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Alternating sign hypermatrix decompositions of Latin-like squares

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ABSTRACT

To any $n \times n$ Latin square L, we may associate a unique sequence of mutually orthogonal permutation matrices $P = P_1, P_2, ..., P_n$ such that $L = L(P) = \sum kP_k$. Brualdi and Dahl (2018) described a generalisation of a Latin square, called an *alternating sign hypermatrix Latin-like square (ASHL)*, by replacing P with an *alternating sign hypermatrix (ASHM)*. An ASHM is an $n \times n \times n$ (0,1,-1)-hypermatrix in which the non-zero elements in each row, column, and vertical line alternate in sign, beginning and ending with 1. Since every sequence of n mutually orthogonal permutation matrices forms the planes of a unique $n \times n \times n$ ASHM, this generalisation of Latin squares follows very naturally, with an ASHM A having corresponding ASHL $L = L(A) = \sum kA_k$, where A_k is the kth plane of A. This paper addresses open problems posed in Brualdi and Dahl's article, firstly by characterising how pairs of ASHMs with the same corresponding ASHL relate to one another and identifying the smallest dimension for which this can happen, and secondly by exploring the maximum number of times a particular integer may occur as an entry of an $n \times n$ ASHL. A construction is given for an $n \times n$ ASHL with the same entry occurring $\lfloor \frac{n^2+4n-19}{2} \rfloor$ times, improving on the previous best of $2n$.

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1. Introduction

An *alternating sign matrix (ASM)*, is a $(0, 1, -1)$-matrix in which the non-zero elements in each row and column alternate in sign, beginning and ending with 1. The $n \times n$ *diamond ASM* is an ASM with the maximum number of non-zero entries for given n. For odd n, there is a unique diamond ASM D_n. For even n, there are two diamond ASMs D_n and D'_n.

Example 1.1. The following are the two 4×4 diamond ASMs D_4 and D'_4, and the 5×5 diamond ASM D_5.

$$\mathbf{D_4}: \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & +1 & -1 & +1 \\ +1 & -1 & +1 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} \quad \mathbf{D'_4}: \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & -1 & +1 & 0 \\ 0 & +1 & -1 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix} \quad \mathbf{D_5}: \begin{pmatrix} 0 & 0 & +1 & 0 & 0 \\ 0 & +1 & -1 & +1 & 0 \\ +1 & -1 & +1 & -1 & +1 \\ 0 & +1 & -1 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \end{pmatrix}$$

It is easily observed that permutation matrices are examples of ASMs, which arise naturally as the unique smallest lattice containing the permutation matrices of the *Bruhat order* [3].

A *Latin square* [4] of order n is an $n \times n$ array containing n symbols such that each symbol occurs exactly once in each row and column. Any $n \times n$ Latin square L with symbols $1, 2, \dots, n$ can be decomposed into a unique sequence P of $n \times n$ mutually orthogonal permutation matrices $P = P_1, P_2, \dots, P_n$ by the following relation.

$$L = L(P) = \sum_k kP_k$$

For example, consider the following Latin square.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

This decomposition leads to a natural extension of the concept of a Latin square, first introduced by Brualdi and Dahl [1], by replacing the sequence of permutation matrices with planes of an *alternating sign hypermatrix (ASHM)*. Before we define these objects, we must first define some features of a hypermatrix.

An $n \times n \times n$ hypermatrix $A = [a_{ijk}]$ has n^2 lines of each of the 3 following types. Each line has n entries.

- *Row lines* $A_{*jk} = [a_{ijk} : i = 1, \dots, n]$, for given $1 \leq j, k \leq n$;
- *Column lines* $A_{i*k} = [a_{ijk} : j = 1, \dots, n]$, for given $1 \leq i, k \leq n$;
- *Vertical lines* $A_{ij*} = [a_{ijk} : k = 1, \dots, n]$, for given $1 \leq i, j \leq n$.

In this paper, we refer to a *plane* $P_k(A)$ of A to be the horizontal plane $A_{**k} = [a_{ijk} : i,j = 1,\dots,n]$, for given $1 \le k \le n$.

An *alternating sign hypermatrix (ASHM)* is a $(0, \pm 1)$-hypermatrix for which the non-zero entries in each row, column, and vertical line of the hypermatrix alternate in sign, starting and ending with $+1$.

For example, the following is a 3×3 ASHM.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \nearrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \nearrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The north-east arrow $P_k(A) \nearrow P_{k+1}(A)$ is used to denote that $P_k(A)$ is below P_{k+1}. For simplicity, for the remainder of this paper, we will omit the zero entries of all ASHMs, and represent all ± 1 entries with $+$ or $-$.

An *ASHL* is an $n \times n$ matrix L constructed from an $n \times n \times n$ ASHM A by

$$L = L(A) = \sum_k k P_k(A).$$

From the previous example, we then have the following ASHL.

$$L(A) = 1 \begin{pmatrix} & & + \\ & + & \\ + & & \end{pmatrix} + 2 \begin{pmatrix} & + & \\ + & - & + \\ & + & \end{pmatrix} + 3 \begin{pmatrix} + & & \\ & + & \\ & & + \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Brualdi and Dahl [1] posed a number problems about ASHLs, the following of which are addressed in this paper.

- Given an $n \times n$ ASHL L, let $A_n(L)$ be the set of all $n \times n \times n$ ASHMs A such that $L(A) = L$. Investigate $A_n(L)$.
- What is the maximum number of times an integer can occur as an entry of an $n \times n$ ASHL?

In this paper, the relationship between two ASHMs that generate the same ASHL is described, and a general construction is given for building an $n \times n$ ASHL containing $\lfloor \frac{n^2+4n-19}{2} \rfloor$ copies of one symbol. This improves on Brualdi and Dahl's lower bound of $2n$ for the maximum number of times that one symbol can appear in an $n \times n$ ASHL.

2. ASHLs with multiple ASHM-decompositions

In Brualdi and Dahl's paper [1], it was proven that for a Latin square L, if $L = L(A)$ for some ASHM A, then A must be a permutation hypermatrix. The following question was then posed.

Problem 2.1. *Given an $n \times n$ ASHL L, let $A_n(L)$ be the set of all $n \times n \times n$ ASHMs A such that $L(A) = L$. Investigate $A_n(L)$.*

This is presented in [1] as a completely open problem, with no examples of distinct ASHMs with the same corresponding ASHL given. This problem is motivated by the observation that in the case of a Latin square L, $A_n(L)$ contains exactly one ASHM whose planes form a set of mutually orthogonal permutation matrices. Before we discuss how two ASHMs in $A_n(L)$ relate to one another, it is useful to introduce the following definition, and more generally examine how any pair of $n \times n \times n$ ASHMs relate to one another.

An $n \times n \times n$ *T-block* $T_{i_1,j_1,k_1:\, i_2,j_2,k_2}$ is a hypermatrix $\pm[t_{ijk}]$ such that

$$t_{ijk} = \begin{cases} 1 & (i,j,k) = (i_1,j_1,k_1), (i_2,j_2,k_1), (i_2,j_1,k_2), \mathrm{or} (i_1,j_2,k_2) \\ -1 & (i,j,k) = (i_2,j_1,k_1), (i_1,j_2,k_1), (i_1,j_1,k_2), \mathrm{or} (i_2,j_2,k_2) \\ 0 & \text{otherwise} \end{cases}$$

where $i_1 < i_2, j_1 < j_2$, and $k_1 < k_2$.

A T-block can be most usefully visualised as an $n \times n \times n$ matrix containing the subhypermatrix

$$\pm\left[\begin{pmatrix} + & - \\ - & + \end{pmatrix} \nearrow \begin{pmatrix} - & + \\ + & - \end{pmatrix}\right]$$

such that these are the only non-zero entries. This is a 3-d extension of a concept defined by Brualdi, Kiernan, Meyer, and Schroeder [2].

The following lemma will be needed to investigate $A_n(L)$.

Lemma 2.2. *Let A and B be two $n \times n \times n$ ASHMs. Then $A - B$ can be expressed as a sum of T-blocks.*

Proof. If $A = B$, this is trivially true.

Assume $A \neq B$. In A and B, the sum of the entries in any row, column, or vertical line is 1. Therefore the sum of any row, column, or vertical line of $D = A - B$ is 0. Now iterate the following step.

- Let k_1 be the least integer for which the plane $P_{k_1}(D)$ contains non-zero entries. For some positive entry $d_{i_1 j_1 k_1}$ of D, we can find entries $d_{i_2 j_1 k_1}$, $d_{i_1 j_2 k_1}$ and $d_{i_1 j_1 k_2}$ with negative sign, where $k_2 > k_1$.
  Let $D \leftarrow D - T_{i_1,j_1,k_1:\, i_2,j_2,k_2}$ and repeat this step if D is not a 0-hypermatrix.

Note that the sum of the absolute values of the entries of $P_{k_1}(D)$ is at least 2 less than the previous step, and that all line sums of D are 0 after each step. We can therefore run this iterative process repeatedly, resulting in $P_{k_1}(D)$ becoming a 0-matrix for the current value of k_1, and eventually resulting in D becoming a 0-hypermatrix. Therefore $A - B$ can be expressed as a sum of T-blocks. $\square$

Let T be a T-block. The *depth* of T, $d(T)$, is defined as follows.

$$d(T) = \begin{cases} k_2 - k_1, & T = T_{i_1,j_1,k_1:\,i_2,j_2,k_2} \\ k_1 - k_2, & T = -T_{i_1,j_1,k_1:\,i_2,j_2,k_2} \end{cases}$$

Two T-blocks, T_1 and T_2, have *opposite depth* if $d(T_1) = -d(T_2)$

Theorem 2.3. *Two ASHMs A and B satisfy $L(A) = L(B)$ if and only if any expression of $A - B$ as a sum of T-blocks satisfies that, in any vertical line V of $A - B$,*

$$\sum_{T \in T_V} d(T) = 0,$$

where T_V is the subset of these T-blocks with non-zero entries in V.

Proof. From Lemma 2.2, we know that $A - B$ can be expressed as a sum of T-blocks. The entries of any vertical line V in $A-B$ can be decomposed into pairs (t_{k_1}, t_{k_2}), where $k_1 < k_2$, such that pairs are non-zero entries in the same T-block. Note that $t_{k_1} = -t_{k_2}$. Therefore

$$\sum kV_k = \sum_{T \in T_V} (k_1 t_{k_1} + k_2 t_{k_2}) = \sum_{T \in T_V} \pm(k_1 - k_2) = \sum_{T \in T_V} d(T).$$

- Suppose that, for any vertical line V in $A - B$,

  $$\sum_{T \in T_V} d(T) = 0.$$

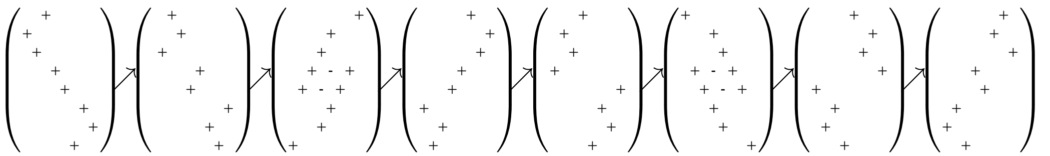
  This means that $\sum kV_k = 0$, which means that $L(A - B) = 0$. Therefore $L(A) = L(B)$.
- Now suppose that $L(A) = L(B)$. Then $L(A - B) = 0$, so for any vertical line V in $A - B$, we have $\sum kV_k = 0$. Therefore

  $$\sum_{T \in T_V} d(T) = 0. \quad \square$$

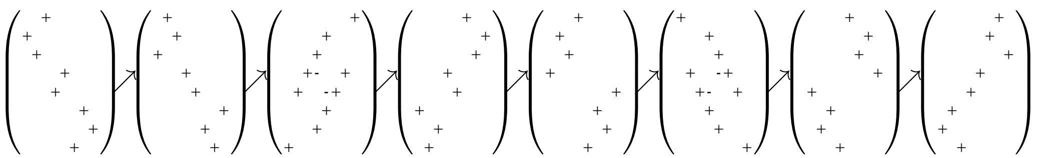
This provides some progress on Problem 2.1, as an ASHM B is contained in $A_n(L)$ if and only if A and B satisfy Theorem 2.3 for all $A \in A_n(L)$. In particular, Theorem 2.3 provides a strategy for constructing an ASHM B for which $L(B) = L(A)$ for some given ASHM A, by adding T-blocks to A in such a way that satisfies $\sum_{T \in T_V} d(T) = 0$. This is by far our most successful method for generating pairs of ASHMs with the same corresponding ASHL. The relationship between elements of $A_n(L)$ can be characterised further, as shown in the following two examples.

Example 2.4. Here, A and B are two ASHMs for which $D = A - B$ has a very natural decomposition as the sum of three T-blocks with depths $1, 1, -2$, respectively, occupying the same vertical lines.

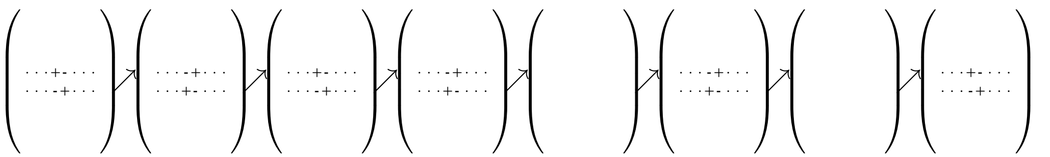
$A =$



$B =$



$D =$



$$D = T_{4,4,1:\,5,5,2} + T_{4,4,3:\,5,5,4} - T_{4,4,6:\,5,5,8}$$

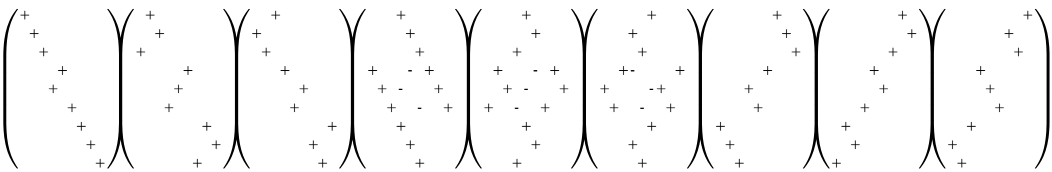
$$L(A) = L(B) = \begin{pmatrix} 6 & 2 & 1 & 5 & 7 & 4 & 8 & 3 \\ 1 & 5 & 2 & 6 & 3 & 8 & 7 & 4 \\ 2 & 1 & 5 & 3 & 6 & 7 & 4 & 8 \\ 5 & 6 & 3 & 3 & 3 & 6 & 3 & 7 \\ 7 & 3 & 6 & 3 & 3 & 3 & 6 & 5 \\ 4 & 8 & 7 & 6 & 3 & 5 & 1 & 2 \\ 8 & 7 & 4 & 3 & 6 & 2 & 5 & 1 \\ 3 & 4 & 8 & 7 & 5 & 1 & 2 & 6 \end{pmatrix}$$

D can also be expressed as the sum of pairs of T-blocks with opposite depth occupying the same vertical lines.

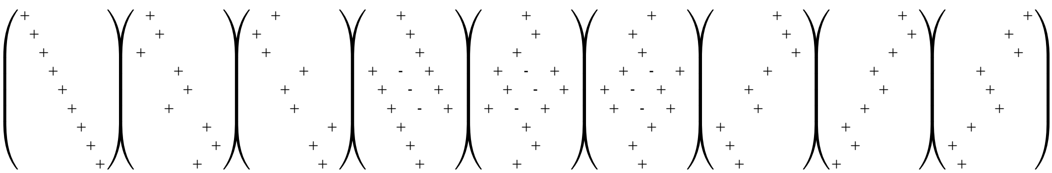
$$D = (T_{4,4,1:\,5,5,2} - T_{4,4,7:\,5,5,8}) + (T_{4,4,3:\,5,5,4} - T_{4,4,6:\,5,5,7})$$

Example 2.5. Here, A and B are two ASHMs for which $D = A - B$ has a very natural decomposition as the sum of three T-blocks such that each pair occupy exactly two of the same vertical lines. (Note that the north-east arrow between each plane of the hypermatrices has been omitted in this example due to space restrictions.)

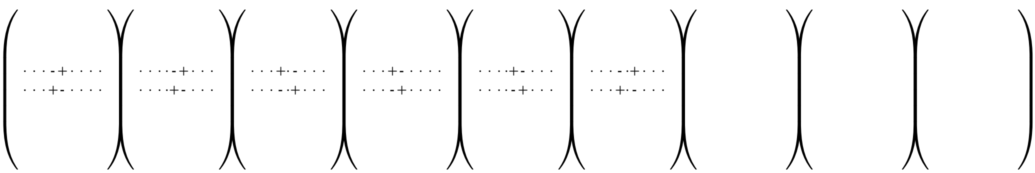
$A =$



$B =$



$D =$



$$D = -T_{4,4,1:\,5,5,4} - T_{4,5,2:\,5,6,5} + T_{4,4,3:\,5,6,6}$$

$$L(A) = L(B) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 \\ 2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 \\ 4 & 5 & 6 & 6 & 5 & 4 & 4 & 5 & 6 \\ 6 & 4 & 5 & 4 & 6 & 5 & 6 & 4 & 5 \\ 5 & 6 & 4 & 5 & 4 & 6 & 5 & 6 & 4 \\ 7 & 8 & 9 & 4 & 5 & 6 & 1 & 2 & 3 \\ 9 & 7 & 8 & 6 & 4 & 5 & 3 & 1 & 2 \\ 8 & 9 & 7 & 5 & 6 & 4 & 2 & 3 & 1 \end{pmatrix}$$

D can also be expressed as the sum of pairs of T-blocks occupying the same vertical lines with opposite depth.

$$D = (T_{4,4,3:\,5,6,6} - T_{4,4,1:\,5,6,4}) + (T_{4,5,1:\,5,6,4} - T_{4,5,2:\,5,6,5})$$

These examples demonstrate an alternative characterisation of two ASHMs with the same corresponding ASHL.

Theorem 2.6. *Two ASHMs A and B satisfy $L(A) = L(B)$ if and only if $A - B$ can be expressed as a sum of pairs of T-blocks with opposite depth occupying the same vertical lines.*

Proof. First assume that two ASHMs A and B satisfy $L(A) = L(B)$. From Theorem 2.3, we know that $D = A - B$ can be decomposed into T-blocks such that in any vertical line V of D,

$$\sum_{T \in T_V} d(T) = 0.$$

Run the following iterative step.

- Let k_1 be the least integer for which the plane $P_{k_1}(D)$ contains non-zero entries. For some positive entry $d_{i_1 j_1 k_1}$ of D, we can find entries $d_{i_2 j_1 k_1}$, $d_{i_1 j_2 k_1}$ and $d_{i_1 j_1 k_2}$ with negative sign, where $k_2 > k_1$. Choose k_2 to be the least integer satisfying this condition.
  As $\sum_{T \in T_{V_{i_1 j_1}}} d(T) = 0$, there must also be another positive entry $d_{i_1 j_1 k_3}$ in $V_{i_1 j_1}$. Choose k_3 to be the largest integer satisfying this condition. Let $k_4 = k_3 - (k_2 - k_1)$, and let

$$D' = D - T_{i_1,j_1,k_1:\,i_2,j_2,k_2} + T_{i_1,j_1,k_4:\,i_2,j_2,k_3}.$$

  Now repeat this step for D'.

Note that the sum of the absolute value of the entries of $P_{k_1}(D')$ is at least 2 less than that of $P_{k_1}(D)$, and that all line sums of D' are 0. Note also that $k_4 > k_1$, because the sum of the absolute value of the negative entries in V must equal the sum of the positive entries in V and the weighted sums of each must also equal. If $k_4 \leq k_1$, this implies that all the negative entries of V are positioned above all positive entries in V, which means that their weighted sum is negative. Therefore $k_4 > k_1$, which means that k_1 remains the lowest integer for which $P_{k_1}(D)$ contains non-zero entries. We can therefore run this iterative process repeatedly, resulting in $P_{k_1}(D')$ becoming a 0-matrix and on the next iteration, k_1 will increase to the new lowest integer for which plane $P_{k_1}(D')$ contains non-zero entries until D' is a 0-hypermatrix. Therefore $A - B$ can be expressed as a sum of pairs of T-blocks with opposite depth occupying the same vertical lines.

Now, assume that $A - B$ can be expressed as a sum of pairs of T-blocks with opposite depth occupying the same vertical lines. This means that, in any vertical line V of $A - B$,

$$\sum_{T \in T_V} d(T) = \sum_{T} (d(T) - d(T)) = 0.$$

Which, by Theorem 2.3, means that $L(A) = L(B)$. □

So, for any ASHM A with ASHL $L = L(A)$, we have that $A_n(L)$ is the set of all ASHMs B for which $A - B$ can be expressed as a sum of pairs of T-blocks with opposite depth occupying the same vertical lines. The following theorem tells us the smallest dimension an ASHL L can have if $A_n(L)$ contains more than one element.

Theorem 2.7. *The minimum n for which two distinct $n \times n \times n$ ASHMs A and B can satisfy $L(A) = L(B)$ is 4.*

Proof. Two ASHMs A and B can satisfy $L(A) = L(B)$ only if both ASHMs contain at least one negative entry [1].

The following are the only two 3×3 ASHMs containing negative entries, and these do not satisfy $L(A) = L(B)$.

$$A = \begin{pmatrix} & & + \\ & + & \\ + & & \end{pmatrix} \nearrow \begin{pmatrix} & + & \\ + & - & + \\ & + & \end{pmatrix} \nearrow \begin{pmatrix} + & & \\ & + & \\ & & + \end{pmatrix}$$

$$B = \begin{pmatrix} + & & \\ & + & \\ & & + \end{pmatrix} \nearrow \begin{pmatrix} & + & \\ + & - & + \\ & + & \end{pmatrix} \nearrow \begin{pmatrix} & & + \\ & + & \\ + & & \end{pmatrix}$$

$$L(A) = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad L(B) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

The following example is a pair of $4 \times 4 \times 4$ ASHMs with the same corresponding ASHL.

$$A = \begin{pmatrix} & & & + \\ & & + & \\ & + & & \\ + & & & \end{pmatrix} \nearrow \begin{pmatrix} & & + & \\ & + & - & + \\ + & - & + & \\ & + & & \end{pmatrix} \nearrow \begin{pmatrix} & + & & \\ + & & & \\ & & & + \\ & & + & \end{pmatrix} \nearrow \begin{pmatrix} + & & & \\ & & + & \\ & + & & \\ & & & + \end{pmatrix}$$

$$B = \begin{pmatrix} & & & + \\ & + & & \\ & & + & \\ + & & & \end{pmatrix} \nearrow \begin{pmatrix} & & + & \\ & & & + \\ + & & & \\ & + & & \end{pmatrix} \nearrow \begin{pmatrix} & + & & \\ + & - & + & \\ & + & - & + \\ & & + & \end{pmatrix} \nearrow \begin{pmatrix} + & & & \\ & + & & \\ & & + & \\ & & & + \end{pmatrix}$$

$$L(A) = L(B) = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Therefore the minimum dimension for which two ASHMs A and B can satisfy $L(A) = L(B)$ is 4. $\square$

Note that $A_n(L)$ can contain ASHMs with different numbers of non-zero elements. In the following example, the number of non-zero entries in A is 68, while the number of non-zero entries in B is 76.

Example 2.8. $A =$

$$\begin{pmatrix} &&&&&+&& \\ &&&&+&&& \\ &+&&&&&& \\ +&&&&&&& \\ &&&&&&&+ \\ &&&&&&+& \\ &&&+&&&& \\ &&+&&&&& \end{pmatrix} \nearrow \begin{pmatrix} &&&+&&&& \\ &&+&&&&& \\ &&&&&&+& \\ &&&&+&&& \\ &+&&&&&& \\ +&&&&&&& \\ &&&&&&&+ \\ &&&&&+&& \end{pmatrix} \nearrow \begin{pmatrix} &&&&&&+& \\ &&&&&&&+ \\ &&&&+&&& \\ &&&&&+&& \\ &&&+&&&& \\ &&+&&&&& \\ &+&&&&&& \\ +&&&&&&& \end{pmatrix} \nearrow \begin{pmatrix} &+&&&&&& \\ &&&+&&&& \\ +&&&&&&& \\ &&+&&&&& \\ &&&&+&&& \\ &&&&&+&& \\ &&&&&&+& \\ &&&&&&&+ \end{pmatrix} \nearrow \begin{pmatrix} &&&&+&&& \\ &&&&&+&& \\ &&&+&&&& \\ &+&&&-&&&+ \\ +&&&-&&&+& \\ &&&&+&&& \\ &&+&&&&& \\ &&&+&&&& \end{pmatrix} \nearrow \begin{pmatrix} +&&&&&&& \\ &+&&&&&& \\ &&+&&&&& \\ &&&+&&&& \\ &&&&&+&& \\ &&&&&&&+ \\ &&&&+&&& \\ &&&&&&+& \end{pmatrix} \nearrow \begin{pmatrix} &&&&&&&+ \\ &&&&&&+& \\ &&&&&+&& \\ &&&&+&&& \\ &&+&&&&& \\ &&&+&&&& \\ +&&&&&&& \\ &+&&&&&& \end{pmatrix} \nearrow \begin{pmatrix} &&+&&&&& \\ +&&&&&&& \\ &&&&&&&+ \\ &&&&&&+& \\ &&&+&&&& \\ &+&&&&&& \\ &&&&&+&& \\ &&&&+&&& \end{pmatrix}$$

$B =$

$$\begin{pmatrix} &&&&&+&& \\ &&&&+&&& \\ &+&&&&&& \\ +&&&&&&& \\ &&&&&&&+ \\ &&&&&&+& \\ &&&+&&&& \\ &&+&&&&& \end{pmatrix} \nearrow \begin{pmatrix} &&&+&&&& \\ &&+&&&&& \\ &&&&&&+& \\ &&&&+&&& \\ &+&&&&&& \\ +&&&&&&& \\ &&&&&&&+ \\ &&&&&+&& \end{pmatrix} \nearrow \begin{pmatrix} &&&&&&+& \\ &&&&&&&+ \\ &&&&+&&& \\ &&&+&-&+&& \\ &&&&+&&& \\ &&+&&&&& \\ &+&&&&&& \\ +&&&&&&& \end{pmatrix} \nearrow \begin{pmatrix} &+&&&&&& \\ &&&+&&&& \\ +&&&&&&& \\ &&+&-&+&&& \\ &&&+&&&& \\ &&&&&+&& \\ &&&&&&+& \\ &&&&&&&+ \end{pmatrix} \nearrow \begin{pmatrix} &&&&+&&& \\ &&&&&+&& \\ &&&+&&&& \\ &+&&&-&&&+ \\ +&&&-&&&+& \\ &&&&+&&& \\ &&+&&&&& \\ &&&+&&&& \end{pmatrix} \nearrow \begin{pmatrix} +&&&&&&& \\ &+&&&&&& \\ &&+&&&&& \\ &&&&+&&& \\ &&&+&-&+&& \\ &&&&&&&+ \\ &&&&+&&& \\ &&&&&&+& \end{pmatrix} \nearrow \begin{pmatrix} &&&&&&&+ \\ &&&&&&+& \\ &&&&&+&& \\ &&&+&&&& \\ &&+&-&+&&& \\ &&&+&&&& \\ +&&&&&&& \\ &+&&&&&& \end{pmatrix} \nearrow \begin{pmatrix} &&+&&&&& \\ +&&&&&&& \\ &&&&&&&+ \\ &&&&&&+& \\ &&&+&&&& \\ &+&&&&&& \\ &&&&&+&& \\ &&&&+&&& \end{pmatrix}$$

$$L(A) = L(B) = \begin{pmatrix} 6 & 4 & 8 & 2 & 5 & 1 & 3 & 7 \\ 8 & 6 & 2 & 4 & 1 & 5 & 7 & 3 \\ 4 & 1 & 6 & 5 & 3 & 7 & 2 & 8 \\ 1 & 5 & 4 & 6 & 4 & 3 & 8 & 5 \\ 5 & 2 & 7 & 6 & 4 & 6 & 5 & 1 \\ 2 & 8 & 3 & 7 & 5 & 4 & 1 & 6 \\ 7 & 3 & 5 & 1 & 6 & 8 & 4 & 2 \\ 3 & 7 & 1 & 5 & 8 & 2 & 6 & 4 \end{pmatrix}$$

3. The maximum number of equal entries of an ASHL

The following question is also posed in Brualdi and Dahl's paper [1].

Problem 3.1. *What is the maximum number of times an integer can occur as an entry of an $n \times n$ ASHL?*

It is shown in their paper that an integer can occur $2n$ times in an $n \times n$ ASHL, and it is asked if the maximum is equal to $2n$. The following example exceeds this bound.

Example 3.2. Here, 4 occurs as an entry in this 7×7 ASHL 29 times.
$A =$

$$\begin{pmatrix} &&+&&&& \\ &+&&&&& \\ +&&&&&& \\ &&&+&&& \\ &&&&&&+ \\ &&&&&+& \\ &&&&+&& \end{pmatrix} \nearrow \begin{pmatrix} &&&&&&+ \\ &&&&&+& \\ &&&+&&& \\ &&+&-&+&& \\ &&&+&&& \\ &+&&&&& \\ +&&&&&& \end{pmatrix} \nearrow \begin{pmatrix} &+&&&&& \\ +&-&&+&&& \\ &&+&-&+&& \\ &+&-&+&-&+& \\ &&+&-&+&& \\ &&&+&&-&+ \\ &&&&&+& \end{pmatrix} \nearrow \begin{pmatrix} &&&+&&& \\ &&+&-&+&& \\ &+&-&+&-&+& \\ +&-&+&-&+&-&+ \\ &+&-&+&-&+& \\ &&+&-&+&& \\ &&&+&&& \end{pmatrix} \nearrow \begin{pmatrix} &&&&&+& \\ &&&+&&-&+ \\ &&+&-&+&& \\ &+&-&+&-&+& \\ &&+&-&+&& \\ +&-&&+&&& \\ &+&&&&& \end{pmatrix} \nearrow \begin{pmatrix} +&&&&&& \\ &+&&&&& \\ &&&+&&& \\ &&+&-&+&& \\ &&&+&&& \\ &&&&&+& \\ &&&&&&+ \end{pmatrix} \nearrow \begin{pmatrix} &&&&+&& \\ &&&&&+& \\ &&&&&&+ \\ &&&+&&& \\ +&&&&&& \\ &+&&&&& \\ &&+&&&& \end{pmatrix}$$

$$L(A) = \begin{pmatrix} 6 & 3 & 1 & 4 & 7 & 5 & 2 \\ 3 & 4 & 4 & 4 & 4 & 4 & 5 \\ 1 & 4 & 4 & 4 & 4 & 4 & 7 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 7 & 4 & 4 & 4 & 4 & 4 & 1 \\ 5 & 4 & 4 & 4 & 4 & 4 & 3 \\ 2 & 5 & 7 & 4 & 1 & 3 & 6 \end{pmatrix}$$

This is the highest possible number of times an entry can be repeated in a 7×7 ASHL, as each number $1, 2, \ldots, n$ must appear exactly once in the first and last rows and columns of an $n \times n$ ASHL. This means that the upper bound for such a construction is $(n-2)^2 + 4$, which is $(7-2)^2 + 4 = 29$, in the $n = 7$ case.

We define the *diamond positions* of an $n \times n$ ASM A to be the positions of A corresponding to non-zero entries of the diamond ASM D_n. Example 3.2 can be generalised in the following way.

Theorem 3.3. *For a given n, there exists an $n \times n$ ASHL such that*

- *$\frac{n+1}{2}$ occurs as an entry $\frac{n^2+4n-19}{2}$ times, if n is odd;*
- *$\frac{n}{2}$ occurs as an entry $\frac{n^2+4n-20}{2}$ times, if n is even.*

Proof. Let $p = \lfloor \frac{n+1}{2} \rfloor$, $m = \lceil \frac{n+1}{2} \rceil$, and note that $p = m$ for odd n. We construct an ASHM A with the required properties as follows.

- $P_p(A) = D_n$, and for $k = 1, 2, \ldots, p-1$, plane $P_{p\pm k}(A)$ contains the diamond ASM D_{n-2k} such that there is a $+$ entry in every position where there is a $-$ entry in the diamond ASM contained in the plane $P_{p\pm(k-1)}$.
- The other non-zero entries of $P_{p-1}(A)$ are a diagonal of $+$ entries from $A_{1,m+2,p-1}$ to $A_{p-2,n,p-1}$, a diagonal of $-$ entries from $A_{2,m+2,p-1}$ to $A_{p-2,n-1,p-1}$, a diagonal of $+$ entries from $A_{m+2,1,p-1}$ to $A_{n,p-2,p-1}$, and a diagonal of $-$ entries from $A_{m+2,2,p-1}$ to $A_{n-1,p-2,p-1}$.
- The other non-zero entries of $P_1(A)$ are a diagonal of $+$ entries from $A_{1,m+1,1}$ to $A_{p-1,n,1}$ and a diagonal of $+$ entries from $A_{m+1,1,1}$ to $A_{n,p-1,1}$.
- The other non-zero entries of $P_2(A)$ are an anti-diagonal of $+$ entries from $A_{2,p-2,2}$ to $A_{p-2,2,2}$, an anti-diagonal of $+$ entries from $A_{m+2,n-1,2}$ to $A_{n-1,m+2,2}$, and $+$ entries in $A_{1,1,2}$ and $A_{n,n,2}$.
- For $k = 2, \ldots, p-3$, the other non-zero entries of $P_{p-k}(A)$ are an anti-diagonal of $+$ entries from $A_{1,k,p-k}$ to $A_{k,1,p-k}$, and an anti-diagonal of $+$ entries from $A_{n-k+1,n,p-k}$ to $A_{n,n-k+1,p-k}$.
- For $k = 1, 2, \ldots, p-1$, the entries of $P_{p+k}(A)$ not containing D_{n-2k} (as outlined in the first step) satisfy $A_{i,j,p+k} = A_{n-i,j,p-k}$.
- If n is even, the non-zero entries of $P_n(A)$ are an anti-diagonal from $A_{p,1,n}$ to $A_{1,p,n}$ and an anti-diagonal from $A_{n,m,n}$ to $A_{m,n,n}$.

We see that p occurs in all diamond positions of $L = L(A)$ because

$$(p-k+1) - (p-k+2) + \cdots - (p+k-2) + (p+k-1) = p.$$

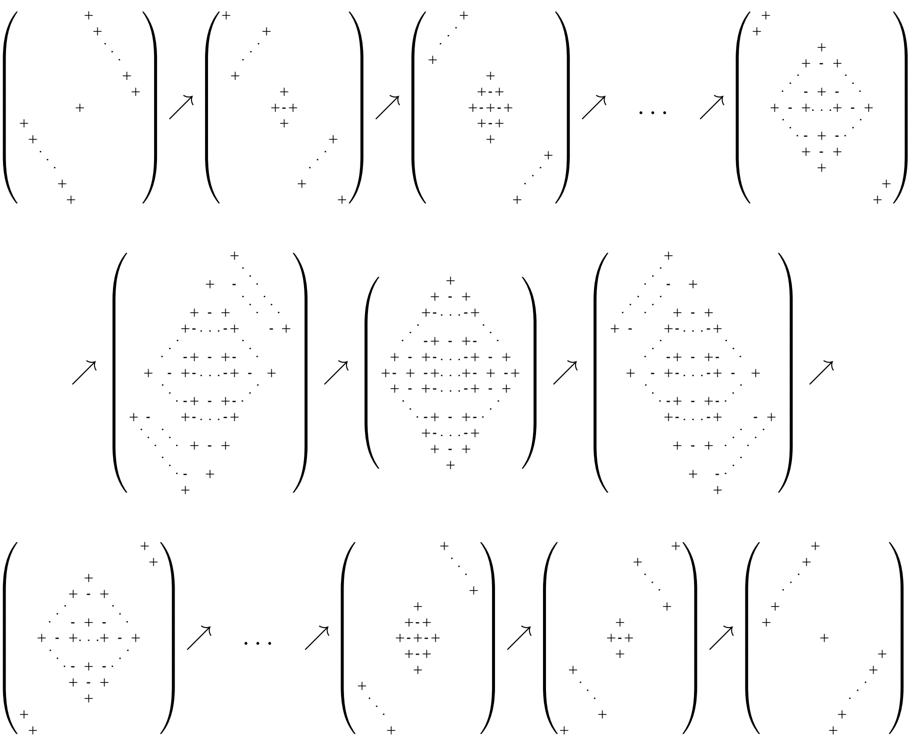
The other occurrences of p as entries of L occur along diagonals from $L_{2,m+2}$ to $L_{p-2,n-1}$ and from $L_{m+2,2}$ to $L_{n-1,p-2}$ by

$$1-(p-1)+(n+p-m-1)=n-m+1=p,$$

and along antidiagonals from $L_{p-2,2}$ to $L_{2,p-2}$ and from $L_{n-1,m+2}$ to $L_{m+2,n-1}$ by

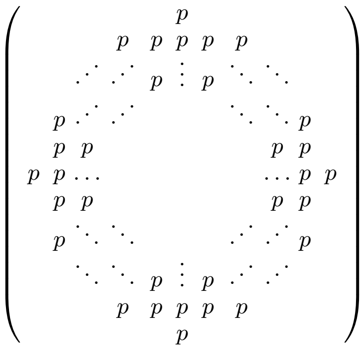
$$2-(p+1)+(n+p-m)=n-m+1=p.$$

<u>The odd case</u>:



It can be easily seen that each plane of this hypermatrix is an ASM. All vertical lines of A corresponding to diamond positions of L clearly have the alternating property. All vertical lines corresponding to the diagonal from $(2,p+2)$ to $(p-2,n-1)$, the diagonal from $(p+2,2)$ to $(n-1,p-2)$, the anti-diagonal from $(2,p-2)$ to $(p-2,2)$, and the anti-diagonal from $(p+2,n-1)$ to $(n-1,p+2)$ have exactly three non-zero entries, which alternate $+,-,+$. All other vertical lines contain exactly one non-zero entry, namely one $+$ entry. Therefore this is an ASHM.

The p entries occur in the following positions of the corresponding ASHL.

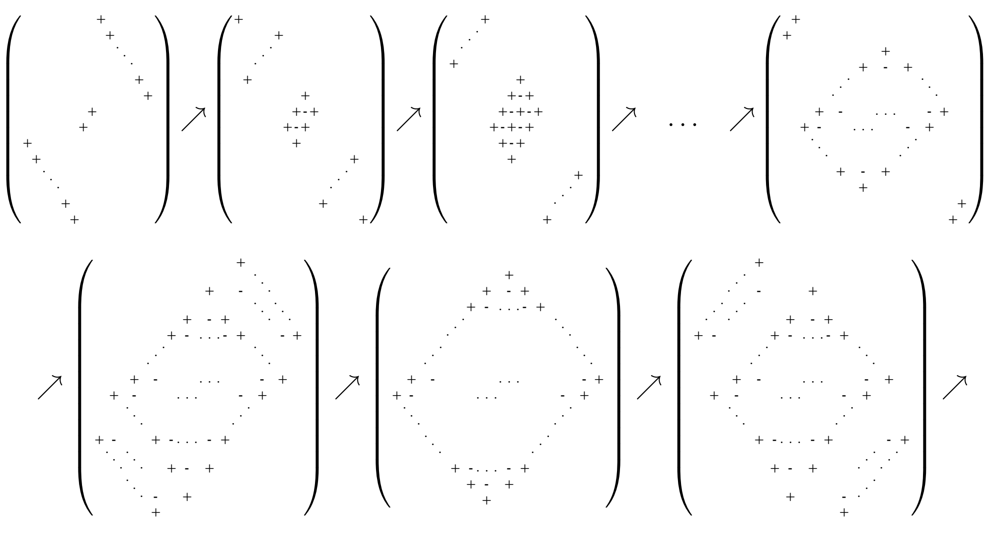


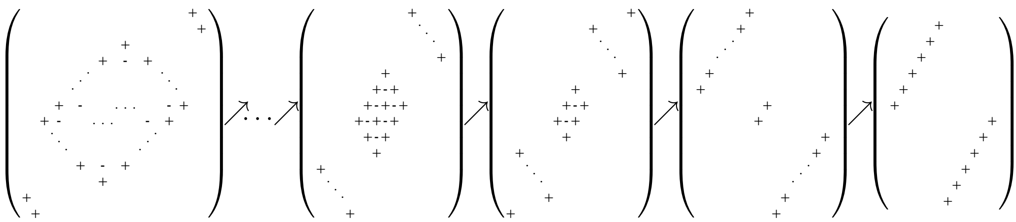
As outlined above, p occurs as an entry in the diamond positions of L, and also occurs as every entry in the diagonal from $L_{2,p+2}$ to $L_{p-2,n-1}$, the diagonal from $L_{p+2,2}$ to $L_{n-1,p-2}$, the anti-diagonal from $L_{2,p-2}$ to $L_{p-2,2}$, and the anti-diagonal from $L_{p+2,n-1}$ to $L_{n-1,p+2}$.

Therefore p occurs as an entry of L a total of $\frac{n^2+4n-19}{2}$ times:

$$(1+3+\cdots+n-2+n+n-2+\cdots+3+1)+4\Big(\frac{n+1}{2}-3\Big)$$
$$=\Big(\frac{n+1}{2}\Big)^2+\Big(\frac{n-1}{2}\Big)^2+(2n-10)=\frac{n^2+4n-19}{2}$$

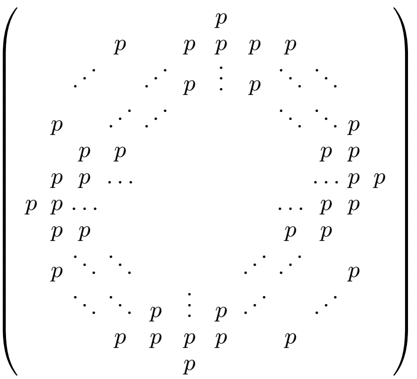
<u>The even case:</u>





It can be easily seen that each plane of this hypermatrix is an ASM. All vertical lines of A corresponding to diamond positions of L clearly have the alternating property. All vertical lines corresponding to the diagonal from $(2, m+2)$ to $(p-2, n-1)$, the diagonal from $(m+2, 2)$ to $(n-1, p-2)$, the anti-diagonal from $(2, p-2)$ to $(p-2, 2)$, and the anti-diagonal from $(m+2, n-1)$ to $(n-1, m+2)$ have exactly three non-zero entries, which alternate $+, -, +$. All other vertical lines contain exactly one non-zero entry, namely one $+$ entry. Therefore this is an ASHM.

The p entries occur in the following positions of the corresponding ASHL.



As outlined above, p occurs as an entry in the diamond positions of L, and also occurs as every entry in the diagonal from $L_{2,m+2}$ to $L_{p-2,n-1}$, the diagonal from $L_{m+2,2}$ to $L_{n-1,p-2}$, the anti-diagonal from $L_{2,p-2}$ to $L_{p-2,2}$, and the anti-diagonal from $L_{m+2,n-1}$ to $L_{n-1,m+2}$.

Therefore p occurs as an entry of L a total of $\frac{n^2+4n-20}{2}$ times:

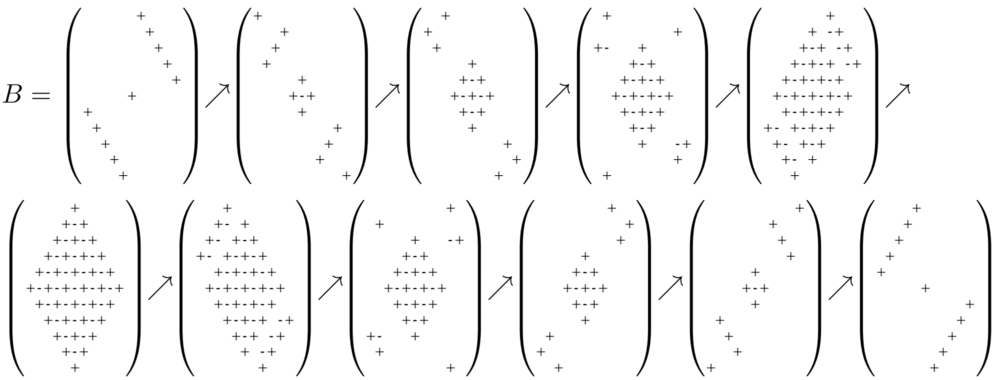
$$2(1+3+\cdots+n-1)+4\Big(\frac{n+1}{2}-3\Big)=2\Big(\frac{n}{2}\Big)^2+(2n-10)=\frac{n^2+4n-20}{2} \qquad \square$$

Note that this bound is not tight. This is currently our best general construction, but we have constructed specific examples narrowly exceeding this bound. This was achieved by adding a small number of T-blocks to an ASHM generated by the above construction.

Example 3.4. In the $n = 11$ case, the construction outlined in the proof of Theorem 3.3 gives an $n \times n \times n$ ASHM A with ASHL $L(A)$ containing the same entry 73 times. The ASHM B, with $L(B)$ containing the same entry 77 times, is obtained by the addition of T-blocks to A as follows.

$$B = A + T_{2,1,3:\,3,2,4} + T_{9,10,3:\,10,11,4} - T_{2,2,4:\,11,10,8} + T_{9,1,8:\,10,2,9} + T_{2,10,8:\,3,11,9}$$

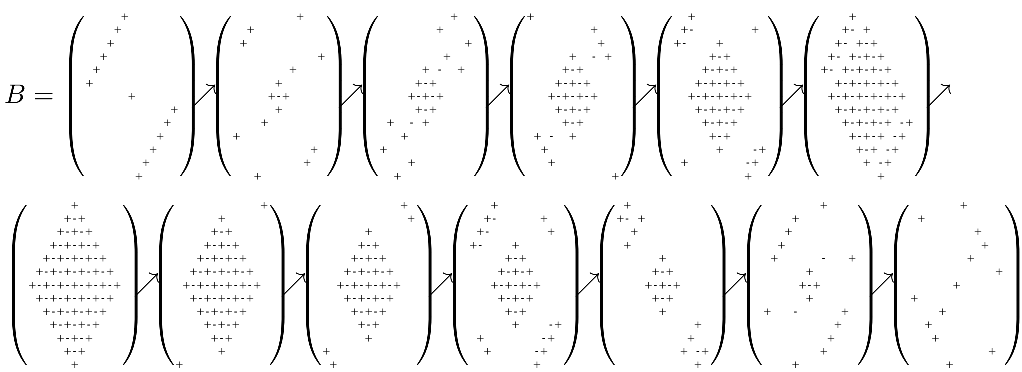
Explicitly, B is the following ASHM.



Which has the following corresponding ASHL.

$$L(B) = \begin{pmatrix} 2 & 4 & 3 & 7 & 11 & 6 & 1 & 5 & 9 & 8 & 10 \\ 3 & 8 & 7 & 6 & 6 & 6 & 6 & 6 & 5 & 4 & 9 \\ 4 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 8 \\ 7 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 5 \\ 11 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 1 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 1 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 11 \\ 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 7 \\ 8 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 4 \\ 9 & 8 & 5 & 6 & 6 & 6 & 6 & 6 & 7 & 4 & 3 \\ 10 & 4 & 9 & 5 & 1 & 6 & 11 & 7 & 3 & 8 & 2 \end{pmatrix}$$

Example 3.5. In the $n = 13$ case, the construction outlined in the proof of Theorem 3.3 gives an $n \times n \times n$ ASHM A with ASHL $L(A)$ containing the same entry 101 times. The following ASHM B exceeds this, with $L(B)$ containing the same entry 103 times.



$$L(B)=\begin{pmatrix}
4 & 11 & 5 & 10 & 6 & 1 & 7 & 13 & 12 & 2 & 3 & 9 & 8\\
11 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 3 & 4 & 10 & 5 & 9\\
5 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 13 & 4 & 10 & 3\\
10 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 6 & 13 & 4 & 2\\
6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 3 & 12\\
1 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 13\\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7\\
13 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 1\\
12 & 3 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 6\\
2 & 4 & 13 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 10\\
3 & 10 & 4 & 13 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 5\\
9 & 5 & 10 & 4 & 3 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 11\\
8 & 9 & 3 & 2 & 12 & 13 & 7 & 1 & 6 & 10 & 5 & 11 & 4
\end{pmatrix}$$

We conclude by posing the following problem.

Problem 3.6. *Is it possible to construct an $n \times n \times n$ ASHM A for which $(n-2)^2+4$ entries of $L(A)$ are equal, for $n > 7$?*

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