Parametric Conditions for a Monotone TSK Fuzzy Inference System to be an *n*-Ary Aggregation Function

Yi Wen Kerk, Chin Ying Teh, Kai Meng Tay, and Chee Peng Lim

***Abstract*—Despite the popularity and practical importance of the fuzzy inference system (FIS), the use of an FIS model as an *n*-ary aggregation function, which is characterized by both the monotonicity and boundary properties, is yet to be established. This is because research on ensuring that FIS models satisfy the monotonicity property, i.e., monotone FIS, is relatively new, not to mention the additional requirement of satisfying the boundary property. The aim of this article, therefore, is to establish the parametric conditions for the Takagi–Sugeno–Kang (TSK) FIS model to operate as an *n*-ary aggregation function (hereafter denoted as *n*-TSK-FIS) via the specifications of fuzzy membership functions and fuzzy rules. An absorption property with fuzzy rules interpretation is outlined, and the use of *n*-TSK-FIS as a uninorm is explained. Exploiting the established parametric conditions, a framework for which an *n*-TSK-FIS model can be constructed from data samples is formulated and analyzed, along with a number of remarks. Synthetic data sets and a benchmark example on education assessment are presented and discussed. To be best of the authors' knowledge, this article serves as the first use of the TSK-FIS model as an *n*-ary aggregation function.**

***Index Terms*—Aggregation function, boundary condition, fuzzy partition, fuzzy rule base, monotonicity, Takagi–Sugeno–Kang (TSK) fuzzy inference system (FIS).**

I. Introduction

A. Background

AGGREGATION functions are important for information fusion, as they serve as numerical functions to combine several numerical values into single representative value [1]–[6]. A function $f : [0,1]^n \to [0,1]$ is known as an n-ary aggregation function for a given n-dimensional input space, i.e., $\boldsymbol{X} = X_1 \times X_2 \times \cdots \times X_n = [0,1]^n$, and an output space, i.e., $\boldsymbol{Y} = [0,1]$, subject to two properties: First, f satisfies the *monotonicity* property, namely $f(\boldsymbol{x}_{(1)} = (x_{1,(1)}, \ldots, x_{i,(1)}, \ldots, x_{n,(1)})) \leq f(\boldsymbol{x}_{(2)} = (x_{1,(2)}, \ldots, x_{i,(2)}, \ldots, x_{n,(2)}))$ for $x_{i,(1)} \leq x_{i,(2)} \in X_i$, $i \in \{1, \ldots, n\}$ is an integer, is always true; and second, f satisfies the *boundary* property, namely $f(0, 0, \ldots, 0) = 0$ and $f(1, 1, \ldots, 1) = 1$ is always true. Some commonly used aggregation functions include minimum/maximum [3], product [3], ordered weighted averaging operator [4], *t*-norms/*t*-conorms [3], and fuzzy integral [5]–[8].

The fuzzy inference system (FIS) is a popular computing paradigm for various modeling and control applications [9]–[11]. A typical issue in FIS modeling is the fulfillment of the monotonicity property at the output of an FIS model [12]–[21]. In this aspect, a number of mathematical conditions for FIS models to satisfy the monotonicity property are available, which include the Mamdani FIS [12], Takagi–Sugeno–Kang (TSK) FIS [13]–[21], single-input rule module FIS [16], hierarchical FIS [17], and interval type-2 FIS [18] models.

B. Aims and Motivations

In this article, we aim to establish the connection between FIS models and aggregation functions. Since the use of FIS as a class of aggregation functions is yet to be established, we propose a monotone TSK FIS model as an n-ary aggregation function, known as *n*-TSK-FIS, which bridges between both areas of FIS modeling and aggregation functions. This article is inspired by the research to adopt numerical functions as aggregation functions [1], [2]. While FIS models use aggregation functions to aggregate inference of the fuzzy rules, we, however, employ the entire TSK FIS model to implement an *n*-ary aggregation function in this article. The use of FIS as an aggregation function is motivated by its advantages for, first, implementing a nonlinear mapping from its input space to output space [22], [23], especially in a complex system; and second, customizing experts' knowledge provided in the form of fuzzy rules to describe the behavior of a complex system [22].

To the best of our knowledge, research to accord FIS models with both the monotonicity and boundary properties is new. Our previous studies [15], [19], [20] only focus on the monotone zero-order TSK FIS model without considering the boundary property. Specifically, we have introduced monotone fuzzy rules relabeling [15] and an interval method [20] to preprocess nonmonotone fuzzy rules. In addition, we have highlighted a

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