

# Multiplicative Consistency Ascertaining, Inconsistency Repairing and Weights Derivation of Hesitant Multiplicative Preference Relations

Yejun Xu, Mengqi Li, Francisco Chiclana, and Enrique Herrera-Viedma

**Abstract**—This paper investigates **multiplicative consistency** ascertaining, inconsistency repairing and weights derivation for hesitant multiplicative preference relations (HMPRs). Firstly, the completely **multiplicative consistency** and weakly **multiplicative consistency** of HMPRs are defined. Based on them, 0-1 mixed programming models and simple algebraic operations are proposed to ascertain the **multiplicative consistency** of HMPRs. Then, some goal programming models are developed to generate the weights from consistent HMPRs and to revise inconsistent HMPRs. An integrated procedure to manage the multiplicative consistencies of HMPRs is designed. The proposed methods are also extended to accommodate incomplete HMPRs, and to estimate missing values. **Finally, some numerical examples, a comparative analysis with existent approaches, and a simulation analysis are included to illustrate the practicality and effectiveness of the developed models.**

**Index Terms**—Consistency ascertaining, HMPRs, inconsistency repairing, missing values, weights derivation.

## I. INTRODUCTION

IN decision making, the following **relations** are widely used to represent the preference information of **decision-makers**: multiplicative preference relation (MPR) [1], fuzzy preference relation [2], interval preference relation [3-5], intuitionistic preference relation [6-9], and linguistic preference relation [10-13]. However, these **preference relations** do not allow to handle situations where **decision-makers** ascertain the membership of elements with a set of values derived from their hesitancy among several different values. To handle these cases, Torra [14] introduced the concept of **hesitant fuzzy set** with elements in the unit interval  $[0, 1]$ . **Based on the concept of hesitant fuzzy set, Xia and Xu [15] used Saaty's Analytic Hierarchy Process (AHP) 1/9-9 scale [1] to further define the concept of hesitant multiplicative preference relation (HMPR), which can vividly simulate both the decision-makers' uncertainty and hesitation by allowing preferences to be expressed with hesitant**

multiplicative elements (HMEs) **using the AHP scale.**

In recent years, HMPR research has become a hot topic [16]. In particular, it is worth mentioning the HMPR research on priority weights derivation [17-20], consistency analysis [21-25] and group consensus [21, 26-29].

Consistency is one of the key and challenging issues that need to be resolved in decision-making processes. Inconsistent preferences can lead to bad decisions. Thus, methods have been developed to deal with the consistencies of the various **preference relations** [10, 11, 30-39]. Consistency of HMPRs, which can help **decision-makers** to derive reasonable weighting values and decision results, has also received great attention recently with regard to the following two aspects: 1) consistency ascertaining: how to measure the consistency level of an HMPR, and 2) inconsistency repairing: how to derive an HMPR with acceptable consistency from an inconsistent HMPR.

So far, research scholars have made some suggestions regarding consistency and the priority derivation of HMPRs. Indeed, Zhang and Wu [21] defined the multiplicatively consistent HMPR and developed a decision support model for group decision making as per the group consensus level. However, their  $\alpha$ -normalization and  $\beta$ -normalization processes reduce or add some additional values to an HME, respectively, which destructs and distorts the **decision-maker's** original judgments. Furthermore, Zhang and Wu [17] introduced the definitions of consistent HMPR and acceptably consistent HMPR, and derived the interval weights from HMPRs based on the  $\beta$ -normalization process but no inconsistency rectification process was proposed. Meng et al. also defined the consistency of HMPRs in [25], which is based on the assumption of any element in the HMEs forming a consistent MPR. In real applications, it is difficult to provide fully consistent MPRs, which is even more difficult in the case of HMPRs. If the uncertain information provided by a **decision-maker** is

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consistent, then this indicates that such **decision-maker** possesses a strong logic and is sure of his/her information [40]. In other words, he/she knows his/her preferences perfectly well, and therefore he/she is not hesitant about his/her judgements in the HMPR. Mou et al. [23] defined the **multiplicative consistency level** of HMPRs and developed a method to repair inconsistency, which included a normalized process, and an artificial threshold of acceptable consistency level. Similarly, Nie's approach is based on a randomly given threshold of the consistency index, which lacks theoretical basis. Zhang and Guo [41] gave some formulae for calculating the weights of incomplete HMPRs, but they only considered the acceptable consistent incomplete HMPRs and ignored the inconsistent cases that occur in practical problems. Lin et al. [42] constructed a linear programming model to obtain priorities from HMPRs. Additionally, HMPRs have been widely utilized to handle various practical issues, such as the allocation of water conservancy investment of river basins [18, 25], logistics service provider selection [42] and city sustainable development evaluation [43].

The above analysis highlights some research achievements with regard to the consistency of HMPRs. However, there are still some issues that remain to be solved. The research motivations of this paper can be summarized as follows:

(1) Existing **multiplicative consistency** research approaches are often hindered with drawbacks related to the changing of the **decision-makers'** original judgments and the optional setting of consistency thresholds. Therefore, it is necessary to answer the following question: what is the **multiplicative consistency** of HMPRs and how can it be verified?

(2) When **decision-makers** hesitate to express their opinions in decision-making problems, a precise priority vector cannot represent the **decision-makers'** hesitation judgments accurately and naturally [17]. Consequently, the following question needs answering: how to generate suitable and realistic weights from an HMPR with **multiplicative consistency**?

(3) **Decision-makers** with allodoxophobia may hesitate to deal with decision-making problems. Thus, the development of models to help **decision-makers** eliminate their illogical, inconsistent or unreasonable information could be really useful to **decision-makers** in general, and to allodoxophobia **decision-makers** in particular. Hence, a question to address is: if an HMPR is inconsistent, how can inconsistency be repaired?

(4) There are few papers in the literature reporting on **multiplicative consistency** measurement, inconsistency level improvement, and weights derivation for incomplete HMPRs, which is addressed in this article.

To answer the above questions, two new **multiplicative consistencies** of HMPRs, **completely multiplicative consistency** and **weakly multiplicative consistency**, respectively, are introduced. Moreover, 0-1 mixed programming models and some algebraic approaches are developed to determine the consistency type for HMPRs. Goal programming methods are proposed to (i) derive priority weights from an HMPR and (ii) find the inconsistent elements in an HMPR. This new approach allows **decision-makers** to assign suitable weights to different stages to reflect their preferences in HMPR problems.

Subsequently, an efficient and flexible integrated algorithm is designed to test consistency, obtain logical weights and repair inconsistency of HMPRs, while a novel method to judge the consistency type, estimate missing values and derive priority vectors from incomplete HMPRs is developed.

The rest of this paper is arranged as follows. Section II introduces the required basic concepts of MPR, **hesitant multiplicative sets (HMS)** and HMPR. Two new definitions of consistency of HMPRs are introduced in Section III. Section IV develops methods to ascertain the consistency of HMPRs. In Section V, a priority weight derivation model and an inconsistency repairing method based on **multiplicative consistency** are proposed. These are used to obtain consistent HMPRs and the reasonable alternatives ranking results. Section VI is devoted to incomplete HMPRs, and two **multiplicative consistency**-based goal programming models are proposed to assess their unknown values and to ascertain their consistency. Section VII provides three examples, a discussion and a simulation analysis to show the effectiveness of the developed approaches. Finally, some conclusions are offered in Section VIII.

## II. PRELIMINARIES

In order to make the paper self-contained, some concepts associated with MPRs, HMSs and HMPRs, which are used throughout the whole paper, are reviewed.

For simplicity, let  $X = \{x_1, \dots, x_n\}$  be a finite set of alternatives, and  $N = \{1, \dots, n\}$ .

*Definition 1* [1]. An MPR  $R = (r_{ij})_{n \times n} \subset X \times X$  is reciprocal if

$$r_{ij} \cdot r_{ji} = 1, r_{ii} = 1, r_{ij} \in [1/9, 9], \forall i, j \in N. \quad (1)$$

*Definition 2* [1]. An MPR  $R = (r_{ij})_{n \times n}$  is perfect consistent if

$$r_{ij} = r_{ik} \cdot r_{kj}, \forall i, j, k \in N. \quad (2)$$

Let  $w = (w_1, \dots, w_n)^T$  be the weight vector of the set of alternatives  $X$ , such that  $w_i > 0$ , and  $\sum_{i=1}^n w_i = 1$ . If MPR  $R$  on  $X$  is perfect consistent, then

$$r_{ij} = \frac{w_i}{w_j}, \forall i, j \in N. \quad (3)$$

An MPR is incomplete when some of its elements are missing.

*Definition 3* [44]. An MPR  $R = (r_{ij})_{n \times n}$  is incomplete when some of its elements cannot be given by the **decision-maker**, while the rest of provided preference values,  $\Omega$ , satisfy the conditions:

$$r_{ij} \cdot r_{ji} = 1, r_{ii} = 1, r_{ij} > 0, \text{ for all } r_{ij} \in \Omega. \quad (4)$$

*Definition 4* [45]. An incomplete MPR  $R = (r_{ij})_{n \times n}$  is consistent if

$$r_{ij} = r_{ik} \cdot r_{kj}, \text{ for all } r_{ij}, r_{ik}, r_{kj} \in \Omega. \quad (5)$$

Motivated by the concepts of hesitant fuzzy set and MPR, Xia and Xu [15] defined the concept of HMS:

*Definition 5* [15]. An HMS  $M$  on  $X$  is mathematically

expressed as:

$$M = \{ \langle x, b_M(x) \rangle \mid x \in X \}, \quad (6)$$

where  $b_M(x)$  is a subset of finite cardinality of set  $[1/9, 9]$ , which denotes all the possible membership degrees of the element  $x \in X$  to the set  $M$ .

For convenience,  $b=b_M(x)$  is often called an HME. Motivated by Torra [14], Zhang and Wu [17] defined the upper and lower bounds of an HME.

**Definition 6 [17].** The upper and lower bounds of an HME are  $h_{ij}^+ = \max \{ h_{ij}^t \mid t = 1, \dots, l_{h_{ij}} \}$  and  $h_{ij}^- = \min \{ h_{ij}^t \mid t = 1, \dots, l_{h_{ij}} \}$ , respectively.

Combining HMSs and MPRs, the concept of **HMPR** is defined:

**Definition 7 [15].** An HMPR  $H=(h_{ij})_{n \times n} \subset X \times X$ , is a preference relation with HMEs,  $h_{ij} = \{ h_{ij}^t \mid t = 1, 2, \dots, l_{h_{ij}} \}$ , indicating all the possible degrees to which alternative  $x_i$  is preferred to alternative  $x_j$  subject to the constraints:

$$h_{ij}^{\sigma(i)} h_{ji}^{\sigma(l_{h_{ij}} - t + 1)} = 1, \quad h_{ii} = \{1\}, \quad l_{h_{ij}} = l_{h_{ji}}, \quad i, j \in N, \quad (7)$$

where  $h_{ij}^{\sigma(i)}$  denotes the  $t$ th smallest element in  $h_{ij}$ .

Similar to the definition of hesitant fuzzy preference relations discussed by Xu et al. [46], the values in each HME are ordered from smallest to largest as per Definition 7, which may result in property (7) not to be verified. At the same time, because of the disorder of sets, there is no need to arrange  $h_{ij}$  in ascending or descending order. Thus, a revised definition of HMPRs is introduced here.

**Definition 8.** An HMPR  $H=(h_{ij})_{n \times n} \subset X \times X$ , is a preference relation with HMEs,  $h_{ij} = \{ h_{ij}^t \mid t = 1, 2, \dots, l_{h_{ij}} \}$ , indicating the possible degrees to which alternative  $x_i$  is preferred to alternative  $x_j$ , subject to the following constraints:

$$h_{ij}^t h_{ji}^{l_{h_{ij}} - t + 1} = 1, \quad h_{ii} = \{1\}, \quad l_{h_{ij}} = l_{h_{ji}}, \quad i, j \in N \quad (8)$$

If some elements of an HMPR cannot be given by a **decision-maker**, then an incomplete HMPR results. Zhang and Guo [41] introduced the concept of acceptable incomplete HMPRs.

**Definition 9 [41].** An HMPR  $H=(h_{ij})_{n \times n} \subset X \times X$  is incomplete when some of its HMEs are unknown while its known HMEs  $h_{ij} = \{ h_{ij}^t \mid t = 1, 2, \dots, l_{h_{ij}} \}$  satisfy the constraints

$$h_{ij}^t h_{ji}^{l_{h_{ij}} - t + 1} = 1, \quad h_{ii} = \{1\}, \quad l_{h_{ij}} = l_{h_{ji}}, \quad i, j \in N \quad (9)$$

TABLE I. NOMENCLATURE

Abbreviations	Illustration
AHP	Analytic Hierarchy Process
MPRs	Multiplicative preference relations
HMPRs	Hesitant multiplicative preference relations
HMSs	Hesitant multiplicative sets
HMEs	Hesitant multiplicative elements
CMC	Completely multiplicative consistent
WMC	Weakly multiplicative consistent
LCR	Length change ratio
NAR	Numerical adjustment ratio
AD	Absolute deviation
LAD	Logarithm absolute deviation
DR	Difference ratio

To improve readability, Table I lists the abbreviations used in this paper.

### III. MULTIPLICATIVE CONSISTENCIES OF HMPRS

This section introduces two **multiplicative consistency concepts** for HMPRs: **completely multiplicative consistency** and **weakly multiplicative consistency**.

**Definition 10.** Let  $H=(h_{ij})_{n \times n}$  be an HMPR. If there is a complete consistent MPR  $R=(r_{ij})_{n \times n}$ , such that

$$r_{ij} = r_{ik} \cdot r_{kj}, \quad r_{ij} \in h_{ij}, \quad \forall i, j, k \in N, \quad (10)$$

then  $H$  is called a completely multiplicative consistent (CMC) HMPR and  $R$  is a complete consistent MPR in  $H$ .

The CMC HMPR concept extracts existing elements from the HMPR to form an MPR that satisfies the multiplicative transitivity property (2). As the information provided by a **decision-maker** is uncertain, our goal is “to find the reasonable information in an HMPR”. Definition 10 does not rely on Zhang’s  $\beta$ -normalization [26]. Therefore, no elements are added to HMEs. In any case, **completely multiplicative consistency** is difficult to be verified by a **an** HMPR. Let us consider the following example: when evaluating a set of three alternatives  $X=\{x_1, x_2, x_3\}$ , a **decision-maker** expresses that alternative  $x_1$  is weakly less important than alternative  $x_2$ , and gives the preference value  $h_{12}=1/2$ ; while  $x_2$  is strongly more important than alternative  $x_3$ , and gives the preference value  $h_{23}=5$ . In the AHP context, if **his/her information** is consistent, then it should be  $h_{13}=h_{12} \times h_{23}=5/2$ . However, the value  $5/2$  is not one of the original scale values in the AHP scale set  $\{1/9, \dots, 1/2, 1, 2, \dots, 9\}$ . In addition to the above, if the **decision-maker** is unsure about the preference of alternative  $x_1$  over alternative  $x_3$  but considers  $x_1$  more important than  $x_3$ , and gives the following HME  $\{2, 3\}$ , then it is obviously that **his/her** preferences are not **CMC** ( $5/2$  is between 2 and 3). In this case, we could regard the **decision-maker**’s information to be close to complete consistent. In our view, because the upper and lower bounds of HMEs produce a range containing all possible **decision-maker**’s preference information, the extraction of a consistent MPR from the upper and lower bounds of HMEs is a viable approach. In order to accommodate this scenario, another consistency property of HMPRs is introduced here.

**Definition 11.** Let  $H=(h_{ij})_{n \times n}$  be an HMPR. If there is a complete consistent MPR  $R=(r_{ij})_{n \times n}$  satisfying

$$r_{ij} = r_{ik} \cdot r_{kj}, \quad h_{ij}^- \leq r_{ij} \leq h_{ij}^+, \quad \forall i, j, k \in N, \quad (11)$$

then  $H$  is called a weakly multiplicative consistent (WMC) HMPR and  $R$  is a complete consistent MPR in  $H$ .

Considering the aforementioned relationship between  $r_{ij}$  and  $w$  as per (3), an equivalent definition of WMC HMPR is given below:

**Definition 12.** An HMPR  $H=(h_{ij})_{n \times n}$  is called a WMC HMPR, if there exists a weight vector  $w=(w_1, \dots, w_n)^T$ , such that

$$h_{ij}^- \leq \frac{w_i}{w_j} \leq h_{ij}^+, \quad \forall i, j \in N. \quad (12)$$

Reciprocity property of HMPRs allows the above definition

to be rewritten equivalently as follows:

**Definition 13.** An HMPR  $H=(h_{ij})_{n \times n}$  is called a WMC HMPR if there is a weighting vector  $w=(w_1, \dots, w_n)^T$ , such that

$$h_{ij}^- \leq \frac{w_i}{w_j} \leq h_{ij}^+, i=1, 2, \dots, n, j=i+1, \dots, n. \quad (13)$$

It is obvious that a CMC HMPR is a special case of WMC HMPR, while a WMC HMPR might not necessarily be a CMC HMPR (see Fig.1).

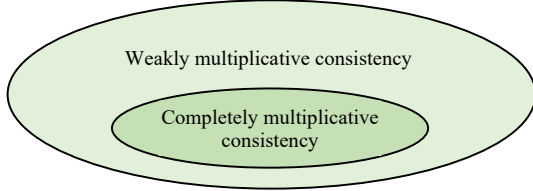


Fig. 1. The relationship of the two multiplicative consistencies for HMPRs.

#### IV. CONSISTENCY ASCERTAINING

An important question to answer is whether an HMPR is CMC or WMC.

The direct verification of the CMC property as per Definition 11 is not an easy task. In order to facilitate calculation, the following 0-1 indicator variables of HMEs  $h_{ij}$  are introduced:

$$\alpha_{ij}^t = \begin{cases} 1, & \text{if } h_{ij}^t \in h_{ij} \text{ is chosen} \\ 0, & \text{otherwise} \end{cases}. \text{ Each element in } h_{ij} \text{ can be}$$

expressed as follows:  $\prod_{t=1}^{l_{h_{ij}}} (h_{ij}^t)^{\alpha_{ij}^t}$  with  $\sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t = 1$ .

According to Definition 10, if  $H$  is a CMC HMPR, then

$$\prod_{t=1}^{l_{h_{ij}}} (h_{ij}^t)^{\alpha_{ij}^t} = \prod_{t=1}^{l_{h_{ik}}} (h_{ik}^t)^{\alpha_{ik}^t} \times \prod_{t=1}^{l_{h_{kj}}} (h_{kj}^t)^{\alpha_{kj}^t}. \quad (14)$$

This is equivalent to:

$$\sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t \log(h_{ij}^t) - \sum_{t=1}^{l_{h_{ik}}} \alpha_{ik}^t \log(h_{ik}^t) - \sum_{t=1}^{l_{h_{kj}}} \alpha_{kj}^t \log(h_{kj}^t) = 0. \quad (15)$$

As aforementioned, (15) does not always hold. We relax (15) appropriately with the introduction of nonnegative deviation numbers  $d_{ijk}^-$  and  $d_{ijk}^+$ ,  $\forall i, j, k \in N$ :

$$\sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t \log(h_{ij}^t) - \sum_{t=1}^{l_{h_{ik}}} \alpha_{ik}^t \log(h_{ik}^t) - \sum_{t=1}^{l_{h_{kj}}} \alpha_{kj}^t \log(h_{kj}^t) - d_{ijk}^+ + d_{ijk}^- = 0. \quad (16)$$

Equation (16) becomes (15) iff  $d_{ijk}^- = d_{ijk}^+ = 0$ . Thus, the following 0-1 mixed programming model is established to ascertain the **completely multiplicative consistency property** of HMPRs.

$$(M-1) J_1 = \min \sum_{k=1}^n \sum_{i=1, i \neq k}^n \sum_{j=1, i \neq j, j \neq k}^n (d_{ijk}^- + d_{ijk}^+)$$

$$\begin{aligned} & \sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t \log(h_{ij}^t) - \sum_{t=1}^{l_{h_{ik}}} \alpha_{ik}^t \log(h_{ik}^t) - \sum_{t=1}^{l_{h_{kj}}} \alpha_{kj}^t \log(h_{kj}^t) - d_{ijk}^+ + d_{ijk}^- = 0, \\ & i, j, k \in N, i \neq k, i \neq j, j \neq k \\ & \sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t \log(h_{ij}^t) + \sum_{t=1}^{l_{h_{ji}}} \alpha_{ji}^{l_{h_{ji}}-t+1} \log(h_{ji}^{l_{h_{ji}}-t+1}) = 0, i, j \in N, i \neq j \\ & \sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t = \sum_{t=1}^{l_{h_{ji}}} \alpha_{ji}^t = 1, i, j \in N, i \neq j \\ & \alpha_{ij}^t = 0 \text{ or } 1, i, j \in N, i \neq j, t = 1, 2, \dots, l_{h_{ij}} \\ & d_{ijk}^-, d_{ijk}^+ \geq 0, i, j, k \in N, i \neq k, i \neq j, j \neq k \end{aligned}$$

By solving (M-1), if  $J_1=0$  for all  $i, j$  with  $i \neq j$  and each  $t=1, 2, \dots, l_{h_{ij}}$ , then  $H$  is CMC; otherwise,  $H$  is not CMC.

Reciprocity of  $H$  means that (M-1) can be equivalently rewritten as:

$$(M-2) J_2 = \min \sum_{i=1}^{n-1} \sum_{k=i+1}^{j-1} \sum_{j=k+1}^n (d_{ijk}^- + d_{ijk}^+)$$

$$\begin{aligned} & \sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t \log(h_{ij}^t) - \sum_{t=1}^{l_{h_{ik}}} \alpha_{ik}^t \log(h_{ik}^t) - \sum_{t=1}^{l_{h_{kj}}} \alpha_{kj}^t \log(h_{kj}^t) - d_{ijk}^+ + d_{ijk}^- = 0, \\ & i, j, k \in N, i < k < j \\ & \sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t \log(h_{ij}^t) + \sum_{t=1}^{l_{h_{ji}}} \alpha_{ji}^{l_{h_{ji}}-t+1} \log(h_{ji}^{l_{h_{ji}}-t+1}) = 0, i, j \in N, i \neq j \\ & \sum_{t=1}^{l_{h_{ij}}} \alpha_{ij}^t = \sum_{t=1}^{l_{h_{ji}}} \alpha_{ji}^t = 1, i, j \in N, i < j \\ & \alpha_{ij}^t = 0 \text{ or } 1, i, j \in N, i < j, t = 1, 2, \dots, l_{h_{ij}} \\ & d_{ijk}^-, d_{ijk}^+ \geq 0, i, j, k \in N, i < k < j \end{aligned}$$

The following result proves the validity of model (M-2) to ascertain the **completely multiplicative consistency** property of HMPRs.

**Theorem 1.** An HMPR  $H$  is a CMC HMPR iff  $J_2=0$ .

**Proof. Sufficiency.** If  $J_2=0$ , then  $d_{ijk}^- = d_{ijk}^+ = 0, \forall i, j, k \in N$  and (16) reduces to (15). Thus,  $H$  is CMC.

**Necessary.** If  $H$  is CMC, (15) holds and  $d_{ijk}^- = d_{ijk}^+ = 0$  in (16), which implies that  $J_2=0$ .  $\square$

When  $H$  is a CMC HMPR, a complete consistent MPR can be derived by solving (M-2). On the contrary, if  $H$  is not a CMC HMPR, in the following, some algebraic methods are proposed to detect whether it is a WMC HMPR.

**Theorem 2.** An HMPR  $H=(h_{ij})_{n \times n}$  is a WMC HMPR iff

$$\max_k \{h_{ij}^-, h_{ik}^-, h_{kj}^-\} \leq \min_k \{h_{ij}^+, h_{ik}^+, h_{kj}^+\}, \forall i, j, k \in N. \quad (17)$$

**Proof.** If  $H$  is a WMC HMPR, then there is a complete consistent MPR  $R=(r_{ij})_{n \times n}$  such that

$$h_{ij}^- \leq r_{ij} \leq h_{ij}^+, \forall i, j \in N, \quad (18)$$

$$h_{ik}^- \leq r_{ik} \leq h_{ik}^+, \forall i, k \in N, \quad (19)$$

$$h_{kj}^- \leq r_{kj} \leq h_{kj}^+, \forall k, j \in N. \quad (20)$$



Multiplying (19) by (20), we have:

$$h_{ik}^- h_{kj}^- \leq r_{ij} \leq h_{ik}^+ h_{kj}^+, \forall i, j, k \in N. \quad (21)$$

Since (21) holds for any  $k \in N$ , it is  $\max_k \{h_{ij}^-, h_{ik}^- h_{kj}^-\} \leq \min_k \{h_{ij}^+, h_{ik}^+ h_{kj}^+\}$  for all  $i, j, k \in N$ .

Conversely, if (17) holds for all  $i, j, k \in N$ , there exists a complete consistent MPD  $R=(r_{ij})_{n \times n}$  satisfying  $r_{ij}=r_{ik} \cdot r_{kj}$ ,  $h_{ij}^- \leq r_{ij} \leq h_{ij}^+$ ,  $\forall i, j, k \in N$ . By Definition 11,  $H$  is a WMC HMPR.  $\square$

As per (17), we have the following equivalence theorem to ascertain the **weakly multiplicative consistency** property of HMPRs.

**Theorem 3.** An HMPR  $H=(h_{ij})_{n \times n}$  is a WMC HMPR iff

$$\bigcap_{k=1}^n [h_{ik}^- h_{kj}^-, h_{ik}^+ h_{kj}^+] \neq \emptyset, \forall i, j, k \in N. \quad (22)$$

*Proof.* We only need to prove that (17) and (22) are equivalent. Suppose that  $H$  is a WMC HMPR, then there is a complete consistent MPD  $R=(r_{ij})_{n \times n}$  such that

$$h_{ij}^- \leq r_{ij} \leq h_{ij}^+, \forall i, j \in N, \quad (23)$$

$$h_{ik}^- \leq r_{ik} \leq h_{ik}^+, \forall i, k \in N, \quad (24)$$

$$h_{kj}^- \leq r_{kj} \leq h_{kj}^+, \forall k, j \in N. \quad (25)$$

Therefore, it is

$$h_{ik}^- h_{kj}^- \leq r_{ij} \leq h_{ik}^+ h_{kj}^+, \forall i, j, k \in N. \quad (26)$$

Since (26) holds for any  $k \in N$ , it is  $r_{ij} \in \bigcap_{k=1}^n [h_{ik}^- h_{kj}^-, h_{ik}^+ h_{kj}^+] \neq \emptyset$ , which is equivalent to  $\max_k \{h_{ij}^-, h_{ik}^- h_{kj}^-\} \leq \min_k \{h_{ij}^+, h_{ik}^+ h_{kj}^+\}$ . By Theorem 2,  $H$  is a WMC HMPR, which completes the proof of Theorem 3.  $\square$

The below interval MPD definition is needed for Theorem 4, which is an equivalent result to Theorem 2 and Theorem 3, to ascertain the **weakly multiplicative consistency** property of HMPRs. Recall that given two interval numbers  $\bar{x}=[x^-, x^+]$  and  $\bar{y}=[y^-, y^+]$  with  $x^-, y^- > 0$ , their product is:  $\bar{x} \cdot \bar{y}=[x^- y^-, x^+ y^+]$ .

**Definition 14** [47]. An **interval MPD**  $\bar{H}=(\bar{h}_{ij})_{n \times n}$  is a preference relation with elements  $\bar{h}_{ij}=[h_{ij}^-, h_{ij}^+]$  verifying:  $0 < h_{ij}^- \leq h_{ij}^+$ ,  $h_{ij}^- h_{ji}^+ = 1$ ,  $h_{ij}^+ h_{ji}^- = 1$ . The element  $\bar{h}_{ij}$  is called the interval preference ratio and denotes that alternative  $x_i$  is between  $h_{ij}^-$  and  $h_{ij}^+$  times as important as alternative  $x_j$ .

Notice that given an HMPR  $H=(h_{ij})_{n \times n}$ , the **interval MPD**  $\bar{H}=(\bar{h}_{ij})_{n \times n}$  with elements  $\bar{h}_{ij}=[h_{ij}^-, h_{ij}^+]$  can be constructed.

**Theorem 4.** An HMPR  $H=(h_{ij})_{n \times n}$  is a WMC HMPR iff  $\bar{H}=(\bar{h}_{ij})_{n \times n}$ ,  $\bar{h}_{ij}=[h_{ij}^-, h_{ij}^+]$ , satisfies

$$\bigcap_{k=1}^n (\bar{h}_{ik} \bar{h}_{kj}) \neq \emptyset, \forall i, j, k \in N. \quad (27)$$

*Proof. Sufficiency.* If  $\bigcap_{k=1}^n (\bar{h}_{ik} \bar{h}_{kj}) \neq \emptyset$  for all  $i, j, k \in N$ , then it

is  $\bigcap_{k=1}^n (\bar{h}_{ik} \bar{h}_{kj}) = [p_{ij}^-, p_{ij}^+]$ . Thus, it is  $\max_k \{h_{ij}^-, h_{ik}^- h_{kj}^-\} \leq p_{ij}^- \leq p_{ij}^+ \leq \min_k \{h_{ij}^+, h_{ik}^+ h_{kj}^+\}$ , i.e., Theorem 2 is true and  $H$  is WMC.

*Necessary.* If  $H$  is a WMC HMPR, then there is a complete consistent MPD  $R=(r_{ij})_{n \times n}$  satisfying  $h_{ij}^- \leq r_{ij} \leq h_{ij}^+$  and

$$r_{ij} = r_{ik} \cdot r_{kj} \in \bar{h}_{ik} \bar{h}_{kj}, \forall i, j, k \in N. \text{ Therefore, } \bigcap_{k=1}^n (\bar{h}_{ik} \bar{h}_{kj}) \neq \emptyset. \quad \square$$

Reciprocity of HMPRs means that when ascertaining the validity of the above results only the elements of the upper or lower part of an HMPR are to be considered.

## V. GOAL PROGRAMMING APPROACH TO PRIORITY WEIGHT DERIVATION AND INCONSISTENCY REPAIRING OF AN HMPR

Consistency is a key property of preference relations, so it is natural to generate priority weights of alternatives from consistent HMPRs. In this section, the following two research questions will be answered: (1) How to generate a priority weight vector from a consistent HMPR? (2) How to rectify the inconsistency of an HMPR?

To answer these questions, effective optimization models based on **multiplicative consistency** are established (i) to test the **weakly multiplicative consistency property**, (ii) to derive priority weights of alternatives, and (iii) to repair the inconsistency of a given HMPR.

To find out whether a given HMPR  $H=(h_{ij})_{n \times n}$  is WMC, non-negative deviation values  $d_{ij}^-$  and  $d_{ij}^+$  are introduced in (13):

$$h_{ij}^- - d_{ij}^- \leq \frac{w_i}{w_j} \leq h_{ij}^+ + d_{ij}^+, i=1,2,\dots,n, j=i+1,\dots,n. \quad (28)$$

Clearly,  $H$  is WMC iff  $d_{ij}^-$  and  $d_{ij}^+$  are 0 in (28), for  $i=1,\dots,n, j=i+1,\dots,n$ . Therefore, the sum of these deviations is used as the objective function of the following optimization model:

$$(M-3) \quad J_3 = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^n (d_{ij}^- + d_{ij}^+)$$

$$\text{s.t.} \quad \begin{cases} \frac{w_i}{w_j} + d_{ij}^- \geq h_{ij}^-, i=1,2,\dots,n-1, j=i+1,\dots,n \\ \frac{w_i}{w_j} - d_{ij}^+ \leq h_{ij}^+, i=1,2,\dots,n-1, j=i+1,\dots,n \\ \sum_{i=1}^n w_i = 1 \\ w_i \geq 0, i=1,2,\dots,n \\ d_{ij}^-, d_{ij}^+ \geq 0, i=1,2,\dots,n-1, j=i+1,\dots,n \end{cases}$$

The following result proves the validity of model (M-3) to ascertain the **weakly multiplicative consistency property** of HMPRs:

**Theorem 5.** An HMPR  $H=(h_{ij})_{n \times n}$  is a WMC HMPR iff  $J_3=0$ .

*Proof. Necessary.* If  $H$  is a WMC HMPR, then (13) holds and it is  $d_{ij}^- = d_{ij}^+ = 0$  in (28), which implies that  $J_3=0$ .

*Sufficiency.* If  $J_3=0$ , then  $d_{ij}^- = d_{ij}^+ = 0$ ,  $\forall i, j \in N$ , and (28) becomes (13). Hence,  $H$  is a WMC HMPR.  $\square$

Model (M-3) provides an alternative way, but equivalent to (13), to ascertain the weakly multiplicative consistency property of HMPRs. Unlike the algebraic operations in Section IV, model (M-3) generates the priority weights of alternatives directly from the HMPR. In any case, when  $J_3=0$ ,  $H$  is WMC but not necessarily CMC, which can be ascertained with model (M-2).

Since a non-linear programming model may have multiple solutions, there may be more than one set of weights  $w_i$  with  $J_3=0$  in (M-3). As a result, (M-3) main aims are to ascertain whether an HMPR has the weakly multiplicative consistency property and to repair inconsistency, but not to derive the priority weight vector. Hence, it is only necessary to observe the value of  $J_3$  to test if the consistency type is WMC. Thus, a more reasonable and reliable method to derive the weights of alternatives is needed.

When an HMPR is consistent, model (M-3) results in priority weights of alternatives as single values in the unit interval. However, following the argument provided in [17], interval priority weights are more natural and reasonable than precise weights for hesitant judgments provided by decision-makers. Therefore, to generate interval priority weights of alternatives from consistent HMPRs, the below lower and upper approximation models are proposed:

(M-4)  $w_i^- = \min w_i$

$$\text{s.t.} \begin{cases} \frac{w_i}{w_j} \geq h_{ij}^-, i = 1, 2, \dots, n-1, j = i+1, \dots, n \\ \frac{w_i}{w_j} \leq h_{ij}^+, i = 1, 2, \dots, n-1, j = i+1, \dots, n \\ \sum_{i=1}^n w_i = 1 \\ w_i \geq 0, i = 1, 2, \dots, n \end{cases}$$

(M-5)  $w_i^+ = \max w_i$

$$\text{s.t.} \begin{cases} \frac{w_i}{w_j} \geq h_{ij}^-, i = 1, 2, \dots, n-1, j = i+1, \dots, n \\ \frac{w_i}{w_j} \leq h_{ij}^+, i = 1, 2, \dots, n-1, j = i+1, \dots, n \\ \sum_{i=1}^n w_i = 1 \\ w_i \geq 0, i = 1, 2, \dots, n \end{cases}$$

Given a WMC HMPR, solving models (M-4) and (M-5) will result in unique optimal interval priority weights of alternatives  $w_i = [w_i^-, w_i^+]$ ,  $\forall i \in N$ . Thus, if an HMPR is not consistent, it is necessary first to repair its inconsistency. In the following, an inconsistency repairing method is proposed. The principles of the modification are two: (i) to reduce the total adjustments of an HMPR, and (ii) not to increase the number of the values in the adjusted HMPR with respect to the original HMPR.

Given an inconsistency HMPR, model (M-3) allows to identify the inconsistent elements. Therefore, it can, guide the inconsistency repairing process as described below:

1. If  $J_3 \neq 0$ , then there are optimal nonzero deviations  $d_{ij}^-$  and

$d_{ij}^+$  when solving (M-3), which corresponds to HMPR inconsistent elements. Indeed, if  $d_{i_0 j_0}^{+(-)} \neq 0$ , then  $h_{i_0 j_0}$  is an inconsistent element. With regard to the inconsistent element, its range changes from  $[h_{ij}^-, h_{ij}^+]$  to  $[h_{ij}^- - d_{ij}^-, h_{ij}^+ + d_{ij}^+]$ . In other words, the upper and lower bounds of HME are replaced by the new values  $h_{ij}^+ + d_{ij}^+$  and  $h_{ij}^- - d_{ij}^-$ , while the other values remain unchanged, that is:

$$\bar{h}_{ij} = \begin{cases} \{h_{ij}^- - d_{ij}^- \text{ or } h_{ij}^+ + d_{ij}^+\}, & l_{h_{ij}} = 1 \\ \{h_{ij}^- - d_{ij}^-, h_{ij}^{\sigma(2)}, \dots, h_{ij}^{\sigma(l_{h_{ij}}-1)}, h_{ij}^+ + d_{ij}^+\}, & l_{h_{ij}} \neq 1 \end{cases} \quad (29)$$

Thus, a new modified HMPR  $\bar{H}$  is obtained:

$$(\bar{h}_{ij}, \bar{h}_{ji}) = \begin{cases} (\bar{h}_{ij}, 1/\bar{h}_{ij}), & \text{if } h_{ij} \text{ is the inconsistent element} \\ (h_{ij}, h_{ji}), & \text{otherwise} \end{cases} \quad (30)$$

2. In (29), there are two cases for adjusting the inconsistent elements. Notice that when there is only one element in the HME, the original value is replaced by the modified value. When there are two or more elements in the HME, the lower and upper bound values are replaced and the rest of values are unchanged. This approach maintains the original number of values in each HME, and preserves most of the decision-maker's original preferences because only the inconsistent elements are adjusted.
3. After improving the consistency of the HMPR, the priority weight vector derived from the newly adjusted HMPR satisfies (13), and it is  $J_3=0$ . Consequently, (29) and (30) convert an inconsistent HMPR into a consistent HMPR.

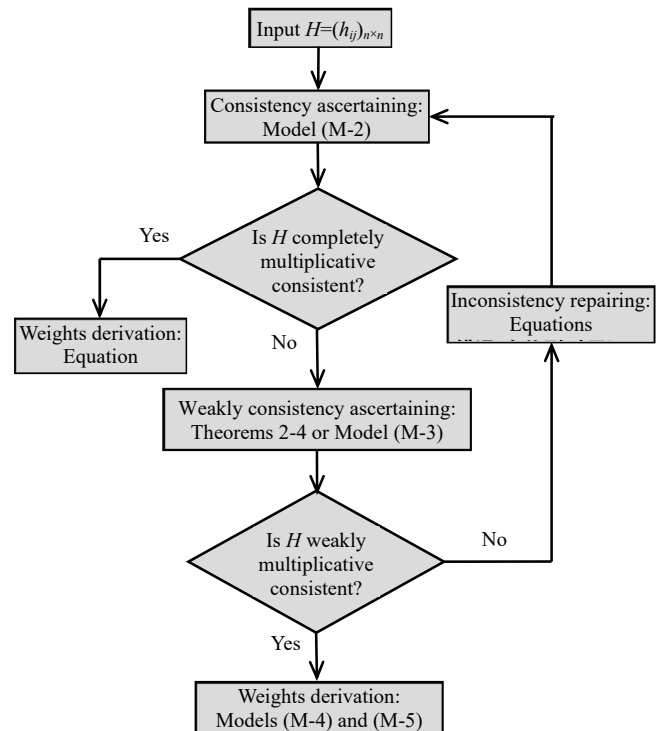


Fig. 2. The process of consistency ascertaining, inconsistency repairing and weights derivation for HMPRs.

In what follows, an integrated algorithm to ascertain consistency, inconsistency repairing and priority weights derivation for HMPRs is proposed, with corresponding flowchart depicted in Fig. 2.

### Algorithm 1.

*Step 1.* Given an HMPR  $H$ , check **completely multiplicative consistency property** with model (M-2). If  $H$  is CMC, go to Step 3A; otherwise, go to next step.

*Step 2.* Check **weakly multiplicative consistency property** by (17), (22), (27) or model (M-3). If  $H$  is WMC, go to Step 3B; otherwise, go to Step 4.

*Step 3.* Priority weights derivation and ranking of alternatives.

*Step 3A.* (Following Step 1).

Derive the priority weights with the **Logarithmic Least Squares Method** [48, 49]:

$$w_i = \left( \prod_{j=1}^n r_{ij} \right)^{1/n} / \left( \sum_{i=1}^n \left( \prod_{j=1}^n r_{ij} \right)^{1/n} \right), i = 1, 2, \dots, n \quad (31)$$

and go to Step 5.

*Step 3B.* (Following Step 2).

Generate the interval weights with models (M-4) and (M-5). Alternatives are ranked according to their priority weights ranking, via the degree of possibility of  $w_i \geq w_j$  [5, 50]:

$$p(w_i \geq w_j) = \frac{\max\{0, w_i^+ - w_j^-\} - \max\{0, w_i^- - w_j^+\}}{w_i^+ - w_i^- + w_j^+ - w_j^-} \quad (32)$$

$$p_i = \sum_{j=1}^n p_{ij}, i \in N \quad (33)$$

and  $p_{ij} = p(w_i \geq w_j)$ .

Interval weights are ranked using  $p_i$  values, i.e.  $w_i \succ w_j$  iff  $p_i > p_j$ . Go to Step 5.

*Step 4.* Solve model (M-3), and repair inconsistency with (29); construct the newly adjusted HMPR with (30). Go to Step 1.

*Step 5.* End.

## VI. INCOMPLETE HMPRS

In a decision-making problem, **decision-makers** may omit some judgements, i.e. some information may be unknown. Hence, a key problem to address is the estimation of missing information. With respect to incomplete HMPRs, this section extends **two multiplicative consistency** concepts of complete HMPRs to the case of incomplete HMPRs and utilizes two **multiplicative consistency**-based goal programming models: (i) to estimate their missing HME preference values, and (ii) to ascertain the type of **multiplicative consistency** property that is verified.

Let  $H = (h_{ij})_{n \times n}$  be an incomplete HMPR. The notation  $h_{ij} = x$  is used to represent that  $h_{ij}$  is not given by the **decision-maker**. To **incorporate (16) into incomplete HMPRs**, the following indicator functions for an incomplete HMPR  $H$  are introduced:

$$\delta_{ij} = \begin{cases} 1, & h_{ij} \neq x \\ 0, & h_{ij} = x \end{cases},$$

$$\delta_{ijk} = \begin{cases} 1, & \delta_{ij}\delta_{ik}\delta_{kj} = 1 \\ 0, & \text{otherwise} \end{cases}.$$

When  $h_{ij}, h_{ik}, h_{kj}$  are all known it is  $\delta_{ijk} = 1$ . Then, (16) for an **incomplete HMPR can be rewritten as:**

$$\delta_{ijk} \left( \sum_{t=1}^{l_{hj}} \alpha_{ij}^t \log(h_{ij}^t) - \sum_{t=1}^{l_{hk}} \alpha_{ik}^t \log(h_{ik}^t) - \sum_{t=1}^{l_{hj}} \alpha_{kj}^t \log(h_{kj}^t) \right) - \varepsilon_{ijk}^+ + \varepsilon_{ijk}^- = 0 \quad (34)$$

Consequently, to ascertain the **completely multiplicative consistency property** of incomplete HMPRs, the following 0-1 mixed programming model is constructed:

$$\begin{aligned} \text{(M-6)} \quad J_6 = \min & \sum_{k=1}^n \sum_{i=1, i \neq k}^n \sum_{j=1, j \neq i, j \neq k}^n (\varepsilon_{ijk}^- + \varepsilon_{ijk}^+) \\ \text{s.t.} \quad & \delta_{ijk} \left( \sum_{t=1}^{l_{hj}} \alpha_{ij}^t \log(h_{ij}^t) - \sum_{t=1}^{l_{hk}} \alpha_{ik}^t \log(h_{ik}^t) - \sum_{t=1}^{l_{hj}} \alpha_{kj}^t \log(h_{kj}^t) \right) - \varepsilon_{ijk}^+ \\ & + \varepsilon_{ijk}^- = 0, i, j, k \in N, i \neq j, i \neq k, j \neq k \\ & \delta_{ij} \left( \sum_{t=1}^{l_{hj}} \alpha_{ij}^t \log(h_{ij}^t) + \sum_{t=1}^{l_{hj}} \alpha_{ji}^{l_{hj}-t+1} \log(h_{ji}^{l_{hj}-t+1}) \right) = 0, i, j \in N, i \neq j \\ & \sum_{t=1}^{l_{hj}} \alpha_{ij}^t = \sum_{t=1}^{l_{ji}} \alpha_{ji}^t = 1, i, j \in N, i \neq j \\ & \alpha_{ij}^t = 0 \text{ or } 1, i, j \in N, i \neq j \\ & \varepsilon_{ijk}^-, \varepsilon_{ijk}^+ \geq 0, i, j, k \in N, i \neq j, i \neq k, j \neq k \\ & \delta_{ij} = \begin{cases} 1, & h_{ij} \neq x \\ 0, & h_{ij} = x \end{cases}, i, j \in N \\ & \delta_{ijk} = \begin{cases} 1, & \delta_{ij}\delta_{ik}\delta_{kj} = 1 \\ 0, & \text{otherwise} \end{cases}, i, j, k \in N \end{aligned}$$

This model can be equivalently simplified as follows:

$$\begin{aligned} \text{(M-7)} \quad J_7 = \min & \sum_{k=1}^n \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\varepsilon_{ijk}^- + \varepsilon_{ijk}^+) \\ \text{s.t.} \quad & \delta_{ijk} \left( \sum_{t=1}^{l_{hj}} \alpha_{ij}^t \log(h_{ij}^t) - \sum_{t=1}^{l_{hk}} \alpha_{ik}^t \log(h_{ik}^t) - \sum_{t=1}^{l_{hj}} \alpha_{kj}^t \log(h_{kj}^t) \right) - \varepsilon_{ijk}^+ \\ & + \varepsilon_{ijk}^- = 0, i, j, k \in N, i < j, k \neq i, k \neq j \\ & \delta_{ij} \left( \sum_{t=1}^{l_{hj}} \alpha_{ij}^t \log(h_{ij}^t) + \sum_{t=1}^{l_{hj}} \alpha_{ji}^{l_{hj}-t+1} \log(h_{ji}^{l_{hj}-t+1}) \right) = 0, i, j \in N, i < j \\ & \sum_{t=1}^{l_{hj}} \alpha_{ij}^t = \sum_{t=1}^{l_{ji}} \alpha_{ji}^t = 1, i, j \in N, i < j \\ & \alpha_{ij}^t = 0 \text{ or } 1, i, j \in N, i \neq j, t = 1, 2, \dots, l_{hj} \\ & \varepsilon_{ijk}^-, \varepsilon_{ijk}^+ \geq 0, i, j, k \in N, i < j, k \neq i, k \neq j \\ & \delta_{ij} = \begin{cases} 1, & h_{ij} \neq x \\ 0, & h_{ij} = x \end{cases}, i, j \in N \\ & \delta_{ijk} = \begin{cases} 1, & \delta_{ij}\delta_{ik}\delta_{kj} = 1 \\ 0, & \text{otherwise} \end{cases}, i, j, k \in N \end{aligned}$$

The following result proves the validity of model (M-7) to ascertain the **completely multiplicative consistency property** of incomplete HMPRs.

**Theorem 6.** An incomplete HMPR  $H$  is CMC iff  $J_7=0$ .

*Proof. Necessary.* If an incomplete HMPR  $H$  is CMC, then (15) holds for all known elements, that is,  $\varepsilon_{ijk}^- = \varepsilon_{ijk}^+ = 0$  in (34). Therefore,  $J_7=0$ .

*Sufficiency.* If an incomplete HMPR  $H$  has  $J_7=0$ , then  $\varepsilon_{ijk}^- = \varepsilon_{ijk}^+ = 0$  for all known elements, and (34) becomes (15). Thus, the incomplete HMPR  $H$  is CMC.  $\square$

The following example illustrates the process of estimating missing values and determining the consistency of incomplete HMPRs with model (M-7).

**Example 1.** Consider the incomplete HMPR  $H_1$  (adapted from [41]):

$$H_1 = \begin{pmatrix} \{\} & \{\frac{1}{2}, 1\} & \{\frac{1}{3}, \frac{1}{2}\} & \{\} \\ \{1, 2\} & \{\} & \{1, 2, 3\} & \{2\} \\ \{2, 3\} & \{\frac{1}{3}, \frac{1}{2}, 1\} & \{\} & x \\ \{\} & \{\frac{1}{2}\} & x & \{\} \end{pmatrix}.$$

Solving model (M-7) gives  $J_7=0$ . Thus, the incomplete HMPR  $H_1$  is CMC. At the same time, the missing HMEs obtained are  $h_{34}=\{2\}$  and  $h_{43}=\{1/2\}$ , and there exists a complete consistent MPR  $R_1$

$$R_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

Thus,  $H_1$  is transformed into the below complete HMPR:

$$H'_1 = \begin{pmatrix} \{\} & \{\frac{1}{2}, 1\} & \{\frac{1}{3}, \frac{1}{2}\} & \{\} \\ \{1, 2\} & \{\} & \{1, 2, 3\} & \{2\} \\ \{2, 3\} & \{\frac{1}{3}, \frac{1}{2}, 1\} & \{\} & \{2\} \\ \{\} & \{\frac{1}{2}\} & \{\frac{1}{2}\} & \{\} \end{pmatrix}.$$

The proposed model is more reasonable and effective than Sahu and Gupta's model [22], since their  $\beta$ -normalization method is superfluous, and no additional elements are added to HMEs. In addition, the proposed model can determine the consistency type of incomplete HMPRs, while Sahu and Gupta's model fails to do so.

When the incomplete HMPR  $H$  is not CMC, its weakly multiplicative consistency property is considered. Similar to Theorems 2-4 of Section IV, the following results for incomplete HMPRs are provided.

**Theorem 7.** An incomplete HMPR  $H=(h_{ij})_{n \times n}$  is WMC iff for all known elements:

$$\max_k \{h_{ij}^-, h_{ik}^-, h_{kj}^-\} \leq \min_k \{h_{ij}^+, h_{ik}^+, h_{kj}^+\} \quad (35)$$

or, equivalently

$$\bigcap_{k=1}^n [h_{ik}^- h_{kj}^-, h_{ik}^+ h_{kj}^+] \neq \emptyset \quad (36)$$

is verified.

Given an incomplete HMPR,  $H=(h_{ij})_{n \times n}$ , its associated incomplete interval MPR  $\bar{H}=(\bar{h}_{ij})_{n \times n}$  has elements:

$\bar{h}_{ij}=[h_{ij}^-, h_{ij}^+]$  if  $\bar{h}_{ij}$  is known, otherwise  $\bar{h}_{ij}=x$  is unknown.

Let  $\Omega$  be the set of all the known elements in  $\bar{H}$ .

**Theorem 8.** An incomplete HMPR  $H=(h_{ij})_{n \times n}$  is WMC iff

$$\bigcap_{k=1}^n (\bar{h}_{ik} \bar{h}_{kj}) \neq \emptyset, \text{ for all } \bar{h}_{ij} \in \Omega \quad (37)$$

The following optimization model for incomplete HMPRs can be established, based on the weakly multiplicative consistency property, to estimate the missing information and to test consistency:

$$(M-8) \quad J_8 = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\varepsilon_{ij}^- + \varepsilon_{ij}^+)$$

$$\text{s.t.} \quad \begin{cases} \delta_{ij} \left( \frac{w_i}{w_j} + \varepsilon_{ij}^- - h_{ij}^- \right) \geq 0, i=1, 2, \dots, n-1, j=i+1, \dots, n \\ \delta_{ij} \left( \frac{w_i}{w_j} - \varepsilon_{ij}^+ - h_{ij}^+ \right) \leq 0, i=1, 2, \dots, n-1, j=i+1, \dots, n \\ \sum_{i=1}^n w_i = 1 \\ w_i \geq 0, i \in N \\ \delta_{ij} = \begin{cases} 1, & h_{ij} \neq x \\ 0, & h_{ij} = x \end{cases}, i, j \in N \\ \varepsilon_{ij}^-, \varepsilon_{ij}^+ \geq 0, i=1, 2, \dots, n-1, j=i+1, \dots, n \end{cases}$$

Solving model (M-8), the incomplete HMPR is WMC when  $J_8=0$ , in which case, via (3), its missing elements can be estimated. As before, an example is provided below to illustrate the weakly multiplicative consistency-based HMPR completion process.

**Example 2.** Consider the incomplete HMPR  $H_2$ :

$$H_2 = \begin{pmatrix} \{\} & \{\frac{3}{7}\} & \{\frac{3}{7}, \frac{3}{2}\} & \{\} \\ \{\frac{7}{3}\} & \{\} & \{\frac{3}{7}, \frac{3}{2}\} & x \\ \{\frac{2}{3}, \frac{7}{3}\} & \{\frac{2}{3}, \frac{7}{3}\} & \{\} & \{\frac{2}{3}, \frac{7}{3}\} \\ \{\} & x & \{\frac{3}{7}, \frac{3}{2}\} & \{\} \end{pmatrix}.$$

Solving model (M-7) gives  $J_7=2.4692 \neq 0$ . Thus,  $H_2$  is not CMC, and Theorem 7 is used to check whether  $H_2$  is WMC. For all known elements, (36) yields:

$$\bigcap_{k=1}^4 [h_{1k}^- h_{k3}^-, h_{1k}^+ h_{k3}^+] = [\frac{3}{7}, \frac{3}{2}] \cap [\frac{9}{49}, \frac{9}{14}] \cap [\frac{3}{7}, \frac{3}{2}] \cap [\frac{3}{7}, \frac{3}{2}] = [\frac{9}{49}, \frac{9}{14}] \neq \emptyset.$$

Thus, the incomplete HMPR  $H_2$  is WMC. Notice that this can also be verified using (35) as shown in Table II.

Solving model (M-8) gives  $J_8=0$ , and the missing HMEs are estimated as  $h_{24}=\{7/3\}$  and  $h_{42}=\{3/7\}$ . The following complete HMPR  $H'_2$  and complete consistent MPR  $R_2$  are obtained

$$H'_2 = \begin{pmatrix} \{\} & \{\frac{3}{7}\} & \{\frac{3}{7}, \frac{3}{2}\} & \{\} \\ \{\frac{7}{3}\} & \{\} & \{\frac{3}{7}, \frac{3}{2}\} & \{\frac{7}{3}\} \\ \{\frac{2}{3}, \frac{7}{3}\} & \{\frac{2}{3}, \frac{7}{3}\} & \{\} & \{\frac{2}{3}, \frac{7}{3}\} \\ \{\} & \{\frac{3}{7}\} & \{\frac{3}{7}, \frac{3}{2}\} & \{\} \end{pmatrix}$$



TABLE II  
CONSISTENCY ASCERTAINING FOR EXAMPLE 2

preferences	$i$	$j$	$k$	$h_{ik}^-h_{kj}^-$	$h_{ik}^+h_{kj}^+$	Consistency test
$\bar{h}_{12}$	1	2	1	$\frac{3}{7}$	$\frac{3}{7}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = \frac{3}{7}$
	1	2	3	$\frac{2}{7}$	$\frac{7}{2}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = \frac{3}{7}$
						passed
$\bar{h}_{13}$	1	3	1	$\frac{3}{7}$	$\frac{3}{2}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = \frac{3}{7}$
	1	3	2	$\frac{9}{49}$	$\frac{9}{14}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = \frac{9}{14}$
	1	3	4	$\frac{3}{7}$	$\frac{3}{2}$	passed
$\bar{h}_{14}$	1	4	1	1	1	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = 1$
	1	4	3	$\frac{2}{7}$	$\frac{7}{2}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = 1$
						passed
$\bar{h}_{23}$	2	3	1	1	$\frac{7}{2}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = 1$
	2	3	2	$\frac{3}{7}$	$\frac{3}{2}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = \frac{3}{2}$
						passed
$\bar{h}_{34}$	3	4	1	$\frac{2}{3}$	$\frac{7}{3}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = \frac{2}{3}$
	3	4	3	$\frac{2}{3}$	$\frac{7}{3}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = \frac{7}{3}$
						passed

and

$$R_2 = \begin{pmatrix} 1 & \frac{3}{7} & \frac{3}{7} & 1 \\ \frac{7}{3} & 1 & 1 & \frac{7}{3} \\ \frac{7}{3} & 1 & 1 & \frac{7}{3} \\ 1 & \frac{3}{7} & \frac{3}{7} & 1 \end{pmatrix}.$$

After estimating the missing values, an incomplete HMPR is converted into a complete HMPR, and Algorithm 1 can be used to generate the priority weights of alternatives. The consistency improving method proposed in Section V can be applied to incomplete HMPRs found to be inconsistent with both models (M-7) and (M-8).

## VII. ILLUSTRATIVE EXAMPLES AND COMPARATIVE ANALYSIS

### A. Illustrative Examples

This section offers three examples that complement the theoretical effectiveness of the approaches presented in previous sections: **Example 3 and Example 4 concern with CMC and WMC HMPRs, respectively**, while Example 5 verifies the practical value of our proposal.

**Example 3.** Consider the following HMPR on  $X = \{x_1, x_2, x_3\}$  (adapted from Zhang [26]):

$$H_3 = \begin{pmatrix} \{1\} & \{\frac{1}{7}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}\} & \{\frac{1}{6}, \frac{1}{3}, 1\} \\ \{3, 4, 5, 7\} & \{1\} & \{2, 3, 5\} \\ \{1, 3, 6\} & \{\frac{1}{5}, \frac{1}{3}, \frac{1}{2}\} & \{1\} \end{pmatrix}.$$

**Step 1.** Solving model (M-2) gives  $J_2 = 0$ . Thus,  $H_3$  is CMC. Meanwhile, the following complete consistent MPR is derived

$$R = \begin{pmatrix} 1 & \frac{1}{3} & 1 \\ 3 & 1 & 3 \\ 1 & \frac{1}{3} & 1 \end{pmatrix}.$$

**Step 2.** From (31), the following priority weight vector of alternatives is obtained:  $w = (0.2, 0.6, 0.2)^T$ , and the alternatives ranking would be:  $x_2 \succ x_1 \sim x_3$ .

**Example 4.** Consider the following HMPR (adapted from Lin and Wang [19]):

$$H_4 = \begin{pmatrix} \{1\} & \{\frac{2}{3}, 2, \frac{5}{2}\} & \{\frac{3}{4}, \frac{5}{2}, \frac{7}{2}\} & \{\frac{1}{3}, \frac{3}{2}\} \\ \{\frac{2}{5}, \frac{1}{2}, \frac{3}{2}\} & \{1\} & \{\frac{1}{2}, 2, 3\} & \{1, 2, \frac{5}{2}\} \\ \{\frac{2}{7}, \frac{2}{5}, \frac{4}{3}\} & \{\frac{1}{3}, \frac{1}{2}, 2\} & \{1\} & \{\frac{1}{3}, \frac{1}{2}, 1\} \\ \{\frac{2}{3}, 3\} & \{\frac{2}{5}, \frac{1}{2}, 1\} & \{1, 2, 3\} & \{1\} \end{pmatrix}.$$

**Step 1.** Solving model (M-2), we have  $J_2 = 0.5754$ , which means that HMPR  $H_4$  is not CMC.

**Step 2.** Expression (17) is used to check the weakly multiplicative consistency property for  $H_4$ , with the corresponding processes shown in Table III. It is concluded that HMPR  $H_4$  is WMC. Notice that this could have been done using (22). Indeed, HMPR  $H_4$  is WMC because

TABLE III CONSISTENCY ASCERTAINING FOR EXAMPLE 4						
preferences	$i$	$j$	$k$	$h_{ik}^-h_{kj}^-$	$h_{ik}^+h_{kj}^+$	Consistency test
$\bar{h}_{12}$	1	2	1	$\frac{2}{3}$	$\frac{5}{2}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = \frac{2}{3}$
	1	2	3	$\frac{1}{4}$	$\frac{7}{2}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = \frac{3}{2}$
	1	2	4	$\frac{2}{15}$	$\frac{3}{2}$	passed
$\bar{h}_{13}$	1	3	1	$\frac{3}{4}$	$\frac{7}{2}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = \frac{3}{4}$
	1	3	2	$\frac{1}{3}$	$\frac{15}{2}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = \frac{7}{2}$
	1	3	4	$\frac{1}{3}$	$\frac{9}{2}$	passed
$\bar{h}_{14}$	1	4	1	$\frac{1}{3}$	$\frac{3}{2}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = \frac{2}{3}$
	1	4	2	$\frac{2}{25}$	$\frac{7}{2}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = \frac{3}{2}$
	1	4	3	$\frac{1}{4}$	$\frac{7}{2}$	passed
$\bar{h}_{23}$	2	3	1	$\frac{3}{10}$	$\frac{21}{4}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = 1$
	2	3	2	$\frac{1}{2}$	3	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = 3$
	2	3	4	1	$\frac{15}{2}$	passed
$\bar{h}_{24}$	2	4	1	$\frac{2}{15}$	$\frac{9}{4}$	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = 1$
	2	4	2	1	$\frac{5}{2}$	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = \frac{9}{4}$
	2	4	3	$\frac{1}{6}$	3	passed
$\bar{h}_{34}$	3	4	1	$\frac{2}{21}$	2	$\max\{h_{ij}^-, h_{ik}^-h_{kj}^-\} = \frac{1}{3}$
	3	4	2	$\frac{1}{3}$	5	$\min\{h_{ij}^+, h_{ik}^+h_{kj}^+\} = 1$
	3	4	3	$\frac{1}{3}$	1	passed

$$\bigcap_{k=1}^4 [h_{1k}^-h_{k2}^-, h_{1k}^+h_{k2}^+] = [\frac{2}{3}, \frac{5}{2}] \cap [\frac{2}{3}, \frac{5}{2}] \cap [\frac{1}{4}, 7] \cap [\frac{2}{15}, \frac{3}{2}] \neq \emptyset,$$

$$\bigcap_{k=1}^4 [h_{1k}^-h_{k3}^-, h_{1k}^+h_{k3}^+] = [\frac{3}{4}, \frac{7}{2}] \cap [\frac{1}{3}, \frac{15}{2}] \cap [\frac{3}{4}, \frac{7}{2}] \cap [\frac{1}{3}, \frac{9}{2}] \neq \emptyset,$$

$$\bigcap_{k=1}^4 [h_{1k}^-h_{k4}^-, h_{1k}^+h_{k4}^+] = [\frac{1}{3}, \frac{3}{2}] \cap [\frac{2}{3}, \frac{25}{4}] \cap [\frac{1}{4}, \frac{7}{2}] \cap [\frac{1}{3}, \frac{3}{2}] \neq \emptyset,$$

$$\bigcap_{k=1}^4 [h_{2k}^-h_{k3}^-, h_{2k}^+h_{k3}^+] = [\frac{3}{10}, \frac{21}{4}] \cap [\frac{1}{2}, 3] \cap [\frac{1}{2}, 3] \cap [1, \frac{15}{2}] \neq \emptyset,$$

$$\bigcap_{k=1}^4 [h_{2k}^- h_{k4}^-, h_{2k}^+ h_{k4}^+] = [\frac{2}{15}, \frac{9}{4}] \cap [1, \frac{5}{2}] \cap [1, \frac{5}{2}] \cap [\frac{1}{6}, 3] \neq \emptyset,$$

$$\bigcap_{k=1}^4 [h_{3k}^- h_{k4}^-, h_{3k}^+ h_{k4}^+] = [\frac{2}{21}, 2] \cap [\frac{1}{3}, 5] \cap [\frac{1}{3}, 1] \cap [\frac{1}{3}, 1] \neq \emptyset.$$

Since HMPR  $H_4$  is WMC, the priority weights of alternatives are derived by solving models (M-3)-(M-5).

Step 3. Solving model (M-3) gives  $J_3=0$ , and  $w=(0.25, 0.25, 0.25, 0.25)^T$ . From (3), we obtain the following complete consistent MPR

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Solving models (M-4) and (M-5), the following interval priority vector of alternatives are obtained from  $H_4$ :  $w(H_4)=[(0.1875, 0.3818], [0.2222, 0.4091], [0.0952, 0.2667], [0.1739, 0.3333]]$ . Using expression (32), the ranking of the

alternatives would be:  $x_2 \succ_{0.5813} x_1 \succ_{0.5878} x_4 \succ_{0.7196} x_3$ . Thus, alternative  $x_2$  is superior to alternative  $x_1$  with 58.13% possibility degree, alternative  $x_1$  is superior to alternative  $x_4$  with 58.78% possibility degree, while alternative  $x_4$  is superior alternative  $x_3$  with 71.96% possibility degree.

Example 5. A practical problem is considered where an investment company is looking to invest a sum of money in the best of the following four possible investment options.

- (1)  $x_1$  is an energy company;
- (2)  $x_2$  is a medical corporation;
- (3)  $x_3$  is a high-tech company;
- (4)  $x_4$  is a food company.

The investment company evaluates the four alternative companies with the help of a third-party evaluation agency, which provides the following HMPR information:

$$H_5 = \begin{pmatrix} \{1\} & \{3\} & \{5, 7\} & \{3\} \\ \{\frac{1}{3}\} & \{1\} & \{\frac{1}{9}, \frac{1}{7}\} & \{5\} \\ \{\frac{1}{7}, \frac{1}{5}\} & \{7, 9\} & \{1\} & \{\frac{1}{7}, \frac{1}{5}\} \\ \{\frac{1}{3}\} & \{\frac{1}{5}\} & \{5, 7\} & \{1\} \end{pmatrix}.$$

Step 1. Solving model (M-2) gives  $J_2=10.3296$ . Thus,  $H_5$  is not CMC.

Step 2. Solving model (M-3) gives  $J_3=5.9238$ . Thus,  $H_5$  is not WMC. This means that we are in the presence of an inconsistent HMPR.

Step 3. The optimal deviation values are  $d_{23}^+ = 1.5238$ ,  $d_{34}^+ = 0.4$ ,  $d_{24}^- = 4$ ; so,  $h_{23}$ ,  $h_{24}$  and  $h_{34}$  are the inconsistent elements of HMPR  $H_5$ . From (29), the adjusted elements are:  $\bar{h}_{23} = \{\frac{1}{9}, 1.667\}$ ,  $\bar{h}_{24} = \{1\}$ ,  $\bar{h}_{34} = \{\frac{1}{7}, \frac{3}{5}\}$ . From (30), the improved HMPR  $\bar{H}_5$  is

$$\bar{H}_5 = \begin{pmatrix} \{1\} & \{3\} & \{5, 7\} & \{3\} \\ \{\frac{1}{3}\} & \{1\} & \{\frac{1}{9}, 1.667\} & \{5\} \\ \{\frac{1}{7}, \frac{1}{5}\} & \{0.6, 9\} & \{1\} & \{\frac{1}{7}, \frac{3}{5}\} \\ \{\frac{1}{3}\} & \{1\} & \{\frac{5}{3}, 7\} & \{1\} \end{pmatrix}.$$

Step 4. Solving model (M-2) implies that  $\bar{H}_5$  is not CMC.

Step 5. Solving model (M-3) gives  $J_3=0$ . Thus,  $\bar{H}_5$  is WMC.

Step 6. Solving models (M-4) and (M-5), the interval priority weight vector of the alternatives for  $\bar{H}_5$  is:  $w(\bar{H}_5)=[(0.5357, 0.5382], [0.1786, 0.1794], [0.1029, 0.1071], [0.1786, 0.1794]]$ . Applying (32) results in the following

ranking of the four alternatives:  $x_1 \succ x_2 \sim x_4 \succ x_3$ . This means that investment options  $x_1$  is superior to investment options  $x_2$  with 100% possibility degree, investment options  $x_2$  is equally preferred to investment options  $x_4$ , and investment options  $x_4$  is superior to investment options  $x_3$  with 100% possibility degree. Therefore, the optimal investment would be  $x_1$ .

In decision-making problems, the consistency problem is closely related to the reliability of preferences provided by decision-makers. The rationality of the judgments determines the reliability of the final decision result. It is worth noting that the consistency improvement of HMPRs in this process plays a role in regulating the logic and rationality of the given preference information. Therefore, in practice, our proposal contributes to achieving reliable decision-making results.

In what follows, a discussion and a simulation analysis are reported to illustrate availability and advantages of the proposed method.

## B. Discussion, Simulation and Comparative Analysis

In this section, we compare the peculiarities of existing methods and discuss the advantages of our proposed methods. A summary of the improvements of the proposed method based on the previous illustrative examples is provided. In addition, a systematic analysis with the help of simulation experiments is carried out, which clearly and intuitively highlights the superior performance of the proposed method.

### 1) Discussion

In view of the evident differences with the existing consistency studies, the proposed approach improvements can be summarized as follows:

(a) As far as we are aware, the proposed approach is the first attempt to study simultaneously both completely multiplicative consistency and weakly multiplicative consistency properties for HMPRs, which represents a more effective and precise way to describe and detect consistency. Meng et al. [25] studied the multiplicative consistency property of HMPRs. They proposed a strict consistency concept for HMPRs that requires the existence of multiplicative consistent MPRs for every value in every HMEs. This means that in practice most HMPRs will fail to verify Meng et al.'s [25] definition of consistency property. For example,  $H_3$  (Example 3), which was judged to be CMC, does not satisfy Meng et al.'s consistency definition. Although theoretically there exist HMPRs that verify Meng et al.'s consistency definition, this is not reasonable in practice. Hesitancy means that a decision-maker is unsure about the

preference values when comparing two alternatives, though he/she can give some possible preference values (hence the hesitation). If for every value in every HME, a consistent MPR exists, then this would imply a level of consistency knowledge by the decision-maker that would make hesitancy improbable and therefore impractical in a hesitancy environment. As the decision-maker is hesitant, we should aim to find the reasonable information (i.e., consistent information) from his/her hesitant information, which is exactly the aim of the proposed method.

(b) Zhang and Wu proposed two consistency improvement methods in [17, 21]. These methods rely on a  $\beta$ -normalization process, which convert the HMEs so that they all have the same number of values, and then the HMMPR is managed as several MPRs. The normalization process obviously distorts the DM's original information and the results obtained could be unrelated to the original information, which makes them unreliable. In this paper, the proposed method does not rely on any normalization process, which translates into minimal changes of the original information of DMs and lower computational cost. Moreover, Xu et al. [46] pointed out that Zhang and Wu's [21] consistency process is artificial, the consistent HMMPR may not be an HMMPR because the improved MPRs will not be arranged in ascending order. Additionally, the smaller the improvement process consistency threshold in [21] is, the larger the number of iterations and the computational cost are.

(c) The priority weights of alternatives derived from the proposed method are of interval nature. As the DM's information is hesitant, it is more logical and natural to derive interval weights from consistent HMMPRs than exact priority weights as proposed by Zhu and Xu [18]. Although Zhang and Wu's [17] weight-derivation algorithm for HMMPRs results in an interval priority weight vector for  $H_5$ ,  $w = ([0.3938, 0.4581], [0.1715, 0.1729], [0.1882, 0.2108], [0.1823, 0.2225])$ , which leads to the optimal choice  $x_1$ , which is consistent with the proposed approach, although it is based on an additional normalization process, which implies higher computational cost.

(d) The proposed approach can be utilized to solve decision-making problems with incomplete HMMPRs via the two multiplicative consistency goal programming models developed to ascertain the consistency property and to estimate the missing values. The existent literature method by Sahu and Gupta [22] requires a normalization process to improve the consistency, and therefore is subjected to the previously mentioned drawbacks. Thus, the proposed approach can deal with incomplete information in HMMPR, which allows DMs or decision organizations to express their preferences more flexibly, and therefore more effectively.

In summary, the above analysis shows that the performance of the proposal approach can compete with other approaches. The comparative analysis, based on eight performance criteria, of these methods is summarized in Table IV. The label "✓" means that the method is very suitable, "-" means that the method is acceptable, while "×" means that the method performs poorly on the given criterion.

TABLE IV  
COMPARISON BETWEEN THE EXISTING STUDIES AND OUR PROPOSAL

	The proposed method	Zhang and Wu [21]	Meng et al. [25]	Zhang and Wu [17]	Sahu and Gupta [22]
Ascertain consistency	✓	✓	✓	✓	✓
Consistency thresholds	×	✓	×	×	✓
$\alpha$ or $\beta$ normalization	×	✓	×	✓	✓
Repair inconsistency	✓	✓	×	✓	×
Minimal deviation from decision-maker's original judgments	✓	×	×	×	×
Ability to maintain decision-makers' hesitation in weights derivation	✓	×	✓	✓	×
Ability to address incomplete HMMPRs	✓	×	✓	×	✓
Acceptable computational complexity	✓	-	-	-	-

From Table IV, it can see that the functionality of the proposed approach is powerful, and that it can help to (i) determine the consistency type without the help of consistency threshold setting and normalization process, (ii) repair inconsistency with lower information distortion and computation, (iii) derive interval weights based on the decision-maker's hesitation, and (iv) solve decision-making problems with incomplete HMMPRs. Consequently, the proposed approach can deal with decision-making problems with HMMPRs more flexibly, reasonably, and effectively.

## 2) Simulation and Comparative Analysis

In order to further show the effectiveness and advantages of the proposed method, Monte Carlo simulation experiments are carried out and analyzed. Further, the proposed method is compared with the methods by Zhang and Wu [17] and Zhang and Wu [21], since their methods also proposed different consistency concepts and consistency improving processes. The  $\beta$ -normalization method is used in [17, 21] with  $\bar{h}_{ij} = (h_{ij}^+)^{\varsigma} \times (h_{ij}^-)^{(1-\varsigma)}$  used to add some values to the HMEs of shorter length to make all the HMEs have the same length. In this paper, we assume  $\varsigma = 0.5$ . Both Zhang and Wu [17] and Zhang and Wu [21] split the HMMPR into several MPRs. Zhang and Wu [17] used Saaty's consistency ratio ( $CR < 0.1$ ) to check whether these MPRs are of acceptable consistency. If any of the MPR is not acceptable consistent, Xu and Wei [51]'s Algorithm I (with  $\lambda=0.5$ ) is used to improving its consistency. Notice that Zhang and Wu [21] proposed another algorithm (referred to as Algorithm 2 in [21]) to improve consistency. In the method, a consistency threshold  $\bar{CI}$  is set in advance ( $\bar{CI}=1.01$ ). Meng et al.'s [25] method only find the consistent MPRs in an HMMPR, with no method to repair the inconsistency proposed when there is no such consistent MPR in an HMMPR. Sahu and Gupta [22]

proposed a method to estimate the missing values in an incomplete HMPR, and Zhang and Wu's [17] is adopted to check whether the complete HMPR is of acceptable consistency. If the complete HMPR is not consistent, no consistency improving method is provided. Therefore, in the following, we only do simulations and compare the proposed method with Zhang and Wu [17] and Zhang and Wu [21]'s methods.

A total of 1000 HMPRs with different dimension, ranging from 3 to 9, are randomly generated. In order to be close to the actual decision-making scenario, we assume that the number of elements in each HME is less than 3. Furthermore, all the randomly generated values are in the Saaty's scale  $\{1/9, 1/8, \dots, 1/2, 1, \dots, 9\}$ . In order to compare the performances of the different methods, we propose the following criteria.

(1) Length change ratio (LCR)

$$LCR = \frac{2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \bar{f}_{ij}}{n(n-1)}$$

where  $\bar{f}_{ij} = \begin{cases} 0, & l_{h_{ij}^{(0)}} = l_{h_{ij}^*} \\ 1, & \text{otherwise} \end{cases}$  denotes whether the length of an

HME is changed;  $H^{(0)} = (h_{ij}^{(0)})$  and  $H^* = (h_{ij}^*)$  are the original and the final adjusted HMPRs, respectively.

(2) Numerical adjustment ratio (NAR)

$$NAR = \frac{2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n f_{ij}}{n(n-1)}$$

where  $f_{ij} = \begin{cases} 0, & h_{ij}^{(0)} = h_{ij}^* \\ 1, & \text{otherwise} \end{cases}$  denotes whether the values in an

HME  $h_{ij}^{(0)}$  is adjusted.

(3) Absolute deviation (AD)

$$AD = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n |h_{ij}^{(0)} - h_{ij}^*|$$

AD measures the average numerical difference between the original HMPR  $H^{(0)}$  and the final improved HMPR  $H^*$ . Since the lengths in each  $h_{ij}$  between  $H^{(0)}$  and  $H^*$  are different in Zhang and Wu [17] and Zhang and Wu [21], the  $\beta$ -normalization HMPR is used on the original HMPR to compute AD for the proposed approach.

(4) Logarithm absolute deviation (LAD)

$$LAD = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\ln(h_{ij}^{(0)}) - \ln(h_{ij}^*))^2$$

(5) Difference ratio (DR). Li et al. [52] introduced a ratio-based concept to gauge the difference between two interval multiplicative comparison matrices. Based on this idea, the below DR is proposed to measure the difference between the original HMPR and the improved HMPR.

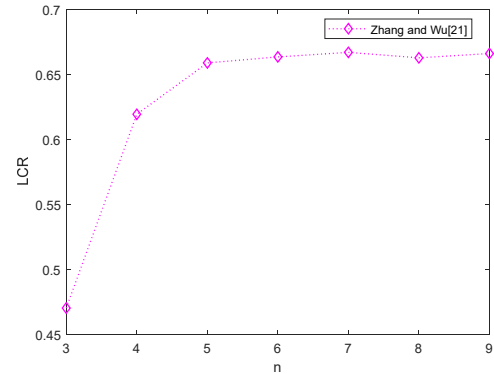
$$DR(H^{(0)}, H^*) = \left( \prod_{i < j} \left( \frac{\max\{h_{ij}^{(0)-}, h_{ij}^{*-}\}}{\min\{h_{ij}^{(0)-}, h_{ij}^{*-}\}} \right) \left( \frac{\max\{h_{ij}^{(0)+}, h_{ij}^{*+}\}}{\min\{h_{ij}^{(0)+}, h_{ij}^{*+}\}} \right) \right)^{\frac{1}{n(n-1)}}$$

Obviously,  $DR(H^{(0)}, H^*) \geq 1$ . The smaller the ratio  $DR(H^{(0)}, H^*)$ , the closer  $H^{(0)}$  is to  $H^*$ . In particular, if  $DR(H^{(0)}, H^*) = 1$ ,  $H^{(0)} = H^*$ .

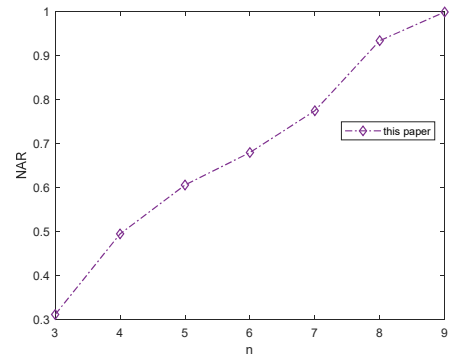
Table V lists the average values of LCR, NAR, AD, LAD and DR for each of the three considered methods, which are represented in Fig. 3 to help visualize the different methods' performance.

TABLE V  
THE AVERAGE LCR, NAR, AD, LAD, DR AND ITERATION VALUES OF  
DIFFERENT METHODS

$n$	methods	LCR	NAR	AD	LAD	DR	Iterations
3	This paper	0	0.149	0.122	0.440	1.227	1
	Zhang and Wu [21]	0.4863	1	3.867	1.908	2.076	2.973
	Zhang and Wu [17]	0.4863	1	2.703	1.319	1.726	3.078
4	This paper	0	0.301	0.283	0.986	1.4223	1
	Zhang and Wu [21]	0.6213	1	4.685	2.967	2.3226	3.033
	Zhang and Wu [17]	0.6213	1	3.965	1.992	2.0718	4.557
5	This paper	0	0.436	0.407	1.46	1.603	1
	Zhang and Wu [21]	0.6565	1	5.013	3.599	2.5201	3.051
	Zhang and Wu [17]	0.6565	1	4.477	2.504	2.2276	5.257
6	This paper	0	0.518	0.524	1.848	1.743	1
	Zhang and Wu [21]	0.6684	1	5.370	4.098	2.7102	3.078
	Zhang and Wu [17]	0.6684	1	4.928	2.913	2.4092	5.652
7	This paper	0	0.636	0.603	1.963	1.7717	1
	Zhang and Wu [21]	0.6606	1	5.552	4.369	2.8243	3.123
	Zhang and Wu [17]	0.6606	1	5.114	3.120	2.5068	5.848
8	This paper	0	0.836	0.699	2.464	1.9121	1
	Zhang and Wu [21]	0.6669	1	5.654	4.611	2.9306	3.153
	Zhang and Wu [17]	0.6669	1	5.212	3.294	2.5901	5.939
9	This paper	0	0.991	0.775	2.888	2.0213	1
	Zhang and Wu [21]	0.6661	1	5.740	4.779	3.0114	3.2580
	Zhang and Wu [17]	0.6661	1	5.265	3.377	2.6375	5.972



(a)



(b)



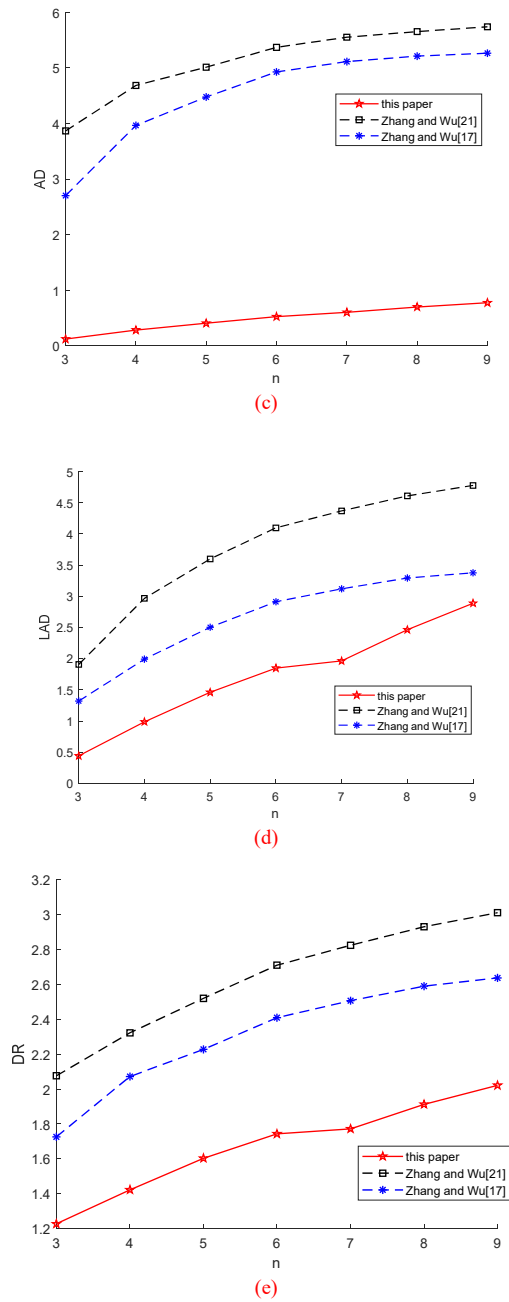


Fig 3. (a) LCR of Zhang and Wu [17] (b) NAR of this paper (c) AD (d) LAD (e) DR

In Fig. 3(a), since the length of HMEs is not changed by the proposed method, the corresponding LCR value is always equal to 0 at every one of the considered dimensions. Zhang and Wu [21] and Zhang and Wu [17] used the same normalization methods, thus their LCR values coincide and therefore there is only need to draw the LCR values for one of them. The LCR values increase drastically from 3 to 5, while they change little when  $n$  is from 5 to 9. In Fig. 3(b), the NAR values in Zhang and Wu [17] and Zhang and Wu [21] are always equal to 1, which means that all the values are revised in their consistency improving processes. However, the NAR values increases from 0.149 ( $n=3$ ) to 0.991 ( $n=9$ ) for the proposed method. These two indexes show that the proposed method perform best in

retain the decision-makers' original information as much as possible.

In Fig 3. (c)-(e), the AD, LAD, and DR values all increase with the value of  $n$ . However, in all cases, the proposed method results in the smallest values, with Zhang and Wu [21] resulting in the largest. Therefore, the proposed method produces improved consistent HMPRs closest to the original HMPRs. These results reinforce the achievement of the aim of the proposed method to retain the decision-makers' original information as much as possible.

3) Computational complexity

Regarding computational complexity as measured by the average number of iterations required to complete the overall process, again the proposed method is superior to the method by Zhang and Wu. This information is provided in the last column of Table V and depicted in Fig. 4. From The proposed method requires in all cases 1 iteration to improve consistency, Zhang and Wu's [21] method is stable at 3 iterations on average, while Zhang and Wu's [17] method need 3 to 6 iterations on average increasing with the dimension value.

As mentioned earlier, Zhang and Wu [21] require that all HMEs have the same length before the process of consistency ascertaining. This normalization method, and therefore its associated complicated calculation process, is superfluous for the proposed method. Since the length of HMEs increases with the normalization process, the HMPR will be converted into a high number of MPRs to judge its consistency, which will increase the computation cost when compared to the proposed method. On the other hand, Zhang and Wu [21] preset a consistency threshold in the process of consistency checking and improvement. Decreasing the threshold value implies an increase of the number of iterations and, as a consequence, the computational cost will increase. In contrast, the consistency properties of HMPRs proposed in this paper can directly be ascertain without the need of a normalization process or a consistency threshold, while the inconsistency repairing method only revises the inconsistent elements, and therefore most of the decision-maker's judgments are unchanged. Most importantly, the proposed approach can achieve multiplicative consistency ascertaining, inconsistency repairing, and weights derivation for HMPRs in one iteration.

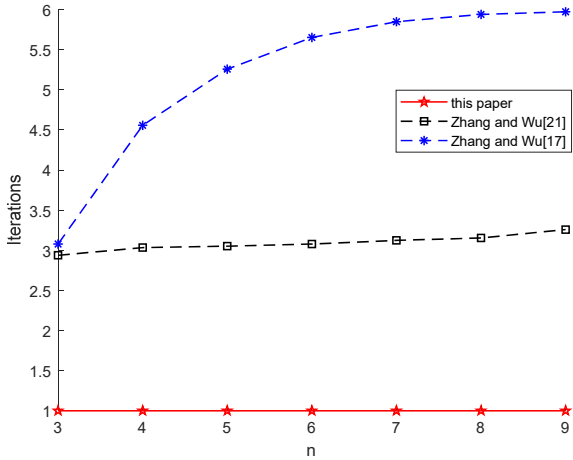


Fig. 4. The average iterations of different methods

Meng et al. [25] also implement the consistency test based on the decision-maker's original HMPR without the normalization process. Their consistency determination and improvement process can also be completed within one iteration. However, Meng et al.'s approach requires to detect that for each value in each HME a multiplicatively consistent MPR needs to be detected. As the length of the HMEs increases, the number of multiplicatively consistent MPRs to be found increases.

Namely, there are a total of  $\prod_{i < j} l_{h_{ij}}$  MPRs that need to be judged, and at least the minimum of  $l_{h_{ij}}$  models to operate.

Hence, this method may not be suitable to be applied in practical decision-making problems due to its high computational cost

Summarizing, when compared with the existing methods, the proposed method has lowest computational complexity and cost. Therefore, the proposed method is a highly functional and computationally convenient method.

## VIII. CONCLUSIONS

In this paper, two types of multiplicative consistency of HMPRs, **completely multiplicative consistency** and **weakly multiplicative consistency**, are investigated simultaneously. A number of 0-1 mixed programming models are established to ascertain these consistency properties. The following cases are addressed:

(1) If an HMPR is CMC, then the corresponding multiplicative consistent MPR can be found.

(2) If an HMPR is not CMC but WMC, then interval priority weights of alternatives are derived, which allows to rank them

(3) If an HMPR is not consistent, only the inconsistent elements are revised to repair the inconsistency, which means that most of the **decision-maker's** judgements are unchanged.

(4) These models have also been extended to the case of incomplete HMPRs by estimating the missing values.

In future, the research areas to focus on include:

1) How to apply the proposed method to other types of **preference relations** [53, 54].

2) In addition to the consistency analysis of individual **decision-makers**, **consensus analysis with HMPRs is also an important research topic in group decision making** [55-59].

3) Investigate new algorithms for **group decision making** problems to tackle practical problems [60-62].

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