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## Abstract

This paper proposes a testing procedure in order to distinguish between the case where the volatility of an asset price is a deterministic function of the price itself and the one where it is a function of one or more (possibly unobservable) factors, driven by not perfectly correlated Brownian motions. Broadly speaking, the objective of the paper is to distinguish between a generic one-factor model and a generic stochastic volatility model. In fact, no specific assumption on the functional form of the drift and variance terms is required.

The proposed tests are based on the difference between two different nonparametric estimators of the integrated volatility process. Building on some recent work by Bandi and Phillips (2003) and Barndorff-Nielsen and Shephard (2004a), it is shown that the test statistics converge to a mixed normal distribution under the null hypothesis of a one factor diffusion process, while diverge in the case of multifactor models. The findings from a Monte Carlo experiment indicate that the suggested testing procedure has good finite sample properties.

|               |                                                                        |
|---------------|------------------------------------------------------------------------|
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|               | local times, occupation densities, mixed normal distribution           |
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# 1 Introduction

In finance the dynamic behavior of underlying economic variables and asset prices has been often described using one-factor diffusion models, where volatility is a deterministic function of the level of the underlying variable.[1]

Since determining the functional form of such diffusion processes is particularly important for pricing contingent claims and for hedging purposes, several specification tests have been proposed, within the class of one-factor models.

Examples include Aït-Sahalia (1996), who compares the parametric density implied by a given null model with a nonparametric kernel density estimator. He rejects most of the commonly employed models and argues that rejections are mainly due to nonlinearity in the drift term.[2] Similar findings to those of Aït-Sahalia (1996) have been also provided by Stanton (1997) and Jiang (1998). Durham (2003) also rejects most of the popular models; in his case rejections are mainly due to misspecification of the volatility term. In particular, he finds implausibly high values for the elasticity parameter in the Constant Elasticity of Variance (CEV) model, implying violation of the stationarity assumption. Bandi (2002) applies fully nonparametric estimation of the drift and variance diffusion terms, based on the spatial methodology of Bandi and Phillips (2003), and finds that the drift term is very close to zero over most of the range of the short term interest rate. Therefore, rejections of a given model seem to be due to failure of the mean reversion property rather than to nonlinearity in the drift term. Qualitatively similar findings are obtained by Conley, Hansen, Luttmer and Scheinkman (1997), using generalized method of moments tests based on the properties of the infinitesimal generator of the diffusion.[3]

Most of the papers cited above have suggested testing and modeling procedures which are valid under the maintained hypothesis of a one-factor diffusion data generating process. Hence, the need of testing for the validity of the whole class of one-factor models.

This is the objective of the paper. Under minimal assumptions, the paper proposes a testing procedure in order to distinguish between the case in which the volatility process is a deterministic function of the level of the underlying variable and the one in which it is a function of one or more

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[1]Although in the financial literature there is a somewhat widespread consensus about the fact that stock prices are better characterized by multifactor stochastic volatility models, short term interest rates are still often modeled as a one-factor diffusion process, in which volatility is a deterministic function of the level of the variable (see e.g. Vasicek, 1977, Brennan and Schwartz, 1979, Cox, Ingersoll and Ross, 1985, Chan, Karolyi, Longstaff and Sanders, 1992, Pearson and Sun, 1994).

[2]Aït-Sahalia (1996) does not reject a generalized version of the Constant Elasticity of Variance model. His results have been revisited by Pritsker (1998), who points out the sensitivity of Aït-Sahalia's test to the degree of dependency in the short interest rate process.

[3]See also the comprehensive review on estimation of one-factor models by Fan (2003).

(possibly unobservable) factors, driven by not perfectly correlated Brownian motions. With a slight abuse of terminology, the former class of models is referred to as one-factor models and the latter as stochastic volatility models.[4] In particular, the paper compares generic classes of one-factor versus stochastic volatility models, without making assumptions on the functional forms of either the drift or the variance component.

If the null hypothesis is not rejected, then one can use the different testing and modeling procedures mentioned above, based on the maintained hypothesis of a one-factor diffusion generating process. Conversely, if the null hypothesis is rejected, then one has to perform model diagnostics within the class of stochastic volatility models, using for example the efficient method of moments (e.g. Chernov, Gallant, Ghysels and Tauchen, 2003), or generalized moment tests based on the properties of the infinitesimal generator of the diffusion (see e.g. Corradi and Distaso, 2004). For example, one can test the validity of multi factor term structure models, suggested by e.g. Duffie and Singleton (1997), Dai and Singleton (2000, 2002).

The suggested test statistics are based on the difference between a kernel estimator of the instantaneous variance, averaged over the sample realization on a fixed time span, and realized volatility. The intuition behind the chosen statistic is the following: under the null hypothesis of a one-factor model, both estimators are consistent for the underlying integrated volatility; under the alternative hypothesis the former estimator is not consistent, while the latter is. More precisely, building on some recent work by Bandi and Phillips (2003) and Barndorff-Nielsen and Shephard (2004a), it is shown that the statistics weakly converge to mixed normal distributions under the null hypothesis and diverge at an appropriate rate under the alternative. The derived asymptotic theory is based on the time interval between successive observations approaching zero, while the time span is kept fixed. As a consequence, the limiting behavior of the statistic is not affected by the drift specification. Also, no stationarity or ergodicity assumption is required.

The proposed testing procedure is derived under the assumptions that the underlying variables are observed without measurement error and that the generating processes belong to the class of continuous semimartingales. Therefore, the provided tests are not robust to the presence of either jumps or market microstructure effects; more precisely, when either of the two occur, the test tends to reject the null hypothesis, even if the volatility process is a deterministic function of the underlying variable. However, as the test is computed over a finite time span, one can first test for the hypotheses of no jumps and no microstructure effects, and then perform the suggested testing procedure over a time span in which neither of the hypotheses above is rejected.

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[4]In the stochastic volatility literature, often by one-factor model one means a model in which volatility is a function of a single stochastic factor, driven by a Brownian motion not perfectly correlated with the one driving the underlying economic variable or the asset price.

The rest of this paper is organized as follows. In Section 2, the testing procedure is outlined and the relevant limit theory is derived. Section 3 reports the findings from a Monte Carlo exercise, in order to assess the finite sample behavior of the proposed tests. Concluding remarks are given in Section 4. All the proof are gathered in the Appendix.

In this paper, $\overset{p}{\longrightarrow}$, $\overset{d}{\longrightarrow}$ and $\overset{a.s.}{\longrightarrow}$ denote respectively convergence in probability, in distribution and almost sure convergence. We write $1_{\{\cdot\}}$ for the indicator function, $\lfloor \varpi \rfloor$ for the integer part of $\varpi$, $\mathbf{I}_J$ for the identity matrix of dimension $J$ and $Z \sim \mathrm{MN}\left(\cdot, \cdot\right)$ to denote that the random variable $Z$ is distributed as a mixed normal.

## 2 Testing for One-Factor vs Stochastic Volatility Models

### 2.1 Set-Up

As discussed above, our objective is to device a data driven procedure for deciding between one-factor diffusion models and stochastic volatility models, under minimal assumptions.

We consider the following class of one-factor diffusion models

$$
\begin{aligned}
\mathrm{d}X_t &= \mu(X_t)\mathrm{d}t + \sigma_t \mathrm{d}W_{1,t} \\
\sigma_t &= \sigma\left(X_t\right)
\end{aligned}
\tag{1}
$$

and the following class of stochastic volatility models

$$
\begin{aligned}
\mathrm{d}X_t &= \mu(X_t)\mathrm{d}t + \sigma_t \mathrm{d}W_{1,t} \\
\sigma_t^2 &= g(f_t) \\
\mathrm{d}f_t &= b(f_t)\mathrm{d}t + \sigma_1(f_t)\mathrm{d}W_{2,t},
\end{aligned}
\tag{2}
$$

where $f_t$ is typically an unobservable state variable driven by a Brownian motion, $W_{2,t}$, possibly but not perfectly correlated with the Brownian motion driving $X_t$, thus allowing for possible leverage effects.

The models in (1) encompass the class of parametric specifications analyzed by Aït-Sahalia (1996), and they also allows for generic nonlinearities. The models in (2) include the square root stochastic volatility of Heston (1993), the Garch diffusion model (Nelson, 1990), the lognormal stochastic volatility model of Hull and White (1987) and Wiggins (1987), and are also related to the class of eigenfuction stochastic volatility models of Meddahi (2001). Note that $f_t$ may be a multidimensional process, thus allowing for multifactor stochastic volatility processes. Also, the one-factor model may be possibly nested within the stochastic volatility model, in the sense that we can allow for the specification $\sigma_t^2 = \sigma^2\left(X_t\right)g(f_t)$. Andersen and Lund (1997) and Durham (2003)

propose to extend the different one-factor models by adding a stochastic volatility term, and suggest models in which volatility depends on both the level of the underlying variable and a latent factor, driven by a different Brownian motion.[5]

In particular, it should be stressed that in our procedure we compare generic classes of one factor versus stochastic volatility models, without any functional form assumption on either the drift or the variance term.

We state the hypothesis of interest as

$$H_0 : \sigma_t^2 = \sigma^2 (X_t), \quad \text{a.s.}$$

versus the alternative

$$H_A : \sigma_t^2 = g (f_t), \quad \text{a.s.}$$

where $\forall \omega \in \Omega^+, \left| \int_0^1 (g (f_s) - g (X_s)) \, ds \right| \neq 0$ and $\Pr (\Omega^+) = 1$, with $\Omega^+ \in \Omega$, and $\Omega$ denotes the probability space on which $(f_t, X_t)$ are defined.

Thus, under the null hypothesis the volatility process is a measurable function of the return process $X_t$. On the other hand, under the alternative, the volatility process is a measurable function of a possibly unobservable process $f_t$. In the paper, we simply require that the occupation densities of the observable process $X_t$ and of the (possibly) unobservable factor $f_t$ do not coincide. In fact, if they do coincide, then the integrated volatility process would be almost surely the same under both hypotheses. Finally, note that the case of $\sigma_t^2 = \sigma^2 (X_t) g(f_t)$ falls under the alternative hypothesis, while the case of a constant variance falls under the null.

In the sequel, we assume that we have data recorded at two different frequencies, over a fixed time span, which for sake of simplicity, but without loss of generality, is assumed equal to 1.[6] More specifically, we assume to have $n$ and $m$ observations, with $m \leq n$, so that the discrete sampling interval is equal respectively to $1/n$ and $1/m$.

The proposed test statistics are based on

$$Z_{n,m,r} = \sqrt{m} \left( \frac{1}{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} S_n^2(X_{i/n}) - RV_{m,r} \right), \tag{3}$$

where $r \in (0, 1]$,

$$S_n^2(X_{i/n}) = \frac{\sum_{j=1}^{n-1} 1_{\left\{ |X_{j/n} - X_{i/n}| < \xi_n \right\}} n \left( X_{(j+1)/n} - X_{j/n} \right)^2}{\sum_{j=1}^{n-1} 1_{\left\{ |X_{j/n} - X_{i/n}| < \xi_n \right\}}} \tag{4}$$

---

[5]Andersen and Lund (1997) find that the inclusion of a stochastic volatility component in a square root model helps the elasticity parameter to fall in the stationary region. Durham (2003) finds that, although the addition of a second factor increases the likelihood, it has very little impact as to what concerns bond pricing.

[6]In Section 3, reporting the results of the simulation study, we will consider a time span equal to five days.

and

$$RV_{m,r} = \sum_{j=1}^{\lfloor (m-1)r \rfloor} \left( X_{(j+1)/m} - X_{j/m} \right)^2. \tag{5}$$

Note that $S_n^2(X_{i/n})$ is a nonparametric estimator of the volatility process evaluated at $X_{i/n}$; Florens-Zmirou (1993) has established consistency and the asymptotic distribution of a scaled version of (4) when the variance process follows (1).[7] Recently, $S_n^2(X_{i/n})$ has been used by Bandi and Phillips (2003), in the context of fully nonparametric estimation of diffusion processes; their asymptotic theory is based on both the time span going to infinity and the discrete interval between successive observations going to zero. This is because they are interested in the joint estimation of the drift and variance diffusion terms.[8]

Conversely, our objective is to distinguish between the cases in which volatility is a measurable function of the observable process, and the one in which it depends on some other state variable. Therefore we remain silent about the drift term, and we only consider asymptotic theory in terms of the discrete interval approaching zero. In fact, on a finite time span the contribution of the drift term is asymptotically negligible.

Notice that $S_n^2(X_{i/n})$ is a consistent estimator of the instantaneous variance only under the null hypothesis. Therefore, also its average over the sample realization of the process on a finite time span, $1/n \sum_{i=1}^{\lfloor (n-1)r \rfloor} S_n^2(X_{i/n})$, is a consistent estimator of integrated volatility only under the null hypothesis.

$RV_{m,r}$, which is known as realized volatility, has been proposed as a measure for volatility concurrently by Andersen, Bollerslev, Diebold and Labys (2001), Andersen, Bollerslev, Diebold and Ebens (2002) and Barndorff-Nielsen and Shephard (2002). The properties of realized volatility have been extensively analyzed by Barndorff-Nielsen and Shephard (2002, 2004a,b), Andersen, Bollerslev, Diebold and Labys (2003), Barndorff-Nielsen, Graversen and Shephard (2004) (see also Andersen, Bollerslev, Meddahi, 2004a,b, and Meddahi, 2002, 2003). Realized volatility is a "model free" estimator of the quadratic variation of the processes defined in (1) and (2), and is consistent for the integrated (daily) volatility under both hypotheses. Barndorff-Nielsen and Shephard (2004a) have shown that a scaled and centered version of $RV_{m,r}$ weakly converges to a mixed normal distribution when the log price process follows a continuous semimartingale, a result which we will use in the proof of our Theroem 1. The reason why we use two different sample frequencies in the

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[7]The estimator $S_n^2(X_{i/n})$ has been also used by Corradi and White (1999) in order provide a test for the correct specification of the variance process, regardless of the drift specification. Within the class of one-factor models, a more general test, also allowing for time non-homogeneity, has been suggested by Dette, Podolskij and Vetter (2004).

[8]Bandi and Phillips (2003) consider a slightly modified version of $S_n^2(X_{i/n})$, with a generic kernel $K(\cdot)$ replacing the indicator function. See also Jiang and Knight (1997).

computation of $S_n^2(X_{i/n})$ and $RV_{m,r}$ will become clear in the next subsection.

In the sequel we shall need the following assumption.

**Assumption 1.**

(a) $\sigma(\cdot)$ and $\mu(\cdot)$, defined in (1), satisfy local Lipschitz and growth conditions. Therefore, for any compact subsets $M$ (under the null hypothesis) and $J$ (under the alternative hypothesis) of the range of the process $X_t$, there exist constants $K_1^M$, $K_2^M$, $K_3^M$, $K_4^M$, $K_1^J$ and $K_2^J$, such that, $\forall (x, y) \in M$ and $\forall (x', y') \in J$,

$$|\sigma(x) - \sigma(y)| \leq K_1^M |x - y|,$$

$$|\sigma(x)|^2 \leq K_2^M (1 + |x|^2),$$

$$|\mu(x) - \mu(y)| \leq K_4^M |x - y|, \quad |\mu(x') - \mu(y')| \leq K_2^J |x' - y'|$$

and

$$x\mu(x) \leq K_3^M (1 + |x|^2), \quad x'\mu(x') \leq K_1^J (1 + |x'|^2).$$

(b) $\sigma_1(\cdot)$ and $b(\cdot)$, defined in (2), satisfy local Lipschitz and growth conditions. Therefore, for any compact subset $L$ of the range of the process $f_t$, there exist constants $K_1^L$, $K_2^L$, $K_3^L$ and $K_4^L$, such that, $\forall (p, q) \in L$,

$$|\sigma_1(p) - \sigma_1(q)| \leq K_1^L |p - q|,$$

$$|\sigma_1(p)|^2 \leq K_2^L (1 + |p|^2),$$

$$|b(p) - b(q)| \leq K_3^L |p - q|$$

and

$$pb(p) \leq K_4^L (1 + |p|^2).$$

(c) $\mu(\cdot), \sigma(\cdot)$ and $g(\cdot)$ are continuously differentiable.

Assumption 1(a) states local Lipschitz and growth conditions for the drift term under both hypotheses and for the variance term under the null hypothesis. Assumption 1(b) states local Lipschitz and growth conditions for the variance term under the alternative. Assumptions 1(a)(b) ensure the existence of a unique strong solution under both hypotheses (see e.g. Chung and Williams, 1990, p.229). Since we are studying the diffusion processes over a fixed time span, we do not need to impose more demanding assumptions, such as stationarity and ergodicity.[9]

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[9]Note that Bandi and Phillips (2001, 2003) allow the time span to approach infinity, and then require the diffusion to be null Harris recurrent.

## 2.2 Limiting Behavior of the Statistic

We can now establish the limiting distribution of the proposed test statistics based on $Z_{n,m,r}$, defined in (3), for both the cases where $n = m$ and $m/n \to 0$, as $m, n \to \infty$.

**Theorem 1.** *Let Assumption 1 hold.*
*Under* $H_0$,

(i)a *if, as $n, m, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$ and for any arbitrarily small $\varepsilon > 0$, $n^{1/2+\varepsilon}\xi_n \to 0$, and if $m = n$, then, pointwise in $r \in (0,1)$*

$$Z_{n,r} \xrightarrow{d} Z_r \sim \mathrm{MN}\left(0, 2\int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r,a)\,(L_X(1,a) - L_X(r,a))}{L_X(1,a)}\mathrm{d}a\right), \tag{6}$$

*where $Z_{n,r} \equiv Z_{n,n,r}$ and*

$$L_X(r,a) = \lim_{\psi \to 0} \frac{1}{\psi}\frac{1}{\sigma^2(a)}\int_0^r \mathbf{1}_{\{X_u \in [a, a+\psi]\}}\sigma^2(X_u)\mathrm{d}u$$

*denotes the standardized local time of the process $X_t$.*

(i)b *Define $Z_n = \max_{j=1,\dots,J}\left|Z_{n,r_j}\right|$ and $Z = \max_{j=1,\dots,J}\left|Z_{r_j}\right|$, where $0 < r_1 < \dots < r_{j-1} < r_j < \dots < r_J < 1$, for $j = 1, \dots, J$, with $J$ arbitrarily large but finite. If, as $n, m, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$, and, for any $\varepsilon > 0$ arbitrarily small, $n^{1/2+\varepsilon}\xi_n \to 0$, and if $m = n$, then*

$$Z_n \xrightarrow{d} Z,$$

*with*

$$\begin{pmatrix} Z_{r_1} \\ Z_{r_2} \\ \vdots \\ Z_{r_J} \end{pmatrix} \sim \mathrm{MN}\left(\mathbf{0}, \begin{pmatrix} V(r_1, r_1) & V(r_1, r_2) & \dots & V(r_1, r_J) \\ V(r_2, r_1) & V(r_2, r_2) & \dots & V(r_2, r_J) \\ \vdots & \vdots & \ddots & \vdots \\ V(r_J, r_1) & V(r_J, r_2) & \dots & V(r_J, r_J) \end{pmatrix}\right), \tag{7}$$

*where $\forall\, r, r'$,*

$$V(r, r') = V(r', r) = 2\int_{-\infty}^{\infty} \sigma^4(a)\frac{L_X(\min(r,r'),a)\,(L_X(1,a) - L_X(\min(r,r'),a))}{L_X(1,a)}\mathrm{d}a.$$

(i)c *If, as $n, m, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$ and $n\xi_n^2 \to 0$, and, for any $\varepsilon > 0$ arbitrarily small, $m/n^{1-\varepsilon} \to 0$, then*

$$Z_{n,m,r} \xrightarrow{d} ZM_r \sim \mathrm{MN}\left(0, 2\int_{-\infty}^{\infty} \sigma^4(a)\,L_X(r,a)\mathrm{d}a\right).$$

*(i)d Define* $Z_{n,m} = \max_{j=1,\dots,J} \left| Z_{n,m,r_j} \right|$ *and* $ZM = \max_{j=1,\dots,J} \left| ZM_{r_j} \right|$, *where* $0 < r_1 < \dots <$ $r_{j-1} < r_j < \dots < r_J < 1$, *for* $j = 1,\dots,J$, *with* $J$ *arbitrarily large but finite. If, as* $n, m, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$ *and* $n\xi_n^2 \to 0$, *and, for any* $\varepsilon > 0$ *arbitrarily small,* $m/n^{1-\varepsilon} \to 0$, *then*

$$Z_{n,m} \xrightarrow{d} ZM,$$

*with*

$$
\begin{pmatrix} ZM_{r_1} \\ ZM_{r_2} \\ \vdots \\ ZM_{r_J} \end{pmatrix} \sim \mathrm{MN} \left( \mathbf{0}, \begin{pmatrix} VM(r_1,r_1) & VM(r_1,r_2) & \dots & VM(r_1,r_J) \\ VM(r_2,r_1) & VM(r_2,r_2) & \dots & VM(r_2,r_J) \\ \vdots & \vdots & \ddots & \vdots \\ VM(r_J,r_1) & VM(r_J,r_2) & \dots & VM(r_J,r_J) \end{pmatrix} \right), \tag{8}
$$

*where that,* $\forall\, r, r'$,

$$VM(r,r') = VM(r',r) = 2 \int_{-\infty}^{\infty} \sigma^4(a)\, L_X(\min(r,r'),a)\mathrm{d}a.$$

*(ii) Under* $\mathrm{H}_A$, *if, as* $n, m, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$ *and* $n\xi_n^2 \to 0$, *and if* $m/n \to \pi \geq 0$, *then, pointwise in* $r \in (0,1]$,

$$\mathrm{Pr} \left( \omega : \frac{1}{\sqrt{m}} |Z_{n,m,r}(\omega)| \geq \varsigma(\omega) \right) \to 1,$$

*where* $\varsigma(\omega) > 0$ *for all* $\omega \in \Omega^+$, *where* $\Omega^+$ *is defined as in the statement of* $H_A$.

Notice that, as shown in the proof in the Appendix, under the alternative hypothesis, and in the case where $f_t$ is a one-dimensional process, the dominant term of the proposed statistic is a scaled version of the absolute value of the difference between the local times of $X_t$ and $f_t$. If instead $f_t$ is a multidimensional process, then the multivariate local time analogue of the $L_f(1,a)$ used in Theorem 1 is not defined, but it can still be interpreted as a occupation density of the multivariate diffusion $f_t$ (see e.g. Geman and Horowitz, 1980 and Bandi and Moloche, 2001). Therefore, in both cases, there exists an (almost surely) strictly positive random variable $\varsigma$, such that $(1/\sqrt{m}) |Z_{m,n,r}| \geq \varsigma$, with probability approaching one.

The following Corollary considers the case where $r = 1$, i.e. when we use the whole span of data in constructing the test statistic.

**Corollary 1.** *Let Assumption 1 hold. Under* $\mathrm{H}_0$, *if, as* $n, m, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$ *and* $n\xi_n^2 \to 0$, *and, for any* $\varepsilon > 0$ *arbitrarily small,* $m/n^{1-\varepsilon} \to 0$, *then*

$$Z_{n,m,1} \xrightarrow{d} \mathrm{MN} \left( 0, 2 \int_{-\infty}^{\infty} \sigma^4(a)\, L_X(1,a)\mathrm{d}a \right).$$

Thus, for $r = 1$, the statistic has a mixed normal limiting distribution for $m/n \to 0$ as $m, n \to \infty$.[10]

The theoretical results derived above provide an unfeasible limit theory, since the variance components have to be estimated. A consistent estimator of the standardized local time is given by

$$\widehat{L}_{X,n}(r, a) = \frac{1}{2n\xi_n} \frac{1}{S_n^2(a)} \sum_{i=1}^{\lfloor (n-1)r \rfloor} 1_{\{|X_{i/n} - a| < \xi_n\}}.$$

Thus an estimator of

$$2 \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r, a)\left(L_X(1, a) - L_X(r, a)\right)}{L_X(1, a)} \mathrm{d}a, \tag{9}$$

i.e. of the quantity resulting in Theorem 1 part (i)a, is given by

$$\int_{\Delta_1}^{\Delta_2} \widehat{\sigma}_n^4(a) \frac{\widehat{L}_{X,n}(r, a)\left(\widehat{L}_{X,n}(1, a) - \widehat{L}_{X,n}(r, a)\right)}{\widehat{L}_{X,n}(1, a)} \mathrm{d}a, \tag{10}$$

where

$$\widehat{\sigma}_n^4(a) = \frac{\sum_{i=1}^{n-1} 1_{\{|X_{i/n} - a| < \xi_n\}} n^2 \left(X_{(i+1)/n} - X_{i/n}\right)^4}{\sum_{i=1}^{n-1} 1_{\{|X_{i/n} - a| < \xi_n\}}}.$$

In order to implement the estimator in (10), we need to choose the interval of integration, $\Delta = (\Delta_1, \Delta_2)$. Now, if we choose $\Delta$ too small, then we may run the risk of getting an inconsistent estimator of the term in (9). On the other hand, if we choose $\Delta$ too large, then for some $a \in \Delta$, $\widehat{L}_{X,n}(r, a)$ and $\widehat{L}_{X,n}(1, a)$ would be very close to zero, and the estimator in (10) will result in a ratio of two terms approaching zero.

Of course, when computing (10) we can exclude all $a \in \Delta$ for which, say, $\widehat{L}_{X,n}(1, a) \le \delta_n$, where $\delta_n \to 0$ as $n \to \infty$. However, devicing a data-driven procedure for choosing $\delta_n$ is not an easy task. In order to avoid this problem, we instead propose below an upper bound for the critical values of the limiting distribution in Theorem 1, parts (i)a and (i)b.

In fact, note that almost surely,

$$2 \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r, a)\left(L_X(1, a) - L_X(r, a)\right)}{L_X(1, a)} \mathrm{d}a$$
$$\le 2 \int_{-\infty}^{\infty} \sigma^4(a) L_X(r, a) \mathrm{d}a \equiv 2 \int_0^r \sigma^4(X_s) \mathrm{d}s,$$

where the last equality above follows from Lemma 3 in Bandi and Phillips (2003).

Now, Barndorff-Nielsen and Shephard (2002) have shown that

$$\frac{n}{3} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left(X_{(i+1)/n} - X_{i/n}\right)^4 \xrightarrow{p} \int_0^r \sigma_s^4 \mathrm{d}s, \tag{11}$$

---

[10]When $m = n$ and $r = 1$, the statistic converges to zero in probability.

where $\sigma_s^4 = \sigma^4(X_s)$ under $H_0$ and $\sigma_s^4 = g^2(f_s)$ under $H_A$; in other words the estimator defined in (11) is consistent for the "true" integrated quarticity under both hypotheses and therefore provides an estimator of the upper bound of the term in (9).

On the other hand, we shall provide correct asymptotic critical values for the limiting distribution in Theorem 1, parts (i)c and (i)d and in Corollary 1. In order to obtain asymptotically valid critical values and to make the limit theory derived in Theorem 1 part (i)d feasible, we will use a data-dependent approach. For $s = 1, \ldots, S$, where $S$ denotes the number of replications, let

$$\widehat{\mathbf{d}}_{m,r}^{(s)} = \begin{pmatrix} \widehat{d}_{m,r_1}^{(s)} \\ \widehat{d}_{m,r_2}^{(s)} \\ \vdots \\ \widehat{d}_{m,r_J}^{(s)} \end{pmatrix} = \begin{pmatrix} \widehat{C}_m(r_1, r_1) & \widehat{C}_m(r_1, r_1) & \widehat{C}_m(r_1, r_1) & \widehat{C}_m(r_1, r_1) \\ \widehat{C}_m(r_1, r_1) & \widehat{C}_m(r_2, r_2) & \widehat{C}_m(r_2, r_2) & \widehat{C}_m(r_2, r_2) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{C}_m(r_1, r_1) & \widehat{C}_m(r_2, r_2) & \ldots & \widehat{C}_m(r_J, r_J) \end{pmatrix}^{1/2} \begin{pmatrix} \eta_1^{(s)} \\ \eta_2^{(s)} \\ \vdots \\ \eta_J^{(s)} \end{pmatrix}, \quad (12)$$

where

$$\widehat{C}_m(r_j, r_j) = \frac{2}{3} \sum_{i=1}^{\lfloor (m-1)r_j \rfloor} m \left( X_{(i+1)/m} - X_{i/m} \right)^4$$

is a consistent estimator of twice the integrated quarticity and, for each $s$, $\left( \eta_1^{(s)} \; \eta_2^{(s)} \ldots \eta_J^{(s)} \right)'$ is drawn from a $N(\mathbf{0}, \mathbf{I}_J)$. Then compute $\max_{j=1,\ldots,J} \left| \widehat{\mathbf{d}}_{m,r}^{(s)} \right|$, repeat this step $S$ times, and construct the empirical distribution. As $S \to \infty$, the empirical distribution of $\max_{j=1,\ldots,J} \left| \widehat{\mathbf{d}}_{m,r}^{(s)} \right|$ will converge the distribution of a random variable defined as

$$\max_{j=1,\ldots,J} \left| \mathrm{MN} \left( 0, 2 \int_{-\infty}^{\infty} \sigma^4(a) L_X(r_j, a) \mathrm{d}a \right) \right|.$$

Therefore an asymptotically valid critical value for the limit theory in Theorem 1 part (i)d will be given by $CV_\alpha^S$, which denotes the $(1-\alpha)-$quantile of the empirical distribution of $\max_{j=1,\ldots,J} |\widehat{d}_{m,r_j}^{(s)}|$, computed using $S$ replications. Given the discussion above, $CV_\alpha^S$ will provide an upper bound for the critical values of the limiting distribution derived in Theorem 1, part i(b). The implied rules for deciding between $H_0$ and $H_A$ are outlined in the following Proposition.

**Proposition 1.** *Let Assumption 1 hold.*

(a) *Let $S \to \infty$. Suppose that as $n, m, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$ and, for any $\varepsilon > 0$ arbitrarily small, $n^{1/2+\varepsilon}\xi_n \to 0$. If $m = n$, then do not reject $H_0$ if*

$$Z_n \leq CV_\alpha^S$$

*and reject otherwise. This rule provides a test with asymptotic size smaller than $\alpha$ and asymptotic unit power.*

*(b) Let $S \to \infty$. Suppose that, as $n, m, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$ and $n\xi_n^2 \to 0$, and, for any $\varepsilon > 0$ arbitrarily small, $m/n^{1-\varepsilon} \to 0$; then do not reject $\mathrm{H}_0$ if*

$$Z_{n,m} \leq CV_\alpha^S$$

*and reject otherwise. This rule provides a test with asymptotic size equal to $\alpha$ and asymptotic unit power.*

As mentioned above, our test is designed to compare two classes of models, namely the one-factor diffusion models and the stochastic volatility models, regardless of the specification of the drift term. Therefore, if for example model (1) is augmented by adding another factor into the drift term (see e.g. Hull and White, 1994), our test will still fail to reject the null hypothesis considered, because the drift term is, over a fixed time span, of a smaller order of probability than the diffusion term and so is asymptotically negligible.

## 2.3   Market Microstructures and jumps

The asymptotic theory derived in the previous subsection relies on the fact that the underlying process is a continuous semi-martingale. However, some recent financial literature has pointed out the effects of possible jumps and market microstructure error on realized volatility (see e.g. Barndorff-Nielsen and Shephard, 2004c,d, Corradi and Distaso, 2004, Andersen, Bollerslev and Diebold, 2003 for jumps, and Aït-Sahalia, Mykland and Zhang, 2003, Zhang, Mykland and Aït-Sahalia, 2003, Bandi and Russell, 2003, Hansen and Lunde, 2004 for microstructure noise).

We begin by analyzing the contribution of large and rare jumps. Suppose that the generating process in (1) is augmented by a jump component,

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \mathrm{d}z_t + \sigma_t \mathrm{d}W_{1,t},$$

where $\sigma_t = \sigma(X_t)$, and $z_t$ is a pure jump process.

The test statistics based on $Z_{n,m,r}$ are not robust to the presence of jumps. The intuitive reason is that jumps have a different impact on the two components of the statistics, namely

$$n^{-1} \sum_{i=1}^{\lfloor (n-1)r \rfloor} S_n^2(X_{i/n}) \text{ and } RV_{m,r}.$$

In fact, in the presence of jumps, $RV_{m,r}$ converges to the integrated volatility process plus the sum of the squared magnitudes of the jumps (see Barndorff-Nielsen and Shephard, 2004c). Conversely, $n^{-1} \sum_{i=1}^{\lfloor (n-1)r \rfloor} S_n^2(X_{i/n})$ converges to integrated volatility plus the weighted sum of the squared magnitudes of the jumps, where the weights depend on the local time of $X_t$. Broadly speaking, a

jump occurring at time $\bar{j}/n$ has a larger effect on the component $n^{-1}\sum_{i=1}^{\lfloor(n-1)r\rfloor}S_n^2(X_{i/n})$ if there are many observations in the neighborhood of $X_{\bar{j}/n}$.

However, since our test is carried over a fixed time span, we can pretest for the presence of no jumps, following for example Barndorff-Nielsen and Shephard (2004c,d); they proposed a test based on the properly scaled difference between realized volatility and bipower variation, which is a consistent estimator of integrated volatility in the presence of large and rare jumps in the log price process. If the null hypothesis is not rejected, we can apply our methodology. Huang and Tauchen (2004) also suggest a variety of Hausman type tests for jumps and find evidence of a relatively small number of jumps in the log price process. A similar finding is reported by Andersen, Bollerslev and Diebold (2003).

As for the presence of microstructure effects, suppose that the observed price of an asset can be decomposed into

$$X_{j/m} = Y_{j/m} + \epsilon_{j/m}.$$

Here $\epsilon_{j/m}$ is interpreted as a noise capturing the market microstructure effect. The contribution of the microstructure noise on realized volatility has already been analyzed in a series of recent papers (see e.g. Aït-Sahalia, Mykland and Zhang, 2003, Zhang, Mykland and Aït-Sahalia, 2003, Bandi and Russell, 2003 and Hansen and Lunde, 2004). For example, if the microstructure noise has a constant variance, i.e. independent of the sampling interval, then

$$m^{-1}RV_{m,r} \xrightarrow{p} 2r\nu$$

where $\nu$ denotes the variance of the microstructure noise (see Zhang, Mykland and Aït-Sahalia, 2003). As for $n^{-1}\sum_{i=1}^{\lfloor(n-1)r\rfloor}S_n^2(X_{i/n})$, due to the discreteness of the measurement error component, the behavior of $(n\xi_n)^{-1}\sum_{j=1}^{n-1}1_{\left\{|X_{j/n}-X_{i/n}|<\xi_n\right\}}$ is not easy to assess. Therefore, our procedure will not be valid if the log price process is contaminated by microstructure noise.

Similarly to the case of large and rare jumps, it is possible to pretest the series under investigation for the absence of microstructure noise. In fact, Awartani, Corradi and Distaso (2004) have suggested a simple test for the null hypothesis of no market microstructure, based on the appropriate scaled difference between two realized volatility measures constructed over different sampling frequencies.[11] We can then apply our procedure over a time span for which neither the null hypothesis of no jumps nor the null hypothesis of no microstructure noise has been rejected.

---

[11] Awartani, Corradi and Distaso (2004) also propose a specification test of the null hypothesis of microstructure noise with constant variance. See also Barndorff-Nielsen and Shephard (2004c) for an alternative model of the market microstructure noise, where the variance of the noise is allowed to depend on the sampling frequency of the data.

# 3    A Simulation Experiment

In this section, the small sample performance of the testing procedure proposed in the previous section will be assessed through a Monte-Carlo experiment. Under the null hypothesis, we consider a version of the Cox, Ingersoll and Ross (1985) model with a mean reverting component in the drift,

$$dX_t = (\kappa + \mu X_t)dt + \eta\sqrt{X_t}dW_{1,t}. \tag{13}$$

We first simulate a discretized version of the continuous trajectory of $X_t$ under (13). We use a Milstein scheme in order to approximate the trajectory, following Pardoux and Talay (1985), who provide conditions for uniform, almost sure convergence of the discrete simulated path to the continuous path, for given initial conditions and over a finite time span. In order to get a very precise approximation to the continuous path, we choose a very small time interval between successive observations (1/5760); moreover, the initial value is drawn from the gamma marginal distribution of $X_t$, and the first 1000 observations are then discarded.

We then sample the simulated process at two different frequencies, $1/n$ and $1/m$, and compute the different test statistics. In particular, the time span has been fixed to five days and five different values have been chosen for the number of intradaily observations $n$, ranging from 144 (corresponding to data recorded every ten minutes) to 1440 (corresponding to data recorded every minute). Therefore, the total number of observations ranges from $\overline{T}n = 720$ to $\overline{T}n = 7200$, where $\overline{T}$ denotes the fixed time span expressed in days. Also, the experiment has been conducted for six different values for $m$ (namely $\left\lfloor (\overline{T}n)^{.7}/\overline{T} \right\rfloor$, $\left\lfloor (\overline{T}n)^{.75}/\overline{T} \right\rfloor$, $\left\lfloor (\overline{T}n)^{.8}/\overline{T} \right\rfloor$, $\left\lfloor (\overline{T}n)^{.9}/\overline{T} \right\rfloor$, $\left\lfloor (\overline{T}n)^{.95}/\overline{T} \right\rfloor$ and then the limiting case $m = n$). The process is repeated for a total of 10000 replications.

Results are reported for two test statistics, namely

$$Z_{n,m} = \max_{j=1,\ldots,J} \sqrt{m}\left| \frac{1}{n} \sum_{i=1}^{\lfloor (n-1)r_j \rfloor} S_n^2(X_{i/n}) - RV_{m,r_j} \right|$$

and

$$Z_{n,m,1} = \sqrt{m}\left( \frac{1}{n} \sum_{i=1}^{n-1} S_n^2(X_{i/n}) - RV_{m,1} \right).$$

Under the conditions stated in Theorem 1, we know that for $m/n \to 0$,

$$Z_{n,m} \xrightarrow{d} ZM = \max_{j=1,\ldots,J} \left| ZM_{r_j} \right|,$$

and for $m = n$,

$$Z_n \xrightarrow{d} Z = \max_{j=1,\ldots,J} \left| Z_{r_j} \right|,$$

where the vectors $(ZM_{r_1} ZM_{r_2} \dots ZM_{r_J})'$ and $(Z_{r_1} Z_{r_2} \dots Z_{r_J})'$ are defined respectively in (8) and (7). In the simulation experiment, $J = 16$, with $r$ starting from $r_1 = .15$ and then increasing by .05 until $r_{16} = .85$. The critical values defined in (12) have been obtained with $S = 1000$.

Similarly, under the conditions stated in Corollary 1, we have that for $m/n \to 0$,

$$ Z_{n,m,1} \xrightarrow{d} \mathrm{MN} \left( 0, 2 \int_{-\infty}^{\infty} \sigma^4(a) L_X(1, a) \mathrm{d}a \right) $$

The empirical sizes (at 5% and 10% level) of the tests discussed above are reported in Table 1, for $\kappa = 0$, $\eta = 1$, $\mu = -.8$, $\xi_n = n^{-10/13}$. The results for different values of the parameters needed to generate (13) and the bandwidth $\xi_n$ display a virtually identical pattern and therefore are omitted for space reasons. Inspection of the Table reveals an overall good small sample behaviour of the considered test statistics. The reported empirical sizes are everywhere very close to the nominal ones, with a slight tendency to underreject for the test based on $Z_{n,m}$. The zeros appearing in the rows when $n = m$ are not surprising; in fact, when using the statistic $Z_n$, the critical values used in the simulation exercise are just an upper bound of the true ones, and therefore one should expect an undersized test.

Under the alternative hypothesis, the following model has been considered,

$$ \begin{aligned} \mathrm{d}X_t &= (\kappa + \mu X_t)\mathrm{d}t + \eta \sqrt{\exp\left(\sigma_t^2\right)} \left( \sqrt{1 - \rho^2} \mathrm{d}W_{1,t} + \rho \mathrm{d}W_{2,t} \right) \\ \mathrm{d}\sigma_t^2 &= (\kappa_1 + \mu_1 \sigma_t^2)\mathrm{d}t + \eta_1 \sqrt{\sigma_t^2} \mathrm{d}W_{2,t}. \end{aligned} \tag{14} $$

A discretized version of (14) has been simulated using a Milstein scheme as above, with $\kappa_1 = 1$, $\eta_1 = 1$, $\mu_1 = -.2$. Then, using the obtained values of $\sigma_t^2$, the series for $X_t$ has been generated, with $\rho = 0$ and keeping the remaining parameters at the values used to generate $X_t$ under (13). The findings for the power of the tests based on $Z_{n,m}$ and $Z_{n,m,1}$ are reported in Table 2. The experiment reveals that the proposed tests has good power properties. The test based on $Z_{n,m}$ is more powerful than the one based on $Z_{n,m,1}$; this is not surprising, given that $Z_{n,m}$ is specifically constructed to highlight the differences between the local times of $X_t$ and $f_t$. In fact, in the case of $Z_{m,n}$ the term driving the power is $\max_r \left| \int_0^r (L_X(r, a) - L_f(r, a)) \, \mathrm{d}a \right|$, which is in general larger than $\left| \int_0^1 (L_X(1, a) - L_f(1, a)) \, \mathrm{d}a \right|$, the term driving the power of $Z_n$. Also, the power of the test based on $Z_{n,m}$ is generally increasing in $n$ and $m$, as one should expect. In some cases, however, the power remains constant or even decreases when $m$ approaches $n$ (namely, the cases when $n = 144, 288, 576$); this is due to the fact that, when $n = m$, we are not using the correct critical values for the test, but just an upper bound, and this may decrease the resulting power of the test.

# 4    Concluding remarks

This paper provides a testing procedure which allows to discriminate between one-factor and stochastic volatility models. Hence, it allows to distinguish between the case in which the volatility of an asset is a function of the asset itself (and therefore the volatility process is Markov and predictable in terms of its own past), and the case in which it is a diffusion process driven by a Brownian motion, which is not perfectly correlated with the Brownian motion driving the asset. The suggested test statistics are based on the difference between a kernel estimator of the instantaneous variance, averaged over the sample realization on a fixed time span, and realized volatility. The intuition behind is the following: under the null hypothesis of a one-factor model, both estimators are consistent for the true underlying integrated (daily) volatility; under the alternative hypothesis the former estimator is not consistent, while the latter is. More precisely, we show that the proposed statistics weakly converge to well defined distributions under the null hypothesis and diverge at an appropriate rate under the alternative. The derived asymptotic theory is based on the time interval between successive observations approaching zero, while the time span is kept fixed. As a consequence, the limiting behavior of the statistic is not affected by the drift specification. Also, no stationarity or ergodicity assumption is required. The finite sample properties of the suggested statistic are analyzed via a small Monte Carlo study. Under the null hypothesis, the asset process is modelled as a version of the Cox, Ingersoll and Ross (1985) model with a mean reverting component in the drift. Thus, volatility is a square root function of the asset itself. Under the alternative, the asset and volatility processes are generated according to a stochastic volatility model, where volatility is modelled as a square root diffusion. The empirical sizes and powers of the proposed tests are reasonably good across various $m/n$ ratios.

Table 1: Actual sizes of the tests based on $Z_{n,m,r}$ for different values of $m$ and $n$

| | $Z_{n,m}$ | | $Z_{n,m,1}$ | |
|---|---|---|---|---|
| | 5% nominal size | 10% nominal size | 5% nominal size | 10% nominal size |
| | $n = 144$ | | | |
| $m = 20$ | 0.03 | 0.07 | 0.05 | 0.07 |
| $m = 27$ | 0.03 | 0.07 | 0.03 | 0.07 |
| $m = 38$ | 0.04 | 0.08 | 0.04 | 0.09 |
| $m = 74$ | 0.04 | 0.08 | 0.05 | 0.10 |
| $m = 103$ | 0.03 | 0.07 | 0.04 | 0.09 |
| $m = n$ | 0.00 | 0.00 | | |
| | $n = 288$ | | | |
| $m = 32$ | 0.03 | 0.07 | 0.05 | 0.12 |
| $m = 46$ | 0.03 | 0.07 | 0.07 | 0.10 |
| $m = 67$ | 0.03 | 0.07 | 0.07 | 0.10 |
| $m = 139$ | 0.03 | 0.07 | 0.06 | 0.09 |
| $m = 200$ | 0.03 | 0.07 | 0.05 | 0.12 |
| $m = n$ | 0.00 | 0.00 | | |
| | $n = 576$ | | | |
| $m = 52$ | 0.02 | 0.07 | 0.08 | 0.11 |
| $m = 78$ | 0.02 | 0.07 | 0.05 | 0.10 |
| $m = 117$ | 0.03 | 0.08 | 0.08 | 0.13 |
| $m = 259$ | 0.03 | 0.08 | 0.04 | 0.08 |
| $m = 386$ | 0.03 | 0.07 | 0.06 | 0.11 |
| $m = n$ | 0.00 | 0.00 | | |
| | $n = 720$ | | | |
| $m = 61$ | 0.03 | 0.08 | 0.07 | 0.10 |
| $m = 92$ | 0.03 | 0.08 | 0.04 | 0.09 |
| $m = 139$ | 0.03 | 0.08 | 0.08 | 0.13 |
| $m = 317$ | 0.03 | 0.08 | 0.04 | 0.09 |
| $m = 478$ | 0.02 | 0.07 | 0.05 | 0.10 |
| $m = n$ | 0.00 | 0.00 | | |
| | $n = 1440$ | | | |
| $m = 100$ | 0.03 | 0.07 | 0.07 | 0.12 |
| $m = 156$ | 0.03 | 0.08 | 0.08 | 0.12 |
| $m = 243$ | 0.06 | 0.10 | 0.07 | 0.12 |
| $m = 592$ | 0.05 | 0.10 | 0.07 | 0.12 |
| $m = 923$ | 0.05 | 0.11 | 0.08 | 0.13 |
| $m = n$ | 0.00 | 0.00 | | |

Table 2: Actual powers of the tests based on $Z_{n,m,r}$ for different values of $m$ and $n$

| | $Z_{n,m}$ | | $Z_{n,m,1}$ | |
|---|---|---|---|---|
| | 5% nominal size | 10% nominal size | 5% nominal size | 10% nominal size |
| | $n = 144$ | | | |
| $m = 20$ | 0.12 | 0.22 | 0.14 | 0.17 |
| $m = 27$ | 0.10 | 0.14 | 0.10 | 0.17 |
| $m = 38$ | 0.14 | 0.18 | 0.09 | 0.15 |
| $m = 74$ | 0.34 | 0.42 | 0.09 | 0.16 |
| $m = 103$ | 0.38 | 0.44 | 0.19 | 0.25 |
| $m = n$ | 0.42 | 0.44 | | |
| | $n = 288$ | | | |
| $m = 32$ | 0.20 | 0.26 | 0.11 | 0.21 |
| $m = 46$ | 0.22 | 0.36 | 0.10 | 0.16 |
| $m = 67$ | 0.44 | 0.48 | 0.12 | 0.16 |
| $m = 139$ | 0.54 | 0.56 | 0.33 | 0.45 |
| $m = 200$ | 0.56 | 0.64 | 0.26 | 0.30 |
| $m = n$ | 0.54 | 0.54 | | |
| | $n = 576$ | | | |
| $m = 52$ | 0.22 | 0.26 | 0.11 | 0.16 |
| $m = 78$ | 0.44 | 0.54 | 0.12 | 0.14 |
| $m = 117$ | 0.48 | 0.54 | 0.16 | 0.19 |
| $m = 259$ | 0.56 | 0.70 | 0.13 | 0.14 |
| $m = 386$ | 0.84 | 0.88 | 0.70 | 0.76 |
| $m = n$ | 0.76 | 0.82 | | |
| | $n = 720$ | | | |
| $m = 61$ | 0.44 | 0.52 | 0.12 | 0.18 |
| $m = 92$ | 0.40 | 0.54 | 0.14 | 0.18 |
| $m = 139$ | 0.58 | 0.66 | 0.21 | 0.25 |
| $m = 317$ | 0.68 | 0.72 | 0.25 | 0.33 |
| $m = 478$ | 0.72 | 0.82 | 0.22 | 0.29 |
| $m = n$ | 0.76 | 0.88 | | |
| | $n = 1440$ | | | |
| $m = 100$ | 0.50 | 0.60 | 0.21 | 0.29 |
| $m = 156$ | 0.55 | 0.75 | 0.20 | 0.24 |
| $m = 243$ | 0.65 | 0.70 | 0.60 | 0.62 |
| $m = 592$ | 0.90 | 0.90 | 0.88 | 0.89 |
| $m = 923$ | 0.90 | 0.95 | 0.65 | 0.70 |
| $m = n$ | 0.95 | 0.95 | | |

# A  Proofs

Before proving Theorem 1, we need the following Lemmas.

**Lemma 1.** *Let Assumption 1 hold. Then*

$$\sup_{s \in [0,1]} |\mu(X_s)| = O_{a.s.}(n^{\varepsilon/4}),$$

$$\sup_{s \in [0,1]} |\sigma^2(X_s)| = O_{a.s.}(n^{\varepsilon/2}),$$

$$\sup_{s \in [0,1]} |g(f_s)| = O_{a.s.}(n^{\varepsilon/2}),$$

*for any $\varepsilon > 0$, arbitrarily small.*

## A.1  Proof of Lemma 1

We start from the case when $X_t$ follows (1). Define $R_l = \{\inf \ t \ : \ |X_t| > l\}$. Thus, $R_l$ is an $\mathcal{F}_t-$measurable stopping time. Let

$$X_{\min(t,R_l)} = \int_0^{\min(t,R_l)} \mu(X_s)\mathrm{d}s + \int_0^{\min(t,R_l)} \sigma^2(X_s)\mathrm{d}W_{1,s}.$$

Obviously, for all $t \le R_l$, $X_{\min(t,R_l)} = X_t$. Now let $\Omega_l = \{\omega : R_l > 1\}$ and $l = l_n = n^{\varepsilon/4}$. Thus, given the growth conditions in Assumption 1(a), $X_t$ is a non-explosive diffusion, and so $\Pr(\Omega_{l_n} \to 1) = 1$. By a similar argument, given Assumptions 1(a), 1(b), the same holds when the volatility process follows (2). Therefore, the statement follows. ∎

**Lemma 2.** *Let Assumption 1 hold. Under $H_0$, if, as $n \to \infty$, $n\xi_n \to \infty$, $n\xi_n^2 \to 0$ and, for any $\varepsilon > 0$ arbitrarily small, $m/n^{1-\varepsilon} \to 0$, then, pointwise in $r$,*

$$\frac{\sqrt{m}}{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left(S_n^2(X_{i/n}) - \sigma^2(X_{i/n})\right) \xrightarrow{p} 0.$$

## A.2  Proof of Lemma 2

By Ito's formula

$$\underbrace{\frac{\sqrt{m}}{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left(S_n^2(X_{i/n}) - \sigma^2(X_{i/n})\right)}_{A_{n,m,r}}$$

$$= \frac{\sqrt{m}}{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( \frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}} n \left(X_{(j+1)/n} - X_{j/n}\right)^2}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}} - \sigma^2(X_{i/n}) \right)$$

$$
= \quad \frac{\sqrt{m}}{n} \underbrace{\sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( \frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}} 2n \int_{j/n}^{(j+1)/n} \left( X_s - X_{j/n} \right) \sigma(X_s) \mathrm{d}W_{1,s}}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}} \right)}_{G_{n,m,r}}
$$

$$
+ \frac{\sqrt{m}}{n} \underbrace{\sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( \frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}} 2n \int_{j/n}^{(j+1)/n} \left( X_s - X_{j/n} \right) \mu(X_s) \mathrm{d}s}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}} \right)}_{H_{n,m,r}}
$$

$$
+ \frac{\sqrt{m}}{n} \underbrace{\sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( \frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}} n \left( \int_{j/n}^{(j+1)/n} \left( \sigma^2(X_s) - \sigma^2(X_{i/n}) \right) \mathrm{d}s \right)}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}} \right)}_{D_{n,m,r}}. \quad (15)
$$

Thus, we need to show that $G_{n,m,r}$, $H_{n,m,r}$ and $D_{n,m,r}$ are $o_P(1)$.

Now, because of Lemma 1,

$$
\begin{aligned}
D_{n,m,r} &\leq \quad \sqrt{m} \sup_{|X_s-X_\tau|\leq\xi_n} \left| \sigma^2(X_s) - \sigma^2(X_\tau) \right| \\
&\leq \quad \sqrt{m} \sup_{\tau\in[0,1]} \left| \nabla\sigma^2(X_\tau) \right| \sup_{|X_s-X_\tau|\leq\xi_n} |X_s - X_\tau| \\
&= \quad O\left(\sqrt{m}\right) O_{a.s.}(n^{\varepsilon/2}) O_{a.s}(\xi_n) = o_{a.s.}(1), \quad (16)
\end{aligned}
$$

provided that $m^{1/2}n^{\varepsilon/2}\xi_n \to 0$. Since $m = o(n^{1-\varepsilon})$, then

$$
O_{a.s}(\sqrt{m}n^{\varepsilon/2}\xi_n) = o_{a.s}(n^{1/2}\xi_n),
$$

which approaches zero almost surely.

As for $G_{n,m,r}$, by the proof of Step 1 of Theorem 1, part(i)a, below, $(\sqrt{n}/\sqrt{m})G_{n,m,r} = G_{n,r}$ converges in distribution and so it's $O_P(1)$; therefore $G_{n,m,r} = o_P(1)$, given that $m/n \to 0$, as $m,n \to \infty$.

Finally, given the continuity of $\mu(\cdot)$,

$$
\begin{aligned}
|H_{n,m,r}| &\leq \quad \sqrt{m} \sup_{s\in[0,1]} |\mu(X_s)| \sup_{\substack{|i/n-s|\leq 1/n \\ s\in[0,1]}} \left| X_s - X_{i/n} \right| \\
&= \quad \sqrt{m} O_{a.s.}(n^{\varepsilon/4}) O_{a.s.} \left( n^{-1/2} \log n \right) = o_{a.s.}(1). \quad (17)
\end{aligned}
$$

In fact, because of the modulus of continuity of a diffusion (see McKean, 1969, p.96),

$$
\sup_{\substack{|i/n-s|\leq i/n \\ s\in[0,1]}} \left| X_s - X_{i/n} \right| = O_{a.s.} \left( n^{-1/2} \log n \right),
$$

and $n^{1/2-\varepsilon+\varepsilon/4}n^{-1/2} \log n = n^{-3\varepsilon/4} \log n \to 0$. Therefore, the statement follows. ∎

We can now prove Theorem 1.

## A.3 Proof of Theorem 1

(i)a

$$Z_{n,r} = \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( S_n^2(X_{i/n}) - \sigma^2(X_{i/n}) \right)}_{A_{n,r}}$$

$$-\underbrace{\sqrt{n} \left( \sum_{j=1}^{\lfloor (n-1)r \rfloor} \left( X_{(j+1)/n} - X_{j/n} \right)^2 - \int_0^r \sigma^2(X_s)\mathrm{d}s \right)}_{B_{n,r}}$$

$$+\underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \sigma^2(X_{i/n}) - \sqrt{n} \int_0^r \sigma^2(X_s)\mathrm{d}s}_{C_{n,r}}. \tag{18}$$

The proof of the statement is based on the four steps below.

Step 1: $A_{n,r} \xrightarrow{d} \mathrm{MN}\left(0, 2\int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r,a)^2}{L_X(1,a)}\mathrm{d}a\right).$

Step 2: $B_{n,r} \xrightarrow{d} \mathrm{MN}\left(0, 2\int_{-\infty}^{\infty} \sigma^4(a)L_X(1,a)\mathrm{d}a\right).$

Step 3: Let $< A_n, B_n >_r$ define the discretized quadratic covariation process.

$$\mathrm{plim}_{n\to\infty} < A_n, B_n >_r -2\int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r,a)^2}{L_X(1,a)}\mathrm{d}a = 0.$$

Step 4: $C_{n,r} = o_P(1).$

Proof of Step 1: First note that using Ito's formula

$$A_{n,r} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( \frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}} n \left( X_{(j+1)/n} - X_{j/n} \right)^2}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}} - \sigma^2(X_{i/n}) \right)$$

$$= \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( \frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}} 2n \int_{j/n}^{(j+1)/n} \left( X_s - X_{j/n} \right) \sigma(X_s)\mathrm{d}W_{1,s}}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}} \right)}_{G_{n,r}}$$

$$+ \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( \frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}} 2n \int_{j/n}^{(j+1)/n} \left( X_s - X_{j/n} \right) \mu(X_s)\mathrm{d}s}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}} \right)}_{H_{n,r}}$$

20

$$+\frac{1}{\sqrt{n}}\underbrace{\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(\frac{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}n\left(\int_{j/n}^{(j+1)/n}\left(\sigma^2(X_s)-\sigma^2(X_{i/n})\right)\mathrm{d}s\right)}{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)}_{D_{n,r}}.$$

Now, given Lemma 1, $D_{n,r}=o_{a.s.}(1)$, provided that $n^{1/2+\varepsilon}\xi_n\to0$, as $n\to\infty$. It is immediate to see that $H_{n,r}$ is of a smaller order of probability than $G_{n,r}$.

Let $<G_n>_r$ denote the discretized quadratic variation process of $F_{n,r}$. By a similar argument as in Bandi and Phillips (2003, pp.271-272),

$$\mathrm{plim}_{n\to\infty}<G_n>_r-2\int_{-\infty}^{\infty}\sigma^4(a)\frac{L_X(r,a)^2}{L_X(1,a)}\mathrm{d}a=0.$$

Thus, by the same argument as in the proof of Theorem 3 in Bandi and Phillips (2003), the statement in Step 1 follows.

Proof of Step 2: It follows from Theorem 1 in Barndorff-Nielsen and Shephard (2004a).

Proof of Step 3: The discretized covariation process $<A_n,B_n>_r$,

$$
\begin{aligned}
&<A_n,B_n>_r\\
=\quad&4n\sum_{i=1}^{\lfloor(n-1)r\rfloor}\sum_{j=1}^{\lfloor(n-1)r\rfloor}\left(\frac{1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}\left(\int_{j/n}^{(j+1)/n}\left(X_s-X_{j/n}\right)\sigma(X_s)\mathrm{d}W_s\right)^2}{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)\\
=\quad&2n\sum_{i=1}^{\lfloor(n-1)r\rfloor}\sum_{j=1}^{\lfloor(n-1)r\rfloor}\left(\frac{1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}\sigma^4(X_{j/n}+o_{a.s.}(1))}{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)\qquad(19)\\
=\quad&2\int_0^r\left(\int_0^r\frac{1_{\{|X_u-X_a|<\xi_n\}}\sigma^4(X_u)\mathrm{d}u}{\int_0^11_{\{|X_u-X_a|<\xi_n\}}\mathrm{d}u}\right)\mathrm{d}a+o_{a.s.}(1)\\
=\quad&2\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{1_{\{|u-a|<\xi_n\}}\sigma^4(u)L_X(r,u)\mathrm{d}u}{\int_{-\infty}^{\infty}1_{\{|u-a|<\xi_n\}}L_X(1,u)\mathrm{d}u}\right)L_X(r,a)\mathrm{d}a+o_{a.s.}(1),
\end{aligned}
$$

where the 2 (instead of 4) on right hand side of (19) comes from Lemma 5.3 in Jacod and Protter (1998). Along the lines of Bandi and Phillips (2001, 2003), by the change of variable

$$\frac{u-a}{\xi_n}=z,$$

we have that

$$<A_n,B_n>_r$$

$$= 2 \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{1_{\{|u-a|<\xi_n\}} \sigma^4(u) L_X(r,u) \mathrm{d}u}{\int_{-\infty}^{\infty} 1_{\{|u-a|<\xi_n\}} L_X(1,u) \mathrm{d}u} \right) L_X(r,a) \mathrm{d}a + o_{a.s.}(1)$$

$$= 2 \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{1_{\{|z\xi_n|<\xi_n\}} \sigma^4(a+z\xi_n) L_X(r,a+z\xi_n) \mathrm{d}z}{\int_{-\infty}^{\infty} 1_{\{|z\xi_n|<\xi_n\}} L_X(1,a+z\xi_n) \mathrm{d}z} \right) L_X(r,a) \mathrm{d}a + o_{a.s.}(1)$$

$$\xrightarrow{a.s.} 2 \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r,a)^2}{L_X(1,a)} \mathrm{d}a. \tag{20}$$

Proof of Step 4:

$$
\begin{aligned}
C_{n,r} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \sigma^2(X_{i/n}) - \sqrt{n} \int_0^r \sigma^2(X_s) \mathrm{d}s \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \sigma^2(X_{i/n}) - \sqrt{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \int_{i/n}^{(i+1)/n} \sigma^2(X_s) \mathrm{d}s \\
&= \sqrt{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \int_{i/n}^{(i+1)/n} \left( \sigma^2(X_{i/n}) - \sigma^2(X_s) \right) \mathrm{d}s \tag{21}
\end{aligned}
$$

and, given the Lipschitz assumption on $\sigma^2(\cdot)$, the last line in (21) is $o_P(1)$ by the same argument as the one used in Step 1.

Given Steps 1-4 above, it follows that the quadratic variation process of $Z_{n,r}$ is given by

$$2 \int_{-\infty}^{\infty} \sigma^4(a) L_X(r,a) \mathrm{d}a + 2 \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r,a)^2}{L_X(1,a)} \mathrm{d}a - 4 \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r,a)^2}{L_X(1,a)} \mathrm{d}a$$

$$= 2 \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r,a)(L_X(1,a) - L_X(r,a))}{L_X(1,a)} \mathrm{d}a. \tag{22}$$

The statement in the theorem then follows.

(i)b Without loss of generality, suppose that $r < r'$. By noting that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r \rfloor} S_n^2(X_{i/n}) - \sqrt{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( X_{i+1/n} - X_{i/n} \right)^2$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor (n-1)r' \rfloor} S_n^2(X_{i/n}) - \sqrt{n} \sum_{i=1}^{\lfloor (n-1)r' \rfloor} \left( X_{i+1/n} - X_{i/n} \right)^2,$$

with $S_n^2(X_{i/n}) = 0$ and $\left( X_{i+1/n} - X_{i/n} \right)^2 = 0$ for $i > \lfloor (n-1)r \rfloor$, the result then follows by the continuous mapping theorem.

(i)c The statistic $Z_{n,m,r}$ can be rewritten as

$$Z_{n,m,r} = \underbrace{\frac{\sqrt{m}}{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \left( S_n^2(X_{i/n}) - \sigma^2(X_{i/n}) \right)}_{A_{n,m,r}}$$

$$\underbrace{-\sqrt{m} \left( \sum_{j=1}^{\lfloor (m-1)r \rfloor} \left( X_{(j+1)/m} - X_{j/m} \right)^2 - \int_0^r \sigma^2(X_s) \mathrm{d}s \right)}_{B_{m,r}}$$

$$\underbrace{+ \frac{\sqrt{m}}{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \sigma^2(X_{i/n}) - \sqrt{m} \int_0^r \sigma^2(X_s) \mathrm{d}s}_{C_{n,m,r}}. \tag{23}$$

Note that $A_{n,m,r} = o_P(1)$ by Lemma 2.

We first need to show that $C_{n,m,r} = o_{a.s.}(1)$. Given Assumption 1(a), Lemma 1, and recalling the modulus of continuity of a diffusion (see McKean, 1969, pp.95-96),

$$\left| \frac{\sqrt{m}}{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \sigma^2(X_{i/n}) - \sqrt{m} \int_0^r \sigma^2(X_s) \mathrm{d}s \right|$$

$$= \left| \frac{\sqrt{m}}{n} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \sigma^2(X_{i/n}) - \sqrt{m} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \int_{i/n}^{(i+1)/n} \sigma^2(X_s) \mathrm{d}s \right|$$

$$= \left| \sqrt{m} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \int_{i/n}^{(i+1)/n} \left( \sigma^2(X_{i/n}) - \sigma^2(X_s) \right) \mathrm{d}s \right|$$

$$\leq \sqrt{m} \sum_{i=1}^{\lfloor (n-1)r \rfloor} \int_{i/n}^{(i+1)/n} \left| \sigma^2(X_{i/n}) - \sigma^2(X_s) \right| \mathrm{d}s$$

$$\leq \sqrt{m} \sup_{\substack{|s-\tau| \leq 1/n \\ s \in [0,r]}} \left| \sigma^2(X_s) - \sigma^2(X_\tau) \right| \leq \sqrt{m} \sup_{\tau \in [0,r]} |\nabla \sigma^2(X_\tau)| \sup_{\substack{|s-\tau| \leq 1/n \\ s \in [0,r]}} |X_s - X_\tau|$$

$$= \sqrt{m} O_{a.s.}(n^{\varepsilon/2}) O_{a.s.}(n^{-1/2} \log n) = o_{a.s.}(1),$$

as $n^{1/2-\varepsilon/2} n^{-1/2} \log n \to 0$. Thus,

$$Z_{n,m,r} = -B_{m,r} + o_{a.s.}(1).$$

The statement then follows from the proof of Step 2 in part i(a).

(i)d The statement follows by the same argument as the one used in part (i)b and by the continuous mapping theorem.

(ii) We will prove the Theorem for the case analyzed in part (i)c; in the other cases the proof follows straightforwardly and is therefore omitted. Under $H_A$, we have that

$$
\begin{aligned}
\mathrm{d}X_t &= \mu(X_t)\mathrm{d}t + \sqrt{\sigma_t^2}\mathrm{d}W_{1,t} \\
\sigma_t^2 &= g(f_t) \\
\mathrm{d}f_t &= b(f_t)\mathrm{d}t + \sigma_1(f_t)\mathrm{d}W_{2,t}.
\end{aligned}
$$

Pointwise in $r$, we can rewrite $Z_{n,m,r}$ as

$$
\begin{aligned}
Z_{n,m,r} &= \frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(S_n^2(X_{i/n}) - g\left(f_{i/n}\right)\right) - \sqrt{m}\sum_{j=1}^{\lfloor(m-1)r\rfloor}\left(X_{(j+1)/m} - X_{j/m}\right)^2 \\
&\quad + \frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor} g\left(f_{i/n}\right) \\
&= \underbrace{\frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(S_n^2(X_{i/n}) - g\left(f_{i/n}\right)\right)}_{E_{n,m,r}} \\
&\quad \underbrace{- \sqrt{m}\left(\sum_{j=1}^{\lfloor(m-1)r\rfloor}\left(X_{(j+1)/m} - X_{j/m}\right)^2 - \int_0^r g\left(f_s\right)\mathrm{d}s\right)}_{F_{m,r}} \\
&\quad + \underbrace{\frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor} g\left(f_{i/n}\right) - \sqrt{m}\int_0^r g\left(f_s\right)\mathrm{d}s}_{L_{n,m,r}}.
\end{aligned}
\tag{24}
$$

By the same argument used in the proof of part (i)a, Step 4 and Step 2 (respectively) $L_{n,m,r} = o_P(1)$ and $F_{m,r} = O_P(1)$.

We can expand $E_{n,m,r}$ as

$$
\begin{aligned}
&E_{n,m,r} \\
&= \frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(\frac{\sum_{j=1}^{n-1}1_{\left\{|X_{j/n}-X_{i/n}|<\xi_n\right\}}n\left(X_{(j+1)/n} - X_{j/n}\right)^2}{\sum_{j=1}^{n-1}1_{\left\{|X_{j/n}-X_{i/n}|<\xi_n\right\}}} - g\left(f_{i/n}\right)\right) \\
&= \frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\underbrace{\left(\frac{\sum_{j=1}^{n-1}1_{\left\{|X_{j/n}-X_{i/n}|<\xi_n\right\}}2n\int_{j/n}^{(j+1)/n}\left(X_s - X_{j/n}\right)\sqrt{g\left(f_s\right)}\mathrm{d}W_{1,s}}{\sum_{j=1}^{n-1}1_{\left\{|X_{j/n}-X_{i/n}|<\xi_n\right\}}}\right)}_{P_{n,m,r}}
\end{aligned}
$$

$$+\frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\underbrace{\left(\frac{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}2n\int_{j/n}^{(j+1)/n}\left(X_s-X_{j/n}\right)\mu(X_s)\mathrm{d}s}{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)}_{S_{n,m,r}}$$

$$+\frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\underbrace{\left(\frac{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}n\left(\int_{j/n}^{(j+1)/n}\left(g\left(f_s\right)-g\left(f_{i/n}\right)\right)\mathrm{d}s\right)}{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)}_{Q_{n,m,r}}. \qquad (25)$$

Now, $P_{n,m,r}$ is $O_p(1)$, again for the same argument used in the proof of part (i)a Step 1, and similarly $S_{n,m,r}=o_P(1)$, since it has the same behavior under both $\mathrm{H}_0$ and $\mathrm{H}_A$. Then, we need to show that $Q_{n,m,r}$, in absolute value, diverges. Now,

$$\frac{1}{\sqrt{m}}Q_{n,m,r} = \underbrace{\frac{1}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(\frac{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}n\left(\int_{j/n}^{(j+1)/n}\left(g\left(X_s\right)-g\left(X_{i/n}\right)\right)\mathrm{d}s\right)}{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)}_{T_{n,m,r}}$$

$$+\underbrace{\frac{1}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(\frac{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}n\left(\int_{j/n}^{(j+1)/n}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)}_{U_{n,m,r}}$$

$$-\underbrace{\frac{1}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(\frac{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}\left(g\left(f_{i/n}\right)-g\left(X_{i/n}\right)\right)}{\sum_{j=1}^{n-1}1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)}_{V_{n,m,r}}. \qquad (26)$$

Note that $T_{n,m,r}$ is $o_P(1)$, by the same argument as the one used in part (i)a, Step 1. As for $V_{n,m,r}$, it can be rewritten as

$$-\frac{1}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(g\left(f_{i/n}\right)-g\left(X_{i/n}\right)\right) = \left(\int_0^r g(X_s)\mathrm{d}s-\int_0^r g(f_s)\mathrm{d}s\right)+O_P(n^{-1/2}), \qquad (27)$$

where the first term of the right hand side of (27) is almost surely different from 0, given that $X_s$ and $f_s$ have different occupation density. Also, in the case in which $f_s$ is one-dimensional, we have that

$$-\frac{1}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(g\left(f_{i/n}\right)-g\left(X_{i/n}\right)\right) = \int_{-\infty}^{\infty}g(a)L_X(r,a)\mathrm{d}a-\int_{-\infty}^{\infty}g(a)L_f(r,a)\mathrm{d}a+O_P(n^{-1/2}),$$

where $L_f(r,a)$ (resp. $L_X(r,a)$) denotes the standardized local time of the process $f_t$ (resp. $X_t$) evaluated at time $r$ and at point $a$, that is it denotes the amount of time spent by the

process $f_t$ (resp. $X_t$) around point $a$, over the period $[0,1]$. Thus,

$$-\frac{\sqrt{m}}{n}\sum_{i=1}^{\lfloor (n-1)r\rfloor}\left(g\left(f_{i/n}\right)-g\left(X_{i/n}\right)\right)$$

diverges (to either $-\infty$ or to $\infty$), at rate $\sqrt{m}$, provided that $L_X(r,a)-L_f(r,a)\neq 0$ (almost surely) for all $a\in A$, with $A$ having non-zero Lebesgue measure, that is provided that $f_t$ and $X_t$ have different occupation densities over a non-negligible set.

Finally, $U_{n,m,r}$ can be written as

$$\frac{1}{n}\sum_{i=1}^{\lfloor (n-1)r\rfloor}\left(\frac{\frac{1}{n\xi_n}\sum_{j=1}^{n-1}1_{\left\{\left|X_{j/n}-X_{i/n}\right|<\xi_n\right\}}n\left(\int_{j/n}^{(j+1)/n}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{i/n})+o_P(1)}\right). \qquad (28)$$

Expanding the sums, (28) can be rewritten as

$$\frac{1}{n\xi_n}\left(\frac{n\left(\int_{1/n}^{2/n}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{1/n})+o_P(1)}\right.$$

$$+\frac{1_{\left\{\left|X_{1/n}-X_{2/n}\right|<\xi_n\right\}}n\left(\int_{1/n}^{2/n}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{2/n})+o_P(1)}$$

$$\left.\ldots+\ldots\frac{1_{\left\{\left|X_{1/n}-X_{[(n-1)r/n]}\right|<\xi_n\right\}}n\left(\int_{1/n}^{2/n}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{(n-1)/n})+o_P(1)}\right)$$

$$+\ldots$$

$$+\frac{1}{n\xi_n}\left(\frac{1_{\left\{\left|X_{j/n}-X_{1/n}\right|<\xi_n\right\}}n\left(\int_{j/n}^{(j+1)/n}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{1/n})+o_P(1)}\right.$$

$$+\frac{1_{\left\{\left|X_{j/n}-X_{2/n}\right|<\xi_n\right\}}n\left(\int_{j/n}^{(j+1)/n}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{2/n})+o_P(1)}$$

$$\left.\ldots+\ldots\frac{1_{\left\{\left|X_{j/n}-X_{[(n-1)r/n]}\right|<\xi_n\right\}}n\left(\int_{j/n}^{(j+1)/n}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{(n-1)/n})+o_P(1)}\right)$$

$$+\ldots$$

$$+\frac{1}{n\xi_n}\left(\frac{1_{\left\{\left|X_{(n-1)/n}-X_{1/n}\right|<\xi_n\right\}}n\left(\int_{(n-1)/n}^{1}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{1/n})+o_P(1)}\right.$$

$$+\frac{1_{\left\{\left|X_{(n-1)/n}-X_{2/n}\right|<\xi_n\right\}}n\left(\int_{(n-1)/n}^{1}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{2/n})+o_P(1)}$$

$$\left.\ldots+\ldots\frac{1_{\left\{\left|X_{(n-1)/n}-X_{[(n-1)r/n]}\right|<\xi_n\right\}}n\left(\int_{(n-1)/n}^{1}\left(g\left(f_s\right)-g\left(X_s\right)\right)\mathrm{d}s\right)}{L_X(1,X_{(n-1)/n})+o_P(1)}\right).$$

Thus,

$$\frac{\frac{1}{n}\sum_{j=1}^{n-1} L_X(1, X_{j/n})\left(g\left(f_{j/n}\right) - g\left(X_{j/n}\right)\right)}{\sup_i L_X(1, X_{i/n})} + o_P(1)$$

$$\leq \frac{1}{n}\sum_{i=1}^{\lfloor(n-1)r\rfloor}\left(\frac{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}} n\left(\int_{j/n}^{(j+1)/n}\left(g\left(f_s\right) - g\left(X_s\right)\right)\mathrm{d}s\right)}{\sum_{j=1}^{n-1} 1_{\{|X_{j/n}-X_{i/n}|<\xi_n\}}}\right)$$

$$\leq \frac{\frac{1}{n}\sum_{j=1}^{n-1} L_X(1, X_{j/n})\left(g\left(f_{j/n}\right) - g\left(X_{j/n}\right)\right)}{\inf_i L_X(1, X_{i/n})} + o_P(1). \tag{29}$$

Note that the numerator in the lower and upper bounds of the inequality in (29) approaches zero if and only if $L_X(1, a) - L_f(1, a) = 0$ (almost surely) for all $a \in A$, with $A$ having non-zero Lebesgue measure, or in the multidimensional case, if $X_s$ and $f_s$ have the same occupation density, which is indeed ruled out under the alternative hypothesis. Therefore, $(1/\sqrt{m})Z_{n,m,r}$ consists of the sum of two nondegenerate random variables which do not cancel out each other. Thus, $Z_{n,m,r}$ diverges at rate $\sqrt{m}$ with probability approaching one.

Therefore, the statement follows. ∎

## A.4 Proof of Corollary 1

It follows directly from Theorem 1, part (i)c. ∎

## A.5 Proof of Proposition 1

(a) From equation (12), it follows that, for $r_1 < r_2 < ... < r_J$,

$$\begin{pmatrix} \widehat{d}_{m,r_1}^{(s)} \\ \widehat{d}_{m,r_2}^{(s)} \\ \vdots \\ \widehat{d}_{m,r_J}^{(s)} \end{pmatrix} \xrightarrow{d} \mathrm{MN}\left(\mathbf{0}, \begin{pmatrix} 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s & 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s & \ldots & 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s \\ 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s & 2\int_0^{r_2}\sigma^4(X_s)\mathrm{d}s & \ldots & 2\int_0^{r_2}\sigma^4(X_s)\mathrm{d}s \\ \vdots & \vdots & \ddots & \vdots \\ 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s & 2\int_0^{r_2}\sigma^4(X_s)\mathrm{d}s & \ldots & 2\int_0^{r_J}\sigma^4(X_s)\mathrm{d}s \end{pmatrix}\right).$$

Also, note that

$$\begin{pmatrix} 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s & 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s & \ldots & 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s \\ 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s & 2\int_0^{r_2}\sigma^4(X_s)\mathrm{d}s & \ldots & 2\int_0^{r_2}\sigma^4(X_s)\mathrm{d}s \\ \vdots & \vdots & \ddots & \vdots \\ 2\int_0^{r_1}\sigma^4(X_s)\mathrm{d}s & 2\int_0^{r_2}\sigma^4(X_s)\mathrm{d}s & \ldots & 2\int_0^{r_J}\sigma^4(X_s)\mathrm{d}s \end{pmatrix}$$

$$
- \begin{pmatrix}
V(r_1, r_1) & V(r_1, r_2) & \ldots & V(r_1, r_J) \\
V(r_2, r_1) & V(r_2, r_2) & \ldots & V(r_2, r_J) \\
\vdots & \vdots & \ddots & \vdots \\
V(r_J, r_1) & V(r_J, r_2) & \ldots & V(r_J, r_J)
\end{pmatrix}
$$

is positive semi-definite, where the latter matrix above is defined in the statement of Theorem 1, part (i)b. Given Theorem 1, part (i)b, the statement follows directly.

(b) Immediate from Theorem 1, part (i)d.

In both cases, the unit asymptotic power of the proposed tests follows from Theorem 1, part (ii). ∎

# References

Aït-Sahalia, Y. (1996). Testing Continuous Time Models of the Spot Interest Rate. *Review of Financial Studies*, 9, 385-426.

Aït-Sahalia, Y., P.A. Mykland and L. Zhang (2003). How Often to Sample a Continuous Time Process in the Presence of Market Microstructure Noise. *Review of Financial Studies*, forthcoming.

Andersen, T.G., T. Bollerslev and F.X. Diebold (2003). Some Like it Smooth and Some Like it Rough: Untangling Continuous and Jump Components in Measuring, Modelling and Forecasting Asset Return Volatility. Working Paper, Duke University.

Andersen, T.G., T. Bollerslev, F.X. Diebold and H. Ebens (2001). The Distribution of Realized Stock Return Volatility. *Journal of Financial Economics*, 61, 43-76.

Andersen, T.G., T. Bollerslev, F.X. Diebold and P. Labys (2001). The Distribution of Realized Exchange Rate Volatility. *Journal of the American Statistical Association*, 96, 42-55.

Andersen, T.G., T. Bollerslev, F.X. Diebold and P. Labys (2003). Modelling and Forecasting Realized Volatility. *Econometrica*, 71, 579-625.

Andersen, T.G., T. Bollerslev and N. Meddahi (2004a). Analytic Evaluation of Volatility Forecasts. *International Economic Review*, forthcoming.

Andersen, T.G., T. Bollerslev and N. Meddahi (2004b). Correcting the Errors: Volatility Forecast Evaluation Using High Frequency Data and Realized Volatilities, *Econometrica*, forthcoming.

Andersen, T.G. and J. Lund (1997). Estimating Continuous-Time Stochastic Volatility Models of the Short-Term Interest Rate. *Journal of Econometrics*, 77, 343-377.

Awartani, B., V. Corradi and W. Distaso (2004). Testing and Modelling Market Microstructure Effects with an Application to the Dow Jones Industrial Average. Queen Mary, University of London.

Bandi, F.M. (2002). Short Term Interest Rate Dynamics: A Spatial Approach. *Journal of Financial Economics*, 65, 73-110.

Bandi, F.M. and G. Moloche (2001). On the Functional Estimation of Multivariate Diffusion Processes. MIT.

Bandi, F.M. and P.C.B. Phillips (2001). A Simple Approach to the Parametric Estimation of Potentially Nonstationary Diffusions. Yale University.

Bandi, F.M. and P.C.B. Phillips (2003). Fully Nonparametric Estimation of Scalar Diffusion Models. *Econometrica*, 71, 241-283.

Bandi, F.M. and J.R. Russell (2003). Microstructure Noise, Realized Volatility, and Optimal Sampling. Working Paper, University of Chicago, Graduate School of Business.

Barndorff-Nielsen, O.E., S.E. Graversen and N. Shephard (2004). Power Variation and Stochastic Volatility: A Review and Some New Results. *Journal of Applied Probability*, 41A, 133-143.

Barndorff-Nielsen, O.E. and N. Shephard (2002). Econometric Analysis of Realized Volatility and Its Use in Estimating Stochastic Volatility Models. *Journal of the Royal Statistical Society*, Series B, 64, 253-280.

Barndorff-Nielsen, O.E. and N. Shephard (2004a). A Feasible Central Limit Theory for Realized Volatility Under Leverage. Manuscript, Nuffield College, Oxford.

Barndorff-Nielsen, O.E. and N. Shephard (2004b). Econometric Analysis of Realized Covariation: High Frequency Realized Covariance, Regression and Correlation in Financial Economics, *Econometrica,* 72, 885-925.

Barndorff-Nielsen, O.E. and N. Shephard (2004c). Power and Bipower Variation with Stochastic Volatility and Jumps (with Discussion). *Journal of Financial Econometrics*, 2, 1-48.

Barndorff-Nielsen, O.E. and N. Shephard (2004d). The Econometrics of Testing for Jumps in Financial Economics Using Bipower Variation. Manuscript, Nuffield College, Oxford.

Brennan, M.J. and E.S. Schwartz (1979). A Continuous Time Approach to the Pricing of Bonds. *Journal of Banking and Finance*, 3, 135-155.

Chan, K.C., G.A. Karolyi, F.A. Longstaff and R.K. Sundaram (1992). An Empirical Comparison of Alternative Models of the Short-Term Interest Rate. *Journal of Finance*, 47, 1209-1227.

Chernov, M., A.R. Gallant, E. Ghysels and G. Tauchen (2003). Alternative Models for Stock Price Dynamics. *Journal of Econometrics*, 116, 225-257.

Chung, K.L. and R.J. Williams (1990). *Introduction to Stochastic Integration.* Birkhauser, Boston.

Conley, T.G., E.G.J. Luttmer, L.P. Hansen and J.A. Scheinkman (1997). Short-Term Interest Rates as Subordinated Diffusions. *Review of Financial Studies*, 10, 525-577.

Corradi, V. and W. Distaso (2004). Estimating and Testing Stochastic Volatility Models via Realized Measures. Manuscript, Queen Mary, University of London.

Corradi, V. and H. White (1999). Specification Tests for the Variance of a Diffusion. *Journal of Time Series Analysis,* 20, 253-270.

Cox, J.C., J.E. Ingersoll and S.A. Ross (1985). A Theory of the Term Structure of Interest Rates. *Econometrica*, 53, 385-407.

Dai, Q. and K.J. Singleton (2000). Specification Analysis of Affine Term Structure Models. *Journal of Finance*, 55, 1943-1978.

Dai, Q. and K.J. Singleton (2003). Expectation Puzzles, Time Varying Risk Premia, and Affine Models of the Term Structure. *Journal of Financial Economics*, 63, 415-441.

Dette, H., M. Podolskij and M. Vetter (2004). Estimation of Integrated Volatility in Continuous Time Financial Models with Application to Goodness-of-Fit Testing. Manuscript, Ruhr-University, Bochum.

Duffie, D. and K.J. Singleton (1997). An Econometric Model of the Term Structure of Interest-Rate Swap Yields. *Journal of Finance*, 52, 1287-1321.

Durham, G.B. (2003). Likelihood-Based Specification Analysis of Continuous-Time Models of the Short-Term Interest Rate. *Journal of Financial Economics*, 70, 463-487.

Fan, J. (2003). A Selective Overview of Nonparametric Methods in Financial Econometrics. Manuscript, Princeton University.

Florens-Zmirou, D. (1993). On Estimating the Diffusion Coefficient from Discrete Observations. *Journal of Applied Probability*, 30, 790-804.

Geman, D. and J. Horowitz (1980). Occupation Densities. *Annals of Probability*, 8, 1-67.

Hansen, P.R. and A. Lunde (2004). An Unbiased Version of Realized Variance. Working Paper, Stanford University.

Heston, S.L. (1993). A Closed Form Solution for Option with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies*, 6, 327-344.

Huang, X. and G. Tauchen (2004). The Relative Contribution of Jumps to Total Price Variance. Duke University.

Hull, J. and A. White (1987). The Pricing of Options on Assets with Stochastic Volatility. *Journal of Finance*, 42, 281-300.

Hull, J. and A. White (1994). Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models. *Journal of Derivatives*, 2, 37-48.

Jacod, J. and P. Protter (1998). Asymptotic Error Distribution for the Euler Method for Stochastic Differential Equations. *Annals of Probability*, 26, 267-307.

Jiang, G.J. (1998). Nonparametric Modelling of US Term Structure of Interest Rates and Implications on the Prices of Derivative Securities. *Journal of Financial and Quantitative Analysis*, 33, 465-497.

Jiang, G.J. and J.L. Knight (1997). A Nonparametric Approach to the Estimation of Diffusion Processes, with an Application to a Short-Term Interest Rate Model. *Econometric Theory*, 13, 615-645.

McKean, H. (1969). *Stochastic Integrals*. Academic Press, New York.

Meddahi, N. (2001). An Eigenfunction Approach for Volatility Modeling, Manuscript, University of Montreal.

Meddahi, N. (2002). A Theoretical Comparison between Integrated and Realized Volatility. *Journal of Applied Econometrics*, 17, 475-508.

Meddahi, N. (2003). ARMA Representation of Integrated and Realized Variances. *Econometrics Journal*, 6, 334–355.

Nelson, D.B. (1990). ARCH Models as Diffusion Approximations. *Journal of Econometrics*, 45, 7-38.

Pardoux, E. and D. Talay (1985). Discretization and Simulation of Stochastic Differential Equations, *Acta Applicandae Mathematicae*, 3, 23-47.

Pearson, N.D. and T.S. Sun (1994). Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll and Ross Model. *Journal of Finance*, 49, 1278-1304.

Pritsker, M. (1998). Nonparametric Density Estimators and Tests of Continuous Time Interest Rate Models. *Review of Financial Studies*, 11, 449-487.

Stanton, R. (1997). A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk. *Journal of Finance*, 52, 1973-2002.

Vasicek, O.A. (1977). An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5, 177-188.

Wiggings, J.B. (1987). Options Values under Stochastic Volatility: Theory and Empirical Estimates. *Journal of Financial Economics*, 10, 351-372.

Zhang, L., P.A. Mykland and Y. Aït-Sahalia (2003). A Tale of Two Time Scales: Determining Integrated Volatility with Noisy High Frequency Data. Working Paper, Carnegie Mellon University.

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