Application of Lie symmetries to Solving Partial Differential Equations associated with the Mathematics of Finance

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Abstract

In financial markets one is sometimes confronted with a complicated system of partial differential equations arising from some physical important problem, and the discovery of the explicit solution of the problem can result with very useful information. That is, the explicit solutions of the financial market models can be used as benchmarks for testing numerical methods of physical experiments. This fact is evidenced by the work of economists Black and Scholes, the Black-Scholes model, whereby they deduced the financial models from solving a linear parabolic partial differential equation that were then used in the finance literature as the main vehicle for pricing contingent claims such as call and put options, together with all other financial derivatives. Due to their work a rich arsenal of methods of theory of partial differential equations were suddenly available for mathematicians working in the area of mathematical finance. Adopting their approach of deducing prices of contingent claim via solving the associated PDE models, we apply the algorithmic quantitative theory of Lie, the Lie symmetry analysis, to derive and solve the models associated with interest rate derivatives whose price dynamics comprise of partial differential equations in their set up.

The interest rate derivative model that we consider is of great importance because it deviates from the usual models that are depended on the usual Vasicek model which has a disadvantage of producing negative interest rates. Our interest rate derivative PDE model is depended on the functional interest rate model that satisfies all properties of an interest rate model and produces positive interest rates upon certain restriction put on the co-domain. We obtain their Lie point symmetries and transformations that we then use to deduce their exact group-invariant solutions. In particular, we analyse a zero-coupon bond pricing PDE model and obtain its various reductions that we then use to solve and produce the pricing models for the aforementioned contingent claim. A systematic reductions on optimal Lie algebra is further performed to obtain optimal invariant solutions of the model as well. The resulting analytical

expressions in both cases can then be used to add to the minute number of pricing models for the interest rate derivatives instruments in the literature; also play a vital role as benchmarks to verify real world data that is analysed numerically by numerical methods in financial markets.

Declaration

I declare that, *Application of Lie symmetries to solving partial differential equations associated with the mathematics of finance* is my work, it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Bosiu Clement Kaibe June 2020

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I take this opportunity to start first by thanking God the Almighty for being with me through this tough trying times. He guided and gave me the strength to hold on until the last moment. To my parents, wife and kids, find in this work the expression of my sincere love to you.

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Index of Abbreviations

SDE, Stochastic Differential Equation;
PDE, Partial Differential Equation;
ODE, Ordinary Differential Equation;
CIR, Cox Ingersoll Ross;
KPP, Kolmogorov Petrovskii Piskunov.

List of Notations

n-dimensional Brownian motion at time $t : dZ$.

Declaration of Publications

Details of published and submitted for publication research work that forms part of the thesis.

Chapter 4
Symmetry Analysis of an Interest Rate Derivatives PDE Model in Financial Mathematics
Bosiu C. Kaibe and John G. O'Hara
Symmetry2019

Chapter 5
The Optimal System of Reductions for the Interest Rate Derivatives Pricing Model
Bosiu C. Kaibe and John G. O'Hara
Symmetry2021

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Chapter 1

General Introduction

1.1 Introduction

The movement in the world markets normally leave financial institutions exposed to different sources of financial risks. Financial institutions as a way of protecting their businesses against these risks devote part of their time developing financial instruments known as derivatives to manage these risks. A financial derivative is defined as a financial contract whose value at maturity T is determined by the price of the underlying financial asset at time T. That is, it is a financial instrument whose value depends on, or derives from, the values of other; more basic underlying variables [12]. Financial derivatives are also used by some institutions to develop financial products to meet the demand of their customers in order to remain competitive in the market. Pricing of financial derivatives have been one of a major concern to financial markets practitioners as discrepancies in financial products prices can lead to great profits to market participants. As Hull [12] points out, we have now reached a stage where those who work in finance, and many outside finance, need to understand how derivatives work, how they are used, and how they are priced. One of the pioneering breakthrough in the pricing of financial derivatives was the work by economists Black and Scholes [13]. In their framework, with the variable σ as the volatility of the stock price, μ as its expected rate of return and r as the risk-free interest rate; the price $u(S,t)$ of a derivative contingent on stock price S following the process

$$dS = \mu S dt + \sigma S dZ; \tag{1.1}$$

the authors have shown that this price can be obtained from solving the following linear parabolic differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 u}{\partial S^2} + r\left(S\frac{\partial u}{\partial S} - u\right) = 0. \tag{1.2}$$

The Black Scholes Merton model which was later called due to the contribution made by Merton [29], have been used as the main vehicle for pricing many financial products whose price dynamics comprise of partial differential equations in their formation. Black-Scholes pricing models have been extended in many ways to price several financial products such as interest rate derivatives, bond options to be precise. Pricing derivatives is one of the central concern in derivatives markets. Pricing securities involves setting up a riskless portfolio and then arguing that the return on the portfolio earns the risk-free rate of interest. This then suggests that the risk-free rate plays an important role in the pricing of derivatives. Black-Scholes model being one of the main vehicle to price contingent claims, it has been extended in many ways to deduce pricing models of many instruments. In the extension to bond options, a challenge that was met was the issue of convergence of the bond price to par at maturity. Therefore, correct modelling procedures of stochastic behaviours of interest rate especially the term structure of the interest rate through time should carefully be implemented in order to construct realistic and reliable valuation models of interest rate derivatives. Analytic approaches and numerical methods have been mostly favoured in solving differential equations eventhough analytic approaches seems to be more preferred as they provide much information on the pricing models. This is evidenced by the increasing number of research articles published in recent years using this approach, see [13]-[20]. Among analytical approaches our focus is on those that apply Lie symmetry analysis to solve PDE models arising in the field of mathematical finance. We recognise among others the work by Ibragimov and Gazizov [15] who introduced the idea of Lie symmetry analysis in finance problems when analysing the Black-Scholes pricing equation. Goard [18] also contributed by proposing new and simple solutions to the bond pricing equation (1.4) via symmetry analysis. Pooe *et al.* [19] using transformations to reduce bond-pricing equation to heat equation, obtained the solution to the zero-coupon bond via usage of those transformations. Sinkala and colleagues in [20] computed new prices for bond PDE model with special consideration given to Vasicek and CIR models. In recent years, Khalique *et al.* [21] proposed new invariant solutions and conservation laws for Vasicek pricing equation model. Lie symmetry analysis as originated in studies by mathematician Sophus Lie, have proved in studies to be one of the prominent tool for obtaining analytical solutions for differential equations.

Kaibe *et al.* [22] deduced the pricing models for the zero-coupon bond PDE model that is depended on the functional interest rate model. This part of research is the original contribution by the author of the thesis in this field. Application of Lie symmetries analysis to interest rate derivatives under functional interest rate models was for the first time introduced in the finance literature.

1.2 Research objective

Majority of the interest rate models most of them do not satisfy the following properties: mean-reversion and positivity of interest rate almost surely. A well known interest rate model which does not satisfy the second property is the Vasicek interest rate model as its interest rate can be negative. Luo *et al.* [27] using the combination of Ornstein-Unlenbeck process [39] (satisfying mean-reversion property) and Bessel process [40] (satisfying positivity property), derived and proposed the following functional interest rate model

$$dX_t = \left(-\eta(t) X_t + \frac{\epsilon(t)}{X_t} \right) dt + \sigma(t) dZ_t, \tag{1.3}$$

where $\{Z_t\}_{t\geq 0}$ is a Brownian motion, and ϵ, η, σ are given functions of t. In this setup $r_t = f(X_t, t)$ is modelled as a function of Markov state variable X_t and time t. If the range of f is confined to only positive real values, then this enables avoidance of negative interest rate which are usually occuring in Vasicek models. This interest rate model embeds most known interest rate models which can be deduced for different choices of f, η, ϵ and σ. Using this functional interest rate model it can be shown that the price dynamics of a zero-coupon bond are described in terms of the following partial differential equation

$$v_t(x,t) + \frac{\sigma^2}{2} v_{xx}(x,t) + \left(-\eta x + \frac{\epsilon}{x}\right) v_x(x,t) - rv(x,t) = 0. \tag{1.4}$$

Numerical methods such as binomial trees, Monte-Carlo simulation, and finite-difference methods are normally tools used to value financial products such as interest rate derivatives. In this work we use analytical approach of Lie symmetry analysis to derive four Lie point symmetries plus an additional infinite subalgebra; and we make use of these symmetries to deduce three types of closed-form solutions for this interest rate derivative pricing equation associated with the functional interest rate model. We further analyse the obtained solutions by investigating their application to the Vasicek interest rate model numerically. The last part of this work will focus on obtaining the corresponding group of adjoint representations and this will be used to obtain an optimal system of one-dimensional subalgebras which is used to construct a family of closed-form solutions of the zero-coupon bond pricing equation.

1.3 Structure of the thesis

The thesis is organised as follows: In chapter 2 we give a review of Lie symmetry analysis of differential equations. The next chapter, presents a brief review of the fundamentals of mathematics of finance. In chapter 4 application of Lie symmetry analysis is performed to an interest rate derivative PDE model in order to deduce its pricing models. In chapter 5 we obtain an optimal system and group-invariant solutions of an interest rate derivatives PDE model. In the final chapter, we present the conclusion and suggest some further venues of research in this area.

Chapter 2

Lie symmetry analysis of differential equations

Lie symmetry analysis is one of the powerful methods for computing analytical solutions of differential equations. Central to this algorithmic procedure, as originated in studies by Lie, is the idea of invariance of differential equation which leads to the solution of differential equation by means of Lie groups. There are many good books in the literature which give excellent introduction to the subject [1]-[6]. The terminologies and definitions which will be used in this thesis are mostly from these books. In this chapter basic concepts of Lie groups of transformations crucial in the study of invariance properties of differential equations are introduced and defined in preparation to apply them in later chapters to analyse and solve real-world problems in mathematical finance whose price dynamics comprise of partial differential equations in their setup.

2.1 Lie group properties and Definitions

In order to begin a construction process of Lie point symmetries we do so by giving few description of few concepts. We start first by introducing the idea of a group.

Definition 2.1.1: Suppose a set of elements, S, has a law of composition $\phi(\epsilon, \delta)$ between the elements ϵ and δ in S; then set S is a group, G, if the following properties are satisfied:

1. Closure: For any elements ϵ and δ in S their composition $\phi(\epsilon, \delta)$ is also in S.

2. Identity: For any element $\epsilon \in S$, there exists an identity element $\epsilon_0 \in S$ such that $\phi(\epsilon, \epsilon_0) = \epsilon = \phi(\epsilon_0, \epsilon)$.

3. Inverse: In S, for any element ϵ there exists a unique element ϵ^{-1} also in S such that $\phi(\epsilon, \epsilon^{-1}) = \epsilon_0 = \phi(\epsilon^{-1}, \epsilon)$, where ϵ_0 is the identity in S.

4. Associative: For any elements ϵ, δ and ω in S, the relation $\phi(\epsilon, \phi(\delta, \omega)) = \phi(\phi(\epsilon, \delta), \omega)$ holds.

Definition 2.1.2: If it happens that $\phi(\epsilon, \delta) = \phi(\delta, \epsilon)$ for all ϵ and δ in Group G, then the group is said to be *Abelian*.

Definition 2.1.3: A subset of elements of G that abide by the same law of composition ϕ, and forms a group, is a *subgroup* of G.

Definition 2.1.4: Let $x = (x_1, ..., x_n)$ lie in region $D \subset \Re^n$. The set of transformation

$$\tilde{x} = X(x; \epsilon), \tag{2.1}$$

defined for each x in D depending on parameter ϵ lying in the set $S \subset \Re$ and obeying the law of composition $\phi(\epsilon, \delta)$, δ also in S; forms a group of transformations on D if:

(i) For each ϵ in S the transformations are one-to-one and onto D and $\tilde{x}$ lies in D.

(ii) The set S abiding by the law of composition ϕ forms a group G.

(iii) For any identity element $\epsilon_0 \in S$, $X(x; \epsilon_0)$=x.

(iv) If $\tilde{x}$=X(x, ϵ) and $\tilde{\tilde{x}}$=X$(\tilde{x}; \delta)$, then $\tilde{\tilde{x}}$=X$(x,\ \phi(\epsilon, \delta))$.

Definition 2.1.5: A group of transformations defines a *one-parameter Lie group of transformations* if together with satisfying properties in Definition 2.1.4,

(i) The parameter ϵ in S, where S is an interval subset of $\Re$, is a continuous parameter and without no loss of generality $\epsilon = 0$ is the identity element, ϵ_0.

(ii) The function X is an analytic function of ϵ in S that is also infinitely differentiable with respect to x in D.

(iii) The law of composition $\phi(\epsilon, \delta)$ is an analytic function of ϵ and δ, where ϵ and $\delta \in S$.

In what follows, we focus on one-parameter Lie group of transformations

$$\tilde{x} = X(x; \epsilon); \tag{2.2}$$

which from it important concepts such as $infinitesimal\ transformation$, *Lie algebra* and *invariant solutions* of models of interest will be deduced. In definition 2.1.4 if properties (i) to (iv) are the only one satisfied for the transformations in equation (2.1) then we refer to the transformations as group of transformations, whereas if all properties hold we refer to them as *Lie group* $of\ transformations$.

Suppose we expand the one-parameter Lie group of transformation $\tilde{x}$ in equation (2.2) about the identity, $\epsilon = 0$, then we get (for some neighborhood of $\epsilon = 0$)

$$\begin{aligned}\tilde{x} &= x + \epsilon\left(\frac{\partial X}{\partial \epsilon}(x;\epsilon)|_{\epsilon=0}\right) + \frac{\epsilon^2}{2}\left(\frac{\partial^2 X}{\partial \epsilon^2}(x;\epsilon)|_{\epsilon=0}\right) + \ldots \\ &= x + \epsilon\left(\frac{\partial X}{\partial \epsilon}(x;\epsilon)|_{\epsilon=0}\right) + O(\epsilon^2). \end{aligned} \tag{2.3}$$

Since the expansion is done in the neighborhood of $\epsilon = 0$, higher terms of ϵ are replaced by $O(\epsilon^2)$ as they are very close to zero. Setting,

$$\xi(x) = \frac{\partial X}{\partial \epsilon}(x;\epsilon)|_{\epsilon=0}, \tag{2.4}$$

this reduces to,

$$\tilde{x} = x + \epsilon\xi(x) + O(\epsilon^2). \tag{2.5}$$

The transformation $x + \epsilon\xi(x)$ is then called the infinitesimal transformation of the Lie group of transformations $\tilde{x}$ in equation (2.2) and the components of $\xi(x)$ are called the infinitesimals.

First Fundamental theorem of Lie (Bluman *et al*. [2]): There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations (2.2) is equivalent to the solution of the initial value problem for the system of first order differential equations

$$\frac{d\tilde{x}}{d\tau} = \xi(\tilde{x}), \tag{2.6}$$

with $\tilde{x} = x$ when $\tau = 0$.

In view of Lie's first fundamental theorem, without loss of generality, it is assumed that the one-parameter (ϵ) Lie group of transformations is parametrized such that its law of composition is $\phi(\epsilon, \delta) = \epsilon+\delta$ so that $\epsilon^{-1} = -\epsilon$ and $X(\epsilon) \equiv 1$. Due to this, the terms of the infinitesimals $\xi(x)$ are now deducible from the differential equation whose dependent variable $\tilde{x}$ depends on the independent variable ϵ. That is,

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x}), \tag{2.7}$$

with $\tilde{x} = x$ when $\epsilon = 0$.

Lie in his work, has shown that an alternative way to solving associated differential equations to get infinitesimals is to introduce an operator known as the infinitesimal generator. We deduce the infinitesimals from solving the resulting system of determining equations.

Definition 2.1.6: An operator such that

$$X = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \tag{2.8}$$

where the $\xi_i(x)$ are the components of $\xi(x)$ for each $i = 1, ..., n$, is called an $infinitesimal\ generator$ of the one-parameter Lie group of transformations in equation (2.2). Important relation between one-parameter Lie group of transformations and its infinitesimal transformations is that they are equivalent [2]. Due to Lie's First Fundamental theorem, this also suggests that the one-parameter Lie group of transformations is also equivalent to its infinitesimal generator.

The infinitesimal generator is algorithmically obtained. But before detailing how is computed, we need to learn about surfaces and action of the symmetry operator on the surface.

Definition 2.1.7: Given a one-parameter Lie group of transformations in equation (2.2), a function $F(x)$ which is such that $F(x) = 0$ is referred to as an $invariant\ surface$ for a one-parameter Lie group of transformations (2.2) provided $F(\tilde{x}) = 0$ when $F(x) = 0$.

Definition 2.1.8: Suppose a function can be written in a solved form $F(x) = x_n - f(x_1, x_2, ..., x_{n-1}) = 0$, then taking into consideration one-parameter Lie group of transformations (2.2), the surface is referred to as the invariant surface of (2.2) if and only if

$$XF(x) = 0 \qquad \text{when} \qquad F(x) = 0. \tag{2.9}$$

Important point from definition 2.1.8 is that the infinitesimals $\xi_i(x)$ may now be computed from the invariant surface as a result deduce the “equivalent” group of transformations admitted by the differential equation in consideration. The obtained group due to the invariance principle maps any solution curve of the differential equation into another of the same equation. We now detail an algorithm behind obtaining the transformation groups given an invariant surface.

2.2 Lie Point Symmetries

A set of Lie symmetries (infinitesimal generators) admitted by an equation defines a (Lie) group of transformations under which the equation is invariant. Let us consider the differential equation with one dependent, given by u, and n independent variables, $x = (x_1, x_2, ..., x_n)$,

$$F(x, u, \overset{1}{u}, \overset{2}{u}, ..., \overset{k}{u}) = 0, \tag{2.10}$$

where $\overset{k}{u}$ represents all the partial differential coordinates of the kth order of u with respect to each x_i. The one-parameter Lie group of transformations

$$\begin{aligned} \tilde{x} &= X(x, u; \epsilon) = x + \epsilon\xi(x, u) + O(\epsilon^2) \\ \tilde{u} &= U(x, u; \epsilon) = x + \epsilon\eta(x, u) + O(\epsilon^2) \end{aligned} \tag{2.11}$$

acting on (x, u)-space and depending on a continuous parameter, ϵ are said to be Lie point symmetry group of equation (2.10) if the equation has the same form in the new variables $\tilde{x}$, $\tilde{u}$ as in the original variables and has as its infinitesimal

$$\chi(x, u) = (\xi(x, u), \eta(x, u)) \tag{2.12}$$

with corresponding infinitesimal generator

$$X = \sum_i \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u}, \tag{2.13}$$

where $u = u(x) = u(x_1, ..., x_n)$. We note from text such as Bluman and Kumei [2] that the kth prolongation of the symmetry is given by

$$X^{[k]} = X + \eta_i^{(1)} \frac{\partial}{\partial u_i} + ... + \eta^{(k)}_{u_{i_1,i_2,...,i_k}} \frac{\partial}{\partial_{u_{i_1,i_2,...,i_k}}} \tag{2.14}$$

where $k = 1, 2, ...$ and the extended infinitesimals $\eta^{(k)}$ are given by

$$\begin{aligned} \eta_i^{(1)} &= D_i\eta - (D_i\xi_j)u_j, \\ \eta^{(k)}_{i_1,i_2...i_k} &= D_{i_k}\eta^{(k-1)}_{i_1 i_2 ... i_{k-1}} - (D_{i_k}\xi_j)u_{i_1,i_2...i_{k-1}j} \end{aligned} \tag{2.15}$$

where $i = 1, 2, ..., n$ and $i_l = 1, 2, 3, ..., n$ for $l = 1, ..., k$ with $k = 2, 3, ...,$ and D_i is the total derivative operator given explicitly by

$$D_i = \frac{d}{dx_i} = \frac{\partial}{\partial_{x_i}} + u_i \frac{\partial}{\partial_u} + u_{ij}\frac{\partial}{\partial_{u_j}} + ... + u_{i_1 i_2 ... i_n}\frac{\partial}{\partial_{u_{i_1,i_2...i_n}}} + ... \tag{2.16}$$

We aim to obtain X which is obtained from the following invariance principle of a PDE. Suppose the operator X in equation (2.13) is the *infinitesimal generator* of the transformations in equation (2.11). Then the *infinitesimal transformations* in (2.11) are said to be admitted by the PDE of the form of equation (2.10)

$$F(x, u, \overset{1}{u}, \overset{2}{u}, ..., \overset{k}{u}) = 0, \tag{2.17}$$

provided

$$X^{[k]}(F(x, u, \overset{1}{u}, \overset{2}{u}, ..., \overset{k}{u}))|_{F=0} = 0 \tag{2.18}$$

where $X^{[k]}$ is the k-prolongation of X and $|_{F=0}$ means evaluated at the solved form of equation (2.17). Solving the invariance condition in equation (2.18) using the fact that it is independent of the derivatives of u, one can deduce the system of undetermined partial differential equations in ξ and η, which when solved we result with Lie symmetries admitted by the given surface. We illustrate this concept with a way of example.

Let us consider a nonlinear potential *Burgers′ equation*

$$u_t = u_{xx} + u_x^2. \tag{2.19}$$

In this case the symmetry generator in equation (2.13) will have the form

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.20}$$

Extending this symmetry generator to second prolongation results with,

$$X^{[2]} = X + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}} \tag{2.21}$$

where η^t, η^x, η^{tt}, η^{xt} and η^{xx} are respectively deduced from equation 2.15 as

$$\eta^t = \eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u, \tag{2.22}$$

$$\eta^x = \eta_x + u_x\eta_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u, \tag{2.23}$$

$$\begin{aligned}
\eta^{tt} &= \eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + u_t^2\eta_{uu} - 2u_{tt}\tau_t - u_t\tau_{tt} - 2u_t^2\tau_{tu} - 3u_t u_{tt}\tau_u \quad (2.24)\\
&- u_t^3\tau_{uu} - 2u_{tx}\xi_t - u_x\xi_{tt} - 2u_t u_x\xi_{tu} - u_t^2 u_x\xi_{uu} - (u_x u_{tt} + 2u_t u_{tx})\xi_u,\\
\eta^{tx} &= \eta_{tx} + u_x\eta_{tu} + u_t\eta_{xu} + u_{xt}\eta_u + u_t u_x\eta_{uu} - u_{tx}(\tau_t + \xi_x) - u_t\tau_{tx} - u_{tt}\tau_x\\
&- u_t u_x(\tau_{tu} + \xi_{xu}) - u_t^2\tau_{xt} - (2u_t u_{tx} + u_x u_{tt})\tau_u - u_t^2 u_x\tau_{uu} - u_x\xi_{tx}\\
&- u_{xx}\xi_t - u_x^2\xi_{tu} - (2u_x u_{tx} + u_t u_{xx})\xi_u - u_t u_x^2\xi_{uu}, \quad (2.25)\\
\eta^{xx} &= \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi_x - u_x\xi_{xx} - 2u_x^2\xi_{xu} - 3u_x u_{xx}\xi_u\\
&- u_x^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} - 2u_t u_x\tau_{xu} - (u_t u_{xx} + 2u_x u_{tx})\tau_u - u_t u_x^2\tau_{uu}.
\end{aligned}$$

$$(2.26)$$

Applying the second prolongation in equation (2.21) on a potential $Burgers'$ *equation* (2.19) this results with ξ, τ and η satisfying the following symmetry conditions

$$\eta^t = \eta^{xx} + 2u_x\eta^x \tag{2.27}$$

If we respectively substitute η^t, η^{xx} and η^x from equations (2.22), (2.26) and (2.23) in equation (2.27) (and also substituting $u_{xx}+u_x^2$ for u_t and simplifying), results with the following infinitesimals

$$
\begin{aligned}
\xi &= c_1 + c_4 x + 2c_5 t + 4c_6 xt, \\
\tau &= c_2 + 2c_4 t + 4c_6 t^2, \\
\eta &= \rho(x,t)e^{-u} + c_3 - c_5 x - 2c_6 t - c_6 x^2,
\end{aligned}
\tag{2.28}
$$

where $c_1, ..., c_6$ are any arbitrary constants and $\rho(x,t)$ is an arbitrary solution to the *heat equation*: $\rho_t = \rho_{xx}$. The symmetry algebra is thus generated by

$$X_1 = \partial_x, \tag{2.29}$$

$$X_2 = \partial_t, \tag{2.30}$$

$$X_3 = \partial_u, \tag{2.31}$$

$$X_4 = x\partial_x + 2t\partial_t, \tag{2.32}$$

$$X_5 = 2t\partial_x - x\partial_u, \tag{2.33}$$

$$X_6 = 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)\partial_u, \tag{2.34}$$

which are deduced from the following symmetry generator

$$
\begin{aligned}
X &= (c_1 + c_4 x + 2c_5 t + 4c_6 xt)\partial_x + (c_2 + 2c_4 t + 4c_6 t^2)\partial_t + \rho(x,t)e^{-u}\partial_u \\
&+ (c_3 - c_5 x - 2c_6 t - c_6 x^2)\partial_u.
\end{aligned}
\tag{2.35}
$$

That is, choosing $c_1 = 1$ and all other $c's$ as zero, we result with

$$X_1 = \partial_x, \tag{2.36}$$

If we choose $c_2 = 1$ and all other $c's$ as zero, we result with

$$X_2 = \partial_t. \tag{2.37}$$

Continuing in this manner choosing $c_3 = 1$ and all other $c's$ as zeros, $c_4 = 1$ and all other $c's$ as zeros, $c_5 = 1$ and all other $c's$ as zeros, $c_6 = 1$ and all other $c's$ as zeros, and all $c's$ zeros; we respectively obtain the remaining generators as

$$X_3 = \partial_u, \tag{2.38}$$

$$X_4 = x\partial_x + 2t\partial_t, \tag{2.39}$$

$$X_5 = 2t\partial_x - x\partial_u, \tag{2.40}$$

$$X_6 = 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)\partial_u, \tag{2.41}$$

and

$$X_\rho = \rho(x,t)e^{-u}\partial_u \tag{2.42}$$

where ρ is any solution to the heat equation. Nowadays there are many computational computer packages the algorithm is build in to compute the Lie point symmetries. In this thesis the computer packages Sym [24] run in conjunction with Mathematica have been used to compute the Lie point symmetries. The symmetries in equation (2.29) to (2.34) constitute the *Lie algebra* of the potential $Burgers'$ *equation* and they are very useful for computing the invariant solutions for the model.

2.3 Lie Algebras

Suppose infinitesimal generator X_a, corresponding to the parameter ϵ_a of the $r-$parameter Lie group of transformations (2.2), is

$$X_a = \sum_{j=1}^{n} \xi_{aj}(x)\frac{\partial}{\partial x^i}, \qquad a = 1, 2, ..., r \tag{2.43}$$

then the *commutator* of X_a and X_b, also known as the *Lie bracket* $[X_a, X_b]$; is another first order operator

$$\begin{aligned} [X_a, X_b] &= X_a(X_b) - X_b(X_a) \\ &= \sum_{i,j=1}^{n} \left[\left(\xi_{ai}(x)\frac{\partial}{\partial x_i}\right)\left(\xi_{bj}(x)\frac{\partial}{\partial x_j}\right) - \left(\xi_{bi}(x)\frac{\partial}{\partial x_i}\right)\left(\xi_{aj}(x)\frac{\partial}{\partial x_j}\right)\right] \\ &= \sum_{j=1}^{n} \eta_j(x)\frac{\partial}{\partial x_j}, \end{aligned} \tag{2.44}$$

where

$$\eta_j(x) = \sum_{i=1}^{n} \left[\xi_{ai}(x)\frac{\partial \xi_{bj(x)}}{\partial x_i} - \xi_{bi}(x)\frac{\partial \xi_{aj(x)}}{\partial x_i}(x)\right] \tag{2.45}$$

and

$$\xi_{aj}(x) = \frac{\partial \tilde{x}_j}{\partial \epsilon_a}|_{\epsilon=0} = \frac{\partial X_j(x;\epsilon)}{\partial \epsilon_a}|_{\epsilon=0} \tag{2.46}$$

$a = 1, 2, ..., r$; $j = 1, 2, ..., n$.

Definition 2.3.1 The operator above together with a vector space L is said to be a *Lie algebra* if a Lie bracket $[X_a, X_b]$ of the infinitesimal generator X_a and X_b satisfies the following properties:

1. It is *antisymmetric* $[X_a, X_b]$ = -$[X_b, X_a]$,

2. It is *bilinear*: $[c_1X_a + c_2X_b, X_c]$=$c_1[X_a, X_c] + c_2[X_b, X_c]$,

3. It satisfies the *Jacobi identity*: $[X_a, [X_b, X_c]]$+$[X_b, [X_c, X_a]]$+$[X_c, [X_a, X_b]] = 0$.

Definition 2.3.2

If for all vectors X_a and X_b in L the commutator $[X_a, X_b] = 0$, then a Lie algebra L is called *Abelian*.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	X_1	$-X_3$	$2X_5$
X_2	0	0	0	$2X_2$	$2X_1$	$4X_4 - 2X_3$
X_3	0	0	0	0	0	0
X_4	$-X_1$	$-2X_2$	0	0	X_5	$2X_6$
X_5	X_3	$-2X_1$	0	$-X_5$	0	0
X_6	$-2X_5$	$2X_3 - 4X_4$	0	$-2X_6$	0	0

Table 2.1: Lie bracket of a potential Burgers' Equation

.

Using the generators in equations (2.29) to (2.34), the commutators are computed and presented in Table 2.1 above and we illustrate how the Lie Bracket or the commutator $[X_1, X_6]$ is computed:

$$
\begin{aligned}
[X_1, X_6] &= X_1X_6 - X_6X_1 \\
&= \partial_x(4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)\partial_u) \\
&- (4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)\partial_u)\partial_x \\
&= (4t\partial_x - 2x\partial_u) - 0 \\
&= 2(2t\partial_x - x\partial_u) \\
&= 2X_5.
\end{aligned}
$$

2.4 Group-Invariant Solutions

In the situation when one is confronted with a complicated system of partial differential equations arising from some physically important problem, the discovery of any explicit solution whatsoever is always of great interest. Explicit solutions can more often than not be used as models for physical experiments, as benchmarks for testing numerical methods. The method used to find group-invariant solutions, generalizing the well-known techniques for finding similarity solutions, provide a systematic computational method for determining a large class of special solutions. The group-invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations. The fundamental concept on group-invariant solutions is that the solutions which are invariant under a given r-parameter symmetry group of the system can all be found by solving a system of differential equations involving r fewer independent variables than the original system.

2.4.1 Reduction using Lie Point Symmetries

Once the Lie point symmetries have been obtained an important step is to reduce the number of independent variables of a given PDE using Lie methods. The resulting solution deduced from the reduction is the resulting group invariant solution. Suppose $u = \theta(x)$ is an invariant solution of equation (2.10) resulting from its invariance under symmetry (2.13) then $u = \theta(x)$ satisfies [2]

$$X(u - \theta(x)) = 0 \qquad \text{when} \qquad u = \theta(x). \tag{2.47}$$

Equation (2.47) is referred to as the *invariance surface condition* for invariant solution corresponding to the symmetry in equation (2.13). A procedure to solve this is to solve the invariance condition. That is, the following corresponding Lagrange equations associated with $u = \theta(x)$ are obtained and solved,

$$\frac{dx_1}{\xi_1(x,u)} = \frac{dx_2}{\xi_2(x,u)} = ... = \frac{dx_n}{\xi_n(x,u)} = \frac{du}{\eta(x,u)}. \tag{2.48}$$

If $(X_1(x,u), X_2(x,u), ..., X_{n-1}(x,u))$, $v(x,u)$ are the obtained n independent invariants of equation (2.48) with $\frac{\partial v}{\partial u} \neq 0$, then resulting solution is given implicitly by the invariant form

$$v(x,u) = \phi(X_1(x,u), X_2(x,u), ..., X_{n-1}(x,u)). \tag{2.49}$$

2.5 Optimal Systems of Lie Symmetries

Given a group that leaves a PDE invariant, it is always desirable to minimize the search for group-invariant solutions to that of finding inequivalent branches of solutions, which leads to the concept of the optimal systems. Consequently, the problem of determining the optimal system of subgroups is reduced to the corresponding problems for subalgebras. In application, one often constructs the optimal system of subalgebras, from which the optimal systems of subgroup and group invariant solutions are reconstructed.

2.5.1 Adjoint representation

An optimal system of a Lie algebra is a set of l-dimensional subalgebras such that every l-dimensional subalgebra is equivalent to a unique element of the set under some element of the adjoint representation. To construct the conjugacy classes we utilise the adjoint action. In particular, for Lie algebras this adjoint action may be described by the Baker-Campbell-Hausdorf formula [38],

$$Ad(exp(\epsilon X_a))X_b = X_b + \epsilon[X_a, X_b] + \frac{\epsilon^2}{2!}[X_a, [X_a, X_b]] + \frac{\epsilon^3}{3!}[X_a, [X_a, [X_a, X_b]]] + ... \tag{2.50}$$

where $[X_a, X_b]$ denotes the commutator of generators X_a and X_b. Utilising this formula, it is possible to construct a table to summarise the adjoint representation of adjoint operators of each of the symmetries admitted by any model under consideration.

Let us consider the Achdou et al. [28] knowledge diffusion model in macroeconomics below

$$f_t(x,t) - \frac{\sigma^2}{2} f_{xx}(x,t) + \alpha f(x,t)(1 - f(x,t)) = 0, \tag{2.51}$$

where $f(x,t)$ is the distribution of productivity and $f(x,0) = f_0(x)$ being the initial productivity distribution. In this problem setup, the economy comprise of population of continuum of individuals indexed by their production or knowledge $z \in \Re^+$. The economy is described by its distribution of production with cdf $G(z,t)$. The evolution of G is modelled as a process of individuals meeting others from the same economy, comparing ideas, and improving their own productivity. Meetings between individuals are set to happen at Poisson intensity α, and from the view point of an individual, the meeting is a random draw from the distribution G. When a meeting does occur, person z compares his or her productivity with person he or she meets and leaves the meeting with the best of the two productivities $\max\{z, z'\}$. Individual productivities fluctuate also in the absence of a meeting. In particular individuals "experiment" and their productivity increase or decrease according to the process dlog $z_t = \sigma dW_t, \sigma > 0$. When log productivity x=log z, then the distribution of productivity $f(x,t)$ satisfies the partial differential equation in equation (2.51). This model as pointed out by [37] is known as Fisher-KPP-type equation and it has been applied in many subjects such as Biology [35] and Mathematics [36]. In this present case we consider its application in Macroeconomics.

2.5.2 Lie point symmetries admitted by the Fisher production model

In this section we compute the Lie point symmetries associated with the Fisher production model. Suppose equation (2.20) is an infinitesimal generator of a symmetry group of equation (2.51) expressed in solved form and F is replaced by variable u in (2.51) to have

$$u_t - \frac{\sigma^2}{2}u_{xx} + \alpha u(1-u) = 0, \tag{2.52}$$

then the invariance condition dictated by the production model is as follows:

$$X^{[2]}\left\{u_t - \frac{\sigma^2}{2}u_{xx} + \alpha u(1-u)\right\}\Bigg|_{(2.52)} = 0. \tag{2.53}$$

Using the computer package Sym run in conjunction with Mathematica we solve equation (2.53) to result with the following infinitesimals

$$\begin{aligned} \tau &= c_1, \\ \xi &= c_2, \\ \eta &= 0, \end{aligned} \tag{2.54}$$

where c_1, c_2 are arbitrary constants.

From equation (2.54), the two infinitesimal generators of equation (2.52) are deduced as

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}. \tag{2.55}$$

Using the Lie's equations in equation (2.7) and the infinitesimal generators in equation (2.55) the one-parameter groups of symmetries are obtained as:

$$\begin{aligned} X_1 &: \qquad \widetilde{x} = x, \qquad \widetilde{t} = t + \epsilon_1, \qquad \widetilde{u} = u \\ X_2 &: \qquad \widetilde{x} = x + \epsilon_2, \qquad \widetilde{t} = t, \qquad \widetilde{u} = u \end{aligned} \tag{2.56}$$

2.5.3 Exact Solutions of Eq. (2.52)

After one has determined the infinitesimal generators as done in equation (2.55), the similarity variables and newly explicit solutions for the model, equation (2.52), can be found by solving the associated characteristic equations as stated earlier in this chapter. The solutions obtained in this section are trivial solutions we therefore report solutions for X_2 and considers optimal system of equation (2.52) in the next section.

Example. We consider the infinitesimal generator X_2. Solving the characteristic equations

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} \tag{2.57}$$

associated with this generator yields the invariants

$$\zeta = t \qquad \text{and} \qquad u(x,t) = \omega(\zeta) \tag{2.58}$$

where $\zeta = t$ is the similarity variable and $\omega(\zeta)$ satisfy the following similarity reduction equation

$$\omega^{'} + \alpha\omega(1-\omega) = 0, \tag{2.59}$$

which solves to

$$\omega(t) = \frac{1}{1+Ce^{\alpha t}}, \tag{2.60}$$

where C is an arbitrary constant. Therefore the invariant solutions resulting from X_2 are reported as

$$u(x,t) = \frac{1}{1+Ce^{\alpha t}}. \tag{2.61}$$

2.5.4 One-Dimensional Optimal System of Subalgebras and Exact Solutions of Eq. (2.52)

The reduction of the independent variables by one is possible under the construction of any linear combination of our generators of symmetry X_i, $i = 1$, 2. We therefore construct a set of minimal known as optimal systems. From the generators in equation (2.55), we obtain the commutators as

$$[X_1, X_1] = [X_1, X_2] = [X_2, X_1] = [X_2, X_2] = 0. \tag{2.62}$$

Using equation (2.50) formula and these obtained commutators, we result we result with the following adjoint representations

$$\begin{aligned} Ad(exp(\epsilon X_1))X_2 &= Ad(exp(\epsilon X_2))X_2 = X_2 \\ Ad(exp(\epsilon X_1))X_1 &= Ad(exp(\epsilon X_2))X_1 = X_1. \end{aligned} \tag{2.63}$$

Following the method in [3] to compute an optimal system of subalgebras, let us consider a linear combination of the symmetry generators

$$X = a_1 X_1 + a_2 X_2. \tag{2.64}$$

The aim is to simplify as many coefficients a_i as possible through application of adjoints maps to X. Now let $a_2 \neq 0$ in equation (2.64) and rescale a_2 such that a_2=1. Acting on X by $Ad(exp(\epsilon X_1))$ we obtain

$$\begin{aligned} Ad(exp(\epsilon X_1))X &= Ad(exp(\epsilon X_1))(a_1X_1 + X_2) \\ &= a_1Ad(exp(\epsilon X_1))X_1 + Ad(exp(\epsilon X_1))X_2 \qquad (2.65) \\ &= a_1X_1 + X_2. \end{aligned}$$

If now $a_2 = 0$, then there is no more simplification since we have X_1 when rescaling $a_1 = 1$. Therefore, an optimal system of subalgebras is given by the following set $\{X_1, a_1X_1 + X_2\}$.

From this optimal system of one-dimensional subalgebras we compute invariant solutions of equation (2.52). X_1 does not provide any solution. In subalgebra $a_1X_1+X_2$ if we replace a_1 by constant γ then the corresponding characteristic equation

$$\frac{dt}{0} = \frac{dx}{\gamma} = \frac{du}{0} \qquad (2.66)$$

yields the invariants

$$\zeta = x - \gamma t, \qquad u(x,t) = \omega(\zeta). \qquad (2.67)$$

Substituting in equation (2.52) results with $\omega(\zeta)$ satisfying the following similarity reduction equation

$$-\gamma\omega'(\zeta) - \beta\omega''(\zeta) + \alpha\omega(\zeta)(1 - \omega(\zeta)) = 0, \qquad (2.68)$$

where $\beta = \frac{\sigma^2}{2}$. Using the hyperbolic tangent method, and setting $\sigma = \sqrt{2}$, equation (2.68) solves to the exact solution

$$\omega(\zeta) = \frac{1}{4}\left\{1 - \tanh\left[-c_1 + \frac{1}{12}\sqrt{6\alpha}\zeta\right]\right\}^2, \tag{2.69}$$

where $\gamma = -\frac{5\sqrt{\alpha}}{\sqrt{6}}$ and c_1 is a constant. Therefore the exact invariant solutions are given by

$$u(x,t) = \frac{1}{4}\left\{1 - \tanh\left[-c_1 + \frac{1}{12}\sqrt{6\alpha}(x - \gamma t)\right]\right\}^2. \tag{2.70}$$

We notice from equation (2.70) that the Fisher production model in equation (2.52) admits the "travelling wave" solution. That is, a solution of the form

$$u(x,t) = \phi(x - \gamma t). \tag{2.71}$$

Chapter 3

Mathematics of Finance Preliminaries

In financial markets financial products known as derivatives have become increasingly important. This is due to the fact that derivatives serve several purposes, namely, they are normally added to bond issues and also used by some companies in compensation plans of their executives. More importantly, they can be used to transfer risk in mortgages from one original lender to another investor. It has reached a point that those who work in finance, and many those who work outside finance, need to understand how derivatives work, how they are used, and how they are priced. In order to shed more light into these issues, especially the pricing of derivatives that is fundamental in this thesis; we start first by giving some definitions of mathematical tools that have been used throughout the thesis. We define concepts such as Brownian motion, stochastic process, filtration, random variables, stochastic differential equation, etc., and give some basic results. Our main references on such basics are Etheridge [9], Wilmott *et al.* [7], Oksendal [26] and Grimmett and Stirzaker [8].

3.1 Random Variables and Stochastic Processes

The triple $(\Omega, \mathcal{F}, \mathbb{P})$, comprising of a set Ω, a $\sigma-$ field $\mathcal{F}$ of subsets of Ω, and a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$, is called a *probability space*. The collection $\mathcal{F}$ is a σ-field, that is, $\Omega \in \mathcal{F}$ and $\mathcal{F}$ are closed under the operations of countable union and taking complements. A *probability measure* $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ satisfying

1. $\mathbb{P}(\emptyset) = 0$,
2. $\mathbb{P}(\Omega) = 1$,
3. if $A_1, A_2, ...$ is a collection of disjoint members of $\mathbb{F}$, in that $A_i \cap A_j = \emptyset$ for all pairs i, j satisfying $i \neq j$, then

$$\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Definition 3.1.1

Let Ω be a nonempty set. A random variable is a function $X : \Omega \longrightarrow \Re$ with the property that $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for each $x \in \Re$. Such a function is said to be $\mathcal{F}-$ measurable.

Definition 3.1.2

The cumulative distribution function of a random variable X is the function $F : \Re \longrightarrow [0, 1]$ given by $F(x) = \mathbb{P}(X \leq x)$.

Definition 3.1.3

Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}_t$. Assume further that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s < t < \infty$ and $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$.

Then we call the collection $\mathcal{F}_t$ of σ-algebras a filtration and $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ is called a filtered probability space.

$\mathcal{F}_t$ is taken to be the set of information available to the observer (e.g. the investor or the bank manager) up to time t. More specifically, $\{\mathcal{F}_t\}_{t>0}$ is considered to be the flow of information over certain time and it is assumed that the bank does not lose information as time passes (hence why we say $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$).

Definition 3.1.4

A real-valued stochastic process is an indexed family of real-valued functions, $\{X_t\}_{t \geq 0}$ on Ω. $\{X_t\}_{t \geq 0}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if X_s is $\mathcal{F}_t$-measurable for each $t \geq s$.

3.2 Brownian Motion

This is the process $W = \{W(t) : t \geq 0\}$, indexed by continuous time and taking values in the real line $\Re$, which is time-homogeneous with independent increments, and with the vital extra property that $W(t)$ has the normal distribution with mean 0 and variance $\sigma^2 t$ for some constant σ^2. Brownian motion is regarded as the cornerstone of the modern theory of random processes [8]. Historically in the year 1827, Robert Brown observed the complex and erratic motion of grains of pollen suspended in a liquid. It was later discovered that such irregular motion comes from extremely large number of collisions of the suspended pollen grains with the molecules of the liquid. Norbert Wiener presented a mathematical model for this motion based on the theory of stochastic processes. The position of a particle at each time $t \geq 0$ is a three dimensional random vector W_t.

Definition 3.2.1

A real-valued stochastic process $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion (or a $\mathbb{P}$-Wiener process) if for some real constant σ, under $\mathbb{P}$,

1. for each $s \geq 0$ and $t > 0$ the random variable $W_{t+s} - W_s$ has the normal distribution with mean zero and variance $\sigma^2 t$,

2. for each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq t_2 \leq ... \leq t_n$, the random variables $\left\{W_{t_r} - W_{t_{r-1}}\right\}$ are independent,

3. $W_0 = 0$,

4. W_t is continuous in $t \geq 0$.

3.3 Stochastic Integration

Brownian motion [23], has played a vital role in stochastic integration of risky asset prices and their modelling. Wiener and other researcher expanded the discovery by Brown by deriving and proving many of the properties associated with the paths of Brownian motion. The following two key properties relates the stochastic integration: (1) the paths of Brownian motion have a non-zero finite quadratic variation, such that on an interval (s, t), the quadratic variation is $(t-s)$, and (2) the paths of Brownian motion have infinite variation on compact time intervals, almost surely. Processes used to model stock price are usually functions of one or more Brownian motions. In this regard, suppose that the stock price is of the form $S_t = f(t, Z_t)$. Using Taylor's theorem, we can write

$$\begin{aligned} f(t+\delta t, Z_{t+\delta t}) - f(t, Z_t) &= \delta t \dot{f}(t, Z_t) + O(\delta t^2) + (Z_{t+\delta_t} - Z_t) f^{'}(t, Z_t) \\ &+ \frac{1}{2!}(Z_{t+\delta t} - Z_t)^2 f^{''}(t, Z_t) + ... \end{aligned} \tag{3.1}$$

where the notation $\dot{f}$, $f^{'}$ and $f^{''}$ must be interpreted as $\dot{f}(t, x) = \frac{\partial f}{\partial t}(t, x)$, $f^{'}(t, x) = \frac{\partial f}{\partial x}(t, x)$ and $f^{''}(t, x) = \frac{\partial^2 f}{\partial x^2}(t, x)$. The dynamics of a stock price is commonly modelled by way of a stochastic differential equation as follows (see for example Etheridge [9, p 75]):

$$dS_t = \dot{f}(t, Z_t)dt + f^{'}(Z_t)dZ_t + \frac{1}{2}f^{''}(Z_t)dt. \tag{3.2}$$

The differential equation above is also resembled in integrated form as follows,

$$S_t = S_0 + \int_0^t \dot{f}\ (s, Z_s)ds + \int_0^t f^{'}(Z_s)dZ_s + \int_0^t \frac{1}{2}f^{''}(Z_s)ds. \tag{3.3}$$

3.3.1 Itô Process

Itô process (or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form [26]

$$X_t = X_0 + \int_0^t a(X_t, t)ds + \int_0^t b(X_t, t)dZ_s. \tag{3.4}$$

If X_t is an Itô process of the form (3.4) the equation is sometimes written in the shorter differential form

$$dX_t = a(X_t, t)dt + b(X_t, t)dZ_t, \tag{3.5}$$

where $a(X_t, t)$ is the drift rate, $b(X_t, t)$ is the variance rate or diffusion and Z_t is a standard Wiener process.

3.3.2 Itô Formula

Let X_t be an Itô process given by

$$dX_t = adt + bdZ_t, \tag{3.6}$$

and $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$ (i.e., *g is twice continuously differentiable on* $[0, \infty) \times \mathbb{R}$), then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t).(dX_t)^2, \tag{3.7}$$

where differentials are multiplied according to the rules

$$dt.dt = dt.dZ_t = dZ_t.dt = 0, \qquad dZ_t.dZ_t = dt. \tag{3.8}$$

3.4 Standard Vanilla Option

A standard option is a financial contract that gives its holder the right to buy or sell the underlying asset at a certain specified price at a specified future date. Finding the appropriate price of this financial contract is crucial in options trading as any discrepancies in the pricing could lead to profitable opportunities for other parties. That is, if the market price of a financial contract is smaller than the true value of the contract, it is said that the contract is undervalued, and it is profitable to buy the contract; and if its market price is greater than the true value of the contract, it is said that the contract is overvalued, and it is profitable to write or sell the contract. Therefore to determine the fair price of all kinds of financial derivatives securities is important in financial markets.

Several approaches can be used to price financial contracts such as, the Black-Scholes pricing model, the arbitrage arguments, the Monte-Carlo simulations and solving the partial differential equation associated with the price of the financial contract. The Black-Scholes pricing model was the one more preferred to be used as benchmark for pricing many financial securities.

The Black-Scholes model concerns an economy which comprises of two assets, a "bond" (or money market account) whose value grows at a continuously compounded constant interest rate r, and a stock price per unit is a stochastic process $S = S_t : t \geq 0$ indexed by time t. Upon using arbitrage arguments and Ito's lemma one is able to deduce Black-Scholes PDE model in equation (1.1). In this work though still abiding by Black-Scholes model analysis we focus on interest rate derivatives.

3.4.1 Bond Options

Interest rate derivatives are instruments whose payoffs are dependent in some way on the level of interest rates. Valuation of interest rate derivatives is more complex as compared to equity and foreign exchange derivatives due to complicated behaviour of an individual interest rate as compared to that of stock price or an exchange rate. The interest rate derivative we consider in this case is the bond option. A bond is a contract, paid for up front, which yields a known amount on a known date in the future, the maturity date, $t = T$. A bond may also pay a known cash dividend known as coupon at fixed times during the contract life. If there are no coupon payment then we result with a zero-coupon bond. The bond's main use is to raise capital by governments and companies where the up-front premium is regarded as a loan to the government or company.

3.4.1.1 Deterministic bond model

Suppose F represents the price of our contract, the bond in this context. If the interest rate $r(t)$ and coupon $\lambda(t)$ are known functions of time, the bond value is therefore a function of time as well. That is, $F = F(t)$. Now let us consider a portfolio that comprise of one bond. The value of the bond at time intervals dt changes as follows,

$$\frac{dF}{dt}dt. \tag{3.9}$$

Suppose during this time changes, the coupon payments $\lambda(t)$ are received, then our portfolio holdings will then change as follows:

$$\left(\frac{dF}{dt} + \lambda(t)\right) dt. \tag{3.10}$$

The portfolio should earn risk-free rate $r(t)$ due to arbitrage considerations, so that

$$\left(\frac{dF}{dt} + \lambda(t)\right) dt = r(t)Fdt. \tag{3.11}$$

Thus,

$$\frac{dF}{dt} + \lambda(t) = r(t)F. \tag{3.12}$$

Equation (3.12) is a linear first order differential equation with its right hand side reflecting the cash that one would receive if the bond was to be converted to cash at time t. Solving it, with the integrating factor as

$$I.F. = e^{-\int r(\tau)d\tau} \tag{3.13}$$

results with the following deterministic bond model,

$$F(t) = e^{-\int_t^T r(\tau)d\tau} \left(Z + \int_t^T \lambda(\tau) e^{\int_t^T r(s)ds} d\tau \right) \tag{3.14}$$

with $F(T) = Z$ as the constant of integration. We notice therefore that the bond value is the sum of the present face value and the coupon stream. If there exists the zero-coupon bonds with all possible maturity dates and interest still deterministic then, $\lambda = 0$, so that

$$F(t,T) = Ze^{-\int_t^T r(\tau)d\tau}. \tag{3.15}$$

If the bond prices were to be quoted today, at time t, for all values dated T then

$$\ln\left(\frac{F(t,T)}{Z}\right) = -\int_t^T r(\tau)d\tau. \tag{3.16}$$

One important point that can be deduced from equation (3.16) is that, if the market prices of the zero-coupon bonds genuinely reflect the deterministic interest rate, which is known, then the interest rate at future dates is deduced from equation (3.16) as,

$$r(T) = \frac{-1}{F(t,T)} \frac{\partial F}{\partial T}, \tag{3.17}$$

taking into consideration that $\frac{\partial F}{\partial T} < 0$ since the interest rate r is positive.

3.4.1.2 Bond Equation under Stochastic Model

In this previous section the interest rate has been taken to be deterministic. If there is uncertainty about the future course of interest rate, then the interest rate is normally modelled as a random variable, such as shown in equation (1.3).

In order to derive stochastic bond model, suppose that the interest rate satisfies the same SDE in equation (1.3) and our portfolio contains bonds with different maturity dates. That is,

$$\Pi = F(t, r, T_1) - \Delta F(t, r, T_2) \equiv F_1 - \Delta F_2. \tag{3.18}$$

The value of the portfolio $d\Pi$ will change as follows under the time step dt,

$$d\Pi = dF_1 - \Delta dF_2 \tag{3.19}$$

with

$$\begin{aligned} \frac{dF_i}{F_i} &= a_i dt + b_i dZ, & (3.20) \\ a_i &= \frac{1}{F_i}\left(F_{i,t} + aF_{i,r} + \frac{1}{2}b^2 F_{i,rr}\right), & (3.21) \\ b_i &= \frac{1}{F_i} bF_{i,r}. & (3.22) \end{aligned}$$

F_i denotes the price associated with maturity T_i for $i = 1, 2$; while the subscripts t, r and rr in F_i represents the derivatives of F_i with respect to t and r.

If in $d\Pi$ we choose $\Delta = \frac{F_{1,r}}{F_{2,r}}$, then this cancel the random term. Due to arbitrage argument the portfolio must earn risk-free rate of interest. That is, $d\Pi = r\Pi dt$ so that

$$a_1 F_1 dt - \Delta a_2 F_2 dt = r(F_1 - \Delta F_2)dt. \tag{3.23}$$

Rearranging we have,

$$(a_1 - r)\frac{F_1}{F_{1,r}} = (a_2 - r)\frac{F_2}{F_{2,r}}. \tag{3.24}$$

This is equivalent to,

$$\frac{a_1 - r}{b_1} = \frac{a_2 - r}{b_2}. \tag{3.25}$$

The left hand side of equation (3.25) is a function of T_1 whereas its right hand side is a function of T_2. Therefore the equation is independent of T. If we express it as known function of $\gamma(r, t)$ we have

$$\frac{a - r}{b} = \gamma(r, t). \tag{3.26}$$

Lastly, plugging a_i and b_i back in the equation and dropping the index i, result with the following bond equation

$$F_t + \frac{b^2}{2}F_{rr} + (a - \gamma b)F_r - rF = 0 \tag{3.27}$$

The function $\gamma(r,t) = \frac{a-r}{b}$ is called the market price of risk. Solving this model result with the price of the bond model under stochastic interest rate dependent on the three parameter functions, namely, drift $a(r,t)$, volatility $b(r,t)$ and market price of risk $\gamma(r,t)$. In the next chapter we focus our attention on the functional interest rate model which has great advantages of satisfying all properties of interest rate and produce positive interest rate upon certain restriction applied to the range of the aforementioned model.

Chapter 4

Symmetry Analysis of an Interest Rate Derivatives PDE Model in Financial Mathematics

Symmetry analysis by Lie have played vital role in deducing explicit solutions of complex models whose formation comprise of partial differential equations. This chapter is largely based on [22]. Contributing to the literature in this field we apply Lie symmetry analysis to solve the zero-coupon bond pricing equation whose price dynamics are described in terms of a partial differential equation (PDE). We use computer software package SYM run in conjunction with Mathematica to compute new complete Lie symmetry group and infinitesimal generators of a one-dimensional zero-coupon bond. We furthermore exercise our skills to solve to obtain a family of exact invariant solutions that constitute the pricing models of the aforementioned contingent claim. The solutions are computed through solving the corresponding similarity reduction equations associated with the derived infinitesimal generators. We conclude this chapter by further applying Lie's theory to generate more solutions via group point transformations. Our findings are presented by way of graphs, with application made to Vasicek interest rate model.

4.1 Introduction

Companies and governments raise capital by issuing financial instruments known as bonds. A bond is a financial contract under which the issuer promises to pay the other party certain amount of money in intervals together with a lump sum of money at the end of agreed time. The money paid in intervals is called coupons while the lump sum paid at the end of the contract is called the principal. If there are no interim payments, the contract is called a zero-coupon bond and the lump sum paid is the face value of the contract. Bond price depends on interest rate which is used in most cases in financial markets for discounting as well as for defining payoff of the interest rate derivative such as bond option. The valuation of this contract over a specific term depends crucially on the random fluctuations of the interest rate market. The construction of the valuation models for this type of financial security should therefore incorporate the stochastic movement of interest rates into consideration. In the case the contract has a short life span, deterministic interest rates are considered whereas for long life span stochastic interest rates should be considered. Several model structures have been considered in the past years to denote interest rate models. As point out by Goard [18], many of these stochastic interest rate models can be embedded in the form

$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dZ, \tag{4.1}$$

where α, β, γ, σ are constants and dZ is a standard Brownian motion. Of these models, the Vasicek model [30] and Cox-Ingersoll-Ross (CIR) model [31] have both proved to be tractable and empirically relevant in capturing properties of interest rates, namely, mean- reversion and positivity of interest rate. That is, as shown by Wilmott et al. [10] when the spot rate follows the SDE

$$dr = a(r,t)dt + b(r,t)dZ, \tag{4.2}$$

the price of the zero-coupon bond is obtained from solving the following PDE

$$V_t + \frac{b^2}{2} V_{rr} + (a - \kappa b)V_r - rV = 0, \tag{4.3}$$

with terminal condition $V(r,T) = 1$. A family of interest rate derivatives comprise of minute number of well-known tractable numerical methods to calibrate its model parameters. This include binomial trees, Monte-Carlo simulation, and finite-difference approximation of PDE. Using Lie's classical method of group invariants as defined in chapter 2, Goard [18] have derived new and simple solutions to the bond-pricing partial differential equation (4.3). She has significantly expanded the class of analytical solvable models for bond options. Sinkala *et al*. [20] have also computed new prices for bond PDE model with special consideration given to Vasicek and CIR models. They have determined the symmetries of the valuation partial differential equation that are compatible with terminal condition and then seeked the desired solution among the invariant solutions arising from the obtained symmetries. Analytic approaches and numerical methods have therefore seemed to be most favoured in solving differential equations. This fact is supported by research work such as that of Ibragimov and Gazizov [15], who introduced the idea of Lie symmetry analysis in finance problems when analysing the Black-Scholes pricing equation. Pooe *et al*. [19] using transformations to reduce bond-pricing equation to heat equation, obtained the solution to the zero-coupon bond via usage of obtained transformations. Khalique *et al*. [21] proposed new invariant solutions and conservation laws for Vasicek pricing equation model. Lie's work have proved in studies therefore to be one of the prominent analytical solver for obtaining analytical solutions for differential equations.

Majority of the PDE models associated with pricing bond options incorporate either the Vasicek interest rate model [30] or Cox-Ingersoll-Ross (CIR) model [31] due to them proving to be tractable and empirically relevant to pricing interest rate derivatives. But it is a known fact that Vasicek model has a drawback of admitting negative interest rate and does violating the positivity property of interest rate. In this work a functional interest rate model by Luo *et al*. [27] satisfying the following SDE

$$dX_t = \left(-\eta(t)X_t + \frac{\epsilon(t)}{X_t} \right) dt + \sigma(t)dZ_t, \tag{4.4}$$

with $\{Z_t\}_{t\geq 0}$ being a Brownian motion, and ϵ, η, σ being functions of t; is explored and used to derive the PDE model of the zero-coupon bond. In this framework $r_t = f(X_t, t)$ is modelled as a function of Markov state variable X_t and time t so that when $f(X_t, t)$ is twice continuously differentiable in x and continuously differentiable in t, when applying Itô formula, the following stochastic differential equation associated with r_t is deduced:

$$\begin{aligned}
dr_t &= df(t, X_t) \\
&= f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2 \qquad (4.5)\\
&= \left\{ f_t(t, X_t) + \frac{\sigma(t)^2}{2} f_{xx}(t, X_t) + \left(\frac{\epsilon(t)}{X_t} - \eta(t)X_t \right) f_x(t, X_t) \right\} dt \\
&+ \sigma(t) f_x(t, X_t) dZ_t.
\end{aligned}$$

If we confine f's co-domain to only positive real values, this enables one to avoid the drawback of negative interest rate that make Vasicek interest rate model less attractive. This interest rate model embeds most known interest rate models which can be deduced for different choices of f, η, ϵ and σ. Numerical methods such as binomial trees, Monte-Carlo simulation, and finite-difference methods are the normal tools used to value financial products such as interest rate derivatives. In this work we deviate from these approaches and we use analytical approach of Lie symmetry analysis to derive four Lie point symmetries plus an additional infinite sub-algebra; and these symmetries are then used to obtain three types of closed-form solutions for the aforementioned pricing equation associated with the functional interest rate model. We illustrate with an example by making application to the Vasicek interest rate model.

4.2 Bond pricing equation derivation and Symmetry Analysis

Let us consider a zero-coupon bond that pays $h(r_T)$ at maturity time T and the functional interest rate model in equation (4.4). The payoff of this zero-coupon bond associated with the functional interest rate model is expressed as $h(f(X_T, t))$ and its price is given by $v(X_t, t)$, where

$$v(x,t) = E\left(e^{\int_t^T f(X_s,s)ds} h(f(X_T,T))|X_t = x\right), \qquad t < T. \tag{4.6}$$

Applying Feynman-Kac formula on equation (4.6), this results with $v(x,t)$ satisfying the following PDE

$$v_t(x,t) + \frac{\sigma^2}{2} v_{xx}(x,t) + \left(-\eta x + \frac{\epsilon}{x}\right) v_x(x,t) - rv(x,t) = 0. \tag{4.7}$$

4.3 Bond pricing equation symmetry analysis

The process of computing Lie symmetries of a PDE model is algorithmic in nature. But due to tedious computations which are sometime very time consuming there are computer software packages readily available to assist in the calculations. In this present chapter computer package SYM [24] run in conjunction with Mathematica, and Maple 2020 [25] have been our main tools for computations. As we have noted in chapter 2 that after one has obtained the invariance condition of the associated model it is important to solve to deduce the resulting infinitesimals. That is, constructing the symmetry group as pointed out by Lie, is equivalent to the determination of the infinitesimal generator

$$\Gamma = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{4.8}$$

where the infinitesimals ξ, τ and ϕ are functions of variables (x,t,u). These infinitesimals are obtained from solving Lie's invariance condition. That is, if $\Gamma^{[2]}$ is the second extension or prolongation of Γ given by

$$\Gamma^{[2]} = \Gamma + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \phi^{xx}\partial_{u_{xx}} + \phi^{xt}\partial_{u_{xt}} + \phi^{tt}\partial_{u_{tt}}, \tag{4.9}$$

where

$$\begin{aligned}
\phi^t &= D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt},\\
\phi^x &= D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt},\\
\phi^{xx} &= D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt},\\
&\ldots
\end{aligned}$$

and "D" represents the total derivative, *i.e.*

$$D_x R = \frac{\partial}{\partial x}R + u_x\frac{\partial}{\partial u}R + u_{xx}\frac{\partial}{\partial u_x}R + u_{xt}\frac{\partial}{\partial u_t}R + ...,$$

then the invariance condition constituted by the zero-coupon bond pricing model in equation (4.7) is

$$\Gamma^{[2]}\left\{u_t(x,t)+\frac{\sigma^2}{2}u_{xx}(x,t)+\left(-\eta x+\frac{\epsilon}{x}\right)u_x(x,t)-ru(x,t)\right\}\Bigg|_{(4.7)}=0. \tag{4.10}$$

Making use of SYM to solve equation (4.10), $\beta=\frac{\sigma^2}{2}$, the infinitesimals are obtained as

$$\begin{aligned}\tau &= \frac{e^{2\eta t}c_1}{\eta}-\frac{e^{-2\eta t}c_2}{\eta}+c_3\\ \xi &= xe^{2\eta t}c_1+xe^{-2\eta t}c_2\\ \phi &= \left(-1-\frac{\epsilon}{\beta}+\frac{r}{\eta}+\frac{x^2\eta}{\beta}\right)e^{2\eta t}uc_1-\frac{r}{\eta}e^{-2\eta t}uc_2+uc_4+B(x,t),\end{aligned} \tag{4.11}$$

where c_1, c_2, c_3, c_4 are any constants and $B(x,t)$ is a solution of equation (4.7). The arbitrary constants in equation (4.11) constitute an infinite dimensional Lie algebra of symmetries and they are given as

$$\begin{aligned}\Gamma_1 &= \frac{\partial}{\partial t},\\ \Gamma_2 &= -\frac{e^{-2\eta t}}{\eta}\frac{\partial}{\partial t}+xe^{-2\eta t}\frac{\partial}{\partial x}-\frac{re^{-2\eta t}}{\eta}u\frac{\partial}{\partial u},\\ \Gamma_3 &= \frac{e^{2\eta t}}{\eta}\frac{\partial}{\partial t}+xe^{2\eta t}\frac{\partial}{\partial x}+\left(-1-\frac{\epsilon}{\beta}+\frac{r}{\eta}+\frac{x^2\eta}{\beta}\right)ue^{2\eta t}\frac{\partial}{\partial u},\\ \Gamma_4 &= u\frac{\partial}{\partial u},\text{ and}\\ \Gamma_B &= B(x,t)\frac{\partial}{\partial u},\end{aligned} \tag{4.12}$$

where $B(x,t)$ is an arbitrary solution of our model in equation (4.7). Using the Lie point symmetries in equation (4.12), we deduce the Lie point transformations or one-parameter groups of symmetries $\psi : (x,t,u) \rightarrow (\widetilde{x},\widetilde{t},\widetilde{u})$ of the zero-coupon bond model in equation (4.7) with $\widetilde{u}(\widetilde{x},\widetilde{t})$ as its solution. Using the five obtained infinitesimal generators in equation (4.12) and solving the following associated ordinary differential equations (with κ as *one-parameter*):

$$\frac{d\tilde{x}}{d\kappa} = \xi(\tilde{x},\tilde{t},\tilde{u}), \qquad \frac{d\tilde{t}}{d\kappa} = \tau(\tilde{x},\tilde{t},\tilde{u}), \qquad \frac{d\tilde{u}}{d\kappa} = \phi(\tilde{x},\tilde{t},\tilde{u}) \tag{4.13}$$

subject to the initial conditions

$$\tilde{x}|_{\kappa=0} = x, \qquad \tilde{t}|_{\kappa=0} = t, \qquad \tilde{u}|_{\kappa=0} = u, \tag{4.14}$$

we result with the corresponding five one-parameter groups of symmetry for the zero-coupon bond pricing equation (4.7)

$$\begin{aligned}
\psi_1 : (x,t,u) &\rightarrow (x, t+\kappa_1, u) \\
\psi_2 : (x,t,u) &\rightarrow \left[xe^{(e^{-2\eta t}\kappa_2)}, \frac{1}{2\eta}\ln\left\{-2\eta\left(\frac{\kappa_2}{\eta} - \frac{e^{2\eta t}}{2\eta}\right)\right\}, e^{\frac{-r\kappa_2}{\eta e^{2\eta t}}} u \right] \\
\psi_3 : (x,t,u) &\rightarrow \left[xe^{\left(e^{2\eta t}\kappa_3\right)}, \frac{-1}{2\eta}\ln\left\{-2\eta\left(\frac{\kappa_3}{\eta} - \frac{e^{2\eta t}}{2\eta}\right)\right\}, ue^{\left(1+\frac{\epsilon}{\beta}-\frac{r}{\eta}-\frac{x^2\eta}{\beta}\right)} e^{2\eta t}\kappa_3 \right] \\
\psi_4 : (x,t,u) &\rightarrow (x, t, ue^{\kappa_4}) \\
\psi_5 : (x,t,u) &\rightarrow (x, t, u + B(x,t)e^{\kappa_5})
\end{aligned} \tag{4.15}$$

where κ_i, i=1, 2, ...,5 are arbitrary constants.

4.4 Exact Invariant Solutions of Eq. (4.7)

Once the infinitesimal generators have been obtained, we then use them to compute the similarity variables from the corresponding Lagrange equations. Solving the Lagrange equations results with a family of new invariant solutions for the zero-coupon bond pricing model associated with the functional interest rate model. It has been shown in chapter 2 that given any partial differential equation of the form

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}...) = 0, \tag{4.16}$$

a function $u = \theta(x, t)$ that results from its invariance under the corresponding infinitesimal generator Γ constitute an invariant solution of the PDE provided:

$$\Gamma(u - \theta(x, t)) = 0 \qquad \text{when} \qquad u = \theta(x, t). \tag{4.17}$$

Therefore applying this important concept we solve the associated Lagrange equations deduced from the associated infinitesimal generators in equation (4.12). This results with the following three types of close-form invariant solutions of the zero-coupon bond pricing equation (4.7) and they are presented in Examples 1, 2 and 3 below.

Example 1: Let us consider the infinitesimal generator Γ_1. Solving its associated Lagrange equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}, \tag{4.18}$$

we result with the following two invariants $J_1 = x$, and $J_2 = u$. Therefore the invariant solution is given by $J_2 = \omega(J_1)$. That is, $u = \omega(x)$.

Substituting $u = \omega(x)$ in equation (4.7) and simplifying, this results with the model satisfying the following ODE:

$$rx\omega(x) + (\epsilon - x^2\eta)\omega'(x) + \beta x\omega''(x) = 0, \tag{4.19}$$

which then solves to

$$\omega(x) = \left(c_1 M\left(m, n, \frac{\eta x^2}{2\beta}\right) + c_2 U\left(m, n, \frac{\eta x^2}{2\beta}\right)\right) x^{\rho}, \tag{4.20}$$

where

$$m = \frac{1}{2}\frac{(\eta + r)\beta - \eta\epsilon}{\eta\beta}, \qquad n = \frac{1}{2}\frac{3\beta - \epsilon}{\beta}, \qquad \rho = \frac{\beta - \epsilon}{\beta} \tag{4.21}$$

and c_1, c_2 are any chosen constants. $M(a, b, .)$ and $U(a, b, .)$ are special types of the Kummer M and U described in more depth in reference [33]. Therefore the derived explicit exact invariant solutions for the zero-coupon bond pricing equation (4.7) associated with Γ_1 are reported as

$$u(x, t) = \left(c_1 M\left(m, n, \frac{\eta x^2}{2\beta}\right) + c_2 U\left(m, n, \frac{\eta x^2}{2\beta}\right)\right) x^{\rho} \tag{4.22}$$

where m, n, ρ are as above.

Example 2: Let us consider the infinitesimal generator Γ_2. Solving its associated Lagrange equations

$$-\eta e^{2\eta t}\frac{dt}{1} = e^{2\eta t}\frac{dx}{x} = -\eta e^{2\eta t}\frac{du}{ru}, \tag{4.23}$$

we obtain the following invariants

$$\zeta = t + \frac{\ln x}{\eta}, \qquad u(x,t) = x^{-\frac{r}{\eta}}\omega\left(t + \frac{\ln x}{\eta}\right) = x^{-\frac{r}{\eta}}\omega\left(\zeta\right). \tag{4.24}$$

The similarity function $\omega = \omega(\zeta)$ satisfies the following similarity reduction equation:

$$r(r\beta + (\beta - \epsilon)\eta)\omega(\zeta) + (-2r\beta - \beta\eta + \epsilon\eta)\omega'(\zeta) + \beta\omega''(\zeta) = 0, \tag{4.25}$$

that solves to

$$\omega(\zeta) = e^{\zeta(r\beta+\beta\eta-\epsilon\eta)}c_1 + e^{r\zeta}c_2. \tag{4.26}$$

Therefore the derived explicit exact invariant solutions of the zero-coupon bond equation (4.7) associated with Γ_2 are reported as

$$u(x,t) = x^{-\frac{r}{\eta}}\left(e^{\zeta(r\beta+\beta\eta-\epsilon\eta)}c_1 + e^{r\zeta}c_2\right) \tag{4.27}$$

where c_1, c_2 are arbitrary constants and ζ is as define in equation (4.24).

Example 3: Lastly let us consider the infinitesimal generator Γ_3. The invariants are computed from the associated Lagrange equations and they are obtained as

$$\zeta = t - \frac{\ln x}{\eta}, \qquad u(x,t) = e^{\frac{x^2\eta}{2\beta}} x^{\left(-1-\frac{\epsilon}{\beta}+\frac{r}{\eta}\right)} \omega\left(t - \frac{\ln x}{\eta}\right), \tag{4.28}$$

with the similarity function $\omega = \omega(\zeta)$ satisfying the following similarity reduction equation:

$$(r-2\eta)(r\beta - (\beta+\epsilon)\eta)\omega(\zeta) + (-2r\beta + 3\beta\eta + \epsilon\eta)\omega'(\zeta) + \beta\omega''(\zeta) = 0. \tag{4.29}$$

When solving this reduction equation, we result with the following solution

$$\omega(\zeta) = e^{\frac{\zeta(r\beta - \beta\eta - \epsilon\eta)}{\beta}} c_1 + e^{\zeta(r-2\eta)} c_2. \tag{4.30}$$

Therefore the derived explicit exact invariant solutions of the zero-coupon bond equation (4.7) associated with Γ_3 are reported as

$$u(x,t) = e^{\frac{x^2\eta}{2\beta}} x^{\left(-1-\frac{\epsilon}{\beta}+\frac{r}{\eta}\right)} \left(e^{\frac{\zeta(r\beta - \beta\eta - \epsilon\eta)}{\beta}} c_1 + e^{\zeta(r-2\eta)} c_2\right) \tag{4.31}$$

where c_1, c_2 are arbitrary constants and ζ is as defined in equation (4.28). Note that Γ_4 does not provide any invariants. This then concludes the computations of the invariant solutions associated with the zero-coupon bond pricing equation (4.7).

4.4.1 New solutions via group point transformations

One of the important concept about Lie's group theory is the ability to transform known solutions of a differential equation into other new solutions of the same equation using Lie point transformations. In this section we generate more solutions using obtained solutions in equations (4.27) and (4.31) to compute new solutions from them using group point transformations in equation (4.15). If Γ in equation (4.8) is the group generator admitted by equation (4.7) and $u = w(x, t)$ is any of the zero-coupon bond pricing equation solutions, then $\tilde{u} = \tilde{w}(\tilde{x}, \tilde{t})$ will still form part other solution of equation (4.7) obtained from group point transformations. That is, if the following substitutions $\tilde{x}, \tilde{t}$, and $\tilde{u}$ are made in the transformations in equation (4.15) and then solving to make u subject this then results with the new solutions for equation (4.7). We give an illustration of this point by making use of ψ_1 in equation (4.15). With this transformation since $\tilde{x} = x$ and $\tilde{u} = u$, then both solutions in equations (4.27) and (4.31) are transformed to new solutions whereby t in both equations is replaced by $(t - \kappa_1)$. That is, equations (4.27) and (4.31) will only be affected on ζ as it now changes to $(t - \kappa_1) + \frac{\ln x}{\eta}$ and $(t - \kappa_1) - \frac{\ln x}{\eta}$, respectively for equations (4.27) and (4.31).

4.5 Results Discussion

In order to emphasise the novelty of Lie's group theory, numerical solutions are deduced and we make application to the Vasicek interest rate model. We make use of the newly obtained explicit solutions in equations (4.27) and (4.31). Chern [11] have shown that with parameters: $x = r + \alpha$, $\epsilon = 0$ and $\eta = \kappa$ functional interest rate model can be transformed to Vasicek model. Using the parameters: $\beta = 2\eta$, $\alpha = (\sigma^2 + 2\epsilon)/(8\eta)$, $r = \frac{1}{4}x^2$ a functional interest rate model can be transformed to CIR model; and so on. The parameters chosen for illustration are as follows

- interest rate (risk-free) r= 0.90,
- volatility $\sigma = 0.80$,
- parameter $\alpha = 0.01$,
- parameter $\eta = 0.5$,
- constant 1 $c_1 = 1$,
- constant 2 $c_2 = 0.5$,
- time to expiration T= 14 years.

We illustrate the solutions in Figure 4.1 and 4.2 for the newly explicit solutions associated with Γ_2 and Γ_3.

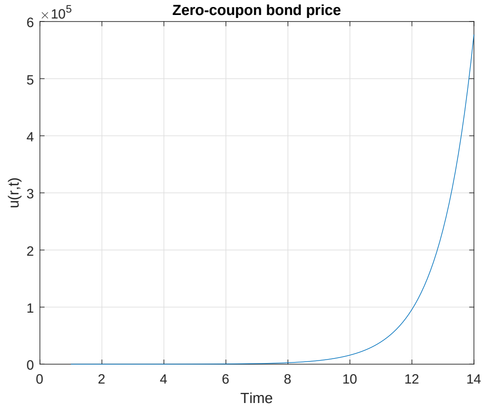

Figure 4.1: Explicit exact invariant solutions associated with Γ_2

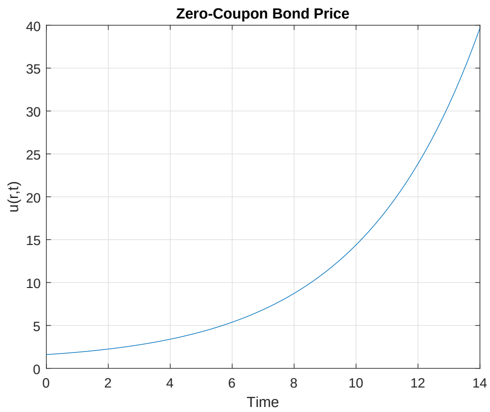

Figure 4.2: Explicit exact invariant solutions associated with Γ_3

An important question that one may ask is what happens to the prices of the bond (zero-coupon bond in our case) as time passes? Bodie et al. [34] on page 433 has given an illustration by a form of an example, to show that it is expected for a zero-coupon bond to sell for par value at maturity eventhough before then it should sell at a discount from par value due to the time value of money. In fact when the interest rate is constant, a zero-coupon bond price should increase at exactly the rate of interest. We observe that the curves of the explicit exact invariant solutions obtained in section 4.5, analysis made with Vasicek model, are smooth and monotonically increasing with respect to time towards the maturity of the bond. Section 4.4.1 using group ψ_1 to illustrate the novelty of the method, we see that the graphs in Figures 4.1 and 4.2 will still remain the same shape under point transformation ψ_1 but they will be horizontally shifted to the right (or translated) by κ_i units ($\kappa_i > 0$). These results shows some resemblance to the graphs by Bodie et al. [34], and Goard [18]; and thus emphasise the important point that as time passes the price of the zero-coupon bond indeed increase at the rate of interest. We further observe that solutions of the zero-coupon bond pricing equation have a direct mapping with one another through point transformations and this then emphasise an important concept of Lie symmetry analysis as mentioned in section 4.4.1.

4.6 Conclusion

In this chapter an application of Lie symmetry analysis have been performed to the bond option model, the zero-coupon bond pricing equation in mathematical finance. Our findings have gathered that the zero-coupon bond pricing equation under consideration admits four point symmetries plus an additional infinite dimensional subalgebra Γ_B. With application made to the obtained infinitesimal generators, explicit exact invariant solutions of the zero-coupon bond pricing model have been computed and verified in section 4.4.1, they are indeed invariant. The novelty of the solutions have been presented in the form of graphs where an exponential growth by prices of the zero-coupon bond equation was observed. With graphs proving invariant under point transformations, this then suggests that there is a direct relationship between solutions of bond options under functional interest rate modelling and solutions under usual interest rate models which one believes direct mappings via Lie symmetry analysis could link them. Therefore a good model depends entirely on one's choices made from parameters r, ϵ, σ and η.

Interest rate derivatives are much more complex to price than equity and foreign exchange derivatives because of their behaviour of an interest rate being more complicated as compared to that of stock price or an exchange rate. Therefore, numerical computations have seems to be more appreciated in the valuation of interest rate derivatives as there are not many analytical expressions for interest rate derivatives. The zero-coupon bond model in equation (4.7) can therefore be classified as a functional PDE model due to the interest rate associated with it modelled as a certain functional transformation of the underlying state variable. The aforementioned functional interest rate model as we have noted does not only embeds the known single factor interest rate models, but also provides a flexible approach and analytical scheme for constructing many more new models to expand the existing family of interest rate financial models. Again as pointed out in [27], this model can provide great benefits in numerical computations as well.

Using the solutions in equations (4.27) and (4.31), the important concept of Lie symmetry analysis have been verified in section 4.4.1. That is, through group point transformations it has been possible to use known solutions to generate more solutions which were unknown. From this fact, it is then safe to conclude that there is existence of a mapping between functional interest rate derivative PDE models into well known ordinary models like Vasicek pricing equation and other model equations. Furthermore, also a mapping of solutions of functional interest derivative PDE models to solutions of known interest rate models PDE equations. Thus with these facts, we want to believe that the contribution made by Kaibe *et al.* [22] to the pricing of interest rate derivatives models can be explored further to increase the minute number of analytical expressions that are currently available for pricing interest rate derivatives. Again with the co-domain of $f(X_t, t) = r_t$ restricted to positive real values, this then helps to avoid the drawback of negative interest rate that is normally found in Vasicek interest rate model.

The combination of the Ornstein-Unlenbeck process and the Bessel process in the model in equation (4.4) enables both two important properties of the interest rate models as mentioned in the introduction to be satisfied, and this then can as a result enable one to construct new models with ease due to its flexibility. The CIR models which was designed as an effort to correct the drawback of negative interest rate by Vasicek interest rate model, satisfies both properties of the interest rate model. Since it is also embedded in the functional interest rate model, it can be shown that a mapping connecting it and the functional interest rate exist as it has been shown in Chern [11]. Chern [11] states that literature regarding this approach is still minute, but we will like to believe that this model due to its functional approach to modelling interest rates and its ability to provide a unified framework for representing existing single factor interest rate models, it can play a vital role in finance literature for pricing interest rate derivatives to expand existing analytical expressions for debt securities.

Chapter 5

The Optimal System of Reductions for the Interest Rate Derivatives Pricing Model

In this chapter a zero-coupon bond pricing model described in terms of a partial differential equation (PDE) is analysed by means of Lie symmetry analysis. We extend further the work of the previous chapter and from the obtained one-parameter Lie point symmetries we deduce the corresponding group of adjoint representations of the model. Furthermore the optimal system of the one-dimensional subalgebras is computed and used to construct a family of closed-form invariant solutions of the aforementioned bond option pricing model via solving the associated reduced ordinary differential equations (ODEs).

5.1 Introduction

Interest rates play a vital role in the valuation of interest rate derivatives, such as caps, bond options and swap options. It is always assumed that the short-term interest rates behaves more or less like a stock price. This assumption is not ideal due to the fact that interest rates appear to be pulled back to some long-run average level over time [12]. A phenomenon known in the finance literature as *mean reversion*. When the interest rate r is high, the mean reversion tends to cause it to have a negative drift. But when r is low, the mean reversion tends to cause it to have a positive drift. If we consider the Vasicek's model with the risk-neutral process for r

$$dr = a(b - r)dt + \sigma dz \tag{5.1}$$

where a, b, and σ are constants; the short rate is pulled to a level b at a rate a. Thus, as explained above, if r is high then drift $a(b - r)$ tends to be negative, whereas if low it tends to be positive. Also imposed upon this "pull" is normally a distributed stochastic term σdz. The Vasicek's model therefore incorporates mean reversion. The only drawback of Vasicek's model is that its short-term interest rate, r, can be negative. Cox, Ingersoll, and Ross in a way to cater for this drawback, proposed an alternative model where the rates are always non-negative [31]. The risk-neutral process for r in their model is

$$dr = a(b - r)dt + \sigma\sqrt{r}dz. \tag{5.2}$$

The mean-reverting drift of this model is the same as that of Vasicek model, but the standard deviation of the change in the short rate in a short period of time is proportional to $\sqrt{r}$. This then means that as the short-term interest rate increase, so does its standard deviation. It is therefore important for interest rate model to satisfy the two properties of interest rate, namely, mean-reversion and positivity of interest rate. Luo *et al.* [27] using the combination of Ornstein-Unlenbeck process and Bessel process, derived and proposed the following functional interest rate model that embeds most known interest rate models for different choices of η, ϵ and σ:

$$dX_t = \left(-\eta(t)X_t + \frac{\epsilon(t)}{X_t} \right) dt + \sigma(t)dZ_t, \tag{5.3}$$

where $\{Z_t\}_{t\geq 0}$ is a Brownian motion. The parameters ϵ, η, σ are given functions of t. In this setup $r_t = f(X_t, t)$ is modeled as a function of Markov state variable X_t and time t. The range of f confined to only positive real values, then this enables one to avoid the drawback of negative interest rate that are normally found in Vasicek model. Using this functional interest rate model, as shown in Kaibe *et al.* [22] the price dynamics for the zero-coupon bond under functional interest rate models are described in terms of the following partial differential equation

$$v_t(x,t) + \frac{\sigma^2}{2} v_{xx}(x,t) + \left(-\eta x + \frac{\epsilon}{x} \right) v_x(x,t) - rv(x,t) = 0. \tag{5.4}$$

Due to lack of analytical expressions for interest rate derivative models, numerical methods such as binomial trees, Monte-Carlo simulation, and finite-difference methods are normally the tools commonly used to value these financial products. Kaibe *et al.* [22] have computed and proposed that the pricing model in equation (5.4) admits four Lie symmetries and an additional infinite dimensional subalgebra. In this chapter, we expand their work further by using the obtained one-parameter Lie point symmetries to deduce the corresponding group of adjoint representations for the zero-coupon bond pricing equation in order to derive an optimal system of the one-dimensional subalgebras. The optimal system is then used to construct a family of closed-form solutions for the aforementioned pricing equation.

It is well known that a one-dimensional list such as $\{\Gamma_\rho\}_{\rho\in A}$ is called an optimal system provided the following two conditions are satisfied: (1) completeness, meaning that any one dimensional subalgebra is equivalent for some Γ_ρ; (2) in-equivalence, meaning that Γ_ρ and Γ_π are in-equivalent for distinct ρ and π. In computing an optimal system one normally determines an invariant which normally gives sort of a restriction on how far one can simplify the Lie algebra. We see the application of this in the work of Sinkala *et al.* [20] and Khalique *et al.* [21] who obtain the associated invariants to the models they were working on in order to deduce the invariant solutions of their models of choice. Hu *et al.* [32] use an algorithmic approach to compute one-dimensional optimal system which still involves computing the invariants together with adjoint transformation matrix before classification of Lie algebras. In this chapter augmenting the work of Kaibe *et al.* [22], we follow Olver'approach [3] to deduce an optimal system of the aforementioned model in order to compute its associated explicit invariant solutions.

5.2 Classification of group-invariant solutions of the zero-coupon bond pricing model

In order to compute group-invariant solutions in which all solutions can be deduced, a set of equivalent classes is always vital. This is because the combinations of symmetries to construct group-invariant solutions are too many. It is therefore, usually not feasible to list the entire family of all possible group-invariant solutions of certain differential equation. A systematic approach of classifying the solutions that leads to an optimal system of group-invariant solutions is always required. In what follows we start first by computing the commutator of the Lie-Bracket and then adjoint representation associated with the zero-coupon bond pricing equation.

5.2.1 Computation of Lie-Brackets

In chapter 2, it has been defined that the Lie-Bracket or Commutator of any two symmetries or generators Γ_A and Γ_B, is computed from the following relation

$$[\Gamma_A, \Gamma_B] = \Gamma_A \Gamma_B - \Gamma_B \Gamma_A. \tag{5.5}$$

The relation is termed skew-symmetric if $[\Gamma_A, \Gamma_B] = -[\Gamma_B, \Gamma_A]$. Using the Lie symmetries

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}, \\
\Gamma_2 &= -\frac{e^{-2\eta t}}{\eta}\frac{\partial}{\partial t} + xe^{-2\eta t}\frac{\partial}{\partial x} - \frac{re^{-2\eta t}}{\eta}u\frac{\partial}{\partial u}, \\
\Gamma_3 &= \frac{e^{2\eta t}}{\eta}\frac{\partial}{\partial t} + xe^{2\eta t}\frac{\partial}{\partial x} + \left(-1 - \frac{\lambda}{\beta} + \frac{r}{\eta} + \frac{x^2\eta}{\beta}\right)ue^{2\eta t}\frac{\partial}{\partial u}, \\
\Gamma_4 &= u\frac{\partial}{\partial u},
\end{aligned} \tag{5.6}$$

in equation (4.12) of the zero-coupon bond model we compute the associated commutators with ϵ replaced by λ. The commutators associated with the zero-coupon bond model are computed and presented in Table 5.1.

$[\Gamma_i, \Gamma_j]$	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	0	$-2\eta\Gamma_2$	$2\eta\Gamma_3$	0
Γ_2	$2\eta\Gamma_2$	0	$-\frac{4}{\eta}\Gamma_1 + 2\left(\frac{2\lambda}{\sigma^2} - \frac{2r}{\eta} + 1\right)\Gamma_4$	0
Γ_3	$-2\eta\Gamma_3$	$\frac{4}{\eta}\Gamma_1 - 2\left(\frac{2\lambda}{\sigma^2} - \frac{2r}{\eta} + 1\right)\Gamma_4$	0	0
Γ_4	0	0	0	0

Table 5.1: Lie Brackets of the zero-coupon bond pricing model

.

As an example, we illustrate how the commutator $[\Gamma_1, \Gamma_2]$ have been computed

$$\begin{aligned}
[\Gamma_1, \Gamma_2] &= \Gamma_1(\Gamma_2) - \Gamma_2(\Gamma_1) \\
&= \frac{\partial}{\partial t}\left(-\frac{e^{-2\eta t}}{\eta}\frac{\partial}{\partial t} + xe^{-2\eta t}\frac{\partial}{\partial x} - \frac{re^{-2\eta t}}{\eta}u\frac{\partial}{\partial u}\right) \\
&- \left(-\frac{e^{-2\eta t}}{\eta}\frac{\partial}{\partial t} + xe^{-2\eta t}\frac{\partial}{\partial x} - \frac{re^{-2\eta t}}{\eta}u\frac{\partial}{\partial u}\right)\frac{\partial}{\partial t} \\
&= 2e^{-2\eta t}\frac{\partial}{\partial t} - 2\eta xe^{-2\eta t}\frac{\partial}{\partial x} + 2re^{-2\eta t}u\frac{\partial}{\partial u} \\
&= -2\eta\left(-\frac{e^{-2\eta t}}{\eta}\frac{\partial}{\partial t} + xe^{-2\eta t}\frac{\partial}{\partial x} - \frac{re^{-2\eta t}}{\eta}u\frac{\partial}{\partial u}\right) \\
&= -2\eta\Gamma_2.
\end{aligned} \tag{5.7}$$

5.2.2 Adjoint representation

An optimal system of a Lie algebra is a set of l-dimensional subalgebras such that every l-dimensional subalgebra is equivalent to a unique element of the set under some element of the adjoint representation [3],

$$\begin{aligned} Ad(exp(\epsilon\Gamma_A))\Gamma_B &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}(ad\Gamma_A)^n\Gamma_B = \Gamma_B - \epsilon[\Gamma_A, \Gamma_B] \\ &+ \frac{\epsilon^2}{2!}[\Gamma_A, [\Gamma_A, \Gamma_B]] - ... \end{aligned} \tag{5.8}$$

where $[\Gamma_A, \Gamma_B]$ denotes the commutator of generators Γ_A and Γ_B. Using the formula in equation (5.7) and the commutators in Table 5.1, the adjoint representations are computed and presented in tabular form in Table 5.2.

Ad	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	Γ_1	$e^{2\epsilon\eta}\Gamma_2$	$e^{-2\epsilon\eta}\Gamma_3$	Γ_4
Γ_2	$\Gamma_1 - 2\epsilon\eta\Gamma_2$	Γ_2	M	Γ_4
Γ_3	$\Gamma_1 + 2\epsilon\eta\Gamma_3$	N	Γ_3	Γ_4
Γ_4	Γ_1	Γ_2	Γ_3	Γ_4

Table 5.2: Adjoint representation of subalgebras of the zero-coupon bond model

$$M = \frac{4\epsilon\Gamma_1}{\eta} - 4\epsilon^2\Gamma_2 + \Gamma_3 + 2\epsilon\left(\frac{2r}{\eta} - \frac{2\lambda}{\sigma^2} - 1\right)\Gamma_4,$$

and

$$N = -\frac{4\epsilon\Gamma_1}{\eta} + \Gamma_2 - 4\epsilon^2\Gamma_3 + 2\epsilon\left(\frac{2\lambda}{\sigma^2} - \frac{2r}{\eta} + 1\right)\Gamma_4.$$

5.2.3 Optimal system of the zero-coupon bond pricing model

Suppose the n-dimensional symmetry algebra ζ of a differential system is generated by the vector fields $\{\Gamma_1, ..., \Gamma_n\}$ and the symmetry group of ζ is denoted by G. A real function Φ on the Lie algebra ζ is called an invariant if $\Phi(Ad_g(\Gamma)) = \Phi(\Gamma)$ for all vector fields $\Gamma \in \zeta$ and all $g \in G$. Here Ad_g is the adjoint representation g and $Ad_g(\Gamma) = g^{-1}\Gamma g$. The determination of the invariant of the full adjoint action is vital as it places restriction on how far one can simplify Γ [3]. Important fact about Lie algebra L_n spanned by symmetries of the model under consideration except the infinite dimensional subalgebra is that it provides the possibility of determining the invariant solutions of the model. In light of this fact we formulate an arbitrary operator that comprise of a linear combinations of our subalgebras. That is,

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 \tag{5.9}$$

which depends on the four arbitrary constants a_1,...,a_4. Now to deduce the optimal system of one-dimensional subalgebra adopting Olver'approach [3], we start first by computing the invariant of the full adjoint map to have a restriction on how far we are to simplify the operator Γ. Composing the adjoint Γ_2 and Γ_3 against Γ produce the desired invariant through few manipulations. That is,

$$\widetilde{\Gamma} = \sum_{i=1}^{4} \widetilde{a}_i\Gamma_i = Ad(e^{\alpha\Gamma_2}) \circ Ad(e^{\beta\Gamma_3})\Gamma. \tag{5.10}$$

Computing first the $Ad(e^{\beta\Gamma_3})\Gamma$ by making use of the adjoints in Table 5.2, we result with

$$\begin{aligned}
\overline{\Gamma} &= Ad(e^{\beta\Gamma_3})\Gamma \\
&= a_1(\Gamma_1 + 2\beta\eta\Gamma_3) + a_3\Gamma_3 + a_4\Gamma_4 \\
&+ a_2\left[-\frac{4\beta}{\eta}\Gamma_1 + \Gamma_2 - 4\beta^2\Gamma_3 + 2\beta(\frac{2\lambda}{\sigma^2} - \frac{2r}{\eta} + 1)\Gamma_4\right] \\
&= \left(a_1 - \frac{4}{\eta}\beta a_2\right)\Gamma_1 + a_2\Gamma_2 + [a_3 + 2a_1\beta\eta - 4a_2\beta^2]\Gamma_3 \\
&+ \left[a_4 + 2\beta\left(\frac{2\lambda}{\sigma^2} - \frac{2r}{\eta} + 1\right)\right]\Gamma_4.
\end{aligned} \tag{5.11}$$

Secondly, if we act by $Ad(e^{\alpha\Gamma_2})$ on $\overline{\Gamma}$ we obtain

$$Ad(e^{\alpha\Gamma_2})\overline{\Gamma} = \widetilde{a_1}\Gamma_1 + \widetilde{a_2}\Gamma_2 + \widetilde{a_3}\Gamma_3 + \widetilde{a_4}\Gamma_4, \tag{5.12}$$

which reduces to

$$
\begin{aligned}
\widetilde{\Gamma} &= \left[a_1 - \frac{4}{\eta}\beta a_2 + \frac{4\alpha}{\eta}(a_3 + 2a_1\beta\eta - 4a_2\beta^2)\right]\Gamma_1 \\
&+ \left[a_2 - 2\alpha\eta(a_1 - \frac{4}{\eta}\beta a_2) - 4\alpha^2(a_3 + 2a_1\beta\eta - 4a_2\beta^2)\right]\Gamma_2 \\
&+ \left[a_3 + 2a_1\beta\eta - 4a_2\beta^2\right]\Gamma_3 \\
&+ \left[a_4 + (2\lambda\eta - 2r\sigma^2 + \eta\sigma^2)\left(\frac{2\beta}{\eta\sigma^2} - \frac{2\alpha}{\eta\sigma^2}(2\beta\eta a_1 - 4a_2\beta^2 + a_3)\right)\right]\Gamma_4.
\end{aligned} \tag{5.13}
$$

Now to obtain α and β we notice that solving the quadratic equation $a_3 + 2a_1\beta\eta - 4a_2\beta^2 = 0$ in equation (5.13) produces β, which from it we are able to obtain the invariant (the radicand in β) of the full adjoint action, namely, $\kappa = a_1^2\eta^2 + 4a_2a_3$. The coefficients $\widetilde{a_1}$, $\widetilde{a_2}$, $\widetilde{a_3}$ and $\widetilde{a_4}$ in (5.13) of the subalgebras Γ_1, Γ_2, Γ_3 and Γ_4 are respectively,

$$\widetilde{a_1} = a_1 - \frac{4}{\eta}\beta a_2 + \frac{4\alpha}{\eta}(a_3 + 2a_1\beta\eta - 4a_2\beta^2), \tag{5.14}$$

$$\widetilde{a_2} = a_2 - 2\alpha\eta(a_1 - \frac{4}{\eta}\beta a_2) - 4\alpha^2(a_3 + 2a_1\beta\eta - 4a_2\beta^2), \tag{5.15}$$

$$\widetilde{a_3} = a_3 + 2a_1\beta\eta - 4a_2\beta^2, \tag{5.16}$$

$$\widetilde{a_4} = a_4 + (2\lambda\eta - 2r\sigma^2 + \eta\sigma^2)\left(\frac{2\beta}{\eta\sigma^2} - \frac{2\alpha}{\eta\sigma^2}(2\beta\eta a_1 - 4a_2\beta^2 + a_3)\right) \tag{5.17}$$

To this end, in order to compute the optimal system of our bond model in equation (5.4) we need to consider the following three cases of the value of the invariant: $\kappa > 0$, $\kappa < 0$ and $\kappa = 0$.

Case 1. $\kappa > 0$

Suppose β is the real root of $a_3 + 2a_1\beta\eta - 4a_2\beta^2 = 0$, then this implies $\widetilde{a_3} = 0$ in equation (5.16) and $\widetilde{a_2} = 0$ in equation (5.15) when

$$\alpha = \frac{a_2}{2\eta(a_1 - \frac{4}{\eta}\beta a_2)}. \tag{5.18}$$

For invariant $\kappa = a_1^2\eta^2 + 4a_2a_3 > 0$ to be satisfied since $\widetilde{a_3} = 0$ and $\widetilde{a_2} = 0$, $\widetilde{a_1} = \sqrt{\kappa} \neq 0$. Rescaling $\widetilde{a_1} = 1$ results with,

$$\Gamma = \Gamma_1 + \widetilde{a_4}\Gamma_4, \tag{5.19}$$

where

$$\widetilde{a_4} = a_4 + (2\beta\eta - 2r\beta^2 + \eta\sigma^2)\frac{2\beta}{\eta\sigma^2}. \tag{5.20}$$

Thus every one-dimensional subalgebra generated by Γ with $\kappa > 0$ is equivalent to the subalgebra spanned by

$$\Gamma = \Gamma_1 + b\Gamma_4, \qquad b \in \Re. \tag{5.21}$$

Case 2. $\kappa < 0$

We set $\beta = 0$ and $\alpha = \frac{-a_1\eta}{4a_3}$ to make $\widetilde{a_1} = 0$. Then,

$$\begin{aligned}
\widetilde{a_2} &= a_2 - 2\eta a_1\left(\frac{-a_1\eta}{4a_3}\right) - 4a_3\left(\frac{-a_1\eta}{4a_3}\right)^2 && (5.22)\\
\widetilde{a_3} &= a_3 \neq 0 && (5.23)\\
\widetilde{a_4} &= a_4 + \frac{a_1}{2\sigma^2}(2\lambda\eta - 2r\sigma^2 + \eta\sigma^2) && (5.24)
\end{aligned}$$

This then implies

$$\Gamma^{'} = \Gamma_3 + \widetilde{a_2}\Gamma_2 + \widetilde{a_4}\Gamma_4 \tag{5.25}$$

$\widetilde{a_3} = 1$ rescaling. Acting on $\Gamma^{'}$ by the group generated by Γ_1, this leads to

$$\Gamma^{''} = e^{-2\epsilon\eta}\Gamma_3 + \widetilde{a_2}e^{2\epsilon\eta}\Gamma_2 + \widetilde{a_4}\Gamma_4 \tag{5.26}$$

which is a scalar multiple of

$$\Gamma^{'''} = \Gamma_3 + \widetilde{a_2}e^{4\epsilon\eta}\Gamma_2 + \widetilde{a_4}e^{2\epsilon\eta}\Gamma_4. \tag{5.27}$$

Depending on the sign of $\widetilde{a_2}$ and $\widetilde{a_4}$ we can make the coefficient of Γ_2 and Γ_4 either +1, -1 or 0. Therefore, any one-dimensional subalgebra spanned by Γ with $a_1 = 0$, $a_3 \neq 0$ is equivalent to the one spanned by either $\Gamma_3 \pm \Gamma_2 \pm \Gamma_4$ or $\Gamma_3 \pm \Gamma_2$ or $\Gamma_3 \pm \Gamma_4$ or Γ_3.

Case 3. $\kappa = 0$

If not all are zero, we can choose α and β in equation (5.14)-(5.17) such that $\widetilde{a_3} \neq 0$ but $\widetilde{a_1} = \widetilde{a_2} = 0$. That is,

$$\widetilde{a_1} = \frac{4\alpha}{\eta}a_3, \tag{5.28}$$

imply $\widetilde{a_1} = 0$ choosing $\alpha = 0$. Also

$$\widetilde{a_2} = -4\alpha^2 a_3 \tag{5.29}$$

imply $\widetilde{a_2} = 0$. Furthermore

$$\widetilde{a_3} = a_3, \tag{5.30}$$

and

$$\widetilde{a_4} = a_4 + (2\lambda\eta - 2r\sigma^2 + \eta\sigma^2)\left(\frac{2\beta}{\eta\sigma^2} - \frac{2\alpha}{\eta\sigma^2}a_3\right) \tag{5.31}$$

so that $\widetilde{a_4} = 0$ when

$$\beta = \frac{-a_4\eta\sigma^2}{2(2\lambda\eta - 2r\sigma^2 + \eta\sigma^2)}. \tag{5.32}$$

Then the one-dimensional subalgebra generated by Γ with $\kappa = 0$ is equivalent to the subalgebra spanned by Γ_3. If we consider also a case where $\widetilde{a_3} = 0$ but $\widetilde{a_4} \neq 0$. This then results with $\Gamma = \Gamma_4$ when scaling $\widetilde{a_4} = 1$. This subalgebra does not produce any invariant solutions, we therefore discard it.

Put together, we therefore result with an optimal system of one-dimensional subalgebras of the zero-coupon bond pricing equation (5.4) given as,

1. $\Gamma_1 + b\Gamma_4, \qquad b \in \Re$
2. $\Gamma_3 + \Gamma_2 + \Gamma_4$
3. $\Gamma_3 - \Gamma_2 - \Gamma_4$
4. $\Gamma_3 + \Gamma_2$
5. $\Gamma_3 - \Gamma_2$
6. $\Gamma_3 + \Gamma_4$
7. $\Gamma_3 - \Gamma_4$

8. Γ_3

5.2.4 Solutions of the zero-coupon bond model deduced from one-dimensional subgroups

Using the members of the constructed optimal system of the one-dimensional subalgebras we perform some reductions to deduce the group-invariant solutions of equation (5.4). The procedure for performing symmetry reduction is well known in the literature [1], [2], [3], [4]. The subalgebras $\Gamma_3 \pm \Gamma_2$ and $\Gamma_3 \pm \Gamma_2 \pm \Gamma_4$ generates much more complicated invariant solutions which we are still working on and their results will be reported later somewhere. We also believe that discrete symmetries can reduce this list of subalgebras of optimal system and this will also be taken into consideration in our report.

Case 1. We consider the subalgebra Γ_3 and solve its associated Lagrange equations

$$\frac{\eta dt}{e^{2\eta t}} = \frac{dx}{xe^{2\eta t}} = \frac{du}{u\left(\left(\frac{r}{\eta} - 1 - \frac{2\epsilon}{\sigma^2} + \frac{2\eta x^2}{\sigma^2}\right) e^{2\eta t}\right)} \tag{5.33}$$

to result with the following invariants

$$\zeta = t - \frac{\ln x}{\eta}, \qquad u(x,t) = e^{\frac{x^2\eta}{2\beta}} x^{\left(-1-\frac{\epsilon}{\beta}+\frac{r}{\eta}\right)} \omega\left(t - \frac{\ln x}{\eta}\right), \tag{5.34}$$

where the similarity function $\omega = \omega(\zeta)$ satisfies the following similarity reduction equation:

$$(r - 2\eta)(r\beta - (\beta + \epsilon)\eta)\omega(\zeta) + (-2r\beta + 3\beta\eta + \epsilon\eta)\omega'(\zeta) + \beta\omega''(\zeta) = 0. \quad (5.35)$$

Solving this reduction equation, we obtain the solution as

$$\omega(\zeta) = e^{\frac{\zeta(r\beta - \beta\eta - \epsilon\eta)}{\beta}} c_1 + e^{\zeta(r-2\eta)} c_2. \quad (5.36)$$

Therefore the derived explicit exact invariant solutions of the zero-coupon bond equation (5.4) associated with Γ_3 are reported as

$$u(x,t) = e^{\frac{x^2\eta}{2\beta}} x^{\left(-1-\frac{\epsilon}{\beta}+\frac{r}{\eta}\right)} \left(e^{\frac{\zeta(r\beta - \beta\eta - \epsilon\eta)}{\beta}} c_1 + e^{\zeta(r-2\eta)} c_2\right) \quad (5.37)$$

where c_1, c_2 are arbitrary constants and ζ is as defined in equation (5.34).

Case 2. We consider the subalgebra $\Gamma_1 + b\Gamma_4$ and solve its associated Lagrange equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{bu}, \quad (5.38)$$

to result with the invariants $J_1 = x$, and $J_2 = ue^{-bt}$. The invariant solution is given by $J_2 = e^{bt}\omega(J_1)$, i.e. $u = e^{bt}\omega(x)$.

Substituting $u = e^{bt}\omega(x)$ in equation (5.4), this reduces the bond model to the following ODE:

$$e^{bt}\left(2(b-r)x\omega(x)+(-2\eta x^2+2\epsilon)\omega'(x)+x\sigma^2\omega''(x)\right)=0. \tag{5.39}$$

Solving this similarity reduction equation we result with the following solution

$$\omega(x)=c_1M\left(m,n,\frac{\eta x^2}{\sigma^2}\right)+c_2U\left(m,n,\frac{\eta x^2}{\sigma^2}\right) \tag{5.40}$$

where

$$m=\frac{r-b}{2\eta}, \qquad n=\frac{\sigma^2+2\epsilon}{2\sigma^2} \tag{5.41}$$

and c_1, c_2 are arbitrary constants. $M(a,b,.)$ and $U(a,b,.)$ are the Kummer M and U special functions [33]. Therefore, the derived explicit exact invariant solutions for the aforementioned zero-coupon bond model in equation (5.4) associated with the subalgebra $\Gamma_1+b\Gamma_4$ we report them as

$$u(x,t)=e^{bt}\left(c_1M\left(m,n,\frac{\eta x^2}{\sigma^2}\right)+c_2U\left(m,n,\frac{\eta x^2}{\sigma^2}\right)\right), \tag{5.42}$$

where m and n are as defined above.

Case 3. We consider the subalgebra $\Gamma_3+\Gamma_4$ and solve its associated Lagrange equations

$$\frac{\eta dt}{e^{2\eta t}}=\frac{dx}{xe^{2\eta t}}=\frac{du}{u\left(\left(\frac{r}{\eta}-1-\frac{2\epsilon}{\sigma^2}+\frac{2\eta x^2}{\sigma^2}\right)e^{2\eta t}+1\right)} \tag{5.43}$$

to result with the invariants

$$\zeta = t - \frac{\ln x}{\eta}, \qquad u(x,t) = e^{\frac{\eta x^2}{\sigma^2} - \frac{e^{-2\eta t}}{2}} x^{-1+\frac{r}{\eta}-\frac{2\epsilon}{\sigma^2}} \omega(\zeta) \tag{5.44}$$

where $\omega(\zeta)$ satisfies the following similarity reduction equation:

$$\begin{aligned} &\left(2n^3 + e^{2\eta\zeta}(2\eta - r)(2\epsilon\eta + (-r+\eta)\sigma^2)\right)\omega(\zeta) \quad + \\ &e^{\eta\zeta}\left((2\eta\epsilon + (3\eta - 2r)\sigma^2)\omega'(\zeta) + \sigma^2\omega''(\zeta)\right) = 0. \end{aligned} \tag{5.45}$$

Solving this differential equation, we result with the following solution

$$\begin{aligned} \omega(\zeta) &= c_1 e^{\dfrac{\left(\left(r - \frac{3\eta}{2}\right)\sigma^2 - \epsilon\eta\right)\zeta}{\sigma^2}} \times BesselJ(N, M) \\ &+ c_2 e^{\dfrac{\left(\left(r - \frac{3\eta}{2}\right)\sigma^2 - \epsilon\eta\right)\zeta}{\sigma^2}} \times BesselY(N, M) \end{aligned} \tag{5.46}$$

where

$$N = \frac{-|\sigma^2 - 2\epsilon|}{2\sigma^2} \qquad \text{and} \qquad M = \frac{e^{-\zeta\eta}\sqrt{2\eta}}{\sigma}.$$

The Bessel functions are special functions as defined in [33]. Therefore, the derived explicit exact invariant solutions for the aforementioned zero-coupon bond model in equation (5.4) associated with the subalgebra $\Gamma_3 + \Gamma_4$ we report them as

$$u(x,t) = e^{\frac{\eta x^2}{\sigma^2} - \frac{e^{-2\eta t}}{2}} x^{-1+\frac{r}{\eta}-\frac{2\epsilon}{\sigma^2}} \omega(\zeta) \tag{5.47}$$

where $\omega(\zeta)$ is as obtained in equation (5.46).

Case 4. We consider the subalgebra $\Gamma_3 - \Gamma_4$ and solve its associated Lagrange equations

$$\frac{\eta dt}{e^{2\eta t}} = \frac{dx}{xe^{2\eta t}} = \frac{du}{u\left(\left(\frac{r}{\eta} - 1 - \frac{2\epsilon}{\sigma^2} + \frac{2\eta x^2}{\sigma^2}\right) e^{2\eta t} - 1\right)} \tag{5.48}$$

to result with the invariants

$$\zeta = t - \frac{\ln x}{\eta}, \qquad u(x,t) = e^{\frac{\eta x^2}{\sigma^2} - \frac{e^{-2\eta t}}{2}} x^{-1+\frac{r}{\eta}-\frac{2\epsilon}{\sigma^2}} \omega(\zeta) \tag{5.49}$$

where $\omega(\zeta)$ satisfies the following similarity reduction equation:

$$\begin{aligned} &\left(-2n^3 + e^{2\eta\zeta}(2\eta - r)(2\epsilon\eta + (-r+\eta)\sigma^2)\right)\omega(\zeta) \quad + \\ &e^{\eta\zeta}\left((2\eta\epsilon + (3\eta - 2r)\sigma^2)\omega'(\zeta) + \sigma^2\omega''(\zeta)\right) = 0. \end{aligned} \tag{5.50}$$

Solving this differential equation, we result with the following solution

$$\begin{aligned} \omega(\zeta) &= c_1 e^{\frac{\left(\left(r - \frac{3\eta}{2}\right)\sigma^2 - \epsilon\eta\right)\zeta}{\sigma^2}} \times BesselJ(R,S) \\ &+ c_2 e^{\frac{\left(\left(r - \frac{3\eta}{2}\right)\sigma^2 - \epsilon\eta\right)\zeta}{\sigma^2}} \times BesselY(R,S) \end{aligned} \tag{5.51}$$

where

$$R = \frac{-|\sigma^2 - 2\epsilon|}{2\sigma^2} \qquad \text{and} \qquad S = \frac{I\sqrt{2\eta}e^{-\zeta\eta}}{\sigma}.$$

Therefore, the derived explicit exact invariant solutions for the aforementioned zero-coupon bond model in equation (5.4) associated with the subalgebra $\Gamma_3 - \Gamma_4$ we report them as

$$u(x,t) = e^{\frac{\eta x^2}{\sigma^2} - \frac{e^{-2\eta t}}{2}} x^{-1+\frac{r}{\eta}-\frac{2\epsilon}{\sigma^2}} \omega(\zeta) \tag{5.52}$$

where $\omega(\zeta)$ is as obtained in equation (5.51).

5.3 Conclusion

In this chapter Lie symmetry analysis has been carried out to the PDE model describing a zero-coupon bond pricing equation associated with the functional interest rate model. This model admits four Lie point symmetries plus an additional infinite-dimensional subalgebra. The four one-dimensional Lie algebra were then used to compute the optimal system of one-dimensional subalgebras. Using the obtained optimal system we performed symmetry reduction in order to deduce new group-invariant solutions for the zero-coupon bond model in equation (5.4). In as much as the PDE models associated with finance are rarely solvable and Monte Carlo methods are usually applied to solve them, we have managed to deduce nontrivial closed-form solutions for the zero-coupon bond model.

Chapter 6

Conclusion

Financial market models in their setup some comprise of a complicated system of partial differential equations arising from physical important problems. The discovery of any explicit solutions whatsoever of these models can always be of great interest. Our studies have focused mostly on obtaining non-trivial results in the finance literature through the application of Lie procedures. We have demonstrated the applications of Lie analysis which includes the computation of Lie point symmetries, Lie point transformations, reduction, optimal systems, and explicit computation of group invariant solutions.

In chapter 4, Lie symmetry analysis have been performed on a zero-coupon bond pricing equation in mathematical finance. It has been shown that the zero-coupon bond pricing equation admits four point symmetries plus an additional infinite dimensional subalgebra. The obtained infinitesimal generators have been used to obtain the associated explicit exact invariant solutions of the zero-coupon bond model and they have also been verified indeed as invariant solutions. The novelty of Lie symmetry analysis was explored further in the case where the solutions were resembled as graphs and an exponential growth by prices of the zero-coupon bond equation was discovered. The graphs also verified an important concept of invariant solutions of the zero coupon bond model in the sense that under the application of the point transformations the graphs seemed to experience a translation shift by certain units without changing the shape of the graphs. Therefore, there exists a relationship between solutions of bond options under functional interest rate modelling and solutions under ordinary interest rate models and this research have shown that a direct mappings via Lie symmetry analysis could link them.

The finance literature comprise of minute number of numerical schemes for valuing interest rate derivatives models. In chapter 5, using the deduced Lie point symmetries derived in chapter 4 we computed the adjoint representation of the zero coupon bond pricing equation. The adjoint representation were then used to compute an optimal system associated with the zero-coupon bond pricing equations under the functional interest rate model. Interest rate derivatives due to the behaviour of an interest rate being more complicated as compared to that of stock price or an exchange rate, have proved to be more complex to value than ordinary equity and foreign exchange derivatives. In this chapter with the help of derived optimal systems we further computed more explicit invariant solutions associated with the above-mentioned contingent claim to add more analytical pricing models for valuing bond options models concentration given to those dependent on functional interest rate models. The pricing model in equation (5.4), one can therefore refer to it as a functional PDE model due to functional interest rate model associated with it. This functional interest rate model as noted in chapters 4 and 5 embeds all known single factor interest rate models and this then suggests a flexibility in constructing many more new models to add to the existing family of analytical pricing models for interest rate derivatives, especially bond options. Also as [27] indicates, the functional interest rate model can provide great benefits in numerical computations as well.

Chern [11] states that literature regarding this approach is still minute. Based on the findings of this research, the author believe that the functional interest rate model due to its functionality approach to modelling interest rates and its ability to provide a unified framework for representing existing single factor interest rate models, can play a vital role in finance to produce valuable pricing models for more complex derived interest rate derivatives in order to add more analytical expressions for the contingent claims. Through the application of group point transformations the known solutions have been used to obtain unknown solutions and this indeed verified the important concept of Lie symmetry analysis. Again with the restriction of positive real values on the co-domain of $f(X_t, t) = r_t$, this enables one to avoid the normal drawback of negative interest rate that is normally found in the Vasicek model.

Future work plans to extend this research further to apply Lie symmetry analysis to the pricing equations associated with swaps, caps and floors whose price evolution are described in terms of a functional interest rate PDE model. The Heath-Jarrow-Morton (HJM) models are also used to model interest rate derivatives models. Future extension of this work also plans to find a relationship mapping between the HJM models and the functional interest rate PDE models.

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