proof is to take the limit $\ell \longrightarrow 0$ on both sides of (2.10.1). As v is finite we can exchange the limit with the sup and study the limit inside. The analysis of all the above terms, except the error term $\lceil \tau - \sigma/\ell \rceil \varepsilon\ell$, is carried out below. Regarding the error term, we would like to make the following remarks as it is clear that simply letting ε go to zero will not suffice for our purpose. As ℓ_ε is bounded and monotone in ε, we consider ℓ_0 given by $\ell_\varepsilon \longrightarrow \ell_0$ as $\varepsilon \longrightarrow 0$. We must consider two cases for ℓ_0: when $\ell_0 = 0$ the analysis in the next paragraph suffices to obtain the result; in the case $\ell_0 > 0$, we can then take at the beginning of this proof $\ell < \ell_0 \leq \ell_\varepsilon$, in which case all the sums in (2.10.1) are independent of ε, we then first let ε go to zero so that $\lceil \tau - \sigma/\ell \rceil \varepsilon\ell \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, and then study the limit $\ell \longrightarrow 0$ as in the following. In both scenarii the conclusion of the theorem holds.

We now carry out the analysis of the remaining terms. To this end, and in order to prevent enforcing unnecessary time regularity on the action process, we will restrict our class of actions to piecewise constant actions, *i.e.* $\nu_t := \sum_{k\geq 1} \nu_k \mathbf{1}_{(\varrho_{k-1},\varrho_k]}(t)$ for a sequence of non-decreasing $\mathbb{F}$–stopping times $(\varrho_k)_{k\geq 0}$, and random variables $(\nu_k)_{k\geq 1}$, such that for any $k \geq 1$, ν_k is $\mathcal{F}^X_{\varrho_{k-1}}$-measurable. We will denote by $\mathcal{A}^{\mathrm{pw}}(t,x)$ the corresponding subclass of actions. By [77] the supremum over $\mathcal{A}(t,x)$ and $\mathcal{A}^{\mathrm{pw}}(t,x)$ coincide. Indeed, under Assumption E.(iv) and E.(v), we can apply [77, Theorem 3.1 and Lemma 4.3]. Assumption E.(iv), *i.e.* the Lipschitz-continuity of $a \longmapsto f_t(t,x,\mathrm{n},a)$, ensures the continuity of the drift coefficient when the space is extended to include the running reward, see [86, Remark 3.8]. Without lost of generality we assume $(\varrho_k)_{k\geq 0} \subseteq \Pi^\ell$, this is certainly the case as we can always refine Π^ℓ so that $\nu_r = \nu_i$ for $\tau_i^\ell \leq r \leq \tau_{i+1}^\ell$.

In the following, we fix $\omega \in \Omega$. A first-order Taylor expansion of the first summation term in (2.10.1) guarantees the existence of $\gamma_i^\ell \in (\tau_i^\ell, \tau_{i+1}^\ell)$, $i \in \{0, \dots, n_\ell\}$ such that

$$
\begin{aligned}
&\bigg| \sum_{i=0}^{n_\ell - 1} \mathsf{R}(\tau_{i+1}^\ell) - \mathsf{R}(\tau_i^\ell) - \Delta\tau_{i+1}^\ell \Big(f_{\tau_i^\ell}(\tau_i^\ell, X, \mathsf{f}_{\tau_i^\ell}(X), \nu_i) + \int_\sigma^{\tau_{i+1}^\ell} \partial_s f_r(\tau_{i+1}^\ell, X, \mathsf{f}_r(X), \nu_r)\mathrm{d}r \Big) \bigg| \qquad (2.10.2)\\
&= \bigg| \sum_{i=0}^{n_\ell - 1} \Delta\tau_{i+1}^\ell \Big(f_{\gamma_i^\ell}(\gamma_i^\ell, X, \mathsf{f}_{\gamma_i^\ell}(X), \nu_i) - f_{\tau_i^\ell}(\tau_i^\ell, X, \mathsf{f}_{\tau_i^\ell}(X), \nu_i) \\
&\qquad\qquad + \sum_{k=0}^{i} \int_{\tau_k^\ell}^{\tau_{k+1}^\ell \wedge \gamma_i^\ell} \partial_s f_r(\gamma_i^\ell, X, \mathsf{f}_{\gamma_i^\ell}(X), \nu_k)\mathrm{d}r - \int_{\tau_k^\ell}^{\tau_{k+1}^\ell} \partial_s f_r(\tau_{i+1}^\ell, X, \mathsf{f}_r(X), \nu_k)\mathrm{d}r \Big) \bigg| \\
&\leq \sum_{i=0}^{n_\ell - 1} |\Delta\tau_{i+1}^\ell| \Big(\rho_f(|\Delta\tau_{i+1}^\ell|) + C\rho_{\mathrm{f}}(|\Delta\tau_{i+1}^\ell|) + \sum_{k=0}^{i} \int_{\tau_k^\ell}^{\tau_{k+1}^\ell} \rho_{\partial_s f}(|\Delta\tau_{i+1}^\ell|) + C\rho_{\mathrm{f}}(|\Delta\tau_{i+1}^\ell|)\mathrm{d}r \Big) \\
&\leq C\big(\rho_f(\ell) + \rho_{\mathrm{f}}(\ell) + \rho_{\partial_s f}(\ell)\big) \xrightarrow{\ell \to 0} 0.
\end{aligned}
$$

The equality follows by replacing the expansion of the terms $\mathsf{R}(\tau_{i+1}^\ell)$ and the fact ν is constant between any two terms of the partition. The first inequality follows from Assumption E, where ρ_f, ρ_{f} and $\rho_{\partial_s f}$ are the modulus of continuity of the maps $t \longmapsto f_t(t,x,a)$, $t \longmapsto \mathrm{f}_t(x)$ and $s \longmapsto \partial_s f_r(s,x,a)$. The limits follows by bounded convergence as in light of Assumption E the constant in the last inequality and the last term are independent of ω. Thus, both expressions on the first line have the same limit for every $\omega \in \Omega$. We claim that for a well chosen sequence of partitions of the interval $[\sigma, \tau]$

$$
\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell - 1} \mathsf{R}(\tau_{i+1}^\ell) - \mathsf{R}(\tau_i^\ell) \bigg] \\
&\xrightarrow{\ell \longrightarrow 0} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^\tau f_r(r, X, \mathsf{f}_r(X), \nu_r)\mathrm{d}r + \int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}\Big[\int_\sigma^r \partial_s f_u(r, X, \mathsf{f}_u(X), \nu_u)\mathrm{d}u \Big]\mathrm{d}r \bigg],
\end{aligned} \qquad (2.10.3)
$$

where the integrals on the right-hand side are w.r.t the Lebesgue measure on $[0,T]$, and we recall the term inside $\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}$ is $\mathcal{F}_r^X$-measurable. Indeed, following McShane [182], for $\ell > 0$ fixed there exists, ω-by-ω, $\widehat{\Pi}^\ell := (\hat{\tau}_i^\ell)_{i \in [n_\ell]}$ a partition of $[\sigma, \tau]$ such that the Riemann sum in (2.10.2) evaluated at $\widehat{\Pi}^\ell$ converges to the Lebesgue integral ω-by-ω. With this, we are left to argue (2.10.3). Recall so far our analysis was for $\omega \in \Omega$ fixed, therefore one has to be careful about, for instance, the measurability of the partition $(\hat{\tau}_i^\ell)_{1 \leq i \leq n_\ell}$. An application of Galmarino's test, see Dellacherie and Meyer [69, Ch. IV. 99–101], guarantees that $\hat{\tau}_i^\ell \in \mathcal{T}_{0,T}$ for all i, *i.e.* the random times $\hat{\tau}_i^\ell$ are in fact stopping times. Finally, (2.10.3) follows by the bounded convergence theorem.

We now move to the second term in (2.10.1). Similarly, there are $\gamma_i^\ell \in (\tau_i^\ell, \tau_{i+1}^\ell)$, $[0,1]$-valued random processes θ^i, $i \in \{0, \dots, n_\ell\}$, such that for $\eta_t^{\tau_i^\ell, \star} := \theta_t^i f_t(X) + (1-\theta_t^i)\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell, \cdot}}[\mathsf{f}_t(X)]$, defined $\mathrm{d}t \otimes \mathrm{d}\overline{\mathbb{P}}^\nu$–a.e., it holds that

$$\begin{aligned}
&\bigg|\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\int_\sigma^{\tau_{i+1}^\ell} f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell, \cdot}}[\mathsf{f}_r(X)], \nu_r) - f_r(\tau_{i+1}^\ell, X, \mathsf{f}_r(X), \nu_r) + \Delta\tau_{i+1}^\ell \partial_s f_r(\tau_{i+1}^\ell, X, \mathsf{f}_r(X), \nu_r)\mathrm{d}r\bigg]\bigg| \\
&= \bigg|\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\int_\sigma^{\tau_{i+1}^\ell} \Delta\tau_{i+1}^\ell\Big(\partial_s f_r(\gamma_i^\ell, X, \mathsf{f}_r(X), \nu_r) - \partial_s f_r(\tau_{i+1}^\ell, X, \mathsf{f}_r(X), \nu_r)\Big)\mathrm{d}r\bigg] \\
&\quad + \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\int_\sigma^{\tau_{i+1}^\ell} \partial_{\mathrm{n}} f_r(\tau_i^\ell, X, \eta_r^{\tau_i^\ell}, \nu_r)\Big(\mathsf{f}_r(X) - \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell, \cdot}}[\mathsf{f}_r(X)]\Big)\mathrm{d}r\bigg]\bigg| \\
&\leq \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\int_\sigma^{\tau_{i+1}^\ell} |\Delta\tau_{i+1}^\ell|\rho_{\partial f}(|\ell|)\mathrm{d}r\bigg] \\
&\quad + \bigg|\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\int_{\tau_i^\ell}^{\tau_{i+1}^\ell} \partial_{\mathrm{n}} f_r(\tau_i^\ell, X, \eta_r^{\tau_i^\ell}, \nu_r)\Big(\mathsf{f}_r(X) - \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell, \cdot}}[\mathsf{f}_r(X)]\Big)\mathrm{d}r\bigg]\bigg| \\
&\leq C\rho_{\partial f}(|\ell|) + C\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\int_{\tau_i^\ell}^{\tau_{i+1}^\ell}\Big|\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_{i+1}^\ell, \cdot}}[\mathsf{f}_r(X)] - \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell, \cdot}}[\mathsf{f}_r(X)]\Big|\mathrm{d}r\bigg] \xrightarrow{\ell \to 0} 0,
\end{aligned}$$

where in the first inequality we used Assumption E and the fact that $\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\zeta, \cdot}}[\mathsf{f}_t(X)] = \mathsf{f}_t(X)$ for $t \in [\sigma, \zeta]$. The same argument together with the fact that $\mathrm{n} \longmapsto f_t(s, x, \mathrm{n}, a)$ has bounded derivatives, which holds by Assumption E, yields the second inequality. The limit follows in light of the continuity of $x \longmapsto f_t(x)$, and of $(t,x) \longmapsto \mathbb{P}^\nu_{t,x}$ for the weak topology. Since the limits agree, we find the second term converges to

$$\begin{aligned}
&- \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\Delta\tau_{i+1}^\ell\int_\sigma^{\tau_{i+1}^\ell}\partial_s f_r(\tau_{i+1}^\ell, X, \mathsf{f}_r(X), \nu_r))\mathrm{d}r\bigg] \\
&\qquad \xrightarrow{\ell \longrightarrow 0} - \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{r,X}}\bigg[\int_\sigma^r \partial_s f_u(r, X, \mathsf{f}_u(X), \nu_u)\mathrm{d}u\bigg]\mathrm{d}r\bigg].
\end{aligned} \tag{2.10.4}$$

Similarly, we can write the third term in (2.10.1), namely

$$\sum_{i=0}^{n_\ell-1}\mathbb{E}^{\overline{\mathbb{P}}^\nu, \overline{\mathbb{P}}^{\nu^\star}}_{\tau_{i+1}^\ell}\bigg[\int_{\tau_{i+1}^\ell}^T f_r(\tau_i^\ell, X, \mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell, \cdot}}[N_r^{\tau_{i+1}^\ell, \star}], \nu_r^\star) - f_r(\tau_{i+1}^\ell, X, N_r^{\tau_{i+1}^\ell, \star}, \nu_r^\star)\mathrm{d}r\bigg],$$

as the sum of

$$
\begin{aligned}
I_1 &:= \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_{\tau^\ell_{i+1}}^T f_r(\tau^\ell_i,X,N_r^{\tau^\ell_i,\star},\nu^\star_r) - f_r(\tau^\ell_{i+1},X,N_r^{\tau^\ell_{i+1},\star},\nu^\star_r)\mathrm{d}r\bigg],\\
I_2 &:= \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_{\tau^\ell_{i+1}}^T f_r(\tau^\ell_i,X,\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau^\ell_i,\cdot}}[N_r^{\tau^\ell_{i+1},\star}],\nu^\star_r) - f_r(\tau^\ell_i,X,N_r^{\tau^\ell_i,\star},\nu^\star_r)\mathrm{d}r\bigg].
\end{aligned}
\tag{2.10.5}
$$

As for I_1, we now there are $\gamma^\ell_i\in(\tau^\ell_i,\tau^\ell_{i+1})$, $[0,1]$-valued random processes θ_i, $i\in\{0,\dots,n_\ell\}$, such that for $\eta_t^{\tau^\ell_{i+1},\star}:=\theta^i_t N_t^{\tau^\ell_i,\star}+(1-\theta^i_t)N_t^{\tau^\ell_{i+1},\star}$

$$
\begin{aligned}
&\bigg|\sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_{\tau^\ell_{i+1}}^T \Big(f_r(\tau^\ell_i,X,N_r^{\tau^\ell_i,\star},\nu^\star_r) - f_r(\tau^\ell_{i+1},X,N_r^{\tau^\ell_{i+1},\star},\nu^\star_r)\Big)\mathrm{d}r\\
&\qquad\qquad + \int_{\tau^\ell_{i+1}}^T \Delta\tau^\ell_{i+1}\partial_s f_r(\tau^\ell_{i+1},X,N_r^{\tau^\ell_{i+1},\star},\nu^\star_r)\mathrm{d}r\\
&\qquad\qquad + \int_{\tau^\ell_{i+1}}^T \Big(\partial_{\mathrm{n}}\hat f_r(\tau^\ell_i,N_r^{\tau^\ell_i,\star})\Delta N_r^{\tau^\ell_{i+1},\star} + \frac12\partial^2_{\mathrm{nn}}\hat f_r(\tau^\ell_i,N_r^{\tau^\ell_i,\star})|\Delta N_r^{\tau^\ell_{i+1},\star}|^2\Big)\mathrm{d}r\bigg]\bigg|\\
&= \bigg|\sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_{\tau^\ell_{i+1}}^T \Delta\tau^\ell_{i+1}\Big(\partial_s f_r(\tau^\ell_{i+1},X,N_r^{\tau^\ell_{i+1},\star},\nu^\star_r) - \partial_s f_r(\gamma^\ell_i,X,N_r^{\tau^\ell_{i+1},\star},\nu^\star_r)\Big)\mathrm{d}r\\
&\qquad\qquad + \frac12\int_{\tau^\ell_{i+1}}^T |\Delta N_r^{\tau^\ell_{i+1},\star}|^2\Big(\partial^2_{\mathrm{nn}}\hat f_r(\tau^\ell_i,N_r^{\tau^\ell_i,\star}) - \partial^2_{\mathrm{nn}}\hat f_r(\tau^\ell_i,\eta_r^{\tau^\ell_{i+1},\star})\Big)\mathrm{d}r\bigg]\bigg|\\
&\leq T^2\rho_{\partial f}(|\ell|) + C\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^T \sup_i|\Delta N_r^{\tau^\ell_{i+1},\star}|\sum_{i=0}^{n_\ell-1}|\Delta N_r^{\tau^\ell_{i+1},\star}|^2\mathrm{d}r\bigg] \xrightarrow{\ell\to 0} 0,
\end{aligned}
\tag{2.10.6}
$$

where in the first equality we used the fact that the first order term in the expansion for n cancels. The inequality follows from Assumption E, the fact that $\theta^i\in[0,1]$ and the $\mathcal{F}^X_{\tau_{i+1}}$-measurability of $\Delta N_r^{\tau^\ell_{i+1},\star}$. Lastly, the limit follows in light of the continuity of $x\longmapsto f_t(x)$ and the continuity of $(t,x)\longmapsto\mathbb{P}^\nu_{t,x}$ for the weak topology. Therefore the limits in (2.10.6) agree.

We now study the first order term, in n, in (2.10.6) together with I_2, as defined in (2.10.5). We claim the limit is zero. Indeed, for $\eta_t^{\tau^\ell_i,\star}:=\theta^\ell_i N_t^{\tau^\ell_i,\star}+(1-\theta^\ell_i)\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau^\ell_i,\cdot}}[N_t^{\tau^\ell_{i+1},\star}]$ we have that there exists $C>0$ such that

$$
\begin{aligned}
&\sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_{\tau^\ell_{i+1}}^T\Big(f_r(\tau^\ell_i,X,\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau^\ell_i,\cdot}}[N_r^{\tau^\ell_{i+1},\star}],\nu^\star_r) - f_r(\tau^\ell_i,X,N_r^{\tau^\ell_i,\star},\nu^\star_r) - \partial_{\mathrm{n}}\hat f_r(\tau^\ell_i,N_r^{\tau^\ell_i,\star})\Delta N_r^{\tau^\ell_{i+1},\star}\Big)\mathrm{d}r\bigg]\bigg|\\
&= \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau^\ell_{i+1}}\bigg[\int_{\tau^\ell_{i+1}}^T \Big(\partial_{\mathrm{n}}\hat f_r(\tau^\ell_i,\eta_r^{\tau^\ell_i})\Big(\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau^\ell_i,\cdot}}[N_r^{\tau^\ell_{i+1},\star}] - N_r^{\tau^\ell_i,\star}\Big) - \partial_{\mathrm{n}}\hat f_r(\tau^\ell_i,N_r^{\tau^\ell_i,\star})\Delta N_r^{\tau^\ell_{i+1},\star}\Big)\mathrm{d}r\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_{\tau_{i+1}^\ell}^T \Big(\partial_{\mathrm{n}}\hat{f}_r(\tau_i^\ell,\eta_r^{\tau_i^\ell})\Big(\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[N_r^{\tau_{i+1}^\ell,\star}]-N_r^{\tau_i^\ell,\star}\Big)-\partial_{\mathrm{n}}\hat{f}_r(\tau_i^\ell,N_r^{\tau_i^\ell,\star})\Delta N_r^{\tau_{i+1}^\ell,\star}\Big)\mathrm{d}r\bigg]\\
&= \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_{\tau_{i+1}^\ell}^T \Big(\partial_{\mathrm{n}}\hat{f}_r(\tau_i^\ell,\eta_r^{\tau_i^\ell})-\partial_{\mathrm{n}}\hat{f}_r(\tau_i^\ell,N_r^{\tau_i^\ell,\star})\Big)\Big(\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[N_r^{\tau_{i+1}^\ell,\star}]-N_r^{\tau_i^\ell,\star}\Big)\mathrm{d}r\bigg]\\
&\leq C\sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_{\tau_{i+1}^\ell}^T \Big|\mathbb{E}^{\overline{\mathbb{P}}^\nu_{\tau_i^\ell,\cdot}}[N_r^{\tau_{i+1}^\ell,\star}]-N_r^{\tau_i^\ell,\star}\Big|^2\mathrm{d}r\bigg] \xrightarrow{\ell\longrightarrow 0} 0,
\end{aligned}
$$

where the inequality follows again from Assumption E.(i) and the fact that $\theta^i\in[0,1]$. The limit follows from Assumption E.(iii).

We are only left left to compute the limit of

$$
\sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau_{i+1}^\ell}\bigg[\int_{\tau_{i+1}^\ell}^T \Big(\Delta\tau_{i+1}^\ell\partial_s f_r(\tau_{i+1}^\ell,X,N_r^{\tau_{i+1}^\ell,\star},\nu_r^\star)+\frac{1}{2}\partial^2_{\mathrm{nn}}\hat{f}_r(\tau_i^\ell,N_r^{\tau_i^\ell,\star})|\Delta N_r^{\tau_{i+1}^\ell,\star}|^2\Big)\mathrm{d}r\bigg].
$$

Regarding the first term, note that for an appropriate choice of Π^ℓ, as in [182],

$$
\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\Delta\tau_{i+1}^\ell\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1},\cdot}}\bigg[\int_{\tau_{i+1}^\ell}^T \partial_s f_r(\tau_{i+1}^\ell,X,N_r^{\tau_{i+1}^\ell,\star},\nu_r^\star)\mathrm{d}r\bigg]\bigg]\\
&\xrightarrow{\ell\longrightarrow 0}\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_u}\bigg[\int_u^T \partial_s f_r(u,X,N_r^{u,\star},\nu_r^\star)\mathrm{d}r\bigg]\mathrm{d}u\bigg].
\end{aligned}
$$

As for the second term let us note that, ignoring the factor 1/2,

$$
\begin{aligned}
&\sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^\nu,\overline{\mathbb{P}}^{\nu^\star}}_{\tau_{i+1}^\ell}\bigg[\int_{\tau_{i+1}^\ell}^T \partial^2_{\mathrm{nn}}\hat{f}_r(\tau_i^\ell,N_r^{\tau_i^\ell,\star})|\Delta N_r^{\tau_{i+1}^\ell,\star}|^2\mathrm{d}r\bigg]\\
&=\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\int_{\tau_{i+1}^\ell}^T \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1},\cdot}}\Big[\partial^2_{\mathrm{nn}}\hat{f}_r(\tau_i^\ell,N_r^{\tau_i^\ell,\star})|\Delta N_r^{\tau_{i+1}^\ell,\star}|^2\Big]\mathrm{d}r\bigg].
\end{aligned}
$$

Then, we can use the fact that $\Delta N_t^{\tau_{i+1},\star}=0$ for $t\in[0,\tau_i]$, to express it as the sum of

$$
\begin{aligned}
I_3 &:= \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^T\sum_{i=0}^{n_\ell-1}\mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1},\cdot}}\Big[\partial^2_{\mathrm{nn}}\hat{f}_r(\tau_i^\ell,N_r^{\tau_i^\ell,\star})|\Delta N_r^{\tau_{i+1}^\ell,\star}|^2\Big]\mathrm{d}r\bigg]\\
I_4 &:= \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\sum_{i=0}^{n_\ell-1}\int_{\tau_i}^{\tau_{i+1}}\partial^2_{\mathrm{nn}}\hat{f}_r(\tau_i^\ell,N_r^{\tau_i^\ell,\star})|\Delta N_r^{\tau_{i+1}^\ell,\star}|^2\mathrm{d}r\bigg].
\end{aligned}
$$

In light of Assumption E.(i), we obtain that I_4 converges to zero, whereas

$$\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^T \sum_{i=0}^{n_\ell-1} \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{\tau_{i+1},\cdot}}\Big[\partial^2_{\rm nn}\hat f_r(\tau_i^\ell, N_r^{\tau_i^\ell,\star})|\Delta N_r^{\tau_{i+1}^\ell,\star}|^2\Big]\mathrm{d}r\bigg]$$
$$\xrightarrow{\ell\longrightarrow 0} \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^T\int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{u,\cdot}}\Big[\partial^2_{\rm nn}\hat f_r(u, N_r^{u\,2})\widehat n_r^{u,\star}\Big]\mathrm{d}u\mathrm{d}r\bigg],$$

at last we use the fact that, by definition, $\widehat n_r^{u,\star}=0$, $\mathrm{d}t\otimes\mathrm{d}\overline{\mathbb{P}}^{\nu^\star}$–a.e., whenever $r\in[0,u]$ so that

$$\mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^T\int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{u,\cdot}}\Big[\partial^2_{\rm nn}\hat f_r(u, N_r^{u,\star})\widehat n_r^{u\,2}\Big]\mathrm{d}u\mathrm{d}r\bigg] = \mathbb{E}^{\overline{\mathbb{P}}^\nu}\bigg[\int_\sigma^\tau \mathbb{E}^{\overline{\mathbb{P}}^{\nu^\star}_{u,\cdot}}\Big[\int_u^T \partial^2_{\rm nn} f_r(u, X, N_r^{u,\star},\nu_r^\star)\widehat n_r^{u\,2}\mathrm{d}r\Big]\mathrm{d}u\bigg].$$

The argument for the fourth term in (2.10.1) is analogous, but simpler, to that of the third term in (2.10.1) exploiting the corresponding properties in Assumption E.(ii)–E.(iii). □

We close this section with the proof of the necessity result in the extension section, namely Theorem 2.4.4.

Proof of Theorem 2.4.4. This proof is analogous to the proof in the standard case. We just have to verify the assumptions allow us to obtain the new version of the results. Indeed, let us result that the proof of Theorem 2.2.10 in Section 2.7 is divided into two parts.

The first part consist of an application of the comparison theorem for the BSDE with terminal condition $\xi + G(T, g(X))$ and generator h for every $\mathbb{P}\in\mathcal{P}(t,x)$. Recall that as a function of z, h is still Lipschitz. So all we are left to verify is that the term in F outside of the sup has the appropriate integrability, *i.e.* that for every $(t,x)\in[0,T]\times\mathcal{X}$, $\mathbb{P}\in\mathcal{P}(t,x)$, $\partial Y\in\mathbb{H}^p_{s,x}(\mathbb{R},\mathbb{F}_+^{X,\mathbb{P}},\mathbb{P})$ and $\widehat m^2\in\mathbb{H}^p_{s,x}(\mathbb{R},\mathbb{F}_+^{X,\mathbb{P}},\mathbb{P})$, recall that by assumption $\partial^2_{\rm nn}G$ is bounded.The first result is immediate from Assumption F.(i). The result for $\widehat m^2$ follow from the integrability assumption on $g(X)$.

The second part of the proof is argued in the same way by introducing the regularised version of the value function v^+. □

2.10.3 Auxiliary lemmata

To being with we present a result that justifies our choice of the class $\mathcal{A}(t,x)$ in our definition of equilibrium.

Lemma 2.10.3.1. *Let $\tau \in \mathcal{T}_{t,T}$, $\nu \in \mathcal{A}(\mathbf{x})$ and $\tilde{\nu} \in \mathcal{A}(t,x)$. Then, $\mathbb{P}^{\nu \otimes_\tau \tilde{\nu}} = \mathbb{P}^\nu \otimes_\tau \mathbb{P}^{\tilde{\nu}}_{\tau,\cdot}$, $\nu \otimes_\tau \tilde{\nu} \in \mathcal{A}(t,x)$, $\mathcal{A}(t,x,\tau) = \mathcal{A}(t,x)$.*

Proof. The latter two results follows from the first. Let $\tilde{\nu} \in \mathcal{A}(t,x)$, we claim that $\mathbb{P}^\nu \otimes_\tau \mathbb{P}^{\tilde{\nu}}_{\tau,X}$ is well defined and solves the martingale problem associated with $\nu \otimes_\tau \nu^\star$. Indeed, by [232, Exercise 6.7.4, Theorem 6.2.2] we have that $\omega \longmapsto \mathbb{P}^{\nu^\star}_{\tau(\omega),X(\tau(\omega),\omega)}[A]$ is $\mathcal{F}_\tau$-measurable for any $A \in \mathcal{F}$, and $\mathbb{P}^{\nu^\star}_{\tau(\omega),X(\tau(\omega),\omega)}[\Omega^\omega_\tau] = 1$ for all $\omega \in \Omega$. Therefore, Theorem 2.1.1 guarantees $\mathbb{P}^\nu \otimes_{\tau(\cdot)} \mathbb{P}^{\tilde{\nu}}_{\tau(\cdot),X(\cdot)}$ is well defined and $\mathbb{P}^\nu \otimes_{\tau(\cdot)} \mathbb{P}^{\nu}_{\tau(\cdot),X(\cdot)}$ equals $\mathbb{P}^\nu$ on $\mathcal{F}_\tau$ and $(\delta_\omega \otimes_{\tau(\omega)} \mathbb{P}^{\nu^\star}_{\tau(\omega),x})_{\omega \in \Omega}$ is an r.c.p.d. of $\mathbb{P}^\nu \otimes_\tau \mathbb{P}^\nu_{\tau,X}$ given $\mathcal{F}_\tau$. In combination with [232, Theorem 1.2.10] this yields M^φ is a $(\mathbb{F}, \mathbb{P}^\nu \otimes_\tau \mathbb{P}^{\nu,\tau,X})$–local martingale on $[t,T]$ with control $\nu \otimes_\tau \nu^\star$. $\square$

Lemma 2.10.3.2. *Given $(t,x) \in [0,T] \times \mathcal{X}$, $\tau \in \mathcal{T}_{t,T}$, $(\mathbb{M}, \widetilde{\mathbb{M}}) \in \mathfrak{M}(t,x) \times \mathfrak{M}(t,x)$ such that $\nu \otimes_\tau \tilde{\nu} \in \mathcal{A}(t,x,\mathbb{P})$. Then $\overline{\mathbb{P}}^\nu \otimes_\tau \overline{\mathbb{P}}^{\tilde{\nu}}_{\tau,\cdot} = \overline{\mathbb{P}}^{\nu \otimes_\tau \tilde{\nu}}$.*

Proof. This follows from Lemma 2.10.3.1 and the fact we can commute changes of measure and concatenation. $\square$

Lemma 2.10.3.3. *For $\ell > 0$ let $(\gamma^\ell_i)_{i \in [n_\ell]}$ be sample points as in* Theorem 2.2.2. *$(\hat{\gamma}^\ell_i)_{i \in [n_\ell]}$ are $\mathbb{F}$–stopping times.*

Proof. We study $(\hat{\gamma}^\ell_i)_{i \in [n_\ell]}$ as the argument for the other sequences in the proof is similar. The result follows from a direct application of Galmarino's test, see [69, CH. IV. 99–101], we recall it next for completeness: Let ϱ be $\mathcal{F}_T$-measurable function with values in $[0,T]$. ϱ is a stopping time if and only if for every $t \in [0,T]$ we have that $\varrho(\omega) \leq t$, $(X_r, W_r, \Delta_r[\varphi])(\omega) = (X_r, W_r, \Delta_r[\varphi])(\tilde{\omega})$ for all $(r,\varphi) \in [0,t] \times \mathcal{C}_b([0,T] \times A)$ implies $\varrho(\omega) = \varrho(\tilde{\omega})$.

Now, in the context of Theorem 2.2.2 we are given $\Pi^\ell = (\tau^\ell_i)_{i \in [n_\ell]}$ a collection of stopping times that partitions the interval $[\sigma,\tau] \subseteq [0,T]$. As $\ell > 0$ is fixed we drop the dependence of the partition on ℓ and write $\Pi = (\tau_i)_{i \in [n]}$. Without loss of generality we consider the case of a partition of $[0,T]$. For $\omega \in \Omega$ we can use [240, Theorem 1] to obtain a sequence of sample points $(\varrho_i(\omega))_{i \in [n]}$ of the intervals prescribed by Π, *i.e.* $\varrho_i(\omega) \in [\tau_{i-1}(\omega), \tau_i(\omega)]$ for all i. We want to show $\varrho_i \in \mathcal{T}_{0,T}$ for all i.

A close inspection to the proof of [241, Theorem 1] allows us to see that for fixed $\omega \in \Omega$ the choice of $\varrho_i(\omega)$ depends solely on $[\tau_{i-1}(\omega), \tau_i(\omega)]$ and the application $t \longmapsto f_t(t,x,a)$ for $(t,x,a) \in$

$[0,T] \times \mathcal{X} \times A$. We recall that as the supremum is taken over $\mathcal{A}^{\mathrm{pw}}(t,x)$ the action process is a fix value $a \in A$ over the interval $[\tau_{i-1}, \tau_i]$. We first note that for $\tilde{\omega} \in \Omega$ as above the fact that $\Pi \subset \mathcal{T}_{0,T}$, *i.e.* $\{\tau_i \leq t\} \in \mathcal{F}_t$, implies $\Pi(\tilde{\omega}) = \Pi(\omega)$. Next, we observe that $f_t(t, X, \nu_i)(\omega) = f_t(t, x_{\cdot \wedge t}, a) = f_t(t, X, \nu_i)(\tilde{\omega})$ as is $f_t(s, x, a)$ is optional for every $(s, a) \in [0,T] \times A$. These two facts imply $\varrho_i(\omega) = \varrho_i(\tilde{\omega})$ and the result follows. $\square$

Part III

Backward stochastic Volterra integral equations

Chapter 3

Lipschitz backward stochastic Volterra integral equations

We saw in Chapter 2 that backward stochastic Volterra integral equations, BSVIEs for short, appear naturally in the study of time-inconsistent stochastic control problems from a game theoretic point of view, see Section 2.2.3 for details. This chapter is concerned with introducing a unified method to address the well-posedness of these objects in the case of Lipschitz data.

In particular, we study a novel general class of multidimensional type-I backward stochastic Volterra integral equations. Toward this goal, we introduce an infinite family of standard backward SDEs and establish its well-posedness, and we show that it is equivalent to that of a type-I backward stochastic Volterra integral equation. We also establish a representation formula in terms of non-linear semi-linear partial differential equation of Hamilton–Jacobi–Bellman type. As an application, we consider the study of time- inconsistent stochastic control from a game-theoretic point of view. We show the equivalence of two current approaches to this problem from both a probabilistic and an analytic point of view.

3.1 On Volterra BSDEs

In this chapter we want to build upon the strategy devised in Chapter 2 and address the well-posedness of a general and novel class of type-I BSVIEs. We let X be the solution to a drift-less stochastic differential equation (SDE, for short) under a probability measure $\mathbb{P}$, and $\mathbb{F}$ be the $\mathbb{P}$-augmentation of the filtration generated by X, see Section 3.2 for details, and consider a tuple

$(Y^{\cdot}_{\cdot}, Z^{\cdot}_{\cdot}, N^{\cdot}_{\cdot})$, of appropriately $\mathbb{F}$-adapted processes, satisfying for any $s \in [0,T]$

$$Y_t^s = \xi(s) + \int_t^T g_r(s, X, Y_r^s, Z_r^s, Y_r^r, Z_r^r)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r - \int_t^T \mathrm{d}N_r^s,\ t \in [0,T],\ \mathbb{P}\text{–a.s.} \tag{3.1.1}$$

We remark that the additional process N corresponds to a martingale process which is $\mathbb{P}$-orthogonal to X. This is a consequence of the fact that we work with a general filtration $\mathbb{F}$. To the best of our knowledge, a theory for type-I BSVIEs, as general as the ones introduced above, remains absent in the literature. Moreover, such class of type-I BSVIEs has only been mentioned in [114, Remark 3.8] as an interesting generalisation of a classic type-I BSVIEs.

Our approach is based on the following class of infinite families of BSDEs

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T, X_{\cdot\wedge T}) + \int_t^T h_r(X_{\cdot\wedge r}, \mathcal{Y}_r, \mathcal{Z}_r, Y_r^r, Z_r^r, \partial Y_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\\
Y_t^s &= \eta(s, X_{\cdot\wedge T}) + \int_t^T g_r(s, X_{\cdot\wedge r}, Y_r^s, Z_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T V_r^{s\top} \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\\
\partial Y_t^s &= \partial_s \eta(s, X_{\cdot\wedge T}) + \int_t^T \nabla g_r(s, X_{\cdot\wedge r}, \partial Y_r^s, \partial Z_r^s, Y_r^s, Y_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial Z_r^{s\top} \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.}
\end{aligned}$$

where $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, Y, Z, N, \partial Y, \partial Z, \partial N)$ are unknown, and required to have appropriate integrability, see Section 3.3 and Equation $(\mathcal{S})$.

We first establish the well-posedness of $(\mathcal{S})$, see Theorem 3.3.5. For this it is important to be able to identify the proper spaces to carry out the analyses, see Remark 3.3.3. Moreover, we show that, for an appropriate choice of data for $(\mathcal{S})$, its well-posedness is equivalent to that of the type-I BSVIE (3.1.1), see Theorem 3.4.3. Noticeably, our approach can naturally be specialised to obtain the well-posedness of previous BSVIEs considered in the literrature in the classic spaces, see Remark 3.4.4. Moreover, as our results provide an alternative approach to BSVIEs, it may allow for the future design of new numerical schemes to solve type-I BSVIEs, which to the best of our knowledge, remain limited to [21]. In addition, we recover classical results for this general class of multidimensional type-I BSVIEs. We provide *a priori* estimates, show the stability of solutions as well as a representation formula in terms of a semilinear PDEs, see Proposition 3.5.1. Given our multidimensional setting, we refrained from considering comparison results, see Wang and Yong [254] for the one-dimensional case.

As an application of our results, we consider the game-theoretic approach to time-inconsistent stochastic control problems. We recall this approach studies the problem faced by the, so-called, *sophisticated agent* who aware of the inconsistency of its preferences seeks for *consistent plans*, *i.e.* equilibria. We show that as a consequence of Theorem 3.4.3, one can reconcile two recent probabilistic approaches to this problem. Moreover, we provide, see Proposition 3.5.3, an equivalent result for two earlier analytic approaches, based on semi-linear PDEs. We believe this helps to elucidate connections between the different takes on the problem available in the literature.

The rest of the chapter is structured as follows. Section 3.2 introduces the stochastic basis on a canonical space as well as the integrability spaces necessary to our analysis. Section 3.3 precisely formulates the class of infinite families of BSDEs ($\mathcal{S}$), which is the crux of our approach, and provides the statement of its well-posedness, while the proof is deferred to Section 3.6. Section 3.4 introduces the class of type-I BSVIEs which are the main object of this paper, and establishes the equivalence of its well-posedness with that of ($\mathcal{S}$) for a particular choice of data. Section 3.5 deals with the representation formula for the class of type-I BSVIEs considered, and presents the application of our results in the context of time-inconsistent stochastic control. Finally, Section 3.6 includes the analysis of ($\mathcal{S}$).

3.2 The stochastic basis on the canonical space

As a note to the reader, we mention that in the spirit of keeping the formulation of each chapter self-contained we will present it at the beginning of each of the remaining chapter of the thesis. This will create a modest level of repetition.

We fix two positive integers n and m, which represent respectively the dimension of the martingale which will drive our equations, and the dimension of the Brownian motion appearing in the dynamics of the former. We consider the canonical space $\mathcal{X} = \mathcal{C}^n$, with canonical process X. We let $\mathcal{F}$ be the Borel σ-algebra on $\mathcal{X}$ (for the topology of uniform convergence), and we denote by $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\in[0,T]}$ the natural filtration of X. We fix a bounded Borel-measurable map $\sigma : [0,T] \times \mathcal{X} \longrightarrow \mathbb{R}^{n\times m}$, $\sigma.(X) \in \mathcal{P}_{\text{meas}}(\mathbb{R}^{n\times m}, \mathbb{F}^X)$, and an initial condition $x_0 \in \mathbb{R}^n$. We assume there is $\mathbb{P} \in \text{Prob}(\mathcal{X})$ such that $\mathbb{P}[X_0 = x_0] = 1$ and X is martingale, whose quadratic variation,

$\langle X\rangle = (\langle X\rangle_t)_{t\in[0,T]}$, is absolutely continuous with respect to Lebesgue measure, with density given by $\sigma\sigma^\top$. Enlarging the original probability space, see Stroock and Varadhan [232, Theorem 4.5.2], there is an $\mathbb{R}^m$-valued Brownian motion B with

$$X_t = x_0 + \int_0^t \sigma_r(X_{\cdot\wedge r})\mathrm{d}B_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

We now let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be the (right-limit) of the $\mathbb{P}$-augmentation of $\mathbb{F}^X$. We stress that we will not assume $\mathbb{P}$ is unique. In particular, the predictable martingale representation property for $(\mathbb{F},\mathbb{P})$-martingales in terms of stochastic integrals with respect to X might not hold.

Remark 3.2.1. *We remark that the previous formulation on the canonical is by no means necessary. Indeed, any probability space supporting a Brownian motion B and a process X satisfying the previous* SDE *will do, and this can be found whenever that equation has a weak solution.*

3.2.1 Functional spaces and norms

We now introduce our spaces. In the following, $(\Omega,\mathcal{F}_T,\mathbb{F},\mathbb{P})$ is as in Section 3.2. We are given a finite-dimensional Euclidean space, *i.e.* $E=\mathbb{R}^k$ for some non-negative integer k and $|\cdot|$ denotes the Euclidean norm. For any $(p,q)\in(1,\infty)^2$, we introduce the spaces

- $\mathcal{L}^p(E)$ of $\mathcal{F}_T$-measurable, E-valued random variables ξ, with $\|\xi\|^p_{\mathcal{L}^p} := \mathbb{E}^{\mathbb{P}}[|\xi|^p] < \infty$;
- $\mathbb{S}^p(E)$ of $Y\in\mathcal{P}_{\mathrm{opt}}(E,\mathbb{F})$, with $\mathbb{P}$–a.s. càdlàg paths on $[0,T]$, with $\|Y\|^p_{\mathbb{S}^p} := \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}|Y_t|^p\Big] < \infty$;
- $\mathbb{L}^{q,p}(E)$ of $Y\in\mathcal{P}_{\mathrm{opt}}(E,\mathbb{F})$, with $\|Y\|^p_{\mathbb{L}^{q,p}} := \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_0^T |Y_r|^q\mathrm{d}r\bigg)^{\frac{p}{q}}\bigg] < \infty$;
- $\mathbb{H}^p(E)$ of $Z\in\mathcal{P}_{\mathrm{pred}}(E,\mathbb{F})$, defined $\sigma\sigma_t^\top\mathrm{d}t$–a.e., with $\|Z\|^p_{\mathbb{H}^p} := \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_0^T |\sigma_t Z_r|^2\mathrm{d}r\bigg)^{\frac{p}{2}}\bigg] < \infty$;
- $\mathbb{M}^p(E)$ of martingales $M\in\mathcal{P}_{\mathrm{opt}}(E,\mathbb{F})$ which are $\mathbb{P}$-orthogonal to X (that is the product XM is an $(\mathbb{F},\mathbb{P})$-martingale), with $\mathbb{P}$–a.s. càdlàg paths, $M_0=0$ and $\|M\|^p_{\mathbb{M}^p} := \mathbb{E}^{\mathbb{P}}\Big[[M]_T^{\frac{p}{2}}\Big] < \infty$;
- $\mathcal{L}^{p,2}(E)$ denotes the space of families $(\xi(s))_{s\in[0,T]}$ of $\mathcal{F}_T$-measurable E-valued random variables such that the mapping $([0,T]\times\Omega,\mathcal{B}([0,T])\otimes\mathcal{F}) \longrightarrow (\mathcal{L}^p(E),\|\cdot\|_{\mathcal{L}^p}) : s\longmapsto \xi(s)$ is continuous, $\|\xi\|^p_{\mathcal{L}^{p,2}} := \sup_{s\in[0,T]}\|\xi\|^p_{\mathcal{L}^p} < \infty$;

- $\mathcal{P}^2_{\text{meas}}(E,\mathcal{F}_T)$ of two parameter processes $(U^s_t)_{(s,t)\in[0,T]^2} : ([0,T]^2\times\Omega,\mathcal{B}([0,T]^2)\otimes\mathcal{F}_T) \longrightarrow (E,\mathcal{B}(E))$ measurable.

  Finally, given an arbitrary integrability space $(\mathbb{I}^p(E),\|\cdot\|_{\mathbb{I}})$, we introduce the space

- $\mathbb{I}^{p,2}(E)$ of $(U^s_t)_{(s,t)\in[0,T]^2}\in\mathcal{P}^2_{\text{meas}}(E,\mathcal{F}_T)$ such that the mapping $([0,T],\mathcal{B}([0,T]))\longrightarrow(\mathbb{I}^p(E),\|\cdot\|_{\mathbb{I}^p}) : s\longmapsto U^s$ is continuous and $\|U\|^p_{\mathbb{I}^{p,2}} := \sup_{s\in[0,T]}\|U^s\|^p_{\mathbb{I}^p}<\infty$.

  For example, $\mathbb{H}^{p,2}(E)$ denotes the space of $(Z^s_t)_{(s,t)\in[0,T]^2}\in\mathcal{P}^2_{\text{meas}}(E,\mathcal{F}_T)$, $([0,T],\mathcal{B}([0,T]))\longrightarrow(\mathbb{H}^p(E),\|\cdot\|_{\mathbb{H}^p}) : s\longmapsto Z^s$ is continuous and $\|Z\|^p_{\mathbb{H}^{p,2}} := \sup_{s\in[0,T]}\|Z^s\|^p_{\mathbb{H}^p}<\infty$.

  Lastly we introduce the space, see Remark 3.2.2 for further details,

- $\overline{\mathbb{H}}^{p,2}(E)$ of $(Z^s_t)_{(s,t)\in[0,T]^2}\in\mathcal{P}^2_{\text{meas}}(E,\mathcal{F}_T)$ such that $([0,T],\mathcal{B}([0,T]))\longrightarrow(\mathbb{H}^p(E),\|\cdot\|_{\mathbb{H}^p}) : s\longmapsto Z^s$ is absolutely continuous with respect to the Lebesgue measure[1], $\mathcal{Z}\in\mathbb{H}^2(E)$, where $\mathcal{Z}:=(Z^t_t)_{t\in[0,T]}$ is given by

$$Z^t_t := Z^T_t - \int_t^T \eth Z^r_t \mathrm{d}r, \text{ and, } \|Z\|^2_{\overline{\mathbb{H}}^{p,2}} := \|Z\|^2_{\mathbb{H}^{2,2}} + \|\mathcal{Z}\|^2_{\mathbb{H}^2}<\infty.$$

Remark 3.2.2. *When $p=q$, we will write $\mathbb{L}^p(E)$ $\big($resp. $\mathbb{L}^{p,2}(E)\big)$ for $\mathbb{L}^{q,p}(E)$ $\big($resp. $\mathbb{L}^{q,p,2}(E)\big)$. With this convention, $\mathbb{L}^2(E)$ $\big($resp. $\mathbb{L}^{2,2}(E)\big)$ will be $\mathbb{L}^{2,2}(E)$ $\big($resp. $\mathbb{L}^{2,2,2}(E)\big)$. Also, $\mathbb{S}^{p,2}(E)$, $\mathbb{L}^{q,p,2}(E)$ and $\mathbb{H}^{p,2}(E)$ are Banach spaces.*

In addition, we remark that the space $\mathbb{H}^2(E)$ being closed implies $\overline{\mathbb{H}}^{p,2}(E)$ is a closed subspace of $\mathbb{H}^{p,2}(E)$ and thus a Banach space. The space $\overline{\mathbb{H}}^{p,2}(E)$ allows us to define a good candidate for $(Z^t_t)_{t\in[0,T]}$ as an element of $\mathbb{H}^2(E)$.[2] Let $\widetilde{\Omega}:=[0,T]\times\mathcal{X}$ and $\tilde{\omega}:=(t,x)\in\widetilde{\Omega}$. By the Radon–Nikodým property and Fubini's theorem, we may define

$$\mathfrak{Z}_s(\tilde{\omega}) := Z^T_t(x) - \int_s^T \eth Z^r_t(x)\mathrm{d}r,\ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e. } \tilde{\omega}\in\widetilde{\Omega},\ s\in[0,T],$$

[1] We recall that $\mathbb{H}^2$, being a Hilbert space and in particular a reflexive Banach space, has the so-called Radon–Nikodým property, see [216, Corollary 5.45]. Thus the absolute continuity of the $\mathbb{H}^2$-valued mapping $s\longmapsto Z^s$ is equivalent to the existence of the density $(\eth Z^s)_{s\in[0,T]}$, which is unique for Lebesgue–a.e. $s\in[0,T]$. The fact $\mathcal{Z}$ is well-defined is argued in Remark 3.2.2

[2] This is based on [114, Section 2.1]. The mapping $s\longmapsto Z^s$ is assumed absolutely continuous, as opposed to continuously differentiable. We are indebted to Yushi Hamaguchi for pointing out inconsistencies in the definition of $\overline{\mathbb{H}}^{2,2}$ on an earlier version of this chapter.

so that $\mathfrak{Z}_s = Z^s, \mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e., $s \in [0,T]$. *Lastly, as* $\tilde{\omega} \in \widetilde{\Omega}$, $s \longmapsto \mathfrak{Z}_s(\tilde{\omega})$ *is continuous, we define*

$$Z_t^t := Z_t^T - \int_t^T \eth Z_t^r \mathrm{d}r, \text{ for } \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e. } (t,x) \text{ in } [0,T] \times \mathcal{X}.$$

3.3 An infinite family of Lipschitz BSDEs

We are given jointly measurable mappings h, g, ξ and η, such that for any $(y,z,u,v,\mathrm{u}) \in \mathbb{R}^{d_1} \times \mathbb{R}^{n\times d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{n\times d_2} \times \mathbb{R}^{d_2}$

$$h : [0,T] \times \mathcal{X} \times \mathbb{R}^{d_1} \times \mathbb{R}^{n\times d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{n\times d_2} \times \mathbb{R}^{d_2} \longrightarrow \mathbb{R}^{d_1},\ h_\cdot(\cdot,y,z,u,v,\mathrm{u}) \in \mathcal{P}_{\mathrm{prog}}(\mathbb{R}^{d_1},\mathbb{F}),$$
$$g : [0,T]^2 \times \mathcal{X} \times \mathbb{R}^{d_2} \times \mathbb{R}^{n\times d_2} \times \mathbb{R}^{d_1} \times \mathbb{R}^{n\times d_1} \longrightarrow \mathbb{R}^{d_2},\ g_\cdot(s,\cdot,u,v,y,z) \in \mathcal{P}_{\mathrm{prog}}(\mathbb{R}^{d_2},\mathbb{F}),$$
$$\xi : [0,T] \times \mathcal{X} \longrightarrow \mathbb{R}^{d_1},\ \eta : [0,T] \times \mathcal{X} \longrightarrow \mathbb{R}^{d_2}.$$

Moreover, we work under the following set of assumptions.

Assumption G. *(i)* $(s,u,v) \longmapsto g_t(s,x,u,v,y,z)$ *(resp.* $s \longmapsto \eta(s,x)$*) is continuously differentiable, uniformly in* (t,x,y,z) *(resp. in* x*). Moreover, the mapping* $\nabla g : [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^{d_2} \times \mathbb{R}^{n\times d_2})^2 \times \mathbb{R}^{d_1} \times \mathbb{R}^{n\times d_1} \longrightarrow \mathbb{R}^{d_2}$ *defined by*

$$\nabla g_t(s,x,\mathrm{u},\mathrm{v},u,v,y,z) := \partial_s g_t(s,x,u,v,y,z) + \partial_u g_t(s,x,u,v,y,z)\mathrm{u} + \sum_{i=1}^n \partial_{v_{:i}} g_t(s,x,u,v,y,z)\mathrm{v}_{i:},$$

satisfies $\nabla g_\cdot(s,\cdot,\mathrm{u},\mathrm{v},u,v,y,z) \in \mathcal{P}_{\mathrm{prog}}(\mathbb{R}^{d_2},\mathbb{F})$;

(ii) $(y,z,u,v,\mathrm{u}) \longmapsto h_t(x,y,z,u,v,\mathrm{u})$ *is uniformly Lipschitz-continuous,* i.e. $\exists L_h > 0$, *such that for all* $(t,x,y,\tilde{y},\ z,\tilde{z},u,\tilde{u},v,\tilde{v},\mathrm{u},\tilde{\mathrm{u}})$

$$\begin{aligned}&|h_t(x,y,z,u,v,\mathrm{u}) - h_t(x,\tilde{y},\tilde{z},\tilde{u},\tilde{v},\tilde{\mathrm{u}})|\\ &\leq L_h\big(|y-\tilde{y}| + |\sigma_t(x)^\top(z-\tilde{z})| + |u-\tilde{u}| + |\sigma_t(x)^\top(v-\tilde{v})| + |\mathrm{u}-\tilde{\mathrm{u}}|\big);\end{aligned}$$

(iii) for $\varphi \in \{g, \partial_s g\}$, $(u,v,y,z) \longmapsto \varphi_t(s,x,u,v,y,z)$ *is uniformly Lipschitz-continuous,* i.e.

$\exists L_\varphi > 0$, *such that for all* $(s,t,x,u,\tilde u,v,\tilde v,y,\tilde y,z,\tilde z)$

$$|\varphi_t(s,x,u,v,y,z) - \varphi_t(s,x,\tilde u,\tilde v,\tilde y,\tilde z)| \leq L_\varphi(|u-\tilde u| + |\sigma_t(x)^\top(v-\tilde v')| + |y-\tilde y| + |\sigma_t(x)^\top(z-\tilde z)|);$$

(iv) *for* $\mathbf{0} := (u,v,y,z)|_{(0,\dots,0)}$, $\big(\tilde h_\cdot, \tilde g_\cdot(s), \nabla\tilde g_\cdot(s)\big) := \big(h_\cdot(\cdot,\mathbf{0},0), g_\cdot(s,\cdot,\mathbf{0}), \partial_s g_\cdot(s,\cdot,\mathbf{0})\big) \in \mathbb{L}^{1,2}(\mathbb{R}^{d_1}) \times \big(\mathbb{L}^{1,2,2}(\mathbb{R}^{d_2})\big)^2$.

Remark 3.3.1. *We comment on the set of requirements in* Assumption G*. Of particular interest is* Assumption G.(i)*, the other being the standard Lipschitz assumptions on the generators as well as their integrability at zero. Anticipating the introduction of* $(\mathcal{S})$ *below and the discussion in* Remark 3.3.3*,* Assumption G.(i) *will allow us to identify the second* BSDE *in the system as the antiderivative of the third one, see* Remark 3.3.3*.*

We define the space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$, whose elements we denote $\mathfrak{h} = (\mathcal{Y},\mathcal{Z},\mathcal{N},U,V,M,\partial U,\partial V,\partial M)$, where

$$\begin{aligned}
\mathcal{H} &:= \mathbb{S}^2(\mathbb{R}^{d_1}) \times \mathbb{H}^2(\mathbb{R}^{n\times d_1}) \times \mathbb{M}^2(\mathbb{R}^{d_1}) \times \mathbb{S}^{2,2}(\mathbb{R}^{d_2}) \times \overline{\mathbb{H}}^{2,2}(\mathbb{R}^{n\times d_2}) \times \mathbb{M}^{2,2}(\mathbb{R}^{d_2})\\
&\qquad \times \mathbb{S}^{2,2}(\mathbb{R}^{d_2}) \times \mathbb{H}^{2,2}(\mathbb{R}^{n\times d_2}) \times \mathbb{M}^{2,2}(\mathbb{R}^{d_2}),\\
\|\mathfrak{h}\|^2_{\mathcal{H}} &:= \|\mathcal{Y}\|^2_{\mathbb{S}^2} + \|\mathcal{Z}\|^2_{\mathbb{H}^2} + \|\mathcal{N}\|^2_{\mathbb{M}^2} + \|U\|^2_{\mathbb{S}^{2,2}} + \|V\|^2_{\overline{\mathbb{H}}^{2,2}} + \|M\|^2_{\mathbb{M}^{2,2}}\\
&\qquad + \|\partial U\|^2_{\mathbb{S}^{2,2}} + \|\partial V\|^2_{\mathbb{H}^{2,2}} + \|\partial M\|^2_{\mathbb{M}^{2,2}}.
\end{aligned}$$

We are now ready to precise the class of systems subject to our study. Given $(\xi,\eta,\partial_s\eta) \in \mathcal{L}^2(\mathbb{R}^{d_1}) \times (\mathcal{L}^{2,2}(\mathbb{R}^{d_2}))^2$, $\partial_s\eta$ as in Assumption G, we consider the system, which for any $s\in[0,T]$ holds, $\mathbb{P}$–a.s., for every $t\in[0,T]$

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T,X_{\cdot\wedge T}) + \int_t^T h_r(X,\mathcal{Y}_r,\mathcal{Z}_r,U_r^r,V_r^r,\partial U_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r - \int_t^T \mathrm{d}\mathcal{N}_r,\\
U_t^s &= \eta(s,X_{\cdot\wedge T}) + \int_t^T g_r(s,X,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\mathrm{d}r - \int_t^T V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}M_r^s,\\
\partial U_t^s &= \partial_s\eta(s,X_{\cdot\wedge T}) + \int_t^T \nabla g_r(s,X,\partial U_r^s,\partial V_r^s,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s.
\end{aligned} \tag{$\mathcal{S}$}$$

Definition 3.3.2. *We say* $\mathfrak{h}$ *is a solution to* $(\mathcal{S})$ *if* $\mathfrak{h}\in\mathcal{H}$ *and* $(\mathcal{S})$ *holds.*

Remark 3.3.3. *We now expound on our choice for the set-up and the structure of* $(\mathcal{S})$*.*

(i) *We first highlight two aspects which are crucial to establish the connection between* $(\mathcal{S})$ *and* type-I BSVIE (3.1.1). *The first is the presence of* ∂U *in the generator of the first equation. This causes the system to be fully coupled but is nevertheless necessary in our methodology, this will be clear from the proof of* Theorem 3.4.3 *in* Section 3.4. *The second relates to our choice to write three equations instead of two. In fact, our approach is based on being able to identify* ∂U *as the derivative with respect to the* s *variable of* U *in an appropriate sense and, at least formally, it is clear that the third equation allows us to do so, see* Lemma 3.6.1 *for details. Alternatively, we could have chosen not to write the third equation and consider the system which for any* $s \in [0,T]$, *holds* $\mathbb{P}$–a.s. *for any* $t \in [0,T]$

$$
\begin{aligned}
\mathcal{Y}_t &= \xi(T, X_{\cdot\wedge T}) + \int_t^T h_r(X, \mathcal{Y}_r, \mathcal{Z}_r, U_r^r, V_r^r, \partial U_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r - \int_t^T \mathrm{d}\mathcal{N}_r,\\
U_t^s &= \eta(s, X_{\cdot\wedge T}) + \int_t^T g_r(s, X, U_r^s, V_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T V_r^{s\top} \mathrm{d}X_r - \int_t^T \mathrm{d}M_r^s,\\
\partial U_t^s &:= \frac{\mathrm{d}}{\mathrm{d}s} U^s_{|_{(s,t)}},
\end{aligned}
$$

where $\frac{\mathrm{d}}{\mathrm{d}s}U^s$ *corresponds to the density with respect to the Lebesgue measure of* $s \longmapsto U^s$. *Nevertheless, for the proof of well-posedness of* $(\mathcal{S})$ *that we present in* Section 3.6, *we have to derive appropriate estimates for* $(\partial U_t^t)_{t\in[0,T]}$, *and for this it is advantageous to do the identification by adding the third equation in* $(\mathcal{S})$ *and work on the space* $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$.

(ii) *We also emphasise that the presence of* $(V_t^t)_{t\in[0,T]}$ *in the generator of the first equation requires us to reduce the space of the solution from the classic* $(\mathfrak{H}, \|\cdot\|_{\mathfrak{H}})$ *to* $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ *where*

$$
\begin{aligned}
\mathfrak{H} := {} & \mathbb{S}^2(\mathbb{R}^{d_1}) \times \mathbb{H}^2(\mathbb{R}^{n\times d_1}) \times \mathbb{M}^2(\mathbb{R}^{d_1}) \times \mathbb{S}^{2,2}(\mathbb{R}^{d_2}) \times \mathbb{H}^{2,2}(\mathbb{R}^{n\times d_2}) \times \mathbb{M}^{2,2}(\mathbb{R}^{d_2})\\
& \times \mathbb{S}^{2,2}(\mathbb{R}^{d_2}) \times \mathbb{H}^{2,2}(\mathbb{R}^{n\times d_2}) \times \mathbb{M}^{2,2}(\mathbb{R}^{d_2}),
\end{aligned}
$$

and $\|\cdot\|_{\mathfrak{H}}$ *denotes the norm induced by* $\mathfrak{H}$. *Ultimately, this is due to the presence of* $(Z_t^t)_{t\in[0,T]}$ *in the* type-I BSVIE (3.1.1). *On this matter, we stress that to the best of our knowledge, our results constitute the first comprehensive study of* type-I BSVIEs *as general as* (3.1.1). *We remark our identification of the appropriate space to carry out the analysis is based on* [114, Section 2.1]. *In the case where* $(V_t^t)_{t\in[0,T]}$ *(resp.* $(Z_t^t)_{t\in[0,T]}$*) does not appear in the generator of the first* BSDE *in*

$(\mathcal{S})$ (*resp.* type-I BSVIE (3.1.1)), Proposition 3.6.5 (*resp.* Remark 3.4.4) *provide the arguments for how one can adapt our approach to yield a solution in the classical space. This shows that our methodology recovers existing results on* type-I BSVIE (1.3.3) *as well as the so-called* extended type-I BSVIE (1.3.5).

Remark 3.3.4. *In addition, we highlight two features of* $(\mathcal{S})$ *that will come into play in the setting of* type-I BSVIE (3.1.1)*, and differ from the one in the classic literature. They are related to the fact we work under the general filtration* $\mathbb{F}$*. The first is the fact that the stochastic integrals in* $(\mathcal{S})$ *are with respect to the canonical process* X*. Recall that* σ *is not assumed to be invertible* (*it is not even a square matrix in general and can vanish*)*, therefore the filtration generated by* X *is different from the one generated by* B*. This yields more general results and it allows for extra flexibility necessary in some applications, see* Chapter 2 *for an example. The second difference is the presence of the processes* $(N, M, \partial M)$*. As it was mentioned in* Section 3.2*, we work with a probability measure for which the martingale representation property for* $\mathbb{F}$*–local martingales in terms of stochastic integrals with respect to* X *does not necessarily hold. Therefore, we need to allow for orthogonal martingales in the representation. Certainly, there are known properties which are equivalent to the orthogonal martingales vanishing,* i.e. $N = M = \partial M = 0$*, for example when* $\mathbb{P}$ *is an extremal point of the convex hull of the probability measures that satisfy the properties in* Section 3.2*, see* [142, Theorem 4.29].

Assumption G provides an appropriate framework to derive the well-posedness of $(\mathcal{S})$. The following is the main theorem of this section whose proof we postpone to Section 3.6.

Theorem 3.3.5. *Let* Assumption G *hold. Then* $(\mathcal{S})$ *admits a unique solution in* $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$*. For any* $\mathfrak{h} \in \mathcal{H}$ *solution to* $(\mathcal{S})$ *there exists* $C > 0$ *such that*

$$\|\mathfrak{h}\|_{\mathcal{H}}^2 \leq C\Big(\|\xi\|_{\mathcal{L}^2}^2 + \|\eta\|_{\mathcal{L}^{2,2}}^2 + \|\partial_s\eta\|_{\mathcal{L}^{2,2}}^2 + \|\tilde{h}\|_{\mathbb{L}^{1,2}}^2 + \|\tilde{g}\|_{\mathbb{L}^{1,2,2}}^2 + \|\nabla\tilde{g}\|_{\mathbb{L}^{1,2,2}}^2\Big).$$

Moreover, if for $i \in \{1, 2\}$ $\mathfrak{h}^i \in \mathcal{H}$ *denotes the solution to* $(\mathcal{S})$ *with data* $(\xi^i, h^i, \eta^i, g^i, \partial_s\eta^i, \nabla g^i)$*, then*

$$\|\delta\mathfrak{h}\|_{\mathcal{H}}^2 \leq C\Big(\|\delta\xi\|_{\mathcal{L}^2}^2 + \|\delta\eta\|_{\mathcal{L}^{2,2}}^2 + \|\delta\partial_s\eta\|_{\mathcal{L}^{2,2}}^2 + \|\delta_1 h\|_{\mathbb{L}^{1,2}}^2 + \|\delta_1 g\|_{\mathbb{L}^{1,2,2}}^2 + \|\delta_1\nabla g\|_{\mathbb{L}^{1,2,2}}\Big),$$

where for $\varphi \in \{\mathcal{Y}, \mathcal{Z}, \mathcal{N}, U, V, M, \partial U, \partial V, \partial M, \xi, \eta, \partial_s\eta\}$ *and* $\Phi \in \{h, g, \nabla g\}$

$\delta\varphi := \varphi^1 - \varphi^2$, and $\delta_1\Phi_t := \Phi^1_t(\mathcal{Y}^1_r, \mathcal{Z}^1_t, U^{1t}_t, V^{1t}_t) - \Phi^2_t(Y^1_r, Z^1_t, U^{1t}_t, V^{1t}_t)$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e. on $[0,T] \times \mathcal{X}$.

Remark 3.3.6. *The reader may wonder about our choice to leave out the diagonal of* ∂V *in the generator of the first equation in* $(\mathcal{S})$*. As we will argue below, this would require us to consider an auxiliary infinite family of quadratic* BSDEs*. Since the main purpose of this paper is to relate the well-posedness of* $(\mathcal{S})$ *to that of the* type-I BSVIE (3.1.1)*, and inasmuch as we do not need to consider this case to establish* Theorem 3.4.3*, we have refrained to pursue it in this document. Nevertheless, this case is covered as part of the study of the extension of* $(\mathcal{S})$ *to the quadratic case in* Chapter 4*. If we were to study the system, which for any* $s \in [0,T]$*, holds* $\mathbb{P}$–a.s. *for any* $t \in [0,T]$

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T, X_{\cdot\wedge T}) + \int_t^T h_r(X, \mathcal{Y}_r, \mathcal{Z}_r, U^r_r, V^r_r, \partial U^r_r, \partial V^r_r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r - \int_t^T \mathrm{d}N_r,\\
U^s_t &= \eta(s, X_{\cdot\wedge T}) + \int_t^T g_r(s, X, U^s_r, V^s_r, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T V^{s\top}_r \mathrm{d}X_r - \int_t^T \mathrm{d}M^s_r,\\
\partial U^s_t &= \partial_s\eta(s, X_{\cdot\wedge T}) + \int_t^T \nabla g_r(s, X, \partial U^s_r, \partial V^s_r, U^s_r, V^s_r, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial V^{s\top}_r \mathrm{d}X_r - \int_t^T \mathrm{d}\partial M^s_r,
\end{aligned}$$

and as it is clear from our analysis in Section 3.6*, its well-posedness requires both having a rigorous method to define the mapping* $t \longmapsto \partial V^t_t$*, as well as deriving a priori estimates for the norm of* ∂V^t_t*. In analogy with* Lemma 3.6.1 *and* Remark 3.3.3*, both tasks require us to make sense of the family of* BSDEs *with terminal condition* $\partial_{ss}\eta$ *and generator*

$$\nabla^2 g_t(s, x, \tilde{\mathrm{u}}, \tilde{\mathrm{v}}, \mathrm{u}, \mathrm{v}, u, v, y, z) := \nabla g_t(s, x, \tilde{\mathrm{u}}, \tilde{\mathrm{v}}, u, v, y, z) + \sum_{\substack{(\pi_i, \pi_j, \tilde{\pi}_i, \tilde{\pi}_j)\\ \in \Pi^2 \times \widetilde{\Pi}^2}} \tilde{\pi}_i^\top \partial^2_{\pi_i\pi_j} g_t(s, x, u, v, y, z)\tilde{\pi}_j,$$

where $\Pi := (s, u, v_{1:}, ..., v_{n:})$, $\tilde{\Pi} := (1, \mathrm{u}, \mathrm{v}_{1:}, ..., \mathrm{v}_{n:})$ *and* $\partial^2_{\pi_i\pi_j} g_t(s, x, u, v, y, z)$ *denote the second order derivatives of* g*. Even though we could add assumptions ensuring that the second order derivatives are bounded, it is clear from the second term in the generator that we would necessarily need to consider a quadratic framework.*

3.4 Well-posedness of Lipschitz type-I BSVIEs

We now address the well-posedness of type-I BSVIEs. Let d be a non-negative integer, and f and ξ be jointly measurable functionals such that for any $(s,y,z,u,v)\in[0,T]\times(\mathbb{R}^d\times\mathbb{R}^{n\times d})^2$

$$
\begin{aligned}
&f:[0,T]^2\times\mathcal{X}\times(\mathbb{R}^d\times\mathbb{R}^{n\times d})^2\longrightarrow\mathbb{R}^d,\ f_\cdot(s,\cdot,y,z,u,v)\in\mathcal{P}_{\rm prog}(\mathbb{R}^d,\mathbb{F}),\\
&\xi:[0,T]\times\mathcal{X}\longrightarrow\mathbb{R}^d,\ \xi(s,\cdot)\text{ is }\mathcal{F}\text{-measurable.}
\end{aligned}
$$

To derive the main result in this section, we will exploit the well-posedness of $(\mathcal{S})$. Therefore, we work under the following set of assumptions.

Assumption H. *(i)* $(s,y,z)\longmapsto f_t(s,x,y,z,u,v)$ *(resp.* $s\longmapsto\xi(s,x)$*) is continuously differentiable, uniformly in* (t,x,u,v) *(resp. in* x*). Moreover, the mapping* $\nabla f:[0,T]^2\times\mathcal{X}\times(\mathbb{R}^d\times\mathbb{R}^{n\times d})^3\longrightarrow\mathbb{R}^d$ *defined by*

$$
\nabla f_t(s,x,\mathrm{u},\mathrm{v},y,z,u,v):=\partial_s f_t(s,x,y,z,u,v)+\partial_y f_t(s,x,y,z,u,v)\mathrm{u}+\sum_{i=1}^n\partial_{z_{:i}}f_t(s,x,y,z,u,v)\mathrm{v}_{i:},
$$

satisfies $\nabla f_\cdot(s,\cdot,y,z,u,v,\mathrm{u},\mathrm{v})\in\mathcal{P}_{\rm prog}(\mathbb{R}^d,\mathbb{F})$ *for all* $s\in[0,T]$;

(ii) for $\varphi\in\{f,\partial_s f\}$, $(u,v,y,z)\longmapsto\varphi_t(s,x,y,z,u,v)$ *is uniformly Lipschitz-continuous,* i.e. $\exists L_\varphi>0$, *such that for all* $(s,t,x,y,\tilde y,z,\tilde z,u,\tilde u,v,\tilde v)$

$$
|\varphi_t(s,x,y,z,u,v)-\varphi_t(s,x,\tilde y,\tilde z,\tilde u,\tilde v)|\leq L_\varphi\big(|y-\tilde y|+|\sigma_t(x)^\top(z-\tilde z)|+|u-\tilde u|+|\sigma_t(x)^\top(v-\tilde v)|\big).
$$

(iii) $\big(\tilde f_\cdot,\tilde f_\cdot(s),\nabla\tilde f_\cdot(s)\big):=\big(f_\cdot(\cdot,\cdot,\mathbf{0}),f_\cdot(s,\cdot,\mathbf{0}),\partial_s f_\cdot(s,\cdot,\mathbf{0})\big)\in\mathbb{L}^{1,2}(\mathbb{R}^d)\times\big(\mathbb{L}^{1,2,2}(\mathbb{R}^d)\big)^2$.

Let $(\mathcal{H}^\star,\|\cdot\|_{\mathcal{H}^\star})$ denote the space of $(Y,Z,N)\in\mathcal{H}^\star$ such that $\|(Y,Z,N)\|_{\mathcal{H}^\star}<\infty$ where

$$
\mathcal{H}^\star:=\mathbb{S}^{2,2}(\mathbb{R}^d)\times\overline{\mathbb{H}}^{2,2}(\mathbb{R}^{n\times d})\times\mathbb{M}^{2,2}(\mathbb{R}^d),\ \|\cdot\|_{\mathcal{H}^\star}:=\|Y\|^2_{\mathbb{S}^{2,2}}+\|Z\|^2_{\overline{\mathbb{H}}^{2,2}}+\|N\|^2_{\mathbb{M}^{2,2}}.
$$

We consider the n-dimensional type-I BSVIE on $(\mathcal{H}^\star,\|\cdot\|_{\mathcal{H}^\star})$ which for any $s\in[0,T]$, holds $\mathbb{P}$–a.s. for any $t\in[0,T]$

$$
Y_t^s=\xi(s,X)+\int_t^T f_r(s,X,Y_r^s,Z_r^s,Y_r^r,Z_r^r)\mathrm{d}r-\int_t^T Z_r^{s\top}\mathrm{d}X_r-\int_t^T\mathrm{d}N_r^s. \tag{3.4.1}
$$

We work under the following notion of solution.

Definition 3.4.1. *We say* (Y,Z,N) *is a solution to the* type-I BSVIE (3.4.1) *if* $(Y,Z,N)\in\mathcal{H}^\star$ *verifies* (3.4.1)*.*

Defining $h_t(x,y,z,u,v,\mathrm{u}):=f_t(t,x,y,z,u,v)-\mathrm{u}$, we may consider the system, which for any $s\in[0,T]$, holds $\mathbb{P}$–a.s. for any $t\in[0,T]$

$$\begin{aligned}\mathcal{Y}_t&=\xi(T,X)+\int_t^T h_r(X,\mathcal{Y}_r,\mathcal{Z}_r,Y_r^r,Z_r^r,\partial Y_r^r)\mathrm{d}r-\int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r-\int_t^T \mathrm{d}\mathcal{N}_r,\\ Y_t^s&=\xi(s,X)+\int_t^T f_r(s,X,Y_r^s,Z_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\mathrm{d}r-\int_t^T Z_r^{s\top}\mathrm{d}X_r-\int_t^T \mathrm{d}N_r^s,\\ \partial Y_t^s&=\partial_s\xi(s,X)+\int_t^T \nabla f_r(s,X,\partial Y_r^s,\partial Z_r^s,Y_r^s,Z_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\mathrm{d}r-\int_t^T \partial Z_r^{s\top}\mathrm{d}X_r-\int_t^T \mathrm{d}\partial N_r^s.\end{aligned}\tag{$\mathcal{S}_f$}$$

Remark 3.4.2. *We now make a few comments on our set-up for the study* type-I BSVIE (3.4.1)*.*

(i) *The necessity of the set of assumptions in* Assumption H *to our approach, based on the systems introduced in* Section 3.3*, is clear. Compared to the set of assumption made by recent works on* BSVIEs *in the literature we notice the main difference is the regularity with respect to the s variable we imposed on the data of the problem,* i.e. Assumption H.(i)*. In particular, we highlight that* type-I BSVIE (1.3.5)*, in which the diagonal of Y, but not of Z is allowed in the generator, had been considered in* [114; 244]*. In such a scenario, the authors assumed* $(\xi,f)\in\mathcal{L}^{2,2}(\mathbb{R}^d)\times\mathbb{L}^{1,2,2}(\mathbb{R}^d)$*, and no additional condition is required to obtain the well-posedness of* (1.3.5)*. As it will be clear from* Proposition 3.6.5 *and* Remark 3.4.4 *our procedure can be adapted to work under such set of assumptions provided the diagonal of Z is not considered in the generator.*

(ii) *Moreover, the spaces of the solution considered in* [114; 244] *also differ, echoing the absence of the diagonal of Z in the generator. The authors work with the notion of* C-solution*, that is, Y is assumed to be a jointly measurable process, such that $s\longmapsto Y^s$ is continuous in* $\mathbb{L}^{1,p}(\mathbb{R}^d)$*, $p\geq 2$, and for every $s\in[0,T]$, Y^s is $\mathbb{F}$-adapted with $\mathbb{P}$–*a.s. *continuous paths. This coincides with our definition of the space* $\mathbb{L}^{1,p,2}(\mathbb{R}^d)$*. Similarly, Z belongs to the space* $\mathbb{H}^{2,2}(\mathbb{R}^{n\times d})$*. On the other hand,* [245] *provides a representation formula for* type-I BSVIEs *for which the driver allows for the diagonal of Z, but not of Y. More precisely, they introduce a* PDE*, similar to the one we will introduce in* Section 3.5*, prove its well-posedness, and then a Feynman–Kac formula. Naturally, in* *this case* (Y,Z) *inherits the regularity of the underlying* PDE*.*

(iii) *The main contribution of our methodology to the field of* BSVIEs *is to be able to accommodate* type-I BSVIEs *for which the diagonal of* Z *appears in the generator. For this, the definition of the space* $(\mathcal{H}^\star, \|\cdot\|_{\mathcal{H}^\star})$, *notably the space* $\overline{\mathbb{H}}^{2,2}(\mathbb{R}^{n\times d})$, *and* Assumption H.$(i)$ *play a central role. As first noticed in* Chapter 2*, see* Lemma 3.6.1*, under this assumption one can identify the density, with respect to the Lebesgue measure, of the* $\mathbb{H}^2$*-valued mapping* $s \longmapsto Z^s$, *namely the* $\mathbb{H}^2$*-valued mapping* $s \longmapsto \partial Z^s$. *This allows us to define the diagonal of* Z *following* [114, Section 2.1]*, and consequently, introduce the space* $\overline{\mathbb{H}}^{2,2}(\mathbb{R}^{n\times d})$*.*

(iv) *We highlight that* Theorem 3.4.3 *below guarantees a unique solution to* (3.4.1) *in the space* $\mathcal{H}^\star$, *for which the mapping* $s \longmapsto Z^s$ *is assumed to be absolutely continuous, cf.* [114, Section 2.1]*. This allows for extra generality compare to assuming such mapping is,* a priori*, continuously differentiable. Nevertheless, as a by-product of* Assumption H.(i)*, the unique solution ensured by* Theorem 3.4.3 *is automatically* $\mathcal{C}^1$ *with derivative* $s \longmapsto \partial Z^s$*. Moreover, not only does our approach identify the dynamics of* $(\partial Y, \partial Z, \partial N)$*, but also, in the terminology of* [114, Section 2.1]*, it does guarantee* $(Z_t^t)_{t\in[0,T]}$ *is the unique process that satisfies the* (D)*-property with respect to* Z*.*

(v) *Let us remark that* Assumption H.(i)*, being an assumption on the data of the* BSVIE*, is easier to verify in practice compare to the regularity required in* [114]*. Certainly, our results would still hold true if we require the differentiability of data* (ξ, f) *with respect to the parameter* s *in the* $\mathcal{L}^2$ *(resp.* $\mathbb{L}^{1,2}$*) sense, or, even better, absolute continuity,.*

(vi) *Lastly, we stress that the above* type-I BSVIE *is defined for* $(s,t) \in [0,T]^2$*, as opposed to* $0 \leq s \leq t \leq T$*. However, anticipating the result of* Theorem 3.4.3*, this could be handled by first solving on* $(s,t) \in [0,T]^2$ *and then consider the restriction to* $0 \leq s \leq t \leq T$*.*

We are now in position to prove the main result of this paper. The next result shows that under the previous choice of data for $(\mathcal{S}_f)$, its solution solves the type-I BSVIE with data (ξ, f) and *vice versa*.

Theorem 3.4.3. *Let* Assumption H *hold. Then*

(i) *the well-posedness of* $(\mathcal{S}_f)$ *is equivalent to that of the* type-I BSVIE (3.4.1);

(ii) *the* type-I BSVIE (3.4.1) *is well-posed, and for any* $(Y, Z, N) \in \mathcal{H}^\star$ *solution to* type-I BSVIE (3.4.1) *there exists* $C > 0$ *such that*

$$\|(Y, Z, N)\|_{\mathcal{H}^\star} \leq C\Big(\|\xi\|^2_{\mathcal{L}^{2,2}} + \|\partial_s \xi\|^2_{\mathcal{L}^{2,2}} + \|\tilde{f}\|^2_{\mathbb{L}^{1,2,2}} + \|\nabla \tilde{f}\|^2_{\mathbb{L}^{1,2,2}}\Big). \tag{3.4.2}$$

Moreover, if $(Y^i, Z^i, N^i) \in \mathcal{H}^\star$ *denotes the solution to* type-I BSVIE (3.4.1) *with data* (ξ^i, f^i) *for* $i \in \{1, 2\}$*, we have*

$$\|(\delta Y, \delta Z, \delta N)\|^2_{\mathcal{H}^\star} \leq C\Big(\big\|\delta\xi\big\|^2_{\mathcal{L}^2} + \big\|\delta\partial_s\xi\big\|^2_{\mathcal{L}^{2,2}} + \|\delta_1 f\|^2_{\mathbb{L}^{1,2}} + \|\delta_1 \nabla f\|^2_{\mathbb{L}^{1,2,2}}\Big).$$

Proof. (ii) is a consequence of (i). Indeed, (3.4.2) follows from Proposition 3.6.3, and the well-posedness of type-I BSVIE (3.4.1) from that of $(\mathcal{S}_f)$, which holds by Assumption H and Theorem 3.3.5. We now argue (i).

Let $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, Y, Z, N, \partial Y, \partial Z, \partial N) \in \mathcal{H}$ be a solution to $(\mathcal{S}_f)$. It then follows from Lemma 3.6.2 that letting $\widetilde{N}_t := N^t_t - \int_0^t \partial N^r_r \mathrm{d}r$, $t \in [0, T]$, we have that $\mathbb{P}$–a.s. for any $t \in [0, T]$

$$Y^t_t = \xi(T, X) + \int_t^T h_r(X, Y^r_r, Z^r_r, \mathcal{Y}_r, \mathcal{Z}_r, \partial Y^r_r)\mathrm{d}r - \int_t^T Z^{r\top}_r \mathrm{d}X_r - \int_t^T \mathrm{d}\widetilde{N}_r. \tag{3.4.3}$$

This shows that $\big((Y^t_t)_{t\in[0,T]}, (Z^t_t)_{t\in[0,T]}, \mathcal{Y}_\cdot, \mathcal{Z}_\cdot, (\widetilde{N}_t)_{t\in[0,T]}\big)$, solves the first BSDE in $(\mathcal{S}_f)$. It then follows from the well-posedness of $(\mathcal{S}_f)$, which holds by Assumption H and Theorem 3.3.5, that $\big((Y^t_t)_{t\in[0,T]}, (Z^t_t)_{t\in[0,T]}, (\widetilde{N}_t)_{t\in[0,T]}\big) = (\mathcal{Y}_\cdot, \mathcal{Z}_\cdot, \mathcal{N}_\cdot)$ in $\mathbb{S}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{n\times d}) \times \mathbb{M}^2(\mathbb{R}^d)$. Consequently

$$Y^s_t = \xi(s, X) + \int_t^T f_r(s, X, Y^s_r, Z^s_r, Y^r_r, Z^r_r)\mathrm{d}r - \int_t^T Z^{s\top}_r \mathrm{d}X_r - \int_t^T \mathrm{d}N^s_r,\ t \in [0, T],\ \mathbb{P}\text{–a.s.},\ s \in [0, T].$$

We are left to show the converse result. Let $(Y, Z, N) \in \mathcal{H}^\star$ be a solution to type-I BSVIE (3.4.1). We begin by noticing that the processes $\mathcal{Y} := (Y^t_t)_{t\in[0,T]}, \mathcal{Z} := (Z^t_t)_{t\in[0,T]}, \mathcal{N} := (N^t_t)_{t\in[0,T]}$ are well-defined. In particular, $\mathcal{Z} \in \mathbb{H}^2(\mathbb{R}^{n\times d})$ is well-defined as $Z \in \overline{\mathbb{H}}^{2,2}(\mathbb{R}^{n\times d})$. Moreover, $\mathcal{Y} \in \mathbb{L}^{2,2}(\mathbb{R}^d)$ follows from

$$\|\mathcal{Y}\|^2_{\mathbb{L}^2} = \mathbb{E}\bigg[\int_0^T |Y^r_r|^2 \mathrm{d}r\bigg] \leq \mathbb{E}\bigg[\int_0^T \sup_{t\in[0,T]} |Y^r_t|^2 \mathrm{d}r\bigg] = \int_0^T \|Y^r\|_{\mathbb{S}^2} \mathrm{d}r < \infty.$$

Then, since Assumption H holds and $(\mathcal{Y},\mathcal{Z},Y,Z,N)\in\mathbb{L}^2(\mathbb{R}^d)\times\mathbb{H}^2(\mathbb{R}^{n\times d})\times\mathbb{S}^{2,2}(\mathbb{R}^d)\times\mathbb{H}^{2,2}(\mathbb{R}^{n\times d})\times\mathbb{M}^{2,2}(\mathbb{R}^d)$, we obtain, from Lemma 3.6.1, there is $(\partial Y,\partial Z,\partial N)\in\mathbb{S}^{2,2}(\mathbb{R}^d)\times\mathbb{H}^{2,2}(\mathbb{R}^{n\times d})\times\mathbb{M}^{2,2}(\mathbb{R}^d)$ such that for $s\in[0,T]$, $\mathbb{P}$–a.s. for any $t\in[0,T]$

$$\partial Y_t^s=\partial_s\xi(s,X)+\int_t^T\nabla f_r(s,X,\partial Y_r^s,\partial Z_r^s,Y_r^s,Z_r^s,Y_r^r,Z_r^r)\mathrm{d}r-\int_t^T\partial Z_r^{s\top}\mathrm{d}X_r-\int_t^T\mathrm{d}\partial N_r^s.$$

We claim that $\mathfrak{h}:=(\mathcal{Y},\mathcal{Z},\widetilde{N},Y,Z,N,\partial Y,\partial Z,\partial N)$ is a solution to $(\mathcal{S}_f)$, where $\widetilde{N}_t:=N_t^t-\int_0^t\partial N_r^r\mathrm{d}r$, $t\in[0,T]$. For this, we first note that in light of Lemmata 3.6.1 and 3.6.2 we have that $\mathbb{P}$–a.s. for any $t\in[0,T]$

$$\mathcal{Y}_t=\xi(T,X)+\int_t^T h_r(X,\mathcal{Y}_r,\mathcal{Z}_r,Y_r^r,Z_r^r,\partial Y_r^r)\mathrm{d}r-\int_t^T\mathcal{Z}_r{}^\top\mathrm{d}X_r-\int_t^T\mathrm{d}\widetilde{N}_r,\tag{3.4.4}$$

and $\widetilde{\mathcal{N}}\in\mathbb{M}^{2,2}(\mathbb{R}^d)$. We are only left to argue $\mathcal{Y}\in\mathbb{S}^2(\mathbb{R}^d)$. Note that by Assumption H and Equation (I.1) there exists $C>0$ such that

$$\begin{aligned}|\mathcal{Y}_t|^2\leq C\bigg(&|\xi|^2+\Big(\int_0^T|\tilde f_r|\mathrm{d}r\Big)^2+\int_0^T(|\mathcal{Y}_r|^2+|\sigma_r^\top\mathcal{Z}_r|^2+|Y_r^r|^2+|\sigma_r^\top Z_r^r|^2+|\partial Y_r^r|^2)\mathrm{d}r\\&+\bigg|\int_t^T\mathcal{Z}_r^\top\mathrm{d}X_r\bigg|^2+\bigg|\int_t^T\mathrm{d}\widetilde{N}_r\bigg|^2\bigg).\end{aligned}$$

Moreover, by Doob's inequality $\mathbb{E}\bigg[\sup_{t\in[0,T]}\bigg|\int_0^t\mathcal{Z}^\top\mathrm{d}X_r\bigg|^2\bigg]\leq 4\|\mathcal{Z}\|_{\mathbb{H}^2}^2$ and Equation (3.6.15) yields

$$\begin{aligned}\|\mathcal{Y}\|_{\mathbb{S}^2}^2\leq C\Big(&\|\xi\|_{\mathcal{L}^{2,2}}^2+\|\partial_s\xi\|_{\mathcal{L}^{2,2}}^2+\|\tilde f\|_{\mathbb{L}^{1,2,2}}^2+\|\nabla\tilde f\|_{\mathbb{L}^{1,2,2}}^2\\&+\|\mathcal{Y}\|_{\mathbb{L}^2}^2+\|Y\|_{\mathbb{L}^{2,2}}^2+\|Z\|_{\mathbb{H}^{2,2}}^2+\|\partial Z\|_{\mathbb{H}^{2,2}}^2\Big)<\infty.\end{aligned}$$

We conclude $\|\mathfrak{h}\|_{\mathcal{H}}<\infty$, $\mathfrak{h}\in\mathcal{H}$ and thus $\mathfrak{h}$ solves $(\mathcal{S}_f)$. $\square$

Remark 3.4.4. *There are two noticeable differences between* Theorem 3.4.3 *and the results in the literature on* type-I BSVIEs (1.3.3), (1.3.4) *and* (1.3.5)*, as previously studied in* Chapter 2, [269], [245] *and* [114; 244]*, respectively. The first is the additional terms, involving the derivative with respect to the parameter s of the data, appearing in the* a priori *estimates and the stability. The second one, is the space where the solution lives. Both differences are due to the fact that we are* *handling the diagonal term for Z in the generator.*

In fact, in light of Proposition 3.6.5 *for the case of* type-I BSVIEs (1.3.5)*,* i.e. *where only the diagonal of Y is allowed in the generator, one can work in the more standard* (*compared to the existing literature*) *space* $(\mathfrak{H}^\star, \|\cdot\|_{\mathfrak{H}^\star})$ *given by*

$$\mathfrak{H}^\star := \mathbb{S}^{2,2}(\mathbb{R}^d) \times \mathbb{H}^{2,2}(\mathbb{R}^{n\times d}) \times \mathbb{M}^{2,2}(\mathbb{R}^d),\ \|(Y,Z,N)\|^2_{\mathfrak{H}^\star} := \|Y\|^2_{\mathbb{S}^{2,2}} + \|Z\|^2_{\mathbb{H}^{2,2}} + \|N\|^2_{\mathbb{M}^{2,2}}.$$

Then, the a priori *estimate* (3.4.2) *simplifies to*

$$\|(Y,Z,N)\|_{\mathfrak{H}^\star} \leq C\big(\|\xi\|^2_{\mathcal{L}^{2,2}} + \|\tilde{f}\|^2_{\mathbb{L}^{1,2,2}}\big),$$

and for (Y^i, Z^i, N^i) *the solution to* type-I BSVIE (1.3.5) *with data* (ξ^i, f^i) *for* $i \in \{1,2\}$*, we obtain*

$$\|(\delta Y, \delta Z, \delta N)\|^2_{\mathfrak{H}^\star} \leq C\Big(\|\delta\xi\|^2_{\mathcal{L}^2} + \|\delta_1 f\|^2_{\mathbb{L}^{1,2}}\Big).$$

3.5 Time-inconsistency, type-I BSVIEs and parabolic PDEs

This section is devoted to the application of our results in Section 3.4 to the problem of time-inconsistent control for sophisticated agents. Moreover, we also reconcile seemingly different approaches to the study of this problem.

3.5.1 A representation formula for adapted solutions of type-I BSVIEs

Building upon the fact that second-order, parabolic, semilinear PDEs of HJB type admit a non-linear Feynman–Kac representation formula, we can identify the family of PDEs associated to Type-I BSVIEs. This is similar to the representation of forward backward stochastic differential equations, FBSDEs for short, see [255].

For $(s,t,x,u,y,v,z,\gamma,\Sigma) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^2 \times (\mathbb{R}^{n\times d})^2 \times (\mathbb{R}^{n\times n})^d \times \mathbb{R}^{n\times m}$, define $\mathrm{Tr}[\Sigma\Sigma^\top\gamma] \in$

$\mathbb{R}^d$ by $\big(\mathrm{Tr}[\Sigma\Sigma^\top\gamma]\big)_i := \mathrm{Tr}[\Sigma\Sigma^\top\gamma_i]$, $i \in \{1,\dots,d\}$. Let f and σ be as in the preceding section, and $b : [0,T]\times\mathcal{X} \longrightarrow \mathbb{R}^m$ be bounded, $b.(X) \in \mathcal{P}_{\mathrm{meas}}(\mathbb{R}^m, \mathbb{F})$ and

$$G(s,t,x,u,v,y,z,\gamma) := v^\top\sigma(t,x)b(t,x) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)\gamma] + f(s,t,x,u,v,y,z).$$

Proposition 3.5.1. *Suppose $\varphi_t(X,\cdot) = \varphi_t(X_t,\cdot)$ for $\varphi \in \{b,\sigma,f,\partial_s f\}$, and $\varphi(s,X) = \varphi(s,X_T)$ for $\varphi \in \{\xi, \partial_s\xi\}$. For $(\mathcal{O} := [0,T)\times[0,T]\times\mathbb{R}^n =$, let $\mathcal{V} \in \mathcal{C}^d_{1,1,2}([0,T]^2\times\mathbb{R}^n)$ be a classical solution to*

$$\begin{aligned}&\partial_t\mathcal{V}(s,t,x) + G(s,t,x,\mathcal{V}(s,t,x),\partial_x\mathcal{V}(s,t,x),\mathcal{V}(t,t,x),\partial_x\mathcal{V}(t,t,x),\partial^2_{xx}\mathcal{V}(s,t,x)) = 0,\ (s,t,x)\in\mathcal{O},\\ &\mathcal{V}(s,T,x) = \xi(s,x),\ (s,x)\in[0,T]\times\mathbb{R}^n,\end{aligned}$$

for which $\mathcal{V}(s,t,x)$ and $\partial_x\mathcal{V}(s,t,x)$ have uniform exponential growth in x[3], i.e.

$$\exists C>0, \forall(s,t,x)\in[0,T]^2\times\mathcal{X}, |\mathcal{V}(t,x)| + |\partial_x\mathcal{V}(t,x)| \leq C\exp(C|x|_1).$$

Then, $Y^s_t := \mathcal{V}(s,t,X_t)$, and $Z^s_t := \partial_x\mathcal{V}(s,t,X_t)$ define a solution to the type-I BSVIE *which for every $s\in[0,T]$ holds* $\mathbb{P}$–a.s. *for any $t\in[0,T]$*

$$Y^s_t = Y^s_T + \int_t^T \big(f_r(s,X_r,Y^s_r,Z^s_r,Y^r_r,Z^r_r) + Z^{s\top}_r\sigma_r(X_r)b_r(X_r)\big)\mathrm{d}r - \int_t^T Z^{s\top}_r\mathrm{d}X_r. \tag{3.5.1}$$

Proof. Let $s\in[0,T]$ and $\mathbb{P}$ as in Section 3.2. Applying Itô's formula to Y^s_t we find that $\mathbb{P}$–a.s.

$$\begin{aligned}Y^s_t &= Y^s_T - \int_t^T \Big(\partial_t\mathcal{V}(s,r,X_r) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma_r^\top(X_r)\partial^2_{xx}\mathcal{V}(s,r,X_r)]\Big)\mathrm{d}r - \int_t^T Z^{s\top}_r\mathrm{d}X_r\\ &= Y^s_T + \int_t^T \big(f_r(s,X_r,Y^s_r,Z^s_r,Y^r_r,Z^r_r) + Z^{s\top}_r\sigma_r(X_r)b_r(X_r)\big)\mathrm{d}r - \int_t^T Z^{s\top}_r\mathrm{d}X_r.\end{aligned}$$

We verify the integrability of (Y,Z). As σ is bounded, X_t has exponential moments of any order which are bounded on $[0,T]$, *i.e.* $\exists C>0$, such that $\sup_{t\in[0,T]}\mathbb{E}^{\mathbb{P}}[\exp(c|X_t|_1)] \leq C < \infty$, for any $c>0$, where C depends on T and the bound on σ. This together with the growth condition on $\mathcal{V}(s,t,x)$ and $\partial_x\mathcal{V}(s,t,x)$ yield the integrability. □

Remark 3.5.2. *In the previous result the* type-I BSVIEs *has an additional term linear in z. This*

[3] Here, $|\cdot|_1$ denotes the ℓ_1 norm in $\mathbb{R}^n$ as in Theorem 2.2.8

is a consequence of the dynamics of X under $\mathbb{P}$, see Section 3.2*. Nevertheless, as b is bounded, we can define $\mathbb{P}^b \in \mathrm{Prob}(\mathcal{X})$, equivalent to $\mathbb{P}$, given by*

$$\frac{\mathrm{d}\mathbb{P}^b}{\mathrm{d}\mathbb{P}} := \exp\bigg(\int_0^T b_r(X_r)\cdot \mathrm{d}B_r - \int_0^T |b_r(X_r)|^2 \mathrm{d}r\bigg).$$

By Girsanov's theorem $B^b := B - \int_0^\cdot b_r(X_r)\mathrm{d}r$ is a $\mathbb{P}^b$–Brownian motion and

$$X_t = x_0 + \int_0^t \sigma_r(X_r) b_r(X_r)\mathrm{d}r + \int_0^t \sigma_r(X_r)\mathrm{d}B^b_r,\ t\in[0,T],\ \mathbb{P}^b\text{–a.s.},$$

and consequently

$$Y^s_t = Y^s_T + \int_t^T f_r(s, X_r, Y^s_r, Z^s_r, Y^r_r, Z^r_r)\mathrm{d}r - \int_t^T Z_r^\top \mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.},\ s\in[0,T].$$

3.5.2 On equilibria and their value function in time-inconsistent control problems

The game theoretic approach to time-inconsistent stochastic control problems in continuous-time started with the Markovian setting, and is grounded in the notion of equilibrium first proposed in Ekeland and Pirvu [82], Ekeland and Lazrak [81], and the infinite family of PDEs, or Hamilton–Jacobi–Bellman equation, provided by Björk, Khapko, and Murgoci [38], see Equation (2.2.9) below. Soon after, Wei, Yong, and Yu [260] presented a verification argument via a one dimensional PDE, but over an extended domain, see Equation (3.5.3) below. Both approaches have generated independent lines of research in the community, including both analytic and probabilistic methods, but no compelling connections have been established, as far as we know.

BSDEs and BSVIEs appear naturally as part of the probabilistic study of these problems. This approach allows extensions to a non-Markovian framework, and to reward functionals given by recursive utilities. Indeed, the approaches in Chapter 2 and [245] address these directions, and can be regarded as extensions of [38] and [260], respectively. As such, it is not surprising that in order to characterise an equilibrium and its associated value function, both Chapter 2 and [245] lay down an infinite family of BSDEs, and a type-I BSVIEs, respectively. In fact, Theorem 2.2.8 and [245, Theorem 5.1] establish representation formulae for the analytic, *i.e.* PDEs, counterparts.

Moreover, in Chapter 2 our approach through BSDEs led to the well-posedness of a BSVIE. This is nothing but a manifestation of Theorem 3.4.3 which reconciles, at the probabilistic level, the findings of Chapter 2 and [245]. Moreover, we also include Proposition 3.5.3 which does the same at the PDE level. To sum up, we can visualise this in the next picture.

$$
\begin{tikzcd}
[38] \arrow[r, "\text{Theorem 2.2.8}"] \arrow[d, dotted, leftrightarrow, "\text{Proposition 3.5.3}"'] & \text{Chapter 2} \arrow[d, leftrightarrow, "\text{Theorem 3.4.3}"] \\
[260] \arrow[r, "{[245,\ \text{Theorem 5.1}]}"'] & [245]
\end{tikzcd}
$$

Let $A \subseteq \mathbb{R}^p$ be a compact set, we introduce the mappings

$$\bar{f} : [0,T]^2 \times \mathbb{R}^n \times A \longrightarrow \mathbb{R},\ b : [0,T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^m,\ \text{bounded},$$

with $f_\cdot(s,\cdot,a) \in \mathcal{P}_{\text{prog}}(\mathbb{R},\mathbb{F})$ and $b_\cdot(\cdot,a) \in \mathcal{P}_{\text{meas}}(\mathbb{R}^m,\mathbb{F})$ for $(s,a) \in [0,T] \times A$. With this we may define

$$\begin{aligned}
&\bar{g}(s,t,x,a,v) := \bar{f}(s,t,x,a) + v \cdot \sigma(t,x) b(t,x,a),\ H(s,t,x,v) := \sup_{a \in A} \bar{g}(s,t,x,a,v), \\
&\nabla \bar{g}(s,t,x,a,\mathrm{v}) := \partial_s \bar{f}(s,t,x,a) + \mathrm{v} \cdot \sigma(t,x) b(t,x,a),
\end{aligned}$$

and assume there is $a^\star : [0,T]^2 \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow A$, measurable, such that, $\bar{g}(s,t,x,a^\star(s,t,x,v),v) = H(s,t,x,v)$.

Following the approach of [245], let us assume that given an admissible A-valued strategy ν over the interval $[s,T]$, the reward at $s \in [0,T]$ is given by the value at s of the Y coordinate of the solution to the type-I BSVIE given by

$$Y_t^\nu = \xi(t, X_T) + \int_t^T \bar{g}_r(t, X_r, \nu_r, Z_r^t)\mathrm{d}r - \int_t^T Z_r^t \cdot \mathrm{d}X_r,\ t \in [s,T],\ \mathbb{P}\text{–a.s.}$$

[245] finds that the value along the *equilibrium policy* coincides with the Y coordinate of the

following type-I BSVIE

$$Y_t = \xi(t, X_T) + \int_t^T \bar{g}_r\big(t, X_r, a^\star(r, r, X_r, Z_r^r), Z_r^t\big)\mathrm{d}r - \int_t^T Z_r^t \cdot \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},$$

where the diagonal of Z appears in the generator. However, decoupling the dependence between the time variable and the variable source of time-inconsistency, we can define for any $s \in [0,T]$, $\mathbb{P}$–a.s., for any $t \in [0,T]$

$$Y_t^s := \xi(s, X_T) + \int_t^T \bar{g}_r\big(s, X_r, a^\star(r, r, X_r, Z_r^r), Z_r^s\big)\mathrm{d}r - \int_t^T Z_r^s \cdot \mathrm{d}X_r.$$

It then follows from Theorem 3.4.3 that this approach is equivalent to that of Chapter 2 based on the system which for any $s \in [0,T]$, holds $\mathbb{P}$–a.s. for any $t \in [0,T]$

$$\begin{aligned}
Y_t &= \xi(T, X_T) + \int_t^T \big(H_r(r, X_r, Z_r) - \partial Y_r^r\big)\mathrm{d}r - \int_t^T Z_r \cdot \mathrm{d}X_r,\\
\partial Y_t^s &= \partial_s \xi(s, X_T) + \int_t^T \nabla \bar{g}_r\big(s, X_r, a^\star(r, r, X_r, Z_r^r), \partial Z_r^s\big)\mathrm{d}r - \int_t^T \partial Z_r^s \cdot \mathrm{d}X_r.
\end{aligned}$$

We now move on to establish the connection of the analyses at the PDE level. The original result of [38] is based on the semi-linear PDE system of HJB type given for $\big(V(t,x), \mathcal{J}(s,t,x)\big) \in \mathcal{C}_{1,2}([0,T] \times \mathbb{R}^n) \times \mathcal{C}_{1,1,2}([0,T]^2 \times \mathbb{R}^n)$ and $(s,t,x) \in \mathcal{O}$ by

$$\begin{aligned}
&\partial_t V(t,x) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)\partial_{xx}V(t,x)] + H(t,t,x,\partial_x V(t,x)) - \partial_s \mathcal{J}(t,t,x) = 0,\\
&\partial_t \mathcal{J}(s,t,x) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)\partial_{xx}\mathcal{J}(s,t,x)] + \bar{g}\big(s,t,x,\partial_x\mathcal{J}(s,t,x), a^\star(t,t,x,\partial_x V(t,x))\big) = 0,\\
&V(T,x) = \xi(T,x),\ \mathcal{J}(s,T,x) = \xi(s,x),\ (s,x) \in [0,T] \times \mathbb{R}^d.
\end{aligned} \tag{3.5.2}$$

On the other hand, [260] considers the equilibrium HJB equation for $\mathcal{J}(s,t,x) \in \mathcal{C}_{1,1,2}([0,T]^2 \times \mathbb{R}^n)$ and $(s,t,x) \in \mathcal{O}$ given by

$$\begin{aligned}
&\partial_t \mathcal{V}(s,t,x) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)\partial_{xx}\mathcal{V}(s,t,x)] + \bar{g}\big(s,t,x,\partial_x\mathcal{V}(t,t,x), a^\star(t,t,x,\partial_x\mathcal{V}(t,t,x))\big) = 0,\\
&\mathcal{V}(s,T,x) = \xi(s,x),\ (s,x) \in [0,T] \times \mathbb{R}^d.
\end{aligned} \tag{3.5.3}$$

By setting $\mathcal{V}(s,t,x) = \mathcal{J}(s,t,x)$, it is immediate that a solution to (2.2.9) defines a solution to

(3.5.3). The next proposition establishes the converse result.

Proposition 3.5.3. *Suppose* (3.5.2) *and* (3.5.3) *are well-posed.*

(i) Let $\mathcal{J}(s,t,x)$ *solve* (3.5.2) *Then* $\mathcal{V}(s,t,x) := \mathcal{J}(s,t,x)$ *solves* (3.5.3).

(ii) Let $\mathcal{V}(s,t,x)$ *solve* (3.5.3). *Then* $\big(V(t,x), \mathcal{J}(s,t,x)\big) := \big(\mathcal{V}(t,t,x), \mathcal{V}(s,t,x)\big)$ *solves* (3.5.2).

Proof. We are only left to argue (ii). It is clear $\big(V(t,x), \mathcal{J}(s,t,x)\big) \in \mathcal{C}_{1,2}([0,T]\times\mathbb{R}^n)\times\mathcal{C}_{1,1,2}([0,T]^2\times\mathbb{R}^n)$, the results follows as $-\partial_t V(t,x) + \partial_s \mathcal{J}(t,t,x) = -\partial_t \mathcal{V}(t,t,x)$. □

3.6 Analysis of the BSDE system

In order to alleviate notations, and as it is standard in the literature, we suppress the dependence on ω, *i.e.* on X, in the functionals, and, write $\mathbb{E}$ instead of $\mathbb{E}^{\mathbb{P}}$ as the underlying probability measure $\mathbb{P}$ is fixed. Moreover, we will write $\mathbb{I}^2$ instead of $\mathbb{I}^2(E)$ for any of the integrability spaces involved, the specific space E is fixed and understood without ambiguity.

3.6.1 Regularity of the system and the diagonal processes

In preparation to the proof of Theorem 3.3.5, we present next a couple of lemmata from which we will benefit in the following. As a historical remark, we mention the following is in the spirit of the analysis in Protter [211, Section 3] and Pardoux and Protter [202] of forward Volterra integral equations.

A technical detail in our analysis is to identify appropriate spaces so that given $(\partial U, U, V, M)$ one can rigorously define the processes $\big((U^t_t)_{t\in[0,T]}, (V^t_t)_{t\in[0,T]}, (M^t_t)_{t\in[0,T]}, (\partial U^t_t)_{t\in[0,T]}\big)$. It is known that for $U \in \mathbb{S}^{2,2}$, the diagonal process $(U^t_t)_{t\in[0,T]}$ is well-defined. Indeed, this follows from the pathwise regularity of U^s for $s \in [0,T]$ and has been noticed since [114; 256] and Chapter 2. The same argument works for $(\partial U, M) \in \mathbb{S}^{2,2} \times \mathbb{M}^{2,2}$. Unfortunately, the same reasoning cannot be applied for arbitrary $V \in \mathbb{H}^{2,2}$ and motives the introduction of the space $\overline{\mathbb{H}}^{2,2}$, see Remark 3.3.3.

Lemma 3.6.1. *Let* Assumption G *hold and* $(\mathcal{Y}, \mathcal{Z}) \in \mathbb{L}^2 \times \mathbb{H}^2$. *Let* $(U, V, M) \in \mathbb{L}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{M}^{2,2}$

be a solution to

$$U_t^s = \eta(s) + \int_t^T g_r(s, U_r^s, V_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}M_r^s,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\ s \in [0,T];$$

(i) *there exist* $(\partial U, \partial V, \partial M) \in \mathbb{S}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{M}^{2,2}$ *unique solution to the equation, which for* $s \in [0,T]$ *holds* $\mathbb{P}$–a.s. *for any* $t \in [0,T]$

$$\partial U_t^s = \partial_s \eta(s) + \int_t^T \nabla g_r(s, \partial U_r^s, \partial V_r^s, U_r^s, V_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s; \tag{3.6.1}$$

(ii) *there exists* $C > 0$*, such that for all* $c > 4L_g$ *and* $(s,t) \in [0,T]^2$

$$\begin{aligned} \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top \partial V_r^s|^2\mathrm{d}r\bigg] \leq C\mathbb{E}\bigg[\mathrm{e}^{cT}\big|\partial_s\eta(s)\big|^2 + \bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}|\nabla\tilde{g}_r(s)|\mathrm{d}r\bigg)^2 \\ + \int_t^T \mathrm{e}^{cr}\big(|U_r^s|^2 + |\sigma_r^\top V_r^s|^2 + |\mathcal{Y}_r|^2 + |\sigma_r^\top \mathcal{Z}_r|^2\big)\mathrm{d}r\bigg]; \end{aligned} \tag{3.6.2}$$

(iii) *for any* $s \in [0,T]$*, in* $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2$*,*

$$\bigg(\int_s^T \partial U^r \mathrm{d}r, \int_s^T \partial V^r \mathrm{d}r, \int_s^T \partial M^r \mathrm{d}r\bigg) = (U^T - U^s, V^T - V^s, M^T - M^s); \tag{3.6.3}$$

(iv) $V \in \overline{\mathbb{H}}^{2,2}$*. Moreover, for* $\mathcal{V} := (V_t^t)_{t\in[0,T]}$ *and* $\varepsilon > 0$*,* $\mathbb{P}$–a.s.

$$\int_t^T |\sigma_u^\top \mathcal{V}_u|^2\mathrm{d}u \leq \int_t^T |\sigma_u^\top V_u^t|^2\mathrm{d}u + \int_t^T\int_r^T \varepsilon|\sigma_u^\top V_u^r|^2 + \varepsilon^{-1}|\sigma_u^\top \partial V_u^r|^2\mathrm{d}u\mathrm{d}r,\ t \in [0,T]. \tag{3.6.4}$$

Proof. Note that in light of Assumption G.(i), for $(t,s,x,u,v,y,z) \in [0,T]^2 \times \mathcal{X} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_2\times n} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_1\times n}$, $(\mathrm{u},\mathrm{v}) \longmapsto \nabla g_t(s,x,\mathrm{u},\mathrm{v},u,v,y,z)$ is linear. Therefore, for any $s \in [0,T]$ the second equation defines a linear BSDE, in $(\partial U^s, \partial V^s)$, whose generator at zero, by Assumption G.(iii), is in $\mathbb{L}^{1,2}$. Therefore, its solution $(\partial U^s, \partial V^s, \partial M^s) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2$ is well-defined from classic results, see for instance Zhang [273] or [88]. The continuity of the applications $s \longmapsto (\partial U^s, \partial V^s, \partial M^s)$, e.g. $([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{S}^2, \|\cdot\|_{\mathbb{S}^2}) : s \longmapsto \partial U^s$, follows from the classical stability results of BSDE, and that by assumption $(U,V,Y,Z) \in \mathbb{L}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{L}^2 \times \mathbb{H}^2$ and $s \longmapsto (\partial_s\eta(s), \nabla\tilde{g}(s))$ is continuous. This establishes $(\partial U, \partial V, \partial M) \in \mathbb{S}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{M}^{2,2}$.

(ii) follows from classic *a priori* estimates, but when the norms are considered over $[t,T]$ instead of $[0,T]$. Indeed, following the argument in [88, Proposition 2.1], applying Itô's formula to $\mathrm{e}^{ct}|\partial U_t^s|^2$ we may find $C>0$ such that for any $c>4L_g$ and $(s,t)\in[0,T]^2$

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{r\in[t,T]}\Big\{\mathrm{e}^{cr}|\partial U_r^s|^2\Big\}+\int_t^T \mathrm{e}^{cr}|\sigma_r^\top\partial V_r^s|^2\mathrm{d}r+\int_t^T \mathrm{e}^{cr-}\mathrm{d}[\partial M^s]_r\bigg]\\
&\leq C\mathbb{E}\bigg[\mathrm{e}^{cT}|\partial_s\eta(s)|^2+\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}|\nabla g_t(s,0,0,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r)|\mathrm{d}r\bigg)^2\bigg]\\
&=C\mathbb{E}\bigg[\mathrm{e}^{cT}|\partial_s\eta(s)|^2+\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}\Big|\partial_s g_r(s,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\Big|\mathrm{d}r\bigg)^2\bigg]\\
&\leq C\mathbb{E}\bigg[\mathrm{e}^{cT}|\partial_s\eta(s)|^2+\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}|\nabla\tilde{g}_r(s)|\mathrm{d}r\bigg)^2+\int_t^T \mathrm{e}^{cr}(|U_t^s|^2+|\sigma_t^\top V_t^s|^2+|\mathcal{Y}_t|^2+|\sigma_t^\top\mathcal{Z}_t|^2)\mathrm{d}t\bigg],
\end{aligned}$$

where in the second inequality we exploited the fact $(u,v,y,z)\longmapsto\partial_s g(t,s,x,u,v,y,z)$ is Lipschitz, see Assumption G.(iii), and $C>0$ was appropriately updated.

Next, we assume (iii) and show (iv). In light of (i) and (iii), $s\longmapsto\partial V^s$ is the density of $s\longmapsto V^s$ with respect to the Lebesgue measure. Arguing as in Remark 3.2.2, we obtain that we can define

$$V_t^t:=V_t^T-\int_t^T\partial V_t^r\mathrm{d}r,\ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e. in }[0,T]\times\mathcal{X}. \tag{3.6.5}$$

We now verify (3.6.4). By definition of $\mathcal{V}$, Fubini's theorem and Young's inequality we have that for $\varepsilon>0$

$$\begin{aligned}
\int_t^T|\sigma_u^\top V_u^u|^2-|\sigma_u^\top V_u^t|^2\mathrm{d}u&=\int_t^T\int_t^u 2\mathrm{Tr}\Big[V_u^{r\top}\sigma_u\sigma_u^\top\partial V_u^r\Big]\mathrm{d}r\mathrm{d}u\\
&=\int_t^T\int_r^T 2\mathrm{Tr}\Big[V_u^{r\top}\sigma_u\sigma_u^\top\partial V_u^r\Big]\mathrm{d}u\mathrm{d}r\\
&\leq\int_t^T\int_r^T\varepsilon|\sigma_u^\top V_u^r|^2+\varepsilon^{-1}|\sigma_u^\top\partial V_u^r|^2\mathrm{d}u\mathrm{d}r.
\end{aligned}$$

Thus $\|\mathcal{V}\|_{\mathbb{H}^2}<\infty$ and consequently $V\in\overline{\mathbb{H}}^{2,2}$. This proves (iv).

We now argue (iii). We also remark that a similar argument to the one in (i) shows that under Assumption G $U\in\mathbb{S}^{2,2}$. We know the mapping $[0,T]\ni s\longmapsto(\partial Y^s,\partial Z^s,\partial M^s)$ is continuous, in particular integrable. A formal integration with respect to s to (3.6.1) leads to

$$\int_s^T \partial U_t^r \mathrm{d}r = \int_s^T \partial_s \eta(r)\mathrm{d}r + \int_s^T \int_t^T \partial_s g_u(r, U_u^r, V_u^r, \mathcal{Y}_u, \mathcal{Z}_u) + \partial_y g_u(r, U_u^r, V_u^r, \mathcal{Y}_u, \mathcal{Z}_u)\partial U_u^r \mathrm{d}u\mathrm{d}r$$
$$+ \int_t^T \int_s^T \sum_{i=1}^n \partial_{v_i} g_u(r, U_u^r, V_u^r, \mathcal{Y}_u, \mathcal{Z}_u)(\partial V_u^r)_{:i}\mathrm{d}u\mathrm{d}r$$
$$- \int_s^T \int_t^T \partial Z_u^{r\top}\mathrm{d}X_u\mathrm{d}r - \int_t^T \int_s^T \mathrm{d}\partial M_u^r \mathrm{d}r.$$

Fix $s \in [0,T]$ and let $(\Pi^\ell)_\ell$ be a properly chosen sequence of partitions of $[s,T]$, as in van Neerven [240, Theorem 1], $\Pi^\ell = (s_i)_{i=1,\dots,n_\ell}$ with $\|\Pi^\ell\| := \sup_i |s_{i+1} - s_i| \leq \ell$. To ease the notation, we set $\Delta s_i^\ell := s_i^\ell - s_{i-1}^\ell$, and, for a generic family process $(\phi^s)_{s\in[0,T]}$, and a mapping $s \longmapsto \partial_s \eta(s,x)$, we define

$$I^\ell(\phi) := \sum_{i=0}^{n_\ell} \Delta s_i^\ell \phi^{s_i^\ell},\ \delta\phi := \phi^T - \phi^s,\ I^\ell(\partial_s \eta(\cdot,x)) := \sum_{i=0}^{n} \Delta s_i^\ell \partial_s \eta(s_i^\ell, x),\ \delta\eta(x) := \eta(T,x) - \eta(s,x).$$

We then notice that for any $t \in [0,T]$

$$I^\ell(\partial U)_t - (\delta U)_t = I^\ell(\partial_s \eta(\cdot)) - \delta\eta + \int_t^T [I^\ell(\partial_z g_u(\cdot, U_u^\cdot, V_u^\cdot, \mathcal{Y}_u, \mathcal{Z}_u)\partial V_u^\cdot) - \delta g_u(\cdot, U_u^\cdot, V_u^\cdot, \mathcal{Y}_u, \mathcal{Z}_u)]\mathrm{d}u$$
$$+ \int_t^T [I^\ell(\partial_s g_u(\cdot, U_u^\cdot, V_u^\cdot, \mathcal{Y}_u, \mathcal{Z}_u)) + I^\ell(\partial_y g_u(\cdot, U_u^\cdot, V_u^\cdot, \mathcal{Y}_u, \mathcal{Z}_u)\partial U_u^\cdot)]\mathrm{d}u$$
$$- \int_t^T \big(I^\ell(\partial V)_r - (\delta V)_r\big)^\top \mathrm{d}X_u - I^\ell(\partial M_T^\cdot - \partial M_t^\cdot) - \delta(M_T - M_t).$$

We now note that the integrability of $(\partial U, \partial V)$ and (U,V) yields $I^\ell(\partial U) - (\delta U) \in \mathbb{S}^{2,2}$ and $I^\ell(\partial V) - (\delta V) \in \mathbb{H}^{2,2}$. Therefore, Bouchard, Possamaï, Tan, and Zhou [40, Theorem 2.2] yields

$$\|I^\ell(\partial U) - (\delta U)\|_{\mathbb{S}^{2,2}}^2 + \|I^\ell(\partial V) - (\delta V)\|_{\mathbb{H}^{2,2}}^2 + \|I^\ell(\partial M) - (\delta M)\|_{\mathbb{M}^{2,2}}^2$$
$$\leq C\mathbb{E}\bigg[\big|I^\ell(\partial_s \eta(\cdot)) - \delta\eta\big|^2 + \bigg(\int_t^T \Big|I^\ell(\partial_s g_u(\cdot, U_u^\cdot, V_u^\cdot, \mathcal{Y}_u, \mathcal{Z}_u)) - \delta g_u(\cdot, U_u^\cdot, V_u^\cdot, \mathcal{Y}_u, \mathcal{Z}_u)\Big|\mathrm{d}u\bigg)^2\bigg].$$

To conclude, we first note that by our choice of $(\Pi^\ell)_\ell$, $I^\ell(\partial U)$ converges to the Lebesgue integral of ∂U^s. In addition, the uniform continuity of $s \longmapsto \partial_s \eta(s,x)$ and $s \longmapsto \partial_s g(s,x,u,v,y,z)$, see Assumption G.$(i)$, justifies, via bounded convergence, the convergence in $\mathbb{S}^{2,2}$ (resp. $\mathbb{H}^{2,2}$) of $I^\ell(\partial U^s)$ to $U^T - U^s$ (resp. $I^\ell(\partial V^s)$ to $V^T - V^s$) as $\ell \longrightarrow 0$. The result follows in virtue of the uniqueness of (U, V, M). $\square$

The next lemma identifies the dynamics of $(U_t^t)_{t\in[0,T]}$.

Lemma 3.6.2. *Let* $(\mathcal{Y}, \mathcal{Z}) \in \mathbb{L}^2 \times \mathbb{H}^2$ *and* $(U, \partial U, V, \partial V, M, \partial M) \in (\mathbb{L}^{2,2})^2 \times \overline{\mathbb{H}}^{2,2} \times \mathbb{H}^{2,2} \times (\mathbb{M}^{2,2})^2$ *satisfy for any* $s \in [0,T]$, $\mathbb{P}$–a.s. *for any* $t \in [0,T]$

$$
\begin{aligned}
U_t^s &= \eta(s, X_{\cdot\wedge T}) + \int_t^T g_r(s, X, U_r^s, V_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}M_r^s,\\
\partial U_t^s &= \partial_s\eta(s, X_{\cdot\wedge T}) + \int_t^T \nabla g_r(s, X, \partial U_r^s, \partial V_r^s, U_r^s, V_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s.
\end{aligned}
$$

Then

$$
U_t^t = U_T^T + \int_t^T \big(g_r(r, U_r^r, V_r^r, \mathcal{Y}_r, \mathcal{Z}_r) - \partial U_r^r\big)\mathrm{d}r - \int_t^T V_r^r\mathrm{d}X_r - \int_t^T \mathrm{d}\widetilde{M}_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},
$$

where $\widetilde{M} := (\widetilde{M}_t)_{t\in[0,T]}$ *is given by* $\widetilde{M}_t := M_t^t - \int_0^t \partial M_r^r \mathrm{d}r$. *Moreover,* $\widetilde{M} \in \mathbb{M}^2$.

Proof. We show that $\mathbb{P}$–a.s., for any $t \in [0,T]$

$$
\int_t^T \partial U_r^r\mathrm{d}r = U_T^T - U_t^t + \int_t^T g_r(r, U_r^r, V_r^r, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T V_r^r\mathrm{d}X_r - \int_t^T \mathrm{d}\widetilde{M}_r.
$$

Indeed, note that, $\mathbb{P}$–a.s., for any $t \in [0,T]$

$$
\begin{aligned}
\int_t^T \partial U_r^r\mathrm{d}r = \int_t^T \partial_s\eta(r)\mathrm{d}r + \int_t^T \Big(\int_r^T \nabla g_u(r, \partial U_u^r, \partial V_u^r, U_u^r, V_u^r, \mathcal{Y}_u, \mathcal{Z}_u)\mathrm{d}u\Big)\mathrm{d}r\\
- \int_t^T \Big(\int_r^T \partial V_u^{r\top}\mathrm{d}X_u - \int_r^T \mathrm{d}\partial M_u^r\Big)\mathrm{d}r.
\end{aligned}
$$

By Fubini's theorem, the change of variables formula for the Lebesgue integral, [212, Theorem 54], and Lemma 3.6.1 we have that, $\mathbb{P}$–a.s., for any $t \in [0,T]$

$$
\begin{aligned}
&\int_t^T \int_r^T \nabla g_u(r, \partial U_u^r, \partial V_u^r, U_u^r, V_u^r, \mathcal{Y}_u, \mathcal{Z}_u)\mathrm{d}u\mathrm{d}r\\
&= \int_t^T \int_t^u \nabla g_u(r, \partial U_u^r, \partial V_u^r, U_u^r, V_u^r, \mathcal{Y}_u, \mathcal{Z}_u)\mathrm{d}r\mathrm{d}u\\
&= \int_t^T g_u(u, U_u^u, V_u^u, \mathcal{Y}_u, \mathcal{Z}_u) - g_u(t, U_u^t, V_u^t, \mathcal{Y}_u, \mathcal{Z}_u)\mathrm{d}u.
\end{aligned}
$$

Similarly, given $\partial V \in \mathbb{H}^{2,2}$, the version of Fubini's theorem for stochastic integration, see [212, Theorem 65], yields

$$\int_t^T \int_r^T \partial V_u^{r\top} \mathrm{d}X_u \mathrm{d}r = \int_t^T \int_t^u \partial V_u^{r\top} \mathrm{d}r \mathrm{d}X_u = \int_t^T (V_u^u - V_u^t)^\top \mathrm{d}X_u.$$

Now, by Lemma 3.6.1 we have that, $\mathbb{P}$–a.s., for any $t \in [0,T]$

$$\int_t^T \int_r^T \mathrm{d}\partial M_u^r \mathrm{d}r = \int_t^T (\partial M_T^r - \partial M_r^r)\mathrm{d}r = \widetilde{M}_T - \widetilde{M}_t + M_t^t - M_T^t.$$

We are left to verify $\widetilde{M} \in \mathbb{M}^2$. Indeed, note that $\widetilde{M}_0 = 0$ and

(i) $\widetilde{M}$ has càdlàg paths. This follows from the fact that, as $\mathbb{F}$ satisfies the usual conditions, there exist a càdlàg modification of $(M_t^t)_{t\in[0,T]}$, which, abusing notations, we still denote by $(M_t^t)_{t\in[0,T]}$. Indeed for $t \in [0,T]$

$$M_t^t = M_t^T - \int_t^T \partial M_t^r \mathrm{d}r, \text{ } \mathbb{P}\text{–a.s.}$$

(ii) $\widetilde{M}$ is a martingale. Indeed by Lemma 3.6.1, for $0 \leq u \leq t \leq T$, $\mathbb{P}$–a.s.

$$\mathbb{E}[\widetilde{M}_t|\mathcal{F}_u] = \mathbb{E}[M_t^t|\mathcal{F}_u] - \int_0^u \partial M_r^r \mathrm{d}r - \int_u^t \mathbb{E}[\partial M_r^r|\mathcal{F}_u]\mathrm{d}r = M_u^t - \int_0^u \partial M_r^r \mathrm{d}r - \int_u^t \partial M_u^r \mathrm{d}r = \widetilde{M}_u;$$

(iii) $\widetilde{M}$ is orthogonal to X. For $0 \leq u \leq t \leq T$, $\mathbb{P}$–a.s.

$$\begin{aligned}
\mathbb{E}[X_t\widetilde{M}_t|\mathcal{F}_u] &= \mathbb{E}[X_t M_t^t|\mathcal{F}_u] - \mathbb{E}\bigg[X_t \int_0^u \partial M_r^r \mathrm{d}r\bigg|\mathcal{F}_u\bigg] - \int_u^t \mathbb{E}[\mathbb{E}[X_t|\mathcal{F}_r]\partial M_r^r|\mathcal{F}_u]\mathrm{d}r \\
&= X_u M_u^t - X_u \int_0^u \partial M_r^r \mathrm{d}r - X_u \int_u^t \partial M_u^r \mathrm{d}r \\
&= X_u M_u^t - X_u \int_0^u \partial M_r^r \mathrm{d}r - X_u M_u^t + X_u M_u^u \\
&= X_u \widetilde{M}_u,
\end{aligned}$$

where in the second equality we used the fact $\int_0^u \partial M_r^r \mathrm{d}r$ is $\mathcal{F}_u$-measurable, the tower property and the orthogonality of M^s and X and of ∂M^s and X for $s \in [0,T]$. The third equality follows from Lemma 3.6.1.

(iv) $\|\widetilde{M}\|_{\mathbb{M}^2} < \infty$,

$$\begin{aligned}\|\widetilde{M}\|^2_{\mathbb{M}^2} &= \mathbb{E}\bigg[\bigg|M^T_T - \int_0^T \partial M^r_r \mathrm{d}r\bigg|^2\bigg] \\ &\leq 2\bigg(\mathbb{E}\big[[M^T]_T\big] + T\int_0^T \mathbb{E}\big[[\partial M^r]_r\big]\mathrm{d}r\bigg) \leq 2\Big[\|M^T\|^2_{\mathbb{M}^2} + T^2\|\partial M\|^2_{\mathbb{M}^{2,2}}\Big] < \infty.\end{aligned}$$

$\square$

3.6.2 *A priori* estimates

We now establish *a priori* estimates for $(\mathcal{S})$. To ease the readability, recall it holds for any $s \in [0,T]$, holds $\mathbb{P}$–a.s. for any $t \in [0,T]$

$$\begin{aligned}\mathcal{Y}_t &= \xi(T, X_{\cdot\wedge T}) + \int_t^T h_r(X, \mathcal{Y}_r, \mathcal{Z}_r, U^r_r, V^r_r, \partial U^r_r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r - \int_t^T \mathrm{d}\mathcal{N}_r, \\ U^s_t &= \eta(s, X_{\cdot\wedge T}) + \int_t^T g_r(s, X, U^s_r, V^s_r, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T {V^s_r}^\top \mathrm{d}X_r - \int_t^T \mathrm{d}M^s_r \\ \partial U^s_t &= \partial_s\eta(s, X_{\cdot\wedge T}) + \int_t^T \nabla g_r(s, X, \partial U^s_r, \partial V^s_r, U^s_r, V^s_r, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial {V^s_r}^\top \mathrm{d}X_r - \int_t^T \mathrm{d}\partial M^s_r.\end{aligned} \tag{$\mathcal{S}$}$$

Let us introduce $(\mathcal{H}^o, \|\cdot\|_{\mathcal{H}^o})$ and $(\mathfrak{H}^o, \|\cdot\|_{\mathfrak{H}^o})$ where $\|\cdot\|_{\mathcal{H}^o}$ and $\|\cdot\|_{\mathfrak{H}^o}$ denote the norms induced by

$$\begin{aligned}\mathcal{H}^o &:= \mathbb{L}^2 \times \mathbb{H}^2 \times \mathbb{M}^2 \times \mathbb{L}^{2,2} \times \overline{\mathbb{H}}^{2,2} \times \mathbb{M}^{2,2} \times \mathbb{L}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{M}^{2,2}, \\ \mathfrak{H}^o &:= \mathbb{L}^2 \times \mathbb{H}^2 \times \mathbb{M}^2 \times \big(\mathbb{L}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{M}^{2,2}\big)^2.\end{aligned}$$

To obtain estimates between the difference of solutions, it is more convenient to work with norms defined by adding exponential weights. We recall, for instance, that for $c \in [0,\infty)$ the norm $\|\cdot\|_{\mathbb{H}^{2,c}}$ is given by

$$\|\mathcal{Z}\|^2_{\mathbb{H}^{2,c}} = \mathbb{E}\bigg[\int_0^T \mathrm{e}^{ct}|\sigma_t^\top \mathcal{Z}_t|^2 \mathrm{d}t\bigg],$$

and they are equivalent for any two values of c, since $[0,T]$ is compact. With this, we define the norm $\|\cdot\|_{\mathcal{H}^{o,c}}$. In the following, we take the customary approach of introducing arbitrary constants $C > 0$ to our analysis. These constants will typically depend on the data of the problem, e.g. the Lipschitz constants and T and on the value of c unless otherwise stated.

Proposition 3.6.3. *Let* Assumption G *hold and* $(\mathcal{Y},\mathcal{Z},\mathcal{N},U,V,M,\partial U,\partial V,\partial M)\in\mathcal{H}^o$ *satisfy* $(\mathcal{S})$*. Then* $(\mathcal{Y},U,\partial U)\in\mathbb{S}^2\times\mathbb{S}^{2,2}\times\mathbb{S}^{2,2}$*. Furthermore, there exists a constant* $C>0$ *such that for* $\|\cdot\|_{\mathcal{H}}$ *as in* Section 3.3

$$\|\mathfrak{h}\|^2_{\mathcal{H}}\leq C\underbrace{\Big(\|\xi\|^2_{\mathcal{L}^2}+\|\eta\|^2_{\mathcal{L}^{2,2}}+\|\partial_s\eta\|^2_{\mathcal{L}^{2,2}}+\|\tilde h\|^2_{\mathbb{L}^{1,2}}+\|\tilde g\|^2_{\mathbb{L}^{1,2,2}}+\|\nabla\tilde g\|^2_{\mathbb{L}^{1,2,2}}\Big)}_{=:I_0^2}<\infty.$$

Proof. We proceed in several steps. We recall that in light of Assumption G, $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$–a.e.

$$\begin{aligned}|h_r(\mathcal{Y}_r,\mathcal{Z}_r,U_r^r,V_r^r,\partial U_r^r))|&\leq|\tilde h|+L_h\big(|\mathcal{Y}_r|+|\sigma_r^\top\mathcal{Z}_r|+|U_r^r|+|\sigma_r^\top V_r^r|+|\partial U_r^r|\big),\\ |g_r(s,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r))|&\leq|\tilde g(s)|+L_g\big(|U_r^s|+|\sigma_r^\top V_r^s|+|\mathcal{Y}_r|+|\sigma_r^\top\mathcal{Z}_r|\big),\\ |\nabla g_r(s,\partial U_r^s,\partial V_r^s,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r))|&\leq|\nabla\tilde g_r(s)|+L_{\partial_s g}\big(|U_r^s|+|\sigma_r^\top V_r^s|+|\mathcal{Y}_r|+|\sigma_r^\top\mathcal{Z}_r|\big)\\ &\quad+L_g\big(|\partial U_r^s|+|\sigma_r^\top\partial V_r^s|\big).\end{aligned}\tag{3.6.6}$$

Step 1: we start with auxiliary estimates. By Meyer–Itô's formula for $\mathrm{e}^{\frac{c}{2}t}|\partial U_t^s|$, see Protter [212, Theorem 70]

$$\begin{aligned}&\mathrm{e}^{\frac{c}{2}t}|\partial U_t^s|+L_T^0-\int_t^T\mathrm{e}^{\frac{c}{2}r}\,\mathrm{sgn}(\partial U_r^s)\cdot\partial V_r^{s\top}\mathrm{d}X_r-\int_t^T\mathrm{e}^{\frac{c}{2}r-}\,\mathrm{sgn}(\partial U_{r-}^s)\cdot\mathrm{d}\partial M_r^s\\ &=\mathrm{e}^{\frac{c}{2}T}|\partial_s\eta(s)|+\int_t^T\mathrm{e}^{\frac{c}{2}r}\Big(\mathrm{sgn}(\partial U_r^s)\cdot\nabla g_r(s,\partial U_r^s,\partial V_r^s,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r)-\frac{c}{2}|\partial U_r^s|\Big)\mathrm{d}r,\end{aligned}\tag{3.6.7}$$

where $L^0:=L^0(\partial U^s)$ denotes the non-decreasing and pathwise-continuous local time of the semi-martingale ∂U^s at 0, see [212, Chapter IV, pp. 216]. We also notice that for any $s\in[0,T]$ the last two terms on the left-hand side are martingales, recall that $\partial V^s\in\mathbb{H}^2$.

We now take conditional expectation with respect to $\mathcal{F}_t$ in Equation (3.6.7). We may use (3.6.6) and the fact L^0 is non-decreasing to derive that for $t\in[0,T]$ and $c>2L_g$

$$\begin{aligned}\mathrm{e}^{\frac{c}{2}t}|\partial U_t^s|\leq\mathbb{E}_t\Big[&\mathrm{e}^{\frac{c}{2}T}|\partial_s\eta(s)|+\int_t^T\mathrm{e}^{\frac{c}{2}r}\Big(|\partial U_r^s|(L_g-c/2)+|\nabla\tilde g_r(s)|+L_g|\sigma_r^\top\partial V_r^s|\Big)\mathrm{d}r\\ &+\int_t^T\mathrm{e}^{\frac{c}{2}r}L_{\partial_s g}\big(|U_r^s|+|\sigma_r^\top V_r^s|+|\mathcal{Y}_r|+|\sigma_r^\top\mathcal{Z}_r|\big)\mathrm{d}r\Big]\end{aligned}$$

$$\leq \mathbb{E}_t\bigg[\mathrm{e}^{\frac{c}{2}T}|\partial_s\eta(s)| + \int_t^T \mathrm{e}^{\frac{c}{2}r}\Big(|\nabla\tilde{g}_r(s)| + \bar{L}\big(|\sigma_r^\top\partial V_r^s| + |U_r^s| + |\sigma_r^\top V_r^s| + |\mathcal{Y}_r| + |\sigma_r^\top\mathcal{Z}_r|\big)\Big)\mathrm{d}r\bigg]. \tag{3.6.8}$$

where $\bar{L} := \max\{L_g, L_{\partial_s g}\}$. Squaring in (3.6.8), we may use (I.1) and Jensen's inequality to derive for $t \in [0,T]$

$$\frac{\mathrm{e}^{ct}}{7}|\partial U_t^t|^2 \leq \mathbb{E}_t\bigg[\mathrm{e}^{cT}|\partial_s\eta(t)|^2 + T\bar{L}^2\int_t^T \mathrm{e}^{cr}\big(|U_r^t|^2 + |\mathcal{Y}_r|^2 + |\sigma_r^\top\partial V_r^t|^2 + |\sigma_r^\top V_r^t|^2 + |\sigma_r^\top\mathcal{Z}_r|^2\big)\mathrm{d}r\bigg] + \mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}|\nabla\tilde{g}_r(t)|\mathrm{d}r\bigg)^2\bigg].$$

Integrating the previous expression and taking expectation, it follows from the tower property that for any $t \in [0,T]$

$$\begin{aligned}
\frac{1}{7}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}|\partial U_r^r|^2\mathrm{d}r\bigg] &\leq \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cT}|\partial_s\eta(r)|^2\mathrm{d}r\bigg] + \mathbb{E}\bigg[\int_t^T\bigg(\int_r^T \mathrm{e}^{\frac{c}{2}u}|\nabla\tilde{g}_u(r)|\mathrm{d}u\bigg)^2\mathrm{d}r\bigg]\\
&\quad + T\bar{L}^2\mathbb{E}\bigg[\int_t^T\int_r^T \mathrm{e}^{cu}\big(|U_u^r|^2 + |\mathcal{Y}_u|^2 + |\sigma_u^\top\partial V_u^r|^2 + |\sigma_u^\top V_u^r|^2 + |\sigma_u^\top\mathcal{Z}_u|^2\big)\mathrm{d}u\mathrm{d}r\bigg]\\
&\leq T\sup_{r\in[0,T]}\bigg\{\|\mathrm{e}^{cT}|\partial_s\eta(r)|^2]\|_{\mathcal{L}^2} + \mathbb{E}\bigg[\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}u}|\nabla\tilde{g}_u(r)|\mathrm{d}u\bigg)^2\bigg]\bigg\}\\
&\quad + T^2\bar{L}^2\sup_{r\in[0,T]}\bigg\{\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}\big(|U_u^r|^2 + |\sigma_u^\top V_u^r|^2 + |\sigma_u^\top\partial V_u^r|^2\big)\mathrm{d}u\bigg]\bigg\}\\
&\quad + T^2\bar{L}^2\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}\big(|\mathcal{Y}_u|^2 + |\sigma_u^\top\mathcal{Z}_u|^2\big)\mathrm{d}u\bigg].
\end{aligned}$$

Thus, we obtain there is $C_{\partial u} > 0$ such that for any $c > 2L_g$ and $t \in [0,T]$

$$\frac{1}{C_{\partial u}}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}|\partial U_r^r|^2\mathrm{d}r\bigg] \leq \mathrm{e}^{cT}\big(\|\partial_s\eta\|^2_{\mathcal{L}^{2,2}} + \|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,2,2}}\big) + \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|\mathcal{Y}_r|^2 + |\sigma_r^\top\mathcal{Z}_r|^2\big)\mathrm{d}r\bigg] + \sup_{s\in[0,T]}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|U_r^s|^2 + |\sigma_r^\top V_r^s|^2 + |\sigma_r^\top\partial V_r^s|^2\big)\mathrm{d}r\bigg]. \tag{3.6.9}$$

Similarly, we may find $C_u > 0$ such that for any $c > 2L_g$ and $t \in [0,T]$

$$\frac{1}{C_u}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}|U_r^r|^2\mathrm{d}r\bigg] \leq \mathrm{e}^{cT}\big(\|\eta\|^2_{\mathcal{L}^{2,2}} + \|\tilde{g}\|^2_{\mathbb{L}^{1,2,2}}\big) + \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|\mathcal{Y}_r|^2 + |\sigma_r^\top\mathcal{Z}_r|^2\big)\mathrm{d}r\bigg] + \sup_{s\in[0,T]}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top V_r^s|^2\mathrm{d}r\bigg]. \tag{3.6.10}$$

We now estimate the term V_t^t. In light of (3.6.2) and (3.6.4), with $\mathrm{e}^{ct}\sigma_t^\top$ instead of $\sigma_t^\top$, there exists $C>0$ such that for any $\varepsilon>0$, $c>4L_g$ and $t\in[0,T]$

$$\begin{aligned}
&\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}|\sigma_u^\top V_u^u|^2\mathrm{d}u\bigg]\\
&\leq \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}|\sigma_u^\top V_u^t|^2\mathrm{d}u\bigg]+\int_t^T \mathbb{E}\bigg[\int_r^T \mathrm{e}^{cu}\big(\varepsilon|\sigma_u^\top V_u^r|^2+\varepsilon^{-1}|\sigma_u^\top \partial V_u^r|^2\big)\mathrm{d}u\bigg]\mathrm{d}r\\
&\leq \sup_{r\in[0,T]}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}|\sigma_u^\top V_u^r|^2\mathrm{d}u\bigg]+T\sup_{r\in[0,T]}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}\big(\varepsilon|\sigma_u^\top V_u^r|^2+|\sigma_u^\top \partial V_u^r|^2/\varepsilon\big)\mathrm{d}u\bigg]\\
&\leq (1+\varepsilon T)\sup_{r\in[0,T]}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}|\sigma_u^\top V_u^r|^2\mathrm{d}u\bigg]+TC/\varepsilon\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}\big(|\mathcal{Y}_u|^2+|\sigma_u^\top \mathcal{Z}_u|^2\big)\mathrm{d}u\bigg]\\
&\quad +TC\varepsilon^{-1}\sup_{r\in[0,T]}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}\big(|U_u^r|^2+|\sigma_u^\top V_u^r|^2\big)\mathrm{d}u\bigg]\\
&\quad +TC\varepsilon^{-1}\mathrm{e}^{cT}\sup_{r\in[0,T]}\mathbb{E}\bigg[\big|\partial_s\eta(r)\big|^2+\bigg(\int_t^T|\nabla\tilde g_u(r)|\mathrm{d}u\bigg)^2\bigg].
\end{aligned}$$

Thus, taking $\varepsilon=TC$ we may find $C_v>0$ such that for any $c>4L_g$ and $t\in[0,T]$

$$\begin{aligned}
\frac{1}{C_v}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cu}|\sigma_r^\top V_r^r|^2\mathrm{d}r\bigg]\leq \mathrm{e}^{cT}\big(\|\partial_s\eta\|^2_{\mathcal{L}^{2,2}}+\|\nabla\tilde g\|^2_{\mathbb{L}^{1,2,2}}\big)+\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|\mathcal{Y}_r|^2+|\sigma_r^\top \mathcal{Z}_r|^2\big)\mathrm{d}r\bigg]\\
+\sup_{s\in[0,T]}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|U_r^s|^2+|\sigma_r^\top V_r^s|^2\big)\mathrm{d}r\bigg].
\end{aligned}\tag{3.6.11}$$

We emphasise that the constants $(C_{\partial u},C_u,C_v)\in(0,\infty)^3$ depend only of the data of the problem and are universal for any value of $c>4L_g$.

Step 2: Let $s\in[0,T]$, we show that $(\mathcal{Y},U,\partial U)\in\mathbb{S}^2\times\mathbb{S}^{2,2}\times\mathbb{S}^{2,2}$. To alleviate the notation we introduce

$$\mathfrak{Y}:=(\mathcal{Y},U^s,\partial U^s),\ \mathfrak{Z}:=(\mathcal{Z},V^s,\partial V^s),\ \mathfrak{N}:=(\mathcal{N},M^s,\partial M^s),$$

whose elements we may denote with superscripts, e.g. $\mathcal{Y}^1$, $\mathcal{Y}^2$, $\mathcal{Y}^3$ correspond to $\mathcal{Y}$, U^s, ∂U^s.

By (I.1) and (3.6.6), we obtain that there exists $C>0$, which may change value from line to line, such that

$$\begin{aligned}
|U_t^s|^2\leq C\bigg(|\eta(s)|^2+\bigg(\int_0^T|\tilde g_r(s)|\mathrm{d}r\bigg)^2+\int_0^T\big(|U_r^s|^2+|\sigma_r^\top V_r^s|^2+|\mathcal{Y}_r|^2+|\sigma_r^\top\mathcal{Z}_r|^2\big)\mathrm{d}r\\
+\bigg|\int_t^T V_r^{s\top}\mathrm{d}X_r\bigg|^2+\bigg|\int_t^T\mathrm{d}M_r^s\bigg|^2\bigg).
\end{aligned}$$

We note that by Doob's inequality

$$\mathbb{E}\bigg[\sup_{t\in[0,T]}\bigg|\int_0^t V_r^{s\top}\mathrm{d}X_r\bigg|^2\bigg] \leq 4\|V^s\|_{\mathbb{H}^2}^2. \tag{3.6.12}$$

Taking supremum over $t \in [0,T]$ and expectation we obtain for $s \in [0,T]$

$$\begin{aligned} \|U^s\|_{\mathbb{S}^2}^2 \leq C\big(&\|\eta(s)\|_{\mathcal{L}^2}^2 + \|\tilde{g}(s)\|_{\mathbb{L}^{1,2}}^2 \\ &+ \|U^s\|_{\mathbb{L}^2}^2 + \|V^s\|_{\mathbb{H}^2}^2 + \|\mathcal{Y}\|_{\mathbb{L}^2}^2 + \|\mathcal{Z}\|_{\mathbb{H}^2}^2 + \|M^s\|_{\mathbb{M}^2}^2\big) < \infty. \end{aligned} \tag{3.6.13}$$

Similarly, we obtain that there exists $C > 0$ such that for $s \in [0,T]$

$$\|\partial U^s\|_{\mathbb{S}^2}^2 \leq C\bigg(\|\partial_s\eta(s)\|_{\mathcal{L}^2}^2 + \|\nabla\tilde{g}(s)\|_{\mathbb{L}^{1,2}}^2 + \sum_{i=1}^3 \|\mathfrak{Y}^i\|_{\mathbb{L}^2}^2 + \|\mathfrak{Z}^i\|_{\mathbb{H}^2}^2 + \|\partial M^s\|_{\mathbb{M}^2}^2\bigg) < \infty. \tag{3.6.14}$$

Given $(\eta, \partial_s\eta, \tilde{g}, \partial\tilde{g}) \in (\mathcal{L}^{2,2})^2 \times (\mathbb{L}^{1,2,2})^2$, $(U, \partial U, V, \partial V, M, \partial M) \in (\mathbb{L}^{2,2})^2 \times (\mathbb{H}^{2,2})^2 \times (\mathbb{M}^{2,2})^2$, (3.6.13) and (3.6.14), the mapping

$$([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{S}^2, \|\cdot\|_{\mathbb{S}^2}) : s \longmapsto \mathfrak{Y}^{is} \text{ is continuous for } i \in \{2,3\},$$

and $\|\mathfrak{Y}^i\|_{\mathbb{S}^{2,2}} < \infty$. Consequently, $\mathfrak{Y}^i \in \mathbb{S}^{2,2}$ for $i \in \{2,3\}$. Arguing as above we may also derive,

$$\begin{aligned} |\mathcal{Y}_t|^2 \leq C\bigg(&|\xi|^2 + \bigg(\int_0^T |\tilde{h}_r|\mathrm{d}r\bigg)^2 + \int_0^T \big(|\mathcal{Y}_r|^2 + |\sigma_r^\top \mathcal{Z}_r|^2 + |U_r^r|^2 + |\sigma_r^\top V_r^r|^2 + |\partial U_r^r|^2\big)\mathrm{d}r \\ &+ \bigg|\int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r\bigg|^2 + \bigg|\int_t^T \mathrm{d}N_r\bigg|^2\bigg), \end{aligned}$$

which in turn yields, in combination with (3.6.9), (3.6.10) and (3.6.11),

$$\|\mathcal{Y}\|_{\mathbb{S}^2}^2 \leq C\big(I_0^2 + \|\mathcal{Y}\|_{\mathbb{L}^2}^2 + \|\mathcal{Z}\|_{\mathbb{H}^2}^2 + \|U\|_{\mathbb{L}^{2,2}}^2 + \|V\|_{\mathbb{H}^{2,2}}^2 + \|\partial V\|_{\mathbb{H}^{2,2}}^2 + \|N\|_{\mathbb{M}^2}^2\big) < \infty. \tag{3.6.15}$$

Finally, taking sup over $s \in [0,T]$ and adding together (3.6.15) (3.6.13) and (3.6.14) we obtain

$$\|\mathcal{Y}\|_{\mathbb{S}^2}^2 + \|U^s\|_{\mathbb{S}^{2,2}}^2 + \|\partial U\|_{\mathbb{S}^{2,2}}^2 \leq C\big(I_0^2 + \|(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, U, V, M, \partial U, \partial V, \partial M)\|_{\mathcal{H}^o}\big). \tag{3.6.16}$$

Step 3: We obtain the estimate of the norm. To ease the notation we introduce

$$h_r := h_r(\mathcal{Y}_r, \mathcal{Z}_r, U_r^r, V_r^r, \partial U_r^r),\ g_r(s) := g_r(s, U_r^s, V_r^s, \mathcal{Y}_r, \mathcal{Z}_r),$$
$$\nabla g_r(s) := \nabla g_r(s, \partial U_r^s, \partial V_r^s, U_r^s, V_r^s, \mathcal{Y}_r, \mathcal{Z}_r).$$

By applying Itô's formula to $\mathrm{e}^{ct}(|\mathcal{Y}_t|^2 + |U_t^s|^2 + |\partial U_t^s|^2)$ we obtain, $\mathbb{P}$–a.s.

$$\begin{aligned}&\sum_{i=1}^{3} \mathrm{e}^{ct}|\mathfrak{Y}_t^i|^2 + \int_t^T \mathrm{e}^{cr}|\sigma_r^\top \mathfrak{Z}_r^i|^2 \mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\big[\mathfrak{N}^i\big]_r + \mathfrak{M}_t - \mathfrak{M}_T\\ &= \mathrm{e}^{cT}(|\xi(T)|^2 + |\eta(s)|^2 + |\partial_s\eta(s)|^2)\\ &\quad + \int_t^T \mathrm{e}^{cr}(2\mathcal{Y}_r\cdot h_r + 2U_r^s\cdot g_r(s) + 2\partial U_r^s\cdot\nabla g_r(s) - c(|\mathcal{Y}_r|^2 + |U_r^s|^2 + |\partial U_r^s|^2))\mathrm{d}r,\end{aligned}$$

where we used the orthogonality of X and both M^s and N, and we introduced the martingale

$$\mathfrak{M}_t := 2\sum_{i=1}^{3}\int_0^t \mathrm{e}^{cr}\mathfrak{Y}_r^i\cdot{\mathfrak{Z}_r^i}^\top\,\mathrm{d}X_r + \int_0^t \mathrm{e}^{cr-}\mathfrak{Y}_{r-}^i\cdot\mathrm{d}\,\mathfrak{N}_r^i.$$

Indeed, the Burkholder–Davis–Gundy inequality together with the fact that $(\mathcal{Y}, U^s, \partial U^s) \in (\mathbb{S}^2)^3$ shows that $\mathfrak{M}$ is uniformly integrable, and consequently a true martingale. Moreover, we insist on the fact that the integrals with respect to N, M^s and ∂M^s account for possible jumps, see [142, Lemma 4.24].

Moreover, as $(\mathcal{Y}, U, \partial U) \in \mathbb{S}^2\times(\mathbb{S}^{2,2})^2$, from (3.6.6) and with Young's inequality we obtain that for any $(\varepsilon, \tilde\varepsilon) \in (0,\infty)^2$, there is $C(\tilde\varepsilon)\in(0,\infty)$ such that the left-hand side above is smaller than

$$\begin{aligned}&\leq \mathrm{e}^{cT}(|\xi|^2 + |\eta(s)|^2 + |\partial_s\eta(s)|^2) + \int_t^T \mathrm{e}^{cr}(C(\tilde\varepsilon) - c)(|\mathcal{Y}_r|^2 + |U_r^s|^2 + |\partial U_r^s|^2)\mathrm{d}r\\ &\quad + \int_t^T \tilde\varepsilon\mathrm{e}^{cr}\bigg(|\sigma_r^\top\mathcal{Z}_r|^2 + |\sigma_r^\top V_r^s|^2 + |\sigma_r^\top\partial V_r^s|^2 + \frac{1}{C_u}|U_r^r|^2 + \frac{1}{C_v}|V_r^r|^2 + \frac{1}{C_{\partial u}}|\partial U_r^r|^2\bigg)\mathrm{d}r\\ &\quad + \varepsilon\sum_{i=1}^{3}\sup_{r\in[0,T]}\mathrm{e}^{cr}|\mathfrak{Y}_t^i|^2 + \frac{1}{\varepsilon}\bigg(\int_0^T \mathrm{e}^{\frac{c}{2}r}|\tilde h_r|\mathrm{d}r\bigg)^2 + \frac{1}{\varepsilon}\bigg(\int_0^T \mathrm{e}^{\frac{c}{2}r}|\tilde g_r(s)|\mathrm{d}r\bigg)^2 + \frac{1}{\varepsilon}\bigg(\int_0^T \mathrm{e}^{\frac{c}{2}r}|\nabla\tilde g_r(s)|\mathrm{d}r\bigg)^2,\end{aligned}$$

with $(C_{\partial u}, C_u, C_v)$ as in (3.6.9)-(3.6.11). Taking expectation and letting $c > 4L_g$, we find there is $C > 0$ such that

$$\mathbb{E}\Big[\sum_{i=1}^{3} \mathrm{e}^{ct}|\mathfrak{Y}_t^i|^2 + \int_t^T \mathrm{e}^{cr}|\sigma_r^\top \mathfrak{Z}_r^i|^2 \mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}[\mathfrak{N}^i]_r\Big]$$
$$\leq \mathbb{E}\Big[\int_t^T \mathrm{e}^{cr}|\mathcal{Y}_r|^2(C(\tilde\varepsilon)-c)\mathrm{d}r\Big] + \sup_{s\in[0,T]} \mathbb{E}\Big[\int_t^T \mathrm{e}^{cr}|U_r^s|^2(C(\tilde\varepsilon)-c)\mathrm{d}r\Big]$$
$$+ \sup_{s\in[0,T]} \mathbb{E}\Big[\int_t^T \mathrm{e}^{cr}|\partial U_r^s|^2(C(\tilde\varepsilon)-c)\mathrm{d}r\Big] + \varepsilon C\big(\|\mathcal{Y}\|^2_{\mathbb{S}^{2,c}} + \|U\|^2_{\mathbb{S}^{2,2,c}} + \|\partial U\|^2_{\mathbb{S}^{2,2,c}}\big)$$
$$+ (1+\varepsilon^{-1}+\tilde\varepsilon)CI_0^2 + \tilde\varepsilon C\Big(\|\mathcal{Z}\|^2_{\mathbb{H}^{2,c}} + \|V\|^2_{\mathbb{H}^{2,2,c}} + \|\partial V\|^2_{\mathbb{H}^{2,2,c}}\Big).$$

We then let $\tilde\varepsilon = 1/(2^4C)$, $c \geq \max\{4L_g, C(\tilde\varepsilon)\}$, and take sup over $t\in[0,T]$ (resp. $(s,t)\in[0,T]^2$) to each term on the left side separately. Adding these terms up we find there is $C>0$, such that for any $\varepsilon > 0$

$$\begin{aligned}\frac{1}{T}\|\mathfrak{h}\|_{\mathfrak{H}^o} \leq &\sup_{t\in[0,T]} \mathbb{E}\big[\mathrm{e}^{ct}|\mathcal{Y}_t|^2\big] + \sup_{(s,t)\in[0,T]^2} \mathbb{E}\big[\mathrm{e}^{ct}|U_t^s|^2\big] + \sup_{(s,t)\in[0,T]^2} \mathbb{E}\big[\mathrm{e}^{ct}|\partial U_t^s|^2\big] \\ &+ \|\mathcal{Z}\|^2_{\mathbb{H}^2} + \|V\|^2_{\mathbb{H}^{2,2}} + \|\partial V\|^2_{\mathbb{H}^{2,2}} + \|\mathcal{N}\|^2_{\mathbb{M}^2} + \|M\|^2_{\mathbb{M}^{2,2}} + \|\partial M\|^2_{\mathbb{M}^{2,2}} \\ &\leq (1+\varepsilon^{-1})CI_0^2 + \varepsilon C\big(\|\mathcal{Y}\|^2_{\mathbb{S}^2} + \|U\|^2_{\mathbb{S}^{2,2}} + \|\partial U\|^2_{\mathbb{S}^{2,2}}\big).\end{aligned} \tag{3.6.17}$$

We can the use (3.6.17) back in (3.6.16) to find $\varepsilon\in(0,\infty)$ small enough so that[4] $\|\mathfrak{h}\|^2_{\mathfrak{H}} \leq CI_0^2$. The result in terms of the norm $\|\cdot\|_{\mathcal{H}}$ follows from (3.6.11). □

Proposition 3.6.4. *Let* $(\xi^i,\eta^i,\partial_s\eta^i)\in\mathcal{L}^2\times(\mathcal{L}^{2,2})^2$ *and* $(h^i,g^i,\partial_s g^i)$ *for* $i\in\{1,2\}$ *satisfy* Assumption G *and suppose that* $\mathfrak{h}^i\in\mathcal{H}^o$ *is a solution to* $(\mathcal{S})$ *with coefficients* $(\xi^i,h^i,\eta^i,g^i,\partial_s\eta^i,\nabla g^i)$, $i\in\{1,2\}$. *Then*

$$\|\delta\mathfrak{h}\|^2_{\mathcal{H}} \leq C\Big(\|\delta\xi\|^2_{\mathcal{L}^2} + \|\delta\eta\|^2_{\mathcal{L}^{2,2}} + \|\delta\partial\eta\|^2_{\mathcal{L}^{2,2}} + \|\delta_1 h\|^2_{\mathbb{L}^{1,2}} + \|\delta_1 g\|^2_{\mathbb{L}^{1,2,2}} + \|\delta_1\nabla g\|_{\mathbb{L}^{1,2,2}}\Big),$$

where for $\varphi\in\{\mathcal{Y},\mathcal{Z},\mathcal{N},U,V,M,\partial U,\partial V,\partial M,\xi,\eta,\partial_s\eta\}$ *and* $\Phi\in\{h,g,\nabla g\}$

$\delta\varphi := \varphi^1-\varphi^2$, and, $\delta_1\Phi_t := \Phi_t^1(\mathcal{Y}_r^1,\mathcal{Z}_t^1,U_t^{1t},V_t^{1t}) - \Phi_t^2(\mathcal{Y}_r^1,\mathcal{Z}_t^1,U_t^{1t},V_t^{1t})$, $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$–a.e. *in* $[0,T]\times\mathcal{X}$.

Proof. Note that by the Lipschitz assumption on h and g there exist bounded processes with

[4] Recall the norm $\|\cdot\|_{\mathfrak{H}}$ is the norm induced by the space $\mathfrak{H}$ as defined in Remark 3.3.3.(ii)

appropriate dimensions $(\alpha^i,\beta^i,\gamma^i,\varepsilon^i)$, $i\in\{1,2,3\}$, ρ and ϱ such that for every $s\in[0,T]$, $\mathbb{P}$–a.s., $t\in[0,T]$

$$\begin{aligned}
\delta\mathcal{Y}_t &= \delta\xi(T) - \int_t^T \delta\mathcal{Z}_r^\top \mathrm{d}X_r - \int_t^T \mathrm{d}\delta N_r,\\
&\quad + \int_t^T \big(\delta_1 h_r + \gamma_r^1\delta\mathcal{Y}_r + \alpha_r^{1\top}\sigma_r^\top\delta\mathcal{Z}_r + \beta_r^1\delta U_r^r + \varepsilon_r^{1\top}\sigma_r^\top\delta V_r^r\big)\mathrm{d}r\\
\delta U_t^s &= \delta\eta(s) - \int_t^T \delta V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\delta M_r^s\\
&\quad + \int_t^T \big(\delta_1 g_r(s) + \beta_r^2\delta U_r^s + \varepsilon_r^{2\top}\sigma_r^\top\delta V_r^s + \gamma_r^2\delta\mathcal{Y}_r + \alpha_r^{2\top}\sigma_r^\top\delta\mathcal{Z}_r\big)\mathrm{d}r,\\
\delta\partial U_t^s &= \delta\partial_s\eta(s) - \int_t^T \delta\partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\delta\partial M_r^s\\
&\quad + \int_t^T \big(\delta_1\nabla g_r(s) + \rho_r\delta\partial U_r^s + \varrho_r\delta\partial V_r^s + \beta_r^3\delta U_r^s + \varepsilon_r^{3\top}\sigma_r^\top\delta V_r^s + \gamma_r^3\delta\mathcal{Y}_r + \alpha_r^{3\top}\sigma_r^\top\delta\mathcal{Z}_r\big)\mathrm{d}r.
\end{aligned}$$

We can therefore apply Proposition 3.6.3 and the result follows. □

When the data of the system is chosen so as to study the class of type-I BSVIEs considered in Section 3.4, our approach can be specialised so as to enlarge the initial space and simplify the *a priori* estimates obtained in Proposition 3.6.3. We let $(\mathcal{H}^{\star,o},\|\cdot\|_{\mathcal{H}^{\star,o}})$ be

$$\mathcal{H}^{\star,o} := \mathbb{L}^2\times\mathbb{H}^2\times\mathbb{M}^2\times\mathbb{L}^{2,2}\times\mathbb{H}^{2,2}\times\mathbb{M}^{2,2},$$

$$\|(\mathcal{Y},\mathcal{Z},\mathcal{N},Y,Z,N)\|^2_{\mathcal{H}^{\star,o}} := \|\mathcal{Y}\|^2_{\mathbb{L}^2} + \|\mathcal{Z}\|^2_{\mathbb{H}^2} + \|\mathcal{N}\|^2_{\mathbb{M}^2} + \|Y\|^2_{\mathbb{L}^{2,2}} + \|Z\|^2_{\mathbb{H}^{2,2}} + \|N\|^2_{\mathbb{M}^{2,2}}.$$

Proposition 3.6.5. *Let* Assumption H *hold and consider* $(\mathcal{Y},\mathcal{Z},\mathcal{N},Y,Z,N)\in\mathcal{H}^{\star,o}$ *solution to the system which for any* $s\in[0,T]$*, holds* $\mathbb{P}$–a.s. *for any* $t\in[0,T]$

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T,X) + \int_t^T \big(f_r(r,X,\mathcal{Y}_r,\mathcal{Z}_r,Y_r^r) - \partial Y_r^r\big)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top X_r - \int_t^T \mathcal{N}_r,\\
Y_t^s &= \xi(s,X) + \int_t^T f_r(s,X,Y_r^s,Z_r^s,\mathcal{Y}_r)\mathrm{d}r - \int_t^T Z_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}N_r^s,\ t\in[0,T],
\end{aligned}\tag{$\mathcal{S}_f^o$}$$

with ∂Y *given as in* Lemma 3.6.1. *Then* $(\mathcal{Y},Y)\in\mathbb{S}^2\times\mathbb{S}^{2,2}$ *and there exist* $C>0$ *such that*

$$\|(\mathcal{Y},\mathcal{Z},\mathcal{N},Y,Z,N)\|^2_{\mathcal{H}^\star} \le C\Big(\|\xi\|^2_{\mathcal{L}^{2,2}} + \|\tilde f\|^2_{\mathbb{L}^{1,2,2}}\Big) < \infty.$$

Proof. We first note that $Y\in\mathbb{S}^{2,2}$ follows as in Proposition 3.6.3. Thus, in light of Lemma 3.6.1, there exists $(\partial Y, \partial Z, \partial N) \in \mathbb{S}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{M}^{2,2}$ solution to the BSDE with data $(\partial_s \xi, \partial f)$, and, $\big((Y_t^t)_{t\in[0,T]}, (Z_t^t)_{t\in[0,T]}\big) \in \mathbb{S}^2 \times \mathbb{H}^2$ are well-defined. Moreover, ∂Y_r^r is well-defined as an element of $\mathbb{L}^{1,2}$, *i.e.* $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e. on $[0,T] \times \mathcal{X}$, as a consequence of the path-wise continuity of $\partial Y_\cdot^s$. With this, we conclude $\mathcal{Y} \in \mathbb{S}^2$.

Let us now note that given $(Y_t^t)_{t\in[0,T]}, (Z_t^t)_{t\in[0,T]}$ and $(\partial Y_t^t)_{t\in[0,T]}$, the first equation, being a Lipschitz BSDE, admits a unique solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{N})$. In addition, for $\widetilde{N}$ as in Lemma 3.6.2 we obtain

$$Y_t^t = Y_T^T + \int_t^T \big(f_r(r, Y_r^r, Z_r^r, \mathcal{Y}_r) - \partial Y_r^r\big)\mathrm{d}r - \int_t^T Z_r^{r\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\widetilde{N}_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.} \tag{3.6.18}$$

Thus, $(\mathcal{Y}_\cdot, \mathcal{Z}_\cdot, \mathcal{N}_\cdot) = \big((Y_t^t)_{t\in[0,T]}, (Z_t^t)_{t\in[0,T]}, (\widetilde{N}_t)_{t\in[0,T]}\big)$ in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2$ and we obtain

$$\mathcal{Y}_t = \xi(t) + \int_t^T f_r(t, Y_r^t, Z_r^t, \mathcal{Y}_r)\mathrm{d}r - \int_t^T Z_r^{t\top}\mathrm{d}X_r - \int_t^T \mathrm{d}N_r^t,\ t \in [0,T],\ \mathbb{P}\text{–a.s.} \tag{3.6.19}$$

With this equation we can simplify our estimates. Let us introduce the system

$$\begin{aligned}
\mathcal{Y}_t &= \xi(t) + \int_t^T f_r(t, Y_r^t, Z_r^t, \mathcal{Y}_r)\mathrm{d}r - \int_t^T Z_r^{t\top}\mathrm{d}X_r - \int_t^T \mathrm{d}N_r^t,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\\
Y_t^s &= \xi(s) + \int_t^T f_r(s, Y_r^s, Z_r^s, \mathcal{Y}_r)\mathrm{d}r - \int_t^T Z_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}N_r^s,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\ s \in [0,T].
\end{aligned} \tag{$\mathcal{A}$}$$

Then, following the same reasoning of Proposition 3.6.3, *i.e.* applying Itō's formula to $\mathrm{e}^{ct}\big(|\mathcal{Y}_t| + |Y_t^s|\big)$ in combination with Young's inequality, we obtain there is $C > 0$ such that $\|(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, Y, Z, N)\|_{\mathcal{H}^\star}^2 \leq C\big(\|\xi\|_{\mathcal{L}^{2,2}}^2 + \|\tilde{f}\|_{\mathbb{L}^{1,2,2}}^2\big) < \infty$. $\square$

3.6.3 Proof of Theorem 3.3.5

Before we present the proof of Theorem 3.3.5 we recall that in light of Proposition 3.6.3 and Proposition 3.6.4 once the result is obtained for $\mathcal{H}^o$ the existence of a unique solution in $\mathcal{H}$ follows immediately.

Proof of Theorem 3.3.5. Note that uniqueness follows from Proposition 3.6.4. To show existence, let us define the map

$$\mathfrak{T} : \mathcal{H}^o \longrightarrow \mathcal{H}^o$$
$$(y, z, n, u, v, m, \mathrm{u}, \mathrm{v}, \mathrm{m}) \longmapsto (Y, Z, N, U, V, M, \partial U, \partial V, \partial M),$$

with $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, U, V, M, \partial U, \partial V, \partial M)$ given for any $s \in [0,T]$, $\mathbb{P}$–a.s. for any $t \in [0,T]$ by

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T, X_{\cdot\wedge T}) + \int_t^T h_r(X, y_r, z_r, U_r^r, V_r^r, \partial U_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top dX_r - \int_t^T \mathrm{d}N_r,\\
U_t^s &= \eta(s, X_{\cdot\wedge,T}) + \int_t^T g_r(s, X, U_r^s, V_r^s, y_r, z_r)\mathrm{d}r - \int_t^T V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}M_r^s,\\
\partial U_t^s &= \partial_s\eta(s, X_{\cdot\wedge,T}) + \int_t^T \nabla g_r(s, X, \partial U_r^s, \partial V_r^s, U_r^s, V_r^s, y_r, z_r)\mathrm{d}r - \int_t^T \partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s.
\end{aligned}$$

Step 1: We first show $\mathfrak{T}$ is well defined. Let $(y, z, n, u, v, m, \mathrm{u}, \mathrm{v}, \mathrm{m}) \in \mathcal{H}^o$.

(i) Let us first consider the tuples (U, V, M) and $(\partial U, \partial V, \partial M)$. Let us first consider the second equation. Given $(y, z) \in \mathbb{S}^2 \times \mathbb{H}^2$ and Assumption G, this equation is a standard Lipschitz BSDE whose well-posedness follows by classical arguments, see [273; 88]. This yields $(U^s, V^s, M^s) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2$ for all $s \in [0,T]$.

Let us argue the continuity of $([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{L}^2, \|\cdot\|_{\mathbb{L}^2}) : s \longmapsto U^s$. Let $(s_n)_n \subseteq [0,T], s_n \xrightarrow{n\to\infty} s_0 \in [0,T]$ and define for $\varphi \in \{U, V, \eta\}$, $\Delta\varphi^n := \varphi^{s_n} - \varphi^{s_0}$. From the classic stability result for BSDEs we obtain that there is $C > 0$ such that

$$\mathbb{E}\bigg[\int_0^T |\Delta U_t^n|^2 \mathrm{d}r\bigg] \leq 2T\Big(\|\Delta\eta^n\|_{\mathcal{L}^2}^2 + TL_g^2(\rho_g^2(|s_n - s_0|))\Big).$$

We conclude $\|U\|_{\mathbb{L}^{2,2}} < \infty$ and $U \in \mathbb{L}^{2,2}$. Given $(U^s, V^s) \in \mathbb{L}^2 \times \mathbb{H}^2$ together with $(y, z) \in \mathbb{S}^2 \times \mathbb{H}^2$, the argument for $(\partial U^s, \partial V^s, \partial M^s)$ for fixed $s \in [0,T]$ is identical.

(ii) We now show that $(V, \partial V, M, \partial M) \in (\mathbb{H}^{2,2})^2 \times (\mathbb{M}^{2,2})^2$. Again, the argument for $(\partial V, \partial M)$ is completely analogous. Applying Itō's formula to $|U_r^s|^2$ we obtain

$$|U_t^s|^2 + \int_t^T |\sigma_r^\top V_r^s|^2 \mathrm{d}r + \int_t^T \mathrm{d}[M^s]_r = |\eta(s)|^2 + 2\int_t^T U_r^s \cdot g_r(s, U_r^s, V_r^s, y_r, z_r)\mathrm{d}r$$

$$-2\int_t^T U_r^s \cdot V_r^{s\top}\mathrm{d}X_r - 2\int_t^T U_{r-}^s \cdot \mathrm{d}M_r^s.$$

First note $U^s \in \mathbb{S}^2$ guarantees that the last two terms are true martingale for any $s \in [0,T]$. To show that $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{H}^2, \|\cdot\|_{\mathbb{H}^2})$ (resp. $(\mathbb{M}^2, \|\cdot\|_{\mathbb{M}^2})$) $: s \longmapsto V^s$ (resp. M^s) is continuous, let $(s_n)_n \subseteq [0,T]$, $s_n \xrightarrow{n\to\infty} s_0 \in [0,T]$. We then deduce there is $C>0$ such that

$$\mathbb{E}\bigg[\int_0^T |\sigma_r^\top \Delta V_r^n|^2 \mathrm{d}r + [\Delta M^s]_T\bigg] \leq C\Big(\|\Delta\eta\|_{\mathcal{L}^2}^2 + \rho_g^2(|s_n - s_0|)\Big),$$

and, likewise, we obtain

$$\sup_{s\in[0,T]} \mathbb{E}\bigg[\int_0^T |\sigma_r^\top V_r^s|^2 \mathrm{d}r + [M^s]_T\bigg] \leq C\Big(\|\eta\|_{\mathcal{L}^{2,2}}^2 + \|\tilde{g}\|_{\mathbb{L}^{1,2}}^2\Big) < \infty.$$

Since the first term on the right-hand side is finite from Assumption G, we obtain $\|V\|_{\mathbb{H}^{2,2}} + \|M\|_{\mathbb{M}^{2,2}} < \infty$.

We are left to argue $V \in \overline{\mathbb{H}}^{2,2}$. Applying Lemma 3.6.1 to the system, which for any $s \in [0,T]$ satisfies $\mathbb{P}$–a.s. for any $t \in [0,T]$

$$\begin{aligned}
U_t^s &= \eta(s, X_{\cdot\wedge,T}) + \int_t^T g_r(s, X, U_r^s, V_r^s, y_r, z_r)\mathrm{d}r - \int_t^T V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}M_r^s,\\
\partial U_t^s &= \partial_s\eta(s, X_{\cdot\wedge,T}) + \int_t^T \nabla g_r(s, X, \partial U_r^s, \partial V_r^s, U_r^s, V_r^s, y_r, z_r)\mathrm{d}r - \int_t^T \partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s,
\end{aligned}$$

we obtain $s \longmapsto V^s$ is absolutely continuous with density $s \longmapsto \partial V^s$. Consequently, we may define the diagonal of V, denoted by $\mathcal{V} := (V_t^t)_{t\in[0,T]}$, as in Equation (3.6.5), with

$$V_t^t := V_t^T - \int_t^T \partial V_t^r \mathrm{d}r,\ \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e. in } [0,T]\times\mathcal{X}.$$

The fact $\|\mathcal{V}\|_{\mathbb{H}^2} < \infty$ follows as in Lemma 3.6.1.(iv). This yields $\mathcal{V} \in \mathbb{H}^2$ and we conclude $V \in \overline{\mathbb{H}}^{2,2}$.

(iii) We derive an auxiliary estimate. Recall $(U, V, M, \partial U, \partial V, \partial M)$ satisfy (3.6.3). Now, in light of Assumption G and (ii), we may find, as in **Step** 1 in Proposition 3.6.3, a universal constant $C > 0$

such that for any $c > 4L_g$ and $t \in [0,T]$

$$\begin{aligned}&\frac{1}{C}\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|U_r^r|^2 + |\partial U_r^r|^2 + |\sigma_r^\top V_r^r|^2\big)\mathrm{d}r\bigg]\\ &\leq \mathrm{e}^{cT}\big(\|\eta\|^2_{\mathcal{L}^{2,2}} + \|\tilde g\|^2_{\mathbb{L}^{1,2}} + \|\partial_s \eta\|^2_{\mathcal{L}^{2,2}} + \|\nabla \tilde g\|^2_{\mathbb{L}^{1,2,2}}\big) + \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|y_r|^2 + |\sigma_r^\top z_r|^2\big)\mathrm{d}r\bigg].\end{aligned} \tag{3.6.20}$$

(iv) We now argue for the tuple $(\mathcal{Y},\mathcal{Z},\mathcal{N})$. Notice that

$$\widetilde{\mathcal{Y}}_t := \mathbb{E}\bigg[\xi(T) + \int_0^T h_r(y_r, z_r, U_r^r, V_r^r, \partial U_r^r)\mathrm{d}r\bigg|\mathcal{F}_t\bigg],\ \text{is a square integrable } \mathbb{F}\text{-martingale.}$$

Indeed, under Assumption G, h is uniformly Lipschitz in (y,z,u), so (3.6.20) yields for any $t \in [0,T]$

$$\mathbb{E}[|\widetilde{\mathcal{Y}}_t|^2] \leq 6\bigg(\|\xi\|^2_{\mathcal{L}^2} + \|\tilde h\|^2_{\mathbb{L}^{1,2}} + TL_h^2\bigg(\|y\|^2_{\mathbb{H}^2} + \|z\|^2_{\mathbb{H}^2} + \mathbb{E}\bigg[\int_0^T \big(|U_r^r|^2 + |V_r^r|^2 + |\partial U_r^r|^2\big)\mathrm{d}r\bigg]\bigg)\bigg) < \infty.$$

Integrating the above expression, Fubini's theorem implies that $\widetilde{\mathcal{Y}} \in \mathbb{L}^2$, thus the predictable martingale representation property for local martingales guarantees the existence of a unique $(\mathcal{Z},\mathcal{N}) \in \mathbb{H}^2 \times \mathbb{M}^2$ such that $(\mathcal{Y},\mathcal{Z},\mathcal{N})$ satisfies the correct dynamics and Doob's inequality implies $\mathcal{Y} \in \mathbb{S}^2$, where

$$\mathcal{Y} := \widetilde{\mathcal{Y}} - \mathbb{E}\bigg[\int_0^\cdot h_r(y_r, z_r, U_r^r, V_r^r, \partial U_r^r)\mathrm{d}r\bigg].$$

All together, we have shown that $\mathfrak{T}(y,z,n,u,v,m,\partial u,\partial v,\partial m) \in \mathcal{H}^o$.

Step 2: We show $\mathfrak{T}$ is a contraction under the equivalent norm $\|\cdot\|_{\mathcal{H}^{o,c}}$, for some $c > 0$ large enough. Let $(y^i,z^i,n^i,u^i,v^i,m^i) \in \mathcal{H}^o$, $\mathfrak{h}^i = \mathfrak{T}(y^i,z^i,n^i,u^i,v^i,m^i,\partial u^i,\partial v^i,\partial m^i)$ for $i \in \{1,2\}$. We first note that by Lemma 3.6.2

$$\mathcal{U}_t^i = \eta(T) + \int_t^T \big(g_r(r,\mathcal{U}_r^i,\mathcal{V}_r^i,y_r^i,z_r^i) - \partial \mathcal{U}_r^i\big)\mathrm{d}r - \int_t^T \mathcal{V}_r^i \mathrm{d}X_r - \int_t^T \mathrm{d}\widetilde{\mathcal{M}}_r^i,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},$$

where

$$(\mathcal{U}^i,\mathcal{V}^i,\widetilde{\mathcal{M}}^i,\partial\mathcal{U}^i) := \big((U_t^{it})_{t\in[0,T]},(V_t^{it})_{t\in[0,T]},(\widetilde{M}_t^i)_{t\in[0,T]},(\partial U_t^{it})_{t\in[0,T]}\big),$$

and $\widetilde{M}^i$ as in Lemma 3.6.2 for $i \in \{1,2\}$.

To ease the readability we define $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e.

$$
\begin{aligned}
\delta h_r &:= h_r(y^1_r, z^1_r, \mathcal{U}^1_r, \mathcal{V}^1_r, \partial \mathcal{U}^1_r) - h_r(y^2_r, z^2_r, \mathcal{U}^2_r, \mathcal{V}^2_r, \partial \mathcal{U}^2_r), \\
\delta \hat{g}_r &:= g_r(r, \mathcal{U}^1_r, \mathcal{V}^1_r, y^1_r, z^1_r) - \partial \mathcal{U}^1_r - g_r(r, \mathcal{U}^2_r, \mathcal{V}^2_r, y^2_r, z^2_r) + \partial \mathcal{U}^2_r, \\
\delta g_r(s) &:= g_r(s, U^{1s}_r, V^{1s}_r, y^1_r, z^1_r) - g_r(s, U^{2s}_r, V^{2s}_r, y^2_r, z^2_r), \\
\delta \nabla g_r(s) &:= \nabla g_r(s, \partial U^{s1}_r, \partial V^{s1}_r, U^{s1}_r, V^{s1}_r, y^{s1}_r, z^1_r) - \nabla g_r(s, \partial U^{s2}_r, \partial V^{s2}_r, U^{s2}_r, V^{s2}_r, y^{s2}_r, z^2_r),
\end{aligned}
$$

and

$$
\delta \mathfrak{Y} := (\delta \mathcal{Y}, \delta \mathcal{U}, \delta U^s, \delta \partial U^s),\ \delta \mathfrak{Z} := (\delta \mathcal{Z}, \delta \mathcal{V}, \delta V^s, \delta \partial V^s),\ \delta \mathfrak{N} := (\delta \mathcal{N}, \delta \widetilde{\mathcal{M}}, \delta M^s, \delta \partial M^s),
$$

whose elements we may denote with superscripts, e.g. $\delta\mathfrak{Y}^1$, $\delta\mathfrak{Y}^2$, $\delta\mathfrak{Y}^3$, $\delta\mathfrak{Y}^4$ correspond to $\delta\mathcal{Y}$, $\delta\mathcal{U}$, δU^s, $\delta\partial U^s$.

(i) In light of (3.6.3), as in **Step** 1 in Proposition 3.6.3, we may find that for $c > 4L_g$ there exists a universal constant $C \in (0, \infty)$ such that for $t \in [0, T]$

$$
\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|\delta \mathcal{U}_r|^2 + |\delta \partial \mathcal{U}_r|^2 + |\sigma_r^\top \delta \mathcal{V}_r|^2\big)\mathrm{d}r\bigg] \le C\mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|\delta y_r|^2 + |\sigma_r^\top \delta z_r|^2\big)\mathrm{d}r\bigg]. \tag{3.6.21}
$$

(ii) Applying Itô's formula to $\mathrm{e}^{cr}\big(|\delta\mathcal{Y}_r|^2 + |\delta\mathcal{U}_r|^2 + |\delta U^s_r|^2 + |\delta\partial U^s_r|^2\big)$ and noticing that

$$
(\delta \mathcal{Y}_T, \delta \mathcal{U}_T, \delta U_T, \delta \partial U_T) = (0, 0, 0, 0),
$$

we obtain

$$
\begin{aligned}
&\sum_{i=1}^4 \mathrm{e}^{ct}|\delta \mathfrak{Y}^i_t|^2 + \int_t^T \mathrm{e}^{cr}|\sigma_r^\top \delta \mathfrak{Z}^i_r|^2 \mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\big[\delta \mathfrak{N}^i\big]_r + \widetilde{\mathfrak{M}}^s_T - \widetilde{\mathfrak{M}}^s_t \\
&= \int_t^T \mathrm{e}^{cr}\Big(2\delta Y_r \cdot \delta h_r + 2\delta U^s_r \cdot \delta g_r(s) + 2\delta\partial U^s_r \cdot \delta \nabla g_r(s) - c\big(|\delta Y_r|^2 + |\delta U^s_r|^2 + |\delta \partial U^s_r|^2\big)\Big)\mathrm{d}r,
\end{aligned}
$$

where

$$
\widetilde{\mathfrak{M}}^s_t = 2\sum_{i=1}^4 \int_0^t \mathrm{e}^{cr} \delta \mathfrak{Y}^i_r \cdot \delta \mathfrak{Z}^{i\,\top}_r \mathrm{d}X_r + \int_0^t \mathrm{e}^{cr-}\delta \mathfrak{Y}^i_{r-} \cdot \mathrm{d}\, \delta \mathfrak{N}^i_r.
$$

Again, the fact that $(\delta \mathcal{Y}, \delta \mathcal{U}, \delta U, \delta \partial U) \in (\mathbb{S}^2)^2 \times (\mathbb{S}^{2,2})^2$ guarantees, via the Burkholder–Davis–

Gundy inequality, that $\widetilde{\mathfrak{M}}^s$ is a uniformly integrable martingale, and thus a true martingale for all $s \in [0,T]$.

Additionally, under Assumption G.(ii) and G.(iii), $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e.

$$\begin{aligned}
|\delta h_r| &\leq L_h(|\delta y_r| + |\sigma_t^\top \delta z_r| + |\delta \mathcal{U}_r| + |\sigma_r^\top \delta \mathcal{V}_r| + |\delta \partial \mathcal{U}_r^r|),\\
|\delta \hat{g}_r| &\leq L_g(|\delta y_r| + |\sigma_t^\top \delta z_r| + |\delta \mathcal{U}_r| + |\sigma_r^\top \delta \mathcal{V}_r|) + |\delta \partial \mathcal{U}_r^r|,\\
|\delta g_r(s)| &\leq L_g\big(|\delta U_r^s| + |\sigma_r^\top \delta V_r^s| + |\delta y_r| + |\sigma_r^\top \delta z_r|\big),\\
|\delta \nabla g_r(s)| &\leq L_{\partial_s g}\big(|\delta U_r^s| + |\sigma_r^\top \delta V_r^s| + |\delta y_r| + |\sigma_r^\top \delta z_r|\big) + L_g(|\delta \partial U_r^s| + |\sigma_r^\top \delta \partial V_r^s|).
\end{aligned}$$

In turn, this implies together with Young's inequality and (3.6.21), that for any $c > 4L_g$ there exists a universal constant $C > 0$ such that for any $\varepsilon > 0$

$$\begin{aligned}
&\mathbb{E}\bigg[\sum_{i=1}^4 \mathrm{e}^{ct}|\delta \mathfrak{Y}_t^i|^2 + \int_t^T \mathrm{e}^{cr}|\sigma_r^\top \delta \mathfrak{Z}_r^i|^2 \mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\big[\delta \mathfrak{N}^i\big]_r\bigg]\\
&\leq \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\Big(\big(|\delta \mathcal{Y}_r|^2 + |\delta \mathcal{U}_r|^2 + |\delta U_r^s|^2 + |\delta \partial U_r^s|^2\big)(C\varepsilon^{-1} - c)\Big)\mathrm{d}r\\
&\quad + \int_t^T \mathrm{e}^{cr}\Big(\varepsilon\big(|\delta y_r|^2 + |\sigma_r^\top \delta z_r|^2 + |\delta \mathcal{U}_r| + |\sigma_r^\top \delta \mathcal{V}_r| + |\delta \partial \mathcal{U}_r|\big)\Big)\mathrm{d}r\\
&\leq \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}\big(|\delta \mathcal{Y}_r|^2 + |\delta \mathcal{U}_r|^2\big)(C\varepsilon^{-1} - c)\mathrm{d}r\bigg] + \sup_{s\in[0,T]} \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}|\delta U_r^s|^2(C\varepsilon^{-1} - c)\mathrm{d}r\bigg]\\
&\quad + \sup_{s\in[0,T]} \mathbb{E}\bigg[\int_t^T \mathrm{e}^{cr}|\delta \partial U_r^s|^2(C\varepsilon^{-1} - c)\mathrm{d}r\bigg] + \varepsilon C \mathbb{E}\bigg[\int_0^T \mathrm{e}^{cr}\big(|\delta y_r|^2 + |\sigma_r^\top \delta z_r|^2\big)\mathrm{d}r\bigg],
\end{aligned}$$

where in the second inequality C is appropriately updated. Choosing $\varepsilon = Cc^{-1}$ we obtain

$$\mathbb{E}\bigg[\sum_{i=1}^4 \mathrm{e}^{ct}|\delta \mathfrak{Y}_t^i|^2 + \int_t^T \mathrm{e}^{cr}|\sigma_r^\top \delta \mathfrak{Z}_r^i|^2 \mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\big[\delta \mathfrak{N}^i\big]_r\bigg] \leq \frac{C}{c}\Big(\|\delta y\|_{\mathbb{L}^{2,c}} + \|\delta z\|_{\mathbb{H}^{2,c}}\Big),$$

which yields

$$\|\delta \mathfrak{h}\|^2_{\mathcal{H}^{o,c}} \leq \frac{C}{c}\Big(\|\delta y\|_{\mathbb{L}^{2,c}} + \|\delta z\|_{\mathbb{H}^{2,c}}\Big).$$

We conclude $\mathfrak{T}$ has a fixed point as it is a contraction for c large enough. □

Chapter 4

Quadratic backward stochastic Volterra integral equations

We now want to build upon the strategy devised in Chapter 3 and address the well-posedness of a general class of multidimensional type-I BSVIEs with quadratic generators.

The setting is, up to growth condition of the generator, identical to that of Chapter 3. Nevertheless, for the reader's convience we briefly recall it. We let X be the solution to a drift-less SDE under a probability measure $\mathbb{P}$, and $\mathbb{F}$ be the $\mathbb{P}$-augmentation of the filtration generated by X. We consider a tuple $(Y^\cdot_\cdot, Z^\cdot_\cdot, N^\cdot_\cdot)$, of appropriately $\mathbb{F}$-adapted processes, which satisfy for any $s \in [0,T]$, $\mathbb{P}$–a.s. for any $t \in [0,T]$, the equation

$$Y_t^s = \xi(s) + \int_t^T g_r(s, X, Y_r^s, Z_r^s, Y_r^r, Z_r^r)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r - \int_t^T \mathrm{d}N_r^s. \tag{4.0.1}$$

We will consider both the case of linear (in Y) quadratic (in Z) and fully quadratic generators. To be able to cover type-I BSVIEs as general as (4.0.1), following the ideas in Chapter 3, our approach is based on the following class of infinite families of BSDEs

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T) + \int_t^T h_r(X, \mathcal{Y}_r, \mathcal{Z}_r, Y_r^r, Z_r^r, \partial Y_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r \mathrm{d}X_r - \int_t^T \mathrm{d}\mathcal{N}_r, \\
Y_t^s &= \eta(s) + \int_t^T g_r(s, X, Y_r^s, Z_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r - \int_t^T \mathrm{d}N_r^s, \\
\partial Y_t^s &= \partial_s \eta(s) + \int_t^T \nabla g_r(s, X, \partial Y_r^s, \partial Z_r^s, Y_r^s, Z_r^s, \mathcal{Y}_r, \mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial Z_r^s \mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s.
\end{aligned} \tag{4.0.2}$$

where $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, Y, Z, N, \partial Y, \partial Z, \partial N)$ are unknown, and required to have appropriate integrability.

4.1 Practical motivations

(i) The first motivation builds upon the treatment presented in [246] of scalar continuous-time dynamic risk measures as introduced in [88]. By studying multidimensional type-BSVIEs one could extend these ideas to the more realistic situation where the risky portfolio is vector-valued. In the static case, Jouini, Meddeb, and Touzi [144] provides a notion of multidimensional coherent risk measure that renders a convenient extension of the real-valued risk measures intially introduced in Artzner, Delbaen, Eber, and Heath [13]. Building upon these ideas, Kulikov [160] presents a model that takes into account risks of changing currency exchange rates and transaction costs. In particular, exploiting the cone structure induced by a given partial ordering, the author introduces multidimensional coherent risk measures tail V@R, and weighted V@R.

(ii) Systems of the nature of (4.0.2) are known to appear naturally in problems in behavioural economics and applications in contract theory. An immediate practical motivation behind studying systems of the form prescribed by (4.0.2) in a quadratic framework comes from the study of time-inconsistent control problems. In this kind of problems the idea of optimal controls is incompatible with the underlying agent's preferences and thus one approach that leads to a successful concept of solution is that of consistent plans or equilibria, as initially introduced in Ekeland and Lazrak [80, 81]. In Chapter 2, we established this connection via a extended dynamic principle for the problem faced by a *sophisticated* time-inconsistent agent, *i.e.* one seeking for an *equilibria* among the players prescribed by her future preferences. However, their analysis was limited to a standard Lipschitz assumptions on the data of the problem.

To motivate the results of this document, we specialised to the Markovian framework and precise the dynamics of the controlled process X. This is, we let $b_t(x,a) := \frac{a-x}{t-T}$, $\sigma_t(x) := \sigma_t$ bounded, and $A := [a_1, a_2]$ so that

$$X_t = x_0 + \int_0^t \sigma_t \frac{\alpha_r - X_r}{r-T} \mathrm{d}r + \int_0^t \sigma_r \mathrm{d}W_r^\alpha,\ t \in [0,T],\ \mathbb{P}^\alpha\text{–a.s.}$$

for some A-valued process α which the agent controls. $\mathbb{P}^\alpha$ denotes a probability measure governing the distribution of the canonical process X which the agent controls and for simplicity we assume one dimensional. W^α denotes a $\mathbb{P}^\alpha$–Brownian motion. Let us recall that $\mathbb{P}^\alpha$ is guaranteed to exists as the previous Lipschitz SDE has a unique strong solution. Moreover, at $t = T$, we have that $X_T = \alpha_T \in [a_1, a_2]$.

Now, for real valued k, F, and G with appropriate measurability, we let the reward of an agent performing the A-valued action α from time t onwards, with initial condition $x \in \mathcal{X}$, be given by

$$J(t,x,\alpha) := \mathbb{E}^{\mathbb{P}^\alpha}\bigg[\int_0^T k_r(t,X_r,\alpha_r)\mathrm{d}r + F(t,X_T)\bigg|\mathcal{F}_t\bigg] + G\big(t, \mathbb{E}^{\mathbb{P}^\alpha}[X_T|\mathcal{F}_t]\big),$$

where $\mathbb{E}^{\mathbb{P}^\alpha}[\cdot|\mathcal{F}_t]$ denote the classic conditional expectation operator under $\mathbb{P}^\alpha$. The noticeable features of this type of rewards are: (i) the dependence of k, F and G on t which besides the case of exponential discounting is a source of time-inconsistency; (ii) the possible non-linear dependence of G on a conditional expectation of $\mathrm{g}(X_T)$, another source of time-inconsistency, which would allow for mean–variance type of criteria. We remark that we have refrained ourselves to the Markovian setting for the purpose of illustrating the result in this chapter.

Following the analysis in Chapter 2, let $\xi(T,x) := F(T,x) + G(T,\mathrm{g}(x))$, $\eta(s,x) := \partial_s F(s,x)$, $g_t(s,x,z,a) := k_t(s,a) + b_t(x,a)\cdot\sigma_t^\top z$, $\nabla g_t(s,x,v,a) := \partial_s k_t(s,a) + b_t(x,a)\cdot\sigma_t^\top v$,

$$H_t(x,z,u,n,\mathrm{z}) := \sup_{a\in A}\big\{g_t(t,x,z,a)\big\} - u - \partial_s G(t,n) - \frac{1}{2}\,\mathrm{z}^\top\sigma_t\sigma_t^\top\mathrm{z}\;\partial^2_{nn}G(t,n),$$

and denote by $a^\star(t,x,z)$ the A-valued measurable mapping attaining the sup in H. Then, we will find that agent's value function associated to an equilibrium action $\alpha^\star$ correspond to $\mathcal{Y}$ and $a^\star(t,X_t,\mathcal{Z}_t)$, respectively, given by

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T,X_T) + \int_t^T H_r(X_r,\mathcal{Z}_r,\partial Y_r^r,\widetilde{N}_r,\widetilde{Z}_r)\mathrm{d}r - \int_t^T \mathcal{Z}_r\cdot\mathrm{d}X_r,\\
\partial Y_t^s &= \partial_s\eta(s,X_T) + \int_t^T \nabla g_r(s,X_r,\partial Z_r^s,a^\star(r,X_r,\mathcal{Z}_r))\mathrm{d}r - \int_t^T \partial Z_r^s\cdot\mathrm{d}X_r,\\
\widetilde{N}_t &= X_T + \int_t^T b_r(X_r,a^\star(r,X_r,\mathcal{Z}_r))\cdot\sigma_r^\top\widetilde{Z}_r\mathrm{d}r - \int_t^T \widetilde{Z}_r\cdot\mathrm{d}X_r.
\end{aligned}$$

We highlight that it follows from the definition that H is quadratic in z and it is quadratic in n if, for instance, $G(s,n) = \phi(s)n^2$. Moreover, having access to a well-posedness result for the previous system would guarantee the uniqueness of the equilibrium. This becomes particularly important in light of the recent results in [164], where the authors presents an example, stemming from a mean–variance investment problem, in which uniqueness of the equilibrium fails. In light of the dynamics specified above for X we have that terminal conditions of the previous system are bounded. Morever, if in addition $x \longmapsto (\xi(t,x), \partial_s\eta(s,x))$ is monotone and continuous this bound is determined by a_1, a_2. All in all, we will find that the previous system is well-posed and therefore there is a unique equilibrium associated to the time-inconsistent control problem faced by a sophisticated agent.

(iii) Lastly, even in the case $G = 0$ the reward of the agent can be represented via a standard type-I BSVIE. This is due to the time-inconsistency arising from k and F. In complete analogy to the case of a classic time-consistent agent, the need for the understanding of multidimensional BSVIEs such as (4.0.1) would arise in applications to nonzero-sum risk-sensitive stochastic differential game, El Karoui and Hamadène [84] and [133]; financial market equilibrium problems for several interacting agents, Bielagk, Lionnet, and Dos Reis [29], Espinosa and Touzi [96], [103], [102]; price impact models, [158] [157]; and principal agent contracting problems with competitive interacting agents in Élie and Possamaï [90] to mention just a few.

4.2 Preliminaries

4.2.1 The stochastic basis on the canonical space

For the reader's convenience, we recall the formulation on the canonical space.

We take two positive integers n and m, which represent respectively the dimension of the martingale which will drive our equations, and the dimension of the Brownian motion appearing in the dynamics of the former. We consider the canonical space $\mathcal{X} = \mathcal{C}^n$, with canonical process X. We let $\mathcal{F}^X$ be the Borel σ-algebra on $\mathcal{X}$ (for the topology of uniform convergence), and $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\in[0,T]}$ is the natural filtration of X. We fix a bounded Borel-measurable map $\sigma : [0,T] \times \mathcal{X} \longrightarrow \mathbb{R}^{n\times m}$, $\sigma_\cdot(X) \in \mathcal{P}_{\rm meas}(\mathbb{R}^{n\times m}, \mathbb{F}^X)$, and an initial condition $x_0 \in \mathbb{R}^n$. We assume there is $\mathbb{P} \in \mathrm{Prob}(\mathcal{X})$ such that $\mathbb{P}[X_0 = x_0] = 1$ and X is martingale, whose quadratic variation, $\langle X\rangle = (\langle X\rangle_t)_{t\in[0,T]}$, is absolutely continuous with respect to Lebesgue measure, with density given by $\sigma\sigma^\top$. Enlarging the original probability space, see Stroock and Varadhan [232, Theorem 4.5.2], there is an $\mathbb{R}^m$-valued Brownian motion B with

$$X_t = x_0 + \int_0^t \sigma_r(X_{\cdot\wedge r})\mathrm{d}B_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

We now let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be the (right-limit) of the $\mathbb{P}$-augmentation of $\mathbb{F}^X$. We stress that we will not assume $\mathbb{P}$ is unique. In particular, the predictable martingale representation property for $(\mathbb{F},\mathbb{P})$-martingales in terms of stochastic integrals with respect to X might not hold.

Remark 4.2.1. *We remark that the previous formulation on the canonical is by no means necessary. Indeed, any probability space supporting a Brownian motion B and a process X satisfying the previous* SDE *will do, and this can be found whenever that equation has a weak solution.*

4.2.2 Functional spaces and norms

We now introduce the spaces of interest for our analysis. In the following, $(\Omega,\mathcal{F}_T,\mathbb{F},\mathbb{P})$ denotes the filtered probability space as defined in the introduction. We are given a non-negative real number c and $(E,|\cdot|)$ a finite-dimensional Euclidean space, *i.e.* $E=\mathbb{R}^{\tilde d}$ for some non-negative integer $\tilde d$ and $|\cdot|$ denotes the L^2-norm. We also introduce the $\mathcal{L}^\infty$-norm which for an arbitrary E-valued random variable ζ is given by $\|\zeta\|_\infty := \inf\{C\geq 0 : |\zeta|\leq C,\ \mathbb{P}\text{–a.s.}\}$ as well as the spaces

- $\mathcal{L}^{\infty,c}(E)$ of $\xi\in\mathcal{P}_{\rm meas}(E,\mathcal{F}_T)$ $\mathbb{P}$-essentially bounded, such that $\|\xi\|_{\mathcal{L}^{\infty,c}} := \|\mathrm{e}^{\frac{c}{2}T}\xi\|_\infty < \infty$;
- $\mathcal{S}^{\infty,c}(E)$ of $Y\in\mathcal{P}_{\rm opt}(E,\mathbb{F})$, with $\mathbb{P}$–a.s. càdlàg paths on $[0,T]$ and $\|Y\|_{\mathcal{S}^{\infty,c}} := \Big\|\sup_{t\in[0,T]}\mathrm{e}^{\frac{c}{2}t}|Y_t|\Big\|_\infty < \infty.$;
- $\mathbb{S}^{2,c}(E)$ of $Y\in\mathcal{P}_{\rm opt}(E,\mathbb{F})$, with $\mathbb{P}$–a.s. càdlàg paths and $\|Y\|^2_{\mathbb{S}^{2,c}} := \mathbb{E}\Big[\sup_{t\in[0,T]}\mathrm{e}^{\frac{c}{2}t}|Y_t|^2\Big] < \infty.$;
- $\mathbb{L}^{1,\infty,c}(E)$ of $Y\in\mathcal{P}_{\rm opt}(E,\mathbb{F})$ with $\|Y\|_{\mathbb{L}^{1,\infty,c}} := \Big\|\int_0^T \mathrm{e}^{\frac{c}{2}t}|Y_t|\mathrm{d}t\Big\|_\infty < \infty$;
- $\mathbb{H}^{2,c}(E)$ of $Z\in\mathcal{P}_{\rm pred}(E,\mathbb{F})$, defined $\sigma\sigma_t^\top\mathrm{d}t$–a.e., with $\|Z\|^2_{\mathbb{H}^{2,c}} := \mathbb{E}\Big[\int_0^T \mathrm{e}^{cr}|\sigma_r Z_r|^2\mathrm{d}r\Big] < \infty$;
- $\mathrm{BMO}^{2,c}(E)$ of square integrable E-valued $(\mathbb{F},\mathbb{P})$-martingales M with $\mathbb{P}$–a.s. càdlàg paths on $[0,T]$ and

$$\|M\|^2_{\mathrm{BMO}^{2,c}} := \sup_{\tau\in\mathcal{T}_{0,T}}\bigg\|\mathbb{E}^{\mathbb{P}}\bigg[\bigg|\int_{\tau-}^T \mathrm{e}^{\frac{c}{2}r-}\mathrm{d}M_r\bigg|^2\bigg|\mathcal{F}_\tau\bigg]\bigg\|_\infty < \infty;$$

- $\mathbb{H}^{2,c}_{\rm BMO}(E)$ of $Z \in \mathcal{P}_{\rm pred}(E,\mathbb{F})$, defined $\sigma\sigma_t^\top \mathrm{d}t$–a.e., with $\|Z\|^2_{\mathbb{H}^{2,c}_{\rm BMO}} := \bigg\| \int_0^\cdot Z_r \mathrm{d}X_r \bigg\|^2_{\mathrm{BMO}^{2,c}} < \infty$;

- $\mathbb{M}^{2,c}_{\rm BMO}(E)$ of càdlàg martingales $N \in \mathcal{P}_{\rm opt}(E,\mathbb{F})$, $\mathbb{P}$-orthogonal to X (that is the product XN is an $(\mathbb{F},\mathbb{P})$-martingale), with $N_0 = 0$ and $\|N\|^2_{\mathbb{M}^{2,c}_{\rm BMO}} := \|N\|^2_{\mathrm{BMO}^{2,c}} < \infty$;

- $\mathbb{M}^{2,c}(E)$ of càdlàg martingales $N \in \mathcal{P}_{\rm opt}(E,\mathbb{F})$, $\mathbb{P}$-orthogonal to X, $N_0 = 0$ and $\|N\|^2_{\mathbb{M}^{2,c}} := \mathbb{E}\bigg[\int_0^T \mathrm{e}^{cr-} \mathrm{d}\mathrm{Tr}[N]_r \bigg] < \infty$[1];

- $\mathcal{P}^2_{\rm meas}(E,\mathcal{F}_T)$ of two parameter processes $(U^s_t)_{(s,t)\in[0,T]^2} : ([0,T]^2 \times \Omega, \mathcal{B}([0,T]^2) \otimes \mathcal{F}_T) \longrightarrow (E,\mathcal{B}(E))$ measurable.

- $\mathcal{L}^{\infty,2,c}(E)$ denotes the space of collections $\eta := (\eta(s))_{s\in[0,T]} \in \mathcal{P}^2_{\rm meas}(E,\mathcal{F})$ such that the mapping $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}) \longrightarrow (\mathcal{L}^{\infty,c}(\mathcal{F},\mathbb{P})), \|\cdot\|_{\infty,\mathbb{P}})) : s \longmapsto \eta(s)$ is continuous and $\|\eta\|_{\mathcal{L}^{\infty,2,c}} := \sup_{s\in[0,T]} \|\eta(s)\|_{\mathcal{L}^{\infty,c}} < \infty$.

- Given a Banach space $(\mathbb{I}^c(E), \|\cdot\|_{\mathbb{I}^c})$, we define $(\mathbb{I}^{2,c}(E), \|\cdot\|_{\mathbb{I}^{2,c}})$ the space of $U \in \mathcal{P}^2_{\rm meas}(E,\mathcal{F}_T)$ such that $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{I}^c(E), \|\cdot\|_{\mathbb{I}^c}) : s \longmapsto U^s$ is continuous and $\|U\|_{\mathbb{I}^{2,c}} := \sup_{s\in[0,T]} \|U^s\|_{\mathbb{I}} < \infty$. For example, $\mathbb{L}^{1,\infty,2,c}(E)$ denotes the space of $U \in \mathcal{P}^2_{\rm meas}(E,\mathcal{F}_T)$ such that $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{L}^{1,\infty,c}(E), \|\cdot\|_{\mathbb{L}^{1,\infty,c}}) : s \longmapsto U^s$ is continuous and $\|U\|_{\mathbb{L}^{1,\infty,2,c}} := \sup_{s\in[0,T]} \|U^s\|_{\mathbb{L}^{1,\infty,c}} < \infty$;

- $\overline{\mathbb{H}}^{2,2}(E)$ of $(Z^s_t)_{(s,t)\in[0,T]^2} \in \mathcal{P}^2_{\rm meas}(E,\mathcal{F}_T)$ such that $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{H}^p(E), \|\cdot\|_{\mathbb{H}^p}) : s \longmapsto Z^s$ is absolutely continuous with respect to the Lebesgue measure,[2] $\mathcal{Z} \in \mathbb{H}^{2,c}(E)$, where $\mathcal{Z} := (Z^t_t)_{t\in[0,T]}$ is given by

$$Z^t_t := Z^T_t - \int_t^T \partial Z^r_t \mathrm{d}r, \text{ and, } \|Z\|^2_{\overline{\mathbb{H}}^{2,c,2}} := \|Z\|^2_{\mathbb{H}^{2,2}} + \|\mathcal{Z}\|^2_{\mathbb{H}^2} < \infty.$$

- $\overline{\mathbb{H}}^{2,2}_{\rm BMO}(E)$ of $(Z^s_t)_{(s,t)\in[0,T]^2} \in \mathcal{P}^2_{\rm meas}(E,\mathcal{F}_T)$ such that $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{H}^p_{\rm BMO}(E), \|\cdot\|_{\mathbb{H}^p_{\rm BMO}}) : s \longmapsto Z^s$ is absolutely continuous with respect to the Lebesgue measure, $\mathcal{Z} \in \mathbb{H}^{2,c}_{\rm BMO}(E)$, where $\mathcal{Z} := (Z^t_t)_{t\in[0,T]}$ is given by

$$Z^t_t := Z^T_t - \int_t^T \partial Z^r_t \mathrm{d}r, \text{ and, } \|Z\|^2_{\overline{\mathbb{H}}^{2,c,2}_{\rm BMO}} := \|Z\|^2_{\mathbb{H}^{2,c,2}_{\rm BMO}} + \|\mathcal{Z}\|^2_{\mathbb{H}^{2,c}_{\rm BMO}} < \infty.$$

[1] We remark the use of $[\cdot]$ instead of $\langle\cdot\rangle$ as N is only assumed to be càdlàg

[2] We recall that $\mathbb{H}^2$, being a Hilbert space and in particular a reflexive Banach space, has the so-called Radon–Nikodým property, see [216, Corollary 5.45]. Thus the absolute continuity of the $\mathbb{H}^2$-valued mapping $s \longmapsto Z^s$ is equivalent to the existence of the density $(\partial Z^s)_{s\in[0,T]}$, which is unique for Lebesgue–a.e. $s \in [0,T]$. The fact $\mathcal{Z}$ is well-defined is argued in Remark 4.2.2

Remark 4.2.2. (i) *We remark that the first set of spaces in the previous list, but* $\mathbb{M}^{2,c}_{\rm BMO}(E)$, *are the corresponding weighted version of the classic spaces in the literature for* BSDEs, *which are recovered by taking* $c = 0$. *Such weighted spaces are known to be more suitable to handle existence results. Moreover, given our assumption of finite time horizon these spaces are known to be isomorphic for any value of* c.

(ii) *The second set of these spaces are weighted versions of suitable extensions of the classical ones, whose norms are tailor-made to the analysis of the systems we will study. Some of these spaces have been previously considered in the literature on* BSVIEs, *see* Chapter 3 , [267] *and* [245].*Of particular interest are the spaces* $\overline{\mathbb{H}}^{2,2}(E)$ *and* $\overline{\mathbb{H}}^{2,2}_{\rm BMO}(E)$. *Indeed, the space* $\mathbb{H}^{2,c}(E)$ *being closed implies* $\overline{\mathbb{H}}^{2,2,c}(E)$ *is a closed subspace of* $\mathbb{H}^{2,2,c}(E)$ *and thus a Banach space. Let us recall that the space* $\overline{\mathbb{H}}^{2,2,c}(E)$ *allows us to define a good candidate for* $(Z^t_t)_{t\in[0,T]}$ *as an element of* $\mathbb{H}^{2,c}(E)$. *Let* $\widetilde\Omega := [0,T]\times\mathcal{X}$, $\tilde\omega := (t,x)\in\widetilde\Omega$ *and*

$$\mathfrak{Z}_s(\tilde\omega) := Z^T_t(x) - \int_s^T \partial Z^r_t(x)\mathrm{d}r,\ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e. } \tilde\omega\in\widetilde\Omega,\ s\in[0,T],$$

so that the Radon–Nikodým property and Fubini's theorem imply $\mathfrak{Z}_s = Z^s, \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e.}$, $s\in[0,T]$. *Lastly, as for* $\tilde\omega\in\widetilde\Omega$, $s\longmapsto\mathfrak{Z}_s(\tilde\omega)$ *is continuous, we may define*

$$Z^t_t := Z^T_t - \int_t^T \partial Z^r_t\mathrm{d}r,\ \textit{for}\ \ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e. } (t,x) \text{ in } [0,T]\times\mathcal{X}.$$

(iii) *Lastly, we comment on our choice to introduce the spaces* $\mathbb{M}^{2,c}_{\rm BMO}(E)$ *and* $\mathbb{M}^{2,2,c}_{\rm BMO}(E)$. *Those familiar with the theory of* BSDEs *would recognise the integrability in* $\mathbb{M}^{2,c}(E)$ *as the typical one for orthogonal martingales. However, given the setting of this paper, one might argue whether it would be more natural to require a* BMO*-type of integrability, as the space* $\mathbb{M}^{2,c}_{\rm BMO}(E)$ *does. This had been noticed since* [238]. *Therefore, a natural question is how requiring one specific type of integrability would quantitatively affect our well–posedness results.*

4.2.3 Auxiliary inequalities

Finally, to ease the readability we recall a particularly useful inequality in our setting. It is obtained from the so–called energy inequality, see Meyer [185, Chapter VII. Section 6]. For a positive integer p and a potential X, *i.e.* a positive right-continuous super-martingale s.t. $\mathbb{E}[X_t] \longrightarrow 0$, $t \longrightarrow \infty$, the pth-energy is defined by

$$\mathrm{e}_p(X_t) := \frac{1}{p!}\mathbb{E}[(A_\infty)^p],$$

where A is the increasing, right–continuous process appearing in the Doob–Meyer decomposition of X. The p–energy inequality states that

$$\mathrm{e}_p(X_t) \leq C^p, \text{ whenever } |X_t| \leq C.$$

In the context of this document, we can use it to obtain the following auxiliary inequalities, whose proof is available in Section 4.7.1.

Lemma 4.2.3. *Let $\tilde{d}$ be a positive integer.*

(i) Let $Z \in \mathbb{H}^2_{\mathrm{BMO}^{2,c}}(\mathbb{R}^{n\times\tilde{d}})$. Then,

$$\mathbb{E}\bigg[\bigg(\int_0^T \mathrm{e}^{cr}|\sigma_r^\top Z_r|^2\mathrm{d}r\bigg)^p\bigg] \leq p!\|Z\|^{2p}_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}. \tag{4.2.1}$$

(ii) Let $Z \in \overline{\mathbb{H}}^{2,2}(\mathbb{R}^{n\times\tilde{d}})$, $\mathcal{Z} = (Z^t_t)_{t\in[0,T]}$ and $c > 0$, $\varepsilon > 0$. Then, $\mathbb{P}$–a.s.

$$\int_t^T \mathrm{e}^{cu}|\sigma_u^\top \mathcal{Z}_u|^2\mathrm{d}u \leq \int_t^T \mathrm{e}^{cu}|\sigma_u^\top Z^t_u|^2\mathrm{d}u + \int_t^T\int_r^T \varepsilon\mathrm{e}^{cu}|\sigma_u^\top Z^r_u|^2 + \varepsilon^{-1}\mathrm{e}^{cu}|\sigma_u^\top \partial Z^r_u|^2\mathrm{d}u\mathrm{d}r,\ t\in[0,T].$$

Moreover, for any $t \in [0,T]$

$$\mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{cu}|\sigma_u^\top \mathcal{Z}_u|^2\mathrm{d}u\bigg)^2\bigg] \leq 6\Big((1+T^2)\|Z\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + T^2\|\partial Z\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\Big),$$
$$\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cu}|\sigma_u^\top \mathcal{Z}_u|^2\mathrm{d}u\bigg] \leq (1+T)\|Z\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + T\|\partial Z\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}.$$

4.3 The BSDE system

For a fixed integers d_1 and d_2 we are given jointly measurable mappings h, g, ξ and η, such that for any $(y,z,u,v,\mathrm{u})\in\mathbb{R}^{d_1}\times\mathbb{R}^{n\times d_1}\times\mathbb{R}^{d_2}\times\mathbb{R}^{n\times d_2}\times\mathbb{R}^{d_2}$

$$
\begin{aligned}
&h:[0,T]\times\mathcal{X}\times\mathbb{R}^{d_1}\times\mathbb{R}^{n\times d_1}\times\mathbb{R}^{d_2}\times\mathbb{R}^{n\times d_2}\times\mathbb{R}^{d_2}\longrightarrow\mathbb{R}^{d_1},\ h.(\cdot,y,z,u,v,\mathrm{u})\in\mathcal{P}_{\mathrm{prog}}(\mathbb{R}^{d_1},\mathbb{F}),\\
&g:[0,T]^2\times\mathcal{X}\times\mathbb{R}^{d_2}\times\mathbb{R}^{n\times d_2}\times\mathbb{R}^{d_1}\times\mathbb{R}^{n\times d_1}\longrightarrow\mathbb{R}^{d_2},\ g.(s,\cdot,u,v,y,z)\in\mathcal{P}_{\mathrm{prog}}(\mathbb{R}^{d_2},\mathbb{F}),\\
&\xi:[0,T]\times\mathcal{X}\longrightarrow\mathbb{R}^{d_1},\ \eta:[0,T]\times\mathcal{X}\longrightarrow\mathbb{R}^{d_2}.
\end{aligned}
$$

Moreover, throughout this section we assume the following condition on (η,g).

Assumption I. *$(s,u,v)\longmapsto g_t(s,x,u,v,y,z)$ (resp. $s\longmapsto\eta(s,x)$) is continuously differentiable, uniformly in (t,x,y,z) (resp. in x). Moreover, $\nabla g.(s,\cdot,\mathrm{u},\mathrm{v},u,v,y,z)\in\mathcal{P}_{\mathrm{prog}}(\mathbb{R}^{d_2},\mathbb{F}), s\in[0,T]$ where the mapping $\nabla g:[0,T]^2\times\mathcal{X}\times(\mathbb{R}^{d_2}\times\mathbb{R}^{n\times d_2})^2\times\mathbb{R}^{d_1}\times\mathbb{R}^{n\times d_1}\longrightarrow\mathbb{R}^{d_2}$ is defined by*

$$
\nabla g_t(s,x,\mathrm{u},\mathrm{v},u,v,y,z):=\partial_s g_t(s,x,u,v,y,z)+\partial_u g_t(s,x,u,v,y,z)\mathrm{u}+\sum_{i=1}^n\partial_{v_{:i}}g_t(s,x,u,v,y,z)\mathrm{v}_{i:}.
$$

Set $(\tilde h.,\tilde g.(s),\nabla\tilde g.(s)):=(h.(\cdot,\mathbf{0},0),g.(s,\cdot,\mathbf{0}),\partial_s g.(s,\cdot,\mathbf{0}))$, for $\mathbf{0}:=(u,v,y,z)|_{(0,\dots,0)}$.

To ease the readability, in the rest of the document we will remove the dependence of the previous spaces on the underlying Euclidean spaces where the processes take value, *i.e.* we will write $\mathcal{S}^{\infty,c}$ for $\mathcal{S}^{\infty,c}(\mathbb{R}^{d_1})$.

Given $\mathcal{F}_T$-measurable (ξ,η) and h and g, with $\partial_s\eta$ and ∇g given by Assumption I, which we will refer as the data, we consider the following infinite family of BSDEs defined for $s\in[0,T]$, $\mathbb{P}$–a.s. for any $t\in[0,T]$ by

$$
\begin{aligned}
\mathcal{Y}_t&=\xi(T,X_{\cdot\wedge T})+\int_t^T h_r(X,\mathcal{Y}_r,\mathcal{Z}_r,U_r^r,V_r^r,\partial U_r^r)\mathrm{d}r-\int_t^T\mathcal{Z}_r^\top\mathrm{d}X_r-\int_t^T\mathrm{d}\mathcal{N}_r,\\
U_t^s&=\eta(s,X_{\cdot\wedge T})+\int_t^T g_r(s,X,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\mathrm{d}r-\int_t^T {V_r^s}^\top\mathrm{d}X_r-\int_t^T\mathrm{d}M_r^s,\\
\partial U_t^s&=\partial_s\eta(s,X_{\cdot\wedge T})+\int_t^T\nabla g_r(s,X,\partial U_r^s,\partial V_r^s,U_r^s,V_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\mathrm{d}r-\int_t^T\partial {V_r^s}^\top\mathrm{d}X_r-\int_t^T\mathrm{d}\partial M_r^s,
\end{aligned}
\tag{$\mathcal{S}$}
$$

Under the above assumption, we are interested in studying the well-posedness of $(\mathcal{S})$. For $c>0$, we define the space $(\mathcal{H}^c, \|\cdot\|_{\mathcal{H}^c})$, whose elements we denote $\mathfrak{h} = (Y, Z, N, U, V, M, \partial U, \partial V, \partial M)$, where

$$\mathcal{H}^c := \mathcal{S}^{\infty,c} \times \mathbb{H}^{2,c}_{\text{BMO}} \times \mathbb{M}^{2,c} \times \mathcal{S}^{\infty,2,c} \times \overline{\mathbb{H}}^{2,2}_{\text{BMO}} \times \mathbb{M}^{2,2,c} \times \mathcal{S}^{\infty,2,c} \times \mathbb{H}^{2,2,c}_{\text{BMO}} \times \mathbb{M}^{2,2,c}$$

and $\|\cdot\|_{\mathcal{H}^c}$ denotes the respective induced norm.

Remark 4.3.1. *(i) We highlight that system $(\mathcal{S})$ is fully coupled. This means that a solution to the system has to be determined simultaneously. Moreover, the reader might notice our choice to preven the generator of the first* BSDE *to depend on the diagonal of ∂V. This is due to our interest, as in* Chapter 3*, to establish the connection between these systems and* type-I BSVIEs (4.0.1)*. For this, the presence of the diagonal ∂U plays a key role. It should be clear from our arguments and* Lemma 4.2.3 *that these can be easily extended to accommodate this case.*

(ii) We remark that for any $c \geq 0$, the space $(\mathcal{H}^c, \|\cdot\|_{\mathcal{H}^c})$ is a Banach space. Indeed, it is clearly a normed space. Moreover, the fact that the spaces $\mathcal{S}^{\infty,c}$ and $\mathbb{M}^{2,c}$ are complete is classical in the literature. The completeness of the spaces $\mathbb{H}^{2,c}_{\text{BMO}}$ and $\mathbb{M}^{2,c}_{\text{BMO}}$, which are endowed with BMO*-type norms, follows from* Dellacherie and Meyer [70, CH. VII, Theorem 88]*. Indeed, a dual space is always complete. Finally, the completeness extends clearly to spaces of the form $\mathcal{S}^{\infty,2,c}, \mathbb{H}^{2,2,c}_{\text{BMO}}$ and $\mathbb{M}^{2,2,c}_{\text{BMO}}$.*

(iii) We also remark that implicit in the definition of $\mathcal{H}^c$ is the fact that ∂V coincides, $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$–a.e.*, with the density with respect to the Lebesgue measure of the application $s \longmapsto V^s$ which appears in the definition of the space $\overline{\mathbb{H}}^{2,2}_{\text{BMO}}$. This is for any $s \in [0,T]$*

$$V^s - V^0 = \int_0^s \partial V^r \mathrm{d}r, \text{ in } \mathbb{H}^{2,c}_{\text{BMO}}.$$

This contrasts with the result in the Lipschitz case obtained in Chapter 3 *where this was a consequence of the result, see also the remark after* Definition 3.4.1*. The reason for this is that the quadratic growth of the generator is incompatible with the contraction specified in the proof of* Theorem 3.4.3*.*

We now state precisely what we mean by a solution to $(\mathcal{S})$.

Definition 4.3.2. *A tuple* $\mathfrak{h} = (\mathcal{Y}, \mathcal{Z}, \mathcal{N}, U, V, M, \partial U, \partial V, \partial M)$ *is said to be a solution to* $(\mathcal{S})$ *with terminal condition* (ξ, η) *and generators* (f, g) *under* $\mathbb{P}$*, if* $\mathfrak{h}$ *satisfies* $(\mathcal{S})$ $\mathbb{P}$–a.s.*, and,* $\mathfrak{h} \in \mathcal{H}^c$ *for some* $c > 0$*.*

For $c > 0$ and $R > 0$, we define $\mathcal{B}_R \subseteq \mathcal{H}^c$ to be the subset of $\mathcal{H}^c$ of processes $(Y, Z, N, U, V, M) \in \mathcal{H}^c$ such that

$$\|(Y, Z, N, U, V, M)\|^2_{\mathcal{H}^c} \leq R^2.$$

The need to introduce $\mathcal{B}_R$ is inherent to the quadratic growth nature of $(\mathcal{S})$. Systems of the type of $(\mathcal{S})$ were studied in a Lipschitz framework in Chapter 2. By choosing a weight c large enough, and exploiting the fact that all weighted norms are equivalent, in Chapter 2 we obtained the well-posedness of $(\mathcal{S})$ in the space $\mathfrak{H}^2 := \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2 \times \mathbb{S}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{M}^{2,2} \times \mathbb{S}^{2,2} \times \mathbb{H}^{2,2} \times \mathbb{M}^{2,2}$. In the setting of this paper, the Lipschitz assumption for the generators is replaced by some kind of local quadratic growth. As a consequence, one cannot recover a contraction by simply choosing a weight large enough. In fact, as noticed in [238] and [153], given our growth assumptions, our candidate for providing a contractive map is no longer Lipschitz-continuous, but only locally Lipschitz-continuous. The idea, initially proposed in [238], is then to localise the usual procedure to a ball, thus making the application Lipschitz continuous again, and then to choose the radius of such a ball so as to recover a contraction. The crucial contribution of [238] is to show that such controls can be obtained by taking the data of the system small enough in norm.

Our procedure is inspired by this idea and incorporates it into the strategy devised in Chapter 3 to address the well–posedness of this kind of systems. We have decided to work on weighted spaces as, in our opinion, it does significantly simplify the arguments in the proof. We also mention that we tried to estimate the greatest ball, *i.e.* the largest radius R, for which such a localisation procedure leads to a contraction. Details are found in the proof. As we work on weighted spaces, we will find $c > 0$ large enough so that, given data with sufficiently small norm, $(\mathcal{S})$ has a unique solution in $\mathcal{B}_R \subseteq \mathcal{H}^c$. In words, throughout the proof, we will accumulate conditions on any candidate value for c that allows to verify the necessary steps to obtain the result. As such, an appropriate value of c must satisfy all such conditions. This should be clear from the statement of the result.

4.3.1 The Lipschitz–quadratic case

Assumption J. *(i)* $\exists \tilde c \in (0,\infty)$ *such that* $(\xi, \eta, \partial_s \eta, \tilde f, \tilde g, \nabla \tilde g) \in \mathcal{L}^{\infty,\tilde c} \times \mathcal{L}^{\infty,2,\tilde c} \times \mathbb{L}^{1,\infty,\tilde c} \times \mathbb{L}^{1,\infty,2,\tilde c}$.

(ii) $\exists (L_y, L_u, L_{\mathrm{u}}) \in (0,\infty)^3$ *s.t.* $\forall (s,t,x,y,\tilde y,u,\tilde u,\mathrm{u},\tilde{\mathrm{u}},z,v,\mathrm{v}) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^{d_1})^2 \times (\mathbb{R}^{d_2})^4 \times \mathbb{R}^{n\times d_1} \times (\mathbb{R}^{n\times d_2})^2$

$$\begin{aligned}&|h_t(x,y,z,u,v,\mathrm{u}) - h_t(x,\tilde y,z,\tilde u,v,\tilde{\mathrm{u}})| + |g_t(s,x,u,v,y,z) - g_t(s,x,\tilde u,v,\tilde y,z)|\\ &+ |\nabla g_t(s,x,\mathrm{u},\mathrm{v},u,v,y,z) - g_t(s,x,\tilde{\mathrm{u}},\mathrm{v},\tilde u,v,\tilde y,z)| \leq L_y|y-\tilde y| + L_u|u-\tilde u| + L_{\mathrm{u}}|\mathrm{u}-\tilde{\mathrm{u}}|;\end{aligned}$$

(iii) $\exists (L_z, L_v, L_{\mathrm{v}}) \in (0,\infty)^3$, $\phi \in \mathbb{H}^{2,\tilde c}_{\mathrm{BMO}}$ *s.t.* $\forall (s,t,x,y,u,\mathrm{u},z,\tilde z,v,\tilde v,\mathrm{v},\tilde{\mathrm{v}}) \in [0,T]^2 \times \mathcal{X} \times \mathbb{R}^{d_1} \times (\mathbb{R}^{d_2})^2 \times (\mathbb{R}^{n\times d_1})^2 \times (\mathbb{R}^{n\times d_2})^4$

$$\begin{aligned}&|h_t(x,y,z,u,v,\mathrm{u}) - h_t(x,y,\tilde z,u,\tilde v,\mathrm{u}) - (z-\tilde z)^\top \sigma_r(x)\phi_t|\\ &+ |g_t(s,x,u,v,y,z) - g_t(s,x,u,\tilde v,y,\tilde z) - (v-\tilde v)^\top \sigma_r(x)\phi_t|\\ &+ |\nabla g_t(s,x,\mathrm{u},\mathrm{v},u,v,y,z) - \nabla g_t(s,x,\mathrm{u},\tilde{\mathrm{v}},u,\tilde v,y,\tilde z) - (\mathrm{v}-\tilde{\mathrm{v}})^\top \sigma_r(x)\phi_t|\\ \leq\, & L_z\big(|\sigma_r^\top(x)z| + |\sigma_r^\top(x)\tilde z|\big)|\sigma_r^\top(x)(z-\tilde z)| + L_v\big(|\sigma_r^\top(x)v| + |\sigma_r^\top(x)\tilde v|\big)|\sigma_r^\top(x)(v-\tilde v)|\\ &+ L_{\mathrm{v}}\big(|\sigma_r^\top(x)\mathrm{v}| + |\sigma_r^\top(x)\tilde{\mathrm{v}}|\big)|\sigma_r^\top(x)(\mathrm{v}-\tilde{\mathrm{v}})|.\end{aligned}$$

Remark 4.3.3. *We now comment on the previous set of assumptions.* Assumption J.*(i)* *imposes integrability on the data of the system. We highlight that in our setting we require the integral with respect to the time variable of the generators* $(\tilde f, \tilde g, \nabla \tilde g)$ *to be bounded. This is in contrast to requiring the generators itself to be bounded. On the other hand,* Assumption J.*(ii)* *imposes a classic uniformly Lipschitz growth assumption on the* $(\mathcal{Y}, U, \partial U)$ *terms for the system. Finally,* Assumption J.*(iii)* *imposes a slight generalisation of a local Lipschitz–quadratic growth, this corresponds to the presence of the process* ϕ*, and is similar to the one found in* [238]. *This property is almost equivalent to saying that the underlying function is quadratic in* z. *The two properties would be equivalent if the process* ϕ *was bounded. Here we allow something a bit more general by letting* ϕ *be unbounded but in* $\mathbb{H}^2_{\mathrm{BMO}}$. *As we will see next, since this assumption allow us to apply the Girsanov transformation, we do not need to bound the processes and BMO type conditions are sufficient.*

Lastly, we also remark that ϕ is common for the three generators and do not depend on s. This is certainly a limitation in terms of the system $(\mathcal{S})$. Yet, as we are working towards establishing the well-posedness of the BSVIE (4.4.1) *we will see in* Section 4.4*, namely $(\mathcal{S}_f)$, that will chose h and g so that such condition is sensible.*

As a preliminary to our analysis, we note that Assumption J.(iii) can be simplified. This is the purpose of the next lemma. Therefore, without lost of generality in the rest of this section we assume $\phi = 0$.

Lemma 4.3.4. *Let*

$$\begin{aligned}
\hat{h}_t(x, y, z, u, v, \mathrm{u}) &:= h_t(x, y, z, u, v, \mathrm{u}) - z^\top \sigma_r(x)\phi_t,\\
\hat{g}_t(s, x, u, v, y, z) &:= g_t(s, x, u, v, y, z) - v^\top \sigma_r(x)\phi_t,\\
\nabla\hat{g}_t(s, x, \mathrm{u}, \mathrm{v}, u, v, y, z) &:= g_t(s, x, \mathrm{u}, \mathrm{v}, u, v, y, z) - \mathrm{v}^\top \sigma_r(x)\phi_t.
\end{aligned}$$

Then (Y, Z, N, U, V, N) is a solution to $(\mathcal{S})$ with terminal condition (ξ, η) and generator (f, g) under $\mathbb{P}$ if and only if (Y, Z, N, U, V, N) is a solution to $(\mathcal{S})$ with terminal condition (ξ, η) and generator $(\hat{f}, \hat{g})$ under $\mathbb{Q} \in \mathrm{Prob}(\Omega)$ given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\bigg(\int_0^T \phi_t \cdot \mathrm{d}X_t\bigg).$$

Proof. We first verify that $\mathbb{Q}$ above is well-defined. Indeed, from the fact that $\phi \in \mathbb{H}^{2,c}_{\mathrm{BMO}}(\mathbb{R}^m)$ we have that the process defined above is a uniformly integrable martingale and Girsanov's theorem holds, see [152, Section 1.3]. To verify the assertion of the lemma we note, for instance, that

$$\mathcal{Y}_t = \xi + \int_t^T \hat{h}_r(X, \mathcal{Y}_r, \mathcal{Z}_r, U_r^r, V_r^r, \partial U_r^r)\mathrm{d}r - \int_t^T Z_t^\top(\mathrm{d}X_r - \sigma_r\phi_r\mathrm{d}r) + \int_t^T \mathrm{d}N_r,\ t \in [0, T],\ \mathbb{Q}\text{–a.s.}$$

□

To ease the presentation of our result in the Lipschitz–quadratic case we introduce some notation. For $(\gamma, c, R) \in (0, \infty)^3$, $\varepsilon_i \in (0, \infty)$, $i \in \{1, ..., 11\}$, and $\kappa \in \mathbb{N}$, we define

$$I_0^\varepsilon := \|\xi\|^2_{\mathcal{L}^{\infty,c}} + 2\|\eta\|^2_{\mathcal{L}^{\infty,2,c}} + (1 + \varepsilon_1 + \varepsilon_2)\|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}}$$

$$+\varepsilon_3\|\tilde{h}\|^2_{\mathbb{L}^{1,\infty,c}}+(\varepsilon_4+\varepsilon_5)\|\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}}+(\varepsilon_1+\varepsilon_2+\varepsilon_6)\|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}},$$

$$L_\star := \max\{L_z, L_v, L_{\rm v}\},\ \mathcal{U}(\kappa) := \frac{1}{168\kappa L_\star^2},$$

and,

$$\begin{aligned} c^\varepsilon := \max\{&2L_y+\varepsilon_1^{-1}7TL_{\rm u}^2+(\varepsilon_1+\varepsilon_2)TL_y^2+\varepsilon_7^{-1}L_u^2+\varepsilon_8+\varepsilon_9+\varepsilon_{10},\\ &2L_u+\varepsilon_2^{-1}7T+\varepsilon_7+\varepsilon_8^{-1}L_y^2,\ 2L_{\rm u}+\varepsilon_{10}^{-1}L_y^2+\varepsilon_{11}^{-1}L_u^2\\ &2L_u+(\varepsilon_1+\varepsilon_2)TL_u^2+\varepsilon_9^{-1}L_y^2+\varepsilon_{11},\ 8L_y+2TL_y+2TL_{\rm u}L_y,\\ &4L_u+2TL_u+2TL_{\rm u}L_u\}. \end{aligned}$$

Remark 4.3.5. *Let us mention that the previous expressions arise in the analysis given our goal of finding the largest ball over which we can guarantee a contraction. In particular, for this reason there are several degrees of freedom, ε_i's, that determine c^ε. In particular, let us mention that there are many simplifying choices that can be made, at the risk of loosing some flexibility. For instance, letting $\tilde{\varepsilon}_i\in(0,\infty)^5$, $i\in\{1,\dots,5\}$, $\varepsilon_1=\varepsilon_2=\varepsilon_6=\tilde{\varepsilon}_1$, $\varepsilon_3=\tilde{\varepsilon}_2$, $\varepsilon_4=\varepsilon_5=\tilde{\varepsilon}_3$, $\varepsilon_7=\varepsilon_{11}=\tilde{\varepsilon}_4$ and $\varepsilon_8=\varepsilon_9=\varepsilon_{10}=\tilde{\varepsilon}_5$ we only need to choose 5 variables and*

$$\begin{aligned} I_0^{\tilde{\varepsilon}} &= \|\xi\|^2_{\mathcal{L}^{\infty,c}}+2\|\eta\|^2_{\mathcal{L}^{\infty,2,c}}+(1+2\tilde{\varepsilon}_1)\|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}}+\tilde{\varepsilon}_2\|\tilde{h}\|^2_{\mathbb{L}^{1,\infty,c}}+2\tilde{\varepsilon}_3\|\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}}+3\tilde{\varepsilon}_1\|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}},\\ c^\varepsilon &= \max\{2L_y+\tilde{\varepsilon}_1^{-1}7TL_{\rm u}^2+2\tilde{\varepsilon}_1TL_y^2+\tilde{\varepsilon}_4^{-1}L_u^2+3\tilde{\varepsilon}_5, 2L_u+\tilde{\varepsilon}_1^{-1}7T+\tilde{\varepsilon}_4+\tilde{\varepsilon}_5^{-1}L_y^2, 2L_{\rm u}+\tilde{\varepsilon}_5^{-1}L_y^2+\tilde{\varepsilon}_4^{-1}L_u^2\\ &\quad 2L_u+2\tilde{\varepsilon}_1TL_u^2+\tilde{\varepsilon}_5^{-1}L_y^2+\tilde{\varepsilon}_4,\ 8L_y+2TL_y+2TL_{\rm u}L_y,\ 4L_u+2TL_u+2TL_{\rm u}L_u\}. \end{aligned}$$

Assumption K. *Let $(\gamma,c,R)\in(0,\infty)^3$, $\varepsilon_i\in(0,\infty)$, $i\in\{1,\dots,11\}$, $\kappa\in\mathbb{N}$. We say* Assumption K *holds for κ if*

$$(\sqrt{\varepsilon_1+\varepsilon_2+3\kappa}+\sqrt{3\kappa})^2\leq 28\kappa,\ I_0^\varepsilon\leq\gamma R^2/\kappa,\ R^2<\mathcal{U}(\kappa),\ c\geq c^\varepsilon.$$

Theorem 4.3.6. *Let* Assumption J *holds. Suppose* Assumption K *holds for $\kappa=10$. Then, there exists a unique solution to $(\mathcal{S})$ in $\mathcal{B}_R\subseteq\mathcal{H}^c$ with*

$$R^2<\frac{1}{168\kappa L_\star^2}.$$

Remark 4.3.7. *Before we proceed with the proof, we would like to comment on both the qualitative and quantitative statements in* Theorem 4.3.6*.*

(*i*) *As a result of our procedure, and consistent with the results available in the literature, we require the data of* $(\mathcal{S})$ *to be sufficiently small in order to obtain the well–posedness of* $(\mathcal{S})$*. We stress this property on the data is influenced not only by the value of* γ *and* R*, but also, by the the value of* c *which determines the norms.*

(*ii*) *In addition, we introduced* Assumption K *which depends on a parameter* $\kappa \in \mathbb{N}$*. This parameter is related with the number of processes for which we have to keep track of the integrability of. In particular, we mention that, in the proof we present below, in addition to the 9 elements that prescribe a solution to the system we control the norm of the diagonal process* $(V_t^t)_{t\in[0,T]}$ *which appears in the definition of* $\overline{\mathbb{H}}^{2,2}$*. An alternative line of reasoning is available by leaving out the norm of process* $(V_t^t)_{t\in[0,T]}$ *in the definition of* $\overline{\mathbb{H}}^{2,2}$ *and exploit the inequality derived in* Lemma 4.2.3*.*

(*iii*) *We also want to point out that we tried to obtain the largest ball for which the whole procedure goes through. This can be appreciated in* Equation (4.5.8) *which introduces an upper bound on* R *for which our candidate map is well-defined. A word of caution nonetheless, since our bound of* R *cannot be directly compared to the ones obtained in* [238] *or* [153]*. Indeed, we recall that the norms involved depend on our choice of* c*. As such, the radius of the ball,* $\mathcal{B}_R \subseteq \mathcal{H}^c$*, in* Theorem 4.3.6 *depends on the choice of* c*. Consequently, if we were to exploit the equivalence between all the spaces* $\mathcal{H}^c$ *the radius would have to be appropriately adjusted.*

Remark 4.3.8. *Alternative versions of* Theorem 4.3.6 *are available. These are related to the orthogonal martingales.*

(*i*) *We first highlight that we are able to show, a posteriori, that the* $(\mathcal{N}, M, \partial M)$ *part of the solution to system* $(\mathcal{S})$ *in* $\mathcal{H}^c$ *actually has a finite* BMO*-type norm. This is proved in* Theorem 4.3.9*. We recall that in this quadratic setting with bounded terminal condition, this might be considered as a more natural type of integrability for the martingale part of the solutions.*

(*ii*) *Furthermore, one could wonder about how the statement of* Theorem 4.3.6 *changes in the case where* $(\mathcal{N}, M, \partial M)$ *are required, a priori, to have finite* BMO*-norm. This would allow, for instance, to keep a control on the* BMO*-norm inside the resulting ball. We cover this in* Theorem 4.3.10*.*

(iii) *Finally, we consider the case when the representation property for* $(\mathbb{F},\mathbb{P})$*-martingales in terms of stochastic integrals with respect to* X *holds. This is the case, for example, if* $\mathbb{P}$ *is an extremal point of the convex hull of the set of probability measures satisfying the requirements in* Section 4.2.1*, see* [142, Theorem 4.29]*. In such a scenario, we have that* $\mathcal{N}=M=\partial M=0$*. This case is covered in* Theorem 4.3.10*.*

Theorem 4.3.9. *Let* Assumption J *hold and* $\mathfrak{h}$ *be a solution to* $(\mathcal{S})$ *in the sense of* Definition 4.3.2*. Then*

$$\|\mathcal{N}\|^2_{\mathbb{M}^{2,c}_{\mathrm{BMO}}}+\|M\|^2_{\mathbb{M}^{2,2,c}_{\mathrm{BMO}}}+\|\partial M\|^2_{\mathbb{M}^{2,2,c}_{\mathrm{BMO}}}<\infty.$$

We now the address the well-posedness of $(\mathcal{S})$ in the following two scenarii

(i) when $(\mathcal{N},M,\partial M)\in\mathbb{M}^{2,c}_{\mathrm{BMO}}\times\mathbb{M}^{2,2,c}_{\mathrm{BMO}}\times\mathbb{M}^{2,2,c}_{\mathrm{BMO}}$, *i.e.* $(\mathcal{N},M,\partial M)$ are required to have finite BMO-norm;

(ii) when the predictable martingale representation property for $(\mathbb{F},\mathbb{P})$-martingales in term of stochastic integral with respect to X hold, *i.e.* both $\mathcal{N}$, M and ∂M vanish.

For this, let us introduce the spaces $(\widehat{\mathcal{H}}^c,\|\cdot\|_{\widehat{\mathcal{H}}^c})$ and $(\overline{\mathcal{H}}^c,\|\cdot\|_{\overline{\mathcal{H}}^c})$ for $c>0$ where

$$\widehat{\mathcal{H}}^c:=\mathcal{S}^{\infty,c}\times\mathbb{H}^{2,c}_{\mathrm{BMO}}\times\mathbb{M}^{2,c}_{\mathrm{BMO}}\times\mathcal{S}^{\infty,2,c}\times\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}\times\mathbb{M}^{2,2,c}_{\mathrm{BMO}}\times\mathcal{S}^{\infty,2,c}\times\mathbb{H}^{2,2,c}_{\mathrm{BMO}}\times\mathbb{M}^{2,2,c}_{\mathrm{BMO}},$$
$$\overline{\mathcal{H}}^c:=\mathcal{S}^{\infty,c}\times\mathbb{H}^{2,c}_{\mathrm{BMO}}\times\mathcal{S}^{\infty,2,c}\times\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}\times\mathcal{S}^{\infty,2,c}\times\mathbb{H}^{2,2,c}_{\mathrm{BMO}}.$$

and $\|\cdot\|_{\widehat{\mathcal{H}}^c}$ and $\|\cdot\|_{\overline{\mathcal{H}}^c}$ denote the associated norms. Moreover, $\widehat{\mathcal{B}}_R\subseteq\widehat{\mathcal{H}}^c$ (resp. $\overline{\mathcal{B}}_R\subseteq\overline{\mathcal{H}}^c$) denotes the ball of radius R for the norm $\|\cdot\|_{\widehat{\mathcal{H}}^c}$ (resp. $\|\cdot\|_{\overline{\mathcal{H}}^c}$).

Theorem 4.3.10. *Let* Assumption J *hold. If in addition*

(i) Assumption K *holds for* $\kappa=11$*, then, there exists a unique solution to* $(\mathcal{S})$ *in* $\widehat{\mathcal{B}}_R\subseteq(\widehat{\mathcal{H}}^c,\|\cdot\|_{\widehat{\mathcal{H}}^c})$;

(ii) Assumption K *holds for* $\kappa=8$*, and the predictable martingale representation property for* $(\mathbb{F},\mathbb{P})$*-martingales in term of stochastic integral with respect to* X *hold. Then, there exists a unique solution to* $(\mathcal{S})$ *in* $\overline{\mathcal{B}}_R\subseteq(\overline{\mathcal{H}}^c,\|\cdot\|_{\overline{\mathcal{H}}^c})$.

4.3.2 The quadratic case

Our approach allows us to consider also the case of quadratic generators as follows.

Assumption L. *(i)* $\exists \bar c > 0$ *s.t* $(\xi, \eta, \partial\eta, \tilde f, \tilde g, \nabla \tilde g) \in \mathcal{L}^{\infty,\bar c} \times \mathcal{L}^{\infty,\bar c,2} \times \mathcal{L}^{\infty,\bar c,2} \times \mathbb{L}^{1,\infty,\bar c} \times \mathbb{L}^{1,\infty,\bar c,2} \times \mathbb{L}^{1,\infty,\bar c,2}$.

(ii) $\exists (L_y, L_u, L_{\mathrm{u}}) \in (0,\infty)^3,$ *s.t.* $\forall (s,t,x,y,\tilde y, u, \tilde u, \mathrm{u}, \tilde{\mathrm{u}}, z, v, \mathrm{v}) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^{d_1})^2 \times (\mathbb{R}^{d_2})^4 \times \mathbb{R}^{n\times d_1} \times (\mathbb{R}^{n\times d_2})^2$

$$\begin{aligned}
&|h_t(x,y,z,u,v,\mathrm{u}) - h_t(x,\tilde y, z, \tilde u, v, \tilde{\mathrm{u}})| \leq L_y|y-\tilde y|(|y|+|\tilde y|) + L_u|u-\tilde u|(|u|+|\tilde u|) + L_{\mathrm{u}}|\mathrm{u}-\tilde{\mathrm{u}}|,\\
&|g_t(s,x,u,v,y,z) - g_t(s,x,\tilde u, v, \tilde y, z)| + |\nabla g_t(s,x,\mathrm{u},\mathrm{v},u,v,y,z) - \nabla g_t(s,x,\tilde{\mathrm{u}},\mathrm{v},\tilde u, v\tilde y, z)|\\
\leq\, & L_{\mathrm{u}}|\mathrm{u}-\tilde{\mathrm{u}}|\big||\mathrm{u}|+|\tilde{\mathrm{u}}|\big| + L_u|u-\tilde u|\big||u|+|\tilde u|\big| + L_y|y-\tilde y|\big||y|+|\tilde y|\big|;
\end{aligned}$$

(iii) $\exists (L_z, L_v, L_{\mathrm{v}}) \in (0,\infty)^3,\ \phi \in \mathbb{H}^{2,\bar c}_{\mathrm{BMO}}$ *s.t.* $\forall (s,t,x,y,u,\mathrm{u},z,\tilde z, v, \tilde v, \mathrm{v}, \tilde{\mathrm{v}}) \in [0,T]^2 \times \mathcal{X} \times \mathbb{R}^{d_1} \times (\mathbb{R}^{d_2})^2 \times (\mathbb{R}^{n\times d_1})^2 \times (\mathbb{R}^{n\times d_2})^4$

$$\begin{aligned}
&|h_t(x,y,z,u,v,\mathrm{u}) - h_t(x,y,\tilde z,u,\tilde v,\mathrm{u}) - (z-\tilde z)^\top \sigma_r(x)\phi_t|\\
&+ |g_t(s,x,u,v,y,z) - g_t(s,x,u,\tilde v,y,\tilde z) - (v-\tilde v)^\top \sigma_r(x)\phi_t|\\
&+ |\nabla g_t(s,x,\mathrm{u},\mathrm{v},u,v,y,z) - \nabla g_t(s,x,\mathrm{u},\tilde{\mathrm{v}},u,\tilde v,y,\tilde z) - (\mathrm{v}-\tilde{\mathrm{v}})^\top \sigma_r(x)\phi_t|\\
\leq\, & L_z\big||\sigma_r^\top(x)z| + |\sigma_r^\top(x)\tilde z|\big||\sigma_r^\top(x)(z-\tilde z)| + L_v\big||\sigma_r^\top(x)v| + |\sigma_r^\top(x)\tilde v|\big||\sigma_r^\top(x)(v-\tilde v)|\\
&+ L_{\mathrm{v}}\big||\sigma_r^\top(x)\mathrm{v}| + |\sigma_r^\top(x)\tilde{\mathrm{v}}|\big||\sigma_r^\top(x)(\mathrm{v}-\tilde{\mathrm{v}})|.
\end{aligned}$$

Remark 4.3.11. *In the previous set of assumptions we have allowed the generators to have quadratic growth in all of the terms, with the exception of h on the term* u*. The reason for this are twofold: (i) for the kind of systems that motivate our analysis the term $(\partial U^t_t)_{t\in[0,T]}$ always appears linearly in the generator, (ii) when making the connection with the* type-I BSVIE (4.0.1) *$(\partial U^t_t)_{t\in[0,T]}$ plays the auxiliary role of keeping track of the diagonal processes $(U^t_t)_{t\in[0,T]}$ and $(V^t_t)_{t\in[0,T]}$ and for this it suffices that it appears linearly in the generator. This is, the Lipschitz assumption on this variable will suffice for the purposes of our results and the problems that motivated our study.*

As before, we need to introduce some auxiliary notation. For $(\gamma, c, R) \in (0,\infty)^3$, $\varepsilon_i \in (0,\infty)$, $i \in \{1,...,6\}$, and $\kappa \in \mathbb{N}$, we let I_0^ε be as in Section 4.3.1 and define

$$L_\star := \max\{L_y, L_u, L_{\mathrm{u}}, L_z, L_v, L_{\mathrm{v}}\},\ c^\varepsilon := \max\{\varepsilon_1^{-1}7TL_{\mathrm{u}}^2, \varepsilon_2^{-1}7T, 2L_{\mathrm{u}}\}, \mathcal{U}(\kappa) := \frac{1}{336\kappa L_\star^2 \max\{2, T^2\}}.$$

Assumption M. *Let* $(\gamma, c, R) \in (0,\infty)^3$, $\varepsilon_i \in (0,\infty)$, $i \in \{1,\dots,6\}$, *and* $\kappa \in \mathbb{N}$. *We say* Assumption M *holds for* κ *if*

$$(\sqrt{\varepsilon_1 + \varepsilon_2 + 3\kappa} + \sqrt{3\kappa})^2 \leq 56\kappa,\ I_0^\varepsilon \leq \gamma R^2/\kappa,\ R^2 < \mathcal{U}(\kappa),\ c \geq c^\varepsilon.$$

Theorem 4.3.12. *Let* Assumption L *holds. Suppose* Assumption M *holds for* $\kappa = 10$. *Then, there exists a unique solution to* $(\mathcal{S})$ *in* $\mathcal{B}_R \subseteq \mathcal{H}^c$ *with*

$$R^2 < \frac{1}{336\kappa L_\star^2 \max\{2, T^2\}}.$$

Remark 4.3.13. *We remark that the best way to appreciate the previous result is by contrasting it with our well-posedness result in the linear–quadratic case. For simplicity, let us assume* $L_y = L_u = L_{\mathrm{u}} = L_z = L_v = L_{\mathrm{v}}$. *Regarding the constraint on the weight parameter of the resulting norm, the result is pretty much in the same order of magnitude. Nevertheless, the most noticeable feature is that by allowing the system to have quadratic growth the greatest radius under which our argument is able to guarantee the well-posedness of the solution decreases by a factor* $2\max\{2, T^2\}$. *This quantity can be significant in light of its dependence on the time horizon* T.

4.4 Muldimensional quadratic type-I BSVIEs

We now address the well-posedness of multidimensional linear–quadratic and quadratic type-I BSVIEs. Let d be a non-negative integer, and f and ξ be jointly measurable functionals such that for any $(s, y, z, u, v) \in [0,T] \times (\mathbb{R}^d \times \mathbb{R}^{n\times d})^2$

$$f : [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^d \times \mathbb{R}^{n\times d})^2 \longrightarrow \mathbb{R}^d,\ f_\cdot(s,\cdot,y,z,u,v) \in \mathcal{P}_{\mathrm{prog}}(\mathbb{R}^d, \mathbb{F}),$$
$$\xi : [0,T] \times \mathcal{X} \longrightarrow \mathbb{R}^d,\ \xi(s,\cdot) \text{ is } \mathcal{F}\text{-measurable.}$$

The main result in this section follows exploiting the well-posedness of $(\mathcal{S})$. Therefore, we work under the following set of assumptions.

Assumption N. $(s, y, z) \longmapsto f_t(s, x, y, z, u, v)$ *(resp.* $s \longmapsto \xi(s, x)$*) is continuously differentiable, uniformly in* (t, x, u, v) *(resp. in* x*). Moreover,* $\nabla f_\cdot(s, \cdot, y, z, u, v, \mathrm{u}, \mathrm{v}) \in \mathcal{P}_{\mathrm{prog}}(\mathbb{R}^d, \mathbb{F}), s \in [0,T]$, *where the mapping* $\nabla f : [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^d \times \mathbb{R}^{n\times d})^3 \longrightarrow \mathbb{R}^d$ *is defined by*

$$\nabla f_t(s,x,\mathrm{u},\mathrm{v},y,z,u,v) := \partial_s f_t(s,x,y,z,u,v) + \partial_y f_t(s,x,y,z,u,v)\mathrm{u} + \sum_{i=1}^n \partial_{z_{:i}} f_t(s,x,y,z,u,v)\mathrm{v}_{i:}.$$

Set $(\tilde f_\cdot, \tilde f_\cdot(s), \nabla \tilde f_\cdot(s)) := \big(f_\cdot(\cdot,\mathbf{0},0), f_\cdot(s,\cdot,\mathbf{0}), \partial_s f_\cdot(s,\cdot,\mathbf{0})\big)$, *for* $\mathbf{0} := (u,v,y,z)|_{(0,\dots,0)}$.

Let $(\mathcal{H}^{\star,c}, \|\cdot\|_{\mathcal{H}^{\star,c}})$ denote the space of $(Y,Z,N) \in \mathcal{H}^{\star,c}$ such that $\|(Y,Z,N)\|_{\mathcal{H}^{\star,c}} < \infty$ where

$$\mathcal{H}^{\star,c} := \mathcal{S}^{\infty,2,c} \times \overline{\mathbb{H}}^{2,2}_{\mathrm{BMO}} \times \mathbb{M}^{2,2,c}_{\mathrm{BMO}},\ \|\cdot\|_{\mathcal{H}^{\star,c}} := \|Y\|^2_{\mathcal{S}^{\infty,2,c}} + \|Z\|^2_{\overline{\mathbb{H}}^{2,2}} + \|N\|^2_{\mathbb{M}^{2,2}_{\mathrm{BMO}}}.$$

We consider the n-dimensional type-I BSVIE on $(\mathcal{H}^\star, \|\cdot\|_{\mathcal{H}^\star})$, which for any $s \in [0,T]$ satisfies

$$Y_t^s = \xi(s,X) + \int_t^T f_r(s,X,Y_r^s,Z_r^s,Y_r^r,Z_r^r)\mathrm{d}r - \int_t^T Z_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}N_r^s,\ t\in[0,T],\ \mathbb{P}\text{–a.s.} \tag{4.4.1}$$

We work under the following notion of solution.

Definition 4.4.1. *We say* (Y,Z,N) *is a solution to the* type-I BSVIE (4.4.1) *if* $(Y,Z,N) \in \mathcal{H}^\star$ *verifies* (4.4.1).

We may consider the system, given for any $s \in [0,T]$ by

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T,X) + \int_t^T\Big(f_r(r,X,\mathcal{Y}_r,\mathcal{Z}_r,Y_r^r,Z_r^r) - \partial Y_r^r\Big)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top \mathrm{d}X_r - \int_t^T \mathrm{d}\mathcal{N}_r,\\
Y_t^s &= \xi(s,X) + \int_t^T f_r(s,X,Y_r^s,Z_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\mathrm{d}r - \int_t^T Z_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}N_r^s,\\
\partial Y_t^s &= \partial_s\xi(s,X) + \int_t^T \nabla f_r(s,X,\partial Y_r^s,\partial Z_r^s,Y_r^s,Z_r^s,\mathcal{Y}_r,\mathcal{Z}_r)\mathrm{d}r - \int_t^T \partial Z_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial N_r^s.
\end{aligned} \tag{$\mathcal{S}_f$}$$

Remark 4.4.2. *Let us briefly comment that our set-up for the study* type-I BSVIE (4.4.1) *is based on the systems introduced in* Section 4.3*. As such, the necessity of the set of assumptions in* Assumption N *is clear.*

We are now in position to prove the main result of this paper. The next result shows that under the previous choice of data for $(\mathcal{S}_f)$, its solution solves the type-I BSVIE with data (ξ, f) and *vice versa* in both the linear–quadratic and quadratic case. For this we introduce the following set of assumptions.

Assumption O. *(i)* $\exists \tilde{c} \in (0,\infty)$ *such that* $(\xi, \eta, \partial_s \eta, \tilde{f}, \tilde{g}, \nabla \tilde{g}) \in \mathcal{L}^{\infty,\tilde{c}} \times \mathcal{L}^{\infty,2,\tilde{c}} \times \mathbb{L}^{1,\infty,\tilde{c}} \times \mathbb{L}^{1,\infty,2,\tilde{c}}$;

(ii) $\exists (L_y, L_u, L_{\mathrm{u}}) \in (0,\infty)^3$ *s.t.* $\forall (s,t,x,y,\tilde{y},u,\tilde{u},\mathrm{u},\tilde{\mathrm{u}},z,v,\mathrm{v}) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^6 \times \mathbb{R}^{n\times d} \times (\mathbb{R}^{n\times d})^3$

$$\begin{aligned} & |f_t(s,x,y,z,u,v) - f_t(s,x,\tilde{y},z,\tilde{u},v)| + |\nabla f_t(s,x,\mathrm{u},\mathrm{v},y,z,u,v) - \nabla f_t(s,x,\tilde{\mathrm{u}},\mathrm{v},\tilde{y},z,\tilde{u},v)| \\ \leq\, & L_y|y-\tilde{y}| + L_u|u-\tilde{u}| + L_{\mathrm{u}}|\mathrm{u}-\tilde{\mathrm{u}}|; \end{aligned}$$

(iii) $\exists (L_z, L_v, L_{\mathrm{v}}) \in (0,\infty)^3,\ \phi \in \mathbb{H}^{2,\tilde{c}}_{\mathrm{BMO}}$ *s.t.* $\forall (s,t,x,y,u,\mathrm{u},z,\tilde{z},v,\tilde{v},\mathrm{v},\tilde{\mathrm{v}}) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^3 \times (\mathbb{R}^{n\times d})^6$

$$\begin{aligned} & |f_t(s,x,y,z,u,v) - f_t(s,x,y,\tilde{z},u,\tilde{v}) - (z-\tilde{z})^\top \sigma_r(x)\phi_t| \\ & + |\nabla f_t(s,x,\mathrm{u},\mathrm{v},y,z,u,v) - \nabla f_t(s,x,\mathrm{u},\tilde{\mathrm{v}},y,\tilde{z},u,\tilde{v}) - (\mathrm{v}-\tilde{\mathrm{v}})^\top \sigma_r(x)\phi_t| \\ \leq\, & L_z\big(|\sigma_r^\top(x)z| + |\sigma_r^\top(x)\tilde{z}|\big)|\sigma_r^\top(x)(z-\tilde{z})| + L_v\big(|\sigma_r^\top(x)v| + |\sigma_r^\top(x)\tilde{v}|\big)|\sigma_r^\top(x)(v-\tilde{v})| \\ & + L_{\mathrm{v}}\big(|\sigma_r^\top(x)\mathrm{v}| + |\sigma_r^\top(x)\tilde{\mathrm{v}}|\big)|\sigma_r^\top(x)(\mathrm{v}-\tilde{\mathrm{v}})|. \end{aligned}$$

Assumption P. *(i)* $\exists \tilde{c} \in (0,\infty)$ *such that* $(\xi, \eta, \partial_s \eta, \tilde{f}, \tilde{g}, \nabla \tilde{g}) \in \mathcal{L}^{\infty,\tilde{c}} \times \mathcal{L}^{\infty,2,\tilde{c}} \times \mathbb{L}^{1,\infty,\tilde{c}} \times \mathbb{L}^{1,\infty,2,\tilde{c}}$.

(ii) $\exists (L_y, L_u, L_{\mathrm{u}}) \in (0,\infty)^3$ *s.t.* $\forall (s,t,x,y,\tilde{y},u,\tilde{u},\mathrm{u},\tilde{\mathrm{u}},z,v,\mathrm{v}) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^6 \times \mathbb{R}^{n\times d} \times (\mathbb{R}^{n\times d_2})^3$

$$\begin{aligned} & |f_t(s,x,y,z,u,v) - f_t(s,x,\tilde{y},z,\tilde{u},v)| + |\nabla f_t(s,x,\mathrm{u},\mathrm{v},y,z,u,v) - \nabla f_t(s,x,\tilde{\mathrm{u}},\mathrm{v},\tilde{y},z,\tilde{u},v)| \\ \leq\, & L_y|y-\tilde{y}|\big(|y|+|\tilde{y}|\big) + L_u|u-\tilde{u}|\big(|u|+|\tilde{u}|\big) + L_{\mathrm{u}}|\mathrm{u}-\tilde{\mathrm{u}}|\big(|\mathrm{u}|+|\tilde{\mathrm{u}}|\big); \end{aligned}$$

(iii) $\exists (L_z, L_v, L_{\mathrm{v}}) \in (0,\infty)^3,\ \phi \in \mathbb{H}^{2,\tilde{c}}_{\mathrm{BMO}}$ *s.t.* $\forall (s,t,x,y,u,\mathrm{u},z,\tilde{z},v,\tilde{v},\mathrm{v},\tilde{\mathrm{v}}) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^3 \times (\mathbb{R}^{n\times d})^6$

$$\begin{aligned} & |f_t(s,x,y,z,u,v) - f_t(s,x,y,\tilde{z},u,\tilde{v}) - (z-\tilde{z})^\top \sigma_r(x)\phi_t| \\ & + |\nabla f_t(s,x,\mathrm{u},\mathrm{v},y,z,u,v) - \nabla f_t(s,x,\mathrm{u},\tilde{\mathrm{v}},y,\tilde{z},u,\tilde{v}) - (\mathrm{v}-\tilde{\mathrm{v}})^\top \sigma_r(x)\phi_t| \\ \leq\, & L_z\big(|\sigma_r^\top(x)z| + |\sigma_r^\top(x)\tilde{z}|\big)|\sigma_r^\top(x)(z-\tilde{z})| + L_v\big(|\sigma_r^\top(x)v| + |\sigma_r^\top(x)\tilde{v}|\big)|\sigma_r^\top(x)(v-\tilde{v})| \\ & + L_{\mathrm{v}}\big(|\sigma_r^\top(x)\mathrm{v}| + |\sigma_r^\top(x)\tilde{\mathrm{v}}|\big)|\sigma_r^\top(x)(\mathrm{v}-\tilde{\mathrm{v}})|. \end{aligned}$$

Theorem 4.4.3. *Let* Assumption N *hold. Then, the well-posedness of* $(\mathcal{S}_f)$ *is equivalent to that* *of the* type-I BSVIE (4.4.1) *if either:*

(i) Assumption O *holds and* Assumption K *holds for* $\kappa = 7$. *In such case, there exists a unique solution to the* type-I BSVIE (4.4.1) *in* $\mathcal{B}_R \subseteq \mathcal{H}^{\star,c}$ *with*

$$R^2 < \frac{1}{168\kappa L_\star^2}.$$

(ii) Assumption P *holds and* Assumption M *holds for* $\kappa = 7$. *In such case, there exists a unique solution to the* type-I BSVIE (4.4.1) *in* $\mathcal{B}_R \subseteq \mathcal{H}^{\star,c}$ *with*

$$R^2 < \frac{1}{336\kappa L_\star^2 \max\{2, T^2\}}.$$

Proof. Let us first note that the second part of the statements in (ii) and (iii) are a direct consequente of Theorem 4.3.6 and Theorem 4.3.12, respectively.

Let us first argue why it suffices to have Assumption K and Assumption M hold for $\kappa = 7$ instead of 10. This follows from the specification of data for $(\mathcal{S}_f)$. Indeed, following the notation of the proof of Theorem 4.3.6 in this case we have that $\mathcal{Y} = \mathcal{U}$, $\mathcal{Z} = \mathcal{V}$ and $\mathcal{N} = \mathcal{M}$ so the auxiliary equation introduced in **step 1** (ii) is not necessary and, as from (4.5.6), the argument in the proof holds with 7 instead on 10.

We are only left to argue the equivalence of the solutions. Let us argue (i), the argument for (ii) is analogous. For this we follow Theorem 3.4.3. Let $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, Y, Z, N, \partial Y, \partial Z, \partial N) \in \mathcal{B}_R \subseteq \mathcal{H}^c$ be a solution to $(\mathcal{S}_f)$. It then follows from Lemma 3.6.2 that

$$Y_t^t = \xi(T, X) + \int_t^T \Big(f_r(r, X, Y_r^r, Z_r^r, \mathcal{Y}_r, \mathcal{Z}_r) - \partial Y_r^r \Big) \mathrm{d}r - \int_t^T Z_r^{r\top} \mathrm{d}X_r - \int_t^T \mathrm{d}\widetilde{N}_r,\ t \in [0, T],\ \mathbb{P}\text{–a.s.},$$

where $\widetilde{N}_t := N_t^t - \int_0^t \partial N_r^r \mathrm{d}r$, $t \in [0, T]$, and $\widetilde{N} \in \mathbb{M}^{2,c}$. As in Theorem 4.3.9, we obtain that $\widetilde{N} \in \mathbb{M}^{2,c}_{\mathrm{BMO}}$. This shows that $\big((Y_t^t)_{t\in[0,T]}, (Z_t^t)_{t\in[0,T]}, \mathcal{Y}_\cdot, \mathcal{Z}_\cdot, (\widetilde{N}_t)_{t\in[0,T]}\big)$, solves the first BSDE in $(\mathcal{S}_f)$. It then follows from the well-posedness of $(\mathcal{S}_f)$, which holds by Assumption N, P, M and Theorem 4.3.6, that $\big((Y_t^t)_{t\in[0,T]}, (Z_t^t)_{t\in[0,T]}, (\widetilde{N}_t)_{t\in[0,T]}\big) = (\mathcal{Y}_\cdot, \mathcal{Z}_\cdot, \mathcal{N}_\cdot)$ in $\mathcal{S}^{2,c} \times \mathbb{H}^{2,c}_{\mathrm{BMO}} \times \mathbb{M}^{2,c}_{\mathrm{BMO}}$ and the result follows.

We show the converse result. Let $(Y, Z, N) \in \mathcal{B}_R \subseteq \mathcal{H}^{\star,c}$ be a solution to type-I BSVIE (4.4.1).

It is clear that the processes $\mathcal{Y} := (Y_t^t)_{t\in[0,T]}, \mathcal{Z} := (Z_t^t)_{t\in[0,T]}, \mathcal{N} := (N_t^t)_{t\in[0,T]}$ are well-defined. Then, since Assumption N holds and $(\mathcal{Y},\mathcal{Z},Y,Z,N) \in \mathbb{L}^{1,\infty,c}\times\mathbb{H}^{2,c}_{\mathrm{BMO}}\times\mathcal{S}^{2,2,c}\times\overline{\mathbb{H}}^{2,2}_{\mathrm{BMO}}\times\mathbb{M}^{2,2,c}_{\mathrm{BMO}}$, we can apply Lemma 3.6.1 and obtain the existence of $(\partial Y,\partial Z,\partial N)\in\mathcal{S}^{2,2,c}\times\mathbb{H}^{2,2,c}\times\mathbb{M}^{2,2,c}$ such that for $s\in[0,T]$

$$\partial Y_t^s = \partial_s\xi(s,X) + \int_t^T \nabla f_r(s,X,\partial Y_r^s,\partial Z_r^s,Y_r^s,Z_r^s,Y_r^r,Z_r^r)\mathrm{d}r - \int_t^T \partial Z_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial N_r^s.$$

Moreover, from the fact that Assumption K holds for $\kappa=7$ we obtain that $\|\partial Y\|^2_{\mathcal{S}^{2,2,c}}+\|\partial Z\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}+\|\partial N\|^2_{\mathbb{M}^{2,2,c}}\leq R^2$.

Let us claim that $\mathfrak{h} := (\mathcal{Y},\mathcal{Z},\widetilde{N},Y,Z,N,\partial Y,\partial Z,\partial N)$ is a solution to $(\mathcal{S}_f)$, where $\widetilde{N}_t := N_t^t - \int_0^t \partial N_r^r\mathrm{d}r$, $t\in[0,T]$. For this, we first note that in light of Lemma 3.6.1 and 3.6.2 we have that

$$\mathcal{Y}_t = \xi(T,X) + \int_t^T h_r(X,\mathcal{Y}_r,\mathcal{Z}_r,Y_r^r,Z_r^r,\partial Y_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r{}^\top\mathrm{d}X_r - \int_t^T \mathrm{d}\widetilde{N}_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

Now, $\widetilde{\mathcal{N}}\in\mathbb{M}^{2,2,c}_{\mathrm{BMO}}$ follows as in Theorem 4.3.9. As in **step 1** (*iii*) in the proof of Theorem 4.3.6, we obtain that $\mathcal{Y}\in\mathcal{S}^{2,c}$. We are only left to argue $\|\mathfrak{h}\|\leq R$. This follows readily following **step 1** (*iv*) in the proof of Theorem 4.3.6 and the fact that $\|\mathcal{Z}\|^2_{\mathbb{H}^{2,c}}+\|Z\|^2_{\overline{\mathbb{H}}^{2,2}_{\mathrm{BMO}}}+\|\partial Z\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\leq R^2$. □

4.5 Proof of the linear–quadratic case

Proof of Theorem 4.3.6. For $c>0$, let us introduce the mapping

$$\begin{aligned}\mathfrak{T} : (\mathcal{B}_R,\|\cdot\|_{\mathcal{H}^c}) &\longrightarrow (\mathcal{B}_R,\|\cdot\|_{\mathcal{H}^c})\\ (y,z,n,u,v,m,\partial u,\partial v,\partial m) &\longmapsto (Y,Z,N,U,V,M,\partial U,\partial V,\partial M),\end{aligned}$$

with $(\mathcal{Y},\mathcal{Z},\mathcal{N},U,V,M,\partial U,\partial V,\partial M)$ given for any $s\in[0,T]$, $\mathbb{P}$–a.s. for any $t\in[0,T]$ by

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T,X_{\cdot\wedge T}) + \int_t^T h_r(X,\mathcal{Y}_r,z_r,U_r^r,v_r^r,\partial U_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top dX_r - \int_t^T \mathrm{d}\mathcal{N}_r,\\
U_t^s &= \eta(s,X_{\cdot\wedge,T}) + \int_t^T g_r(s,X,U_r^s,v_r^s,\mathcal{Y}_r,z_r)\mathrm{d}r - \int_t^T V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}M_r^s,\\
\partial U_t^s &= \partial_s\eta(s,X_{\cdot\wedge,T}) + \int_t^T \nabla g_r(s,X,\partial U_r^s,\partial v_r^s,U_r^s,v_r^s,\mathcal{Y}_r,z_r)\mathrm{d}r - \int_t^T \partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s.
\end{aligned}$$

Step 1: We first argue that $\mathfrak{T}$ is well-defined.

(i) In light of Assumption J, there is $c>0$ such that $(\xi,\eta,\partial\eta,\tilde f,\tilde g,\nabla\tilde g)\in\mathcal{L}^{\infty,c}\times\big(\mathcal{L}^{\infty,2,c}\big)^2\times\mathbb{L}^{1,\infty,c}\times\big(\mathbb{L}^{1,\infty,2,c}\big)^2$ and $(z,v,\partial v)\in\mathbb{H}^{2,c}_{\rm BMO}\times\overline{\mathbb{H}}^{2,2}_{\rm BMO}\times\mathbb{H}^{2,2,c}_{\rm BMO}$, thus, we may use (4.2.1) to obtain

$$\begin{aligned}
&\mathbb{E}\bigg[|\xi(T)|^2+\bigg|\int_0^T|h_t(0,z_t,0,v_t^t,0)|\mathrm{d}t\bigg|^2\bigg]+\sup_{s\in[0,T]}\mathbb{E}\bigg[|\eta(s)|^2+\bigg|\int_0^T|g_t(s,0,v_t^s,0,z_t)|\mathrm{d}t\bigg|^2\bigg]\\
&\quad+\sup_{s\in[0,T]}\mathbb{E}\bigg[|\partial\eta(s)|^2+\bigg|\int_0^T|\nabla g_t(s,0,\partial v_t^s,0,v_t^s,0,z_t)|\mathrm{d}t\bigg|^2\bigg]\\
&\leq\mathbb{E}\bigg[|\xi(T)|^2+3\bigg|\int_0^T|\tilde h_t|\mathrm{d}t\bigg|^2+10L_z^2\bigg|\int_0^T|z_t|^2\mathrm{d}t\bigg|^2+3L_v^2\bigg|\int_0^T|v_t^t|^2\mathrm{d}t\bigg|^2\bigg]\\
&\quad+\sup_{s\in[0,T]}\mathbb{E}\bigg[|\eta(s)|^2+3\bigg|\int_0^T|\tilde g_t(s)|\mathrm{d}t\bigg|^2+7L_v^2\bigg|\int_0^T|v_t^s|^2\mathrm{d}t\bigg|^2\bigg]\\
&\quad+\sup_{s\in[0,T]}\mathbb{E}\bigg[|\partial\eta(s)|^2+4\bigg|\int_0^T|\nabla\tilde g_t(s)|\mathrm{d}t\bigg|^2+4L_{\rm v}^2\bigg|\int_0^T|\partial v_t^s|^2\mathrm{d}t\bigg|^2\bigg]\\
&\leq\|\xi\|^2_{\mathcal{L}^{\infty,c}}+3\|\tilde h\|^2_{\mathbb{L}^{1,\infty,c}}+\|\eta\|^2_{\mathcal{L}^{\infty,2,c}}+3\|\tilde g\|^2_{\mathbb{L}^{1,\infty,2,c}}+\|\partial\eta\|^2_{\mathcal{L}^{\infty,2,c}}+4\|\nabla\tilde g\|^2_{\mathbb{L}^{1,\infty,2,c}}\\
&\quad+20L_z^2\|z\|^4_{\mathbb{H}^{2,c}_{\rm BMO}}+14L_v^2\|v\|^4_{\overline{\mathbb{H}}^{2,2,c}_{\rm BMO}}+8L_{\rm v}^2\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\rm BMO}}<\infty.
\end{aligned}$$

Therefore, by Theorem 3.3.5, $\mathfrak{T}$ defines a well-posed system of BSDEs with unique solution in the space $\mathfrak{H}^2$. We recall the spaces involved in the definition of $\mathfrak{H}^2$, and their corresponding norms, were introduced in Section 4.2.2.

(ii) Arguing as in Lemma 3.6.1 and 3.6.2, we may use Assumption J and $v\in\overline{\mathbb{H}}^{2,2}_{\rm BMO}$, *i.e.* ∂v is the density with respect to the Lebesgue measure of $s\longmapsto v^s$, to obtain that $(\mathcal{U},\mathcal{V},\mathcal{M})\in\mathbb{S}^2\times\mathbb{H}^2\times\mathbb{M}^2$ given by

$$\mathcal{U}_t:=U_t^t,\ \mathcal{V}_t:=V_t^t,\ \mathcal{M}_t:=M_t^t-\int_0^t\partial M_r^r\mathrm{d}r,\ t\in[0,T],$$

satisfy the equation

$$\mathcal{U}_t=\eta(T,X_{\cdot\wedge T})+\int_t^T\big(g_r(r,X,\mathcal{U}_r,v_r^r,\mathcal{Y}_r,z_r)-\partial U_r^r\big)\mathrm{d}r-\int_t^T\mathcal{V}_r{}^\top\mathrm{d}X_r-\int_t^T\mathrm{d}\mathcal{M}_r,\ t\in[0,T],\mathbb{P}\text{–a.s.}$$

(iii) We show $(\mathcal{Y},\mathcal{U})\in\mathcal{S}^{\infty,c}\times\mathcal{S}^{\infty,c}$ and $\|U\|_{\mathcal{S}^{\infty,2,c}}+\|\partial U\|_{\mathcal{S}^{\infty,2,c}}<\infty$.

To alleviate the notation we introduce

$$h_r := h_r(\mathcal{Y}_r, z_r, \mathcal{U}_r, v_r^r, \partial U_r^r),\ g_r := g_r(r, \mathcal{U}_r, v_r^r, \mathcal{Y}_r, z_r) - \partial U_r^r,$$
$$g_r(s) := g_r(s, U_r^s, v_r^s, \mathcal{Y}_r, z_r),\ \nabla g_r(s) := \nabla g_r(s, \partial U_r^s, \partial v_r^s, U_r^s, v_r^s, \mathcal{Y}_r, z_r),$$

and

$$\mathfrak{Y} := (\mathcal{Y}, \mathcal{U}, U^s, \partial U^s),\ \mathfrak{Z} := (\mathcal{Z}, \mathcal{V}, V^s, \partial V^s),\ \mathfrak{Z} := (\mathcal{N}, \mathcal{M}, M^s, \partial M^s),$$

whose elements we may denote with superscripts, e.g. $\mathfrak{Y}^1, \mathfrak{Y}^2, \mathfrak{Y}^3, \mathfrak{Y}^4$ correspond to $\mathcal{Y}, \mathcal{U}, U^s, \partial U^s$.

In light of Assumption J, $\mathrm{d}r \otimes \mathrm{d}\mathbb{P}$–a.e.

$$\begin{aligned}
|h_r| &\leq L_y|\mathcal{Y}_r| + L_z|\sigma_r^\top z_r|^2 + L_u|\mathcal{U}_r| + L_v|\sigma_r^\top v_r^r|^2 + L_{\mathrm{u}}|\partial U_r^r| + |\tilde{h}_r|, \\
|g_r| &\leq L_u|\mathcal{U}_r| + L_v|\sigma_r^\top v_r^r|^2 + L_y|\mathcal{Y}_r| + L_z|\sigma_r^\top z_r|^2 + |\partial U_r^r| + |\tilde{g}_r|, \\
|g_r(s)| &\leq L_u|U_r^s| + L_v|\sigma_r^\top v_r^s|^2 + L_y|\mathcal{Y}_r| + L_z|\sigma_r^\top z_r|^2 + |\tilde{g}_r(s)|, \\
|\nabla g_r(s)| &\leq L_{\mathrm{u}}|\partial U_r^s| + L_{\mathrm{v}}|\sigma_r^\top \partial v_r^s|^2 + L_u|U_r^s| + L_v|\sigma_r^\top v_r^s|^2 + L_y|\mathcal{Y}_r| + L_z|\sigma_r^\top z_r|^2 + |\nabla \tilde{g}_r(s)|.
\end{aligned} \tag{4.5.1}$$

Applying Meyer–Itô's formula to $\mathrm{e}^{\frac{c}{2}t}(|\mathcal{Y}_t| + |\mathcal{U}_t| + |U_t^s| + |\partial U_t^s|)$ we obtain

$$\begin{aligned}
&\mathrm{e}^{\frac{c}{2}t}(|\mathcal{Y}_t| + |\mathcal{U}_t| + |U_t^s| + |\partial U_t^s|) + \mathfrak{M}_t - \mathfrak{M}_T + \widehat{L}_T^0 \\
&= \mathrm{e}^{\frac{c}{2}T}(|\xi| + |\eta(T)| + |\eta(s)| + |\partial\eta(s)|) \\
&\quad + \int_t^T \mathrm{e}^{\frac{c}{2}r}\bigg(\mathrm{sgn}(\mathcal{Y}_r)\cdot h_r + \mathrm{sgn}(\mathcal{U}_r)\cdot g_r + \mathrm{sgn}(U_r^s)\cdot g_r(s) + \mathrm{sgn}(\partial U_r^s)\cdot \nabla g_r(s) - \frac{c}{2}\sum_{i=1}^4 |\mathfrak{Y}_r^i|\bigg)\mathrm{d}r,
\end{aligned} \tag{4.5.2}$$

where $\widehat{L}_t^0 := \widehat{L}_t^0(\mathcal{Y}, \mathcal{U}, U^s, \partial U^s)$ denotes the non-decreasing and pathwise-continuous local time of the semimartingale $(\mathcal{Y}, \mathcal{U}, U^s, \partial U^s)$ at 0, see [212, Theorem 70], and, we introduced the martingale (recall that (i) and (ii) guarantee $(Z, \mathcal{V}, \mathcal{N}, \mathcal{M}, V, \partial V, M, \partial M) \in (\mathbb{H}^2)^2 \times (\mathbb{M}^2)^2 \times (\mathbb{H}^{2,2})^2 \times (\mathbb{M}^{2,2})^2$)

$$\mathfrak{M}_t := \sum_{i=1}^4 \int_0^t \mathrm{e}^{\frac{c}{2}r} \mathfrak{Z}_r^i\, \mathrm{sgn}(\mathfrak{Y}_r^i) \cdot \mathrm{d}X_r + \int_0^t \mathrm{e}^{\frac{c}{2}r-}\, \mathrm{sgn}(\mathfrak{Y}_{r-}) \cdot \mathrm{d}\mathfrak{Z}_r^i,\ t \in [0, T].$$

Again, we take conditional expectations with respect to $\mathcal{F}_t$ in Equation (4.5.2) and exploit the fact $\widehat{L}_T^0$ is non-decreasing. Moreover, in combination with (4.5.1) and Lemma 4.7.1, we obtain back in (4.5.2) that for $C_1 := 4L_y + TL_y + TL_{\mathrm{u}}L_y$, $C_2 := 2L_u$, $C_3 := 2L_u + TL_u + TL_{\mathrm{u}}L_u$, and $C_4 := L_{\mathrm{u}}$

$$
\begin{aligned}
&\mathrm{e}^{\frac{c}{2}t}(|\mathcal{Y}_t| + |\mathcal{U}_t| + |U_t^s| + |\partial U_t^s|) + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{\frac{c}{2}r}|\mathcal{Y}_r|(c/2 - C_1)\mathrm{d}r\bigg] + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{\frac{c}{2}r}|\mathcal{U}_r|(c/2 - C_2)\mathrm{d}r\bigg]\\
&\quad + \sup_{s\in[0,T]} \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{\frac{c}{2}r}|U_r^s|(c/2 - C_3)\mathrm{d}r\bigg] + \sup_{s\in[0,T]} \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{\frac{c}{2}r}|\partial U_r^s|(c/2 - C_4)\Big)\mathrm{d}r\bigg]\\
&\leq \mathbb{E}_t\big[\mathrm{e}^{\frac{c}{2}T}(|\xi| + |\eta(T)| + |\eta(s)| + |\partial_s\eta(s)|)\big] + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{\frac{c}{2}r}\Big(|\tilde{h}_r| + |\tilde{g}_r| + |\tilde{g}_r(s)| + |\nabla\tilde{g}_r(s)|\Big)\mathrm{d}r\bigg]\\
&\quad + (T + L_{\mathrm{u}}T)\Big(\|\partial_s\eta\|_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde{g}\|_{\mathbb{L}^{1,\infty,2,c}} + L_\star\Big(\|\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big)\Big)\\
&\quad + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}\Big(4L_z|\sigma_r^\top z_r|^2 + 2L_v|\sigma_r^\top v_r^r|^2 + 2L_v|\sigma_r^\top v_r^s|^2 + L_{\mathrm{v}}|\sigma_r^\top \partial v_r^s|^2\Big)\mathrm{d}r\bigg],
\end{aligned}
$$

where we recall the notation $L_\star = \max\{L_z, L_v, L_{\mathrm{v}}\}$. Thus, for

$$
\begin{aligned}
c &\geq 2\max\{4L_y + TL_y + TL_{\mathrm{u}}L_y, 2L_u, 2L_u + TL_u + TL_{\mathrm{u}}L_u, L_{\mathrm{u}}\}\\
&= \max\{8L_y + 2TL_y + 2TL_{\mathrm{u}}L_y, 4L_u + 2TL_u + 2TL_{\mathrm{u}}L_u, 2L_{\mathrm{u}}\},
\end{aligned}
\tag{4.5.3}
$$

we obtain

$$
\begin{aligned}
\max\{\mathrm{e}^{\frac{c}{2}t}|\mathcal{Y}_t|, \mathrm{e}^{\frac{c}{2}t}|\mathcal{U}_t|, \mathrm{e}^{\frac{c}{2}t}|U_t^s|, \mathrm{e}^{\frac{c}{2}t}|\partial U_t^s|\} &\leq \mathrm{e}^{\frac{c}{2}t}(|\mathcal{Y}_t| + |\mathcal{U}_t| + |U_t^s| + |\partial U_t^s|)\\
&\leq \|\xi\|_{\mathcal{L}^{\infty,c}} + \|\tilde{h}\|_{\mathbb{L}^{1,\infty,c}} + 2(\|\eta\|_{\mathcal{L}^{\infty,2,c}} + \|\tilde{g}\|_{\mathbb{L}^{1,\infty,2,c}})\\
&\quad + (1 + T + TL_{\mathrm{u}})(\|\partial_s\eta\|_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde{g}\|_{\mathbb{L}^{1,\infty,2,c}})\\
&\quad + (4 + T + L_{\mathrm{u}}T)L_\star\Big(\|\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big).
\end{aligned}
$$

(iv) We show $(\mathcal{Z}, \mathcal{V}, \mathcal{N}, \mathcal{M}) \in (\mathbb{H}^{2,c}_{\mathrm{BMO}})^2 \times (\mathbb{M}^{2,c})^2$, $\|V\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|M\|^2_{\mathbb{M}^{2,2,c}} + \|\partial V\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\partial M\|^2_{\mathbb{M}^{2,2,c}} < \infty$.

Applying Itô's formula to $\mathrm{e}^{ct}(|\mathcal{Y}_t|^2 + |\mathcal{U}_t|^2 + |U_t^s|^2 + |\partial U_t^s|^2)$ we obtain

$$
\begin{aligned}
&\sum_{i=1}^4 \mathrm{e}^{ct}|\mathfrak{Y}_t^i|^2 + \int_t^T \mathrm{e}^{cr}|\sigma_r^\top \mathfrak{Z}_r^i|^2\mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\mathrm{Tr}[\mathfrak{Z}^i]_r + \widetilde{\mathfrak{M}}_t - \widetilde{\mathfrak{M}}_T\\
&= \mathrm{e}^{cT}(|\xi|^2 + |\eta(T)|^2 + |\eta(s)|^2 + |\partial_s\eta(s)|^2)\\
&\quad + \int_t^T \mathrm{e}^{cr}\bigg(2\mathcal{Y}_r \cdot h_r + 2\mathcal{U}_r \cdot g_r + 2U_r^s \cdot g_r(s) + 2\partial U_r^s \cdot \nabla g_r(s) - c\sum_{i=1}^4 |\mathfrak{Y}_r^i|^2\bigg)\mathrm{d}r,
\end{aligned}
\tag{4.5.4}
$$

where for any $s \in [0,T]$ we introduced the martingale

$$\widetilde{\mathfrak{M}}_t := \sum_{i=1}^4 \int_0^t \mathrm{e}^{cr} \mathfrak{Z}_r^i \mathfrak{Y}_r^i \cdot \mathrm{d}X_r + \int_0^t \mathrm{e}^{cr-} \mathfrak{Y}_{r-}^i \cdot \mathrm{d}\mathfrak{Z}_r^i,\ t \in [0,T].$$

Indeed, Burkholder–Davis–Gundy's inequality and the fact that $(Y, \mathcal{U}, Z, \mathcal{V}, U, \partial U, V, \partial V) \in (\mathbb{S}^2)^2 \times (\mathbb{H}^2)^2 \times (\mathbb{S}^{2,2})^2 \times (\mathbb{H}^{2,2})^2$, recall (i) and (ii), implies that there exists $C > 0$ such that

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \bigg| \int_0^t \mathrm{e}^{cr} \mathfrak{Z}_r^i \mathfrak{Y}_r^i \cdot \mathrm{d}X_r \bigg|\bigg] \leq C\mathbb{E}\bigg[\bigg| \int_0^T \mathrm{e}^{cr} |\mathfrak{Y}_r^i \sigma_r^\top \mathfrak{Z}_r^i|^2 \mathrm{d}r \bigg|^{\frac{1}{2}}\bigg] \leq C\mathrm{e}^{cT} \|\mathfrak{Y}^i\|_{\mathbb{S}^2} \|\mathfrak{Z}^i\|_{\mathbb{H}^2},\ i \in \{1,\dots,4\},$$

which guarantees each of the processes in $\widetilde{\mathfrak{M}}$ is an uniformly integrable martingale. Thus, taking conditional expectations with respect to $\mathcal{F}_t$ in Equation (4.5.4), we obtain

$$\begin{aligned}
&\sum_{i=1}^4 \mathrm{e}^{ct} |\mathfrak{Y}_t^i|^2 + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr} |\sigma_r^\top \mathfrak{Z}_r^i|^2 \mathrm{d}r + \int_t^T \mathrm{e}^{cr-} \mathrm{d}\mathrm{Tr}[\mathfrak{Z}^i]_r\bigg] \\
&= \mathbb{E}_t\Big[\mathrm{e}^{cT}\big(|\xi|^2 + |\eta(T)|^2 + |\eta(s)|^2 + |\partial_s \eta(s)|^2\big)\Big] \\
&\quad + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}\bigg(2\mathcal{Y}_r \cdot h_r + 2\mathcal{U}_r \cdot g_r + 2U_r^s \cdot g_r(s) + 2\partial U_r^s \cdot \nabla g_r(s) - c\sum_{i=1}^4 |\mathfrak{Y}_r^i|^2\bigg)\mathrm{d}r\bigg],\ t \in [0,T].
\end{aligned}$$

From (iii), Assumption J.(ii) and J.(iii), together with Young's inequality, yield that, for any $\varepsilon_i > 0$, and defining $\widetilde{C}_{\varepsilon_{1,7}} := 2L_y + \varepsilon_1^{-1} 7TL_\mathrm{u}^2 + \varepsilon_7^{-1} L_u^2$, $\widetilde{C}_{\varepsilon_{2,8}} := 2L_u + \varepsilon_2^{-1} 7T + \varepsilon_8^{-1} L_y^2$, $\widetilde{C}_{\varepsilon_9} := 2L_u + \varepsilon_9^{-1} L_y^2$, as well as, $\widetilde{C}_{\varepsilon_{10,11}} := 2L_\mathrm{u} + \varepsilon_{11}^{-1} L_u^2 + \varepsilon_{10}^{-1} L_y^2$ we have

$$\begin{aligned}
2\mathcal{Y}_r \cdot h_r - c|\mathcal{Y}_r|^2 &\leq (\widetilde{C}_{\varepsilon_{1,7}} - c)|\mathcal{Y}_r|^2 + 2\|\mathcal{Y}\|_{\mathcal{S}^{\infty,c}}\big(L_z|\sigma_r^\top z_r|^2 + L_v|\sigma_r^\top v_r^r|^2 + |\tilde{h}_r|\big) \\
&\quad + \varepsilon_1(7T)^{-1}|\partial U_r^r|^2 + \varepsilon_7|\mathcal{U}_r|^2, \\
2\mathcal{U}_r \cdot g_r - c|\mathcal{U}_r|^2 &\leq (\widetilde{C}_{\varepsilon_{2,8}} - c)|\mathcal{U}_r|^2 + 2\|\mathcal{U}\|_{\mathcal{S}^{\infty,c}}\big(L_z|\sigma_r^\top z_r|^2 + L_v|\sigma_r^\top v_r^r|^2 + |\tilde{g}_r|\big) \\
&\quad + \varepsilon_2(7T)^{-1}|\partial U_r^r|^2 + \varepsilon_8|\mathcal{Y}_r|^2, \\
2U_r^s \cdot g_r(s) - c|U_r^s|^2 &\leq (\widetilde{C}_{\varepsilon_9} - c)|U_r^s|^2 + 2\|U\|_{\mathcal{S}^{\infty,2,c}}\big(L_v|\sigma_r^\top v_r^s|^2 + L_z|\sigma_r^\top z_r|^2 + |\tilde{g}_r(s)|\big) + \varepsilon_9|\mathcal{Y}_r|^2, \\
2\partial U_r^s \cdot \nabla g_r(s) - c|\partial U_r^s|^2 &\leq (\widetilde{C}_{\varepsilon_{10,11}} - c)|\partial U_r^s|^2 + \varepsilon_{11}|U_r^s|^2 + \varepsilon_{10}|Y_r|^2 \\
&\quad + 2\|\partial U\|_{\mathcal{S}^{\infty,c,2}}\big(L_\mathrm{v}|\sigma_r^\top \partial v_r^s|^2 + L_v|\sigma_r^\top v_r^s|^2 + L_z|\sigma_r^\top z_r|^2 + |\nabla \tilde{g}_r(s)|\big).
\end{aligned}$$

These inequalities in combination with Lemma 4.7.1, and Young's inequality, show that if we define $C_1^\varepsilon := \widetilde{C}_{\varepsilon_{1,7}} + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10} + (\varepsilon_1+\varepsilon_2)TL_y^2$, $C_2^\varepsilon := \widetilde{C}_{\varepsilon_{2,8}} + \varepsilon_7$, $C_3^\varepsilon := \widetilde{C}_{\varepsilon_9} + \varepsilon_{11} + (\varepsilon_1+\varepsilon_2)TL_u^2$, $C_4^\varepsilon := \widetilde{C}_{\varepsilon_{10,11}}$, then for any $\varepsilon_i > 0$, $i \in \{7, ..., 15\}$

$$
\begin{aligned}
&\sum_{i=1}^4 \mathrm{e}^{ct}|\mathfrak{Y}_t^i|^2 + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top \mathfrak{Z}_r^i|^2\mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{dTr}[\mathfrak{Z}^i]_r + \int_t^T \mathrm{e}^{cr}\big(|\mathcal{Y}_r|^2(c-C_1^\varepsilon) + |\mathcal{U}_r|^2(c-C_2^\varepsilon)\big)\mathrm{d}r\bigg] \\
&\quad + \sup_{s\in[0,T]}\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|U_r^s|^2(c-C_3^\varepsilon)\mathrm{d}r\bigg] + \sup_{s\in[0,T]}\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|\partial U_r^s|^2(c-C_4^\varepsilon)\mathrm{d}r\bigg] \\
&= (\varepsilon_1+\varepsilon_2)\Big(\|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}} + 2L_\star^2\Big(\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big)\Big) \\
&\quad + \mathbb{E}_t\Big[\mathrm{e}^{cT}\big(|\xi|^2 + |\eta(T)|^2 + |\eta(s)|^2 + |\partial_s\eta(s)|^2\big)\Big] + (\varepsilon_3^{-1} + \varepsilon_{12}^{-1} + \varepsilon_{13}^{-1})\|\mathcal{Y}\|^2_{\mathcal{S}^{\infty,c}} \\
&\quad + (\varepsilon_4^{-1} + \varepsilon_{14}^{-1} + \varepsilon_{15}^{-1})\|\mathcal{U}\|^2_{\mathcal{S}^{\infty,c}} + (\varepsilon_5^{-1} + \varepsilon_{16}^{-1} + \varepsilon_{17}^{-1})\|U\|^2_{\mathcal{S}^{\infty,c,2}} + (\varepsilon_6^{-1} + \varepsilon_{18}^{-1} + \varepsilon_{19}^{-1} + \varepsilon_{20}^{-1})\|\partial U\|^2_{\mathcal{S}^{\infty,c,2}} \\
&\quad + \varepsilon_3\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\tilde{h}_r|\mathrm{d}r\bigg|^2\bigg] + \varepsilon_4\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\tilde{g}_r|\mathrm{d}r\bigg|^2\bigg] \\
&\quad + \varepsilon_5\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\tilde{g}_r(s)|\mathrm{d}r\bigg|^2\bigg] + \varepsilon_6\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\nabla\tilde{g}_r|\mathrm{d}r\bigg|^2\bigg] \\
&\quad + (\varepsilon_{12}+\varepsilon_{14}+\varepsilon_{16}+\varepsilon_{18})L_z^2\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\sigma_r^\top z_r|^2\mathrm{d}r\bigg|^2\bigg] + (\varepsilon_{13}+\varepsilon_{15})L_v^2\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\sigma_r^\top v_r^r|^2\mathrm{d}r\bigg|^2\bigg] \\
&\quad + (\varepsilon_{17}+\varepsilon_{19})L_v^2\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\sigma_r^\top v_r^s|^2\mathrm{d}r\bigg|^2\bigg] + \varepsilon_{20}L_{\mathrm{v}}^2\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\sigma_r^\top \partial v_r^s|^2\mathrm{d}r\bigg|^2\bigg].
\end{aligned}
$$

We now let $\tau \in \mathcal{T}_{0,T}$. In light of (4.5.3), for

$$
\begin{aligned}
c \geq \max\,\{&2L_y + \varepsilon_1^{-1}7TL_{\mathrm{u}}^2 + \varepsilon_1 TL_y^2 + \varepsilon_2 TL_y^2 + \varepsilon_7^{-1}L_u^2 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10}, \\
&2L_u + \varepsilon_2^{-1}7T + \varepsilon_7 + \varepsilon_8^{-1}L_y^2,\ 2L_u + \varepsilon_1 TL_u^2 + \varepsilon_2 TL_u^2 + \varepsilon_9^{-1}L_y^2 + \varepsilon_{11}, \\
&2L_{\mathrm{u}} + \varepsilon_{11}^{-1}L_u^2 + \varepsilon_{10}^{-1}L_y^2,\ 8L_y + 2TL_y + 2TL_{\mathrm{u}}L_y,\ 4L_u + 2TL_u + 2TL_{\mathrm{u}}L_u\},
\end{aligned}
\tag{4.5.5}
$$

(4.2.1) yields

$$
\begin{aligned}
&\sum_{i=1}^4 \mathrm{e}^{ct}|\mathfrak{Y}_t^i|^2 + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top \mathfrak{Z}_r^i|^2\mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{dTr}[\mathfrak{Z}^i]_r\bigg] \\
&= I_0^\varepsilon + 2L_\star^2(\varepsilon_1+\varepsilon_2+\varepsilon_{12}+\varepsilon_{14}+\varepsilon_{16}+\varepsilon_{18})\|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + (\varepsilon_3^{-1} + \varepsilon_{12}^{-1} + \varepsilon_{13}^{-1})\|Y\|^2_{\mathcal{S}^{\infty,c}} \\
&\quad + 2L_\star^2\Big((\varepsilon_1+\varepsilon_2+\varepsilon_{13}+\varepsilon_{15}+\varepsilon_{17}+\varepsilon_{19})\|v\|^4_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + (\varepsilon_1+\varepsilon_2+\varepsilon_{20})\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\Big) \\
&\quad + (\varepsilon_4^{-1} + \varepsilon_{14}^{-1} + \varepsilon_{15}^{-1})\|\mathcal{U}\|^2_{\mathcal{S}^{\infty,c}} + (\varepsilon_5^{-1} + \varepsilon_{16}^{-1} + \varepsilon_{17}^{-1})\|U\|^2_{\mathcal{S}^{\infty,c,2}} + (\varepsilon_6^{-1} + \varepsilon_{18}^{-1} + \varepsilon_{19}^{-1} + \varepsilon_{20}^{-1})\|\partial U\|^2_{\mathcal{S}^{\infty,c,2}}.
\end{aligned}
$$

which in turn leads to

$$
\begin{aligned}
&\frac{1}{10}\Big(\|\mathcal{Y}\|^2_{\mathcal{S}^{\infty,c}}+\|\mathcal{U}\|^2_{\mathcal{S}^{\infty,c}}+\|U\|^2_{\mathcal{S}^{\infty,2,c}}+\|\partial U\|^2_{\mathcal{S}^{\infty,2,c}}+\|\mathcal{Z}\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\\
&\quad+\|V\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}+\|\partial V\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}+\|\mathcal{N}\|^2_{\mathbb{M}^{2,c}}+\|M\|^2_{\mathbb{M}^{2,2,c}}+\|\partial M\|^2_{\mathbb{M}^{2,2,c}}\Big)\\
&\leq \|\xi\|^2_{\mathcal{L}^{\infty,c}}+2\|\eta\|^2_{\mathcal{L}^{\infty,2,c}}+(1+\varepsilon_1+\varepsilon_2)\|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}}+\varepsilon_3\|\tilde{h}\|^2_{\mathbb{L}^{1,\infty,c}}+(\varepsilon_4+\varepsilon_5)\|\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}}\\
&\quad+(\varepsilon_1+\varepsilon_2+\varepsilon_6)\|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}}+2L^2_\star(\varepsilon_1+\varepsilon_2+\varepsilon_{12}+\varepsilon_{14}+\varepsilon_{16}+\varepsilon_{18})\|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\\
&\quad+2L^2_\star(\varepsilon_1+\varepsilon_2+\varepsilon_{13}+\varepsilon_{15}+\varepsilon_{17}+\varepsilon_{19})\|v\|^4_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}+2L^2_\star(\varepsilon_1+\varepsilon_2+\varepsilon_{20})\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\\
&\quad+\big(\varepsilon_3^{-1}+\varepsilon_{12}^{-1}+\varepsilon_{13}^{-1}\big)\|Y\|^2_{\mathcal{S}^{\infty,c}}+\big(\varepsilon_4^{-1}+\varepsilon_{14}^{-1}+\varepsilon_{15}^{-1}\big)\|\mathcal{U}\|^2_{\mathcal{S}^{\infty,c}}\\
&\quad+\big(\varepsilon_5^{-1}+\varepsilon_{16}^{-1}+\varepsilon_{17}^{-1}\big)\|U\|^2_{\mathcal{S}^{\infty,c,2}}+\big(\varepsilon_6^{-1}+\varepsilon_{18}^{-1}+\varepsilon_{19}^{-1}+\varepsilon_{20}^{-1}\big)\|\partial U\|^2_{\mathcal{S}^{\infty,c,2}}.
\end{aligned}
\tag{4.5.6}
$$

From (4.5.6) we conclude $(Z,N)\in\mathbb{H}^{2,c}_{\mathrm{BMO}}\times\mathbb{M}^{2,c}$,

$$
\|V\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}+\|\partial V\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}+\|M\|^2_{\mathbb{M}^{2,2,c}}+\|\partial M\|^2_{\mathbb{M}^{2,2,c}}<\infty.
$$

At this point, we can highlight a crucial step in this approach. It is clear from (4.5.6) that the norm of $\mathfrak{T}$ does not have a linear growth in the norm of the input. In the following, we will see that choosing the data of the system small enough and localising $\mathfrak{T}$ will bring us back to the linear growth scenario. For this, we observe that if we define

$$
\begin{aligned}
C_\varepsilon:=\min\big\{&1-10(\varepsilon_3^{-1}+\varepsilon_{12}^{-1}+\varepsilon_{13}^{-1}),\ 1-10(\varepsilon_4^{-1}+\varepsilon_{14}^{-1}+\varepsilon_{15}^{-1}),\\
&1-10(\varepsilon_5^{-1}+\varepsilon_{16}^{-1}+\varepsilon_{17}^{-1}),\ 1-10(\varepsilon_6^{-1}+\varepsilon_{18}^{-1}+\varepsilon_{19}^{-1}+\varepsilon_{20}^{-1})\big\},
\end{aligned}
$$

for $1-10(\varepsilon_3^{-1}+\varepsilon_{12}^{-1}+\varepsilon_{13}^{-1})\in(0,1]$, $1-10(\varepsilon_4^{-1}+\varepsilon_{14}^{-1}+\varepsilon_{15}^{-1})\in(0,1]$, $1-10(\varepsilon_5^{-1}+\varepsilon_{16}^{-1}+\varepsilon_{17}^{-1})\in(0,1]$, $1-10(\varepsilon_6^{-1}+\varepsilon_{18}^{-1}+\varepsilon_{19}^{-1}+\varepsilon_{20}^{-1})\in(0,1]$, and for some $\gamma\in(0,\infty)$,

$$
I_0^\varepsilon\leq\gamma R^2/10,
$$

we obtain back in (4.5.6)

$$
\|(Y,Z,N,U,V,M,\partial U,\partial V,\partial M)\|^2_{\mathcal{H}^c}
$$

$$
\begin{aligned}
&\leq C_\varepsilon^{-1}\Big(10I_0^\varepsilon + 20L_\star^2((\varepsilon_1+\varepsilon_2+\varepsilon_{12}+\varepsilon_{14}+\varepsilon_{16}+\varepsilon_{18})\|z\|^4_{\mathbb{H}^{2,c}_{\rm BMO}}\\
&\qquad\qquad + (\varepsilon_1+\varepsilon_2+\varepsilon_{13}+\varepsilon_{15}+\varepsilon_{17}+\varepsilon_{19})\|v\|^4_{\overline{\mathbb{H}}^{2,2,c}_{\rm BMO}} + (\varepsilon_1+\varepsilon_2+\varepsilon_{20})\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\rm BMO}})\Big)\\
&\leq C_\varepsilon^{-1}R^2\Big(\gamma + 20L_\star^2((\varepsilon_1+\varepsilon_2+\varepsilon_{12}+\varepsilon_{14}+\varepsilon_{16}+\varepsilon_{18})\|z\|^2_{\mathbb{H}^{2,c}_{\rm BMO}}\\
&\qquad\qquad + (\varepsilon_1+\varepsilon_2+\varepsilon_{13}+\varepsilon_{15}+\varepsilon_{17}+\varepsilon_{19})\|v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\rm BMO}} + (\varepsilon_1+\varepsilon_2+\varepsilon_{20})\|\partial v\|^2_{\mathbb{H}^{2,2,c}_{\rm BMO}})\Big)\\
&\leq C_\varepsilon^{-1}R^2\bigg(\gamma + 20L_\star^2R^2\bigg(\varepsilon_1+\varepsilon_2+\sum_{i=12}^{20}\varepsilon_i\bigg)\bigg).
\end{aligned}
$$

Therefore, to obtain $\mathfrak{T}(\mathcal{B}_R)\subseteq\mathcal{B}_R$, that is to say that the image under $\mathfrak{T}$ of the ball of radius R is contained in the ball of radius R, it is necessary to find R^2 such that the term in parentheses above is less or equal than C_ε, *i.e.*

$$
R^2 \leq \frac{1}{20L_\star^2}\frac{C_\varepsilon-\gamma}{\varepsilon_1+\varepsilon_2+\sum_{i=12}^{20}\varepsilon_i}. \tag{4.5.7}
$$

Clearly, there are many choices of ε's so that the above holds. Among such, we wish to choose γ, ε_i, so that the expression to the right in (4.5.7) is maximal. In light of Lemma 4.7.2, we have that $\|(Y,Z,N,U,V,M)\|_{\mathcal{H}^c}\leq R$ provided that

$$
R^2 < \frac{1}{2^5\cdot 3\cdot 5^2\cdot 7\cdot L_\star^2}. \tag{4.5.8}
$$

(v) Lastly, we are left to argue $(U,V,M,\partial U,\partial V,\partial M)\in\mathcal{S}^{\infty,2,c}\times\mathbb{H}^{2,2,c}_{\rm BMO}\times\mathbb{M}^{2,2,c}\times\mathcal{S}^{\infty,2,c}\times\mathbb{H}^{2,2,c}_{\rm BMO}\times\mathbb{M}^{2,2,c}$. The argument for (U,V,M) is analogous to that of $(\partial U,\partial V,\partial M)$, thus we argue the continuity of the applications $([0,T],\mathcal{B}([0,T]))\longrightarrow(\mathcal{S}^{\infty,c},\|\cdot\|_{\mathcal{S}^{\infty,c}})$(resp. $(\mathbb{H}^{2,c}_{\rm BMO},\|\cdot\|_{\mathbb{H}^{2,c}_{\rm BMO}})$, $(\mathbb{M}^{2,c},\|\cdot\|_{\mathbb{M}^{2,c}})$) : $s\longmapsto\varphi^s$ for $\varphi=U^s$ (resp. V^s,M^s).

Recall ρ_g denotes the modulus of continuity of g. Let $(s_n)_n\subseteq[0,T], s_n\xrightarrow{n\to\infty}s_0\in[0,T]$ and define for $\varphi\in\{U,V,M,u,v,\eta\}$, $\Delta\varphi^n:=\varphi^{s_n}-\varphi^{s_0}$. Applying Itô's formula to $\mathrm{e}^{ct}|\Delta U_t^n|^2$, proceeding as in **Step 1** (iv) we obtain

$$
\|\Delta U^n\|^2_{\mathcal{S}^{\infty,c}}+\|\Delta V^n\|^2_{\mathbb{H}^{2,c}_{\rm BMO}}+\|\Delta M^n\|^2_{\mathbb{M}^{2,c}}\leq 4\Big(\|\Delta\eta^n\|^2_{\mathcal{L}^{\infty,c}}+4L_v^2\|\Delta v^n\|^4_{\mathbb{H}^{2,c}_{\rm BMO}}+\rho_g(|s_n-s_0|)^2\Big).
$$

We conclude, $\mathfrak{T}(\mathcal{B}_R)\subseteq\mathcal{B}_R$ for all R satisfying (4.5.8).

Step 2: We now argue that $\mathfrak{T}$ is a contraction in $\mathcal{B}_R \subseteq \mathcal{H}$ for the norm $\|\cdot\|_{\mathcal{H}^c}$.

Let $(Y^i, Z^i, N^i, U^i, V^i, M^i, , U^i, V^i, M^i) := \mathfrak{T}(y^i, z^i, n^i, u^i, v^i, m^i)$ for $(y^i, z^i, n^i, u^i, v^i, m^i, \partial u^i, \partial v^i, \partial m^i) \in \mathcal{B}_R$, $i = 1, 2$.

For $\varphi \in \{y, z, n, u, v, m, \partial u, \partial v, \partial m, \mathcal{Y}, \mathcal{Z}, \mathcal{N}, \mathcal{U}, \mathcal{V}, \mathcal{M}, U, V, M, \partial U, \partial V, \partial M\}$, we denote $\delta\varphi := \varphi^1 - \varphi^2$ and

$$
\begin{aligned}
\delta h_t &:= h_t(Y_t^1, z_t^1, \mathcal{U}_t^1, v_t^{1,t}, \partial U_t^{1,t}) - h_t(Y_t^2, z_t^2, \mathcal{U}_t^2, v_t^{2,t}, \partial U_t^{2,t}),\\
\delta g_t &:= g_t(t, \mathcal{U}_t^1, v_t^{1,t}, Y_t^1, z_t^1) - \partial U_t^{1,t} - g_t(t, \mathcal{U}_t^2, v_t^{2,t}, Y_t^2, z_t^2) + \partial U_t^{2,t},\\
\delta g_t(s) &:= g_t(s, U_t^{1,s}, v_t^{1,s}, Y_t^1, z_t^1) - g_t(s, U_t^{2,s}, v_t^{2,s}, Y_t^2, z_t^2),\\
\delta\nabla g_t(s) &:= \nabla g_t(s, \partial U_t^{1,s}, \partial v_t^{1,s}, U_t^{1,s}, v_t^{1,s}, Y_t^1, z_t^1) - g_t(s, \partial U_t^{2,s}, \partial v_t^{2,s}, U_t^{2,s}, v_t^{2,s}, Y_t^2, z_t^2),\\
\delta\tilde{h}_t &:= h_t(Y_t^2, z_t^1, \mathcal{U}_t^2, v_t^{1,t}, \partial U_t^{2,t}) - h_t(Y_t^2, z_t^2, \mathcal{U}_t^2, v_t^{2,t}, \partial U_t^{2,t}),\\
\delta\tilde{g}_t &:= g_t(t, \mathcal{U}_t^2, v_t^{1,t}, Y_t^2, z_t^1) - \partial U_t^{2,t} - g_t(t, \mathcal{U}_t^2, v_t^{2,t}, Y_t^2, z_t^2) + \partial U_t^{2,t},\\
\delta\tilde{g}_t(s) &:= g_t(s, U_t^{2,s}, v_t^{1,s}, Y_t^2, z_t^1) - g_t(s, U_t^{2,s}, v_t^{2,s}, Y_t^2, z_t^2),\\
\delta\nabla\tilde{g}_t(s) &:= \nabla g_t(s, \partial U_t^{2,s}, \partial v_t^{1,s}, U_t^{2,s}, v_t^{1,s}, Y_t^2, z_t^1) - g_t(s, \partial U_t^{2,s}, \partial v_t^{2,s}, U_t^{2,s}, v_t^{2,s}, Y_t^2, z_t^2).
\end{aligned}
$$

Applying Itô's formula to $\mathrm{e}^{ct}\big(|\delta Y_t|^2 + |\delta\mathcal{U}_t|^2 + |\delta U_t^s|^2 + |\delta\partial U_t^s|^2\big)$, we obtain that for any $t \in [0, T]$

$$
\begin{aligned}
&\sum_{i=1}^4 \mathrm{e}^{ct}|\delta\mathfrak{Y}_t^i|^2 + \int_t^T \mathrm{e}^{cr}|\sigma_r^\top \delta\mathfrak{Z}_r^i|^2\mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\mathrm{Tr}[\delta\mathfrak{Z}^i]_r + \delta\widetilde{\mathfrak{M}}_t - \delta\widetilde{\mathfrak{M}}_T\\
&= \int_t^T \mathrm{e}^{cr}\bigg(2\delta\mathcal{Y}_r\cdot\delta h_r + 2\delta\mathcal{U}_r\cdot\delta g_r + 2\delta U_r^s\cdot\delta g_r(s) + 2\delta\partial U_r^s\cdot\delta\nabla g_r(s) - c\sum_{i=1}^4|\delta\mathfrak{Y}_r^i|^2\bigg)\mathrm{d}r\\
&\leq \int_t^T \mathrm{e}^{cr}\bigg(2|\delta\mathcal{Y}_r|\big(L_y|\delta\mathcal{Y}_r| + L_u|\delta\mathcal{U}_r| + L_{\mathrm{u}}|\delta\partial U_r^r| + |\delta\tilde{h}_r|\big) - c|\delta\mathcal{Y}_r|^2\\
&\qquad + 2|\delta\mathcal{U}_r|\big(L_u|\delta\mathcal{U}_r| + L_y|\delta\mathcal{Y}_r| + |\delta\partial U_r^r| + |\delta\tilde{g}_r|\big) - c|\delta\mathcal{U}_r|^2\\
&\qquad + 2|\delta U_r^s|\big(L_u|\delta U_r^s| + L_y|\delta\mathcal{Y}_r| + |\delta\tilde{g}_r(s)|\big) - c|\delta U_r^s|^2\\
&\qquad + 2|\delta\partial U_r^s|\big(L_{\mathrm{u}}|\delta\partial U_r^s| + L_u|\delta U_r^s| + L_y|\delta\mathcal{Y}_r| + |\delta\nabla\tilde{g}_r(s)|\big) - c|\delta\partial U_r^s|^2\bigg)\mathrm{d}r,
\end{aligned}
$$

where $\delta\widetilde{\mathfrak{M}}$ denotes the corresponding martingale term. Let $\tau \in \mathcal{T}_{0,T}$, as in Lemma 4.7.1 we obtain for $c > 2L_u$

$$\mathbb{E}_\tau\bigg[\int_\tau^T \frac{\mathrm{e}^{cr}}{3T}|\delta\partial U_r^r|^2\mathrm{d}r\bigg] \leq \sup_{s\in[0,T]} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\nabla g_r(s,0,\partial v_r^s,0,v_r^s,0,z_r)|\mathrm{d}r\bigg]\bigg|^2 \\ + TL_y^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta Y_r|^2\mathrm{d}r\bigg] + TL_u^2 \sup_{s\in[0,T]} \mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta U_r^s|^2\mathrm{d}r\bigg]. \tag{4.5.9}$$

We now take conditional expectation with respect to $\mathcal{F}_\tau$ in the expression above. In addition, we use Assumption J in combination with (4.5.9), exactly as in **Step 1** (iv). We then obtain from Young's inequality that for any $\tilde\varepsilon_i \in (0,\infty)$, $i \in \{1,...,11\}$, and

$$c \geq \max\,\{2L_y + \tilde\varepsilon_1^{-1}3TL_{\mathrm{u}}^2 + \tilde\varepsilon_8 + \tilde\varepsilon_9 + \tilde\varepsilon_{10} + \tilde\varepsilon_7^{-1}L_u^2 + (\tilde\varepsilon_1+\tilde\varepsilon_2)TL_y^2,\ 2L_u + \tilde\varepsilon_7 + 3T\tilde\varepsilon_2^{-1} + \tilde\varepsilon_8^{-1}L_y^2, \\ 2L_u + \varepsilon_9^{-1}L_y^2\tilde{+}\tilde\varepsilon_{11} + (\tilde\varepsilon_1+\tilde\varepsilon_2)TL_u^2,\ 2L_{\mathrm{u}} + \tilde\varepsilon_{11}^{-1}L_u^2 + \tilde\varepsilon_{10}^{-1}L_y^2\}, \tag{4.5.10}$$

it follows that

$$\sum_{i=1}^4 \mathrm{e}^{ct}|\delta\mathfrak{Y}_t^i|^2 + \mathbb{E}_\tau\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top\delta\mathfrak{Z}_r^i|^2\mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\mathrm{Tr}[\delta\mathfrak{Z}^i]_r\bigg] \\ \leq \tilde\varepsilon_3^{-1}\|\delta Y\|^2_{\mathcal{S}^{\infty,c}} + \tilde\varepsilon_4^{-1}\|\delta\mathcal{U}\|^2_{\mathcal{S}^{\infty,2,c}} + \tilde\varepsilon_5^{-1}\|\delta U\|^2_{\mathcal{S}^{\infty,2,c}} + \tilde\varepsilon_6^{-1}\|\delta\partial U\|^2_{\mathcal{S}^{\infty,2,c}} \\ + (\tilde\varepsilon_1+\tilde\varepsilon_2) \sup_{s\in[0,T]} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\nabla g_r(s,0,\partial v_r^s,0,v_r^s,0,z_r)|\mathrm{d}r\bigg]\bigg|^2 \\ + \tilde\varepsilon_3 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde h_t|\mathrm{d}r\bigg]\bigg|^2 + \tilde\varepsilon_4 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde g_t|\mathrm{d}r\bigg]\bigg|^2 \\ + \tilde\varepsilon_5 \sup_{s\in[0,T]} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde g_t(s)|\mathrm{d}r\bigg]\bigg|^2 + \tilde\varepsilon_6 \sup_{s\in[0,T]} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\nabla\tilde g_t(s)|\mathrm{d}r\bigg]\bigg|^2. \tag{4.5.11}$$

We now estimate the terms on the right side of (4.5.11). Note that in light of Assumption J.(iii) we have

$$\max\bigg\{\bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\nabla g_r(s,0,\partial v_r^s,0,v_r^s,0,z_r)|\mathrm{d}r\bigg]\bigg|^2,\ \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\nabla\tilde g_t(s)|\mathrm{d}r\bigg]\bigg|^2\bigg\} \\ \leq \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\Big(L_{\mathrm{v}}|\sigma_r^\top\delta\partial v_r^s|(|\sigma_r^\top\partial v_r^1| + |\sigma_r^\top\partial v_r^2|) + L_v|\sigma_r^\top\delta v_r^s|(|\sigma_r^\top v_r^1| + |\sigma_r^\top v_r^2|) \\ + L_z|\sigma_r^\top\delta z_r|(|\sigma_r^\top z_r^1| + |\sigma_r^\top z_r^2|)\Big)\mathrm{d}r\bigg]\bigg|^2$$

$$
\begin{aligned}
&\leq 3L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top \delta\partial v_r^s|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|\sigma_r^\top \partial v_r^1| + |\sigma_r^\top \partial v_r^2|\big)^2\mathrm{d}r\bigg]\\
&\quad + 3L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top \delta v_r^s|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|\sigma_r^\top v_r^1| + |\sigma_r^\top v_r^2|\big)^2\mathrm{d}r\bigg]\\
&\quad + 3L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top \delta z_r|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|\sigma_r^\top z_r^1| + |\sigma_r^\top z_r^2|\big)^2\mathrm{d}r\bigg]\\
&\leq 6L_\star^2R^2\bigg(\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top \delta\partial v_r^s|^2\mathrm{d}r\bigg] + \mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top \delta v_r^s|^2\mathrm{d}r\bigg] + \mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top \delta z_r|^2\mathrm{d}r\bigg]\bigg)\\
&\leq 6L_\star^2R^2\Big(\|\delta\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big),
\end{aligned}
$$

where in the second inequality we used (I.1) and Cauchy–Schwarz's inequality. Similarly

$$
\begin{aligned}
&\max\bigg\{\bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde h_r|\mathrm{d}r\bigg]\bigg|^2, \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde g_r(s)|\mathrm{d}r\bigg]\bigg|^2, \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta g_r|\mathrm{d}r\bigg]\bigg|^2\bigg\}\\
&\leq 4L_\star^2R^2\Big(\|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,c}_{\mathrm{BMO}}}\Big).
\end{aligned}
$$

Overall, we obtain back in (4.5.11) that

$$
\begin{aligned}
&\sum_{i=1}^4 \mathrm{e}^{ct}|\delta\mathfrak{Y}_t^i|^2 + \mathbb{E}_\tau\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top\delta\mathfrak{Z}_r^i|^2\mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\mathrm{Tr}[\delta\mathfrak{Z}^i]_r\bigg]\\
&\leq \tilde\varepsilon_3^{-1}\|\delta Y\|^2_{\mathcal{S}^{\infty,c}} + \tilde\varepsilon_4^{-1}\|\delta\mathcal{U}\|^2_{\mathcal{S}^{\infty,2,c}} + \tilde\varepsilon_5^{-1}\|\delta U\|^2_{\mathcal{S}^{\infty,2,c}} + \tilde\varepsilon_6^{-1}\|\delta\partial U\|^2_{\mathcal{S}^{\infty,2,c}}\\
&\quad + 6L_\star^2R^2(\tilde\varepsilon_1+\tilde\varepsilon_2+\tilde\varepsilon_6)\Big(\|\delta\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big)\\
&\quad + 4L_\star^2R^2(\tilde\varepsilon_3+\tilde\varepsilon_4+\tilde\varepsilon_5)\ \Big(\|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,c}_{\mathrm{BMO}}}\Big).
\end{aligned}
$$

If we define, for $\tilde\varepsilon_i > 10, i\in\{3,4,5,6\}$, $C_{\tilde\varepsilon} := \min\{1-10/\tilde\varepsilon_3,\ 1-10/\tilde\varepsilon_4,\ 1-10/\tilde\varepsilon_5,\ 1-10/\tilde\varepsilon_6\}$, we deduce,

$$
\begin{aligned}
&\|(\delta Y,\delta Z,\delta N,\delta U,\delta V,\delta M,\delta\partial U,\delta\partial V,\delta\partial M)\|^2_{\mathcal{H}^c}\\
&\leq 10C_{\tilde\varepsilon}^{-1}L_\star^2R^2\Big(6(\tilde\varepsilon_1+\tilde\varepsilon_2+\tilde\varepsilon_6)\Big(\|\delta\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big)\\
&\qquad + 4(\tilde\varepsilon_3+\tilde\varepsilon_4+\tilde\varepsilon_5)\ \Big(\|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,c}_{\mathrm{BMO}}}\Big) \qquad (4.5.12)\\
&\leq 20C_{\tilde\varepsilon}^{-1}L_\star^2R^2(3\tilde\varepsilon_1+3\tilde\varepsilon_2+2\tilde\varepsilon_3+2\tilde\varepsilon_4+2\tilde\varepsilon_5+3\tilde\varepsilon_6)\Big(\|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|\delta\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\Big).
\end{aligned}
$$

By minimising the previous upper bound for $\tilde\varepsilon_1$ and $\tilde\varepsilon_2$ fixed, see Lemma 4.7.3, and in light of

(4.5.8) and (4.5.10), we find that letting

$$
\begin{aligned}
R^2 &< \frac{1}{2^5 \cdot 3 \cdot 5^2 \cdot 7 \cdot L_\star^2}, \text{ and} \\
c &\geq \max\{2L_y + \tilde{\varepsilon}_1^{-1} 3TL_{\mathrm{u}}^2 + \tilde{\varepsilon}_8 + \tilde{\varepsilon}_9 + \tilde{\varepsilon}_{10} + \tilde{\varepsilon}_7^{-1} L_u^2 + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2) TL_y^2,\ 2L_u + \tilde{\varepsilon}_7 + 3T\tilde{\varepsilon}_2^{-1} + \tilde{\varepsilon}_8^{-1} L_y^2, \\
&\quad 2L_u + \varepsilon_9^{-1} L_y^2 \tilde{+} \tilde{\varepsilon}_{11} + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2) TL_u^2,\ 2L_{\mathrm{u}} + \tilde{\varepsilon}_{11}^{-1} L_u^2 + \tilde{\varepsilon}_{10}^{-1} L_y^2\},
\end{aligned}
$$

we have that

$$
\begin{aligned}
&\|(\delta Y, \delta Z, \delta N, \delta U, \delta V, \delta M, \delta\partial U, \delta\partial V, \delta\partial M)\|_{\mathcal{H}^c}^2 \\
&< \frac{20}{2^3 \cdot 3 \cdot 7 \cdot 10^2} 3\big(\sqrt{30 + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2)} + \sqrt{30}\big)^2 \Big(\|\delta z\|_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}^2 + \|\delta v\|_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}^2 + \|\delta v\|_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}^2 \Big) \\
&= \frac{\big(\sqrt{30 + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2)} + \sqrt{30}\big)^2}{2^2 \cdot 7 \cdot 10} \Big(\|\delta z\|_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}^2 + \|\delta v\|_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}^2 + \|\delta v\|_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}^2 \Big).
\end{aligned}
$$

Thus, letting choosing $\big(\sqrt{30 + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2)} + \sqrt{30}\big)^2 \leq 2^2 \cdot 7 \cdot 10$, $\mathfrak{T}$ is contractive, i.e

$$
\|(\delta Y, \delta Z, \delta N, \delta U, \delta V, \delta M, \delta\partial U, \delta\partial V, \delta\partial M)\|_{\mathcal{H}^c}^2 < \|\delta z\|_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}^2 + \|\delta v\|_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}^2 + \|\delta\partial v\|_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}^2.
$$

Step 3: We consolidate our results. To begin with, we collect the constraints of the weight of the norms. In light of (4.5.5) and (4.5.10), c must satisfy

$$
\begin{aligned}
c \geq \max\ \{&2L_y + \varepsilon_1^{-1} 7TL_{\mathrm{u}}^2 + \varepsilon_1 TL_y^2 + \varepsilon_2 TL_y^2 + \varepsilon_7^{-1} L_u^2 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10}, \\
&2L_u + \varepsilon_2^{-1} 7T + \varepsilon_7 + \varepsilon_8^{-1} L_y^2 2L_u + \varepsilon_1 TL_u^2 + \varepsilon_2 TL_u^2 + \varepsilon_9^{-1} L_y^2 + \varepsilon_{11}, \\
&2L_{\mathrm{u}} + \varepsilon_{11}^{-1} L_u^2 + \varepsilon_{10}^{-1} L_y^2, ,\ 2L_u + \tilde{\varepsilon}_7 + 3T\tilde{\varepsilon}_2^{-1} + \tilde{\varepsilon}_8^{-1} L_y^2, \\
&2L_y + \tilde{\varepsilon}_1^{-1} 3TL_{\mathrm{u}}^2 + \tilde{\varepsilon}_8 + \tilde{\varepsilon}_9 + \tilde{\varepsilon}_{10} + \tilde{\varepsilon}_7^{-1} L_u^2 + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2) TL_y^2, \\
&2L_u + \varepsilon_9^{-1} L_y^2 \tilde{+} \tilde{\varepsilon}_{11} + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2) TL_u^2,\ 2L_{\mathrm{u}} + \tilde{\varepsilon}_{11}^{-1} L_u^2 + \tilde{\varepsilon}_{10}^{-1} L_y^2, \\
&8L_y + 2TL_y + 2TL_{\mathrm{u}} L_y,\ 4L_u + 2T^2 L_u + 2T^2 L_{\mathrm{u}} L_u\} \\
= \max\ \{&2L_y + \varepsilon_1^{-1} 7TL_{\mathrm{u}}^2 + (\varepsilon_1 + \varepsilon_2) TL_y^2 + \varepsilon_7^{-1} L_u^2 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10}, \\
&2L_u + \varepsilon_2^{-1} 7T + \varepsilon_7 + \varepsilon_8^{-1} L_y^2, \\
&2L_u + \varepsilon_1 TL_u^2 + \varepsilon_2 TL_u^2 + \varepsilon_9^{-1} L_y^2 + \varepsilon_{11},\ 2L_{\mathrm{u}} + \varepsilon_{10}^{-1} L_y^2 + \varepsilon_{11}^{-1} L_u^2, \\
&8L_y + 2TL_y + 2TL_{\mathrm{u}} L_y,\ 4L_u + 2TL_u + 2TL_{\mathrm{u}} L_u\},
\end{aligned}
\tag{4.5.13}
$$

where the equality follows from the choice $\varepsilon_i = \tilde{\varepsilon}_i, i \in \{1, 2, 7, ..., 11\}$.

All together we find that given $\gamma \in (0, \infty)$, $\varepsilon_i \in (0, \infty)$, $i \in \{1, ..., 11\}$, $c \in (0, \infty)$, such that $\varepsilon_1 + \varepsilon_2 \leq (2\sqrt{70} - \sqrt{30})^2 - 30$, $\mathfrak{T}$ is a well–defined contraction in $\mathcal{B}_R \subseteq \mathcal{H}^c$ for the norm $\| \cdot \|_{\mathcal{H}^c}$ provided: (i) γ, ε_i, $i \in \{1, ..., 6\}$, and the data of the problem satisfy $I_0^\varepsilon \leq \gamma R^2/10$; (ii) c satisfies (4.5.13). □

Proof of Theorem 4.3.9. We show for $\|\partial M\|^2_{\mathrm{BMO}^{2,2,c}} < \infty$, the argument for $(\mathcal{N}, M)$ being completely analogous. Without lost of generality we assume $c = 0$, see Remark 4.2.2. In light of Assumption J, we have that $\mathrm{d}r \otimes \mathrm{d}\mathbb{P}$–a.e.

$$|\nabla g_r(s, \partial U_r^s, \partial V_r^s, U_r^s, V_r^s, \mathcal{Y}_r, \mathcal{Z}_r)| \leq L_{\mathrm{u}} |U_r^s| + L_{\mathrm{v}} |\sigma_r^\top V_r^s|^2 + L_u |U_r^s| + L_v |\sigma_r^\top V_r^s|^2 + L_y |\mathcal{Y}_r| + L_z |\sigma_r^\top \mathcal{Z}_r|^2 + |\nabla \tilde{g}_r(s)|.$$

Let $\tau \in \mathcal{T}_{0,T}$. We now note, recall $\partial V \in \mathbb{H}^{2,2}_{\mathrm{BMO}}$

$$\mathbb{E}_\tau\bigg[\bigg(\int_{\tau-}^T \partial V_r^{s\top} \mathrm{d}X_r\bigg)^2\bigg] = \mathbb{E}_\tau\bigg[\bigg(\int_{\tau}^T \partial V_r^{s\top} \mathrm{d}X_r\bigg)^2\bigg] = \mathbb{E}_\tau\bigg[\int_\tau^T |\sigma_r^\top \partial V_r^s|^2 \mathrm{d}r\bigg].$$

All together, it follows from $(\mathcal{S})$ and Jensen's inequality that

$$\begin{aligned}
&\mathbb{E}_\tau\bigg[\bigg|\int_{\tau-}^T \mathrm{d}\partial M_r^s\bigg|^2\bigg] \\
&\leq 10\mathbb{E}_\tau\bigg[|\partial_s \eta(s)|^2 + \bigg|\int_{\tau-}^T |\nabla \tilde{g}_r(s)| \mathrm{d}r\bigg|^2 + |\partial U_{\tau-}^s|^2 + T\int_{\tau-}^T L_{\mathrm{u}} |\partial U_r^s|^2 + L_u |U_r^s|^2 + L_y |\mathcal{Y}_r|^2 \mathrm{d}r \\
&\quad + \bigg|\int_{\tau-}^T L_{\mathrm{v}} |\sigma_r^\top \partial V_r^s|^2 \mathrm{d}r\bigg|^2 + \bigg|\int_{\tau-}^T L_v |\sigma_r^\top V_r^s|^2 \mathrm{d}r\bigg|^2 + \bigg|\int_{\tau-}^T L_z |\sigma_r^\top \mathcal{Z}_r|^2 \mathrm{d}r\bigg|^2 + \bigg|\int_{\tau-}^T V_r^{s\top} \mathrm{d}X_r\bigg|^2\bigg] \\
&\leq 10\Big(\|\partial_s \eta(s)\|^2_{\mathcal{L}^{\infty,2}} + \|\nabla \tilde{g}\|^2_{\mathbb{L}^{1,\infty,2}} + (1 + L_{\mathrm{u}} T^2)\|\partial U\|^2_{\mathcal{S}^{\infty,2}} + L_u T \|U\|^2_{\mathcal{S}^{\infty,2}} + L_y T \|\mathcal{Y}\|^2_{\mathcal{S}^\infty} \\
&\quad + \|\partial V\|^2_{\mathbb{H}^{2,2}_{\mathrm{BMO}}}(1 + 2L_{\mathrm{v}}^2 \|\partial V\|^2_{\mathbb{H}^{2,2}_{\mathrm{BMO}}}) + 2L_v^2 \|V\|^4_{\mathbb{H}^{2,2}_{\mathrm{BMO}}} + 2L_z^2 \|Z\|^4_{\mathbb{H}^2_{\mathrm{BMO}}}\Big).
\end{aligned}$$

□

Proof of Theorem 4.3.10. Upon close inspection of the proof of Theorem 4.3.6, we see that the only stages of the argument where both the presence and the norm of $(\mathcal{N}, M, \partial M)$ plays a role are in (4.5.6), (4.5.7) and (4.5.12). We address each of them in the following. We consider the case (i). The argument for (ii) follows similarly.

If we were to require BMO-norms on $(\mathcal{N}, M, \partial M)$ we see that

(i) In (4.5.6), the presence of the $\operatorname{ess\,sup}^{\mathbb{P}}$ in the BMO-norm would require us to consider the estimate right before (4.5.6) for each of the 9 processes that define the solution to $(\mathcal{S})$. Thus we obtain a factor 11 instead of 10. This yields

$$
\begin{aligned}
C_\varepsilon := \min \{&1 - 11(\varepsilon_3^{-1} + \varepsilon_{12}^{-1} + \varepsilon_{13}^{-1}), 1 - 11(\varepsilon_4^{-1} + \varepsilon_{14}^{-1} + \varepsilon_{15}^{-1}), \\
&1 - 11(\varepsilon_5^{-1} + \varepsilon_{16}^{-1} + \varepsilon_{17}^{-1}), 1 - 11(\varepsilon_6^{-1} + \varepsilon_{18}^{-1} + \varepsilon_{19}^{-1} + \varepsilon_{20}^{-1})\},
\end{aligned}
$$

$$
I_0^\varepsilon \leq \gamma R^2/11,
$$

and

$$
\|(Y, Z, N, U, V, M, \partial U, \partial V, \partial M)\|^2_{\widehat{\mathcal{H}}^c} \leq \ C_\varepsilon^{-1} R^2 \bigg(\gamma + 24 L^2_{z\vee v} R^2 \bigg(\varepsilon_1 + \varepsilon_2 + \sum_{i=12}^{20} \varepsilon_i\bigg)\bigg).
$$

(ii) As a consequence of the previous observation (4.5.7) would be replace by

$$
R^2 \leq \frac{1}{24 L_\star} \frac{C_\varepsilon - \gamma}{\varepsilon_1 + \varepsilon_2 + \sum_{i=12}^{20} \varepsilon_i} < \frac{1}{2^3 \cdot 3 \cdot 7 \cdot 11^2 \cdot L_\star} = \mathcal{U}(11).
$$

where the upper bound comes from the optimisation procedure, *i.e.* Lemma 4.7.2.

(iii) Likewise, with $C_{\tilde{\varepsilon}}$ as in the proof, (4.5.12) is now given by

$$
\begin{aligned}
&\|(Y, Z, N, U, V, M, \partial U, \partial V, \partial M)\|^2_{\widehat{\mathcal{H}}^c} \\
&\leq 24 C_{\tilde{\varepsilon}}^{-1} L^2_{z\vee v} R^2 (3\tilde{\varepsilon}_1 + 3\tilde{\varepsilon}_2 + 2\tilde{\varepsilon}_3 + 2\tilde{\varepsilon}_4 + 2\tilde{\varepsilon}_5 + 3\tilde{\varepsilon}_6) \Big(\|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|\delta \partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} \Big) \\
&< \frac{1}{2^2 \cdot 7 \cdot 11} (\sqrt{33} + \sqrt{33 + \tilde{\varepsilon}_1 + \tilde{\varepsilon}_2})^2 \Big(\|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|\delta v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\delta \partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} \Big),
\end{aligned}
$$

where the second inequality follows from the new version of the optimisation procedure, *i.e.* Lemma 4.7.3.

All things considered, **Step 3** will lead to require Assumption K for $\kappa = 11$. By assumption the result follows. $\square$

4.6 Proof of the quadratic case

Proof of Theorem 4.3.12. For $c > 0$, let us introduce the mapping

$$\mathfrak{T} : (\mathcal{B}_R, \|\cdot\|_{\mathcal{H}^c}) \longrightarrow (\mathcal{B}_R, \|\cdot\|_{\mathcal{H}^c})$$
$$(y, z, n, u, v, m, \partial u, \partial v, \partial m) \longmapsto (Y, Z, N, U, V, M, \partial U, \partial V, \partial M),$$

with $\mathfrak{H} = (\mathcal{Y}, \mathcal{Z}, \mathcal{N}, U, V, M, \partial U, \partial V, \partial M)$ given for any $s \in [0,T]$, $\mathbb{P}$–a.s. for any $t \in [0,T]$ by

$$\begin{aligned}
\mathcal{Y}_t &= \xi(T, X_{\cdot\wedge T}) + \int_t^T h_r(X, y_r, z_r, u_r^r, v_r^r, \partial U_r^r)\mathrm{d}r - \int_t^T \mathcal{Z}_r^\top dX_r - \int_t^T \mathrm{d}\mathcal{N}_r,\\
U_t^s &= \eta(s, X_{\cdot\wedge,T}) + \int_t^T g_r(s, X, u_r^s, v_r^s, y_r, z_r)\mathrm{d}r - \int_t^T V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}M_r^s,\\
\partial U_t^s &= \partial_s\eta(s, X_{\cdot\wedge,T}) + \int_t^T \nabla g_r(s, X, \partial u_r^s, \partial v_r^s, u_r^s, v_r^s, y_r, z_r)\mathrm{d}r - \int_t^T \partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s.
\end{aligned}$$

Step 1: We first argue that $\mathfrak{T}$ is well-defined.

(i) Let us first remark that for $u \in \mathcal{S}^{\infty,2,c}$

$$\mathbb{E}\bigg[\int_0^T |u_t^t|^2\mathrm{d}r\bigg] \leq T\|u\|^2_{\mathcal{S}^{\infty,2,c}}. \tag{4.6.1}$$

In light of Assumption L, there is $c > 0$ such that $(\xi, \eta, \partial\eta, \tilde{f}, \tilde{g}, \nabla\tilde{g}) \in \mathcal{L}^{\infty,c} \times \big(\mathcal{L}^{\infty,2,c}\big)^2 \times \mathbb{L}^{1,\infty,c} \times (\mathbb{L}^{1,\infty,2,c})^2$, thus, we may use (4.2.1) and (4.6.1) to obtain

$$\begin{aligned}
&\mathbb{E}\bigg[|\xi(T)|^2 + \bigg|\int_0^T |h_t(y_t, z_t, u_t^t, v_t^t, 0)|\mathrm{d}t\bigg|^2\bigg] + \sup_{s\in[0,T]}\mathbb{E}\bigg[|\eta(s)|^2 + \bigg|\int_0^T |g_t(s, u_t^s, v_t^s, y_t, z_t)|\mathrm{d}t\bigg|^2\bigg]\\
&\quad + \sup_{s\in[0,T]}\mathbb{E}\bigg[|\partial\eta(s)|^2 + \bigg|\int_0^T |\nabla g_t(s, \partial u_t^s, \partial v_t^s, u_t^s, v_t^s, y_t^s, z_t)|\mathrm{d}t\bigg|^2\bigg]\\
&\leq \mathbb{E}\bigg[|\xi(T)|^2 + 5\bigg|\int_0^T |\tilde{h}_t|\mathrm{d}t\bigg|^2 + 17L_y^2\bigg|\int_0^T |y_t|^2\mathrm{d}t\bigg|^2 + 17L_z^2\bigg|\int_0^T |z_t|^2\mathrm{d}t\bigg|^2 + 5L_u^2\bigg|\int_0^T |u_t^t|^2\mathrm{d}t\bigg|^2\\
&\quad + 5L_v^2\bigg|\int_0^T |v_t^t|^2\mathrm{d}t\bigg|^2\bigg] + \sup_{s\in[0,T]}\mathbb{E}\bigg[|\eta(s)|^2 + 5\bigg|\int_0^T |\tilde{g}_t(s)|\mathrm{d}t\bigg|^2 + 12L_u^2\bigg|\int_0^T |u_t^s|^2\mathrm{d}t\bigg|^2 + 12L_v^2\bigg|\int_0^T |v_t^s|^2\mathrm{d}t\bigg|^2\bigg]\\
&\quad + \sup_{s\in[0,T]}\mathbb{E}\bigg[|\partial\eta(s)|^2 + 7\bigg|\int_0^T |\nabla\tilde{g}_t(s)|\mathrm{d}t\bigg|^2 + 7L_{\mathrm{u}}^2\bigg|\int_0^T |\partial u_t^s|^2\mathrm{d}t\bigg|^2 + 7L_{\mathrm{v}}^2\bigg|\int_0^T |\partial v_t^s|^2\mathrm{d}t\bigg|^2\bigg]\\
&\leq \|\xi\|^2_{\mathcal{L}^{\infty,c}} + 5\|\tilde{h}\|^2_{\mathbb{L}^{1,\infty,c}} + \|\eta\|^2_{\mathcal{L}^{\infty,2,c}} + 5\|\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}} + \|\partial\eta\|^2_{\mathcal{L}^{\infty,2,c}} + 7\|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}}
\end{aligned}$$

$$+ 17L_y^2T^2\|y\|^4_{\mathcal{S}^{\infty,c}} + 34L_z^2\|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + 17L_u^2T^2\|u\|^4_{\mathcal{S}^{\infty,2,c}} + 24L_v^2\|v\|^4_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}$$
$$+ 7L_{\mathrm{u}}^2T^2\|\partial u\|^4_{\mathcal{S}^{\infty,2,c}} + 14L_{\mathrm{v}}^2\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} < \infty.$$

Therefore, by Theorem 3.4.3, $\mathfrak{T}$ defines a well-posed system of BSDEs with unique solution in the space $\mathfrak{H}^2$. We recall the spaces involved in the definition of $\mathfrak{H}^2$, and their corresponding norms, were introduced in Section 4.2.2.

(ii) For $\mathcal{U}_t := U_t^t,\ \mathcal{V}_t := V_t^t,\ \mathcal{M}_t := M_t^t - \int_0^t \partial M_r^r \mathrm{d}r,\ t \in [0,T]$, $(\mathcal{U},\mathcal{V},\mathcal{M}) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2$ and satisfy the equation

$$\mathcal{U}_t = \eta(T, X_{\cdot\wedge T}) + \int_t^T \big(g_r(r,X,u_r^r,v_r^r,y_r,z_r) - \partial U_r^r\big)\mathrm{d}r - \int_t^T \mathcal{V}_r^\top \mathrm{d}X_r - \int_t^T \mathrm{d}\mathcal{M}_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.}$$

(iii) $(\mathcal{Y},\mathcal{U}) \in \mathcal{S}^{\infty,c} \times \mathcal{S}^{\infty,c}$ and $\|U\|_{\mathcal{S}^{\infty,2,c}} + \|\partial U\|_{\mathcal{S}^{\infty,2,c}} < \infty$.

In light of Assumption L, $\mathrm{d}r \otimes \mathrm{d}\mathbb{P}$–a.e.

$$\begin{aligned}
|h_r| &\leq L_y|y_r|^2 + L_z|\sigma_r^\top z_r|^2 + L_u|u_r^r|^2 + L_v|\sigma_r^\top v_r^r|^2 + L_{\mathrm{u}}|\partial U_r^r| + |\tilde{h}_r|,\\
|g_r| &\leq L_u|u_r^r|^2 + L_v|\sigma_r^\top v_r^r|^2 + L_y|y_r|^2 + L_z|\sigma_r^\top z_r|^2 + |\partial U_r^r| + |\tilde{g}_r|,\\
|g_r(s)| &\leq L_u|u_r^s|^2 + L_v|\sigma_r^\top v_r^s|^2 + L_y|y_r|^2 + L_z|\sigma_r^\top z_r|^2 + |\tilde{g}_r(s)|, \qquad (4.6.2)\\
|\nabla g_r(s)| &\leq L_{\mathrm{u}}|\partial u_r^s|^2 + L_{\mathrm{v}}|\sigma_r^\top \partial v_r^s|^2 + L_u|u_r^s|^2 + L_v|\sigma_r^\top v_r^s|^2 + L_y|y_r|^2 + L_z|\sigma_r^\top z_r|^2 + |\nabla\tilde{g}_r(s)|.
\end{aligned}$$

Again, we apply Meyer–Itô's formula to $\mathrm{e}^{\frac{c}{2}t}(|\mathcal{Y}_t| + |\mathcal{U}_t| + |U_t^s| + |\partial U_t^s|)$ and take conditional expectations with respect to $\mathcal{F}_t$ as in Equation (4.5.2). Moreover, in combination with (4.6.2) and Lemma 4.7.1, we obtain back in (4.5.2) that

$$\begin{aligned}
&\mathrm{e}^{\frac{c}{2}t}(|\mathcal{Y}_t| + |\mathcal{U}_t| + |U_t^s| + |\partial U_t^s|) + \mathbb{E}_t\bigg[\int_t^T \frac{c}{2}\mathrm{e}^{\frac{c}{2}r}|\mathcal{Y}_r|\mathrm{d}r\bigg] + \mathbb{E}_t\bigg[\int_t^T \frac{c}{2}\mathrm{e}^{\frac{c}{2}r}|\mathcal{U}_r|\mathrm{d}r\bigg]\\
&\quad + \sup_{s\in[0,T]}\mathbb{E}_t\bigg[\int_t^T \frac{c}{2}\mathrm{e}^{\frac{c}{2}r}|U_r^s|\mathrm{d}r\bigg] + \sup_{s\in[0,T]}\mathbb{E}_t\bigg[\int_t^T \frac{c}{2}\mathrm{e}^{\frac{c}{2}r}|\partial U_r^s|\mathrm{d}r\bigg]\\
&\leq \mathbb{E}_t\big[\mathrm{e}^{\frac{c}{2}T}\big(|\xi| + |\eta(T)| + |\eta(s)| + |\partial_s\eta(s)|\big)\big] + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{\frac{c}{2}r}\Big(|\tilde{h}_r| + |\tilde{g}_r| + |\tilde{g}_r(s)| + |\nabla\tilde{g}_r(s)|\Big)\mathrm{d}r\bigg]\\
&\quad + (T + L_{\mathrm{u}}T)\Big(\|\partial_s\eta\|_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde{g}\|_{\mathbb{L}^{1,\infty,2,c}} + TL_\star\big(\|y\|^2_{\mathcal{S}^{\infty,c}} + \|u\|^2_{\mathcal{S}^{\infty,2,c}} + \|\partial u\|^2_{\mathcal{S}^{\infty,2,c}}\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ (T+L_{\mathrm{u}}T)L_\star\big(\|z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\big)\\
&+ \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}\Big(4L_y|\sigma_r^\top y_r|^2 + 2L_u|\sigma_r^\top u_r^r|^2 + 2L_u|\sigma_r^\top u_r^s|^2\Big)\mathrm{d}r\bigg]\\
&+ \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}\Big(L_{\mathrm{u}}|\sigma_r^\top \partial u_r^s|^2 + 4L_z|\sigma_r^\top z_r|^2 + 2L_v|\sigma_r^\top v_r^r|^2 + 2L_v|\sigma_r^\top v_r^s|^2 + L_{\mathrm{v}}|\sigma_r^\top \partial v_r^s|^2\Big)\mathrm{d}r\bigg],
\end{aligned}
$$

where we recall the notation $L_\star = \max\{L_y, L_u, L_{\mathrm{u}}, L_z, L_v, L_{\mathrm{v}}\}$. Thus, for any $c>0$ we obtain

$$
\begin{aligned}
&\max\{\mathrm{e}^{\frac{c}{2}t}|\mathcal{Y}_t|,\, \mathrm{e}^{\frac{c}{2}t}|\mathcal{U}_t|,\, \mathrm{e}^{\frac{c}{2}t}|U_t^s|,\, \mathrm{e}^{\frac{c}{2}t}|\partial U_t^s|\}\\
&\leq \|\xi\|_{\mathcal{L}^{\infty,c}} + \|\tilde h\|_{\mathbb{L}^{1,\infty,2,c}} + 2\big(\|\eta\|_{\mathcal{L}^{\infty,2,c}} + \|\tilde g\|_{\mathbb{L}^{1,\infty,2,c}}\big) + (1+T+TL_{\mathrm{u}})\big(\|\partial_s\eta\|_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde g\|_{\mathbb{L}^{1,\infty,2,c}}\big)\\
&\quad + (4+T+L_{\mathrm{u}}T)L_\star T\Big(\|y\|^2_{\mathcal{S}^{2\infty,c}} + \|u\|^2_{\mathcal{S}^{\infty,2,c}} + \|\partial u\|^2_{\mathcal{S}^{\infty,2,c}}\Big)\\
&\quad + (4+T+L_{\mathrm{u}}T)L_\star\Big(\|z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\Big),
\end{aligned}
$$

(iv) We show $(\mathcal{Z},\mathcal{V},\mathcal{N},\mathcal{M})\in\big(\mathbb{H}^{2,c}_{\mathrm{BMO}}\big)^2\times\big(\mathbb{M}^{2,c}\big)^2$, $\|V\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|M\|^2_{\mathbb{M}^{2,2,c}} + \|\partial V\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\partial M\|^2_{\mathbb{M}^{2,2,c}} < \infty$.

From (iii), Assumption J.(ii) and J.(iii), together with Young's inequality, yield that, for any $\varepsilon_i>0$, $i\in\{1,2\}$, and defining $C_{\varepsilon_1} := \varepsilon_1^{-1}7TL_{\mathrm{u}}^2$, and $C_{\varepsilon_2} := \varepsilon_2^{-1}7T$, we have

$$
\begin{aligned}
2\mathcal{Y}_r\cdot h_r - c|\mathcal{Y}_r|^2 &\leq 2\|\mathcal{Y}\|_{\mathcal{S}^{\infty,c}}\big(L_y|y_r|^2 + L_z|\sigma_r^\top z_r|^2 + L_u|u_r^r|^2 + L_v|\sigma_r^\top v_r^r|^2 + |\tilde h_r|\big)\\
&\quad + \varepsilon_1(7T)^{-1}|\partial U_r^r|^2 + (\widetilde C_{\varepsilon_1} - c)|\mathcal{Y}_r|^2,\\
2\mathcal{U}_r\cdot g_r - c|\mathcal{U}_r|^2 &\leq 2\|\mathcal{U}\|_{\mathcal{S}^{\infty,c}}\big(L_y|y_r|^2 + L_z|\sigma_r^\top z_r|^2 + |u_r^r|^2 + L_v|\sigma_r^\top v_r^r|^2 + |\tilde g_r|\big)\\
&\quad + \varepsilon_2(7T)^{-1}|\partial U_r^r|^2 + (\widetilde C_{\varepsilon_2} - c)|\mathcal{U}_r|^2,\\
2U_r^s\cdot g_r(s) - c|U_r^s|^2 &\leq 2\|U\|_{\mathcal{S}^{\infty,2,c}}\big(L_u|u_r^s|^2 + L_v|\sigma_r^\top v_r^s|^2 + L_y|y_r|^2 + L_z|\sigma_r^\top z_r|^2 + |\tilde g_r(s)|\big) - c|U_r^s|^2,\\
2\partial U_r^s\cdot\nabla g_r(s) - c|\partial U_r^s|^2 &\leq 2\|\partial U\|_{\mathcal{S}^{\infty,c,2}}\big(L_{\mathrm{u}}|\partial u_r^s|^2 + L_v|\sigma_r^\top\partial v_r^s|^2 + L_u|u_r^s|^2 + L_{\mathrm{v}}|\sigma_r^\top v_r^s|^2 + |\nabla\tilde g_r(s)|\big)\\
&\quad + 2\|\partial U\|_{\mathcal{S}^{\infty,c,2}}\big(L_y|y_r|^2 + L_z|\sigma_r^\top z_r|^2\big) - c|\partial U_r^s|^2.
\end{aligned}
$$

These inequalities in combination with the analogous version of Lemma 4.7.1 (which holds for

$c > 2L_{\mathrm{u}}$), Young's inequality, and Itô's formula, as in (4.5.4), show that for any $\varepsilon_i > 0$, $i \in \{3, ..., 24\}$

$$\begin{aligned}
&\sum_{i=1}^{4} \mathrm{e}^{ct}|\mathfrak{Y}_t^i|^2 + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top \mathfrak{Z}_r^i|^2\mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{dTr}[\mathfrak{Z}^i]_r + \int_t^T \mathrm{e}^{cr}\big(|\mathcal{Y}_r|^2(c - C_{\varepsilon_1}) + |\mathcal{U}_r|^2(c - C_{\varepsilon_2})\big)\mathrm{d}r\bigg] \\
&\quad + \sup_{s\in[0,T]} \mathbb{E}_t\bigg[\int_t^T c\mathrm{e}^{cr}|U_r^s|^2\mathrm{d}r\bigg] + \sup_{s\in[0,T]} \mathbb{E}_t\bigg[\int_t^T c\mathrm{e}^{cr}|\partial U_r^s|^2\mathrm{d}r\bigg] \\
=&\mathbb{E}_t\Big[\mathrm{e}^{cT}(|\xi|^2 + |\eta(T)|^2 + |\eta(s)|^2 + |\partial_s\eta(s)|^2)\Big] + (\varepsilon_1 + \varepsilon_2)\Big(\|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}}\Big) \\
&\quad + (\varepsilon_1 + \varepsilon_2)\Big(L_\star T^2(\|y\|^4_{\mathcal{S}^{\infty,c}} + \|u\|^4_{\mathcal{S}^{\infty,c}} + \|\partial u\|^4_{\mathcal{S}^{\infty,c}}) + 2L_\star^2\Big(\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big)\Big) \\
&\quad + (\varepsilon_3^{-1} + \varepsilon_7^{-1} + \varepsilon_8^{-1} + \varepsilon_9^{-1} + \varepsilon_{10}^{-1})\|\mathcal{Y}\|^2_{\mathcal{S}^{\infty,c}} + (\varepsilon_4^{-1} + \varepsilon_{11}^{-1} + \varepsilon_{12}^{-1} + \varepsilon_{13}^{-1} + \varepsilon_{14}^{-1})\|\mathcal{U}\|^2_{\mathcal{S}^{\infty,c}} \\
&\quad + (\varepsilon_5^{-1} + \varepsilon_{15}^{-1} + \varepsilon_{16}^{-1} + \varepsilon_{17}^{-1} + \varepsilon_{18}^{-1})\|U\|^2_{\mathcal{S}^{\infty,c,2}} + (\varepsilon_6^{-1} + \varepsilon_{19}^{-1} + \varepsilon_{20}^{-1} + \varepsilon_{21}^{-1} + \varepsilon_{22}^{-1} + \varepsilon_{23}^{-1} + \varepsilon_{24}^{-1})\|\partial U\|^2_{\mathcal{S}^{\infty,c,2}} \\
&\quad + \mathbb{E}_t\bigg[\varepsilon_3\bigg|\int_t^T \mathrm{e}^{cr}|\tilde{h}_r|\mathrm{d}r\bigg|^2 + \varepsilon_4\bigg|\int_t^T \mathrm{e}^{cr}|\tilde{g}_r|\mathrm{d}r\bigg|^2\bigg] + \varepsilon_5\bigg|\int_t^T \mathrm{e}^{cr}|\tilde{g}_r(s)|\mathrm{d}r\bigg|^2 + \varepsilon_6\bigg|\int_t^T \mathrm{e}^{cr}|\nabla\tilde{g}_r|\mathrm{d}r\bigg|^2\bigg] \\
&\quad + (\varepsilon_7 + \varepsilon_{11} + \varepsilon_{15} + \varepsilon_{19})L_\star T^2\|y\|^4_{\mathcal{S}^{\infty,c}} + (\varepsilon_9 + \varepsilon_{13} + \varepsilon_{17} + \varepsilon_{21})L_\star T^2\|u\|^4_{\mathcal{S}^{\infty,2,c}} + \varepsilon_{23}L_\star T^2\|\partial u\|^4_{\mathcal{S}^{\infty,2,c}} \\
&\quad + (\varepsilon_8 + \varepsilon_{12} + \varepsilon_{16} + \varepsilon_{20})L_z^2\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\sigma_r^\top z_r|^2\mathrm{d}r\bigg|^2\bigg] + (\varepsilon_{10} + \varepsilon_{14})L_v^2\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\sigma_r^\top v_r^r|^2\mathrm{d}r\bigg|^2\bigg] \\
&\quad + (\varepsilon_{18} + \varepsilon_{22})L_v^2\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\sigma_r^\top v_r^s|^2\mathrm{d}r\bigg|^2\bigg] + \varepsilon_{24}L_{\mathrm{v}}^2\mathbb{E}_t\bigg[\bigg|\int_t^T \mathrm{e}^{cr}|\sigma_r^\top \partial v_r^s|^2\mathrm{d}r\bigg|^2\bigg].
\end{aligned}$$

We now let $\tau \in \mathcal{T}_{0,T}$. In light of (4.5.3), for

$$c \geq \max\, \{\varepsilon_1^{-1}7TL_{\mathrm{u}}^2, \varepsilon_2^{-1}7T, 2L_{\mathrm{u}}\}, \tag{4.6.3}$$

Equation (4.2.1) yields

$$\begin{aligned}
&\sum_{i=1}^{4} \mathrm{e}^{ct}|\mathfrak{Y}_t^i|^2 + \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top \mathfrak{Z}_r^i|^2\mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{dTr}[\mathfrak{Z}^i]_r\bigg] \\
=&\ \|\xi\|^2_{\mathcal{L}^{\infty,c}} + 2\|\eta\|^2_{\mathcal{L}^{\infty,2,c}} + (1 + \varepsilon_1 + \varepsilon_2)\|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}} + \varepsilon_3\|\tilde{h}\|^2_{\mathbb{L}^{1,\infty,c}} \\
&\quad + (\varepsilon_4 + \varepsilon_5)\|\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_6)\|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}} \\
&\quad + L_\star^2T^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_7 + \varepsilon_{11} + \varepsilon_{15} + \varepsilon_{19})\|y\|^4_{\mathcal{S}^{\infty,c}} + L_\star^2T^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_9 + \varepsilon_{13} + \varepsilon_{17} + \varepsilon_{21})\|u\|^4_{\mathcal{S}^{\infty,2,c}} \\
&\quad + L_\star^2T^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_{23})\|\partial u\|^4_{\mathcal{S}^{\infty,2,c}} + 2L_\star^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_8 + \varepsilon_{12} + \varepsilon_{16} + \varepsilon_{20})\|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} \\
&\quad + 2L_\star^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_{10} + \varepsilon_{14} + \varepsilon_{18} + \varepsilon_{22})\|v\|^4_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + 2L_\star^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_{24})\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} \\
&\quad + (\varepsilon_3^{-1} + \varepsilon_7^{-1} + \varepsilon_8^{-1} + \varepsilon_9^{-1} + \varepsilon_{10}^{-1})\|\mathcal{Y}\|^2_{\mathcal{S}^{\infty,c}} + (\varepsilon_4^{-1} + \varepsilon_{11}^{-1} + \varepsilon_{12}^{-1} + \varepsilon_{13}^{-1} + \varepsilon_{14}^{-1})\|\mathcal{U}\|^2_{\mathcal{S}^{\infty,c}}
\end{aligned}$$

$$+ \big(\varepsilon_5^{-1} + \varepsilon_{15}^{-1} + \varepsilon_{16}^{-1} + \varepsilon_{17}^{-1} + \varepsilon_{18}^{-1}\big)\|U\|^2_{\mathcal{S}^{\infty,c,2}}$$

$$+ \big(\varepsilon_6^{-1} + \varepsilon_{19}^{-1} + \varepsilon_{20}^{-1} + \varepsilon_{21}^{-1} + \varepsilon_{22}^{-1} + \varepsilon_{23}^{-1} + \varepsilon_{24}^{-1}\big)\|\partial U\|^2_{\mathcal{S}^{\infty,c,2}},$$

which in turn leads to

$$\begin{aligned}
&\frac{1}{10}\Big(\|\mathcal{Y}\|^2_{\mathcal{S}^{\infty,c}} + \|\mathcal{U}\|^2_{\mathcal{S}^{\infty,c}} + \|U\|^2_{\mathcal{S}^{\infty,2,c}} + \|\partial U\|^2_{\mathcal{S}^{\infty,2,c}} + \|\mathcal{Z}\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\\
&\quad + \|V\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|\partial V\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\mathcal{N}\|^2_{\mathbb{M}^{2,c}} + \|M\|^2_{\mathbb{M}^{2,2,c}} + \|\partial M\|^2_{\mathbb{M}^{2,2,c}}\Big)\\
&\leq \|\xi\|^2_{\mathcal{L}^{\infty,c}} + 2\|\eta\|^2_{\mathcal{L}^{\infty,2,c}} + (1+\varepsilon_1+\varepsilon_2)\|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}} + \varepsilon_3\|\tilde{h}\|^2_{\mathbb{L}^{1,\infty,c}}\\
&\quad + (\varepsilon_4+\varepsilon_5)\|\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}} + (\varepsilon_1+\varepsilon_2+\varepsilon_6)\|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}}\\
&\quad + L_\star^2T^2(\varepsilon_1+\varepsilon_2+\varepsilon_7+\varepsilon_{11}+\varepsilon_{15}+\varepsilon_{19})\|y\|^4_{\mathcal{S}^{\infty,c}}\\
&\quad + L_\star^2T^2(\varepsilon_1+\varepsilon_2+\varepsilon_9+\varepsilon_{13}+\varepsilon_{17}+\varepsilon_{21})\|u\|^4_{\mathcal{S}^{\infty,2,c}}\\
&\quad + L_\star^2T^2(\varepsilon_1+\varepsilon_2+\varepsilon_{23})\|\partial u\|^4_{\mathcal{S}^{\infty,2,c}} + 2L_\star^2(\varepsilon_1+\varepsilon_2+\varepsilon_8+\varepsilon_{12}+\varepsilon_{16}+\varepsilon_{20})\|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\\
&\quad + 2L_\star^2(\varepsilon_1+\varepsilon_2+\varepsilon_{10}+\varepsilon_{14}+\varepsilon_{18}+\varepsilon_{22})\|v\|^4_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + 2L_\star^2(\varepsilon_1+\varepsilon_2+\varepsilon_{24})\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\\
&\quad + \big(\varepsilon_3^{-1}+\varepsilon_7^{-1}+\varepsilon_8^{-1}+\varepsilon_9^{-1}+\varepsilon_{10}^{-1}\big)\|\mathcal{Y}\|^2_{\mathcal{S}^{\infty,c}} + \big(\varepsilon_4^{-1}+\varepsilon_{11}^{-1}+\varepsilon_{12}^{-1}+\varepsilon_{13}^{-1}+\varepsilon_{14}^{-1}\big)\|\mathcal{U}\|^2_{\mathcal{S}^{\infty,c}}\\
&\quad + \big(\varepsilon_5^{-1}+\varepsilon_{15}^{-1}+\varepsilon_{16}^{-1}+\varepsilon_{17}^{-1}+\varepsilon_{18}^{-1}\big)\|U\|^2_{\mathcal{S}^{\infty,c,2}}\\
&\quad + \big(\varepsilon_6^{-1}+\varepsilon_{19}^{-1}+\varepsilon_{20}^{-1}+\varepsilon_{21}^{-1}+\varepsilon_{22}^{-1}+\varepsilon_{23}^{-1}+\varepsilon_{24}^{-1}\big)\|\partial U\|^2_{\mathcal{S}^{\infty,c,2}}.
\end{aligned} \tag{4.6.4}$$

From (4.6.4) we conclude $(Z,N) \in \mathbb{H}^{2,c}_{\mathrm{BMO}} \times \mathbb{M}^{2,c}$,

$$\|V\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|\partial V\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|M\|^2_{\mathbb{M}^{2,2,c}} + \|\partial M\|^2_{\mathbb{M}^{2,2,c}} < \infty.$$

Defining C_ε analogously and if for some $\gamma \in (0,\infty)$

$$I_0^\varepsilon \leq \gamma R^2/10, \tag{4.6.5}$$

we obtain back in (4.6.4)

$$\begin{aligned}
&\|(Y,Z,N,U,V,M,\partial U,\partial V,\partial M)\|^2_{\mathcal{H}^c}\\
&\leq C_\varepsilon^{-1}\Big(10I_0^\varepsilon + 10L_\star^2\max\{2,T^2\}\big((\varepsilon_1+\varepsilon_2+\varepsilon_7+\varepsilon_{11}+\varepsilon_{15}+\varepsilon_{19})\|y\|^4_{\mathcal{S}^{\infty,c}}
\end{aligned}$$

$$+(\varepsilon_1+\varepsilon_2+\varepsilon_9+\varepsilon_{13}+\varepsilon_{17}+\varepsilon_{21})\|u\|^4_{\mathcal{S}^{\infty,2,c}}$$
$$+(\varepsilon_1+\varepsilon_2+\varepsilon_{23})\|\partial u\|^4_{\mathcal{S}^{\infty,2,c}}+(\varepsilon_1+\varepsilon_2+\varepsilon_8+\varepsilon_{12}+\varepsilon_{16}+\varepsilon_{20})\|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}$$
$$+(\varepsilon_1+\varepsilon_2+\varepsilon_{10}+\varepsilon_{14}+\varepsilon_{18}+\varepsilon_{22})\|v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}+(\varepsilon_1+\varepsilon_2+\varepsilon_{24})\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\Big)$$
$$\leq C_\varepsilon^{-1}R^2\bigg(\gamma+10L_\star^2\max\{2,T^2\}R^2\bigg(\varepsilon_1+\varepsilon_2+\sum_{i=7}^{24}\varepsilon_i\bigg)\bigg).$$

Therefore, to obtain $\mathfrak{T}(\mathcal{B}_R)\subseteq\mathcal{B}_R$, that is to say that the image under $\mathfrak{T}$ of the ball of radius R is contained in the ball of radius R, it is necessary to find R^2 such that the term in parentheses above is less or equal than C_ε, *i.e.*

$$R^2\leq\frac{1}{10L_\star^2\max\{2,T^2\}}\frac{C_\varepsilon-\gamma}{\varepsilon_1+\varepsilon_2+\sum_{i=7}^{24}\varepsilon_i},$$

which after optimising the choice of ε's renders

$$R^2<\frac{1}{2^6\cdot3\cdot5^2\cdot7\cdot L_\star^2\cdot\max\{2,T^2\}}. \tag{4.6.6}$$

(v) The continuity of the applications $([0,T],\mathcal{B}([0,T]))\longrightarrow(\mathcal{S}^{\infty,c},\|\cdot\|_{\mathcal{S}^{\infty,c}})$ (resp. $(\mathbb{H}^{2,c}_{\mathrm{BMO}},\|\cdot\|_{\mathbb{H}^{2,c}_{\mathrm{BMO}}})$, $(\mathbb{M}^{2,c},\|\cdot\|_{\mathbb{M}^{2,c}})$) : $s\longmapsto\varphi^s$ for $\varphi=U^s,\partial U^s$ (resp. $V^s,\partial V^s,M^s,\partial M^s$) follows analogously as in the proof Theorem 4.3.6.

We conclude, $\mathfrak{T}(\mathcal{B}_R)\subseteq\mathcal{B}_R$ for all R satisfying (4.6.6).

Step 2: We now argue that $\mathfrak{T}$ is a contraction in $\mathcal{B}_R\subseteq\mathcal{H}$ for the norm $\|\cdot\|_{\mathcal{H}^c}$. Let

$$\begin{aligned}
\delta h_t&:=h_t(y_t^1,z_t^1,u_t^{1,t},v_t^{1,t},\partial U_t^{1,t})-h_t(y_t^2,z_t^2,u_t^{2,t},v_t^{2,t},\partial U_t^{2,t}),\\
\delta g_t&:=g_t(t,u_t^{1,t},v_t^{1,t},y_t^1,z_t^1)-\partial U_t^{1,t}-g_t(t,u_t^{2,t},v_t^{2,t},y_t^2,z_t^2)+\partial U_t^{2,t},\\
\delta\tilde h_t&:=h_t(y_t^1,z_t^1,u_t^{1,t},v_t^{1,t},\partial U_t^{2,t})-h_t(y_t^2,z_t^2,u_t^{2,t},v_t^{2,t},\partial U_t^{2,t}),\\
\delta\tilde g_t&:=g_t(t,u_t^{1,t},v_t^{1,t},y_t^1,z_t^1)-g_t(t,u_t^{2,t},v_t^{2,t},y_t^2,z_t^2),\\
\delta\tilde g_t(s)&:=g_t(s,u_t^{1,s},v_t^{1,s},y_t^1,z_t^1)-g_t(s,u_t^{2,s},v_t^{2,s},y_t^2,z_t^2),\\
\delta\nabla\tilde g_t(s)&:=\nabla g_t(s,\partial u_t^{1,s},\partial v_t^{1,s},u_t^{1,s},v_t^{1,s},y_t^1,z_t^1)-g_t(s,\partial u_t^{2,s},\partial v_t^{2,s},u_t^{2,s},v_t^{2,s},y_t^2,z_t^2).
\end{aligned}$$

Applying Itô's formula we obtain that for any $t \in [0,T]$

$$
\begin{aligned}
&\sum_{i=1}^{4} \mathrm{e}^{ct}|\delta\mathfrak{Y}_t^i|^2 + \int_t^T \mathrm{e}^{cr}|\sigma_r^\top \delta\mathfrak{Z}_r^i|^2 \mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\mathrm{Tr}[\delta\mathfrak{Z}^i]_r + \delta\widetilde{\mathfrak{M}}_t - \delta\widetilde{\mathfrak{M}}_T \\
= &\int_t^T \mathrm{e}^{cr}\Big(2\delta\mathcal{Y}_r \cdot \delta h_r + 2\delta\mathcal{U}_r \cdot \delta g_r + 2\delta U_r^s \cdot \delta\tilde{g}_r(s) + 2\delta\partial U_r^s \cdot \delta\nabla\tilde{g}_r(s)\Big)\mathrm{d}r \\
\leq &\int_t^T \mathrm{e}^{cr}\Big(2|\delta\mathcal{Y}_r|\big(L_\mathrm{u}|\delta\partial U_r^r| + |\delta\tilde{h}_r|\big) + 2|\delta\mathcal{U}_r|\big((|\delta\partial U_r^r| + |\delta\tilde{g}_r|\big) \\
&\qquad + 2|\delta U_r^s||\delta\tilde{g}_r(s)| + 2|\delta\partial U_r^s||\delta\nabla\tilde{g}_r(s)| - c\sum_{i=1}^{4}|\delta\mathfrak{Y}_r^i|^2\Big)\mathrm{d}r,
\end{aligned}
$$

where $\delta\widetilde{\mathfrak{M}}$ denotes the corresponding martingale term. Let $\tau \in \mathcal{T}_{0,T}$, as in Lemma 4.7.1 we obtain for $c > 2L_\mathrm{u}$

$$
\mathbb{E}_\tau\bigg[\int_\tau^T \frac{\mathrm{e}^{cr}}{3T}|\delta\partial U_r^r|^2 \mathrm{d}r\bigg] \leq \sup_{s\in[0,T]} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\nabla\tilde{g}_r(s)|\mathrm{d}r\bigg]\bigg|^2. \tag{4.6.7}
$$

We now take conditional expectation with respect to $\mathcal{F}_\tau$ in the expression above and use Assumption J in combination with (4.6.7). We then obtain from Young's inequality that for any $\tilde{\varepsilon}_i \in (0,\infty)$, $i \in \{1,2\}$, and

$$
c \geq \max\,\{\tilde{\varepsilon}_1^{-1}3TL_\mathrm{u}^2,\ 3T\tilde{\varepsilon}_2^{-1},\ 2L_\mathrm{u}\}, \tag{4.6.8}
$$

it follows that

$$
\begin{aligned}
&\sum_{i=1}^{4} \mathrm{e}^{ct}|\delta\mathfrak{Y}_t^i|^2 + \mathbb{E}_\tau\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top \delta\mathfrak{Z}_r^i|^2 \mathrm{d}r + \int_t^T \mathrm{e}^{cr-}\mathrm{d}\mathrm{Tr}[\delta\mathfrak{Z}^i]_r\bigg] \\
\leq\ &\tilde{\varepsilon}_3^{-1}\|\delta Y\|^2_{\mathcal{S}^{\infty,c}} + \tilde{\varepsilon}_4^{-1}\|\delta\mathcal{U}\|^2_{\mathcal{S}^{\infty,2,c}} + \tilde{\varepsilon}_5^{-1}\|\delta U\|^2_{\mathcal{S}^{\infty,2,c}} + \tilde{\varepsilon}_6^{-1}\|\delta\partial U\|^2_{\mathcal{S}^{\infty,2,c}} \\
&+ (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_6) \sup_{s\in[0,T]} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\nabla\tilde{g}_r(s)|\mathrm{d}r\bigg]\bigg|^2 + \tilde{\varepsilon}_3 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde{h}_t|\mathrm{d}r\bigg]\bigg|^2 \\
&+ \tilde{\varepsilon}_4 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde{g}_t|\mathrm{d}r\bigg]\bigg|^2 + \tilde{\varepsilon}_5 \sup_{s\in[0,T]} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}^{\mathbb{P}} \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde{g}_t(s)|\mathrm{d}r\bigg]\bigg|^2.
\end{aligned} \tag{4.6.9}
$$

We now estimate the terms on the right side of (4.6.9). Note that in light of Assumption J.(iii)

we have

$$
\begin{aligned}
&\bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\nabla\tilde{g}_t(s)|\mathrm{d}r\bigg]\bigg|^2\\
&\leq \bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\Big(L_{\mathrm{u}}|\delta\partial u_r^s|\big(|\partial u_r^{1,s}|+|\partial u_r^{2,s}|\big)+L_{\mathrm{v}}|\sigma_r^\top\delta\partial v_r^s|\big(|\sigma_r^\top\partial v_r^{1,s}|+|\sigma_r^\top\partial v_r^{2,s}|\big)\\
&\qquad\qquad\quad +L_u|\delta u_r^s|\big(|u_r^{1,s}|+|u_r^{2,s}|\big)+L_v|\sigma_r^\top\delta v_r^s|\big(|\sigma_r^\top v_r^{1,s}|+|\sigma_r^\top v_r^{2,s}|\big)\\
&\qquad\qquad\quad +L_y|\delta y_r|\big(|y_r^1|+|y_r^2|\big)+L_z|\sigma_r^\top\delta z_r|\big(|\sigma_r^\top z_r^1|+|\sigma_r^\top z_r^2|\big)\Big)\mathrm{d}r\bigg]\bigg|^2\\
&\leq 6L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\partial u_r^s|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|\partial u_r^{1,s}|+|\partial u_r^{2,s}|\big)^2\mathrm{d}r\bigg]\\
&\quad+6L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top\delta\partial v_r^s|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|\sigma_r^\top\partial v_r^{1,s}|+|\sigma_r^\top\partial v_r^{2,s}|\big)^2\mathrm{d}r\bigg]\\
&\quad+6L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta u_r^s|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|u_r^{1,s}|+|u_r^{2,s}|\big)^2\mathrm{d}r\bigg]\\
&\quad+6L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top\delta v_r^s|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|\sigma_r^\top v_r^{1,s}|+|\sigma_r^\top v_r^{2,s}|\big)^2\mathrm{d}r\bigg]\\
&\quad+6L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta y_r|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|y_r^1|+|y_r^2|\big)^2\mathrm{d}r\bigg]\\
&\quad+6L_\star^2\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top\delta z_r|^2\mathrm{d}r\bigg]\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}\big(|\sigma_r^\top z_r^1|+|\sigma_r^\top z_r^2|\big)^2\mathrm{d}r\bigg]\\
&\leq 6L_\star^2R^2\max\{2,T\}\bigg(\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\partial u_r^s|^2\mathrm{d}r\bigg]+\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top\delta\partial v_r^s|^2\mathrm{d}r\bigg]+\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta u_r^s|^2\mathrm{d}r\bigg]\\
&\qquad\qquad +\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top\delta v_r^s|^2\mathrm{d}r\bigg]+\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta y_r|^2\mathrm{d}r\bigg]+\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\sigma_r^\top\delta z_r|^2\mathrm{d}r\bigg]\bigg)\\
&\leq 6L_\star^2R^2\max\{2,T^2\}\Big(\|\delta\partial u\|^2_{\mathcal{S}^{\infty,2,c}}+\|\delta\partial v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}+\|\delta u\|^2_{\mathcal{S}^{\infty,2,c}}+\|\delta v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}}+\|\delta y\|^2_{\mathcal{S}^{\infty,c}}+\|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big),
\end{aligned}
$$

where in the second inequality we used (I.1) and Cauchy–Schwarz's inequality. Similarly

$$
\begin{aligned}
&\max\bigg\{\bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde{h}_r|\mathrm{d}r\bigg]\bigg|^2,\bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta\tilde{g}_r(s)|\mathrm{d}r\bigg]\bigg|^2,\bigg|\mathbb{E}_\tau\bigg[\int_\tau^T \mathrm{e}^{cr}|\delta g_r|\mathrm{d}r\bigg]\bigg|^2\bigg\}\\
&\quad\leq 4L_\star^2R^2\max\{2,T^2\}\Big(\|\delta y\|^2_{\mathcal{S}^{\infty,c}}+\|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}+\|\delta u\|^2_{\mathcal{S}^{\infty,2,c}}+\|\delta v\|^2_{\overline{\mathbb{H}}^{2,c}_{\mathrm{BMO}}}\Big).
\end{aligned}
$$

Overall, we obtain back in (4.6.9) that

$$
\begin{aligned}
&\sum_{i=1}^4 \mathrm{e}^{ct}|\delta\mathfrak{Y}_t^i|^2+\mathbb{E}_\tau\bigg[\int_t^T \mathrm{e}^{cr}|\sigma_r^\top\delta\mathfrak{Z}_r^i|^2\mathrm{d}r+\int_t^T \mathrm{e}^{cr-}\mathrm{dTr}[\delta\mathfrak{Z}^i]_r\bigg]\\
&\leq \tilde{\varepsilon}_3^{-1}\|\delta Y\|^2_{\mathcal{S}^{\infty,c}}+\tilde{\varepsilon}_4^{-1}\|\delta\mathcal{U}\|^2_{\mathcal{S}^{\infty,2,c}}+\tilde{\varepsilon}_5^{-1}\|\delta U\|^2_{\mathcal{S}^{\infty,2,c}}+\tilde{\varepsilon}_6^{-1}\|\delta\partial U\|^2_{\mathcal{S}^{\infty,2,c}}
\end{aligned}
$$

$$+ 6(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_6) L_\star^2 R^2 \max\{2, T^2\} \Big(\|\delta \partial u\|^2_{\mathcal{S}^{\infty,2,c}} + \|\delta \partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|\delta u\|^2_{\mathcal{S}^{\infty,2,c}} \Big)$$
$$+ 6(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_6) L_\star^2 R^2 \max\{2, T^2\} \Big(\|\delta v\|^2_{\overline{\mathbb{H}}^{2,2,c}_{\mathrm{BMO}}} + \|\delta y\|^2_{\mathcal{S}^{\infty,c}} + \|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} \Big)$$
$$+ 4(\tilde{\varepsilon}_3 + \tilde{\varepsilon}_4 + \tilde{\varepsilon}_5) L_\star^2 R^2 \max\{2, T^2\} \Big(\|\delta y\|^2_{\mathcal{S}^{\infty,c}} + \|\delta z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|\delta u\|^2_{\mathcal{S}^{\infty,2,c}} + \|\delta v\|^2_{\overline{\mathbb{H}}^{2,c}_{\mathrm{BMO}}} \Big).$$

If we define, for $\tilde{\varepsilon}_i > 10, i \in \{3,4,5,6\}$, $C_{\tilde{\varepsilon}} := \min\{1 - 10/\tilde{\varepsilon}_3,\ 1 - 10/\tilde{\varepsilon}_4,\ 1 - 10/\tilde{\varepsilon}_5,\ 1 - 10/\tilde{\varepsilon}_6\}$, we deduce,

$$\|\delta \mathfrak{H}\|^2_{\mathcal{H}^c} \leq 20 C_{\tilde{\varepsilon}}^{-1} L_\star^2 R^2 \max\{2, T^2\} (3\tilde{\varepsilon}_1 + 3\tilde{\varepsilon}_2 + 2\tilde{\varepsilon}_3 + 2\tilde{\varepsilon}_4 + 2\tilde{\varepsilon}_5 + 3\tilde{\varepsilon}_6) \|\delta \mathfrak{h}\|_{\mathcal{H}^c}. \tag{4.6.10}$$

Minimising for $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ fixed, we find that letting

$$R^2 < \frac{1}{2^6 \cdot 3 \cdot 5^2 \cdot 7 \cdot L_\star^2 \cdot \max\{2, T^2\}},\ c \geq \max\{\varepsilon_1^{-1} 7 T L_{\mathrm{u}}^2, \varepsilon_2^{-1} 7T, \tilde{\varepsilon}_1^{-1} 3 T L_{\mathrm{u}}^2,\ 3T\tilde{\varepsilon}_2^{-1}, 2L_{\mathrm{u}}\},$$

we have that

$$\|\delta \mathfrak{H}\|^2_{\mathcal{H}^c} < \frac{20}{2^4 \cdot 3 \cdot 7 \cdot 10^2} 3 \big(\sqrt{30 + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2)} + \sqrt{30}\big)^2 \|\delta \mathfrak{h}\|_{\mathcal{H}^c} = \frac{(\sqrt{30 + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2)} + \sqrt{30})^2}{2^3 \cdot 7 \cdot 10} \|\delta \mathfrak{h}\|_{\mathcal{H}^c}.$$

Thus, letting choosing $(\sqrt{30 + (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2)} + \sqrt{30})^2 \leq 2^3 \cdot 7 \cdot 10$, $\mathfrak{T}$ is contractive.

Step 3: We consolidate our results.. In light of (4.6.3) and (4.6.8), taking $\varepsilon_i = \tilde{\varepsilon}_i, i \in \{1,2\}$, c must satisfy

$$c \geq \max\{\varepsilon_1^{-1} 7 T L_{\mathrm{u}}^2, \varepsilon_2^{-1} 7T, \tilde{\varepsilon}_1^{-1} 3 T L_{\mathrm{u}}^2,\ 3T\tilde{\varepsilon}_2^{-1},\ 2L_{\mathrm{u}}\} = \max\{\varepsilon_1^{-1} 7 T L_{\mathrm{u}}^2, \varepsilon_2^{-1} 7T,\ 2L_{\mathrm{u}}\} \tag{4.6.11}$$

All together we find that given $\gamma \in (0,\infty)$, $\varepsilon_i \in (0,\infty)$, $i \in \{1,2\}$, $c \in (0,\infty)$, such that $\varepsilon_1 + \varepsilon_2 \leq (4\sqrt{35} - \sqrt{30})^2 - 30$, $\mathfrak{T}$ is a well-defined contraction in $\mathcal{B}_R \subseteq \mathcal{H}^c$ for the norm $\|\cdot\|_{\mathcal{H}^c}$ provided: (i) γ, ε_i, $i \in \{1,2\}$, and the data of the problem satisfy (4.6.5); (ii) c satisfies (4.6.11).

□

4.7 Auxiliary results

4.7.1 Proofs of Section 4.2

Proof of Lemma 4.2.3. First note that for $Z \in \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{R}^{n\times\tilde d})$, $Z \bullet X_t := \int_0^t Z \mathrm{d}X_r$ is a continuous local martingale, thus we have that

$$\|Z\bullet X\|_{\mathrm{BMO}^{2,c}} = \sup_{\tau\in\mathcal{T}_{0,T}} \Big\| \mathbb{E}\big[\langle \mathrm{e}^{\frac{c}{2}} Z\bullet X\rangle_T - \langle \mathrm{e}^{\frac{c}{2}} Z\bullet X\rangle_\tau |\mathcal{F}_\tau\big]\Big\|_\infty < \infty.$$

Therefore, letting $X_t := \mathbb{E}[\langle \mathrm{e}^{\frac{c}{2}} Z\bullet X\rangle_T - \langle \mathrm{e}^{\frac{c}{2}} Z\bullet X\rangle_t |\mathcal{F}_t]$, we have:

(*i*) $|X_t| \leq \|Z\bullet X\|_{\mathrm{BMO}^{2,c}} = \|Z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}$,

(*ii*) $A = \langle \mathrm{e}^{\frac{c}{2}} Z\bullet X\rangle$. Indeed, note $X_t = \mathbb{E}\big[\langle \mathrm{e}^{\frac{c}{2}} Z\bullet X\rangle_T|\mathcal{F}_t\big] - \langle \mathrm{e}^{\frac{c}{2}} Z\bullet X\rangle_t$.

The result then follows immediately from the energy inequality, *i.e.*

$$\mathbb{E}\bigg[\bigg(\int_0^T \mathrm{e}^{cr}|\sigma_r^\top Z_r|^2\mathrm{d}r\bigg)^p\bigg] = \mathbb{E}[(A)^p_\infty] \leq p!\|Z\|^{2p}_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}.$$

To obtain the second part of the statement, recall that by definition of $\overline{\mathbb{H}}^{2,2}(\mathbb{R}^{n\times\tilde d})$, $s\longmapsto \partial Z^s$ is the density of $s\longmapsto Z^s$ with respect to the Lebesgue measure and $\mathcal{Z}$ is given as in Remark 4.2.2. By definition of $\mathcal{Z}$, Fubini's theorem and Young's inequality we have that for $\varepsilon>0$

$$\begin{aligned}\int_t^T \mathrm{e}^{cu}|\sigma_u^\top Z_u^u|^2 - \mathrm{e}^{cu}|\sigma_u^\top Z_u^t|^2\mathrm{d}u &= \int_t^T\int_t^u 2\mathrm{e}^{cu}\mathrm{Tr}\Big[{Z_u^r}^\top\sigma_u\sigma_u^\top\partial Z_u^r\Big]\mathrm{d}r\mathrm{d}u\\ &= \int_t^T\int_r^T 2\mathrm{e}^{cu}\mathrm{Tr}\Big[{Z_u^r}^\top\sigma_u\sigma_u^\top\partial Z_u^r\Big]\mathrm{d}u\mathrm{d}r\\ &\leq \int_t^T\int_r^T \varepsilon\mathrm{e}^{cu}|\sigma_u^\top Z_u^r|^2 + \varepsilon^{-1}\mathrm{e}^{cu}|\sigma_u^\top\partial Z_u^r|^2\mathrm{d}u\mathrm{d}r.\end{aligned}$$

This proves the first first statement. For the second claim, we may use (I.1) and (4.2.1) to obtain

$$\begin{aligned}&\mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{cu}|\sigma_u^\top\mathcal{Z}_u|^2\mathrm{d}u\bigg)^2\bigg]\\ &\leq 3\bigg(\mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{cu}|\sigma_u^\top Z_u^t|^2\mathrm{d}u\bigg)^2\bigg]\end{aligned}$$

$$+T\int_t^T \mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{cu}|\sigma_u^\top Z_u^r|^2\mathrm{d}u\bigg)^2\bigg]\mathrm{d}r + T\int_t^T \mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{cu}|\sigma_u^\top \partial Z_u^r|^2\mathrm{d}u\bigg)^2\bigg]\mathrm{d}r\bigg)$$
$$\leq 6((1+T^2)\|Z\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + T^2\|\partial Z\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}).$$

The inequality for the $\mathbb{H}^2$ norm is argued similarly taking expectations. □

4.7.2 Proofs of Section 4.5

We next sequence of lemmata helps derive appropriate auxiliary estimates of the terms U_t^t and ∂U_t^t as defined by $\mathfrak{T}$ in Section 4.5.

Lemma 4.7.1. *Let ∂U satisfy the equation*

$$\partial U_t^s = \partial_s \eta(s, X_{\cdot\wedge,T}) + \int_t^T \nabla g_r(s, X, \partial U_r^s, \partial v_r^s, U_r^s, v_r^s, \mathcal{Y}_r, z_r)\mathrm{d}r - \int_t^T \partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{d}\partial M_r^s,$$

and $c \geq \max\{2L_u, 2L_{\mathrm{u}}\}$, the following estimates hold for $t \in [0,T]$

$$\mathbb{E}_t\bigg[\int_t^T \frac{\mathrm{e}^{cr}}{7T}|\partial U_r^r|^2\mathrm{d}r\bigg] \leq \|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}} + 2L_\star^2\Big(\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\Big)$$
$$+ TL_y^2\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|Y_r|^2\mathrm{d}r\bigg] + TL_u^2\sup_{s\in[0,T]}\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|U_r^s|^2\mathrm{d}r\bigg],$$
$$\mathbb{E}_t\bigg[\int_t^T \frac{\mathrm{e}^{\frac{c}{2}r}}{T}|\partial U_r^r|\mathrm{d}r\bigg] \leq \|\partial_s\eta\|_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde{g}\|_{\mathbb{L}^{1,\infty,2,c}} + L_\star\Big(\|\partial v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|v\|^2_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|z\|^2_{\mathbb{H}^{2,c}_{\mathrm{BMO}}}\Big)$$
$$+ L_y\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{\frac{c}{2}r}|Y_r|\mathrm{d}r\bigg] + L_u\sup_{s\in[0,T]}\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|U_r^s|\mathrm{d}r\bigg].$$

Proof. By Meyer–Itô's formula for $\mathrm{e}^{\frac{c}{2}t}|\partial U_t^s|$, see Protter [212, Theorem 70]

$$\mathrm{e}^{\frac{c}{2}t}|\partial U_t^s| + L_T^0 - \int_t^T \mathrm{e}^{\frac{c}{2}r}\operatorname{sgn}(\partial U_r^s)\cdot\partial V_r^{s\top}\mathrm{d}X_r - \int_t^T \mathrm{e}^{\frac{c}{2}r-}\operatorname{sgn}(\partial U_{r-}^s)\cdot\mathrm{d}\partial M_r^s \tag{4.7.1}$$
$$= \mathrm{e}^{\frac{c}{2}T}|\partial_s\eta(s)| + \int_t^T \mathrm{e}^{\frac{c}{2}r}\Big(\operatorname{sgn}(\partial U_r^s)\cdot\nabla g_r(s, \partial U_r^s, \partial v_r^s, U_r^s, v_r^s, \mathcal{Y}_r, z_r) - \frac{c}{2}|\partial U_r^s|\Big)\mathrm{d}r,\ t\in[0,T],$$

where $L^0 := L^0(\partial U^s)$ denotes the non-decreasing and pathwise-continuous local time of the semi-martingale ∂U^s at 0, see [212, Chapter IV, pp. 216]. We also notice that for any $s \in [0,T]$ the last two terms on the left-hand side are martingales, recall that $\partial V^s \in \mathbb{H}^2$ by Theorem 3.4.3.

In light of Assumption J, letting $\nabla g_r(s) := \nabla g_r(s, \partial U_r^s, \partial v_r^s, U_r^s, v_r^s, Y_r, z_r)$, we have that $\mathrm{d}t \otimes$

$\mathrm{d}\mathbb{P}$–a.e.

$$|\nabla g_r(s)| \leq L_{\mathrm{u}}|\partial U^s_r| + L_{\mathrm{v}}|\sigma_r^\top \partial v^s_r|^2 + L_u|U^s_r| + L_v|\sigma_r^\top v^s_r|^2 + L_y|Y_r| + L_z|\sigma_r^\top z_r|^2 + |\nabla \tilde g_r(s)|. \quad (4.7.2)$$

We now take conditional expectation with respect to $\mathcal{F}_t$ in Equation (4.7.1). We may use (4.7.2) and the fact $\tilde L^0$ is non- decreasing to derive that for $c > 2L_{\mathrm{u}}$ and $t \in [0,T]$

$$\begin{aligned} \mathrm{e}^{\frac{c}{2}t}|\partial U^s_t| \leq \mathbb{E}_t\bigg[&\mathrm{e}^{\frac{c}{2}T}|\partial\eta(s)| + \int_t^T \mathrm{e}^{\frac{c}{2}r}\big(|\nabla \tilde g_r(s)| + L_{\mathrm{v}}|\sigma_r^\top \partial v^s_r|^2 + L_u|U^s_r| \\ &+ L_v|\sigma_r^\top v^s_r|^2 + L_y|Y_r| + L_z|\sigma_r^\top z_r|^2\big)\mathrm{d}r\bigg]. \end{aligned} \quad (4.7.3)$$

Squaring in (4.7.3), we may use (I.1) and Jensen's inequality to derive that for $t \in [0,T]$

$$\begin{aligned} \frac{\mathrm{e}^{ct}}{7}|\partial U^t_t|^2 \leq \mathbb{E}_t\bigg[&\mathrm{e}^{cT}|\partial_s\eta(t)|^2 + \bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}|\nabla \tilde g_r(t)|\mathrm{d}r\bigg)^2 + TL_u^2\int_t^T \mathrm{e}^{cr}|U^t_r|^2\mathrm{d}r + TL_y^2\int_t^T \mathrm{e}^{cr}|Y_r|^2\mathrm{d}r \\ &+ L_{\mathrm{v}}^2\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}|\sigma_r^\top \partial v^t_r|^2\mathrm{d}r\bigg)^2 + L_v^2\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}|\sigma_r^\top v^t_r|^2\mathrm{d}r\bigg)^2 + L_z^2\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}r}|\sigma_r^\top z_r|^2\mathrm{d}r\bigg)^2\bigg]. \end{aligned}$$

By integrating the previous expression and taking conditional expectation with respect to $\mathcal{F}_t$, it follows from the tower property that for any $t \in [0,T]$

$$\begin{aligned} &\frac{1}{7}\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|\partial U^r_r|^2\mathrm{d}r\bigg] \\ &\leq \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cT}|\partial_s\eta(r)|^2\mathrm{d}r\bigg] + \mathbb{E}_t\bigg[\int_t^T \bigg[\bigg(\int_r^T \mathrm{e}^{\frac{c}{2}u}|\nabla \tilde g_u(r)|\mathrm{d}u\bigg)^2\mathrm{d}r\bigg] \\ &\quad + TL_u^2\mathbb{E}_t\bigg[\int_t^T\int_r^T \mathrm{e}^{cu}|U^r_u|^2\mathrm{d}u\bigg]\mathrm{d}r + TL_y^2\mathbb{E}_t\bigg[\int_t^T\int_r^T \mathrm{e}^{cu}|Y_u|^2\mathrm{d}u\bigg]\mathrm{d}r\bigg] \\ &\quad + L_{\mathrm{v}}^2\int_t^T \mathbb{E}_t\bigg[\bigg(\int_r^T \mathrm{e}^{\frac{c}{2}u}|\sigma_u^\top \partial v^r_u|^2\mathrm{d}u\bigg)^2\bigg]\mathrm{d}r + L_v^2\int_t^T \mathbb{E}_t\bigg[\bigg(\int_r^T \mathrm{e}^{\frac{c}{2}u}|\sigma_u^\top v^r_u|^2\mathrm{d}u\bigg)^2\bigg]\mathrm{d}r \\ &\quad + L_z^2\int_t^T \mathbb{E}_t\bigg[\bigg(\int_r^T \mathrm{e}^{\frac{c}{2}u}|\sigma_u^\top z_u|^2\mathrm{d}u\bigg)^2\bigg]\mathrm{d}r \\ &\leq T\sup_{r\in[0,T]}\bigg\{\|\mathrm{e}^{cT}|\eta(r)|^2]\|_\infty + \bigg\|\int_r^T \mathrm{e}^{\frac{c}{2}u}|\nabla \tilde g_u(r)|\mathrm{d}u\bigg\|_\infty^2\bigg\} + T^2L_y^2\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cu}|Y_u|^2\mathrm{d}u\bigg] \\ &\quad + T^2L_u^2\sup_{r\in[0,T]}\bigg\{\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cu}|U^r_u|^2\mathrm{d}u\bigg]\bigg\} + TL_{\mathrm{v}}^2\sup_{r\in[0,T]}\bigg\{\mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}u}|\sigma_u^\top \partial v^r_u|^2\mathrm{d}u\bigg)^2\bigg]\mathrm{d}r\bigg\} \\ &\quad + TL_v^2\sup_{r\in[0,T]}\bigg\{\mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}u}|\sigma_u^\top v^r_u|^2\mathrm{d}u\bigg)^2\bigg]\mathrm{d}r\bigg\} + TL_z^2\mathbb{E}_t\bigg[\bigg(\int_t^T \mathrm{e}^{\frac{c}{2}u}|\sigma_u^\top z_u|^2\mathrm{d}u\bigg)^2\bigg]\mathrm{d}r, \end{aligned}$$

and by (4.2.1) we obtain for $c > 2L_u$, and any $t \in [0,T]$

$$\mathbb{E}_t\bigg[\int_t^T \frac{\mathrm{e}^{cr}}{7T}|\partial U_r^r|^2\mathrm{d}r\bigg] \leq \|\partial_s\eta\|^2_{\mathcal{L}^{\infty,2,c}} + \|\nabla\tilde{g}\|^2_{\mathbb{L}^{1,\infty,2,c}} + 2L_\star^2\Big(\|\partial v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}} + \|z\|^4_{\mathbb{H}^{2,c}_{\mathrm{BMO}}} + \|v\|^4_{\mathbb{H}^{2,2,c}_{\mathrm{BMO}}}\Big)$$
$$+ TL_y^2\mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cr}|Y_r|^2\mathrm{d}r\bigg] + +TL_u^2 \sup_{r\in[0,T]} \mathbb{E}_t\bigg[\int_t^T \mathrm{e}^{cu}|U_u^r|^2\mathrm{d}u\bigg].$$

Evaluating at $s = t$ in (4.7.3) and integrating with respect to t we derive the second estimate. $\square$

Lemma 4.7.2 (Optimal upper bound for R)**.** (OPT1) $= 1/(2^45)$, *where*

$$\begin{aligned}
&\sup \frac{\min\{\alpha(\varepsilon_3,\varepsilon_{12},\varepsilon_{13}),\ \alpha(\varepsilon_4,\varepsilon_{14},\varepsilon_{15}),\ \alpha(\varepsilon_5,\varepsilon_{16},\varepsilon_{17}),\ \alpha(\varepsilon_6,\varepsilon_{18},\varepsilon_{19},\varepsilon_{20})\} - \gamma}{\varepsilon_1+\varepsilon_2+\sum_{i=12}^{20}\varepsilon_i}\\
&\text{s.t. } \alpha(\varepsilon_3,\varepsilon_{12},\varepsilon_{13}) = 1 - 10(\varepsilon_3^{-1}+\varepsilon_{12}^{-1}+\varepsilon_{13}^{-1}) \in (0,1],\\
&\quad\ \alpha(\varepsilon_4,\varepsilon_{14},\varepsilon_{15}) = 1 - 10(\varepsilon_4^{-1}+\varepsilon_{14}^{-1}+\varepsilon_{15}^{-1}) \in (0,1],\\
&\quad\ \alpha(\varepsilon_5,\varepsilon_{16},\varepsilon_{17}) = 1 - 10(\varepsilon_5^{-1}+\varepsilon_{16}^{-1}+\varepsilon_{17}^{-1}) \in (0,1],\\
&\quad\ \alpha(\varepsilon_6,\varepsilon_{18},\varepsilon_{19},\varepsilon_{20}) = 1 - 10(\varepsilon_6^{-1}+\varepsilon_{18}^{-1}+\varepsilon_{19}^{-1}+\varepsilon_{20}^{-1}) \in (0,1],\\
&\quad\ \gamma \in (0,\infty);\ \varepsilon_i \in (0,\infty), \forall i \in \{1,\dots,6\}\cup\{12,\dots,20\}.
\end{aligned}$$

Proof. We begin by noticing that as a function of $(\gamma,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6)$ the objective is bounded by the value when $(\gamma,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6) \longrightarrow (0,0,0,\infty,\infty,\infty,\infty)$. Thus, we will maximise

$$\frac{\min\{1-10(\varepsilon_{12}^{-1}+\varepsilon_{13}^{-1}),\ 1-10(\varepsilon_{14}^{-1}+\varepsilon_{15}^{-1}),\ 1-10(\varepsilon_{16}^{-1}+\varepsilon_{17}^{-1}),\ 1-10(\varepsilon_{18}^{-1}+\varepsilon_{19}^{-1}+\varepsilon_{20}^{-1})\}}{\sum_{i=12}^{20}\varepsilon_i}.$$

From this we observe that the optimal value is positive. Indeed, one can find a feasible solution with positive value, and the min in the objective function does not involve common ε_i terms, so the minima is attained at one of the terms. Since the value function is symmetric in each of the variables inside each term of the mean we can assume with out lost of generality

$$\varepsilon_{12}=\varepsilon_{13}=2\alpha_1,\ \varepsilon_{14}=\varepsilon_{15}=2\alpha_2,\ \varepsilon_{16}=\varepsilon_{17}=2\alpha_3,\ \varepsilon_{18}=\varepsilon_{19}=\varepsilon_{20}=3\alpha_4,\ \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}\in(0,\infty)^4.$$

So we can write the objective function as

$$\frac{\min\{1-10\alpha_1^{-1},\ 1-10\alpha_2^{-1},\ 1-10\alpha_3^{-1},\ 1-10\alpha_4^{-1}\}}{4\alpha_1+4\alpha_2+4\alpha_3+9\alpha_4}.$$

Now, without lost of generality the min is attained by the first quantity. This is, the optimisation problem becomes

$$\sup\ \frac{1-10\alpha_1^{-1}}{4\alpha_1+4\alpha_2+4\alpha_3+9\alpha_4}\ \text{s.t. } \alpha_1 \le \min\{\alpha_2,\alpha_3,\alpha_4\}, 1-10\alpha_i^{-1} \in (0,1], \alpha_i \in (0,\infty), i \in \{1,\dots,4\}.$$

Now, as the objective function is decreasing in $\alpha_2, \alpha_3, \alpha_4$, and $\alpha_1 \le \min\{\alpha_2,\alpha_3,\alpha_4\}$,we must have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ and we are left with

$$\sup\ \frac{1-10\alpha_1^{-1}}{21\alpha_1}\ \text{s.t. } 1-10\alpha_1^{-1} \in (0,1], \alpha_1 \in (0,\infty).$$

Let $f(\alpha_1) := \frac{\alpha_1-10}{21\alpha_1^2}$. By first order analysis

$$\partial_{\alpha_1} f(\alpha_1) = \frac{-\alpha_1(\alpha_1-20)}{21\alpha_1^4} = 0,\ \text{yields},\ \alpha_1 \in \{0, 20\}.$$

By inspecting the sign of the derivative, one sees that $\alpha_1 = 0$ corresponds to a minima and $\alpha_1 = 20$ is the maximum and it is feasible. Thus we obtain that

$$f(\alpha_1^\star) = \frac{1}{2^3\cdot 3\cdot 5\cdot 7}.$$

We conclude the maxima happens when $(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \varepsilon_{15}, \varepsilon_{16}, \varepsilon_{17}, \varepsilon_{18}, \varepsilon_{19}, \varepsilon_{20}) = (40, 40, 40, 40, 40, 40, 60, 60, 60)$. Evaluating the value function in these values and letting $(\gamma, \varepsilon_1, \varepsilon_3, \varepsilon_8, \varepsilon_9, \varepsilon_{10}, \varepsilon_{11}) \longrightarrow (0, 0, 0, \infty, \infty, \infty, \infty)$, we obtain this bound. This is, f does not attain its maximum value, but in the feasible region it can get as close as possible. $\square$

Lemma 4.7.3 (Minimal bound for Contraction)**.** (OPT2) $= 3(\sqrt{30 + (\tilde\varepsilon_1 + \tilde\varepsilon_2)} + \sqrt{30})^2$, *where*

$$\text{(OPT2)} :=\ \inf\left\{(3\tilde\varepsilon_1 + 3\tilde\varepsilon_2 + 2\tilde\varepsilon_3 + 2\tilde\varepsilon_4 + 2\tilde\varepsilon_5 + 3\tilde\varepsilon_6)\min\left\{\frac{\tilde\varepsilon_3}{\tilde\varepsilon_3 - 10}, \frac{\tilde\varepsilon_4}{\tilde\varepsilon_4 - 10}, \frac{\tilde\varepsilon_5}{\tilde\varepsilon_5 - 10}, \frac{\tilde\varepsilon_6}{\tilde\varepsilon_6 - 10}\right\}\right\},$$

$$\text{s.t. } 1 - 10\tilde\varepsilon_i^{-1} \in (0,1],\ \tilde\varepsilon_i \in (0,\infty),\ i \in \{3,4,5,6\}.$$

Proof. Without lost of generality let us assume the min is attained by the first quantity, *i.e.* the optimisation problem becomes

$$\begin{aligned}&\inf\ \big(3(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2)+2\tilde{\varepsilon}_3+2\tilde{\varepsilon}_4+2\tilde{\varepsilon}_5+3\tilde{\varepsilon}_6\big)\frac{\tilde{\varepsilon}_3}{\tilde{\varepsilon}_3-10},\\ &\quad \text{s.t. } \tilde{\varepsilon}_8\leq\min\{\tilde{\varepsilon}_4,\tilde{\varepsilon}_5,\tilde{\varepsilon}_6\},\ 1-10\tilde{\varepsilon}_3^{-1}\in(0,1],\ \tilde{\varepsilon}_i\in(0,\infty)\ i\in\{3,4,5,6\}.\end{aligned}$$

As the value function is increasing in $(\tilde{\varepsilon}_4,\tilde{\varepsilon}_5,\tilde{\varepsilon}_6)$, $\tilde{\varepsilon}_3\leq\min\{\tilde{\varepsilon}_4,\tilde{\varepsilon}_5,\tilde{\varepsilon}_6\}$ implies we must have $\tilde{\varepsilon}_3=\tilde{\varepsilon}_4=\tilde{\varepsilon}_5=\tilde{\varepsilon}_6$. Thus, we minimise

$$f(\tilde{\varepsilon}):=3\frac{3\tilde{\varepsilon}^2+\tilde{\varepsilon}(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2)}{\tilde{\varepsilon}-10}.$$

First order analysis gives

$$\partial_{\tilde{\varepsilon}}f(\tilde{\varepsilon})=\frac{9\tilde{\varepsilon}-180\tilde{\varepsilon}-30(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2)}{\tilde{\varepsilon}-10}=0,\ \text{yields, } \tilde{\varepsilon}^{\pm}=10\pm\frac{1}{6}\sqrt{60^2+120(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2)}.$$

The minimum occurs at $\tilde{\varepsilon}^{\star}=10+\frac{1}{6}\sqrt{60^2+120(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2)}$, and

$$f(\tilde{\varepsilon}^{\star})=3\big(\sqrt{30+(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2)}+\sqrt{30}\big)^2.$$

We conclude the minima occurs when $(\tilde{\varepsilon}_3,\tilde{\varepsilon}_4,\tilde{\varepsilon}_5,\tilde{\varepsilon}_6)=(20,20,20,20)$. Evaluating the value function in these values and letting $(\tilde{\varepsilon}_1,\tilde{\varepsilon}_2)\longrightarrow(0,0)$, we obtain this bound. This is, f does not attain its minimum value, but in the feasible region it can get as close as possible. $\square$

Part IV

Time-inconsistent contract theory

Chapter 5

Time-inconsistent contract theory for sophisticated agents

This chapter is devoted to put together all our efforts of Part II and Part III in order to address the moral hazard problem in finite horizon with lump-sum payments, involving a time-inconsistent sophisticated agent and a standard utility maximiser principal. Our main contribution consists of a characterisation of the moral hazard problem faced by the principal. In particular, it shows that under relatively mild technical conditions on the data of the problem, the supremum of the principal's expected utility over the restricted family of contracts is equal to the supremum over all feasible contracts. Nevertheless, this characterisation yields, as far as we know, a novel class of control problems that involve the control of a forward Volterra equation via constrained Volterra type controls. Despite the inherent challenges of these class of problems, we study the solution to this problem under three different specifications of utility functions for both the agent and the principal and draw qualitative implications from the form of the optimal contract. The general case remains the subject of future research.

5.1 Problem statement

Let us recall the formulation. We take two positive integers n and m, which represent respectively the dimension of the process controlled by the agent, and the dimension of the Brownian motion driving this controlled process. We consider the canonical space $\mathcal{X} = \mathcal{C}^n$, endowed with the sup norm and with canonical process X, and whose generic elements we denote x. We reserve the notation x and $\mathbf{x}$ to denote $\mathbb{R}$-valued variables.

We let $\mathcal{F}^X$ be the Borel σ-algebra on $\mathcal{X}$ (for the topology of uniform convergence), and $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\in[0,T]}$ is the natural filtration of X. We let A be a compact subspace of a finite-dimensional Euclidean space (typically A is a subset of $\mathbb{R}^k$ for some positive integer k), where the controls will take values.

5.1.1 Controlled state equation

We fix a bounded Borel-measurable map $\sigma : [0,T] \times \mathcal{X} \longrightarrow \mathbb{R}^{n\times m}$, and an initial condition $x_0 \in \mathbb{R}^n$, and assume that there is a unique solution, denoted by $\mathbb{P}$, to the martingale problem for which X is an $(\mathbb{F}^X, \mathbb{P})$–local martingale, such that $X_0 = x_0$ with probability 1, and $\mathrm{d}\langle X\rangle_t = \sigma_t(X_{\cdot\wedge t})\sigma_t^\top(X_{\cdot\wedge t})\mathrm{d}t$. Enlarging the original probability space if necessary (see Stroock and Varadhan [232, Theorem 4.5.2]), we can find an $\mathbb{R}^m$-valued Brownian motion B such that

$$X_t = x_0 + \int_0^t \sigma_s(X_{\cdot\wedge r})\mathrm{d}B_r,\ t \in [0,T].$$

We now let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be the $\mathbb{P}$–augmentation of $\mathbb{F}^X$ which we assume is right-continuous. We recall that uniqueness of the solution to the martingale problem implies that the predictable martingale representation property holds for $(\mathbb{F},\mathbb{P})$-martingales, which can be represented as stochastic integrals with respect to X (see Jacod and Shiryaev [142, Theorem III.4.29]). We also mention that the right continuity assumption on $\mathbb{F}$ guarantees that $(\mathbb{P},\mathbb{F})$ satisfies the Blumenthal zero–one law.

We can then introduce our drift functional $b : [0,T] \times \mathcal{X} \times A \longrightarrow \mathbb{R}^m$, which is assumed to be Borel-measurable with respect to all its arguments. Let us recall that for any A-valued, $\mathbb{F}$-predictable process α such that

$$\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(\int_0^T b_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}B_r - \frac{1}{2}\int_0^T \big\|b_s(X_{\cdot\wedge s},\alpha_r)\big\|^2\mathrm{d}r\bigg)\bigg] < \infty, \tag{5.1.1}$$

we can define the probability measure $\mathbb{P}^\alpha$ on $(\mathcal{X},\mathcal{F}_T)$, whose density with respect to $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\mathbb{P}^\alpha}{\mathrm{d}\mathbb{P}} := \exp\bigg(\int_0^T b_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}B_r - \frac{1}{2}\int_0^T \big\|b_r(X_{\cdot\wedge r},\alpha_r)\big\|^2\mathrm{d}r\bigg).$$

Moreover, by Girsanov's theorem, the process $B^\alpha := B - \int_0^\cdot b_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r$ is an $\mathbb{R}^m$-valued, $(\mathbb{F},\mathbb{P}^\alpha)$–Brownian motion and we have

$$X_t = x_0 + \int_0^t \sigma_r(X_{\cdot\wedge r})b_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r + \int_0^t \sigma_r(X_{\cdot\wedge r})\mathrm{d}B_r^\alpha,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

Let us emphasise that we are working under the so-called weak formulation of the problem. This means that the state process X is fixed and, in contrast to the typical strong formulation, the Brownian motion and the probability measure are not fixed. Indeed, the choice of α corresponds to the choice of probability measure $\mathbb{P}^\alpha$ and thus the distribution of process X.

5.1.2 The agent's problem

We aim to cover various specifications of time-inconsistent utility functions for the agent. To motivate our formulation, let us start with an informal discussion on the typical nature of the reward functionals assigned to the agent in contract theory. In general terms, and at the formal level, given a contract ξ the value received by a time-inconsistent agent at the beginning of the problem from choosing an action α typically takes the form

$$\mathrm{V}_0^{\mathrm{A}}(\xi,\alpha) := \mathbb{E}^{\mathbb{P}^\alpha}\left[\mathrm{U_A}(0,\xi,C_{0,T}^\alpha)\right],\ \text{with } C_{t,T}^\alpha := \int_t^T c_r(t,X_{\cdot\wedge r},\alpha_r)\mathrm{d}r,$$

where $\mathrm{U_A} : [0,T]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}, (t,x,c)\longmapsto \mathrm{U_A}(t,\mathrm{x},\mathrm{c})$ denotes the agent's utility function, ξ is a real-valued random variable representing the final payment prescribed by the contract and $C_{t,T}^\alpha$ denotes the cumulative cost functional. We highlight that the generic dependence of both $\mathrm{U_A}$ and c on t accounts for the sources of time-inconsistency. In the classic literature, utilities are usually classified under two categories, namely

(i) *separable utility* functions, *i.e.* $\mathrm{U_A}(t,\mathrm{x},\mathrm{c}) = \mathrm{U_A}(t,\mathrm{x}) - \mathrm{c}$,

(ii) *non-separable utility* functions, *i.e.* $\mathrm{U_A}(t,\mathrm{x},\mathrm{c}) = \mathrm{U_A}(t,\mathrm{x}-\mathrm{c})$.

For instance, in the separable case the agent's value takes the familiar form

$$\mathrm{V}_0^{\mathrm{A}}(\xi,\alpha) = \mathbb{E}^{\mathbb{P}^\alpha}\bigg[\mathrm{U_A}(0,\xi) - \int_0^T c_r(0,X_{\cdot\wedge r},\alpha_r)\mathrm{d}r\bigg],$$

which we recall satisfies $\mathrm{V}_0^{\mathrm{A}}(\xi,\alpha) = Y_0^{0,\alpha}$, for $Y_0^{0,\alpha}$ the initial value of the first component of $(Y^{0,\alpha}, Z^{0,\alpha})$ solution to the BSDE

$$Y_t^{0,\alpha} = \mathrm{U}_\mathrm{A}(0,\xi) + \int_t^T \big(\sigma_r(X_{\cdot\wedge r})b_r(X_{\cdot\wedge r},\alpha_r)Z_r^{0,\alpha} - c_r(0,X_{\cdot\wedge r},\alpha_r)\big)\mathrm{d}r - \int_t^T Z_r^{0,\alpha}\cdot \mathrm{d}X_r,\ \mathbb{P}\text{–a.s.}$$

Moreover, in the (time-consistent) case in which the agent discounts exponentially with constant factor ρ, *i.e.* $\mathrm{U}_\mathrm{A}(t,\mathrm{x}) = \mathrm{e}^{-\rho(T-t)}\mathrm{U}_\mathrm{A}(\mathrm{x})$ and $c_t(s,x,a) = \mathrm{e}^{-\rho(t-s)}c_t(x,a)$, it holds that

$$Y_t^{0,\alpha} = \mathrm{U}_\mathrm{A}(\xi) + \int_t^T \big(\sigma_r(X_{\cdot\wedge r})b_r(X_{\cdot\wedge r},\alpha_r)Z_r^{0,\alpha} - c_r(X_{\cdot\wedge r},\alpha_r) - \rho Y_r^{0,\alpha}\big)\mathrm{d}r - \int_t^T Z_r^{0,\alpha}\cdot \mathrm{d}X_r,\ \mathbb{P}\text{–a.s.}$$

The previous representation corresponds to a so-called recursive utility particularly known as *standard additive utility*, see Epstein and Zin [95]. Let us remark that an analogous argument holds in the case of the non-separable exponential utility and refer to El Karoui, Peng, and Quenez [88] for more examples on recursive utilities. Intuitively, a recursive utility can be viewed as an extension of the classic separable or non-separable utilities in which the instantaneous utility depends on the instantaneous action α_t and the future utility via $Y_t^{0,\alpha}$. Extrapolating these ideas, we may arrive at considering rewards functionals of the form $\mathrm{V}_0^{\mathrm{A}}(\xi,\alpha) = Y_0^{0,\alpha}$ where the pair (Y^α, Z^α) satisfies the BSVIE

$$Y_t^{t,\alpha} = \mathrm{U}_\mathrm{A}(t,\xi) + \int_t^T h_r(t, X_{\cdot\wedge r}, Y_r^{t,\alpha}, Z_r^{t,\alpha}, \alpha_r)\mathrm{d}r - \int_t^T Z_r^{t,\alpha}\cdot \mathrm{d}X_r,\ \mathbb{P}\text{–a.s.},\ t\in[0,T]. \tag{5.1.2}$$

By letting both U_A and h depend on t we allow for general discounting structures and incorporate time-inconsistency into the agent's preferences. Moreover, the previous discussion shows that this formulation encompasses time-inconsistent recursive utilities too.

Remark 5.1.1. *In a Markovian framework, the game-theoretic approach to time-inconsistent agents whose reward is given by* (5.1.2) *has been considered in* Wei, Yong, and Yu [260]*,* Wang and Yong [245] *and* Hamaguchi [114]*. In these works, the dynamics of the controlled state process are given in strong formulation and each considered a refinement of the notion of equilibrium in* [82] *that was suitable in their setting. In this work, we use* BSVIEs *to model the agent's reward and build upon the results in* Part III *to extend the non-Markovian framework proposed in* Chapter 2*.*

Let us now present this formulation properly. We define the set of admissible actions, recall A is compact, as

$$\mathcal{A} := \{\alpha \in \mathcal{P}_{\text{pred}}(A, \mathbb{F}) : (5.1.1) \text{ holds}\},$$

and assume we are given jointly measurable mappings $h : [0,T] \times \mathcal{X} \times \mathbb{R} \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}$, $h_\cdot(\cdot, y, z, a) \in \mathcal{P}_{\text{prog}}(\mathbb{R}, \mathbb{F})$ for any $(y, z, a) \in \mathbb{R} \times \mathbb{R}^n \times A$, and $\mathrm{U_A} : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following set of assumptions.

Assumption Q. (i) *For every* $s \in [0,T]$, $\mathrm{x} \longmapsto \mathrm{U_A}(s, \mathrm{x})$ *is invertible,* i.e. *there exists a mapping* $\mathrm{U_A^{(-1)}} : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ *such that* $\mathrm{U_A^{(-1)}}(s, \mathrm{U_A}(s, \mathrm{x})) = \mathrm{x}$;

(ii) $(s, y, z) \longmapsto h_t(s, x, y, z, a)$ *(resp.* $s \longmapsto \mathrm{U^A}(s, \mathrm{x})$*) is continuously differentiable. Morevoer,* $\nabla h_\cdot(s, \cdot, u, v, y, z, a) \in \mathcal{P}_{\text{prog}}(\mathbb{R}, \mathbb{F})$ *for all* $s \in [0,T]$*, where* $\nabla h : [0,T]^2 \times \mathcal{X} \times (\mathbb{R} \times \mathbb{R}^n)^2 \longrightarrow \mathbb{R}$ *is defined by*

$$\nabla h_t(s, x, u, v, y, z, a) := \partial_s h_t(s, x, y, z, a) + \partial_y h_t(s, x, y, z, a) u + \sum_{i=1}^n \partial_{z_i} h_t(s, x, y, z, a) v_i;$$

(iii) *for* $\varphi \in \{h, \partial_s h\}$*,* $(y, z, a) \longmapsto \varphi_t(s, x, y, z, a)$ *is uniformly Lipschitz-continuous with linear growth,* i.e. *there exists some* $C > 0$ *such that* $\forall (s, t, x, y, \tilde{y}, z, \tilde{z}, a, \tilde{a})$*,*

$$|\varphi_t(s, x, y, z, a) - \varphi_t(s, x, \tilde{y}, \tilde{z}, \tilde{a})| \leq C\big(|y - \tilde{y}| + |\sigma_t(x)^\top (z - \tilde{z})| + |a - \tilde{a}|\big),$$
$$|\varphi_t(s, x, y, z, a)| \leq C\big(1 + |y| + |\sigma_t(x)^\top z|\big).$$

Remark 5.1.2. *Let us comment of the previous assumptions. The first condition guarantees we can identify units of utility with contract payments. Indeed, the utility* $\mathrm{U_A}(s, \xi)$ *is sufficient to identify, via* $\mathrm{U_A^{(-1)}}$*, the contract payment* ξ*. The second assumption guarantees sufficient regularity, with respect to the variable source of inconsistency, of the data prescribing the agent's reward*

We assume the agent has a reservation utility $R_0 \in \mathbb{R}$ below which he refuses to take the contract. The agent is hired at time $t = 0$, and the contracts ξ offered by the principal, for which she can only access the information about the state process X, are assumed to provide the agent with a compensation at the terminal time T. Thus, we specify the set of admissible contracts, see

Section 5.3.1 for the definition of the integrability spaces, as

$$\mathcal{C} := \Big\{ \xi : \mathbb{R}\text{-valued and } \mathcal{F}_T\text{-measurable}, \big((\mathrm{U_A}(s,\xi))_{s\in[0,T]}, (\partial_s \mathrm{U_A}(s,\xi))_{s\in[0,T]} \big) \in \big(\mathcal{L}^{2,2}\big)^2 \Big\}. \quad (5.1.3)$$

If hired, the agent chooses an effort strategy $\alpha \in \mathcal{A}$, and at any time $t \in [0,T]$, his value, from time t onwards, from performing α is given by

$$\mathrm{V}_t^{\mathrm{A}}(\xi,\alpha) := Y_t^{t,\alpha}$$

where the pair (Y^α, Z^α) satisfies the BSVIE (5.1.2). We recall $\mathrm{V}^{\mathrm{A}}(\xi,\alpha)$ is commonly referred to in the literature as the *continuation utility*. We always interpret $\mathrm{V}^{\mathrm{A}}(\xi,\alpha)$ as a map from $[0,T]\times\mathcal{X}$ to $\mathbb{R}$.

Given the choice of reward, the problem of the agent is time-inconsistent. We, therefore, assume the agent is a so-called sophisticated time-inconsistent agent who, aware of his inconsistency, can anticipate it, thus making his strategy time-consistent. Consequently, the problem of the agent can be interpreted as an intra-personal game in which he is trying to balance all of his preferences searches for sub-game perfect Nash equilibria. We recall the definition of an equilibrium strategy introduced in Chapter 2, see further comments in Remark 5.1.5. Let $\{\alpha^\star,\alpha\}\subseteq\mathcal{A}$, $t\in[0,T]$, and $\ell\in(0,T-t]$, we define $\nu\otimes_{t+\ell}\nu^\star := \nu\mathbf{1}_{[t,t+\ell)} + \nu^\star\mathbf{1}_{[t+\ell,T]}$.

Definition 5.1.3. *Let $\alpha^\star\in\mathcal{A}$. For $\varepsilon>0$, define*

$$\ell_\varepsilon := \inf\Big\{\ell>0 : \exists\alpha\in\mathcal{A},\ \mathbb{P}\big[\{x\in\mathcal{X} : \exists t\in[0,T],\ \mathrm{V}_t^{\mathrm{A}}(\xi,\alpha^\star) < \mathrm{V}_t^{\mathrm{A}}(\xi,\alpha\otimes_{t+\ell}\alpha^\star) - \varepsilon\ell\}\big] > 0\Big\}.$$

If for any $\varepsilon>0$, $\ell_\varepsilon>0$ then $\alpha^\star$ is an equilibrium. Given $\xi\in\mathcal{C}$, we call $\mathcal{E}(\xi)$ the set of all equilibria associated to ξ.

As such, the agent's goal is, given a contract ξ that is guaranteed by the principal, to choose an effort that aligns with his sophisticated preferences, *i.e.* to find $\alpha^\star\in\mathcal{E}(\xi)$. In contrast to the case of a classic time-consistent utility maximiser, for a time-inconsistent sophisticated agent, there could be more than one equilibria with potentially different rewards, see for instance [164]. In this work, we will restrict our attention to the set for which all equilibria provide the same value to the agent. See additional comments about this point in the following remark.

Definition 5.1.4. *$\mathcal{C}_o$ denotes the family of contracts $\xi \in \mathcal{C}$ such that for any $\{\alpha^\star, \alpha^*\} \subseteq \mathcal{E}(\xi)$, $\mathrm{V}_t^{\mathrm{A}}(\xi, \alpha^\star) = \mathrm{V}_t^{\mathrm{A}}(\xi, \alpha^*), t \in [0, T]$.*

All in all, for $\xi \in \mathcal{C}_o$ we can now define

$$\mathrm{V}_t^{\mathrm{A}}(\xi) := \mathrm{V}_t^{\mathrm{A}}(\xi, \alpha^\star),\ \alpha^\star \in \mathcal{E}(\xi).$$

Remark 5.1.5. (*i*) *In the non-Markovian framework, the strategy devised in* Chapter 2 *builds upon the approach in* [38] *to study rewards given by conditional expectations of non-Markovian functionals. This approach is based on decoupling the sources of inconsistency in the agent's reward and requires to introduce the terms $\partial_s \mathrm{U}_{\mathrm{A}}$ and ∇h into the analysis, see* Section 5.5 *for details. We also mention that* Theorem 5.5.3 *generalises the extended dynamic programming principle obtained in* Chapter 2 *for the case of rewards given by* (5.1.2) *and equilibrium actions as in* Definition 5.1.3.

(*ii*) *The previous definition of equilibrium can be regarded as a reformulation of the classic definition, in* [82]*, via the* $\liminf$*. Indeed, it follows from* Definition 5.1.3 *that given $(\varepsilon, \ell) \in (0, \infty) \times (0, \ell_\varepsilon)$, $\exists \widetilde{\mathcal{X}} \subseteq \mathcal{X}$, $\mathbb{P}[\widetilde{\mathcal{X}}] = 1$, such that*

$$\mathrm{V}_t^{\mathrm{A}}(\xi, \alpha^\star)(x) - \mathrm{V}_t^{\mathrm{A}}(\xi, \alpha \otimes_{t+\ell} \alpha^\star)(x) \geq -\varepsilon\ell,\ \forall (t, x, \alpha) \in [0, T] \times \widetilde{\mathcal{X}} \times \mathcal{A}.$$

(*iii*) *Lastly, we also expand on the necessity to focus our attention on contracts for which all equilibria provide the same value to the agent. The need for said restriction is inherent to contract theory models involving a game-theoretic formulation at the level of the agent. Indeed, in either the case of a finite number of competitive interacting agents seeking for a Nash equilibrium, see* Élie and Possamaï [90]*, or a continuum of players seeking for a mean-field equilibrium, see* Élie, Mastrolia, and Possamaï [91]*, it is generally possible for multiple equilibria to exist. In such cases, the existence of a Pareto-dominating equilibrium, one for which all agents receive no worse reward if deviating from a current equilibrium, is by no means guaranteed. In the context of contract theory, this means that there is no clear rule, at the level of the problem of the agent, to decide which equilibria should be taken for any two equilibria providing different values to different players. As* *giving control on this decision to the principal makes little practical sense, one way to bypass this is to assume that all possible equilibria yield the same value, as we did here.*

Anticipating our analysis in Section 5.3.1*, we highlight that, in the Lipschitz setting of this paper, we can identify conditions on the data of the problem under which this is the case for any $\xi \in \mathcal{C}$, see* Assumption R *and* Remark 5.3.2*. As such, this is not such a stringent assumption in our context. Moreover, in all the examples considered in* Section 5.4 *the agent's participation constraint is saturated,* i.e. *it is held to the reservation level R_0. Hence, one suspects this might be a more general phenomenon beyond this document's scope.*

5.1.3 The principal's problem

We now present the principal's problem. We therefore let $\Xi \subseteq \mathcal{C}_o$ be the set of admissible contracts, defined by

$$\Xi := \{\xi \in \mathcal{C}_o : \mathrm{V}_0^{\mathrm{A}}(\xi) \geq R_0\}.$$

In such manner, any contract $\xi \in \Xi$ is *implementable*, that is, there exists at least one equilibrium strategy, namely $\alpha \in \mathcal{E}(\xi)$, for the agent's problem. Even though in the previous section we focused our attention on equilibria for which the agent gets the same value, we still need a rule for whenever there are more than one such equilibria. Following the standard convention in the literature, we assume that if there is more than one such action, for which the agent is indifferent by assumption, he implements the one that is best for the principal.

The principal has her own utility function $\mathrm{U_P} : \mathcal{X} \times \mathbb{R} \longrightarrow \mathbb{R}$ and solves the problem

$$\mathrm{V}^{\mathrm{P}} := \sup_{\xi \in \Xi} \sup_{\alpha \in \mathcal{E}(\xi)} \mathbb{E}^{\mathbb{P}^\alpha}\big[\mathrm{U_P}(X_{\cdot \wedge T}, \xi)\big].$$

Remark 5.1.6. *We point out that we have assumed the principal is a standard utility maximiser. This is because, in our opinion, the crux of the problem lies in identifying a proper description of the problem of the principal when contracting a time-inconsistent sophisticated agent. In the case of a time-consistent agent,* [64] *identifies this description as a standard stochastic control problem with an additional state variable. Therefore, in the case of a classic time-consistent agent and a time-inconsistent principal, following* [64]*, one expects the problem of the principal to boil down to* *a non-Markovian time-inconsistent control problem with an additional state variable. As studied in* Chapter 2*, these problems are characterised by an infinite family of* BSDEs*, analogue to the* PDE *system in* [38] *in the Markovian case.*

5.2 The first-best problem

In the first-best, or risk-sharing, problem, the principal chooses both the effort and the contract for the agent, and she is simply required to satisfy the participation constraint. To provide appropriate characterisations of the solution to several examples, we will focus on a particular class of reward functionals for the agent. We recall that our goal is to study the second-best problem introduced in the previous section. As such, despite its inherent interest, the results in the current section serve mainly as a reference point for the general analysis we conduct in Section 5.3. Moreover, the following specification is covered by the general formulation presented in Section 5.1, see Remark 5.2.1, and it is yet rich enough to cover examples of both separable and non-separable utilities.

Let us assume the agent has a given increasing and concave utility function $\mathrm{U}^o_\mathrm{A} : \mathbb{R} \longrightarrow \mathbb{R}$ and Borel-measurable discount functions g, and f defined on $[0, T]$, taking values in $(0, +\infty)$, with $g(0) = f(0) = 1$, which are assumed to be continuously differentiable with derivatives g', and f'. Lastly, we have Borel-measurable functionals k and c, defined on $[0, T] \times \mathcal{X} \times A$ and taking values in $\mathbb{R}_+$.

For any $(t, \alpha) \in [0, T] \times \mathcal{A}$, we then specify the agent's continuation utility by

$$\mathrm{V}^\mathrm{A}_t(\xi, \alpha) = \mathbb{E}^{\mathbb{P}^\alpha}\bigg[\mathcal{K}^{t,\alpha}_{t,T} f(T-t)\mathrm{U}^o_\mathrm{A}(g(T-t)\xi) - \int_t^T \mathcal{K}^{t,\alpha}_{t,r} f(r-t) c_r(X_{\cdot\wedge r}, \alpha_r)\mathrm{d}r\bigg|\mathcal{F}_t\bigg], \tag{5.2.1}$$

where

$$\mathcal{K}^{s,\alpha}_{t,T} := \exp\bigg(\int_t^T g(r-s)k_r(X_{\cdot\wedge r}, \alpha_r)\mathrm{d}r\bigg),\ (s, t, \alpha) \in [0, T]^2 \times \mathcal{A}.$$

Regarding the principal, we assume she has her own utility function $\mathrm{U}^o_\mathrm{P} : \mathbb{R} \longrightarrow \mathbb{R}$, which we assume to be concave and strictly increasing so that

$$\mathrm{V}^{\mathrm{P}} = \sup_{(\alpha,\xi)\in\mathcal{A}\times\Xi} \mathbb{E}^{\mathbb{P}^\alpha}\big[\mathrm{U}^o_{\mathrm{P}}(\Gamma(X_T)-\xi)\big],$$

where $\Gamma : \mathbb{R}^n \longrightarrow \mathbb{R}$ denotes a mechanism by which the principal collects the values of the n different coordinates of the state process X.

Remark 5.2.1. (*i*) *As commented above, the previous type of rewards are indeed covered by the formulation via* BSVIEs (5.1.2) *and satisfy* Assumption Q. *Indeed, it corresponds to the choice* $\mathrm{U_A}(s,\mathrm{x}) = f(T-s)\mathrm{U}^o_{\mathrm{A}}(g(T-s)\mathrm{x})$, $\partial_s \mathrm{U_A}(s,\mathrm{x}) = -f'(T-s)\mathrm{U}^o_{\mathrm{A}}(g(T-s)\mathrm{x}) - f(T-s)g'(T-s)\partial_{\mathrm{x}}\mathrm{U}^o_{\mathrm{A}}(g(T-s)\mathrm{x})$, *and*

$$h_t(s,x,y,z,a) = \sigma_t(x)b_t(x,a)\cdot z - f(t-s)c_t(x,a) + g(t-s)k_t(x,a)y,$$
$$\nabla h_t(s,x,u,v,y,z,a) = \sigma_t(x)b_t(x,a)\cdot v + f'(t-s)c_t(x,a) - g'(t-s)k_t(x,a)y + g(t-s)k_t(x,a)u.$$

Regarding the principal, our specification corresponds to $\mathrm{U_P}(x,\mathrm{x}) = \mathrm{U}^o_{\mathrm{P}}(\Gamma(x_T)-\mathrm{x})$. *Let us mention that, to facilitate the resolution of the following examples, we assumed that* $\mathrm{U_P}$ *depends only on the terminal value of* x_T. *This allow us to use the dynamics of* X *as given in* Section 5.1.1. *We highlight this assumption is not necessary in general analysis for the second-best problem we present in* Section 5.3.

We now move on to characterise the solution to the first-best problem in the case of a time-inconsistent agent with both separable and non-separable utility functions. Anticipating the result, we highlight that in the first-best problem, the problem of the principal reduces to solving a standard stochastic control problem.

5.2.1 Non-separable utility

We recall that the CARA utility function, commonly known as the exponential utility, constitutes the stereotypical example of non-separable utility. We then consider (5.2.1) under the choice $c = 0$,

$$\mathrm{U}^o_{\mathrm{P}}(\mathrm{x}) = -\frac{\mathrm{e}^{-\gamma_{\mathrm{P}}\mathrm{x}}}{\gamma_{\mathrm{P}}},\ \mathrm{U}^o_{\mathrm{A}}(\mathrm{x}) := -\frac{\mathrm{e}^{-\gamma_{\mathrm{A}}\mathrm{x}}}{\gamma_{\mathrm{A}}},\ \mathrm{x}\in\mathbb{R},\ \gamma_{\mathrm{A}}>0,\ \gamma_{\mathrm{P}}>0,$$

$k_t(x,a) = \gamma_{\rm A} k^o_t(x,a)$ and assume $a \longmapsto k^o_t(x,a)$ is convex for any $(t,x) \in [0,T] \times \mathcal{X}$. We then have that

$$\mathrm{V}^{\rm A}_t(\xi,\alpha) := \mathbb{E}^{\mathbb{P}^\alpha}\Big[f(T-t)\mathrm{U}^o_{\rm A}\big(g(T-t)\xi - K^{t,\alpha}_{t,T}\big)\Big|\mathcal{F}_t\Big], \text{ for } K^{s,\alpha}_{t,T} := \int_t^T g(r-s)k^o_r(X,\alpha_r)\mathrm{d}r. \quad (5.2.2)$$

The value of principal is thus obtained through the following constrained optimisation problem

$$\mathrm{V}^{\rm P}_{\rm FB} := \sup_{(\alpha,\xi)\in\mathcal{A}\times\mathcal{C}} \mathbb{E}^{\mathbb{P}^\alpha}\big[\mathrm{U}^o_{\rm P}(\Gamma(X_T)-\xi)\big], \text{ s.t. } \mathbb{E}^{\mathbb{P}^\alpha}\bigg[f(T)\mathrm{U}_{\rm A}\bigg(g(T)\xi - \int_0^T g(r)k^o_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r\bigg)\bigg] \geq R_0.$$

Note that, the concavity (resp. convexity) of both $\mathrm{U}^o_{\rm A}$ and $\mathrm{U}^o_{\rm P}$ (resp. $a \longmapsto k^o_t(x,a)$) and the fact $(\mathcal{A},\mathcal{C})$ is a convex set, imply that $\mathrm{V}^{\rm P}_{\rm FB}$ is a concave optimisation problem. The Lagrangian associated to this problem, where $\rho \in \mathbb{R}_+$ denotes the multiplier of the participation constraint, is given, for any $(\alpha,\xi,\rho) \in \mathcal{A}\times\mathcal{C}\times\mathbb{R}_+$, by

$$\mathfrak{L}(\alpha,\xi,\rho) := \mathbb{E}^{\mathbb{P}^\alpha}\bigg[\mathrm{U}^o_{\rm P}(\Gamma(X_T)-\xi) + \rho f(T)\mathrm{U}^o_{\rm A}\bigg(g(T)\xi - \int_0^T g(r)k^o_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r\bigg)\bigg] - \rho R_0.$$

For convenience of the reader, we recall that the dual problem $\mathrm{V}^{\rm P,d}_{\rm FB}$, which is an unconstrained control problem, is in general an upper bound of $\mathrm{V}^{\rm P}_{\rm Fb}$ and is defined by

$$\mathrm{V}^{\rm P}_{\rm FB} = \sup_{(\alpha,\xi)\in\mathcal{A}\times\mathcal{C}} \inf_{\rho\in\mathbb{R}_+} \mathfrak{L}(\alpha,\xi,\rho) \leq \inf_{\rho\in\mathbb{R}_+} \sup_{(\alpha,\xi)\in\mathcal{A}\times\mathcal{C}} \mathfrak{L}(\alpha,\xi,\rho) =: \mathrm{V}^{\rm P,d}_{\rm FB}, \quad (5.2.3)$$

where we used the convention $\sup_\emptyset = -\infty$. As it is commonplace for convex problems, the next result exploits the absence of duality gap, *i.e.* $\mathrm{V}^{\rm P}_{\rm FB} = \mathrm{V}^{\rm P,d}_{\rm FB}$, to compute the value of $\mathrm{V}^{\rm P}_{\rm FB}$. It uses the following notations

$$\bar\gamma := \frac{\gamma_{\rm A}\gamma_{\rm P} g(T)}{\gamma_{\rm A} g(T) + \gamma_{\rm P}},\ C_y := -\frac{1}{\gamma_{\rm P}}\exp\bigg(\frac{\gamma_{\rm P}}{g(T)}\mathrm{U}^{o\,(-1)}_{\rm A}(y)\bigg).$$

Proposition 5.2.2. *Let*

$$\mathrm{V}_{\rm cont} := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha}\bigg[-\frac{1}{\bar\gamma}\exp\bigg(-\bar\gamma\Gamma(X_T) + \int_0^T \frac{\bar\gamma g(r)k^o_r}{g(T)}(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r\bigg)\bigg],$$
$$\rho^\star := \frac{1}{g(T)f(T)}\bigg(\frac{\bar\gamma f(T)}{\gamma_{\rm A} R_0}\mathrm{V}_{\rm cont}\bigg)^{1+\frac{\gamma_{\rm P}}{\gamma_{\rm A} g(T)}}.$$

Suppose $\mathrm{V}_{\mathrm{cont}} < \infty$ *and for any* $(\alpha, \rho) \in \mathcal{A} \times \mathbb{R}_+$, $\xi^\star(\rho, \alpha) \in \mathcal{C}$ *where*

$$\xi^\star(\rho, \alpha) := \frac{1}{g(T)\gamma_{\mathrm{A}} + \gamma_{\mathrm{P}}}\Big(\gamma_{\mathrm{P}}\Gamma(X_T) + \gamma_{\mathrm{A}} K_{0,T}^{0,\alpha} + \log\big(\rho^\star g(T) f(T)\big)\Big).$$

Then

$$\mathrm{V}_{\mathrm{FB}}^{\mathrm{P}} = C_{\frac{R_0}{f(T)}} \mathrm{V}_{\mathrm{cont}}^{\frac{\gamma_{\mathrm{P}}}{\bar{\gamma}}}.$$

Moreover, if $\alpha^\star$ *is an optimal control for* $\mathrm{V}_{\mathrm{cont}}$, *then an optimal contract is given by* $\xi^\star(\rho^\star, \alpha^\star)$.

Proof. Let $(\rho, \alpha) \in \mathbb{R}_+ \times \mathcal{A}$ be fixed and optimise the mapping $\mathcal{C} \ni \xi \longmapsto \mathfrak{L}(\alpha, \xi, \rho) \in \mathbb{R}$. An upper bound of this problem is given by optimising x-by-x. This leads us to define, for any $(\alpha, \rho) \in \mathcal{A} \times \mathbb{R}_+$ fixed, the candidate

$$\xi^\star(\rho, \alpha) = \frac{1}{g(T)\gamma_{\mathrm{A}} + \gamma_{\mathrm{P}}}\Big(\gamma_{\mathrm{P}}\Gamma(X_T) + \gamma_{\mathrm{A}} K_{0,T}^{0,\alpha} + \log\big(\rho g(T) f(T)\big)\Big).$$

To show the upper bound induced by $\xi^\star(\rho, \alpha)$ is attained it suffices to note that $\xi^\star(\rho, \alpha) \in \mathcal{C}$ by assumption. Replacing in $\mathfrak{L}(\rho, \alpha, \xi)$ we obtain

$$\mathrm{V}_{\mathrm{FB}}^{\mathrm{P,d}} = \inf_{\rho \in \mathbb{R}_+}\bigg\{ -\rho R_0 - \frac{1}{\bar{\gamma}}(\rho g(T) f(T))^{\frac{\bar{\gamma}}{g(T)\gamma_{\mathrm{A}}}} \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha}\bigg[\exp\bigg(-\bar{\gamma}\Gamma(X_T) + \frac{\bar{\gamma}}{g(T)} K_{0,T}^{0,\alpha}\bigg)\bigg]\bigg\}.$$

If $\mathrm{V}_{\mathrm{cont}} < \infty$, as the above function is a strictly convex function of ρ, first order conditions gives $\rho^\star$ as in the statement.

We are only left to show that $\mathrm{V}_{\mathrm{FB}}^{\mathrm{P}} = \mathrm{V}_{\mathrm{FB}}^{\mathrm{P,d}}$, *i.e.* that there is no duality gap. For this, it suffices to verify that $(\xi^\star(\rho^\star, \alpha^\star), \alpha^\star)$ is primal feasible, *i.e.* that it satisfy the participation constraint. Indeed

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{\alpha^\star}}\bigg[\mathrm{U}_{\mathrm{A}}^o\bigg(g(T)\xi - \int_0^T g(r) k_r^o(X_{\cdot\wedge r}, \alpha_r^\star)\mathrm{d}r\bigg)\bigg] \\
&= \frac{-1}{\gamma_{\mathrm{A}}}\mathbb{E}^{\mathbb{P}^{\alpha^\star}}\bigg[\exp\bigg(-\bar{\gamma}\Gamma(X_T) - \frac{\gamma_{\mathrm{A}}}{\gamma_{\mathrm{P}}}\bar{\gamma} K_{0,T}^{0,\alpha^\star} - \frac{\bar{\gamma}}{\gamma_{\mathrm{P}}}\log\bigg(\frac{\bar{\gamma} f(T)}{\gamma_{\mathrm{A}} R_0} V_{\mathrm{cont}}\bigg)^{1 + \frac{\gamma_{\mathrm{P}}}{\gamma_{\mathrm{A}} g(T)}} - \gamma_{\mathrm{A}} K_{0,T}^{0,\alpha^\star}\bigg)\bigg] \\
&= \frac{-1}{\gamma_{\mathrm{A}}}\mathbb{E}^{\mathbb{P}^{\alpha^\star}}\bigg[\exp\bigg(-\bar{\gamma}\Gamma(X_T) - \frac{\gamma_{\mathrm{A}}}{\gamma_{\mathrm{P}}}\bar{\gamma} K_{0,T}^{0,\alpha^\star} - \gamma_{\mathrm{A}} K_{0,T}^{0,\alpha^\star}\bigg)\bigg]\bigg(\frac{\bar{\gamma}}{\gamma_{\mathrm{A}}} V_{\mathrm{cont}}\bigg)^{-1}\frac{R_0}{f(T)} = \frac{R_0}{f(T)}.
\end{aligned}$$

$\square$

5.2.2 Separable utility

We consider the case $k = 0$ and $g = 1$ in (5.2.1), and assume $a \longmapsto c(t, x, a)$ is convex for any $(t, x) \in [0, T] \times \mathcal{X}$. The agent's reward from time $t \in [0, T]$ onwards is given, for any $(t, \alpha, \xi) \in [0, T] \times \mathcal{A} \times \mathcal{C}$, by

$$\mathrm{V}_t^{\mathrm{A}}(\xi, \alpha) = \mathbb{E}^{\mathbb{P}^\alpha}\bigg[f(T-t)\mathrm{U}_{\mathrm{A}}^{\mathrm{o}}(\xi) - \int_t^T f(s-t)c_s(X_{\cdot\wedge s}, \alpha_s)\mathrm{d}s\bigg|\mathcal{F}_t\bigg]. \tag{5.2.4}$$

The value of principal is thus obtained through the following constrained optimisation problem

$$\mathrm{V}_{\mathrm{FB}}^{\mathrm{P}} := \sup_{(\alpha,\xi)\in\mathcal{A}\times\mathcal{C}} \mathbb{E}^{\mathbb{P}^\alpha}\big[\mathrm{U}_{\mathrm{P}}^{\mathrm{o}}(\Gamma(X_T) - \xi)\big],\ \text{s.t.}\ \mathbb{E}^{\mathbb{P}^\alpha}\bigg[f(T)\mathrm{U}_{\mathrm{A}}^{\mathrm{o}}(\xi) - \int_0^T f(r)c_r(X_{\cdot\wedge r}, \alpha_r)\mathrm{d}r\bigg] \geq R_0.$$

The Lagrangian associated to this problem is given, for any $(\alpha, \xi, \rho) \in \mathcal{A} \times \mathcal{C} \times \mathbb{R}_+$, by

$$\mathfrak{L}(\alpha, \xi, \rho) := \mathbb{E}^{\mathbb{P}^\alpha}\bigg[\mathrm{U}_{\mathrm{P}}^{\mathrm{o}}(\Gamma(X_T) - \xi) + \rho f(T)\mathrm{U}_{\mathrm{A}}(\xi) - \rho\int_0^T f(r)c_r(X_{\cdot\wedge r}, \alpha_r)\mathrm{d}r\bigg] - \rho R_0.$$

Proposition 5.2.3. (*i*) *Suppose* $\mathrm{U}_{\mathrm{A}}^{\mathrm{o}}$ *and* $\mathrm{U}_{\mathrm{P}}^{\mathrm{o}}$ *are such that mapping* $\xi^\star(\mathrm{x}, \rho)$ *given as the solution to*

$$-\partial_{\mathrm{x}}\mathrm{U}_{\mathrm{P}}^{\mathrm{o}}(\Gamma(\mathrm{x}) - \xi^\star(\mathrm{x}, \rho)) + \rho f(T)\partial_{\mathrm{x}}\mathrm{U}_{\mathrm{A}}^{\mathrm{o}}(\xi^\star(\mathrm{x}, \rho)) = 0,\ (\mathrm{x}, \rho) \in \mathbb{R}^n \times \mathbb{R}_+,$$

is well-defined and $\xi^\star(X_T, \rho) \in \mathcal{C}$ *for any* $\rho \in \mathbb{R}_+$. *For* $\rho \in \mathbb{R}_+$, *let*

$$\mathrm{V}_{\mathrm{cont}}(\rho) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha}\bigg[\mathrm{U}_{\mathrm{P}}^{\mathrm{o}}(\Gamma(X_T) - \xi^\star(X_T, \rho)) + \rho f(T)\mathrm{U}_{\mathrm{A}}^{\mathrm{o}}(\xi^\star(X_T, \rho)) - \rho\int_0^T f(r)c_r(X_{\cdot\wedge r}, \alpha_r)\mathrm{d}r\bigg].$$

Then

$$\mathrm{V}_{\mathrm{FB}}^{\mathrm{P,d}} = \inf_{\rho\in\mathbb{R}_+}\Big\{-\rho R_0 + \mathrm{V}_{\mathrm{cont}}(\rho)\Big\}.$$

Moreover, suppose the pair $\big(\alpha^\star(\rho^\star), \xi^\star(X_T, \rho^\star)\big)$ *is feasible for the primal problem, where* $\alpha^\star(\rho)$ *(resp.* $\rho^\star$*) denote the maximiser in* $\mathrm{V}_{\mathrm{cont}}(\rho)$ *(resp. the above problem), which we assume to exist. Then, there is no duality gap,* i.e.

$$\mathrm{V}_{\mathrm{FB}}^{\mathrm{P}} = \mathrm{V}_{\mathrm{FB}}^{\mathrm{P,d}},$$

the optimal contract is given by $\xi^\star(X_T,\rho^\star)$.

(ii) If $\mathrm{U}^{\mathrm{o}}_{\mathrm{P}}(\mathrm{x})=\mathrm{U}^{\mathrm{o}}_{\mathrm{A}}(\mathrm{x})=\mathrm{x}$, *for* $\alpha\in\mathcal{A}$ *let*

$$\hat{\mathcal{C}}(\alpha)=\bigg\{\xi\in\mathcal{C}:\mathbb{E}^{\mathbb{P}^\alpha}[\xi^\star]=\mathbb{E}^{\mathbb{P}^\alpha}\bigg[\int_0^T\frac{f(r)}{f(T)}c_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r\bigg]+\frac{R_0}{f(T)}\bigg\}.$$

Then, the problem of the principal is given by the solution to the standard control problem

$$\mathrm{V}^{\mathrm{P}}_{\mathrm{FB}}=-\frac{R_0}{f(T)}+\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}^\alpha}\bigg[\Gamma(X_T)-\int_0^T f(r)c_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r\bigg].$$

Moreover, for $\alpha^\star\in\mathcal{A}$ *an optimal control of this problem,* $\hat{\mathcal{C}}(\alpha^\star)$ *contains all the optimal contracts for the principal.e.g. the deterministic contract*

$$\xi^\star:=\frac{R_0}{f(T)}+f(T)^{-1}\mathbb{E}^{\mathbb{P}^{\alpha^\star}}\bigg[\int_0^T f(r)c_r(X_{\cdot\wedge r},\alpha^\star_r)\mathrm{d}r\bigg].$$

Proof. We argue (i). Let $(\rho,\alpha)\in\mathbb{R}_+\times\mathcal{A}$ be fixed and optimise the mapping $\mathcal{C}\ni\xi\longmapsto\mathfrak{L}(\alpha,\xi,\rho)\in\mathbb{R}$. An upper bound of this problem is given by optimising x-by-x. This defines the mapping $\xi^\star(\mathrm{x},\rho)$. As before, the fact that $\xi^\star(\rho,\alpha)\in\mathcal{C}$ guarantees the upper bound is indeed attained. Replacing in $\mathfrak{L}(\rho,\alpha,\xi)$ we obtain $\mathrm{V}_{\mathrm{cont}}(\rho)$ and the corresponding equality for $\mathrm{V}^{\mathrm{P,d}}_{\mathrm{FB}}$. Now, to obtain the absence of duality gap we must verify that there exists a solution to the dual problem that is primal feasible. This is exactly the additional assumption in the statement.

We now consider (ii). In this case, we can solve $\mathrm{V}^{\mathrm{P}}_{\mathrm{FB}}$ directly. In light of $\mathrm{U}_{\mathrm{P}}(\mathrm{x})=\mathrm{U}_{\mathrm{A}}(\mathrm{x})=\mathrm{x}$,

$$\mathrm{V}^{\mathrm{P}}_{\mathrm{FB}}=\sup_{(\alpha,\xi)\in\mathcal{A}\times\mathcal{C}}\mathbb{E}^{\mathbb{P}^\alpha}[\Gamma(X_T)-\xi],\text{ s.t. }\mathbb{E}^{\mathbb{P}^\alpha}\bigg[f(T)\xi-\int_0^T f(r)c_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r\bigg]\geq R_0.$$

Let us note that for fixed $\alpha\in\mathcal{A}$ the principal's reward is linear and strictly decreasing in $\mathbb{E}^{\mathbb{P}^\alpha}[\xi]$ and therefore she is indifferent between contracts that have the same expectation. Therefore, she optimises over the feasible contracts that have the same expectation. Now, for fixed $\alpha\in\mathcal{A}$ any feasibility contract satisfies

$$\mathbb{E}^{\mathbb{P}^\alpha}[\xi^\star]\geq\mathbb{E}^{\mathbb{P}^\alpha}\bigg[\int_0^T\frac{f(r)}{f(T)}c_r(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r\bigg]+\frac{R_0}{f(T)}.$$

Therefore, our previous comment implies that for given α the principal is indifferent between contracts in $\hat{\mathcal{C}}(\alpha)$. Note that $\hat{\mathcal{C}}(\alpha) \neq \emptyset$. Indeed, take the deterministic contract $\xi^\star(\alpha) := \frac{R_0}{f(T)} + f(T)^{-1}\mathbb{E}^{\mathbb{P}^\alpha}\bigg[\int_0^T f(r)c_r(X_{\cdot\wedge r}, \alpha_r)\mathrm{d}r\bigg]$.

Plugging this back into the principal's utility, we get the expression for $\mathrm{V}^{\mathrm{P}}_{\mathrm{FB}}$ in the statement. $\square$

Remark 5.2.4. *We remark that the assumption on the utility functions in* Proposition 5.2.3 *is relatively reasonable. Indeed, it is immediately satisfied, for instance, in either of the following scenarii*

(i) $\mathrm{U}^o_{\mathrm{P}}(\mathrm{x}) = \mathrm{x}$ *and* $\mathrm{U}^o_{\mathrm{A}}(\mathrm{x})$ *is strictly increasing*;

(ii) for $\varphi \in \{\mathrm{U}^o_{\mathrm{A}}, \mathrm{U}^o_{\mathrm{P}}\}$, $x \longmapsto \varphi(\mathrm{x})$ *is concave, strictly increasing and satisfies the following conditions*

$$\lim_{\mathrm{x}\to-\infty} \partial_{\mathrm{x}}\varphi(\mathrm{x}) = \infty,\ \lim_{\mathrm{x}\to\infty} \partial_{\mathrm{x}}\varphi(\mathrm{x}) = 0.$$

5.3 The second-best problem: general scenario

In this section, we bring back our attention to the second-best problem face by the principal

$$\mathrm{V}^{\mathrm{P}} = \sup_{\xi\in\Xi}\ \sup_{\alpha\in\mathcal{E}(\xi)}\ \mathbb{E}^{\mathbb{P}^\alpha}\big[\mathrm{U}_{\mathrm{P}}(X_{\cdot\wedge T}, \xi)\big].$$

We will exploit the theory of type-I BSVIEs, revisited in Part III. We first introduce suitable integrability spaces.

5.3.1 Integrability spaces and Hamiltonian

Following Part III, to carry out the analysis we introduce the spaces

- $\mathcal{L}^2$ of $\xi \in \mathcal{P}_{\mathrm{meas}}(\mathbb{R}, \mathcal{F})$, such that $\|\xi\|^2_{\mathcal{L}^2} := \mathbb{E}\big[|\xi|^2\big] < \infty$;
- $\mathbb{S}^2$ of càdlàg $Y \in \mathcal{P}_{\mathrm{prog}}(\mathbb{R}, \mathbb{F})$ such that $\|Y\|^2_{\mathbb{S}^2} := \mathbb{E}\bigg[\sup_{t\in[0,T]} |Y_t|^2\bigg] < \infty$;
- $\mathbb{L}^2$ of $Y \in \mathcal{P}_{\mathrm{opt}}(\mathbb{R}, \mathbb{F})$, with $\|Y\|^2_{\mathbb{L}^2} := \mathbb{E}\bigg[\bigg(\int_0^T |\mathrm{Y}_r|^2\mathrm{d}r\bigg)\bigg] < \infty$;

- $\mathbb{H}^2$ of $Z \in \mathcal{P}_{\rm pred}(\mathbb{R}^n, \mathbb{F})$ such that $\|Z\|^2_{\mathbb{H}^2} := \mathbb{E}\bigg[\displaystyle\int_0^T |\sigma_r\sigma_r^\top Z_r|^2 \mathrm{d}r\bigg] < \infty$;

To make sense of the class of systems considered in this paper we introduce some extra spaces.

- Given a Banach space $(\mathbb{I}, \|\cdot\|_{\mathbb{I}})$ of E-valued processes, we define $(\mathbb{I}^2, \|\cdot\|_{\mathbb{I}^2})$ the space of $U \in \mathcal{P}^2_{\rm meas}(E, \mathcal{F})$ such that $([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{I}^2, \|\cdot\|_{\mathbb{I}^2}) : s \longmapsto U^s$ is continuous and $\|U\|_{\mathbb{I}^2} := \sup\limits_{s\in[0,T]} \|U^s\|_{\mathbb{I}} < \infty$;

e.g., $\mathbb{S}^{2,2}$ denotes the space of $Y \in \mathcal{P}^2_{\rm meas}(\mathbb{R}, \mathcal{F})$ such that $([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{S}^2(\mathbb{R}), \|\cdot\|_{\mathbb{S}^2}) : s \longmapsto Y^s$ is continuous and $\|Y\|_{\mathbb{S}^{2,2}} := \sup\limits_{s\in[0,T]} \|Y^s\|_{\mathbb{S}^2} < \infty$;

next, we introduce the space

- $\overline{\mathbb{H}}^{2,2}$ of $(Z_\tau)_{\tau\in[0,T]^2} \in \mathcal{P}^2_{\rm meas}(\mathbb{R}^n, \mathcal{F})$ such that $([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{H}^2, \|\cdot\|_{\mathbb{H}^2}) : s \longmapsto Z^s$ is absolutely continuous with respect to the Lebesgue measure with density ∂Z, and $\|Z\|^2_{\overline{\mathbb{H}}^{2,2}} := \|Z\|^2_{\mathbb{H}^{2,2}} + \|\mathcal{Z}\|^2_{\mathbb{H}^2} < \infty$, where $\mathcal{Z} := (Z_t^t)_{t\in[0,T]} \in \mathbb{H}^2$ is given by

$$Z_t^t := Z_t^0 + \int_0^t \partial Z_t^r \mathrm{d}r.$$

Lastly, we introduce the space $\mathfrak{H} := \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{S}^{2,2} \times \overline{\mathbb{H}}^{2,2} \times \mathbb{S}^{2,2} \times \mathbb{H}^{2,2}$.

Remark 5.3.1. *Building upon our analysis in* Part II *and* Part III*, we recall that the second set of these spaces are suitable extensions of the classical ones, whose norms are tailor–made to the analysis of the systems we will study. Some of these spaces have been previously considered in the literature on* BSVIEs*, e.g.* [245]*. Of particular interest is the space $\overline{\mathbb{H}}^{2,2}$, introduced in* Chapter 3*, which allows us to define a good candidate for $(Z_t^t)_{t\in[0,T]}$ as an element of $\mathbb{H}^2$. Indeed, let $\widetilde{\Omega} := [0,T] \times \mathcal{X}$, $\tilde{\omega} := (t,x) \in \widetilde{\Omega}$ and*

$$\mathfrak{Z}_s(\tilde{\omega}) := Z_t^T(x) - \int_s^T \partial Z_t^r(x)\mathrm{d}r,\ \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e. } \tilde{\omega} \in \widetilde{\Omega},\ s \in [0,T],$$

so that the Radon–Nikodým property and Fubini's theorem imply $\mathfrak{Z}_s = Z^s, \mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e., $s \in [0,T]$. Lastly, as for $\tilde{\omega} \in \widetilde{\Omega}$, $s \longmapsto \mathfrak{Z}_s(\tilde{\omega})$ is continuous, we may define

$$Z_t^t := Z_t^T - \int_t^T \partial Z_t^r \mathrm{d}r, \text{ for } \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e. } (t,x) \text{ in } [0,T] \times \mathcal{X}.$$

5.3.2 Characterising equilibria and the BSDE system

Building upon the results in Chapter 2, where only the case of an agent with separable utility was considered, we wish to obtain a characterisation of the equilibria that are associated to any $\xi \in \Xi$. For this we must introduce the Hamiltonian functional $H : [0,T] \times \mathcal{X} \times R \times \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$H_t(x, \mathrm{y}, \mathrm{z}) := \sup_{a \in A} h_t(t, x, \mathrm{y}, \mathrm{z}, a), \; (t, x, \mathrm{y}, \mathrm{z}) \in [0,T] \times \mathcal{X} \times \mathbb{R} \times \mathbb{R}^n.$$

Our standing assumptions on H are the following.

Assumption R. (i) *For any* $(t,x) \in [0,T] \times \mathcal{X}$*, the map* $\mathbb{R} \times \mathbb{R}^n \ni (y,z) \longmapsto H_t(x,y,z)$ *is uniformly Lipschitz-continuous with linear growth,* i.e. *there is* $C > 0$ *such that for any* $(t, x, \mathrm{y}, \tilde{\mathrm{y}}, \mathrm{z}, \tilde{\mathrm{z}}) \in [0,T] \times \mathcal{X} \times \mathbb{R}^2 \times (\mathbb{R}^n)^2$

$$\big|H_t(x, \mathrm{y}, \mathrm{z}) - H_t(x, \tilde{\mathrm{y}}, \tilde{\mathrm{z}})\big| \leq C\big(|\mathrm{y} - \tilde{\mathrm{y}}| + |\sigma_t(x)^\top(\mathrm{z} - \tilde{\mathrm{z}})|\big), \; \big|H_t(x, \mathrm{y}, \mathrm{z})\big| \leq C\big(1 + |\mathrm{y}| + |\sigma_t(x)^\top \mathrm{z}|\big);$$

(ii) *there exists a unique Borel-measurable map* $a^\star : [0,T] \times \mathcal{X} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow A$ *such that*

$$H_t(x, \mathrm{y}, \mathrm{z}) = h_t\big(t, x, \mathrm{y}, \mathrm{z}, a^\star(t, x, \mathrm{y}, \mathrm{z})\big), \; \forall (t, x, \mathrm{y}, \mathrm{z}) \in [0,T] \times \mathcal{X} \times \mathbb{R} \times \mathbb{R}^n.$$

(iii) *For any* $(t,x) \in [0,T] \times \mathcal{X}$*, the map* $\mathbb{R} \times \mathbb{R}^n \ni (\mathrm{y}, \mathrm{z}) \longmapsto a^\star(t, x, \mathrm{y}, \mathrm{z})$ *is uniformly Lipschitz-continuous with linear growth,* i.e. *there is* $C > 0$ *such that for any* $(t, x, \mathrm{y}, \tilde{\mathrm{y}}, \mathrm{z}, \tilde{\mathrm{z}}) \in [0,T] \times \mathcal{X} \times \mathbb{R}^2 \times (\mathbb{R}^n)^2$.

$$\big|a^\star(t, x, \mathrm{y}, \mathrm{z}) - a^\star(t, x, \tilde{\mathrm{y}}, \tilde{\mathrm{z}})\big| \leq C\big(|\mathrm{y} - \tilde{\mathrm{y}}| + |\sigma_t(x)^\top(\mathrm{z} - \tilde{\mathrm{z}})|\big), \; \big|a^\star(t, x, \mathrm{y}, \mathrm{z})\big| \leq C\big(1 + |\mathrm{y}| + |\sigma_t(x)^\top \mathrm{z}|\big);$$

To ease the notation we also introduce $b_t^\star(x, \mathrm{y}, \mathrm{z}) := b_t\big(x, a^\star(t, x, \mathrm{y}, \mathrm{z})\big)$,

$$h_r^\star(s, x, y, z, \mathrm{y}, \mathrm{z}) := h_r\big(s, x, y, z, a^\star(r, x, \mathrm{y}, \mathrm{z})\big),$$
$$\nabla h_r^\star(s, x, u, v, y, z, \mathrm{y}, \mathrm{z}) := \nabla h_r\big(s, x, u, v, y, z, a^\star(r, x, \mathrm{y}, \mathrm{z})\big).$$

Remark 5.3.2. *Let us comment on the previous set of assumptions. Even in the non-Markovian* *setting of this document, the problem faced by a sophisticated agent is related to a system of equations instead of just one, see* Chapter 2*. This raises many issues, among which is the possibility for multiplicity of equilibria with different values.* Assumption R.(i)*,* R.(iii) *guarantee that for a given* $\xi \in \Xi$ *any equilibria* $\alpha^\star \in \mathcal{E}(\xi)$ *provide the agent with the same value, and all correspond to maximisers of the Hamiltonian. Ultimately,* Assumption R.(ii) *guarantees that there is only one maximiser of the Hamiltonian. Let us mention that the existence of* $a^\star$ *is guaranteed under* Assumption Q.(iii) *by* Schäl [221, Theorem 3]*. This conciliates our focus on contracts for which any equilibria provides the same value as we stated in* Section 5.1.2*.*

Under this set of assumptions we are able to show, see Section 5.5, that for any $\xi \in \Xi$

$$\mathcal{E}(\xi) = \big\{(a^\star(t, X_{\cdot\wedge t}, Y_t(\xi), Z_t(\xi)))_{t\in[0,T]}\big\},$$

where the processes $\big(Y(\xi), Z(\xi)\big)$ come from the solution to the following infinite family of BSDEs which for any $s \in [0,T]$ satisfies, $\mathbb{P}$–a.s.

$$\begin{aligned}
Y_t(\xi) &= \mathrm{U_A}(T,\xi) + \int_t^T \Big(H_r\big(X, Y_r(\xi), Z_r(\xi)\big) - \partial Y_r^r(\xi)\Big)\mathrm{d}r - \int_t^T Z_r(\xi)\cdot \mathrm{d}X_r,\\
Y_t^s(\xi) &= \mathrm{U_A}(s,\xi) + \int_t^T h_r^\star\big(s, X, Y_r^s(\xi), Z_r^s(\xi), Y_r(\xi), Z_r(\xi)\big)\mathrm{d}r - \int_t^T Z_r^s(\xi)\cdot \mathrm{d}X_r,\\
\partial Y_t^s(\xi) &= \partial_s \mathrm{U_A}(s,\xi) + \int_t^T \nabla h_r^\star\big(s, X, \partial Y_r^s(\xi), \partial Z_r^s(\xi), Y_r^s(\xi), Z_r^s(\xi), Y_r(\xi), Z_r(\xi)\big)\mathrm{d}r\\
&\quad - \int_t^T \partial Z_r^s(\xi)\cdot \mathrm{d}X_r.
\end{aligned} \tag{5.3.1}$$

Moreover, we have that

$$\begin{gathered}
\mathrm{V}_t^{\mathrm{A}}\big(\xi, a^\star(\cdot, X_\cdot, Y_\cdot(\xi), Z_\cdot(\xi))\big) = Y_t(\xi) = Y_t^t(\xi),\ t\in[0,T],\ \mathbb{P}\text{–a.s.},\\
Z_t(\xi) = Z_t^t(\xi),\ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e.}
\end{gathered} \tag{5.3.2}$$

Given that Assumption Q guarantees that $x \longmapsto \mathrm{U_A}(s,x)$ invertible for every $s \in [0,T]$, we also have that

$$\mathrm{U}_{\mathrm{A}}^{(-1)}\big(s, Y_T^s(\xi)\big) = \xi = \mathrm{U}_{\mathrm{A}}^{(-1)}\big(u, Y_T^u(\xi)\big),\ \mathbb{P}\text{–a.s.},\ (s,u)\in[0,T]^2. \tag{5.3.3}$$

Remark 5.3.3. (i) *We recall that the diagonal process* $(Z_t^t)_{t\in[0,T]}$ *is well-defined for elements in* $\overline{\mathbb{H}}^{2,2}$*, see* Section 5.3.1.

(ii) *Links between time-inconsistent control problems and a broader class of* BSVIEs *have been identified in the past. The first mention of this links appears, as far as we know, in the concluding remarks of* Wang and Yong [255]*. The link was then made rigorous independently by* Chapter 2 *and* Wang and Yong [245]*. In our setting, in light of* (5.3.2)*, such an equation appears as the one satisfied by the reward of the agent along the equilibrium. As such, the pair* $\big(Y_t^s(\xi), Z_t^s(\xi)\big)_{(s,t)\in[0,T]^2}$ *solves a so-called extended* Type-I BSVIE*, which for any* $s\in[0,T]$ *satisfies*

$$Y_t^s(\xi) = \mathrm{U_A}(s,\xi) + \int_t^T h_r^\star(s,X,Y_r^s(\xi),Z_r^s(\xi),Y_r^r(\xi),Z_r^r(\xi))\mathrm{d}r - \int_t^T Z_r^s(\xi)\cdot \mathrm{d}X_r,\ \mathbb{P}\text{–a.s.} \quad (5.3.4)$$

We highlight that this BSVIE *involves the diagonal processes* $\big(Y_t^t(\xi), Z_t^t(\xi)\big)_{(s,t)\in[0,T]^2}$ *and that in light of* Theorem 3.4.3 *the well-posedness of* (5.3.1) *is equivalent to that of* (5.3.4).

5.3.3 The family of restricted contracts

In light of our previous observation, namely (5.3.3), in the following, we will introduce a family of restricted contract payments which we will denote $\overline{\Xi}$. For any contract in this family, we can solve the associate time-inconsistent control problem faced by the agent. Moreover, we will show that any admissible contract available to the principal admits a representation as a contract in $\overline{\Xi}$. Consequently, the principal's optimal expected utility is not reduced if she restricts herself to offer contracts in this family and optimises.

In order to define the family of restricted contracts, we introduce next the process $Y^{y_0,Z}$ which, for a suitable process Z, will represent the value of the agent. This is a preliminary step based on the observation, see (5.3.3), that the value of the agent at the terminal time T coincides with the payment offered by the contract. To alleviate the notation let us set $\mathcal{I} := \{y_0\in\mathcal{C}_1^1 : y_0^0 \geq R_0\}$.

Definition 5.3.4. *We denote by* $\mathcal{H}^{2,2}$ *the collection of processes* $Z\in\overline{\mathbb{H}}^{2,2}$ *satisfying:*

$\|Y^{y_0,Z}\|_{\mathbb{S}^{2,2}}<\infty$*, where for* $y_0\in\mathcal{I}$*,* $Y^{y_0,Z} := (Y^{s,y_0,Z})_{s\in[0,T]}$ *satisfies for every* $s\in[0,T]$*,*

$$Y_t^{s,y_0,Z} = y_0^s - \int_0^t h_r^\star(s,X_{\cdot\wedge r},Y_r^{s,y_0,Z},Z_r^s,Y_r^{r,y_0,Z},Z_r^r)\mathrm{d}r + \int_0^t Z_r^s\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.} \quad (5.3.5)$$

$$\mathrm{U}_{\mathrm{A}}^{(-1)}(s, Y_T^{s,y_0,Z}) = \mathrm{U}_{\mathrm{A}}^{(-1)}(u, Y_T^{u,y_0,Z}),\ \mathbb{P}\text{–a.s.},\ (s,u) \in [0,T]^2. \tag{5.3.6}$$

With this, it is natural to consider the set $\overline{\overline{\Xi}}$ of contracts of the form

$$\mathrm{U}_{\mathrm{A}}^{(-1)}(T, Y_T^{T,y_0,Z}),\ (y_0, Z) \in \mathcal{I} \times \mathcal{H}^{2,2}.$$

The main novelty of our argument, compared to that in the time-consistent case, is the fact that (5.3.6) imposes a constraint on the elements $Z \in \mathcal{H}^{2,2}$. We note that in light of 3.4.3 we have that $\mathcal{H}^{2,2} \neq \emptyset$.

Remark 5.3.5. (i) *The process $Y^{y_0,Z}$ denotes a solution to a so-called forward Volterra integral equation* (FSVIE*, for short*)*. These objects have been studied since* Berger and Mizel [23, 24], Protter [211], Pardoux and Peng [201]*. However, this is not a classic* FSVIE *in the sense that, in addition to $Y^{s,y_0,Z}$, the diagonal processes $\big(\mathrm{Y}_t^{t,y_0,Z}\big)_{t\in[0,T]}$ appears in the generator. For completeness,* Proposition 5.6.4 *below presents a suitable well-posedness result.*

(ii) *We also remark that in light of* (5.3.6)*, for any $\bar{\xi} \in \overline{\overline{\Xi}}$, it holds that $\bar{\xi} = \mathrm{U}_{\mathrm{A}}^{(-1)}(s, Y_T^{s,y_0,Z})$,* $\mathbb{P}$–a.s.*, $s \in [0,T]$.*

(iii) *As mentioned at the beginning of this section, we chose to work with a representation for the agent's value as opposed to the value of the contract itself. This determines the form of the contracts in the definition of $\overline{\overline{\Xi}}$ and provides a quite general and comprehensive approach. For instance, one could have chosen to represent the value of the contract ξ directly for an agent with a time-inconsistent exponential utility. This would have produced a version of* (5.3.1) *whose generators have quadratic growth in Z and whose well-posedness is more delicate than in the Lipschitz case. See for instance,* Briand and Hu [42], Kobylanski [155] *and* Tevzadze [238] *for the study of quadratic* BSDEs*. We recall that in light of* [42]*, taking that approach in the time-consistent scenario requires, at the very least, to assume the contracts have exponential moments of sufficiently large order. Our approach prevents this given our growth assumptions in* Assumption Q*. However, one cannot expect to avoid such restrictions for problems that are inherently quadratic.*

In light of our previous remark, as a preliminary step, we must verify that (5.3.5) uniquely defines $Y^{y_0,Z}$. At the formal level, the following auxiliary lemma says that the integrability condition on $Y^{y_0,Z}$ guarantees this.

Lemma 5.3.6. *Let* Assumption Q *and* Assumption R *hold. Given* $(y_0, Z) \in \mathcal{I} \times \mathcal{H}^{2,2}$ *there exist unique processes* $(Y^Z, \partial Y) \in \mathbb{S}^{2,2} \times \mathbb{S}^{2,2}$ *such that* $Y^{y_0,Z}$ *satisfies* (5.3.5) *and* $\partial Y^{y_0,Z}$ *satisfies*

$$\partial Y_t^{s,y_0,Z} = \partial y_0^s - \int_0^t \nabla h_r^\star(s, X, \partial Y_r^{s,y_0,Z}, \partial Z_r^s, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r + \int_0^t \partial Z_r^s \cdot \mathrm{d}X_r. \quad (5.3.7)$$

Proof. Let us first argue the result for $Y^{y_0,Z}$. Note that the integrability of $Z \in \mathcal{H}^{2,2}$, Assumption Q.(ii) and Assumption R.(iii) yields

$$\sup_{s\in[0,T]} \mathbb{E}\bigg[\bigg(\int_0^T |h_\cdot^\star(s, X_{\cdot\wedge r}, 0, Z_r^s, 0, Z_r^r))|\mathrm{d}r\bigg)^2\bigg] < \infty.$$

The result follows from Proposition 5.6.4. The second part of the statement is a consequence of Proposition 5.6.5. $\square$

We are now ready to state our main result, in words it guarantees that there is no loss of generality for the principal in offering contracts of the form given by $\overline{\Xi}$.

Theorem 5.3.7. (i) *We have* $\overline{\Xi} = \Xi$. *Moreover, for any contract* $\xi \in \overline{\Xi}$, *associated to* $(y_0, Z) \in \mathcal{I} \times \mathcal{H}^{2,2}$, *we have*

$$\mathcal{E}(\xi) = \{a^\star(t, X_{\cdot\wedge t}, Y_t^{t,y_0,Z}, Z_t^t)_{t\in[0,T]}\},\ \mathrm{V}_0^{\mathrm{A}}(\xi) = y_0^0.$$

(ii) *Let* $\mathbb{P}^\star(Z) := \mathbb{P}^{a^\star(\cdot, X_\cdot, Y_\cdot^{\cdot,y_0,Z}, Z_\cdot^\cdot)}$. *The problem of the principal admits the following representation*

$$\mathrm{V}^{\mathrm{P}} = \sup_{y_0\in\mathcal{I}} \underline{V}(y_0),\ \text{where}\ \underline{V}(y_0) := \sup_{Z\in\mathcal{H}^{2,2}} \mathbb{E}^{\mathbb{P}^\star(Z)}\Big[\mathrm{U_P}\Big(X_{\cdot\wedge T}, \mathrm{U_A}^{(-1)}\Big(T, Y_T^{T,y_0,Z}\Big)\Big)\Big]. \quad (5.3.8)$$

Proof. We first argue $\Xi \subseteq \overline{\Xi}$. Let $\xi \in \Xi$. In light of Assumption Q, Theorem 3.3.5 guarantees that for $\xi \in \mathcal{C}$ there exists $(Y(\xi), Z(\xi)) \in \mathbb{S}^{2,2} \times \mathbb{H}^{2,2}$ solution to (5.3.4) and a process $\partial Y(\xi) \in \mathbb{S}^{2,2}$ satisfying that the mapping $([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{S}^2, \|\cdot\|_{\mathbb{S}^2}) : s \longmapsto \partial Y^s(\xi)$ is the derivative of $([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{S}^2, \|\cdot\|_{\mathbb{S}^2}) : s \longmapsto Y^s(\xi)$. Moreover, Assumption Q.(i) guarantees (5.3.6) holds. Moreover, (5.3.2) implies $Y_0(\xi) = \mathrm{V}_0^{\mathrm{A}}(\xi, a^\star(\cdot, X_\cdot, Y_\cdot(\xi), Z_\cdot(\xi))) \geq R_0$, recall $\xi \in \Xi$. From this, taking $y_0(\xi) = Y_0(\xi)$ we have that $(y_0(\xi), Z(\xi)) \in \mathcal{I} \times \mathcal{H}^{2,2}$. Thus $\xi \in \overline{\Xi}$.

To show the reverse inclusion, let $\bar\xi \in \overline{\Xi}$. This is, $\bar\xi = \mathrm{U}_{\mathrm{A}}^{(-1)}(T, Y_T^{T,y_0,Z})$, where, in light of Lemma 5.3.6, $Y^{y_0,Z}$ denotes the unique process, induced by $(y_0, Z) \in \mathcal{I} \times \mathcal{H}^{2,2}$, such that $\|Y^{y_0,Z}\|_{\mathbb{S}^{2,2}} < \infty$ and (5.3.6) holds. In particular,

$$Y_T^{s,y_0,Z} = \mathrm{U}_{\mathrm{A}}(s, \bar\xi),\ \mathbb{P}\text{–a.s.},\ s \in [0,T].$$

Therefore, for any $s \in [0,T]$

$$Y_t^{s,y_0,Z} = \mathrm{U}_{\mathrm{A}}(s,\bar\xi) + \int_t^T h_r^\star(s, X_{\cdot\wedge r}, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r - \int_t^T Z_r^s \cdot \mathrm{d}X_r,\ \mathbb{P}\text{–a.s.} \tag{5.3.9}$$

We now show $\bar\xi \in \mathcal{C}$, as in Definition 5.1.4. We first show $\bar\xi \in \mathcal{C}$, recall (5.1.3). It is immediate to see that

$$\|\mathrm{U}_{\mathrm{A}}(\cdot,\bar\xi)\|_{\mathcal{L}^{2,2}}^2 = \sup_{s\in[0,T]} \mathbb{E}\Big[\big|\mathrm{U}_{\mathrm{A}}(s,\bar\xi)\big|^2\Big] = \sup_{s\in[0,T]} \mathbb{E}\Big[\big|Y_T^{s,y_0,Z}\big|^2\Big] \leq \|Y^{y_0,Z}\|_{\mathbb{S}^{2,2}}^2 < \infty.$$

Now, given $Y^{y_0,Z}$ solution to (5.3.5) and ∂Z by definition of $Z \in \overline{\mathbb{H}}^{2,2}$, Lemma 5.3.6 guarantees there exists $\partial Y^{y_0,Z} \in \mathbb{S}^{2,2}$ such that the pair $(\partial Y^Z, \partial Z)$ satisfies (5.3.7). Moreover, by Proposition 5.6.5 for any $s \in [0,T]$

$$\partial Y_T^{s,Z} = \partial_s \mathrm{U}^{\mathrm{A}}(s,\bar\xi),\ \mathbb{P}\text{–a.s.}$$

Thus, $\|\partial_s \mathrm{U}_{\mathrm{A}}(\cdot,\bar\xi)\|_{\mathcal{L}^{2,2}}^2 \leq \|\partial Y^{y_0,Z}\|_{\mathbb{S}^{2,2}}^2 < \infty$. This shows $\bar\xi \in \mathcal{C}$.

Let us argue $\bar\xi \in \mathcal{C}_o$, *i.e.* that any $\alpha^\star \in \mathcal{E}(\bar\xi)$ provide the agent the same value. In light of Theorem 5.5.4 and Theorem 5.5.5, it suffices to establish (5.3.1) has at most one solution. In light of Assumption Q, $\bar\xi \in \mathcal{C}$,

$$\|\tilde h\|_{\mathbb{L}^{1,2,2}}^2 = \sup_{s\in[0,T]} \mathbb{E}\bigg[\bigg(\int_0^T |h_r^\star(s, X_{\cdot\wedge r}, 0,0,0,0)|\mathrm{d}r\bigg)^2\bigg] < \infty,$$

$$\|\nabla\tilde h\|_{\mathbb{L}^{1,2,2}}^2 = \sup_{s\in[0,T]} \mathbb{E}\bigg[\bigg(\int_0^T |\nabla h_r^\star(s, X_{\cdot\wedge r}, 0,0,0,0,0,0)|\mathrm{d}r\bigg)^2\bigg] < \infty,$$

and Theorem 3.3.5, (5.3.1) is well-posed with solution

$$((Y_t^{t,y_0,Z})_{t\in[0,T]}, (Z_t^t)_{t\in[0,T]}, Y^{y_0,Z}, Z, \partial Y^{y_0,Z}, \partial Z).$$

Thus

$$\mathcal{E}(\bar{\xi}) = \big\{a^\star(t, X_{\cdot\wedge t}, Y_t^{t,y_0,Z}, Z_t^t)_{t\in[0,T]}\big\}.$$

To conclude $\bar{\xi} \in \Xi$, note that by Theorem 5.5.5, $\mathrm{V}_0^{\mathrm{A}}(\bar{\xi}) = y_0^0$, so that $y_0^0 \geq R_0$ guarantees the participation constraint is satisfied. $\square$

The problem of the principal involves controlling, via $Z \in \mathcal{H}^{2,2}$, of the processes (X, Y^Z). The dynamics of X are given, in weak formulation, by

$$X_t = x_0 + \int_0^t \sigma_r(X_{\cdot\wedge r})\Big(b_r^\star(X_{\cdot\wedge r}, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r + \mathrm{d}B_r^\star\Big),\ t\in[0,T],\ \mathbb{P}\text{–a.s.} \tag{5.3.10}$$

where $B^\star := B - \int_0^\cdot b_r^\star(X_{\cdot\wedge r}, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r$ is a $\mathbb{P}^\star(Z)$–Brownian motion. The second state variable, $(Y_t^{t,y_0,Z})_{t\in[0,T]}$, is the first component of the solution to the system

$$\begin{aligned}
Y_t^{t,y_0,Z} &= y_0^0 - \int_0^t \Big(H_r(X_{\cdot\wedge r}, Y_r^{r,y_0,Z}, Z_r^r) - \partial Y_r^{r,y_0,Z}\Big)\mathrm{d}r + \int_0^t Z_r^r \cdot \mathrm{d}X_r,\\
Y_t^{s,y_0,Z} &= y_0^s - \int_0^t h_r^\star(s, X_{\cdot\wedge r}, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r + \int_0^t Z_r^s \cdot \mathrm{d}X_r,\\
\partial Y_t^{s,y_0,Z} &= \partial y_0^s - \int_0^t \nabla h_r^\star(s, X_{\cdot\wedge r}, \partial Y_r^{s,y_0,Z}, \partial Z_r^s, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r + \int_0^t \partial Z_r^s \cdot \mathrm{d}X_r.
\end{aligned}$$

In the general situation, it is unclear—at least to us—how to address this class of problems. In fact, the previous system provides two possible representations for the term $Y_T^{T,y_0,Z}$ appearing in the reward of the principal, each coming with its own challenges:

(i) If we choose to focus on the dynamics of $Y_t^{t,y_0,Z}$ as given by the first equation, one can exploit the fact that the action of the control Z is only through the diagonal $(Z_t^t)_{t\in[0,T]}$. Moreover, given $(\partial Y_t^{t,y_0,Z})_{t\in[0,T]}$, the first equation implies $(Y_t^{t,y_0,Z})_{t\in[0,T]}$ is an Itô process. Nevertheless, as $(\partial Y_t^{t,y_0,Z})_{t\in[0,T]}$ is not given, there is no direct access to the dynamics of $(\partial Y_t^{t,Z})_{t\in[0,T]}$ that is amenable to the analysis, *i.e.* that would allow us to use Itô calculus. Moreover, this process is neither Markov nor a semi-martingale and does not satisfy a flow property

$$\partial Y_s^{s,y_0,Z} \neq \partial Y_t^{t,y_0,Z} - \int_t^s \nabla h_r^\star(s, X_{\cdot\wedge r}, \partial Y_r^{s,y_0,Z}, \partial Z_r^s, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r + \int_t^s Z_r^s \cdot \mathrm{d}X_r$$

(ii) Alternatively, suppose we choose the representation through the second equation. In that case, the problem of the principal involves the control of the family of FSVIEs $Y_t^{s,y_0,Z}$, where the controls consist of a constrained family of processes $Z \in \mathcal{H}^{2,2}$ that impacts the dynamics via $((Z_t^t)_{t\in[0,T]}, (Z_t^s)_{t\in[0,T]})$. The fact that the dependence is through both arguments makes the recent approach in Viens and Zhang [243] for controlled FSVIEs inoperable. Moreover, in general, this seems to be quite a challenging problem. Therefore, we will for now concentrate our attention on simpler examples.

As a motivation for our approach in the following examples, we recall that for classic separable utilities, as in Chapter 2, it is known that in the case of exponential discounting with parameter ρ, it holds that $\partial Y = \rho Y$, in which case the second and third equations are redundant. Indeed, the first equation suffices to describe the dynamics of $(Y_t^t)_{t\in[0,T]}$. This motivates the study of $\mathcal{H}^{2,2}$ under particular specifications of utility functions for both the agent and the principal, hoping to be able to

(a) reduce the complexity of the set $\mathcal{H}^{2,2}$;

(b) exploit its particular structure to formulate an ansatz to the problem of the principal.

This is exactly what we do in the following sections. The general case remains subject of further research.

5.4 The second-best problem: examples

5.4.1 Agent with discounted utility reward

As an initial example, let us consider the scenario in Section 5.2.1 under the additional choice $g = 1$, which implies $K_{t,T}^{s,\alpha}$ does not depend on $s \in [0,T]$. Thus, for any $(t, \alpha, \xi) \in [0,T] \times \mathcal{A} \times \Xi$, we have

$$\mathrm{V}_t^{\mathrm{A}}(\xi,\alpha) = \mathbb{E}^{\mathbb{P}^\alpha}\Big[f(T-t)\mathrm{U}_{\mathrm{A}}^o(\xi - K_{t,T}^\alpha)\Big|\mathcal{F}_t\Big],\ K_{t,T}^\alpha := \int_t^T k_r^o(X_{\cdot\wedge r},\alpha_r)\mathrm{d}r. \tag{5.4.1}$$

Under this specification, (5.3.1) reduces significantly. Indeed

$$h_t(s,x,y,z,a) = \sigma_t(x)b_t(x,a)\cdot z + \gamma_{\mathrm{A}} k_t^o(x,a)y,$$

$$\nabla h_t(s,x,u,v,y,z,a) = \sigma_t(x)b_t(x,a)\cdot v + \gamma_{\mathrm{A}} k_t^o(x,a)u,$$

$$\mathrm{U}_{\mathrm{A}}^o(s,\mathrm{x}) = f(T-s)\mathrm{U}_{\mathrm{A}}^o(\mathrm{x}),\ \partial_s\mathrm{U}_{\mathrm{A}}^o(s,\mathrm{x}) = -f'(T-s)\mathrm{U}_{\mathrm{A}}^o(\mathrm{x}).$$

Remark 5.4.1. (i) *We highlight that the absence of accumulative cost in the agent's reward functional,* i.e. $c=0$*, together with the choice* $g=1$ *makes the driver in the second family of* BSDEs *independent of the variable* s*,* i.e. $\nabla h = h$*. Moreover, it coincides with the functional maximised in the Hamiltonian* H*.*

(ii) *We remark that in this scenario, the non-exponential discount factor,* i.e. *the time-inconsistent preferences, does not add much to the problem. Even though the agent's continuation utility changes by a factor, the optimal/equilibrium control state pair coincides for both problems. Our aim in presenting it is to illustrate how the technique presented in* Section 5.3.3 *is compatible with the results known in the case of a time-consistent agent.*

The next result provides a drastic simplification of the infinite dimensional system introduced in Section 5.3.2. This is due to the particular form of the reward of the agent (5.4.1).

Lemma 5.4.2. (i) *Let* $\xi\in\Xi$ *and the agent's reward be given by* (5.4.1)*. Then,* (5.3.1) *is equivalent to the* BSDE

$$Y_t(\xi) = \mathrm{U}_{\mathrm{A}}^o(\xi) + \int_t^T \Big(H_r(X_{\cdot\wedge r},Y_r(\xi),Z_r(\xi)) + \frac{f'(T-r)}{f(T-r)}Y_r(\xi)\Big)\mathrm{d}r - \int_t^T Z_r(\xi)\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

(ii) *Let* $Z\in\mathcal{H}^{2,2}$*. Then* $(Y_t^{t,y_0,Z})_{t\in[0,T]}$ *solves the* BSDE

$$Y_t^{t,y_0,Z} = Y_T^{T,y_0,Z} + \int_t^T \Big(H_r(X_{\cdot\wedge r},Y_r^{r,y_0,Z},Z_r^r) + \frac{f'(T-r)}{f(T-r)}Y_r^{r,y_0,Z}\Big)\mathrm{d}r - \int_t^T Z_r^r\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

(iii) $\mathcal{H}^{2,2} = \mathcal{H}^2$*, where* $\mathcal{H}^2$ *denotes the family of* $Z \in \mathbb{H}^2$ *satisfying* $\|Y^{y_0,Z}\|_{\mathbb{S}^2} < \infty$ *where, for any* $y_0 \in (R_0, \infty)$*,*

$$Y_t^{y_0,Z} = y_0 - \int_0^t \bigg(H_r(X_{\cdot\wedge r}, Y_r^{y_0,Z}, Z_r) + \frac{f'(T-r)}{f(T-r)} Y_r^{y_0,Z} \bigg) \mathrm{d}r + \int_0^t Z_r \cdot \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.}$$

(iv) $\overline{\Xi} = \{\mathrm{U}_{\mathrm{A}}^{o\,(-1)}(Y_T^{y_0,Z}) : (y_0, Z) \in (R_0,\infty) \times \mathcal{H}^2\}$*. Moreover, for any* $\xi \in \overline{\Xi}$

$$\mathcal{E}(\xi) = \{(a^\star(t, X_{\cdot\wedge t}, Y_t^{y_0,Z}, Z_t)_{t\in[0,T]}\},\ \mathrm{V}_0^{\mathrm{A}}(\xi) = y_0.$$

Proof. It is immediate from (5.3.1) that, $\mathbb{P}$–a.s.

$$\begin{aligned}
Y_t^s(\xi) &= \mathbb{E}^{\mathbb{P}^\star(\xi)}\Big[f(T-s)\mathrm{U}_{\mathrm{A}}^o\Big(\xi - K_{t,T}^{a^\star}\Big)\Big|\mathcal{F}_t\Big],\\
\partial Y_t^s(\xi) &= -\mathbb{E}^{\mathbb{P}^\star(\xi)}\Big[f'(T-s)\mathrm{U}_{\mathrm{A}}^o\Big(\xi - K_{t,T}^{a^\star}\Big)\Big|\mathcal{F}_t\Big],\ \text{and,}\\
Y_t^t(\xi) &= Y_t(\xi).
\end{aligned}$$

Thus

$$Y_t^s(\xi) = \frac{f(T-s)}{f(T-t)} Y_t(\xi),\ \mathbb{P}\text{–a.s.},\ s \in [0,T],\ \partial Y_t^t(\xi) = -\frac{f'(T-t)}{f(T-t)} Y_t^t(\xi) = -\frac{f'(T-t)}{f(T-t)} Y_t(\xi),\ \mathbb{P}\text{–a.s.},$$

and, for any $(s,u) \in [0,T]^2$

$$\mathrm{U}_{\mathrm{A}}^{(-1)}(s, Y_T^s(\xi)) = \mathrm{U}_{\mathrm{A}}^{o\,(-1)}\bigg(\frac{Y_T^s(\xi)}{f(T-s)}\bigg) = \xi = \mathrm{U}_{\mathrm{A}}^{o\,(-1)}\bigg(\frac{Y_T^u(\xi)}{f(T-u)}\bigg) = \mathrm{U}_{\mathrm{A}}^{(-1)}(u, Y_T^u(\xi)),\ \mathbb{P}\text{–a.s.}$$

All together, this shows that (5.3.1) reduces to the equation in the statement. The result then follows as we can trace back the argument and construct a solution to (5.3.1) starting from a solution to the BSDE in the statement.

We now argue *(ii)*. Let $Z \in \mathcal{H}^{2,2}$. Then, there is $(y_0^s)_{s\in[0,T]}$ such that (5.3.6) holds and

$$Y_t^{s,y_0,Z} = y_0^s - \int_0^t h_r^\star(s, X_{\cdot\wedge r}, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r + \int_0^t Z_r^s \cdot \mathrm{d}X_r.$$

Let us note that (5.3.6) implies $Y_T^{s,y_0,Z} = f(T-s)Y_T^{T,y_0,Z}$. Since

$$h_t(s,x,y,z,a) = h_t(u,x,y,z,a), (s,u) \in [0,T]^2,$$

we obtain

$$Y_t^{s,y_0,Z} = f(T-s)Y_T^{T,y_0,Z} + \int_t^T h_r^\star(r, X_{\cdot\wedge r}, Y_r^{s,y_0,Z}, Z_r^s, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r - \int_t^T Z_r^s \cdot \mathrm{d}X_r,$$

so that

$$\begin{aligned} Y_t^{s,y_0,Z} &= \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[f(T-s)Y_T^{T,y_0,Z}\exp\bigg(\gamma_{\mathrm{A}}\int_t^T k_r^{o\star}(X_{\cdot\wedge r}, Y_r^{r,y_0,Z}, Z_r^r)\mathrm{d}r\bigg)\bigg|\mathcal{F}_t\bigg], \\ \partial Y_t^{t,y_0,Z} &= -\frac{f'(T-t)}{f(T-t)}Y_t^{t,y_0,Z}. \end{aligned}$$

Note that $(Y_t^{t,y_0,Z})_{t\in[0,T]} \in \mathbb{S}^2$. Thanks to Theorem 5.3.7, the result follows replacing $\partial Y_t^{t,y_0,Z}$ in the first equation of (5.3.1).

We are left to argue (iii) as (iv) is argued as in Theorem 5.3.7. $\mathcal{H}^{2,2} \subseteq \mathcal{H}^2$ follows by (ii). Indeed, there is y_0^0 such that

$$Y_t^{t,y_0,Z} = y_0^0 - \int_0^t \bigg(H_r(X_{\cdot\wedge r}, Y_r^{r,y_0,Z}, Z_r^r) + \frac{f'(T-r)}{f(T-r)}Y_r^{r,y_0,Z}\bigg)\mathrm{d}r + \int_0^t Z_r^r \cdot \mathrm{d}X_r.$$

Conversely, let $(y_0, Z) \in (R_0, \infty) \times \mathcal{H}^2$ and $Y^{y_0,Z} \in \mathbb{S}^2$ as in the statement. Then, letting

$$Y_t^{s,y_0,Z} := \frac{f(T-s)}{f(T-t)}Y_t^{y_0,Z} = \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[f(T-s)Y_T^{y_0,Z}\exp\bigg(\gamma_{\mathrm{A}}\int_t^T k_r^{o\star}(X_{\cdot\wedge r}, Y_r^{y_0,Z}, Z_r)\mathrm{d}r\bigg)\bigg|\mathcal{F}_t\bigg],$$

$$\partial Y_t^{s,y_0,Z} := -\frac{f'(T-s)}{f(T-t)}Y_t^{y_0,Z},$$

the martingale representation theorem, which holds in light of (S) and the integrability of $(Y^{y_0,Z}, Z)$, guarantees the existence of $(\tilde{Z}, \partial\tilde{Z}) \in \overline{\mathbb{H}}^{2,2} \times \mathbb{H}^{2,2}$ such that, as elements of $\mathbb{H}^2$,

$$\tilde{Z}^s = \tilde{Z}^0 + \int_0^s \partial\tilde{Z}^r\mathrm{d}r, \text{ and } \tilde{Z}_t^t = Z_t, \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e.}$$

It then follows that $(\tilde{Z}^s)_{s\in[0,T]} \in \mathcal{H}^{2,2}$. $\square$

5.4.1.1 Principal's second-best solution

In the following, we will exploit the so-called certainty equivalent, *i.e.* the relation $\xi = \mathrm{U}_{\mathrm{A}}^{o\,(-1)}(V_T^{\mathrm{A}}(\xi,\alpha))$ between the the contract and the terminal value of the value function. The benefits of this are twofold: it lays down an expression that can be replaced directly into the principal's criterion, and it removes $Y^{y_0,Z}$ from the generator of the expression representing the contract in exchange for a term which is quadratic in Z. For this we need to introduce some extra notation.

Let $\widehat{H} : [0,T] \times \mathcal{X} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by

$$\widehat{H}_t(x,\mathrm{z}) := \sup_{a\in A} \widehat{h}_t(x,\mathrm{z},a),\ (t,x,\mathrm{z}) \in [0,T] \times \mathcal{X} \times \mathbb{R}^n,$$

with $\widehat{h}_t(x,\mathrm{z},a) := \sigma_t(x)b_t(x,a)\cdot \mathrm{z} - k_t^o(x,a)$. The mapping $[0,T] \times \mathcal{X} \times \mathbb{R}^n \longmapsto \hat{a}^\star(t,x,z) \in A$ is defined, as before, by the relation $\widehat{H}_t(x,\mathrm{z}) = \widehat{h}_t(x,\mathrm{z},\hat{a}^\star(t,x,\mathrm{z}))$, and $\lambda_t^\star(x,\mathrm{z})$, $k_t^{o\star}(x,\mathrm{z})$ are also defined.

Proposition 5.4.3. *The problem of the principal can be represented as the following standard control problem*

$$\mathrm{V}^{\mathrm{P}} = \sup_{y_0\geq R_0} \underline{V}(y_0),\ \text{with}\ \underline{V}(y_0) = \sup_{Z\in\mathcal{H}^2} \mathbb{E}^{\mathbb{P}^\star(Z)}\Big[\mathrm{U}_{\mathrm{P}}\Big(X_{\cdot\wedge T}, \widehat{Y}_T^{y_0,Z}\Big)\Big],$$

where $\mathbb{P}^\star(Z) := \mathbb{P}^{a^\star(\cdot,X_\cdot,\hat{Z}_\cdot)}$ *and* $\widehat{Y}_T^{y_0,Z}$ *is given by the terminal value of*

$$\widehat{Y}_t^{y_0,Z} := -\frac{1}{\gamma_{\mathrm{A}}}\ln\big(-\gamma_{\mathrm{A}}y_0\big) - \int_0^t \bigg(\widehat{H}_r(X_{\cdot\wedge r},\widehat{Z}_r) - \frac{\gamma_{\mathrm{A}}}{2}|\sigma_r^\top(X_{\cdot\wedge r})\widehat{Z}_r|^2 - \frac{1}{\gamma_{\mathrm{A}}}\frac{f'(T-r)}{f(T-r)}\bigg)\mathrm{d}r + \int_0^t \widehat{Z}_r\cdot \mathrm{d}X_r,$$

where,

$$\widehat{Z}_t := -\frac{1}{\gamma_{\mathrm{A}}}\frac{Z_t}{Y_t^{y_0,Z}}.$$

Proof. We first note that in light Lemma 5.4.2, we may replace the optimisation over $\mathcal{H}^{2,2}$ with $\mathcal{H}^2$. Let $Z \in \mathcal{H}^2$. The result then follows from Lemma 5.4.2 by applying Itô's formula to $\mathrm{U}_{\mathrm{A}}^{(-1)}(Y_t^Z)$. □

Remark 5.4.4. (i) *Let us highlight the main message behind* Proposition 5.4.3. *When the agent's reward is given by* (5.4.1)*, the principal's second-best problem reduces to a standard control problem.*

This is a drastic simplification of the result in Theorem 5.3.7 *and a consequence of the particular form of the agent's reward.*

(ii) In a Markovian setting in which the dependence of the data on the path X is via the current value, we see from the controlled dynamics for X and $\widehat{Y}^{y_0,Z}$ that the problem boilds down to solving $\underline{V}$. Employing the standard dynamic programming approach we obtain that the relevant term for this problem is given for $(t,\mathrm{x},\mathrm{y}) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}$ by

$$\partial_t \underline{V}(t,\mathrm{x},\mathrm{y}) + \mathrm{H}\big(t,\mathrm{x},\mathrm{y},\partial \underline{V}(t,\mathrm{x},\mathrm{y}),\partial^2 \underline{V}(t,\mathrm{x},\mathrm{y})\big) = 0,$$

where

$$\begin{aligned}\mathrm{H}(t,\mathrm{x},\mathrm{y},p,M) := \sup_{z\in\mathbb{R}^n} \bigg\{ & \lambda_t^\star(x,z)\cdot p_{\mathrm{x}} + \bigg(\frac{\gamma_{\mathrm{A}}}{2}|\sigma_r^\top(\mathrm{x})z|^2 - \widehat{H}_r(\mathrm{x},z) + \frac{1}{\gamma_{\mathrm{A}}}\frac{f'(T-t)}{f(T-t)}\bigg)p_{\mathrm{y}} \\ & + \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma_t^\top(\mathrm{x})(M_{\mathrm{xx}} + zz^\top M_{\mathrm{yy}} + 2z\cdot M_{\mathrm{xy}})\big]\bigg\},\end{aligned}$$

for $p := \begin{pmatrix} p_{\mathrm{x}} \\ p_{\mathrm{y}} \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}$, $M := \begin{pmatrix} M_{\mathrm{xx}} & M_{\mathrm{xy}} \\ M_{\mathrm{xy}} & M_{\mathrm{yy}} \end{pmatrix} \in \mathbb{S}^+_{n+1}(\mathbb{R})$, $M_{\mathrm{xx}} \in \mathbb{S}^+_n(\mathbb{R})$, $M_{\mathrm{yy}} \in \mathbb{S}^+_1(\mathbb{R})$, *and* $M_{\mathrm{xy}} \in \mathcal{M}_{n,1}(\mathbb{R})$.

In the following proposition, whose proof is available in Section 5.7, we study the case $n=1$, so that

$$\underline{V}(y_0) = \sup_{Z\in\mathcal{H}^2} \mathbb{E}^{\mathbb{P}^\star(Z)}\Big[\mathrm{U_P}\Big(X_T - \widehat{Y}_T^{y_0,Z}\Big)\Big)\Big].$$

This result is equivalent to solving the HJB equation in Remark 5.4.4.

Proposition 5.4.5. *Let principal and agent have exponential utility with parameters γ_{P} and γ_{A}, respectively. Let $C_y := -\frac{1}{\gamma_{\mathrm{P}}}\mathrm{e}^{-\gamma_{\mathrm{P}}(x_0-y)}$, $\widehat{R}_0 := \mathrm{U}_{\mathrm{A}}^{\mathrm{o}\,(-1)}(R_0)$, and assume that*

(i) the maps σ, $\lambda^\star$ and $k^{o\star}$ do not depend on the x variable;

(ii) for any $t\in[0,T]$, the map $\mathbb{R}\ni z \overset{g}{\longmapsto} \lambda_t^\star(z) - k_t^{o\star}(z) - \frac{\gamma_{\mathrm{A}}}{2}|\sigma_t^\top z|^2 - \frac{\gamma_{\mathrm{P}}}{2}|\sigma_t^\top(1-z)|^2$ has a unique maximiser $z^\star(t)$, such that $[0,T]\ni t \longmapsto z^\star(t)$ is square integrable.

Then

$$\xi^\star := \mathrm{U}_{\mathrm{A}}^{(-1)}\bigg(\frac{R_0}{f(T)}\bigg) - \int_0^T \bigg(\widehat{H}_r(z_r^\star) - \frac{\gamma_{\mathrm{A}}}{2}|\sigma_r^\top z_r^\star|^2\bigg)\mathrm{d}r + \int_0^T z_r^\star \mathrm{d}X_r,$$

is an optimal solution to principal's second-best problem and

$$\mathrm{V}^{\mathrm{P}} = C_{\widehat{R}_0} f(T)^{\frac{\gamma_{\mathrm{P}}}{\gamma_{\mathrm{A}}}} \exp\bigg(- \gamma_{\mathrm{P}} \int_0^T g(z^\star(t))\mathrm{d}t \bigg).$$

Remark 5.4.6. *To close this section we present a few remarks:*

(i) *Comparing the results in* Proposition 5.4.5 *and* Proposition 5.2.2 *we see that, as expected, in general the solution to the second-best and first-best problem are not equal.*

(ii) *If we bring ourselves back to the setting of* [130], i.e. $b_t(x,a) = a/\sigma$, $\sigma_t(x) = \sigma$, $k_t^o(x,a) = ka^2/2$, *we have*

$$z^\star(t) = \frac{1+\sigma^2\gamma_{\mathrm{P}}k}{1+\sigma^2k(\gamma_{\mathrm{A}}+\gamma_{\mathrm{P}})},\ a^\star(t) = \frac{1+\sigma^2\gamma_{\mathrm{P}}k}{c(1+\sigma^2k(\gamma_{\mathrm{A}}+\gamma_{\mathrm{P}}))}.$$

This recovers the result for the case of a risk-neutral principal, i.e. $\gamma_{\mathrm{P}} = 0$, *presented in* [130]*. The optimal contract and the respective rewards differ by a factor which depends on the discount factor and agent's risk aversion parameter.*

(iii) *Following up on the previous comment, we add that the optimal contract takes the form of a Markovian rule. Moreover, it is linear. This is consistent with the seminal work of* [130] *and the conclusion of* [45] *in which the robustness of these policies was studied. Nevertheless, as we will see in* Section 5.4.3*, this appears to be a consequence of the simplicity of the source of time-inconsistency considered in this section.*

5.4.2 Agent with separable utility

We consider the scenario in Section 5.2.2, *i.e.*

$$\mathrm{V}_t^{\mathrm{A}}(\xi,\alpha) = \mathbb{E}^{\mathbb{P}^\alpha}\bigg[f(T-t)\mathrm{U}_{\mathrm{A}}^o(\xi) - \int_t^T f(s-t)c_s(X_{\cdot\wedge s},\alpha_s)\mathrm{d}s \bigg| \mathcal{F}_t \bigg],\ (t,\alpha,\xi)\in[0,T]\times\mathcal{A}\times\Xi,$$

and we have $\mathrm{U}_{\mathrm{A}}(s,\mathrm{x}) = f(T-s)\mathrm{U}_{\mathrm{A}}^o(\mathrm{x})$, $\partial_s\mathrm{U}_{\mathrm{A}}(s,\mathrm{x}) = -f'(T-s)\mathrm{U}_{\mathrm{A}}^o(\mathrm{x})$,

$$h_t(s,x,z,a) = \sigma_t(x)b_t(x,a)\cdot z - f(t-s)c_t(x,a),\ \nabla h_t(s,x,v,a) = \sigma_t(x)b_t(x,a)\cdot v + f'(t-s)c_t(x,a).$$

The mappings $H_t(x,\mathrm{z})$, $a^\star(t,x,\mathrm{z})$, $\lambda_t^\star(x,\mathrm{z})$, $c_t^\star(x,\mathrm{z})$, and the probability $\mathbb{P}^\star(Z)=\mathbb{P}^{a^\star(\cdot,X_\cdot,Z_\cdot^\cdot)}$ are obtained accordingly.

In this section, we are trying to get a deeper understanding of the family $\mathcal{H}^{2,2}$ under the previous specification of preferences for the agent. In particular, we want to understand how the elements of the family $(Y^{s,Z},Z^s)_{s\in[0,T]}$ are related to each other. In light of Assumption Q and Assumption R, for any $Z\in\mathcal{H}^{2,2}$ we denote $M^{s,Z}$ the $(\mathbb{F},\mathbb{P}^\star(Z))$-square integrable martingale

$$M_t^{s,Z}:=\mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\int_0^T\delta_r^\star(s,X_{\cdot\wedge r},Z_r^r)\mathrm{d}r\bigg|\mathcal{F}_t\bigg],\ t\in[0,T],$$

where

$$\delta_r^\star(s,x,\mathrm{z}):=c_r^\star(x,\mathrm{z})\bigg(f(r-s)-\frac{f(T-s)}{f(T)}f(r)\bigg),\ (s,t,x,\mathrm{z})\in[0,T]^2\times\mathcal{X}\times\mathbb{R}^n.$$

We also recall that $\mathbb{P}^\star(Z)$ is the unique solution to the martingale problem for which X has characteristic triplet $(\lambda^\star,\sigma\sigma^\top,0)$. Thus, the representation property holds for $(\mathbb{F},\mathbb{P}^\star(Z))$-martingales (see [142, Theorem III.4.29]) and we can introduce the unique $\mathbb{F}$-predictable process $\widetilde{Z}^{s,Z}$ such that $\sup_{s\in[0,T]}\mathbb{E}^{\mathbb{P}^\star(Z)}\Big[\int_0^T|\sigma_r\sigma_r^\top\widetilde{Z}_r^{s,Z}|^2\mathrm{d}r\Big]<\infty$,[1] and, in light of (5.3.10),

$$M_t^{s,Z}=M_0^{s,Z}+\int_0^t\widetilde{Z}_r^{s,Z}\cdot\big(\mathrm{d}X_r-\lambda_r^\star(X_{\cdot\wedge r},Z_r^r)\mathrm{d}r\big),\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

The next lemma, proved in Section 5.7, presents relationships satisfied by the elements of family, $(Y^{s,Z},Z^s)_{s\in[0,T]}$, and how we can use them to obtain another characterisation of $\mathcal{H}^{2,2}$ and Ξ.

Lemma 5.4.7. *(i) Let $Z\in\mathcal{H}^{2,2}$, for any $s\in[0,T]$*

$$Y_t^{s,Z}=\frac{f(T-s)}{f(T)}Y_t^{0,Z}-\mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\int_t^T\delta_r^\star(s,X_{\cdot\wedge r},Z_r^r)\mathrm{d}r\bigg|\mathcal{F}_t\bigg],\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

(ii) Let $Z\in\mathcal{H}^{2,2}$, for any $s\in[0,T]$

$$Z_t^s=\frac{f(T-s)}{f(T)}Z_t^0-\widetilde{Z}_t^{s,Z},\ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{–a.e.}$$

[1]The integrability for $s\in[0,T]$ fixed is clear. The sup follows as in Theorem 3.3.5 as $\delta^\star$ is uniformly continuous in s.

(iii) $\mathcal{H}^{2,2} = \mathcal{H}^\bullet$*, where* $\mathcal{H}^\bullet$ *denotes the class of* $Z \in \overline{\mathbb{H}}^{2,2}(\mathbb{R}^n)$ *such that* $\|Y^Z\|_{\mathbb{S}^2} < \infty$*, where for* $y_0 \in (R_0, \infty)$

$$Y_t^{y_0,Z} := \frac{y_0}{f(T)} - \int_0^t f(T)^{-1} h_r^\star(0, X_{\cdot\wedge r}, Z_r^0, Z_r^r)\mathrm{d}r + \int_0^t f(T)^{-1} Z_r^0 \cdot \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.}$$

and

$$Z_t^s = \frac{f(T-s)}{f(T)} Z_t^0 - \widetilde{Z}_t^{s,Z},\ \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e.} \tag{5.4.2}$$

(iv) $\overline{\Xi} = \big\{ \mathrm{U}_{\mathrm{A}}^{o\,(-1)}(Y_T^{y_0,Z}) : (y_0, Z) \in (R_0,\infty) \times \mathcal{H}^\bullet \big\}$*. For any* $\xi \in \overline{\Xi}$*,*

$$\mathcal{E}(\xi) = \big\{ (a^\star(t, X_{\cdot\wedge t}, Z_t^t)_{t\in[0,T]} \big\},\ \mathrm{V}_0^{\mathrm{A}}(\xi) = y_0.$$

Remark 5.4.8. *(i) In the exponential discounting case,* i.e. $f(t) := \mathrm{e}^{-\rho t}$ *for some* $\rho > 0$*, we have*

$$f(r-s) - \frac{f(T-s)f(r-u)}{f(T-u)} = \mathrm{e}^{-\rho(r-s)} - \mathrm{e}^{-\rho(r-s)} = 0.\ (r,s,u) \in [0,T]^3.$$

Thus, $\delta^\star = 0$ *and the result of* Lemma 5.4.7 *simplifies to*

$$Y_t^{s,y_0,Z} = \frac{f(T-s)}{f(T)} Y_t^{0,y_0,Z},\ t \in [0,T],\ \mathbb{P}\text{–a.s.}\ Z_t^s = \frac{f(T-s)}{f(T)} Z_t^0,\ \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e.},\ s \in [0,T].$$

Therefore, this implies that in the non-exponential discounting case, the term

$$\mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\int_t^T \delta_r^\star(s, X_{\cdot\wedge r}, Z_r^r) \bigg| \mathcal{F}_t \bigg],$$

is exactly the correction due to time-inconsistency.

(ii) We also remark that the choice Z^0 *in the constraint for the family* Z *is arbitrary. Indeed, it could be replaced by any other element* Z^u *of the family* $Z \in \mathcal{H}^{2,2}$*.*

5.4.2.1 Principal's second-best solution

Thanks to Lemma 5.4.7, we have now proved that

Proposition 5.4.9. *The problem of the principal can be represented as the following control problem*

$$\mathrm{V}^{\mathrm{P}} = \sup_{y_0 \geq R_0} \underline{V}(y_0), \text{ where } \underline{V}(y_0) = \sup_{Z \in \mathcal{H}^{\bullet}} \mathbb{E}^{\mathbb{P}^{\star}(Z)}\Big[\mathrm{U_P}\Big(X_{\cdot\wedge T}, \mathrm{U_A^{(-1)}}(\mathrm{Y}_T^{y_0,Z})\Big)\Big],$$

where $\mathbb{P}^{\star}(Z) = \mathbb{P}^{a^{\star}(\cdot,X_{\cdot},Z_{\cdot}^{\cdot})}$ *and*

$$\mathrm{Y}_t^{y_0,Z} = \frac{y_0}{f(T)} - \int_0^t f(T)^{-1} h_r^{\star}(0, X_{\cdot\wedge r}, Z_r^0, Z_r^r)\mathrm{d}r + \int_0^t f(T)^{-1} Z_r^0 \cdot \mathrm{d}X_r, \; t \in [0,T], \; \mathbb{P}\text{–a.s.}$$

We remark that contrary to the example in Section 5.4.1, Proposition 5.4.9 reduces the problem of the principal to a non-standard control problem. Indeed, we have to optimise over $\mathcal{H}^{\bullet}$, a family of infinite-dimensional controls which has to satisfy a novel type of constraint, namely (5.4.2). Nonetheless, under additional assumptions on the model, we can proceed with the resolution.

As in Section 5.4.1, we focus on the case $n = 1$ so that

$$\underline{V}(y_0) = \sup_{Z \in \mathcal{H}^{\bullet}} \mathbb{E}^{\mathbb{P}^{\star}(Z)}\Big[\mathrm{U_P^o}\Big(X_T - \mathrm{U_A^o}^{(-1)}(\mathrm{Y}_T^{y_0,Z})\Big)\Big]. \tag{5.4.3}$$

Proposition 5.4.10. *Let $n = 1$, the principal and the agent be risk-neutral,* i.e. $\mathrm{U_A^o}(\mathrm{x}) = \mathrm{U_P^o}(\mathrm{x}) = \mathrm{x}$, $\mathrm{x} \in \mathbb{R}$.

(i) Suppose there is a unique measurable map $z^{\star} : [0,T] \times \mathcal{X} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying

$$\mathrm{H}_t(x,v) = v\lambda_t^{\star}(x, z^{\star}(t,x,v)) + \lambda_t^{\star}(x, z^{\star}(t,x,v)) - \frac{f(r)}{f(T)}c^{\star}(x, z^{\star}(t,x,v)), \text{ for any } (t,x) \in [0,T] \times \mathcal{X},$$

where for any $(t,x,v) \in [0,T] \times \mathcal{X} \times \mathbb{R}^n$

$$\mathrm{H}_t(x,v) := \sup_{z \in \mathbb{R}} \Big\{ v\lambda^{\star}(x,z) + \lambda_t^{\star}(x,z) - \frac{f(r)}{f(T)}c^{\star}(x,z)\Big\},$$

Moreover, assume the mapping $\mathbb{R}^n \ni v \longmapsto \mathrm{H}_t(x,v) \in \mathbb{R}$ is Lipschitz-continuous uniformly in (t,x) with linear growth. Then, $\mathrm{V}^{\mathrm{P}} = x_0 - \frac{R_0}{f(T)} + U_0$ *where the pair (U,V) denotes a solution to the* BSDE

$$U_t = \int_t^T \mathrm{H}_r(X_{\cdot\wedge r}, V_r)\mathrm{d}r - \int_t^T V_r \cdot \mathrm{d}X_r.$$

In addition, let

$$y_0^\star := R_0,\ \mathcal{Z}_t := z^\star(t, X_{\cdot\wedge t}, V_t),\ Z_t^{0,\star} := \frac{f(T)}{f(T-t)}\mathcal{Z}_t,\ \mathbb{P}^\star(\mathcal{Z}) := \mathbb{P}^{a^\star(\cdot, X_\cdot, \mathcal{Z}_\cdot)},$$

and suppose $\mathbb{E}^{\mathbb{P}^\star(\mathcal{Z})}\big[\int_0^T |\sigma_r\sigma_r^\top \mathcal{Z}_r|^2 \mathrm{d}r\big] < \infty$. *Then, there exists* $Z^\star \in \overline{\mathbb{H}}^{2,2}$, *such that* $(y_0^\star, Z^\star) \in [R_0, \infty) \times \mathcal{H}^\bullet$ *define a solution to the second-best problem and the optimal contract is given by*

$$\xi^\star := \frac{R_0}{f(T)} - f(T)^{-1}\int_0^T h_r^\star(0, X_{\cdot\wedge r}, Z_r^{0,\star}, \mathcal{Z}_r)\mathrm{d}r + f(T)^{-1}\int_0^T Z_r^{0,\star}\cdot \mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

(ii) Suppose the maps $\lambda^\star$ *and* $c^\star$ *do not depend on the* x *variable and for any* $t \in [0,T]$, *the map* $\mathbb{R} \ni z \xmapsto{g} \lambda_t^\star(z) - f(t)/f(T)c_t^\star(z)$ *has a unique maximiser* $z^\star(t)$, *such that* $[0,T] \ni t \longmapsto z^\star(t)$ *is Lebesgue integrable.*

Then, a solution $(y_0^\star, Z^\star) \in [R_0, \infty) \times \mathcal{H}^\bullet$ *for the second-best problem is given by*

$$y_0^\star = R_0,\ Z_t^s := \frac{f(T-s)}{f(T-t)}z^\star(t),\ \widetilde{Z}_t^s := 0,\ (s,t)\in[0,T]^2,\ \text{and } \mathrm{V}^\mathrm{P} = x_0 - \frac{R_0}{f(T)} + \int_0^T g_t(z^\star(t))\mathrm{d}t.$$

Moreover, the associated optimal contract is given by

$$\xi^\star := \frac{R_0}{f(T)} - f(T)^{-1}\int_0^T \big(\lambda_t^\star(z^\star(t)) - f(t)c_t^\star(z^\star(t))\big)\mathrm{d}t + \int_0^T \frac{z^\star(t)}{f(T-t)}\mathrm{d}X_t.$$

Proof. Let us show (i). As both agent and principal are risk neutral, we have

$$\underline{\mathrm{V}}(y_0) = \sup_{Z\in\mathcal{H}^\bullet} \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\int_0^T \Big(\lambda_t^\star(X_{\cdot\wedge r}, Z_r^r) - \frac{f(r)}{f(T)}c_r^\star(X_{\cdot\wedge r}, Z_r^r)\Big)\mathrm{d}r\bigg]$$

An upper bound $\underline{\mathrm{V}}(y_0)$ is obtained by ignoring (5.4.2). In such scenario, the mapping H in the statement denotes the Hamiltonian and by classical arguments in control, see El Karoui, Peng, and Quenez [88], its value is given by U_0 where (U, V) are as in the statement. We are left to show this bound is attained. For this we must verify $Z^\star \in \mathcal{H}^\bullet$.

On the one side, note that the integrability of $\mathcal{Z}$ together with Assumption Q guarantee

$$\mathbb{E}^{\mathbb{P}^\star(\mathcal{Z})}[|\xi^\star|^2] < \infty.$$

Therefore, by Theorem 3.3.5 and Theorem 3.4.3, there exists a unique solution $(Y^\star, Z^\star) \in \mathbb{S}^{2,2} \times \overline{\mathbb{H}}^{2,2}$ to the BSVIE with data $(\xi^\star, h^\star)$ given by

$$Y_t^{s,\star} = f(T-s)\xi^\star + \int_t^T h_r^\star(s, X_{\cdot\wedge r}, Z_r^{s,\star}, Z_r^{r,\star})\mathrm{d}r - \int_t^T Z_r^{s,\star} \cdot \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\ s \in [0,T].$$

On the other side, under the integrability assumption on $\mathcal{Z}$ we have that for every $s \in [0,T]$

$$\widehat{Y}_t^s := \mathbb{E}^{\mathbb{P}^\star(\mathcal{Z})}\bigg[f(T-s)\xi^\star - \int_t^T f(r-s)c^\star(X_{\cdot\wedge r}, \mathcal{Z}_r)\mathrm{d}r\bigg|\mathcal{F}_t\bigg]$$

defines a $\mathbb{P}^\star(\mathcal{Z})$-square integrable martingale. Thus, there exists a family of process $(\widehat{Z}^s)_{s\in[0,T]}$ such that

$$\widehat{Y}_t^s = f(T-s)\xi^\star + \int_t^T h_r^\star(s, X_{\cdot\wedge r}, \widehat{Z}_r^s, \mathcal{Z}_r)\mathrm{d}r - \int_t^T \widehat{Z}_r^s \cdot \mathrm{d}X_r\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\ s \in [0,T].$$

Moreover, in light of (Q) we have that $\widehat{Z} \in \overline{\mathbb{H}}^{2,2}$. Therefore, by uniqueness of the solution

$$(Y^\star, Z^\star, (Z_t^{t,\star})_{t\in[0,T]}) = (\widehat{Y}, \widehat{Z}, \mathcal{Z}),\ \text{in } \mathbb{S}^{2,2} \times \overline{\mathbb{H}}^{2,2} \times \mathbb{H}^2$$

From this, arguing as in Lemma 5.4.7 we obtain that $Z^\star$ satisfies (5.4.2).

We now argue (ii). Note that we can find an upper bound for V^{P}. Indeed, we have

$$\begin{aligned}\mathrm{V}^{\mathrm{P}} &= x_0 + \sup_{y_0 \geq R_0} \sup_{Z\in\mathcal{H}^\bullet} \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[-\frac{y_0}{f(T)} + \int_0^T \Big(\lambda_t^\star(Z_t^t) - \frac{f(t)}{f(T)}c_t^\star(Z_t^t)\Big)\mathrm{d}t\bigg] \\ &\leq x_0 - \frac{R_0}{f(T)} + \int_0^T g_t(z^\star(t))\mathrm{d}t =: \mathrm{V}^{\mathrm{P},\star}.\end{aligned}$$

We now show that the pair $(y_0^\star, Z^\star)$ given in the statement is a feasible solution that attains $\mathrm{V}^{\mathrm{P},\star}$. To verify feasibility note that, by assumption, $z^\star(\cdot)$ is deterministic, and so is $Z^\star$. Thus, it is straightforward from the definition that $Z^\star \in \mathcal{H}^\bullet$. Lastly, it follows by definition that under $(y_0^\star, Z^\star)$ the upper bound $\mathrm{V}^{\mathrm{P},\star}$ is attained. □

Remark 5.4.11. (i) *Let us now present a formal argument regarding our choice* $\mathrm{U}_{\mathrm{A}}^o(\mathrm{x}) = \mathrm{U}_{\mathrm{P}}^o(\mathrm{x}) = \mathrm{x}$ *in the previous result for solving* (5.4.3). *Suppose for simplicity the maps* σ, $\lambda^\star$ *and* $c^\star$ *do not* *depend on the x variable so that the dynamics of the state variables are given by*

$$X_t = x_0 + \int_0^t \lambda^\star(Z_r)\mathrm{d}r + \int_0^t \sigma_r \cdot \mathrm{d}B_r^\star, \quad Y_t^{y_0,Z} = \frac{y_0}{f(T)} - \int_0^t \frac{f(r)}{f(T)} c_r^\star(Z_r)\mathrm{d}r + \int_0^t \frac{\sigma_r}{f(T)} Z_r^0 \cdot \mathrm{d}B_r^\star.$$

Moreover, suppose the value function $v(t,\mathrm{x},\mathrm{y})$ *is regular enough so that Itô's formula yields,* $\mathbb{P}^\star(Z)$–a.s.

$$v(T, X_T, Y_T^{y_0,Z}) - v(t, X_t, Y_t^{y_0,Z}) + \int_t^T \Big(\sigma_r \partial_x v + \frac{\sigma_r}{f(T)} Z_r^0 \partial_y v\Big)(r, X_r, Y_r^{y_0,Z})\mathrm{d}B_r^\star$$

$$= \int_t^T \Big(\partial_t v - \lambda_r^\star(Z_r^r)\partial_x v - \frac{f(t)}{f(T)} c_r^\star(Z_r^r)\partial_y v + \frac{\sigma_r^2}{2}\partial_{xx} v + \frac{\sigma_r^2}{f(T)} Z_r^0 \partial_{xy} v + \frac{\sigma_r^2}{2f(T)^2}|Z_r^0|^2 \partial_{yy} v\Big)(r, X_r, Y_r^{y_0,Z})\mathrm{d}r,$$

And let us highlight the presence of both Z_t^t *and* Z_t^0 *in the last term. From this we can see, formally, that for general* $\mathrm{U}_\mathrm{A}^\mathrm{o}$ *and* $\mathrm{U}_\mathrm{P}^\mathrm{o}$ *the process* $(Z_t^t)_{t\in[0,T]}$ *alone is not sufficient to obtain the solution of* (5.4.3)*. Moreover, recall we can not take* Z_t^t *and* Z_t^0 *independently due to the constraint* (5.4.2)*. Lastly, under the assumptions of* Proposition 5.4.10 *one expects, intuitively, that* $\partial_{xx} v = \partial_{xy} v = \partial_{yy} v = 0$ *so that the choice* Z^0 *can be made after optimising over* $(Z_t^t)_{t\in[0,T]}$.

Remark 5.4.12. *We close this section with a few remarks.*

(*i*) *It is worth mentioning that even in the setting of* Proposition 5.2.3.(*ii*) *the optimal contract is neither linear nor Markovian. Moreover, from the expression describing the optimal contract we see that this is entirely related to the presence of the discounting structure which is the source of time-inconsistency.*

(*ii*) *It follows from* Proposition 5.2.3 *that for risk-neutral preferences, the utility of the principal is the same for both the first-best and second-best problem and that the optimal second-best contract is also optimal there. This is a typical result for time-consistent risk-neutral agents, and it would certainly be worth studying whether this remains true for more general specifications of* $\mathrm{U}_\mathrm{P}^\mathrm{o}$ *and* $\mathrm{U}_\mathrm{A}^\mathrm{o}$*. In light of* Remark 5.4.11*, this question further motivates the study of the general class of non-standard control problems introduced by* Theorem 5.3.7.

5.4.3 Agent with utility of discounted income

We now consider the scenario in Section 5.2.1 under the additional choice $f = 1$. We then have

$$\mathrm{V}_t^{\mathrm{A}}(\xi,\alpha) := \mathbb{E}^{\mathbb{P}^\alpha}\Big[\mathrm{U}_{\mathrm{A}}^o\Big(g(T-t)\xi - K_{t,T}^{t,\alpha}\Big)\Big|\mathcal{F}_t\Big], \text{ where } K_{t,T}^{s,\alpha} := \int_t^T g(r-s)k_r^o(X,\alpha_r)\mathrm{d}r. \tag{5.4.4}$$

In the context of (5.3.1), this corresponds to

$$h_t(s,x,y,z,a) = \sigma_t(x)b_t(x,a)\cdot z + \gamma_{\mathrm{A}} g(t-s)k_t^o(x,a)y,$$
$$\nabla h_t(s,x,u,v,y,a) = \sigma_t(x)b_t(x,a)\cdot v - \gamma_{\mathrm{A}} g'(t-s)k_t^o(x,a)y + \gamma_{\mathrm{A}} g(t-s)k_t^o(x,a)u,$$

$\mathrm{U}_{\mathrm{A}}(s,\xi) = \mathrm{U}_{\mathrm{A}}^o(g(T-s)\xi)$, and $\partial_s \mathrm{U}_{\mathrm{A}}(s,x) = -g'(T-s)\partial_{\mathrm{x}}\mathrm{U}_{\mathrm{A}}^o(g(T-s)\xi)\mathrm{U}_{\mathrm{A}}^o(g(T-s)\xi)$.

Remark 5.4.13. (*i*) *The problem introduce by* (5.4.4) *is time-inconsistent even in the case of exponential discounting,* i.e. $g(t) = \mathrm{e}^{-\rho t}$, $t \in [0,T]$, *for some* $\rho > 0$. *This is due to the exponential utility* U_{A}. *Indeed, the* BSDE *representation allows us to interpret the reward of the agent as a recursive utility in which the terminal value is discounted at a rate* $e^{g(T-s)}$ *whereas the generator discounts at a rate* $g(t-s)$. *It is known, see* Marín-Solano and Navas [179, Section 4.5]*, that even in the case of exponential discounting the problem becomes time-inconsistent as soon as the rates at which the terminal value and the running reward are discounted differ. We also recall that the case of no discounting,* i.e. $g(t) = 1$, *corresponds to the seminal work* Holmström and Milgrom [130].

(*ii*) *Let us note* h *exhibits both of the features of the examples in* Sections 5.4.1 *and* 5.4.2*, this is, the second term includes the discount factor and the* y *variable.*[2] *We highlight that a key element in* Proposition 5.4.10 *was the fact that the dynamics of* $Y^{s,y_0,Z}$ *were given by* (y_0, Z) *without* $Y^{y_0,Z}$ *on the right hand side. Consequently, the presence of* y *in* h *forces us to begin by changing variables to the certainty equivalent for the problem of the agent,* i.e. *from* Z *to* $\widehat{Z}$ *as we denote below. In this way, we remove* y *in the dynamics of* $Y^{y_0,Z}$ *at the expense of the mapping* $\delta^\star$*, which we use to identify an auxiliary martingale, becoming quadratic in the new variable* $\widehat{Z}$. *On the one hand, this creates a subtle issue when trying to establish a correspondence between the natural integrability of the variables* Z *and* $\widehat{Z}$. *This will ultimately prevent us from obtaining a complete characterisation*

[2] In fact, (5.4.4) covers the situation in Section 5.4.2 in the particular case of a risk-neutral agent, recall $1/\gamma_{\mathrm{A}} - \mathrm{U}_{\mathrm{A}}(\mathrm{x}) \longrightarrow \mathrm{x}$, whenever $\gamma_{\mathrm{A}} \to 0$.

of the family $\mathcal{H}^{2,2}$. On the other hand, the quadratic term does not correspond to the diagonal values of the control variable $\hat{Z}$. This makes the approach in Section 5.4.2, *namely* Proposition 5.4.10, *inoperable and forces us to restrict ourselves to a suitable subclass that is amenable to the analysis.*

As we may probably expect after our analysis in Section 5.4.1, the process $Y^{y_0,Z}$ in the definition of $\mathcal{H}^{2,2}$ becomes more amenable to the analysis by working in terms of the certainty equivalent. For this, we introduce, for $(t,s,x,\mathrm{z},z,v) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^n)^3$,

$$\widehat{H}_t(x,\mathrm{z}) := \sup_{a\in A} \widehat{h}_t(t,x,z,a),\ \widehat{h}_t(s,x,z,a) := \sigma_t(x)b_t(x,a)\cdot z - g(t-s)k_t^o(x,a),$$
$$\nabla\widehat{h}_t(s,x,v,z,a) := \sigma_t(x)b_t(x,a))\cdot v + g'(t-s)k_r^o(x,a) - \gamma_{\mathrm{A}}\sigma_t^\top(x)z\cdot\sigma_t^\top(x)v.$$

The maps $\hat{a}^\star(t,x,\mathrm{z})$, $\lambda_t^\star(x,\mathrm{z})$, $k_t^{o\star}(x,\mathrm{z})$, $\widehat{h}_t^\star(s,x,z,\mathrm{z})$, $\nabla\widehat{h}_t^\star(s,x,v,z,\mathrm{z})$, and the probability $\mathbb{P}^\star(\mathrm{z})$ are defined accordingly.

Moreover, inspired by Section 5.4.2, we introduce the mapping $\delta^\star$ given, for $(s,t,x,\mathrm{z},z,\tilde{z}) \in [0,T]^2 \times \mathcal{X} \times (\mathbb{R}^n)^3$, by

$$\delta_t^\star(s,x,\mathrm{z},z,\tilde{z}) := k_r^{o\star}(x,\mathrm{z})\Big(g(r-s) - \frac{g(T-s)}{g(T)}g(r)\Big) + \frac{\gamma_{\mathrm{A}}}{2}\Big(|\sigma_r^\top(x)z|^2 - \frac{g(T-s)}{g(T-u)}|\sigma_r^\top(x)\tilde{z}|^2\Big).$$

The following result is analogous to Lemma 5.4.7, we defer its proof to Section 5.7.

Lemma 5.4.14. *Let $Z \in \mathcal{H}^{2,2}$.*

(i) There exists family of processes $(\widehat{Y}^{s,y_0,Z}, \widehat{Z}^s)_{s\in[0,T]}$ such that for every $s \in [0,T]$

$$\widehat{Y}_t^{s,y_0,Z} = -\frac{1}{\gamma_{\mathrm{A}}}\ln\big(-\gamma_{\mathrm{A}}y_0^s\big) - \int_0^t \Big(\widehat{h}_r^\star(s,X_{\cdot\wedge r},\widehat{Z}_r^s,\widehat{Z}_r^r) - \frac{\gamma_{\mathrm{A}}}{2}|\sigma_r^\top\widehat{Z}_r^s|^2\Big)\mathrm{d}r + \int_0^t \widehat{Z}_r^s\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

(ii) If $\widehat{Z} \in \mathbb{H}^{2,2}$ then for every $s \in [0,T]$

$$\widehat{Y}_t^{s,y_0,Z} = \frac{g(T-s)}{g(T)}\widehat{Y}_t^{0,y_0,Z} - \mathbb{E}^{\mathbb{P}^\star(Z)}\Big[\int_t^T \delta_r^\star(s,X_{\cdot\wedge r},\widehat{Z}_r^r,\widehat{Z}_r^s,\widehat{Z}_r^0)\mathrm{d}r\Big|\mathcal{F}_t\Big].$$

(iii) Moreover, if the process $M^{s,Z}$ given by

$$M_t^{s,Z} := \mathbb{E}^{\mathbb{P}^\star(Z)}\Big[\int_0^T \delta_r^\star(s,X_{\cdot\wedge r},\widehat{Z}_r^r,\widehat{Z}_r^s,\widehat{Z}_r^0)\mathrm{d}r\Big|\mathcal{F}_t\Big],\ \mathbb{P}\text{–a.s.},\ (s,t)\in[0,T]^2,$$

is a square integrable $(\mathbb{F}, \mathbb{P}^\star(Z))$*-martingale, then*

$$\widehat{Z}_t^s = \frac{g(T-s)}{g(T)} \widehat{Z}_t^0 - \widehat{Z}_t^{s,Z}, \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e.}$$

where $\widehat{Z}^{s,Z}$ *denotes the term in the representation of* $M^{s,Z}$.

Remark 5.4.15. (i) *We highlight that in contrast to the analysis presented in* Sections 5.4.1 *and* 5.4.2*, the previous result does not provide an equivalent representation of the set* $\mathcal{H}^{2,2}$*. This is intimately related to the square integrability condition on the process* $M^{s,Z}$ *required in* (iii) *above, and the fact that* $(z, \tilde z) \longmapsto \delta_t^\star(x, \mathrm{z}, z, \tilde z)$ *is quadratic for* $(t, s, x, \mathrm{z}) \in [0,T]^2 \times \mathcal{X} \times \mathbb{R}^m$ *fixed.*

(ii) *As a sanity check at this point, let us verify the coherence of the previous system in terms of the analysis of the previous section. In the following we omit the dependence on* X *and assume* $\widehat{Z} \in \mathbb{H}^{2,2}(\mathbb{R}^n)$*. Let*

$$(\Delta^s g)(t) := g(t-s) - g(t),\ K_{t,\tau}^{Z,s} := \exp\bigg(\gamma_{\mathrm{A}} \int_t^\tau \Big(k_r^\star(\widehat{Z}_r)(\Delta^s g)(r) - \gamma_{\mathrm{A}} |\sigma_r \widehat{Z}_r^0|^2 + \gamma_{\mathrm{A}} \widehat{Z}_r^{s,\top} \sigma_r \cdot \widehat{Z}_r^{0\top} \sigma_r\Big) \mathrm{d}r\bigg),$$

By applying Itô's formula to $\widetilde{Y}_r^s := K_{t,r}^{Z,s} \mathrm{U}_{\mathrm{A}}(\widehat{Y}_r^{s,Z} - \widehat{Y}_r^{0,y_0,Z})$*, we have that for any* $s \in [0,T]$, $\mathbb{P}$–a.s.

$$\begin{aligned}
&\mathrm{U}_{\mathrm{A}}^o(\widehat{Y}_t^{s,y_0,Z} - \widehat{Y}_t^{0,y_0,Z}) \\
&= \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\mathrm{U}_{\mathrm{A}}^o\bigg(y_0^s - y_0^0 - \int_0^t \Big((\Delta^s g)(r) k_r^\star(\widehat{Z}_r) - \gamma_{\mathrm{A}} |\sigma_r \widehat{Z}_r^0|^2 + \gamma_{\mathrm{A}} \widehat{Z}_r^{s,\top} \sigma_r \cdot \widehat{Z}_r^{0\top} \sigma_r\Big) \mathrm{d}r\bigg) \bigg| \mathcal{F}_t\bigg].
\end{aligned}$$

As $\gamma_{\mathrm{A}}^{-1} - \mathrm{U}_{\mathrm{A}}^o(y) \xrightarrow{\gamma_{\mathrm{A}} \to 0} y$*, we see the previous equation induces the corresponding one in* Lemma 5.4.7.

5.4.3.1 Principal's second-best solution

Let us highlight that in contrast to Section 5.4.2, the analysis in the previous section does not provide a full characterisation of $\mathcal{H}^{2,2}$ in the context to rewards given by (5.2.4). This is principally due to the integrability necessary on the variable $\widehat{Z}$, induced by the certainty equivalent, in order to apply the methodology devised in 5.4.2, see Lemma 5.4.14. Nevertheless, given that current example generalises the previous two, we build upon the structure of the optimal solution that was obtained to propose a family over which the optimisation in the problem of the principal can be carried out.

We will focus on the case $n=1$ and we will pay special attention to the class $\widetilde{\mathcal{H}} \subseteq \mathcal{H}^{2,2}$ of processes $Z \in \mathcal{H}^{2,2}$ for which given the pair $(y_0, Z) \in \mathcal{I}_0 \times \mathcal{H}^{2,2}$, and $Y^{y_0,Z}$ given by (5.3.5), there exists a pair of predictable processes (η, ζ) such that

$$\widehat{Z}_t^s = \frac{1}{-\gamma_{\mathrm{A}}} \frac{Z_t^s}{Y_t^{y_0,s,Z}} = \eta_{s,t}\zeta_t,\ \eta_{t,t} = 1,\ t \in [0,T].$$

Therefore, we have from Theorem 5.3.7 and (5.4.2) that

$$\underline{\mathrm{V}}^{\mathrm{P}} := \sup_{y_0^0 \geq R_0} \sup_{Z \in \widetilde{\mathcal{H}}} \mathbb{E}^{\mathbb{P}^\star(Z)}\Big[\mathrm{U}_{\mathrm{P}}^{\mathrm{o}}\Big(X_T - \widehat{Y}_T^{0,y_0,Z}\Big)\Big] \leq \mathrm{V}^{\mathrm{P}}.$$

Remark 5.4.16. (i) *We remark that the previous definition implicitly requires that for any $t \in [0,T]$ the mapping $s \longmapsto \eta_{s,t}$ is differentiable.*

(ii) *In addition, provided $k^{o\star}(x,z)$ does not depend on x it is easy to verify that that $\widetilde{\mathcal{H}} \neq \emptyset$. In light of the previous lemma, we have that for $Z \in \widetilde{\mathcal{H}}$*

$$M_t^{s,Z} = \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\int_0^T k_r^{o\star}(\zeta_r)\Big(g(r-s) - \frac{g(T-s)}{g(T)}g(r)\Big) + \frac{\gamma_{\mathrm{A}}}{2}|\sigma_r^\top \zeta_r|^2\Big(|\eta_{s,r}|^2 - \frac{g(T-s)}{g(T)}|\eta_{0,r}|^2\Big|^2\Big)\mathrm{d}r\Big|\mathcal{F}_t\bigg],$$

and

$$\widetilde{Z}_t^{s,Z} = \Big(\frac{g(T-s)}{g(T)}\eta_{0,t} - \eta_{s,t}\Big)\zeta_t, \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{–a.e.}$$

This implies that $\widetilde{\mathcal{H}}$ includes, in particular, all the processes Z that are induced by deterministic pairs (ζ, η). Indeed, for such class of processes we have that $M^{s,Z}$ is deterministic, $\widetilde{Z}^{s,Z} = 0$, and consequently, $\eta_{s,t} = g(T-s)/g(T-t)$ provides a non trivial element of $\widetilde{\mathcal{H}}$. The previous argument also holds in the case of exponential discounting, in which we recall that the agent's problem remains time-inconsistent.

The following result characterises the solution to $\underline{\mathrm{V}}^{\mathrm{P}}$. Its proof is available in Section 5.7.

Proposition 5.4.17. *Let principal and agent have exponential utility with parameters γ_{P} and γ_{A}, respectively. Let $C_y := -\frac{1}{\gamma_{\mathrm{P}}}\mathrm{e}^{-\gamma_{\mathrm{P}}(x_0 - y)}$, $\widehat{R}_0 := \mathrm{U}_{\mathrm{A}}^{\mathrm{o}\,(-1)}(R_0)$, and assume that:*

(i) *the maps σ, $\lambda^\star$ and $k^{o\star}$ do not depend on the x variable*;

(ii) for any $(t,\eta)\in[0,T]\times\mathbb{R}$, *the map* $G:\mathbb{R}\longrightarrow\mathbb{R}$ *given by*

$$G(z):=\lambda_t^\star(z)-\frac{g(t)}{g(T)}k_t^{o\star}(z)-\frac{\gamma_A}{2g(T)}\Big|\frac{\eta_0\sigma_t}{\eta}\Big|^2|z|^2-\frac{\gamma_{\rm P}}{2}\sigma_r^2\bigg(1-\frac{\eta_0}{\eta g(T)}z\bigg)^2,$$

has a unique maximiser $z^\star(t,\eta)$, *such that* $[0,T]\ni t\longmapsto z^\star(t,\eta)$ *is square-integrable.*

Then

$$\underline{\rm V}^{\rm P}=\sup_{Z^\eta\in\widetilde{\mathcal{H}}}C_{\widehat{R}_0}\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\gamma_{\rm P}\int_0^T G_r(z^\star(r,\eta_r))\mathrm{d}r\bigg)\bigg],\ \textit{where},\ Z_t^{s,\eta}:=\frac{\eta_s}{\eta_t}z^\star(t,\eta_t).$$

Moreover

(i) let $\underline{\rm V}^{{\rm P},o}$ *denote the restriction of* $\underline{\rm V}^{\rm P}$ *to the subclass of* $\widetilde{\mathcal{H}}$ *with deterministic* η. *Then the optimal deterministic contract is given by the family*

$$Z_t^s:=\frac{g(T-s)}{g(T-t)}z^\star(t,g(T-t)),$$

and

$$\xi=\frac{\widehat{R}_0}{g(T)}-\frac{1}{g(T)}\int_0^t h_r^\star\bigg(0,\frac{g(T)}{g(T-t)}z^\star(t,g(T-t)),z^\star(r,g(T-r))\bigg)\mathrm{d}r+\int_0^t\frac{z^\star(t,g(T-t))}{g(T-t)}\mathrm{d}X_r;$$

(ii) in the case $(\gamma_{\rm A},\gamma_{\rm P})=(0,0)$, i.e. *the case of risk-neutral principal and agent, the solution to* ${\rm V}^{\rm P}$, *and consequently of* $\underline{\rm V}_{\rm P}$, *agrees with the value given by* Proposition 5.4.10 *and the optimal family* Z *is deterministic.*

Remark 5.4.18. *We close this section with a few remarks.*

(i) The solution to the problem of the principal for the general class of restricted contracts induced by $\mathcal{H}^{2,2}$ *escaped the analysis presented above. As detailed in* Remark 5.4.13.*(ii), this is due to subtle integrability issues when trying to identify an appropriate reduction of* $\mathcal{H}^{2,2}$, *and the quadratic nature of the generator when working in term of the certainty equivalent. We believe this echoes the intricacies of the non-standard class of control problem introduced in* Theorem 5.3.7.

(ii) If, as in Remark 5.4.6, *we bring ourselves back to the setting of* [130], i.e. $b_t(x,a)=a/\sigma$,

$\sigma_t(x) = \sigma,\ k_t(x,a) = ka^2/2$, *we have*

$$Z_t^s = \frac{g(T)g(T-s)(g(T-t)+\gamma_{\rm P}\sigma^2 k)}{g(t)g^2(T-t)+\sigma^2 kg(T)(\gamma_{\rm A} g(T)+\gamma_{\rm P})}.$$

We highlight that: (*a*) *in contrast to* [130], *for any type of discounting structure* (*including exponential discounting*) *the previous expression and consequently the optimal action is neither linear nor Markovian. This corroborates our comment in* Remark 5.4.13.(*i*), *in the sense that even in the case of exponential discounting the problem of the agent remains time-inconsistent;* (*b*) *in the case of no discounting,* i.e. $g = 1$, *when we bring ourselves back to the model of* Remark 5.4.6*, the previous expression coincides with the linear contract result specified by* [130]. *This shows that, even if possibly not the best, the optimal contract in the class* $\widetilde{\mathcal{H}}$ *at least captures the optimal contract when the problem becomes time-consistent again.*

5.5 On time-inconsistency for BSVIE-type rewards

Let us start by mentioning that in the context of rewards given by (5.1.2), the methodology devised in Chapter 2, which builds on the approach in the Markovian framework of [38], is based introducing the family of processes $(Y^s, Z^s)_{s\in[0,T]}$ solution to the backward stochastic Volterra integral equation (BSVIEs for short), which satisfies

$$Y_t^{s,\alpha} = \eta(s,\xi) + \int_t^T h_r(s, X_{\cdot\wedge r}, Y_r^{s,\alpha}, Z_r^{s,\alpha}, \alpha_r)\mathrm{d}r - \int_t^T Z_r^{s,\alpha}\cdot \mathrm{d}X_r,\ \mathbb{P}\text{–a.s.},\ s\in[0,T]. \tag{5.5.1}$$

Throughout this section we fix $\xi \in \mathcal{C}$ and $\alpha^\star \in \mathcal{E} := \mathcal{E}(\xi)$. Thus, we identify the agent's reward under $\alpha \in \mathcal{A}$ via $\mathrm{V}_t^{\mathrm{A}}(\alpha) := \mathrm{V}_t^{\mathrm{A}}(\alpha,\xi) = Y_t^{t,\alpha}$. We write $\mathrm{V}_t^{\mathrm{A}}$ for the associated value function under $\alpha^\star$.

To establish an extended dynamic programming principle we need the following minimal set of assumptions.

Assumption S. (*i*) $(s,y,z) \longmapsto h_t(s,x,y,z,a)$ (*resp.* $s \longmapsto \mathrm{U}^{\mathrm{A}}(s,x)$) *is continuously differentiable, uniformly in* (t,x,a) (*resp. in* x). *Moreover, the mapping* $\nabla h : [0,T]^2 \times \mathcal{X} \times (\mathbb{R}\times\mathbb{R}^n)^2 \longrightarrow \mathbb{R}$ *defined by*

$$\nabla h_t(s,x,u,v,y,z,a) := \partial_s h_t(s,x,y,z,a) + \partial_y h_t(s,x,y,z,a)u + \sum_{i=1}^{n} \partial_{z_i} h_t(s,x,y,z,a)v_i,$$

satisfies $\nabla h_\cdot(s,\cdot,u,v,y,z,a) \in \mathcal{P}_{\rm prog}(\mathbb{R},\mathbb{F})$; *for all* $s \in [0,T]$;

(ii) for $\varphi \in \{h, \partial_s h\}$, $(y,z,a) \longmapsto \varphi_t(s,x,y,z,a)$ *is uniformly Lipschitz-continuous with linear growth,* i.e. *there exists some* $C>0$ *such that* $\forall (s,t,x,y,\tilde{y},z,\tilde{z},a,\tilde{a})$,

$$|\varphi_t(s,x,y,z,a) - \varphi_t(s,x,\tilde{y},\tilde{z},\tilde{a})| \leq C\big(|y-\tilde{y}| + |\sigma_t(x)^\top(z-\tilde{z})| + |a-\tilde{a}|\big);$$

(iii) let $\big(\tilde{h}_\cdot(s), \nabla\tilde{h}_\cdot(s)\big) := \big(h_\cdot(s,\cdot,0,0,0), \nabla h_\cdot(s,\cdot,0,0,0,0,0)\big)$, *then* $\big(\tilde{h},\nabla\tilde{h}\big) := \big(\tilde{h}(\cdot), \nabla\tilde{h}(\cdot)\big) \in \mathbb{L}^{1,2,2} \times \mathbb{L}^{1,2,2}$.

Remark 5.5.1. *We remark that* Assumption S.*(iii) is satisfied if one assumes the classic linear growth condition,* i.e. *there exists some* $C>0$ *such that for all* (s,t,x,y,z,u,v,a)

$$\begin{aligned}
\big|h_t(s,x,y,z,a)\big| &\leq C\big(1+|y|+|\sigma_t(x)^\top z|\big),\\
\big|\nabla h_t(s,x,u,v,y,z,a)\big| &\leq C\big(1+|u|+|\sigma_t(x)^\top v|+|y|+|\sigma_t(x)^\top z|\big).
\end{aligned}$$

Under Assumption S, Lemma 3.6.1 guarantees that for any $\alpha \in \mathcal{A}$ there exists $(\partial Y^\alpha, \partial Z^\alpha) \in \mathbb{S}^2 \times \mathbb{H}^{2,2}$ such that for every $s \in [0,T]$

$$\begin{aligned}
\partial Y_t^{s,\alpha} = \partial_s \eta(s,\xi) &+ \int_t^T \nabla h_r(s, X_{\cdot\wedge r}, \partial Y_r^{s,\alpha}, \partial Z_r^{s,\alpha}, Y_r^{s,\alpha} Z_r^{s,\alpha}, \alpha_r)\mathrm{d}r\\
&- \int_t^T \partial Z_r^{s,\alpha} \cdot \mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.},
\end{aligned} \tag{5.5.2}$$

which ultimately implies the absolute continuity of the mapping $([0,T],\mathcal{B}([0,T])) \longrightarrow (\mathbb{H}^2, \|\cdot\|_{\mathbb{H}^2})$: $s \longmapsto Z^{s,\alpha}$. With this, the process $(Z_t^{t,\alpha})_{t\in[0,T]}$ is well-defined. Moreover, see Lemma 3.6.2, for any

$\alpha \in \mathcal{A}$

$$
\begin{aligned}
Y_t^{t,\alpha} = \eta(T,\xi) + \int_t^T \Big(h_r(r, X_{\cdot\wedge r}, Y_r^{r,\alpha}, Z_r^{r,\alpha}, \alpha_r) - \partial Y_r^{r,\alpha} \Big) \mathrm{d}r \\
- \int_t^T Z_r^{r,\alpha} \cdot \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.}
\end{aligned}
\tag{5.5.3}
$$

We begin stating the following auxiliary result.

Lemma 5.5.2. *Let* Assumption S *hold. For any* $\{\gamma, \gamma'\} \subseteq \mathcal{T}_{0,T}$, $\gamma \leq \gamma'$, *and* $\alpha \in \mathcal{A}$

$$
\mathbb{E}^{\mathbb{P}}\bigg[Y_\gamma^{\gamma,\alpha} - Y_{\gamma'}^{\gamma',\alpha} + \int_\gamma^{\gamma'} \partial Y_r^{r,\alpha} \mathrm{d}r \bigg| \mathcal{F}_\gamma \bigg], \text{ depends only on the value of } \alpha \textit{ on } [\gamma, \gamma'].
$$

Proof. This property is clear for BSDEs whose generator does not depend on (y, z). Indeed,

$$
\mathbb{E}^{\mathbb{P}}\bigg[Y_\gamma^{\gamma,\alpha} - Y_{\gamma'}^{\gamma',\alpha} + \int_\gamma^{\gamma'} \partial Y_r^{r,\alpha} \mathrm{d}r \bigg| \mathcal{F}_\gamma \bigg] = \mathbb{E}^{\mathbb{P}}\bigg[\int_\gamma^{\gamma'} h_r(r, X_{\cdot\wedge r}, \alpha_r) \mathrm{d}r \bigg| \mathcal{F}_\gamma \bigg].
$$

To extend this result to the BSDEs (5.5.1)–(5.5.2) we consider the Picard iteration procedure

$$
\begin{aligned}
Y_t^{s,\alpha,n+1} &= \eta(s,\xi) + \int_t^T h_r(s, X_{\cdot\wedge r}, Y_r^{s,\alpha,n}, Z_r^{s,\alpha,n}, \alpha_r) \mathrm{d}r - \int_t^T Z_r^{s,\alpha,n+1} \cdot \mathrm{d}X_r, \\
\partial Y_t^{s,\alpha,n+1} &= \partial_s \eta(s,\xi) + \int_t^T \nabla h_r(s, X_{\cdot\wedge r}, \partial Y_r^{s,\alpha,n}, \partial Z_r^{s,\alpha,n}, Y_r^{s,\alpha,n}, Z_r^{s,\alpha,n}, \alpha_r) \mathrm{d}r \\
&\quad - \int_t^T \partial Z_r^{s,\alpha,n+1} \cdot \mathrm{d}X_r,
\end{aligned}
$$

and note that, as in (5.5.3)

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\bigg[Y_\gamma^{\gamma,\alpha,n+1} - Y_{\gamma'}^{\gamma',\alpha,n+1} + \int_\gamma^{\gamma'} \partial Y_r^{r,\alpha,n+1} \mathrm{d}r \bigg| \mathcal{F}_\gamma \bigg] \\
&= \mathbb{E}^{\mathbb{P}}\bigg[\int_\gamma^{\gamma'} h_r(r, X_{\cdot\wedge r}, Y_r^{r,\alpha,n}, Z_r^{r,\alpha,n}, \alpha_r) \mathrm{d}r \bigg| \mathcal{F}_\gamma \bigg].
\end{aligned}
\tag{5.5.4}
$$

Then, from the fact that $Y^{\alpha,0} = Z^{\alpha,0} = \partial Y^{\alpha,0} = \partial Z^{\alpha,0} = 0$ we see that (5.5.4) implies the result at the initial step. It is then also clear, again from (5.5.4), that this property is preserved at every iteration and thus in the limit. $\square$

In the following, given $(\sigma, \tau) \in \mathcal{T}_{t,T} \times \mathcal{T}_{t,t+\ell}$, with $\sigma \leq \tau$, we denote by $\Pi^\ell := (\tau_i^\ell)_{i=1,\ldots,n_\ell} \subseteq \mathcal{T}_{t,T}$ a generic partition of $[\sigma, \tau]$ with mesh smaller than ℓ, *i.e.* for $n_\ell := \lceil (\tau - \sigma)/\ell \rceil$, $\sigma =: \tau_0^\ell \leq \cdots \leq$

$\tau^{\ell}_{n^{\ell}} := \tau$, $\forall \ell$, and $\sup_{1\leq i\leq n_\ell} |\tau^\ell_i - \tau^\ell_{i-1}| \leq \ell$. We also let $\Delta\tau^\ell_i := \tau^\ell_i - \tau^\ell_{i-1}$. The previous definitions hold x-by-x.

Theorem 5.5.3 (Dynamic programming principle)**.** *Let* Assumption S *hold. Let* $\alpha^\star \in \mathcal{E}(\xi)$ *and* $\{\sigma,\tau\} \subset \mathcal{T}_{t,T}$*, with* $\sigma \leq \tau$*. Then,* $\mathbb{P}$–a.s.

$$\mathrm{V}^{\mathrm{A}}_\sigma = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}}\ \mathbb{E}^{\mathbb{P}}\Big[\mathrm{V}^{\mathrm{A}}_\tau + \int_\sigma^\tau \Big(h_r(r, X_{\cdot\wedge r}, Y^{r,\alpha}_r, Z^{r,\alpha}_r, \alpha_r) - \partial Y^{r,\alpha^\star}_r\Big)\mathrm{d}r\Big|\mathcal{F}_\sigma\Big],$$

where for every $s \in [0,T]$*,* $\partial Y^{s,\alpha^\star}$ *denotes the solution to* (5.5.2) *with* $\alpha^\star$*. Moreover,* $\alpha^\star$ *attains the* $\operatorname{ess\,sup}^{\mathbb{P}}$.

Proof. We first show the inequality $\geq$. We proceed in 3 steps. Let $\varepsilon > 0$, $0 < \ell < \ell_\varepsilon$, and Π^ℓ be a partition of $[\sigma,\tau]$.

Step 1**:** From the definition of equilibria we have that for any $\alpha \in \mathcal{A}$

$$\mathrm{V}^{\mathrm{A}}_\sigma \geq \mathrm{V}^{\mathrm{A}}_\sigma(\alpha\otimes_{\tau_1}\alpha^\star) - \varepsilon\ell \geq \mathbb{E}^{\mathbb{P}}\Big[Y^{\sigma,\alpha\otimes_{\tau_1}\alpha^\star}_\sigma - Y^{\tau_1,\alpha\otimes_{\tau_1}\alpha^\star}_{\tau_1} + Y^{\tau_1,\alpha\otimes_{\tau_1}\alpha^\star}_{\tau_1}\Big|\mathcal{F}_\sigma\Big] - \varepsilon\ell.$$

Recall that for any $\rho \in \mathcal{T}_{0,T}$, $Y^{\rho,\alpha\otimes_\rho\alpha^\star}_\rho = Y^{\rho,\alpha^\star}_\rho$. In light of the arbitrariness of $\alpha \in \mathcal{A}$ we obtain

$$\mathrm{V}^{\mathrm{A}}_\sigma \geq \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}}\ \mathbb{E}^{\mathbb{P}}\Big[Y^{\sigma,\alpha\otimes_{\tau_1}\alpha^\star}_\sigma - Y^{\tau_1,\alpha\otimes_{\tau_1}\alpha^\star}_{\tau_1} + \mathrm{V}^{\mathrm{A}}_{\tau_1}\Big|\mathcal{F}_\sigma\Big] - \varepsilon\ell,\ \mathbb{P}\text{–a.s.}$$

Step 2**:** Let us note that in light of **Step 1**

$$\begin{aligned}
\mathrm{V}^{\mathrm{A}}_\sigma &\geq \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}}\ \mathbb{E}^{\mathbb{P}}\Big[Y^{\sigma,\alpha\otimes_{\tau_1}\alpha^\star}_\sigma - Y^{\tau_1,\alpha\otimes_{\tau_1}\alpha^\star}_{\tau_1} + \mathrm{V}^{\mathrm{A}}_{\tau_1}\Big|\mathcal{F}_\sigma\Big] - \varepsilon\ell\\
&= \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}}\ \mathbb{E}^{\mathbb{P}}\Big[Y^{\sigma,\alpha\otimes_{\tau_1}\alpha^\star}_\sigma - Y^{\tau_1,\alpha\otimes_{\tau_1}\alpha^\star}_{\tau_1}\\
&\qquad\qquad + \operatorname*{ess\,sup}_{\tilde\alpha\in\mathcal{A}}{}^{\mathbb{P}}\ \mathbb{E}^{\mathbb{P}}\Big[Y^{\tau_1,\tilde\alpha\otimes_{\tau_2}\alpha^\star}_{\tau_1} - Y^{\tau_2,\tilde\alpha\otimes_{\tau_2}\alpha^\star}_{\tau_2} + Y^{\tau_2,\tilde\alpha\otimes_{\tau_2}\alpha^\star}_{\tau_2}\Big|\mathcal{F}_{\tau_1}\Big]\Big|\mathcal{F}_\sigma\Big] - 2\varepsilon\ell\\
&= \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}}\ \mathbb{E}^{\mathbb{P}}\Big[\mathrm{V}^{\mathrm{A}}_{\tau_2} + Y^{\sigma,\alpha\otimes_{\tau_1}\alpha^\star}_\sigma - Y^{\tau_1,\alpha\otimes_{\tau_1}\alpha^\star}_{\tau_1} + Y^{\tau_1,\alpha\otimes_{\tau_2}\alpha^\star}_{\tau_1} - Y^{\tau_2,\alpha\otimes_{\tau_2}\alpha^\star}_{\tau_2}\Big|\mathcal{F}_\sigma\Big] - 2\varepsilon\ell,
\end{aligned}$$

where the second equality holds in light of [210, Lemma 3.5] as [210, Assumption 1.1] holds under

Assumption Q. Iterating the previous argument we obtain that $\mathbb{P}$–a.s.

$$\mathrm{V}_\sigma^{\mathrm{A}} \geq \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}} \ \mathbb{E}^{\mathbb{P}}\bigg[\mathrm{V}_\tau^{\mathrm{A}} + \sum_{i=0}^{n^\ell-1} Y_{\tau_i}^{\tau_i,\alpha\otimes_{\tau_{i+1}}\alpha^\star} - Y_{\tau_{i+1}}^{\tau_{i+1},\alpha\otimes_{\tau_{i+1}}\alpha^\star}\bigg|\mathcal{F}_\sigma\bigg] - n_\ell\varepsilon\ell.$$

Now, we use the fact that for any $i \in \{0, ..., n_\ell - 1\}$ and $\alpha \otimes_{\tau_{i+1}} \alpha^\star$, Lemma 5.5.2 implies

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\bigg[Y_{\tau_i}^{\tau_i,\alpha\otimes_{\tau_{i+1}}\alpha^\star} - Y_{\tau_{i+1}}^{\tau_{i+1},\alpha\otimes_{\tau_{i+1}}\alpha^\star} + \int_{\tau_i}^{\tau_{i+1}} \partial Y_r^{r,\alpha\otimes_{\tau_{i+1}}\alpha^\star}\mathrm{d}r\bigg|\mathcal{F}_\sigma\bigg]\\
&= \mathbb{E}^{\mathbb{P}}\bigg[\int_{\tau_i}^{\tau_{i+1}} h_r(r, X_{\cdot\wedge r}, Y_r^{r,\alpha}, Z_r^{r,\alpha}, \alpha_r)\mathrm{d}r\bigg|\mathcal{F}_\sigma\bigg].
\end{aligned}$$

Replacing in the previous expression we obtain that $\mathbb{P}$–a.s.

$$\begin{aligned}
\mathrm{V}_\sigma^{\mathrm{A}} \geq \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}} \ \mathbb{E}^{\mathbb{P}}\bigg[&\ \mathrm{V}_\tau^{\mathrm{A}} + \int_\sigma^\tau h_r(r, X_{\cdot\wedge r}, Y_r^{r,\alpha}, Z_r^{r,\alpha}, \alpha_r)\mathrm{d}r \\
&- \sum_{i=0}^{n^\ell-1} \int_{\tau_i}^{\tau_{i+1}} \partial Y_r^{r,\alpha\otimes_{\tau_{i+1}}\alpha^\star}\mathrm{d}r - n_\ell\varepsilon\ell\bigg|\mathcal{F}_\sigma\bigg].
\end{aligned} \tag{5.5.5}$$

Step 3: Let $i \in \{0, \ldots, n_\ell - 1\}$. In light of Assumption Q, the stability of the system of BSDE defined by (5.5.1) and (5.5.2), see Proposition 3.6.4, yields there exists a constant $C > 0$ such that

$$\bigg\|\int_{\tau_i}^{\tau_{i+1}} \partial Y_r^{r,\alpha\otimes_{\tau_{i+1}}\alpha^\star} - \partial Y_r^{r,\alpha^\star}\mathrm{d}r\bigg\|_{\mathcal{L}^2} \leq \ell\Big\|\partial Y^{\alpha\otimes_{\tau_{i+1}}\alpha^\star} - \partial Y^{\alpha^\star}\Big\|_{\mathbb{S}^{2,2}} \leq \ell C\mathbb{E}\bigg[\bigg(\int_{\tau_i}^{\tau_{i+1}} |\alpha_r - \alpha_r^\star|\mathrm{d}r\bigg)^2\bigg],$$

which leads to

$$\bigg\|\sum_{i=0}^{n_\ell-1}\int_{\tau_i}^{\tau_{i+1}} \partial Y_r^{r,\alpha\otimes_{\tau_{i+1}}\alpha^\star} - \partial Y_r^{r,\alpha^\star}\mathrm{d}r\bigg\|_{\mathcal{L}^2} \leq \ell C\mathbb{E}\bigg[\bigg(\int_\sigma^\tau |\alpha_r - \alpha_r^\star|\mathrm{d}r\bigg)^2\bigg] \xrightarrow{\ell\to 0} 0.$$

By choosing an appropriate partition Π^ℓ and dominated convergence we obtain that

$$I(n_\ell) := \sum_{i=0}^{n^\ell-1}\int_{\tau_i}^{\tau_{i+1}} \partial Y_r^{r,\alpha\otimes_{\tau_{i+1}}\alpha^\star}\mathrm{d}r \xrightarrow{\ell_\varepsilon\to 0} \int_\sigma^\tau \partial Y_r^{r,\alpha^\star}\mathrm{d}r,\ \mathbb{P}\text{–a.s.}$$

Back in (5.5.5) we obtain

$$\mathrm{V}_\sigma^{\mathrm{A}} = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}}{}^{\mathbb{P}} \ \mathbb{E}^{\mathbb{P}}\bigg[\mathrm{V}_\tau^{\mathrm{A}} + \int_\sigma^\tau \Big(h_r(r, X_{\cdot\wedge r}, Y_r^{r,\alpha}, Z_r^{r,\alpha}, \alpha_r) - \partial Y_r^{r,\alpha^\star}\Big)\mathrm{d}r\bigg|\mathcal{F}_\sigma\bigg],\ \mathbb{P}\text{–a.s.}$$

Lastly, we show that the equality is attained by $\alpha^\star \in \mathcal{E}$. Indeed, note that (5.5.3) implies

$$\mathrm{V}^{\mathrm{A}}_\sigma = \mathbb{E}^{\mathbb{P}}\bigg[\mathrm{V}^{\mathrm{A}}_\tau + \int_\sigma^\tau \Big(h_r\big(r, X_{\cdot\wedge r}, Y_r^{r,\alpha^\star}, Z_r^{r,\alpha^\star}, \alpha_r^\star\big) - \partial Y_r^{r,\alpha^\star}\Big)\mathrm{d}r\bigg|\mathcal{F}_\sigma\bigg],\ \mathbb{P}\text{–a.s.}$$

□

Let us recall that the Hamiltonian associated to h is given by

$$H_t(x,y,z) = \sup_{a\in A} h_t(t,x,y,z,a),\ (t,x,y,z) \in [0,T]\times\mathcal{X}\times\mathbb{R}\times\mathbb{R}^n.$$

Our standing assumptions on H are the following.

Assumption T. *(i) For any $(t,x)\in[0,T]\times\mathcal{X}$, the map $\mathbb{R}\times\mathbb{R}^n \ni (y,z) \longmapsto H_t(x,y,z)$ is uniformly Lipschitz-continuous,* i.e. *there is $C>0$ such that for any $(t,x,\mathrm{y},\tilde{\mathrm{y}},\mathrm{z},\tilde{\mathrm{z}}) \in [0,T]\times\mathcal{X}\times\mathbb{R}^2\times(\mathbb{R}^n)^2$*

$$\big|H_t(x,\mathrm{y},\mathrm{z}) - H_t(x,\tilde{\mathrm{y}},\tilde{\mathrm{z}})\big| \leq C\big(|\mathrm{y}-\tilde{\mathrm{y}}| + |\sigma_t(x)^\top(\mathrm{z}-\tilde{\mathrm{z}})|\big);$$

(ii) there exists a unique Borel-measurable map $a^\star : [0,T]\times\mathcal{X}\times\mathbb{R}\times\mathbb{R}^n \longrightarrow A$ such that

$$H_t(x,\mathrm{y},\mathrm{z}) = h_t\big(t,x,\mathrm{y},\mathrm{z},a^\star(t,x,\mathrm{y},\mathrm{z})\big),\ \forall (t,x,\mathrm{y},\mathrm{z}) \in [0,T]\times\mathcal{X}\times\mathbb{R}\times\mathbb{R}^n;$$

(iii) for any $(t,x)\in[0,T]\times\mathcal{X}$, the map $\mathbb{R}\times\mathbb{R}^n \ni (\mathrm{y},\mathrm{z}) \longmapsto a^\star(t,x,\mathrm{y},\mathrm{z})$ is uniformly Lipschitz-continuous, i.e. *there is $C>0$ such that for any $(t,x,\mathrm{y},\tilde{\mathrm{y}},\mathrm{z},\tilde{\mathrm{z}}) \in [0,T]\times\mathcal{X}\times\mathbb{R}^2\times(\mathbb{R}^n)^2$.*

$$\big|a^\star(t,x,\mathrm{y},\mathrm{z}) - a^\star(t,x,\tilde{\mathrm{y}},\tilde{\mathrm{z}})\big| \leq C\big(|\mathrm{y}-\tilde{\mathrm{y}}| + |\sigma_t(x)^\top(\mathrm{z}-\tilde{\mathrm{z}})|\big);$$

(iv) $(\tilde{H},\tilde{a}^\star) \in \mathbb{L}^{1,2}(\mathbb{R})\times\mathbb{L}^{1,2}(\mathbb{R})$, where $(\tilde{H}_\cdot,\tilde{a}^\star_\cdot) := \big(H_\cdot(\cdot,0,0), a^\star_\cdot(\cdot,0,0)\big)$.

With this we introduce the system defined for any $s\in[0,T]$ by

$$\begin{aligned}
Y_t &= \eta(T,\xi) + \int_t^T \big(H_r(X_{\cdot\wedge r},Y_r,Z_r) - \partial Y_r^r\big)\mathrm{d}r - \int_t^T Z_r\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}\\
Y_t^s &= \eta(s,\xi) + \int_t^T h_r^\star(s,X_{\cdot\wedge r},Y_r^s,Z_r^s,Y_r,Z_r)\mathrm{d}r - \int_t^T Z_r^s\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}\\
\partial Y_t^s &= \partial_s\eta(s,\xi) + \int_t^T \nabla h_r^\star(s,X_{\cdot\wedge r},\partial Y_r^s,\partial Z_r^s,Y_r^s Z_r^s,Y_r,Z_r)\mathrm{d}r - \int_t^T \partial Z_r^s\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}
\end{aligned} \tag{H}$$

We will say $(\mathcal{Y},\mathcal{Z},Y,Z,\partial Y,\partial Z)\in\mathfrak{H}$ is a solution to the system whenever (H) is satisfied. In light of Theorem 5.5.3, given $\alpha^\star\in\mathcal{E}$ it is reasonable to associate the value along the equilibria with a BSDE whose generator is given, partially, by H. This is the purpose of the next result.

Theorem 5.5.4 (Necessity)**.** *Let Assumptions* S *and* T *hold and $\alpha^\star\in\mathcal{E}$. Then, one can construct a solution to* (H)*.*

Proof. Given $\alpha^\star\in\mathcal{E}$, Assumption S guarantees that the processes $(Y^{\alpha^\star},Z^{\alpha^\star})$ and $(\partial Y^{\alpha^\star},\partial Z^{\alpha^\star})$ solution to (5.5.1) and (5.5.2), respectively, are well-defined. Moreover, the diagonal processes $\big((Y_t^{t,\alpha^\star})_{t\in[0,T]},(Z_t^{t,\alpha^\star})_{t\in[0,T]}\big)$ are well-defined as elements of $\mathbb{S}^2\times\mathbb{H}^2$, see Lemma 3.6.2. Given $(\partial Y_t^{t,\alpha^\star})_{t\in[0,T]}\in\mathbb{S}^2$, for any $\alpha\in\mathcal{A}$, we can define the processes $(\mathcal{Y}^\alpha,\mathcal{Z}^\alpha)\in\mathbb{S}^2\times\mathbb{H}^2$ solution to

$$\mathcal{Y}_t^\alpha=\eta(T,\xi)+\int_t^T\Big(h_r(r,X_{\cdot\wedge r},\mathcal{Y}_r^\alpha,\mathcal{Z}_r^\alpha,\alpha_r)-\partial Y_r^{r,\alpha^\star}\Big)\mathrm{d}r-\int_t^T\mathcal{Z}_r^\alpha\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

We now note that under Assumption S the classic comparison result for BSDEs, see for instance [273, Theorem 4.4.1]. Then, it follows from Theorem 5.5.3 that the pair $\big((Y_t^{t,\alpha^\star})_{t\in[0,T]},(Z_t^{t,\alpha^\star})_{t\in[0,T]}\big)$ solves the BSDE

$$Y_t=\eta(T,\xi)+\int_t^T(H_r(X_{\cdot\wedge r},Y_r,Z_r)-\partial Y_r^{r,\alpha^\star})\mathrm{d}r-\int_t^T Z_r\cdot\mathrm{d}X_r,\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

Moreover, the second part of the statement of Theorem 5.5.3 implies that $\alpha^\star=a^\star(\cdot,X,Y_\cdot,Z_\cdot)$, $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$–a.e. Consequently, $(Y^{\alpha^\star},Z^{\alpha^\star})$ and $(\partial Y^{\alpha^\star},\partial Z^{\alpha^\star})$ define a solution to the second and third equations in (H), respectively. □

We close this section with a verification theorem for equilibria.

Theorem 5.5.5 (Verification)**.** *Let Assumptions* S *and* T *hold. Let $(\mathcal{Y},\mathcal{Z},Y,Z,\partial Y,\partial Z)\in\mathfrak{H}$ be a solution to* (H) *with $\alpha^\star:=a^\star(\cdot,X,\mathcal{Y}_\cdot,\mathcal{Z}_\cdot)$. Then, $\alpha^\star\in\mathcal{E}$ and*

$$\mathrm{V}_t^{\mathrm{A}}=\mathcal{Y}_t,\ \mathbb{P}\text{–a.s.}$$

Proof. We verify the definition of an equilibria. Let $\varepsilon>0$, $(t,\ell)\in[0,T]\times(0,\ell_\varepsilon)$ with ℓ_ε to be chosen. Let $(\mathcal{Y}^{\alpha\otimes_{t+\ell}\alpha^\star},\mathcal{Z}^{\alpha\otimes_{t+\ell}\alpha^\star})\in\mathbb{S}^2\times\mathbb{H}^2$ be the solution, which exists in light of Assumption S, to (5.5.1) with action $\alpha \otimes_{t+\ell} \alpha^\star$, that is to say, $\mathbb{P}$–a.s.

$$
\begin{aligned}
\mathcal{Y}_t^{s,\alpha\otimes_{t+\ell}\alpha^\star} = \eta(s,\xi) + \int_t^T h_r\big(s, X_{\cdot\wedge r}, \mathcal{Y}_r^{s,\alpha\otimes_{t+\ell}\alpha^\star}, \mathcal{Z}_r^{s,\alpha\otimes_{t+\ell}\alpha^\star}, (\alpha\otimes_{t+\ell}\alpha^\star)_r\big)\mathrm{d}r \\
- \int_t^T \mathcal{Z}_r^{s,\alpha\otimes_{t+\ell}\alpha^\star}\cdot \mathrm{d}X_r,\ t\in[0,T].
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\mathcal{Y}_t^{t,\alpha\otimes_{t+\ell}\alpha^\star} = \eta(T,\xi) + \int_t^T \Big(h_r(r, X_{\cdot\wedge r}, \mathcal{Y}_r^{\alpha\otimes_{t+\ell}\alpha^\star}, \mathcal{Z}_r^{\alpha\otimes_{t+\ell}\alpha^\star}, (\alpha\otimes_{t+\ell}\alpha^\star)_r) - \partial Y_r^{r,\alpha\otimes_{t+\ell}\alpha^\star}\Big)\mathrm{d}r \\
- \int_t^T \mathcal{Z}_r^{r,\alpha\otimes_{t+\ell}\alpha^\star}\cdot \mathrm{d}X_r.
\end{aligned}
$$

Now

$$
\begin{aligned}
&\mathcal{Y}_t - \mathcal{Y}_t^{t,\alpha\otimes_{t+\ell}\alpha^\star} \\
&= \int_t^T \Big(H_r(X_{\cdot\wedge r}, Y_r, Z_r) - h_r(r, X_{\cdot\wedge r}, \mathcal{Y}_r^{\alpha\otimes_{t+\ell}\alpha^\star}, \mathcal{Z}_r^{\alpha\otimes_{t+\ell}\alpha^\star}, (\alpha\otimes_{t+\ell}\alpha^\star)_r)\Big)\mathrm{d}r \\
&\quad - \int_t^T \partial Y_r^{r,\alpha^\star} - \partial Y_r^{r,\alpha\otimes_{t+\ell}\alpha^\star}\mathrm{d}r - \int_t^T \big(Z_r - \mathcal{Z}_r^{r,\alpha\otimes_{t+\ell}\alpha^\star}\big)\cdot \mathrm{d}X_r \\
&\geq \int_{t+\ell}^T \Big(H_r(X_{\cdot\wedge r}, Y_r, Z_r) - \partial Y_r^{r,\alpha^\star} - h_r(r, X_{\cdot\wedge r}, \mathcal{Y}_r^{\alpha\otimes_{t+\ell}\alpha^\star}, \mathcal{Z}_r^{\alpha\otimes_{t+\ell}\alpha^\star}, \alpha_r^\star) + \partial Y_r^{r,\alpha\otimes_{t+\ell}\alpha^\star}\Big)\mathrm{d}r \\
&\quad - \int_t^{t+\ell} \partial Y_r^{r,\alpha^\star} - \partial Y_r^{r,\alpha\otimes_{t+\ell}\alpha^\star}\mathrm{d}r - \int_t^T \big(Z_r - \mathcal{Z}_r^{r,\alpha\otimes_{t+\ell}\alpha^\star}\big)\cdot \mathrm{d}X_r \\
&= \int_t^{t+\ell} \partial Y_r^{r,\alpha\otimes_{t+\ell}\alpha^\star} - \partial Y_r^{r,\alpha^\star}\mathrm{d}r - \int_t^T \big(Z_r - \mathcal{Z}_r^{r,\alpha\otimes_{t+\ell}\alpha^\star}\big)\cdot \mathrm{d}X_r,
\end{aligned}
$$

where the inequality follows by definition of H and $\alpha^\star$ and Assumption T. The second equality follows from the fact that the first term cancels on $[t+\ell, T]$, see Lemma 5.5.2. Taking expectation we find

$$
\mathrm{V}_t^{\mathrm{A}} - \mathrm{V}_t^{\mathrm{A}}(\alpha\otimes_{t+\ell}\alpha^\star) = \mathbb{E}\Big[\mathcal{Y}_t - \mathcal{Y}_t^{t,\alpha\otimes_{t+\ell}\alpha^\star}\Big|\mathcal{F}_t\Big] \geq \mathbb{E}\bigg[\int_t^{t+\ell} \partial Y_r^{r,\alpha\otimes_{t+\ell}\alpha^\star} - \partial Y_r^{r,\alpha^\star}\mathrm{d}r\bigg|\mathcal{F}_t\bigg].
$$

By Proposition 3.6.4 we find that

$$
\bigg\|\int_t^{t+\ell} \partial Y_r^{r,\alpha\otimes_{t+\ell}\alpha^\star} - \partial Y_r^{r,\alpha^\star}\mathrm{d}r\bigg\|_{\mathcal{L}^2} \leq \ell\Big\|\partial Y^{\alpha\otimes_{t+\ell}\alpha^\star} - \partial Y^{\alpha^\star}\Big\|_{\mathbb{S}^{2,2}} \leq \ell C\mathbb{E}\bigg[\bigg(\int_t^{t+\ell} |\alpha_r - \alpha_r^\star|\mathrm{d}r\bigg)^2\bigg].
$$

By the boundedness of the action set, we may choose ℓ_ε such that the last term above is smaller that $\ell\varepsilon$. With this, we conclude $\alpha^\star \in \mathcal{E}$. The second part of the statement follows from the fact that $\mathcal{Y}_t = Y_t^{t,\alpha^\star}$, $\mathbb{P}$–a.s. $\square$

5.6 On forward stochastic Volterra integral equations

We are given a jointly measurable mapping h, a processes Z, and a family $(Y_0^s)_{s\in[0,T]} \in \mathcal{I}$ such that for any $(y,u) \in \mathbb{R}^2 \times (\mathbb{R}^n)^2$

$$h : [0,T]^2 \times \mathcal{X} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R},\ h_\cdot(\cdot, y, u) \in \mathcal{P}_{\mathrm{prog}}(\mathbb{R}, \mathbb{F}).$$

Moreover, we work under the following set of assumptions.

Assumption U. (i) $(s,y) \longmapsto h_t(s,x,y,u)$ *(resp.* $s \longmapsto Y_0^s(x)$*) is continuously differentiable, uniformly in* (t,x,u) *(resp. in* x*). Moreover, the mapping* $\nabla h : [0,T]^2 \times \mathcal{X} \times (\mathbb{R} \times \mathbb{R}^n)^2 \longrightarrow \mathbb{R}$ *defined by*

$$\nabla h_t(s,x,\mathrm{u},y,u) := \partial_s h_t(s,x,y,u) + \partial_y h_t(s,x,y,u)\mathrm{u},\ \nabla h_\cdot(s,\cdot,\mathrm{u},y,u) \in \mathcal{P}_{\mathrm{prog}}(\mathbb{R},\mathbb{F});$$

(ii) *for* $\varphi \in \{h, \partial_s h\}$, $(y,u) \longmapsto \partial_s \varphi_t(s,x,y,u)$ *is uniformly Lipschitz-continuous,* i.e. *there exists some* $C > 0$ *such that* $\forall (s,t,x,y,\tilde{y},u,\tilde{u})$,

$$|\varphi_t(s,x,y,u) - \varphi_t(s,x,\tilde{y},\tilde{u})| \leq C(|y-\tilde{y}| + |u-\tilde{u}|);$$

(iii) $(Y_0, \partial_s Y_0) \in (\mathcal{I})^2$, $Z \in \overline{\mathbb{H}}^{2,2}$, $(\tilde{h}_\cdot(s), \nabla\tilde{h}_\cdot(s)) := (h_\cdot(s,\cdot,\mathbf{0}), \partial_s h_\cdot(s,\cdot,\mathbf{0})) \in (\mathbb{L}^{1,2,2})^2$, *for* $\mathbf{0} := (y,u)|_{(0,0)}$.

We are interested in establishing the well-posedness of the FSVIE

$$Y_t^s = Y_0^s + \int_0^t h_r(s, X_{\cdot\wedge r}, Y_r^s, Y_r^r)\mathrm{d}r + \int_0^t Z_r^s \cdot \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\ s \in [0,T]. \tag{5.6.1}$$

To alleviate the notation we write Y instead of Y^Z in the previous equation.

Definition 5.6.1. *We say Y is a solution to the* FSVIE (5.6.1) *if Y satisfies equation* (5.6.1) *and* $Y \in \mathbb{S}^{2,2}$.

Remark 5.6.2. *We remark that in light of the pathwise continuity of Y^s for every $s \in [0,T]$ the process $(Y_t^t)_{t\in[0,T]}$ is well defined* $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e. *on $[0,T] \times \mathcal{X}$.*

We begin presenting a priori estimates for solutions of (5.6.1). These can be recover from the arguments in [273].

Lemma 5.6.3. *Let Y be a solution to* (5.6.1)*, there exists a constant $C > 0$ such that*

$$\|Y\|^2_{\mathbb{S}^{2,2}} \leq C\big(\|Y_0\|^2_{\mathcal{L}^{2,2}} + \|\tilde{h}\|^2_{\mathbb{L}^{1,2,2}} + \|Z\|^2_{\mathbb{H}^{2,2}}\big).$$

Moreover, for Y^i solution to (5.6.1) *with data (Y_0^i, h^i, Z^i) satisfying* (U) *for $i \in \{1,2\}$ there exists $C > 0$ such that*

$$\|Y^1 - Y^2\|^2_{\mathbb{S}^{2,2}} \leq C\big(\|Y_0^1 - Y_0^2\|^2_{\mathcal{L}^{2,2}} + \|\tilde{h}^1 - \tilde{h}^2\|^2_{\mathbb{L}^{1,2,2}} + \|Z^1 - Z^2\|^2_{\mathbb{H}^{2,2}}\big).$$

Proof. Let us observe that the continuity of the application $s \longmapsto \|\Delta Y^s\|_{\mathbb{S}^2}$ implies

$$\mathbb{E}\bigg[\int_0^T \mathrm{e}^{-cr}|\Delta Y_r^r|^2 \mathrm{d}r\bigg] \leq \int_0^T \mathbb{E}\bigg[\sup_{u\in[0,T]} \mathrm{e}^{-cu}|\Delta Y_u^r|^2\bigg]\mathrm{d}r \leq T\|\Delta Y\|^2_{\mathbb{S}^{2,2}}. \tag{5.6.2}$$

With this, the proof of both statements can obtained following the line of [273, Theorem 3.2.2 and Theorem 3.2.4]. □

We are now ready to establish the well-posedness of (5.6.1).

Proposition 5.6.4. *Let* Assumption U *hold. There is a unique solution to* (5.6.1)*.*

Proof. Uniqueness follows from Lemma 5.6.3. We use a Picard iteration argument. Let $Y_\cdot^{s,0} = Y_0^s, s \in [0,T]$ and

$$Y_t^{s,n+1} = Y_0^s + \int_0^t h_r(s, X_{\cdot\wedge r}, Y_r^{s,n}, Y_r^{r,n})\mathrm{d}r + \int_0^t Z_r^s \cdot \mathrm{d}X_r,\ t \in [0,T].$$

We note that $Y^n \in \mathbb{S}^{2,2}$ for $n \geq 0$. Indeed, the result holds for Y^0 and the process $(Y_t^{t,0})_{t\in[0,T]} \in \mathbb{L}^{1,2}$ is well-defined. Inductively, in light of Assumption U, the fact that $Z \in \mathbb{H}^{2,2}$ and $(Y_t^{t,n})_{t\in[0,T]} \in \mathbb{L}^{1,2}$, see (5.6.2), yields $Y^{s,n+1} \in \mathbb{S}^2$ for every $s \in [0,T]$. The continuity of $s \longmapsto \|Y^{s,n}\|_{\mathbb{S}^2}$, Assumption U together with Lemma 5.6.3 guarantees $Y^{n+1} \in \mathbb{S}^{2,2}$. Moreover, the pathwise continuity $Y^{s,n+1}$ for any $s \in [0,T]$ guarantees $(Y_t^{t,n+1})_{t\in[0,T]}$ is well-defined $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e.

Let $\Delta Y^n := Y^n - Y^{n-1}$. Then, for any $s \in [0,T]$

$$\Delta Y_t^{s,n+1} = \int_0^t \big(h_r(s, X_{\cdot\wedge r}, Y_r^{s,n}, Y_r^{r,n}) - h_r(s, X_{\cdot\wedge r}, Y_r^{s,n-1}, Y_r^{r,n-1})\big)\mathrm{d}r.$$

The inequality $2ab \leq \varepsilon^{-1}a^2 + \varepsilon b^2$ for any $\varepsilon > 0$ yields that for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\begin{aligned}
&\mathrm{e}^{-ct}|\Delta Y_t^{s,n+1}|^2 \\
&= \int_0^t \mathrm{e}^{-cr}\big(2|\Delta Y_r^{s,n+1}|(h_r(s, X_{\cdot\wedge r}, Y_r^{s,n}, Y_r^{r,n}) - h_r(s, X_{\cdot\wedge r}, Y_r^{s,n-1}, Y_r^{r,n-1})) - c|\Delta Y_r^{s,n+1}|\big)\mathrm{d}r \\
&\leq \int_0^t \mathrm{e}^{-cr}\big(|\Delta Y_r^{s,n+1}|^2(C(\varepsilon) - c) + \varepsilon|\Delta Y_r^{s,n}|^2 + \varepsilon|\Delta Y_r^{r,n}|^2\big)\mathrm{d}r,
\end{aligned}$$

we then find that for $c > C(\varepsilon)$

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \mathrm{e}^{-ct}|\Delta Y_t^{s,n+1}|^2\bigg] \leq \varepsilon\mathbb{E}\bigg[\int_0^T \mathrm{e}^{-cr}\big(|\Delta Y_r^{s,n}|^2 + |\Delta Y_r^{r,n}|^2\big)\mathrm{d}r\bigg]. \tag{5.6.3}$$

Now, as $\Delta Y^{s,n} \in \mathbb{S}^{2,2}$ we may use (5.6.2) back in (5.6.3) and obtain that for $\tilde\varepsilon = \frac{1}{8T}$, $c > C(\tilde\varepsilon)$, we have$\|\Delta Y^{n+1}\|^2_{\mathbb{S}^{2,2,c}} \leq 4^{-1}\|\Delta Y^n\|^2_{\mathbb{S}^{2,2,c}}$. Inductively, we find that for all $n \geq 1$, $\|\Delta Y^n\|^2_{\mathbb{S}^{2,2,c}} \leq C4^{-n}$. Thus, for $m > n$,

$$\|Y^m - Y^n\|_{\mathbb{S}^{2,2,c}} \leq \sum_{k=n+1}^m \|\Delta Y^k\|_{\mathbb{S}^{2,2,c}} \leq \sum_{k=n+1}^m \frac{C}{2^k} \leq \frac{C}{2^n}.$$

Hence there is $Y \in \mathbb{S}^{2,2}$ such that $Y^n \xrightarrow{n\longrightarrow 0} Y$. □

We now establish a result regarding the differentiability of (5.6.1). Recall that for $Z \in \overline{\mathbb{H}}^{2,2}$ there exists by definition, see Section 5.3.1, a process ∂Z which can be interpreted as the derivative of the mapping $([0,T], \mathcal{B}([0,T])) \longrightarrow (\mathbb{H}^{2,2}, \|\cdot\|_{\mathbb{H}^2}) : s \longmapsto Z^s$.

Proposition 5.6.5. *Let* Assumption U *hold and* $Y \in \mathbb{S}^2$ *be the solution to* (5.6.1). *There is a* *unique process* $\partial Y \in \mathbb{S}^{2,2}$ *that satisfies*

$$\partial Y_t^s = \partial_s Y_0^s + \int_0^t \nabla h_r(s, X_{\cdot\wedge r}, \partial Y_r^s, Y_r^s, Y_r^r)\mathrm{d}r + \int_0^t \partial Z_r^s \cdot \mathrm{d}X_r,\ t \in [0,T],\ \mathbb{P}\text{–a.s.},\ s \in [0,T]. \quad (5.6.4)$$

Moreover

$$\int_0^s \partial Y^u \mathrm{d}u = Y^s - Y^0,\ \textit{in } \mathbb{S}^2.$$

Proof. Note that given the pair $(Y,Z) \in \mathbb{S}^{2,2} \times \overline{\mathbb{H}}^{2,2}$, Assumption U guarantees there is $C > 0$ such that

$$\sup_{s\in[0,T]} \bigg(\int_0^T |\nabla \tilde{h}_r(s, X_{\cdot\wedge r}, 0, Y_r^s, Y_r^r)|\mathrm{d}r \bigg)^2 \leq C\Big(\|\partial_s \tilde{h}\|_{\mathbb{L}^{1,2,2}}^2 + \|Y\|_{\mathbb{S}^{2,2}}^2 \Big) < \infty.$$

We now note that Assumption U.(ii) guarantees $\mathrm{u} \longmapsto \nabla h_t(s,x,\mathrm{u},y,u)$ is Lipschitz uniformly in (s,t,x,y,u). Therefore, Proposition 5.6.4 guarantees there is a unique solution $\partial Y \in \mathbb{S}^{2,2}$. The second part of the statement, follows arguing as in Lemma 3.6.1 in light of the stability result in Lemma 5.6.3 and the fact $Z \in \overline{\mathbb{H}}^{2,2}$. $\square$

5.7 Proofs of Section 5.4

5.7.1 Proof of Proposition 5.4.5

We first note that, $\mathbb{P}$–a.s.

$$\begin{aligned}
&X_T - \widehat{Y}_T^{y_0,Z} \\
&= x_0 - \widehat{Y}_0 + \int_0^T \bigg(\lambda_r^\star(X_{\cdot\wedge r}, \widehat{Z}_r)(1-\widehat{Z}_r) + \widehat{H}_r(X_{\cdot\wedge r}, \widehat{Z}_r) - \frac{\gamma_{\mathrm{A}}}{2}|\sigma_r^\top(X_{\cdot\wedge r})\widehat{Z}_r|^2 - \frac{1}{\gamma_{\mathrm{A}}}\frac{f'(T-r)}{f(T-r)} \bigg)\mathrm{d}r \\
&\quad + \int_0^T \sigma_r^\top(X_{\cdot\wedge r})(1-\widehat{Z}_r)\mathrm{d}B_r^{a^\star(\hat{Z})},
\end{aligned}$$

so that,

$$\mathrm{U}_{\mathrm{P}}(X_T - \widehat{Y}_T^{y_0,Z}) = C_{\widehat{Y}_0} M_T \exp\bigg(-\gamma_{\mathrm{P}} \int_0^T G_r(X_{\cdot\wedge r}, \widehat{Z}_r)\mathrm{d}r \bigg),\ \mathbb{P}\text{–a.s.},$$

where

$$G_t(x,z) := \lambda_r^\star(x,z) - k_r^{o\star}(x,z) - \frac{\gamma_A}{2}|\sigma_r^\top(x)z|^2 - \frac{\gamma_\mathrm{P}}{2}|\sigma_r^\top(x)(1-z)|^2 - \frac{1}{\gamma_\mathrm{A}}\frac{f'(T-r)}{f(T-r)},$$

and M denotes the supermartingale

$$M_t := \exp\bigg(- \gamma_\mathrm{P}\int_0^t \sigma_r^\top(X_{\cdot\wedge r})(1-\widehat{Z}_r)\cdot \mathrm{d}B_r^{a^\star(\hat{Z})} - \frac{\gamma_\mathrm{P}^2}{2}\int_0^t |\sigma_r^\top(X_{\cdot\wedge r})(1-\widehat{Z}_r)|^2\mathrm{d}r\bigg),\ t\in[0,T].$$

Indeed, M is a local martingale that is bounded from below and $M_0=1$. Consequently,

$$\mathbb{E}^{\mathbb{P}^\star(Z)}\big[\mathrm{U}_\mathrm{P}\big(X_T - \widehat{Y}_T^{y_0,Z}\big)\big] \leq C_{\widehat{Y}_0}\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(- \gamma_\mathrm{P}\int_0^T G_r(X_{\cdot\wedge r},\widehat{Z}_r)\mathrm{d}r\bigg)\bigg].$$

Now, under assumptions (i) and (ii) in the statement it is clear that

$$\mathrm{V_P} \leq C_{\widehat{R}_0} f(T)^{\frac{\gamma_\mathrm{P}}{\gamma_\mathrm{A}}}\exp\bigg(- \gamma_\mathrm{P}\int_0^T g(z^\star(t))\mathrm{d}t\bigg) =: \mathrm{V}^{\mathrm{P},\star},$$

where the upper bound is given by

$$\sup_{z\in\mathbb{R}}\Big\{\lambda_r^\star(z) - k_r^{o\star}(z) - \frac{\gamma_A}{2}|\sigma_r^\top z|^2 - \frac{\gamma_\mathrm{P}}{2}|\sigma_r^\top(1-z)|^2\Big\}.$$

Let us now show the upper bound is attained. Indeed, letting

$$\begin{aligned} Z_t^\star &:= -\gamma_\mathrm{A}\mathrm{U}_\mathrm{A}^o(\widehat{Y}_t^{\hat{R}_0,z^\star})z^\star(t),\\ \widehat{Y}_t^{\hat{R}_0,z^\star} &:= \widehat{R}_0 - \int_0^t\bigg(\widehat{H}_r(z_r^\star) - \frac{\gamma_\mathrm{A}}{2}|\sigma_r^\top z_r^\star|^2 - \frac{1}{\gamma_\mathrm{A}}\frac{f'(T-r)}{f(T-r)}\bigg)\mathrm{d}r + \int_0^t z_r^\star \mathrm{d}X_r,\ t\in[0,T], \end{aligned}$$

it is easy to verify that the integrability assumption on $z^\star$ guarantees that $Z^\star\in\mathcal{H}^2$. We conclude that $\xi^\star\in\overline{\overline{\Xi}}$, where $\xi^\star$ denotes the contract induced by $\widehat{R}_0$ and $Z^\star$, is optimal as it attains $\mathrm{V}^{\mathrm{P},\star}$. this concludes the proof.

5.7.2 Proof of Lemma 5.4.7

We argue (i). Let $Z \in \mathcal{H}^{2,2}$. Recall

$$Y_t^{s,y_0,Z} = y_0^s - \int_0^t \big(\lambda_r^\star(X_{\cdot\wedge r}, Z_r^r)Z_r^s - f(r-s)c_r^\star(X_{\cdot\wedge r}, Z_r^r)\big)\mathrm{d}r + \int_0^t Z_r^s \cdot \mathrm{d}X_r,$$

and, in light of (5.3.6), we have

$$\mathrm{U}_{\mathrm{A}}^{o\,(-1)}\bigg(\frac{Y_T^{s,y_0,Z}}{f(T-s)}\bigg) = \mathrm{U}_{\mathrm{A}}^{o\,(-1)}\bigg(\frac{Y_T^{u,y_0,Z}}{f(T-u)}\bigg),\ (s,u) \in [0,T].$$

Therefore, for any $s \in [0,T]$

$$\begin{aligned}
0 &= \frac{Y_0^{s,y_0,Z}}{f(T-s)} - \frac{Y_0^{0,y_0,Z}}{f(T)} \\
&\quad - \int_0^T \lambda_r^\star(X_{\cdot\wedge r}, Z_r^r)\bigg(\frac{Z_r^s}{f(T-s)} - \frac{Z_r^0}{f(T)}\bigg) - c_r^\star(X_{\cdot\wedge r}, Z_r^r)\bigg(\frac{f(r-s)}{f(T-s)} - \frac{f(r)}{f(T)}\bigg)\mathrm{d}r \\
&\quad + \int_0^T \bigg(\frac{Z_r^s}{f(T-s)} - \frac{Z_r^0}{f(T)}\bigg) \cdot \mathrm{d}X_r \\
&= \frac{Y_0^{s,y_0,Z}}{f(T-s)} - \frac{Y_0^{0,y_0,Z}}{f(T-u)} + \frac{1}{f(T-s)}\int_0^T \delta_r^\star(s, X_{\cdot\wedge r}, Z_r^r)\mathrm{d}r \\
&\quad + \int_0^T \bigg(\frac{Z_r^s}{f(T-s)} - \frac{Z_r^0}{f(T)}\bigg) \cdot \big(\mathrm{d}X_r - \lambda_r^\star(X_{\cdot\wedge r}, Z_r^r)\mathrm{d}r\big) \\
&= \frac{Y_t^{s,y_0,Z}}{f(T-s)} - \frac{Y_t^{0,y_0,Z}}{f(T)} + \frac{1}{f(T-s)}\int_t^T \delta_r^\star(s, X_{\cdot\wedge r}, Z_r^r)\mathrm{d}r \\
&\quad + \int_t^T \bigg(\frac{Z_r^s}{f(T-s)} - \frac{Z_r^0}{f(T)}\bigg) \cdot \big(\mathrm{d}X_r - \lambda_r^\star(X_{\cdot\wedge r}, Z_r^r)\mathrm{d}r\big).
\end{aligned}$$

The result then follows taking conditional expectation thanks to the integrability of Z^s and Z^0.

We now argue (ii). Let $s \in [0,T]$ be fixed. Note that

$$\begin{aligned}
N_t^{s,Z} &:= \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\int_t^T \delta_r^\star(s, X_{\cdot\wedge r}, Z_r^r)\mathrm{d}r\bigg|\mathcal{F}_t\bigg] \\
&= M_t^{s,Z} - \int_0^t \delta_r^\star(s, X_{\cdot\wedge r}, Z_r^r)\mathrm{d}r \\
&= M_0^{s,Z} + \int_0^t \widetilde{Z}_r^{s,Z} \cdot \big(\mathrm{d}X_r - \lambda_r^\star(X_{\cdot\wedge r}, Z_r^r)\mathrm{d}r\big) - \int_0^t \delta_r^\star(s, X_{\cdot\wedge r}, Z_r^r)\mathrm{d}r. \qquad (5.7.1)
\end{aligned}$$

Therefore, in light of (i), we have that there exists a finite variation process A such that

$$\begin{aligned} Y_t^{s,y_0,Z} &= \frac{f(T-s)}{f(T)} Y_t^{0,y_0,Z} - \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\int_t^T \delta_r^\star(s,X_{\cdot\wedge r},Z_r^r)\mathrm{d}r\bigg|\mathcal{F}_t\bigg] \\ &= A_t - \int_0^t \bigg(\frac{f(T-s)}{f(T)} Z_t^0 - \widetilde{Z}_t^{s,Z}\bigg)\cdot\big(\mathrm{d}X_r - \lambda_r^\star(X_{\cdot\wedge r},Z_r^r)\mathrm{d}r\big),\ \mathbb{P}\text{–a.s.} \end{aligned}$$

The result then follows from the uniqueness of the Itô decomposition of $Y_t^{s,y_0,Z}$.

We are only left to argue (iii) as (iv) is a direct consequence. The inclusion $\mathcal{H}^{2,2}\subseteq\mathcal{H}^\bullet$ follows from (ii) and taking $Y_t^Z := Y_t^{0,y_0,Z}/f(T)$. Conversely, given $Z\in\mathcal{H}^\bullet$ we define for any $s\in[0,T]$

$$Y_t^{0,y_0,Z} := f(T)Y_t^{y_0,Z},\ Y_t^{s,y_0,Z} := \frac{f(T-s)}{f(T)}Y_t^{0,y_0,Z} - N_t^{s,Z},\ t\in[0,T],\ \mathbb{P}\text{–a.s.}$$

Let us note $Y_0^{s,y_0,Z}$ is clearly differentiable and $Y^{s,y_0,Z}$ satisfies (5.3.6). Indeed, as $N_T^{s,Z}=0$, $s\in[0,T]$, we have

$$\frac{Y_T^{s,y_0,Z}}{f(T-s)} = Y_T^{y_0,Z} = \frac{Y_T^{u,y_0,Z}}{f(T-u)},\ (s,u)\in[0,T]^2.$$

We now verify $Y^{y_0,Z}\in\mathbb{S}^{2,2}$. Let us first note that $\|\widetilde{Z}\|^2_{\mathbb{H}^{2,2}}<\infty$. Indeed,

$$\begin{aligned} \|\widetilde{Z}^s\|^2_{\mathbb{H}^2} &= \mathbb{E}^{\mathbb{P}}\bigg[\int_0^t |\sigma_r\sigma_r^\top \widetilde{Z}_r^{s,Z}|^2\mathrm{d}r\bigg] \\ &= \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[M_t\int_0^t |\sigma_r\sigma_r^\top \widetilde{Z}_r^{s,Z}|^2\mathrm{d}r\bigg] \leq \mathbb{E}^{\mathbb{P}^\star(Z)}\bigg[\int_0^t |\sigma_r\sigma_r^\top \widetilde{Z}_r^{s,Z}|^2\mathrm{d}r\bigg]<\infty, \end{aligned}$$

where M denotes the supermartingale given by

$$M_t := \exp\bigg(-\int_0^t b_r^\star(X_{\cdot\wedge r},Z_r^r)\cdot\mathrm{d}B_r^\star - \frac{1}{2}\int_0^t |b_r^\star(X_{\cdot\wedge r},Z_r^r)|^2\mathrm{d}r\bigg),\ t\in[0,T].$$

From this, it follows by Assumption Q that

$$\|N^Z\|^2_{\mathbb{S}^{2,2}} \leq C\bigg(\mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \big|\lambda_r^\star(X,Z_r^r)\big|^2\mathrm{d}r\bigg] + \sup_{s\in[0,T]}\mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \big|\delta_r^\star(s,X,Z_r^r)\big|^2\mathrm{d}r\bigg] + \|\widetilde{Z}\|^2_{\mathbb{H}^{2,2}}\bigg)<\infty.$$

We also note that the continuity of $s\longmapsto f(t-s)$ implies the continuity of $s\longmapsto\|N^{s,Z}\|_{\mathbb{S}^2}$. Moreover,

$\|Y^{0,y_0,Z}\|^2_{\mathbb{S}^2} < \infty$ guarantees, by definition, that $\|Y^{y_0,Z}\|^2_{\mathbb{S}^{2,2}} < \infty$. Moreover, by definition

$$
\begin{aligned}
&Y_t^{s,y_0,Z}\\
&=\frac{f(T-s)}{f(T)}\Big(y_0+\int_0^t c_r^\star(X_{\cdot\wedge r}Z_r^r)f(r)\mathrm{d}r+\int_0^t Z_r^0\cdot(\mathrm{d}X_r-\lambda_r^\star(X_{\cdot\wedge r},Z_r^r)\mathrm{d}r)\Big)\\
&\quad -M_0^{s,Z}-\int_0^t \widetilde{Z}_r^{s,Z}\cdot(\mathrm{d}X_r-\lambda_r^\star(X_{\cdot\wedge r},Z_r^r)\mathrm{d}r)+\int_0^t \delta_r^\star(s,X_{\cdot\wedge r},Z_r^r)\mathrm{d}r\\
&=\frac{f(T-s)}{f(T)}y_0-M_0^{s,Z}+\int_0^t c_r^\star(X_{\cdot\wedge r}Z_r^r)f(r-s)\mathrm{d}r\\
&\quad+\int_0^t\Big(\frac{f(T-s)}{f(T)}Z_r^0-\widetilde{Z}_r^{s,Z}\Big)\cdot(\mathrm{d}X_r-\lambda_r^\star(X_{\cdot\wedge r},Z_r^r)\mathrm{d}r)\\
&=\frac{f(T-s)}{f(T)}y_0-M_0^{s,Z}+\int_0^t c_r^\star(X_{\cdot\wedge r}Z_r^r)f(r-s)\mathrm{d}r+\int_0^t Z_r^s\cdot(\mathrm{d}X_r-\lambda_r^\star(X_{\cdot\wedge r},Z_r^r)\mathrm{d}r)\\
&=y_0^s-\int_0^t h_r^\star(s,X_{\cdot\wedge r},Z_r^s,Z_r^r)\mathrm{d}r+\int_0^t Z_r^s\cdot\mathrm{d}X_r,
\end{aligned}
$$

where the third inequality follows from the fact $Z\in\mathcal{H}^\bullet$. We conclude $Z\in\mathcal{H}^{2,2}$.

5.7.3 Proof of Lemma 5.4.14

Let us note that (ii) and (iii) are argued as in Lemma 5.4.7. We now argue (i). Let $(y_0,Z)\in\mathcal{I}\times\mathcal{H}^{2,2}$. Note that given $Y^{s,y_0,Z}$, in light of the regularity of y_0^s and the generator, it is possible to define $\partial Y^{s,y_0,Z}$ such that

$$
\begin{aligned}
Y_t^{t,y_0,Z}&=y_0^0-\int_0^t\Big(H_r(X_{\cdot\wedge r},Y_r^{r,y_0,Z},Z_r^r)-\partial Y_r^{r,Z}\Big)\mathrm{d}r+\int_0^t Z_r^r\cdot\mathrm{d}X_r,\\
Y_t^{s,y_0,Z}&=y_0^s-\int_0^t h_r^\star(s,X_{\cdot\wedge r},Y_r^{s,y_0,Z},Z_r^s,Y_r^{r,y_0,Z},Z_r^r)\mathrm{d}r+\int_0^t Z_r^s\cdot\mathrm{d}X_r,\\
\partial Y_t^{s,y_0,Z}&=\partial y_0^s-\int_0^t\nabla h_r^\star(s,X_{\cdot\wedge r},\partial Y_r^{s,y_0,Z},\partial Z_r^s,Y_r^{s,y_0,Z},Y_r^{r,y_0,Z},Z_r^r)\mathrm{d}r+\int_0^t\partial Z_r^s\cdot\mathrm{d}X_r.
\end{aligned}
$$

Letting

$$\widehat{Z}_t:=\frac{1}{-\gamma_{\mathrm{A}}}\frac{Z_t^t}{Y_t^{t,y_0,Z}},\ \text{and,}\ \widehat{Z}_t^s:=\frac{1}{-\gamma_{\mathrm{A}}}\frac{Z_t^s}{Y_t^{s,y_0,Z}},$$

we obtain that

$$
\begin{aligned}
Y_t^{t,y_0,Z}&=y_0^0-\int_0^t\Big(-\gamma_{\mathrm{A}}Y_r^{r,y_0,Z}\widehat{H}_r(X_{\cdot\wedge r},Z_r^r)-\partial Y_r^{r,y_0,Z}\Big)\mathrm{d}r+\int_0^t Z_r^r\cdot\mathrm{d}X_r,\\
Y_t^{s,y_0,Z}&=y_0^s-\int_0^t-\gamma_{\mathrm{A}}Y_r^{s.y_0,Z}\widehat{h}_r^\star(s,X_{\cdot\wedge r},\widehat{Z}_r^s,\widehat{Z}_r^r)\mathrm{d}r+\int_0^t Z_r^s\cdot\mathrm{d}X_r,
\end{aligned}
$$

$$\partial Y_t^{s,Z} = \partial y_0^s - \int_0^t \nabla h_r^\star(s, X_{\cdot\wedge r}, \partial Y_r^{s,y_0,Z}, \partial Z_r^s, Y_r^{s,y_0,Z}, \widehat{Z}_r^r)\mathrm{d}r + \int_0^t \partial Z_r^s \cdot \mathrm{d}X_r.$$

The result then follows by Itô's formula introducing

$$\widehat{Y}_t^{s,y_0,Z} := -\frac{1}{\gamma_{\mathrm{A}}}\ln(-\gamma_{\mathrm{A}} Y_t^{s,y_0,Z}),\ \partial \widehat{Y}_t^{s,y_0,Z} := \frac{1}{-\gamma_{\mathrm{A}}}\frac{\partial Y_t^{s,y_0,Z}}{Y_t^{s,y_0,Z}},\ \text{and}$$
$$\partial \widehat{Z}_t^s := \frac{1}{-\gamma_{\mathrm{A}}}\bigg(\frac{\partial Z_t^s}{Y_t^{s,y_0,Z}} + \gamma_{\mathrm{A}}^2 \partial \widehat{Y}_t^{s,y_0,Z}\widehat{Z}_t^s\bigg).$$

5.7.4 Proof of Proposition 5.4.17

Note that it always holds that, $\mathbb{P}^\star(Z)$–a.s.

$$\begin{aligned}
&X_T - \mathrm{U}_{\mathrm{A}}^{o\,(-1)}(Y_T^{0,Z})/g(T)\\
&= x_0 - \frac{\mathrm{U}_{\mathrm{A}}^{(-1)}(Y_0^0)}{g(T)} + \int_0^T \bigg(\lambda_r^\star(X_{\cdot\wedge r}, \widehat{Z}_r^r) - \frac{g(r)}{g(T)}k_r^{o\star}(X_{\cdot\wedge r}, \widehat{Z}_r^r) - \frac{\gamma_{\mathrm{A}}}{2g(T)}|\sigma_r^\top(X_{\cdot\wedge r})\widehat{Z}_r^0|^2\bigg)\mathrm{d}r\\
&\quad + \int_0^T \bigg(1 - \frac{\widehat{Z}_r^0}{g(T)}\bigg)\cdot(\mathrm{d}X_r - \lambda_r^\star(X_{\cdot\wedge r}, \widehat{Z}_r^r)\mathrm{d}r),
\end{aligned}$$

so that

$$\mathrm{U}_{\mathrm{P}}^o\Big(X_T - \mathrm{U}_{\mathrm{A}}^{o\,(-1)}(Y_T^{0,Z})/g(T)\Big) = C_{\widehat{Y}_0^0} M_T \exp\bigg(-\gamma_{\mathrm{P}}\int_0^T G_r(X_{\cdot\wedge r}, \widehat{Z}_r, \widehat{Z}_r^0)\mathrm{d}r\bigg),\ \mathbb{P}\text{–a.s.},$$

where

$$G_t(x,z,v) := \lambda_t^\star(x,z) - \frac{g(t)}{g(T)}k_t^{o\star}(x,z) - \frac{\gamma_A}{2g(T)}|\sigma_t^\top(x)v|^2 - \frac{\gamma_{\mathrm{P}}}{2}\bigg|\sigma_r^\top(x)\bigg(1 - \frac{v}{g(T)}\bigg)\bigg|^2,$$

and M denotes the supermartingale

$$M_t := \exp\bigg(-\gamma_{\mathrm{P}}\int_0^t \sigma_r^\top(X_{\cdot\wedge r})\bigg(1 - \frac{\widehat{Z}_r^0}{g(T)}\bigg)\cdot \mathrm{d}B_r^{a^\star(Z)} - \frac{\gamma_{\mathrm{P}}^2}{2}\int_0^t \bigg|\sigma_r^\top(X_{\cdot\wedge r})\bigg(1 - \frac{\widehat{Z}_r^0}{g(T)}\bigg)\bigg|^2\mathrm{d}r\bigg),\ t\in[0,T].$$

Consequently

$$\mathbb{E}^{\mathbb{P}^\star(Z)}\big[\mathrm{U}_{\mathrm{P}}^o\big(X_T - \mathrm{U}_{\mathrm{A}}^{o\,(-1)}(Y_T^{0,Z})/g(T)\big)\big] \leq C_{\widehat{Y}_0^0}\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\gamma_{\mathrm{P}}\int_0^T G_r(X_{\cdot\wedge r}, \widehat{Z}_r^r, \widehat{Z}_r^0)\mathrm{d}r\bigg)\bigg]. \quad (5.7.2)$$

Now, under the additional assumptions, we have that for any $Z \in \widetilde{\mathcal{H}}$

$$\mathbb{E}^{\mathbb{P}^\star(Z)}\big[\mathrm{U}_{\mathrm{P}}^o\big(X_T - \mathrm{U}_{\mathrm{A}}^{o\,(-1)}(\mathrm{Y}_T^{0,Z})/g(T)\big)\big] \leq C_{\widehat{R}_0}\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\gamma_{\mathrm{P}}\int_0^T G_r\Big(\zeta_r, \eta_{0,r}\zeta_r\Big)\mathrm{d}r\bigg)\bigg].$$

Therefore, as

$$z^\star(t,\eta) \in \operatorname*{arg\,max}_{z\in\mathbb{R}}\bigg\{\lambda_t^\star(z) - \frac{g(t)}{g(T)}k_t^{o\star}(z) - \frac{\gamma_A}{2g(T)}|\sigma_t^\top z|^2|\eta|^2 - \frac{\gamma_{\mathrm{P}}}{2}\bigg|\sigma_r^\top\bigg(1-\frac{z}{g(T)}\eta\bigg)\bigg|^2\bigg\},$$

as longs as η is chosen so that $Z^\eta \in \widetilde{\mathcal{H}}$ the upper bound is attained.

Let us argue the second part of the statement. Since we are now constrained to deterministic choices of η the integrability of $z^\star$ and the boundedness of $[0,T] \ni t \longmapsto g(t)$ guarantee that the constant process $M^{s,Z}$ in Lemma 5.4.2 is finite and thus square integrable. Therefore, as the contract induced by the family

$$Z_t^s := \frac{g(T-s)}{g(T-t)}z^\star(t,\eta_{0,t}^\star),\ \eta_{0,t}^\star = \frac{g(T)}{g(T-t)},$$

attains the upper bound in (5.7.2), the result follows. The last statement follows letting $(\gamma_{\mathrm{A}},\gamma_{\mathrm{P}}) \longrightarrow (0,0)$ and noticing the terms involving $\widehat{Z}^0$ in (5.7.2) vanish. Therefore the upper bound is attained by the maximiser of $G(z) = \lambda_t^\star(z) - g(t)k_t^{o\star}(z)/g(T)$, *i.e.* the deterministic contract given by $Z_t^s = f(T-s)z^\star(t)/f(T-t)$.

Conclusion

This thesis studied the decision-making of agents exhibiting time-inconsistent preferences and its implications in the context of contract theory. In particular, we studied the contracting problem between a standard utility maximiser principal and a sophisticated time-inconsistent agent. We showed that the contracting problem faced by the principal can be reformulated as a novel class of control problems exposing the complications of the agent's preferences. This corresponds to the control of a forward Volterra equation via constrained Volterra type controls. The structure of this problem is inherently related to the representation of the agent's value function via extended type-I backward stochastic differential equations. Despite the inherent challenges of this class of problems, our reformulation allows us to study the solution for different specifications of preferences for the principal and the agent. Regarding the implications of our results we can mention the following:

(i) from a methodological point of view, unlike in the time-consistent case, the solution to the moral hazard problem does not reduce, in general, to a standard stochastic control problem. Nevertheless, the solution to the risk-sharing problem between a utility maximiser principal and a time-inconsistent sophisticated agent does, see Section 2. This suggest a dire difference between the first-best and second-best problems as soon as the agent is allowed to have time-inconsistent preferences;

(ii) a second takeaway from our analysis is associated with the so-called optimality of linear contracts. These are contracts consisting of a constant part and a term proportional to the terminal value of the state process as in the seminal work of [130]. We study two examples that can be regarded as (time-inconsistent) variations of [130], which we refer to as discounted utility, and utility of discounted income. In the former case, by virtue of the simplicity of the source of time-inconsistency, we find that optimal contract is linear. In the latter case, we find that the optimal contract is no longer linear unless there is no discounting (as in [130]). Our point here is that slight deviations of the model in [130] seem to challenge the virtues attributed to linear contracts and this suggests that would typically cease to be optimal in general for time-inconsistent agents;

(*iii*) lastly, we comment on the non-Markovian nature of the optimal contract. It is known that, beyond the realm of the model in [130], the optimal contract in the time-consistent scenario is, in general, non-Markovian in the state process X, see [64]. Indeed, we find the same result in the case of an agent with separable time-inconsistent preferences. Moreover, in our context the non-Markovian structure is also manifestation of the agent's time-inconsistent preferences.

Bibliography

[1] T. Adrian and M.M. Westerfield. Disagreement and learning in a dynamic contracting model. *Review of Financial Studies*, 22(10):3873–3906, 2009.

[2] N. Agram. Dynamic risk measure for BSVIE with jumps and semimartingale issues. *Stochastic Analysis and Applications*, 37(3):361–376, 2019.

[3] N. Agram and B. Djehiche. Reflected backward stochastic Volterra integral equations and related time-inconsistent optimal stopping problems. *ArXiv preprint arXiv:2004.11654*, 2020.

[4] N. Agram, B. Øksendal, and S. Yakhlef. Optimal control of forward–backward stochastic Volterra equations. In F. Gesztesy, H. Hanche-Olsen, E.R. Jakobsen, Y.I. Lyubarskii, N.H. Risebro, and K. Seip, editors, *Non–linear partial differential equations, mathematical physics, and stochastic analysis: the Helge Holden anniversary volume*, number 250768 in EMS Series of Congress Reports, pages 3–35. European Mathematical Society Publishing House, Zürich, Switzerland, 2018.

[5] N. Agram, B. Øksendal, and S. Yakhlef. New approach to optimal control of stochastic Volterra integral equations. *Stochastics: An International Journal of Probability and Stochastic Processes*, 91(6):873–894, 2019.

[6] R. Aïd, D. Possamaï, and N. Touzi. Optimal electricity demand response contracting with responsiveness incentives. *ArXiv preprint arXiv:1810.09063*, 2018.

[7] C. Alasseur, C. Chaton, and E. Hubert. Optimal insurance contract with benefits in kind under adverse selection. *ArXiv preprint arXiv:2001.02099*, 2020.

[8] M. Allais. Le comportement de l'homme rationnel devant le risque : critique des postulats et axiomes de l'école américaine. *Econometrica*, 21(4):503–546, 1953.

[9] M. Amador, I. Werning, and G.-M. Angeletos. Commitment vs. flexibility. *Econometrica*, 74 (2):365–396, 2006.

[10] V.V. Anh and J. Yong. Backward stochastic Volterra integral equations in Hilbert spaces. In P.A. Ravi and P Kanishka, editors, *Differential & difference equations and applications*, pages 57–66. Hindawi, New York, 2006.

[11] V.V. Anh, W. Grecksch, and J. Yong. Regularity of backward stochastic Volterra integral equations in Hilbert spaces. *Stochastic Analysis and Applications*, 29(1):146–168, 2010.

[12] V.I. Arkin and M.T. Saksonov. Necessary conditions of optimality in the problems of control of stochastic differential equations. *Doklady Akademii Nauk SSSR*, 244(1):11–15, 1979.

[13] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.

[14] L. Balbus, K. Reffett, and L. Wozny. Time consistent equilibria in dynamic models with recursive payoffs and behavioral discounting. *SSRN preprint 3722808*, 2020.

[15] B. Baldacci and D. Possamaï. Governmental incentives for green bonds investment. *ArXiv preprint arXiv:2101.00648*, 2021.

[16] B. Baldacci, D. Possamaï, and M. Rosenbaum. Optimal make-take fees in a multi market-maker environment. *SIAM Journal on Financial Mathematics*, 12(1):446–486, 2021.

[17] E.J. Balder. Generalized equilibrium results for games with incomplete information. *Mathematics of Operations Research*, 13(2):265–276, 1988.

[18] R.J. Barro. Ramsey meets Laibson in the neoclassical growth model. *The Quarterly Journal of Economics*, 114(4):1125–1152, 1999.

[19] S. Basak and G. Chabakauri. Dynamic mean–variance asset allocation. *Review of Financial Studies*, 23(8):2970–3016, 2010.

[20] E. Bayraktar, J. Zhang, and Z. Zhou. On the notions of equilibria for time-inconsistent stopping problems in continuous time. *ArXiv preprint arXiv:1909.01112*, 2019.

[21] C. Bender and S. Pokalyuk. Discretization of backward stochastic Volterra integral equations. In T. Gerstner and P. Kloeden, editors, *Recent revelopments in computational finance—foundations, algorithms and applications*, volume 14 of *Interdisciplinary mathematical sciences*, pages 245–278. World Scientific Publishing Company, 2013.

[22] A. Bensoussan. Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions. *Stochastics: An International Journal of Probability and Stochastic Processes*, 9(3):169–222, 1983.

[23] M.A. Berger and V.J. Mizel. Volterra equations with Itô integrals–I. *Journal of Integral Equations*, 2(3):187–245, 1980.

[24] M.A. Berger and V.J. Mizel. Volterra equations with Itô integrals–II. *Journal of Integral Equations*, 2(4):319–337, 1980.

[25] D.P. Bertsekas and S.E. Shreve. *Stochastic optimal control: the discrete time case*, volume 139 of *Mathematics in science and engineering*. Academic Press New York, 1978.

[26] B. Biais, T. Mariotti, G. Plantin, and J.-C. Rochet. Dynamic security design: convergence to continuous time and asset pricing implications. *The Review of Economic Studies*, 74(2): 345–390, 2007.

[27] B. Biais, T. Mariotti, J.-C. Rochet, and S. Villeneuve. Large risks, limited liability, and dynamic moral hazard. *Econometrica*, 78(1):73–118, 2010.

[28] B. Biais, T. Mariotti, and J.-C. Rochet. Dynamic financial contracting. In D. Acemoglu, M. Arellano, and E. Dekel, editors, *Advances in economics and econometrics, 10th world congress of the Econometric Society, volume 1, economic theory*, number 49 in Econometric Society Monographs, pages 125–171. Cambridge University Press, 2013.

[29] J. Bielagk, A. Lionnet, and G. Dos Reis. Equilibrium pricing under relative performance concerns. *SIAM Journal on Financial Mathematics*, 8(1):435–482, 2017.

[30] A. Bisin, A. Lizzeri, and L. Yariv. Government policy with time inconsistent voters. *American Economic Review*, 105(6):1711–1737, 2015.

[31] J.-M. Bismut. *Analyse convexe et probabilités.* PhD thesis, Faculté des sciences de Paris, 1973.

[32] J.-M. Bismut. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*, 44(2):384–404, 1973.

[33] J.-M. Bismut. Contrôle des systèmes linéaires quadratiques : applications de l'intégrale stochastique. *Séminaire de probabilités de Strasbourg*, XII:180–264, 1978.

[34] T. Björk and A. Murgoci. A general theory of Markovian time inconsistent stochastic control problems. *SSRN preprint 1694759*, 2010.

[35] T. Björk and A. Murgoci. A theory of Markovian time-inconsistent stochastic control in discrete time. *Finance and Stochastics*, 18(3):545–592, 2014.

[36] T. Björk, A. Murgoci, and X.Y. Zhou. Mean–variance portfolio optimization with state–dependent risk aversion. *Mathematical Finance*, 24(1):1–24, 2014.

[37] T. Björk, M. Khapko, and A. Murgoci. Time inconsistent stochastic control in continuous time: theory and examples. *ArXiv preprint arXiv:1612.03650*, 2016.

[38] T. Björk, M. Khapko, and A. Murgoci. On time-inconsistent stochastic control in continuous time. *Finance and Stochastics*, 21(2):331–360, 2017.

[39] P. Bond and G. Sigurdsson. Commitment contracts. *The Review of Economic Studies*, 85 (1):194–222, 07 2017.

[40] B. Bouchard, D. Possamaï, X. Tan, and C. Zhou. A unified approach to *a priori* estimates for supersolutions of BSDEs in general filtrations. *Annales de l'institut Henri Poincaré, Probabilités et Statistiques* (B), 54(1):154–172, 2018.

[41] P. Briand and Y. Hu. BSDE with quadratic growth and unbounded terminal value. *Probability Theory and Related Fields*, 136(4):604–618, 2006.

[42] P. Briand and Y. Hu. Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probability Theory and Related Fields*, 141(3-4):543–567, 2008.

[43] P.-B. Brutscher. Payment matters? An exploratory study into the pre-payment electricity metering. Technical report, University of Cambridge, 2011.

[44] A. Capponi and C. Frei. Dynamic contracting: accidents lead to nonlinear contracts. *SIAM Journal on Financial Mathematics*, 6(1):959–983, 2015.

[45] G. Carroll. Robustness and linear contracts. *American Economic Review*, 105(2):536–63, 2015.

[46] D. Cetemen, F. Feng, and C. Urgun. Renegotiation and dynamic inconsistency:contracting with non-exponential discounting. Technical report, Collegio Carlo Alberto, University of Washington and Princeton University, 2021.

[47] S. Chen and J. Yong. A linear quadratic optimal control problem for stochastic Volterra integral equations. In S. Tang and J. Yong, editors, *Control theory and related topics—in memory of professor Xunjing Li, Fudan university, China, 3–5 June 2005*, pages 44–66, 2007.

[48] X. Chen and J. Sung. Managerial compensation and outcome volatility. *SSRN preprint 3140205*, 2018.

[49] P. Cheridito and K. Nam. Multidimensional quadratic and subquadratic BSDEs with special structure. *Stochastics: An International Journal of Probability and Stochastic Processes*, 87 (5):871–884, 2015.

[50] R.J. Chitashvili. Martingale ideology in the theory of controlled stochastic processes. In J.V. Prokhorov and K. Itô, editors, *Probability theory and mathematical statistics. Proceedings of the fourth USSR–Japan symposium, held at Tbilisi, USSR, August 23–29, 1982*, volume 1021 of *Lecture notes in mathematics*, pages 73–92. Springer, 1983.

[51] R.J. Chitashvili and M.G. Mania. Optimal locally absolutely continuous change of measure. Finite set of decisions. Part I. *Stochastics: An International Journal of Probability and Stochastic Processes*, 21(2):131–185, 1987.

[52] R.J. Chitashvili and M.G. Mania. Optimal locally absolutely continuous change of measure. Finite set of decisions. Part II: optimization problems. *Stochastics: An International Journal of Probability and Stochastic Processes*, 21(3):187–229, 1987.

[53] S. Christensen and K. Lindensjö. Moment constrained optimal dividends: precommitment & consistent planning. *ArXiv preprint arXiv:1909.10749*, 2019.

[54] S. Christensen and K. Lindensjö. Time-inconsistent stopping, myopic adjustment & equilibrium stability: with a mean–variance application. *ArXiv preprint arXiv:1909.11921*, 2019.

[55] S. Christensen and K. Lindensjö. On time-inconsistent stopping problems and mixed strategy stopping times. *Stochastic Processes and their Applications*, 130(5):2886–2917, 2020.

[56] J. Claisse, D. Talay, and X. Tan. A pseudo-Markov property for controlled diffusion processes. *SIAM Journal on Control and Optimization*, 54(2):1017–1029, 2016.

[57] R. Cont and D.-A. Fournié. Change of variable formulas for non-anticipative functionals on path space. *Journal of Functional Analysis*, 259(4):1043–1072, 2010.

[58] H. Coulibaly and A. Aman. Backward stochastic Volterra integral equations with time delayed generators. *Journal of Mathematics Research*, 11(3):94–102, 2019.

[59] J. Cvitanić and J. Zhang. *Contract theory in continuous-time models*. Springer, 2012.

[60] J. Cvitanić, X. Wan, and J. Zhang. Optimal contracts in continuous-time models. *International Journal of Stochastic Analysis*, 2006(095203), 2006.

[61] J. Cvitanić, X. Wan, and J. Zhang. Principal–agent problems with exit options. *The B.E. Journal of Theoretical Economics*, 8(1):1–43, 2008.

[62] J. Cvitanić, X. Wan, and J. Zhang. Optimal compensation with hidden action and lump-sum payment in a continuous-time model. *Applied Mathematics and Optimization*, 59(1):99–146, 2009.

[63] J. Cvitanić, D. Possamaï, and N. Touzi. Moral hazard in dynamic risk management. *Management Science*, 63(10):3328–3346, 2017.

[64] J. Cvitanić, D. Possamaï, and N. Touzi. Dynamic programming approach to principal–agent problems. *Finance and Stochastics*, 22(1):1–37, 2018.

[65] C. Czichowsky. Time-consistent mean–variance portfolio selection in discrete and continuous time. *Finance and Stochastics*, 17(2):227–271, 2013.

[66] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 45 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 2014.

[67] M.H.A. Davis and P. Varaiya. Dynamic programming conditions for partially observable stochastic systems. *SIAM Journal on Control and Optimization*, 11(2):226–261, 1973.

[68] F. Delbaen, Y. Hu, and X. Bao. Backward SDEs with superquadratic growth. *Probability Theory and Related Fields*, 150:145–192, 2011.

[69] C. Dellacherie and P.-A. Meyer. *Probabilities and potential*, volume 29 of *Mathematics studies*. North–Holland, 1978.

[70] C. Dellacherie and P.-A. Meyer. *Probabilities and potential B: theory of martingales*, volume 72 of *Mathematics studies*. North–Holland, 1982.

[71] S. Dellavigna. Psychology and economics: evidence from the field. *Journal of Economic Literature*, 47(2):315–372, 2009.

[72] P.M. DeMarzo and Y. Sannikov. Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance*, 61(6):2681–2724, 2006.

[73] P.M. DeMarzo, M.J. Fishman, Z. He, and N. Wang. Dynamic agency and the q theory of investment. *The Journal of Finance*, 67(6):2295–2340, 2012.

[74] L. Denis, M. Hu, and S. Peng. Function spaces and capacity related to a sublinear expectation: application to G–Brownian motion paths. *Potential Analysis*, 34(2):139–161, 2011.

[75] B. Djehiche and P. Helgesson. The principal–agent problem with time inconsistent utility functions. *ArXiv preprint arXiv:1503.05416*, 2015.

[76] M.F. Djete, D. Possamaï, and X. Tan. McKean–Vlasov optimal control: the dynamic programming principle. *ArXiv preprint arXiv:1907.08860*, 2019.

[77] M.F. Djete, D. Possamaï, and X. Tan. McKean–Vlasov optimal control: limit theory and equivalence between different formulations. *ArXiv preprint arXiv:2001.00925*, 2020.

[78] J. Djordjević and S. Janković. On a class of backward stochastic Volterra integral equations. *Applied Mathematics Letters*, 26(12):1192–1197, 2013.

[79] J. Djordjević and S. Janković. Backward stochastic Volterra integral equations with additive perturbations. *Applied Mathematics and Computation*, 265:903–910, 2015.

[80] I. Ekeland and A. Lazrak. Being serious about non-commitment: subgame perfect equilibrium in continuous time. *ArXiv preprint arXiv:0604264*, 2006.

[81] I. Ekeland and A. Lazrak. The golden rule when preferences are time inconsistent. *Mathematics and Financial Economics*, 4(1):29–55, 2010.

[82] I. Ekeland and T.A. Pirvu. Investment and consumption without commitment. *Mathematics and Financial Economics*, 2(1):57–86, 2008.

[83] O. El Euch, T. Mastrolia, M. Rosenbaum, and N. Touzi. Optimal make–take fees for market making regulation. *Mathematical Finance*, 31(1):109–148, 2021.

[84] N. El Karoui and S. Hamadène. Bsdes and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations. *Stochastic Processes and their Applications*, 107(1):145–169, 2003.

[85] N. El Karoui and X. Tan. Capacities, measurable selection and dynamic programming part I: abstract framework. *ArXiv preprint arXiv:1310.3363*, 2013.

[86] N. El Karoui and X. Tan. Capacities, measurable selection and dynamic programming part II: application in stochastic control problems. *ArXiv preprint arXiv:1310.3364*, 2013.

[87] N. El Karoui, D.H. Nguyen, and M. Jeanblanc-Picqué. Compactification methods in the control of degenerate diffusions: existence of an optimal control. *Stochastics*, 20(3):169–219, 1987.

[88] N. El Karoui, S. Peng, and M.-C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.

[89] K. Eliaz and R. Splieger. Contracting with diversely naive agents. *The Review of Economic Studies*, 73(3):689–714, 2006.

[90] R. Élie and D. Possamaï. Contracting theory with competitive interacting agents. *SIAM Journal on Control and Optimization*, 57(2):1157–1188, 2019.

[91] R. Élie, T. Mastrolia, and D. Possamaï. A tale of a principal and many many agents. *Mathematics of Operations Research*, 44(2):440–467, 2019.

[92] R. Élie, E. Hubert, T. Mastrolia, and D. Possamaï. Mean-field moral hazard for optimal energy demand response management. *Mathematical Finance*, 31(1):399–473, 2021.

[93] D. Ellsberg. Risk, ambiguity and the Savage axioms. *The Quarterly Journal of Economics*, 75(4):643–669, 1961.

[94] F. Englmaier, M. Fahn, and M.A. Schwarz. Long-term employment relations when agents are present biased. Technical report, LMU Munich, JKU Linz and University of Innsbruck, 2020.

[95] L.G. Epstein and S.E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework. *Econometrica*, 57(4):937–969, 1989.

[96] G.-É. Espinosa and N. Touzi. Optimal investment under relative performance concerns. *Mathematical Finance*, 25(2):221–257, 2015.

[97] L.C. Evans, C.W. Miller, and I. Yang. Convexity and optimality conditions for continuous time principal–agent problems. Technical report, University of California, Berkeley and Massachusetts Institute of Technology, 2015.

[98] H. Fang and D. Silverman. Time-inconsistency and welfare program participation: evidence from the NLSY. *International Economic Review*, 50(4):1043–1077, 2009.

[99] W.H. Fleming and H.M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic modelling and applied probability*. Springer–Verlag New York, 2nd edition, 2006.

[100] K. Fong. Evaluating skilled experts: optimal scoring rules for surgeons. Technical report, Stanford University, 2009.

[101] S. Frederick, G. Loewenstein, and T. O'Donoghue. Time discounting and time preference: a critical review. *Journal of Economic Literature*, 40(2):351–401, 2002.

[102] C. Frei. Splitting multidimensional BSDEs and finding local equilibria. *Stochastic Processes and their Applications*, 124(8):2654–2671, 2014.

[103] C. Frei and G. Dos Reis. A financial market with interacting investors: does an equilibrium exist? *Mathematics and Financial Economics*, 4(3):161–182, 2011.

[104] M. Friedman and L.J. Savage. The utility analysis of choices involving risk. *Journal of Political Economy*, 56(4):279–304, 1948.

[105] D. Fudenberg and D.K. Levine. A dual-self model of impulse control. *The American Economic Review*, 96(5):1449–1476, 2006.

[106] S. Galperti. Commitment, flexibility, and optimal screening of time inconsistency. *Econometrica*, 83(4):1425–1465, 2015.

[107] Steven M Goldman. Consistent plans. *The Review of Economic Studies*, 47(3):533–537, 1980.

[108] D. Gottlieb. Competition over time-inconsistent consumers. *Journal of Public Economic Theory*, 10(4):673–684, 2008.

[109] D. Gottlieb and X. Zhang. Long-term contracting with time-inconsistent agents. *Econometrica*, 89(2):793–824, 2021.

[110] F. Gul and W. Pesendorfer. Temptation and self-control. *Econometrica*, 69(6):1403–1435, 2001.

[111] M. Halac and P. Yared. Fiscal rules and discretion under persistent shocks. *Econometrica*, 82(5):1557–1614, 2014.

[112] Y. Hamaguchi. Small-time solvability of a flow of forward–backward stochastic differential equations. *Applied Mathematics and Optimization*, 2020.

[113] Y. Hamaguchi. Infinite horizon backward stochastic Volterra integral equations and discounted control problems. *ArXiv preprint arXiv:2105.02438*, 2020.

[114] Y. Hamaguchi. Extended backward stochastic Volterra integral equations and their applications to time-inconsistent stochastic recursive control problems. *Mathematical Control and Related Fields*, 11(3):433–478, 2021.

[115] Y. Hamaguchi. Time-inconsistent consumption–investment problems in incomplete markets under general discount functions. *SIAM Journal on Control and Optimization*, 59(3):2121–2146, 2021.

[116] B. Han and H.Y. Wong. Time-consistent feedback strategies with Volterra processes. *ArXiv preprint arXiv:1907.11378*, 2019.

[117] C. Harris and D. Laibson. Dynamic choices of hyperbolic consumers. *Econometrica*, 69(4): 935–957, 2001.

[118] J. Harter and A. Richou. A stability approach for solving multidimensional quadratic BSDEs. *Electronic Journal of Probability*, 24(4):1–51, 2019.

[119] U.G. Haussmann and J.-P. Lepeltier. On the existence of optimal controls. *SIAM Journal on Control and Optimization*, 28(4):851–902, 1990.

[120] X.D. He and Z. Jiang. On the equilibrium strategies for time-inconsistent problems in continuous time. *SSRN preprint 3308274*, 2019.

[121] X.D. He and X.Y. Zhou. Who are I: time inconsistency and intrapersonal conflict and reconciliation. *ArXiv preprint arXiv:2105.01829*, 2021.

[122] X.D. He, S. Hu, J. Obłój, and X.Y. Zhou. Optimal exit time from casino gambling: strategies of precommitted and naive gamblers. *SIAM Journal on Control and Optimization*, 57(3): 1845–1868, 2019.

[123] X.D. He, M.S. Strub, and T. Zariphopoulou. Forward rank-dependent performance criteria: time-consistent investment under probability distortion. *ArXiv preprint arXiv:1904.01745*, 2019.

[124] Z. He. Optimal executive compensation when firm size follows geometric Brownian motion. *Review of Financial Studies*, 22(2):859–892, 2009.

[125] P. Heidhues. Exploiting naïveté about self-control in the credit market. *The American Economic Review*, 100(5):2279–2303, 2010.

[126] M.F. Hellwig. The role of boundary solutions in principal–agent problems of the Holmström–Milgrom type. *Journal of Economic Theory*, 136(1):446–475, 2007.

[127] M.F. Hellwig and K.M. Schmidt. Discrete-time approximations of the Holmström–Milgrom Brownian-motion model of intertemporal incentive provision. *Econometrica*, 70(6):2225–2264, 2002.

[128] N. Hernández Santibáñez and T. Mastrolia. Contract theory in a VUCA world. *SIAM Journal on Control and Optimization*, 57(4):3072–3100, 2019.

[129] F. Hoffmann and S. Pfeil. Reward for luck in a dynamic agency model. *The Review of Financial Studies*, 23(9):3329–3345, 2010.

[130] B. Holmström and P. Milgrom. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica*, 55(2):303–328, 1987.

[131] Y. Hu and B. Øksendal. Linear Volterra backward stochastic integral equations. *Stochastic Processes and their Applications*, 129(2):626–633, 2019.

[132] Y. Hu and S. Peng. Adapted solution of a backward semilinear stochastic evolution equation. *Stochastic Analysis and Applications*, 9(4):445–459, 1991.

[133] Y. Hu and S. Tang. Multi-dimensional backward stochastic differential equations of diagonally quadratic generators. *Stochastic Processes and their Applications*, 126(4):1066–1086, 2016.

[134] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.

[135] Y. Hu, H. Jin, and X.Y. Zhou. Time-inconsistent stochastic linear–quadratic control. *SIAM Journal on Control and Optimization*, 50(3):1548–1572, 2012.

[136] Y. Hu, H. Jin, and X.Y. Zhou. Time-inconsistent stochastic linear–quadratic control: characterization and uniqueness of equilibrium. *SIAM Journal on Control and Optimization*, 55 (2):1261–1279, 2017.

[137] Y.-J. Huang and Z. Zhou. Strong and weak equilibria for time-inconsistent stochastic control in continuous time. *Mathematics of Operations Research*, 46(2):405–833, 2018.

[138] Y.-J. Huang and Z. Zhou. Optimal equilibria for time-inconsistent stopping problems in continuous time. *Mathematical Finance*, 30(3):1103–1134, 2020.

[139] Y.-J. Huang, A. Nguyen-Huu, and X.Y. Zhou. General stopping behaviors of naïve and noncommitted sophisticated agents, with application to probability distortion. *Mathematical Finance*, 30(1):310–340, 2020.

[140] E. Hubert. Continuous-time incentives in hierarchies. *ArXiv preprint arXiv:2007.10758*, 2020.

[141] E. Hubert, T. Mastrolia, D. Possamaï, and Warin. Incentives, lockdown, and testing: from Thucydides's analysis to the COVID-19 pandemic. *ArXiv preprint arXiv:2009.00484*, 2020.

[142] J. Jacod and A.N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der mathematischen Wissenschaften*. Springer–Verlag Berlin Heidelberg, 2003.

[143] A. Jamneshan, M. Kupper, and P. Luo. Multidimensional quadratic BSDEs with separated generators. *Electronic Communications in Probability*, 22(58):1–10, 2017.

[144] E. Jouini, M. Meddeb, and N. Touzi. Vector-valued coherent risk measures. *Finance and Stochastics*, 8(4):531–552, 2004.

[145] N. Ju and X. Wan. Optimal compensation and pay-performance sensitivity in a continuous-time principal–agent model. *Management Science*, 58(3):641–657, 2012.

[146] Yu.M. Kabanov. On the Pontryiagin maximum principle for the linear stochastic differential equations. In *Probabilistic models and control of economic processes*. TsEMI akademii nauk SSSR, Moscow, 1978.

[147] O. Kadan, P. J Reny, and M. Swinkels, J. Existence of optimal mechanisms in principal–agent problems. *Econometrica*, 85(3):769–823, 2017.

[148] A.K. Karaivanov and F.M. Martin. Markov-perfect risk sharing, moral hazard and limited commitment. *Journal of Economic Dynamics and Control*, 94:1–23, 2018.

[149] R.L. Karandikar. On pathwise stochastic integration. *Stochastic Processes and their Applications*, 57(1):11–18, 1995.

[150] I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate texts in mathematics*. Springer–Verlag New York, 2nd edition, 1998.

[151] C. Karnam, J. Ma, and J. Zhang. Dynamic approaches for some time inconsistent problems. *The Annals of Applied Probability*, 27(6):3435–3477, 2017.

[152] N. Kazamaki. *Continuous exponential martingales and BMO*, volume 1579 of *Lecture notes in mathematics*. Springer–Verlag Berlin Heidelberg, 1994.

[153] N. Kazi-Tani, D. Possamaï, and C. Zhou. Quadratic BSDEs with jumps: a fixed-point approach. *Electronic Journal of Probability*, 20(66):1–28, 2015.

[154] K.L. Keiber. Overconfidence in the continuous-time principal–agent problem. Technical report, WHU Otto Beisheim Graduate School of Management, 2003.

[155] M. Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *The Annals of Probability*, 28(2):558–602, 2000.

[156] B. Kószegi. Behavioral contract theory. *Journal of Economic Literature*, 52(4):1075–1118, 2014.

[157] D.O. Kramkov and S. Pulido. Stability and analytic expansions of local solutions of systems of quadratic BSDEs with applications to a price impact model. *SIAM Journal on Financial Mathematics*, 7(1):567–587, 2016.

[158] D.O. Kramkov and S. Pulido. A system of quadratic BSDEs arising in a price impact model. *The Annals of Applied Probability*, 26(2):794–817, 2016.

[159] E. Kromer and L. Overbeck. Differentiability of BSVIEs and dynamic capital allocations. *International Journal of Theoretical and Applied Finance*, 20(7):1–26, 2017.

[160] A.V. Kulikov. Multidimensional coherent and convex risk measures. *Theory of Probability & Its Applications*, 52(4):614–635, 2008.

[161] M. Kupper, P. Luo, and L. Tangpi. Multidimensional Markovian FBSDEs with super-quadratic growth. *Stochastic Processes and their Applications*, 129(3):902–923, 2019.

[162] J.-J. Laffont and D. Martimort. *The theory of incentives: the principal–agent model*. Princeton University Press, 2002.

[163] D. Laibson. Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics*, 112(2):443–477, 1997.

[164] D. Landriault, B. Li, D. Li, and V.R. Young. Equilibrium strategies for the mean–variance investment problem over a random horizon. *SIAM Journal on Financial Mathematics*, 9(3): 1046–1073, 2018.

[165] J.-P. Lepeltier and J. San Martín. Existence for BSDE with superlinear–quadratic coefficient. *Stochastics: An International Journal of Probability and Stochastic Processes*, 63(3–4):227–240, 1998.

[166] R.C.W. Leung. Continuous-time principal–agent problem with drift and stochastic volatility control: with applications to delegated portfolio management. *SSRN preprint 2482009*, 2014.

[167] C. Li and Z. Qiu. A solvable time-inconsistent principal–agent problem. *Discrete Dynamics in Nature and Society*, 2018(8512608):1–15, 2018.

[168] H. Li, C. Mu, and J. Yang. Optimal contract theory with time-inconsistent preferences. *Economic Modelling*, 52:519–530, 2016.

[169] W. Li, R. Wu, and K. Wang. Existence and uniqueness of M-solutions for backward stochastic Volterra integral equations. *Electronic Journal of Differential Equations*, 2014(178):1–16, 2014.

[170] C.S.H. Lim and A. Yurukoglu. Dynamic natural monopoly regulation: time inconsistency, moral hazard, and political environments. *Journal of Political Economy*, 126(1):263–312, 2018.

[171] J. Lin. Adapted solution of a backward stochastic nonlinear Volterra integral equation. *Stochastic Analysis and Applications*, 20(1):165–183, 2002.

[172] Y. Lin, Z. Ren, N. Touzi, and J. Yang. Random horizon principal–agent problem. *ArXiv preprint arXiv:2002.10982*, 2020.

[173] K. Lindensjö. A regular equilibrium solves the extended HJB system. *Operations Research Letters*, 47(5):427–432, 2019.

[174] B. Liu, C. Mu, and J. Yang. Dynamic agency and investment theory with time-inconsistent preferences. *Finance Research Letters*, 20:88–95, 2017.

[175] W. Liu, W. Huang, B. Liu, and C. Mu. Optimal mortgage contracts with time-inconsistent preferences. *The European Journal of Finance*, 25(18):1834–1855, 2019.

[176] W. Lu. Backward stochastic Volterra integral equations associated with a Lévy process and applications. *ArXiv preprint arXiv:1106.6129*, 2016.

[177] C.-t.A. Ma. Adverse selection in dynamic moral hazard. *The Quarterly Journal of Economics*, 106(1):255–275, 1991.

[178] J. Ma, T.-K.L. Wong, and J. Zhang. Time-consistent conditional expectation under probability distortion. *ArXiv preprint arXiv:1809.08262*, 2018.

[179] J. Marín-Solano and J. Navas. Consumption and portfolio rules for time-inconsistent investors. *European Journal of Operational Research*, 201(3):860–872, 2010.

[180] T. Mastrolia and D. Possamaï. Moral hazard under ambiguity. *Journal of Optimization Theory and Applications*, 179(2):452–500, 2018.

[181] T. Mastrolia and Z. Ren. Principal–agent problem with common agency without communication. *SIAM Journal on Financial Mathematics*, 9(2):775–799, 2018.

[182] E.J. McShane. *Unified Integration.* Pure and applied mathematics. Academic Press Orlando, 1983.

[183] R. Mehra and E.C. Prescoot. The equity premium: a puzzle. *Journal of Monetary Economics*, 15:145–161, 1985.

[184] H. Mei and C. Zhu. Closed-loop equilibrium for time-inconsistent McKean–Vlasov controlled problem. *ArXiv preprint arXiv:2002.06952*, 2020.

[185] P.-A. Meyer. *Probability and potentials.* Blaisdell Publishing Company, 1966.

[186] J.A. Mirrlees and R.C. Raimondo. Strategies in the principal–agent model. *Economic Theory*, 53(3):605–656, 2013.

[187] H.M. Müller. The first-best sharing rule in the continuous-time principal–agent problem with exponential utility. *Journal of Economic Theory*, 79(2):276–280, 1998.

[188] H.M. Müller. Asymptotic efficiency in dynamic principal–agent problems. *Journal of Economic Theory*, 91(2):292–301, 2000.

[189] R.B. Myerson. Leadership, trust, and power: dynamic moral hazard in high office. Technical report, University of Chicago, 2008.

[190] M. Nutz and R. van Handel. Constructing sublinear expectations on path space. *Stochastic Processes and their Applications*, 123(8):3100–3121, 2013.

[191] M. Nutz and Y. Zhang. Conditional optimal stopping: a time-inconsistent optimization. *ArXiv preprint arXiv:1901.05802*, 2019.

[192] T. O'Donoghue and M. Rabin. Doing it now or later. *The American Economic Review*, 89 (1):103–124, 1999.

[193] T. O'Donoghue and M. Rabin. Incentives for procrastinators. *The Quarterly Journal of Economics*, 114(3):769–816, 1999.

[194] M.J. Osborne and A. Rubinstein. *A course in game theory.* MIT press, 1994.

[195] H. Ou-Yang. Optimal contracts in a continuous-time delegated portfolio management problem. *The Review of Financial Studies*, 16(1):173–208, 2003.

[196] L. Overbeck and J.A.L. Röder. Path-dependent backward stochastic Volterra integral equations with jumps, differentiability and duality principle. *Probability, Uncertainty and Quantitative Risk*, 3(4):1–37, 2018.

[197] F.H. Page Jr. The existence of optimal contracts in the principal–agent model. *Journal of Mathematical Economics*, 16(2):157–167, 1987.

[198] H. Pagès. Bank monitoring incentives and optimal ABS. *Journal of Financial Intermediation*, 22(1):30–54, 2013.

[199] H. Pagès and D. Possamaï. A mathematical treatment of bank monitoring incentives. *Finance and Stochastics*, 18(1):39–73, 2014.

[200] A. Papapantoleon, D. Possamaï, and A. Saplaouras. Existence and uniqueness for BSDEs with jumps: the whole nine yards. *Electronic Journal of Probability*, 23(121):1–68, 2018.

[201] É. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *System and Control Letters*, 14(1):55–61, 1990.

[202] É. Pardoux and P.E. Protter. Stochastic Volterra equations with anticipating coefficients. *The Annals of Probability*, 18(4):1635–1655, 1990.

[203] E.S. Phelps and R.A. Pollak. On second-best national saving and game-equilibrium growth. *The Review of Economic Studies*, 35(2):185–199, 1968.

[204] T. Piskorski and A. Tchistyi. Optimal mortgage design. *Review of Financial Studies*, 23(8): 3098–3140, 2010.

[205] T. Piskorski and M.M. Westerfield. Optimal dynamic contracts with moral hazard and costly monitoring. *Journal of Economic Theory*, 166:242–281, 2016.

[206] R.A. Pollak. Consistent planning. *The Review of Economic Studies*, 35(2):201–208, 1968.

[207] A. Popier. Backward stochastic Volterra integral equations with jumps in a general filtration. *ESAIM: Probability and Statistics*, 25:133–203, 2021.

[208] D. Possamaï and X. Tan. Weak approximation of second-order BSDEs. *The Annals of Applied Probability*, 25(5):2535–2562, 2015.

[209] D. Possamaï and N. Touzi. Is there a golden parachute in Sannikov's principal–agent problem? *ArXiv preprint arXiv:2007.05529*, 2020.

[210] D. Possamaï, X. Tan, and C. Zhou. Stochastic control for a class of nonlinear kernels and applications. *The Annals of Probability*, 46(1):551–603, 2018.

[211] P.E. Protter. Volterra equations driven by semimartingales. *The Annals of Probability*, 13 (2):519–530, 1985.

[212] P.E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic modelling and applied probability.* Springer–Verlag Berlin Heidelberg, 2nd edition, 2005.

[213] M. Rabin. Psychology and economics. *Journal of Economic Literature*, 36(1):11–46, 1998.

[214] Y. Ren. On solutions of backward stochastic Volterra integral equations with jumps in Hilbert spaces. *Journal of Optimization Theory and Applications*, 144(2):319–333, 2010.

[215] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.

[216] R.A. Ryan. *Introduction to tensor products of Banach spaces.* Springer monographs in mathematics. Springer London, 2002.

[217] P.A. Samuelson. A note on measurement of utility. *The Review of Economic Studies*, 4(2): 155–161, 1937.

[218] Y. Sannikov. Agency problems, screening and increasing credit lines. Technical report, Princeton University, 2007.

[219] Y. Sannikov. A continuous-time version of the principal–agent problem. *The Review of Economic Studies*, 75(3):957–984, 2008.

[220] Y. Sannikov. Contracts: the theory of dynamic principal–agent relationships and the continuous-time approach. In D. Acemoglu, M. Arellano, and E. Dekel, editors, *Advances in economics and econometrics, 10th world congress of the Econometric Society, volume 1, economic theory*, number 49 in Econometric society monographs, pages 89–124. Cambridge University Press, 2013.

[221] M. Schäl. A selection theorem for optimization problems. *Archiv der Mathematik*, 25(1): 219–224, 1974.

[222] H. Schättler and J. Sung. The first-order approach to the continuous-time principal–agent problem with exponential utility. *Journal of Economic Theory*, 61(2):331–371, 1993.

[223] H. Schättler and J. Sung. On optimal sharing rules in discrete- and continuous-time principal–agent problems with exponential utility. *Journal of Economic Dynamics and Control*, 21(2): 551–574, 1997.

[224] M.D. Schroder, S. Sinha, and S. Levental. The continuous-time principal–agent problem with moral hazard and recursive preferences. Technical report, Michigan State University, 2010.

[225] Y. Shi and T. Wang. Solvability of general backward stochastic Volterra integral equations. *Journal of the Korean Mathematical Society*, 49(6):1301–1321, 2012.

[226] Y. Shi, T. Wang, and J. Yong. Mean-field backward stochastic Volterra integral equation. *Discrete and Continuous Dynamical Systems–Series B*, 18(7):1929–1967, 2013.

[227] Y. Shi, T. Wang, and J. Yong. Optimal control problems of forward–backward stochastic Volterra integral equations. *Mathematical Control and Related Fields*, 5(3):613–649, 2015.

[228] Y. Shi, J. Wen, and J. Xiong. Optimal control problems of forward–backward stochastic Volterra integral equations. *Journal of Differential Equations*, 269(9):6492–6528, 2020.

[229] H.M. Soner, N. Touzi, and J. Zhang. Martingale representation theorem for the G-expectation. *Stochastic Processes and their Applications*, 121(2):265–287, 2011.

[230] H.M. Soner, N. Touzi, and J. Zhang. Wellposedness of second order backward SDEs. *Probability Theory and Related Fields*, 153:149–190, 2012.

[231] S.E. Spear and S. Srivastava. On repeated moral hazard with discounting. *The Review of Economic Studies*, 54(4):599–617, 1987.

[232] D.W. Stroock and S.R.S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der mathematischen Wissenschaften*. Springer–Verlag Berlin Heidelberg, 1997.

[233] R.H. Strotz. Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies*, 23(3):165–180, 1955.

[234] J. Sung. Linearity with project selection and controllable diffusion rate in continuous-time principal–agent problems. *The RAND Journal of Economics*, 26(4):720–743, 1995.

[235] J. Sung. Corporate insurance and managerial incentives. *Journal of Economic Theory*, 74 (2):297–332, 1997.

[236] J. Sung. Optimal contracting under mean-volatility joint ambiguity uncertainties. *Economic Theory*, to appear, 2021.

[237] K.S. Tan, W. Wei, and X.Y. Zhou. Failure of smooth pasting principle and nonexistence of equilibrium stopping rules under time-inconsistency. *ArXiv preprint arXiv: 1807.01785*, 2018.

[238] R. Tevzadze. Solvability of backward stochastic differential equations with quadratic growth. *Stochastic Processes and their Applications*, 118(3):503–515, 2008.

[239] N. Van Long and G. Sorger. A dynamic principal–agent problem as a feedback Stackelberg differential game. *Central European Journal of Operations Research*, 18(4):491–509, 2010.

[240] J. van Neerven. Approximating Bochner integrals by Riemann sums. *Indagationes Mathematicae*, 13(2):197–208, 2002.

[241] M. Veraar. The stochastic Fubini theorem revisited. *Stochastics: An International Journal of Probability and Stochastic Processes*, 84(4):1–9, 2011.

[242] N. Vieille and J.W. Weibull. Multiple solutions under quasi-exponential discounting. *Economic Theory*, 39(3):513–526, 2009.

[243] F. Viens and J. Zhang. A martingale approach for fractional Brownian motions and related path dependent PDEs. *The Annals of Applied Probability*, 29(6):3489–3540, 2019.

[244] H. Wang. Extended backward stochastic Volterra integral equations, quasilinear parabolic equations, and Feynman–Kac formula. *Stochastics and Dynamics*, 21(01):215004, 2020.

[245] H. Wang and J. Yong. Time-inconsistent stochastic optimal control problems and backward stochastic Volterra integral equations. *ESAIM: Control, Optimisation and Calculus of Variations*, 27(22):1–40, 2021.

[246] H. Wang, J. Sun, and J. Yong. Recursive utility processes, dynamic risk measures and quadratic backward stochastic Volterra integral equations. *Applied Mathematics and Optimization*, to appear, 2019.

[247] H. Wang, J. Yong, and J. Zhang. Path dependent Feynman–Kac formula for forward backward stochastic Volterra integral equations. *ArXiv preprint arXiv:2004.05825*, 2020.

[248] T. Wang. Backward stochastic Volterra integral equation under local Lipschitz condition. *Journal of Shandong University* (*Natural Science*), 46(7):112–115, 2011.

[249] T. Wang. L^p solutions of backward stochastic Volterra integral equations. *Acta Mathematica Sinica, English Series*, 28(9):1875–1882, 2012.

[250] T. Wang. Linear quadratic control problems of stochastic Volterra integral equation. *Control, Optimisation and Calculus of Variations*, 24:1849–1879, 2018.

[251] T. Wang and Y. Shi. A maximum principle for forward–backward stochastic Volterra integral equations and applications in finance. *ArXiv preprint arXiv:1004.2206*, 2010.

[252] T. Wang and Y. Shi. Symmetrical solutions of backward stochastic Volterra integral equations and their applications. *Discrete and Continuous Dynamical Systems–Series B*, 14(1):251–274, 2010.

[253] T. Wang and Y. Shi. A class of time inconsistent risk measures and backward stochastic Volterra integral equations. *Risk and Decision Analysis*, 4(1):17–24, 2013.

[254] T. Wang and J. Yong. Comparison theorems for some backward stochastic Volterra integral equations. *Stochastic Processes and their Applications*, 125(5):1756–1798, 2015.

[255] T. Wang and J. Yong. Backward stochastic Volterra integral equations—representation of adapted solutions. *Stochastic Processes and their Applications*, 129(12):4926–4964, 2019.

[256] T. Wang and H. Zhang. Optimal control problems of forward–backward stochastic Volterra integral equations with closed control regions. *SIAM Journal on Control and Optimization*, 55(4):2574–2602, 2017.

[257] Y. Wang, W. Huang, B. Liu, and X. Zhang. Optimal effort in the principal–agent problem with time-inconsistent preferences. *The North American Journal of Economics and Finance*, 52(100909), 2020.

[258] Z. Wang and X. Zhang. Non-Lipschitz backward stochastic Volterra type equations with jumps. *Stochastics and Dynamics*, 7(4):479–496, 2007.

[259] Q. Wei and X. Xiao. An optimal control problem of forward–backward stochastic Volterra integral equations with state constraints. *Abstract and Applied Analysis*, 2014(432718), 2014.

[260] Q. Wei, J. Yong, and Z. Yu. Time-inconsistent recursive stochastic optimal control problems. *SIAM Journal on Control and Optimization*, 55(6):4156–4201, 2017.

[261] M.M. Westerfield. Optimal dynamic contracts with hidden actions in continuous time. Technical report, University of Southern California, 2006.

[262] N. Williams. On dynamic principal–agent problems in continuous time. Technical report, University of Wisconsin, Madison, 2009.

[263] N. Williams. Persistent private information. *Econometrica*, 79(4):1233–1275, 2011.

[264] N. Williams. A solvable continuous time dynamic principal–agent model. *Journal of Economic Theory*, 159(part B):989–1015, 2015.

[265] H. Xing and G. Žitković. A class of globally solvable Markovian quadratic BSDE systems and applications. *The Annals of Probability*, 46(1):491–550, 2018.

[266] T. Yamada and S. Watanabe. On the uniqueness of solutions of stochastic differential equations. *Journal of Mathematics of Kyoto University*, 11(1):155–167, 1971.

[267] J. Yong. Backward stochastic Volterra integral equations and some related problems. *Stochastic Processes and their Applications*, 116(5):779–795, 2006.

[268] J. Yong. Continuous-time dynamic risk measures by backward stochastic Volterra integral equations. *Applicable Analysis*, 86(11):1429–1442, 2007.

[269] J. Yong. Well-posedness and regularity of backward stochastic Volterra integral equations. *Probability Theory and Related Fields*, 142(1–2):21–77, 2008.

[270] J. Yong. Time-inconsistent optimal control problems and the equilibrium HJB equation. *Mathematical Control and Related Fields*, 2(3):271–329, 2012.

[271] M. Yılmaz. Repeated moral hazard with a time-inconsistent agent. *Journal of Economic Behavior & Organization*, 95:70–89, 2013.

[272] M. Yılmaz. Contracting with a naïve time-inconsistent agent: to exploit or not to exploit? *Mathematical Social Sciences*, 77:46–51, 2015.

[273] J. Zhang. *Backward stochastic differential equations—from linear to fully nonlinear theory*, volume 86 of *Probability theory and stochastic modelling*. Springer–Verlag New York, 2017.

[274] Y. Zhang. Dynamic contracting with persistent shocks. *Journal of Economic Theory*, 144(2): 635–675, 2009.

[275] Y. Zhou. Principal–agent analysis in continuous-time. Technical report, The Chinese University of Hong Kong, 2006.

[276] J.Y. Zhu. *Sticky incentives and dynamic agency*. PhD thesis, University of California Berkeley, 2011.