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# $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control design for polytopic continuous-time Markov jump linear systems with uncertain transition rates

Cecília F. Morais, Márcio F. Braga, Ricardo C. L. F. Oliveira and Pedro L. D. Peres*,†

*School of Electrical and Computer Engineering, University of Campinas – UNICAMP, Campinas, SP, Brazil*

## SUMMARY

This paper investigates the problems of $\mathscr{H}_\infty$ and $\mathscr{H}_2$ state feedback control design for continuous-time Markov jump linear systems. The matrices of each operation mode are supposed to be uncertain, belonging to a polytope, and the transition rate matrix is considered partly known. By appropriately modeling all the uncertain parameters in terms of a multi-simplex domain, new design conditions are proposed, whose main advantage with respect to the existing ones is to allow the use of polynomially parameter-dependent Lyapunov matrices to certify the mean square closed-loop stability. Synthesis conditions are derived in terms of matrix inequalities with a scalar parameter. The conditions, which become LMIs for fixed values of the scalar, can cope with $\mathscr{H}_\infty$ and $\mathscr{H}_2$ state feedback control in both mode-independent and mode-dependent cases. Using polynomial Lyapunov matrices of larger degrees and performing a search for the scalar parameter, less conservative results in terms of guaranteed costs can be obtained through LMI relaxations. Numerical examples illustrate the advantages of the proposed conditions when compared with other techniques from the literature. Copyright © 2015 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

Linear matrix inequalities (LMIs) [1], substantiated by the Lyapunov stability theory [2], have been largely employed to investigate the behavior of dynamical systems in the last two decades. Many LMI-based results have been reported for stability analysis and synthesis of controllers and filters in various scenarios, including uncertain linear systems, linear parameter varying systems, time-delay systems, nonlinear systems, and Markov jump linear systems (MJLS) [3–9].

Concerning the applicability, MJLS are a remarkable tool to describe dynamical systems because they can adequately model, for instance, plants subject to random abrupt variation in their operation point or structure, such as Networked Control Systems, manufacturing, aerospace, economy, and stochastic finance (see [10] and references therein). MJLS belong to a class of stochastic hybrid systems with multiple operation modes, where each one corresponds to a particular dynamical system that can be affected by uncertainties. The jumps between the modes are ruled by a transition probability matrix, in the discrete-time case, and a transition rate matrix, in the continuous-time domain, related to a Markov chain. Many researches assume that only partial information about the transition matrices is available because in real world problems, the exact knowledge of the probabilities or rates is difficult or can demand a high cost to be obtained. In this more complex scenario, one of the first approaches to deal with uncertain transition matrices was proposed in [11], where a polytope of two vertices representing the uncertainties is generated by the multiplication of a nominal transition

*Correspondence to: Pedro L. D. Peres, School of Electrical and Computer Engineering, University of Campinas – UNICAMP, Av. Albert Einstein, 400, 13083-852, Campinas, SP, Brazil.
†E-mail: peres@dt.fee.unicamp.br

matrix and a bounded scalar parameter. Usually, in the literature concerned with continuous-time MJLS, the uncertain transition rate matrix is gathered in two classes: elements lying inside an interval that can be known or inferred [9, 12–17] and limitless entries [18, 19]. As far as the authors know, the LMI approaches for analysis and control of continuous-time MJLS do not allow the immediate extension to cope with Lyapunov matrices depending on the uncertain parameters. In general, constant (parameter-independent) Lyapunov matrices are used for each operation mode to assess the MJLS closed-loop stability [11, 20]. The main reason for such limitation is that the resulting robust LMIs do not depend linearly on the uncertain parameters [18, 19].

In the literature, a common strategy is to study MJLS with precisely known transition matrix and uncertainties affecting only the Markovian operation modes [21–24]. Therefore, the major contribution of this paper is to provide a general approach to simultaneously cope with uncertainties affecting both the matrices of the operation modes and the transition rate matrix in the problem of state feedback control design for continuous-time MJLS. The matrices of all the operation modes are supposed to belong to independent polytopes, which covers affine and interval uncertainty as well. The transition rate matrix is considered partly known, that is, each row can be affected by unknown but bounded components or completely unknown elements. Whenever the bounds of a given row can be inferred, the row is described in terms of a polytope. Otherwise, rows with unbounded entries require a special handling, which actually corresponds to the strategy that is currently used in the literature. Therefore, the uncertainties in the system (matrices of the operation modes and transition rates) are combined into a single domain created by the Cartesian product of simplex, called multi-simplex [25]. Using extensions of mean square stability conditions with $\mathscr{H}_\infty$ and $\mathscr{H}_2$ guaranteed costs, new matrix inequalities that include slack variables and a scalar parameter are proposed for the design of a state feedback control gain under the assumptions of complete or null mode availability. For fixed values of the scalars, the conditions become LMIs. Differently from the existing approaches in the literature, the conditions are expressed in terms of inequalities that are linear with respect to the uncertain transition rates. As one of the main advantages of the proposed formulation, polynomially parameter-dependent Lyapunov matrices can be used to certify closed-loop mean square stability with $\mathscr{H}_\infty$ and $\mathscr{H}_2$ bounds. When the Markovian model is not subject to uncertainties and a mode-dependent gain is sought, the conditions become necessary and sufficient for a large enough value of the scalar parameter. If uncertainties affect the system or the control gain does not depend on the mode, LMI relaxations combined with the search of a scalar parameter can provide less conservative results in terms of $\mathscr{H}_\infty$ and $\mathscr{H}_2$ guaranteed costs when compared with other approaches for state feedback control of continuous-time MJLS from the literature, as illustrated by numerical examples.

## 2. PROBLEM STATEMENT

Consider the fundamental probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \Gamma)$ and the following continuous-time MJLS

$$\mathscr{G} = \begin{cases} \dot{x}(t) = A_{\theta_t}(\alpha)x(t) + B_{\theta_t}(\alpha)u(t) + E_{\theta_t}(\alpha)w(t) \\ z(t) = C_{z\theta_t}(\alpha)x(t) + D_{z\theta_t}(\alpha)u(t) + E_{z\theta_t}(\alpha)w(t) \\ w(t) \in \mathscr{L}_2^{n_w}, \ \mathscr{E}\left(\|x(0)\|^2\right) < \infty, \ \theta_0 \sim \mu \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the noise vector, assumed to be an arbitrary signal in Hilbert space $\mathscr{L}_2^{n_w}$, and $z(t) \in \mathbb{R}^{n_z}$ is the controlled output. The random variable $\theta_t$ evolves accordingly to a Markov chain assuming values in the finite set $\mathbb{K} \triangleq \{1, \ldots, \sigma\}$, associated to the stationary transition probabilities

$$\Pr(\theta_{t+h} = j \mid \theta_t = i) = \begin{cases} p_{ij}h + o(h), & \text{with} \quad p_{ij} \geqslant 0, \qquad i \neq j \\ 1 + p_{ij}h + o(h), & \text{with} \quad p_{ij} = -\sum_{j=1,\ j\neq i}^{\sigma} p_{ij}, \quad i = j \end{cases}$$



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where $h > 0$, $\lim_{h\to 0}(o(h)/h) = 0$, and $p_{ij}$ is the transition rate from mode $i$ at time $t$ to mode $j$ at time $t + h$, being grouped in the transition rate matrix $\Gamma = [p_{ij}]$. Additionally, this paper assumes that each element $p_{ij}$ can be completely unknown, that is, $p_{ij} = ?$, or can lie inside the interval $[\underline{p_{ij}}, \overline{p_{ij}}]$, that is, $\underline{p_{ij}} \leqslant p_{ij} \leqslant \overline{p_{ij}}$. For this end, consider the sets

$$\mathscr{I}_K^i \triangleq \left\{ j : \underline{p}_{ij} \leqslant p_{ij} \leqslant \overline{p}_{ij} \right\} \qquad \text{and} \qquad \mathscr{I}_{UK}^i \triangleq \left\{ j : j \notin \mathscr{I}_K^i \right\} \tag{2}$$

Note that $\mathscr{I}_K^i$ covers two cases: completely known $p_{ij}$ and bounded $p_{ij}$. For ease of notation, whenever possible, $\theta_t$ is replaced by $i$, $\forall i \in \mathbb{K}$. The set of matrices $\{A_i(\alpha) \in \mathbb{R}^{n_x \times n_x}, B_i(\alpha) \in \mathbb{R}^{n_x \times n_u}, E_i(\alpha) \in \mathbb{R}^{n_x \times n_w}, C_{zi}(\alpha) \in \mathbb{R}^{n_z \times n_x}, D_{zi}(\alpha) \in \mathbb{R}^{n_z \times n_u}, E_{zi}(\alpha) \in \mathbb{R}^{n_z \times n_w}\}$ can be written as a convex combination of $V_i$ known vertices

$$M_i(\alpha_i) = \sum_{v=1}^{V_i} \alpha_{iv} M_{iv}, \quad \forall i \in \mathbb{K}$$

where $M_i(\alpha_i)$ represents any uncertain system matrix given in (1) and $\alpha_i$ is a time-invariant parameter belonging to the unit simplex

$$\Lambda_\eta \triangleq \left\{ \zeta = (\zeta_1, \ldots, \zeta_\eta) \in \mathbb{R}^\eta : \sum_{j=1}^{\eta} \zeta_j = 1, \quad \zeta_j \geqslant 0, \quad j = 1, \ldots, \eta \right\} \tag{3}$$

being $\eta$ the number of parameters of the set $\Lambda_\eta$.

The aim of this paper is to propose new techniques to $\mathscr{H}_\infty$ and $\mathscr{H}_2$ state feedback control design for MJLS with uncertain transition rates and polytopic system matrices. For this purpose, it is necessary to present a generalization for the concept of stability applied to MJLS, named as mean square stability (MSS) [10, 26, 27], which assures that $\lim_{t\to\infty} \mathscr{E}\|x(t)\|^2 = 0$ for any initial condition $x(0) \in \mathbb{R}^{n_x}$, $\theta_0 \in \mathbb{K}$.

To generically represent the uncertainties affecting both $\Gamma$ and the system matrices associated to the Markov modes, a systematic procedure is proposed. The treatment for the uncertainties associated to the transition rate matrix is split in two cases, $i \in \mathscr{I}_K^i$ and $i \in \mathscr{I}_{UK}^i$. Supposing that $i \in \mathscr{I}_{UK}^i$, that is, when the lower bounds of the diagonal elements are unknown, this paper employs an analogous strategy presented in [19] (see the proof of Theorem 1). On the other hand, if $i \in \mathscr{I}_K^i$, then each one of the $u$ uncertain rows of $\Gamma$ can be described by parameters lying into the unit simplex $\Lambda_{L_r}$, $r = 1, \ldots, u$, by simply imposing that all elements of a row must sum up to zero, that is, $\sum_{j=1}^{\sigma} p_{ij} = 0$, similarly to what was done in the discrete-time case where each row summed up to one [16, 28]. Thus, uncertainties affecting the system matrices and the interval uncertainties associated to the partly unknown rows of the transition rate matrix are grouped into one single domain, generated by the Cartesian product of $m = u + \sigma$ unit simplexes $\Lambda = \Lambda_{L_1} \times \cdots \times \Lambda_{L_u} \times \Lambda_{V_1} \times \cdots \times \Lambda_{V_\sigma}$, called a multi-simplex [25]. The dimension of $\Lambda$ is defined as the index $N = (N_1, \ldots, N_m) = (L_1, \ldots, L_u, V_1, \ldots, V_\sigma)$.

Extensions for the computation of $\mathscr{H}_\infty$ and $\mathscr{H}_2$ guaranteed costs [10, 26] for system (1) with polytopic matrices and uncertain transition rate matrix $\Gamma(\alpha)$ in the multi-simplex representation are, respectively, presented in the next lemmas.

*Lemma 1*
System (1), with $B_i(\alpha)$ and $D_{zi}(\alpha)$ identically null, is MSS and $\|\mathscr{G}\|_\infty < \gamma$ if and only if there exist symmetric positive definite parameter-dependent matrices $X_i(\alpha) \in \mathbb{R}^{n_x \times n_x}$, $\forall i \in \mathbb{K}$, such that the parameter-dependent inequalities[‡]

[‡]The symbol $\star$ represents a symmetric block in the LMI.



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$$\begin{bmatrix} X_{pi}(\alpha) + A_i(\alpha)^T X_i(\alpha) + X_i(\alpha) A_i(\alpha) & \star & \star \\ E_i(\alpha) X_i(\alpha) & -\mathrm{I} & \star \\ C_{zi}(\alpha) & E_{zi}(\alpha) & -\gamma^2 \mathrm{I} \end{bmatrix} < 0 \tag{4}$$

hold for each $i \in \mathbb{K}$ and for all $\alpha \in \Lambda$, being $X_{pi}(\alpha) = \sum_{j=1}^{\sigma} p_{ij}(\alpha) X_j(\alpha)$.

Lemma 1 can be viewed as the bounded real lemma [10] applied to uncertain continuous-time MJLS.

*Lemma 2*
System (1), with $B_i(\alpha)$, $D_{zi}(\alpha)$, and $E_{zi}(\alpha)$ identically null, is MSS and $\|\mathscr{G}\|_2 < \rho$ if and only if there exist symmetric positive definite parameter-dependent matrices $X_i(\alpha) \in \mathbb{R}^{n_x \times n_x}$, $\forall i \in \mathbb{K}$, such that the parameter-dependent inequalities

$$\sum_{i=1}^{\sigma} \mu_i \ \mathrm{Tr}\ \left( E_i(\alpha)^T X_i(\alpha) E_i(\alpha) \right) < \rho^2 \tag{5}$$

$$\begin{bmatrix} A_i(\alpha)^T X_i(\alpha) + X_i(\alpha) A_i(\alpha) + X_{pi}(\alpha) & \star \\ C_{zi}(\alpha) & -I \end{bmatrix} < 0 \tag{6}$$

hold for each $i \in \mathbb{K}$ and for all $\alpha \in \Lambda$.

Lemmas 1 and 2 present infinite dimension problems, since the conditions must be fulfilled for all $\alpha \in \Lambda$. However, as shown in [25, 29], robust (parameter-dependent) LMIs admit a homogeneous polynomial solution of sufficiently large partial degrees, whenever one solution exists.

To derive tractable conditions, the following notation and definitions related to homogeneous polynomials in the multi-simplex domain are required. Any homogeneous polynomial matrix $P_i(\alpha)$ of partial degrees $g = (g_1, \dots, g_m) \in \mathbb{N}^m$ can be generically represented by $P_i(\alpha) = \sum_{k \in \mathscr{K}(g)} \alpha^k P_{ik}$, where $\alpha^k$ are homogeneous monomials of degree $g_r$ in each variable $\alpha_r$, that is, $\alpha^k = \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_m^{k_m}$ and $\alpha_r^{k_r} = \alpha_{r1}^{k_{r1}} \alpha_{r2}^{k_{r2}} \dots \alpha_{rN_r}^{k_{rN_r}}$ where $k_r = (k_{r1}, \dots, k_{rN_r})$, $k_{rt} \in \mathbb{N}$, $t = 1, \dots, N_r$, is such that $k_{r1} + \dots + k_{rN_r} = g_r$, being $N_r$ the number of vertices of each unit simplex and $P_{ik}$ the corresponding matrix-valued coefficients. The indices $k = (k_1, k_2, \dots, k_m)$ are obtained by combining all the $N$-tuples of the sets $\mathscr{K}_{N_r}(g_r)$, $r = 1, \dots, m$. By definition, $\mathscr{K}_{N_r}(g_r)$ is the set of $N$-tuples obtained from all possible combinations of $N_r$ non-negative integers with sum $g_r$, and the set $\mathscr{K}(g)$ is the Cartesian product $\mathscr{K}(g) = \mathscr{K}_{N_1}(g_1) \times \dots \times \mathscr{K}_{N_m}(g_m)$. Operations of summation $k + k'$ and subtraction $k - k'$ (whenever $k' \leqslant k$) are defined componentwise. Additionally, the coefficient $\pi(\cdot)$ represents the product $\pi(k) = (k_{11}!)(k_{12}!) \cdots (k_{1N_1}!) \cdots (k_{m1}!) \cdots (k_{mN_m}!)$, for any index $k$, and $\pi(g) = (g_1!)(g_2!) \cdots (g_m!)$, for any $m$-dimensional vector.

Following the definitions presented earlier, the multi-simplex representation of any polytopic matrix $M_i(\alpha)$ from (1) is given by

$$M_i(\alpha) = \sum_{k \in \mathscr{K}(\varepsilon_i)} \alpha^k M_{ik}$$

being $\varepsilon_i = (\overbrace{0, \dots, 0}^{u}, \overbrace{0, \dots, 1, \dots, 0}^{\sigma})$ a unit vector with $m$ elements, whose component $u + i$ is equal to 1. For instance, consider an MJLS (1) with two modes, whose modes 1 and 2 are described by polytopes with $V_1 = 2$ and $V_2 = 3$ vertices, respectively, and a transition rate matrix with $u = 1$ unknown row ($L_1 = 2$). The representation of any system matrix of the 2nd-mode in the multi-simplex domain is given by

$$M_2(\alpha) = \alpha_{31} M_{21} + \alpha_{32} M_{22} + \alpha_{33} M_{23} = \alpha_{31} M_{20000100} + \alpha_{32} M_{20000010} + \alpha_{33} M_{20000001}$$



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where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \Lambda = \Lambda_2 \times \Lambda_2 \times \Lambda_3$ and each vertex $j$ of a polytopic mode $i$ is represented by $M_{ij} = M_{ik}$, $j = 1, \dots, V_i$, and $k \in \mathscr{K}(\varepsilon_i)$.

## 3. MAIN RESULTS

This section presents a systematic procedure to synthesize robust state feedback gains assuring MSS with an upper bound to the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norms of system (1). The proposed technique is based on LMI problems (for fixed values of the scalar parameter), which provide homogeneous polynomial solutions of arbitrary degree in the multi-simplex [25, 30] that fulfill the infinite dimension conditions of lemmas 1 and 2.

The first theorem presents sufficient LMI conditions to synthesize a mode-dependent state feedback controller assuring robust MSS for system (1) with an $\mathscr{H}_\infty$ guaranteed cost.

*Theorem 1*
If there exist symmetric positive definite matrices $P_{i_k} \in \mathbb{R}^{n_x \times n_x}$, $k \in \mathscr{K}(g)$, $\forall i \in \mathbb{K}$, matrices $G_i \in \mathbb{R}^{n_x \times n_x}$, $Z_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, \sigma$, partial degrees $g = (g_1, \dots, g_m) \in \mathbb{N}^m$, $r_i = max\{g, \varepsilon_i\}$, $w_i = max\{g + \mathbb{1}, \varepsilon_i\}$, and a given scalar parameter $\xi > 0$, such that the following LMIs hold

$$P_{i_k} - P_{j_k} < 0, \ \ \forall k \in \mathscr{K}(g), \quad \forall i, j \in \mathscr{I}^i_{UK}, \quad j \neq i \tag{7}$$

$$\frac{\pi(r_i)}{\pi(k)}\Phi_k + \sum_{\substack{k' \in \mathscr{K}(r_i - g) \\ k \geqslant k'}} \frac{\pi(r_i - g)}{\pi(k')}\Theta_k + \sum_{\substack{\tilde{k} \in \mathscr{K}(r_i - \varepsilon_i) \\ k \geqslant \tilde{k}}} \frac{\pi(r_i - \varepsilon_i)}{\pi(\tilde{k})}\Psi_k < 0, \ \forall k \in \mathscr{K}(r_i), \quad \forall i \in \mathscr{I}^i_{UK} \tag{8}$$

$$\frac{\pi(w_i)}{\pi(k)}\Phi_k + \sum_{\substack{k' \in \mathscr{K}(w_i - g - \mathbb{1}) \\ k \geqslant k'}} \sum_{\substack{\hat{k} \in \mathscr{K}(\mathbb{1}) \\ k \geqslant k' + \hat{k}}} \frac{\pi(w_i - g - \mathbb{1})}{\pi(k')}\Theta_k + \sum_{\substack{\tilde{k} \in \mathscr{K}(w_i - \varepsilon_i) \\ k \geqslant \tilde{k}}} \frac{\pi(w_i - \varepsilon_i)}{\pi(\tilde{k})}\Psi_k < 0 \tag{9}$$

$$\forall k \in \mathscr{K}(w_i), \quad \forall i \in \mathscr{I}^i_K$$

with

$$\Phi_k = \begin{bmatrix} 0 & \star & \star & \star & \star \\ -\xi G_i & -G_i - G_i^T & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & -\gamma^2 \mathrm{I} & \star \\ 0 & 0 & 0 & 0 & -\mathrm{I} \end{bmatrix}, \qquad \Theta_k = \begin{bmatrix} \Theta_{k11} & \star & \star & \star & \star \\ \Theta_{k21} & 0 & \star & \star & \star \\ \Theta_{k31} & 0 & \Theta_{k33} & \star & \star \\ 0 & 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Psi_k = \begin{bmatrix} \xi\left(A_{i_{k-\tilde{k}}} G_i + B_{i_{k-\tilde{k}}} Z_i + G_i^T A^T_{i_{k-\tilde{k}}} + Z_i^T B^T_{i_{k-\tilde{k}}}\right) & \star & \star & \star & \star \\ G_i^T A^T_{i_{k-\tilde{k}}} + Z_i^T B^T_{i_{k-\tilde{k}}} & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ E_{i_{k-\tilde{k}}} & 0 & 0 & 0 & \star \\ \xi\left(C_{zi_{k-\tilde{k}}} G_i + D_{zi_{k-\tilde{k}}} Z_i\right) & C_{zi_{k-\tilde{k}}} G_i + D_{zi_{k-\tilde{k}}} Z_i & 0 & E_{zi_{k-\tilde{k}}} & 0 \end{bmatrix}$$

where

$$\Theta_{k11} = \underline{\lambda^i_K} P_{i_{k-\hat{k}}}, \ \ \Theta_{k21} = P_{i_{k-\hat{k}}}, \ \ \Theta_{k31} = \theta_i^T P_{i_{k-\hat{k}}}, \ \ \Theta_{k33} = -\mathrm{diag}\,(\theta_i)\,\hat{P}_{i_{k-\hat{k}}}, \ \ \text{if } i \in \mathscr{I}^i_{UK}$$



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$$\theta_i = \left[\overline{p}_{ij}\mathrm{I}_n\ \overline{p}_{ij+1}\mathrm{I}_n\ \cdots\ \right], \qquad \hat{P}_{i_k} = \mathrm{diag}\left(P_{j_k},\ P_{j+1_k},\ \ldots\right)$$

with $\mathrm{diag}(\theta_i)\hat{P}_{i_k}$, defined $\forall\, p_{ij} \neq 0$, $\forall i \in \mathscr{I}^i_{UK}$, $\forall j \in \mathscr{I}^i_K$, $\underline{\lambda}^i_K = -\sum_{j\in\mathscr{I}^i_K} \underline{p}_{ij}$, $\forall i \in \mathscr{I}^i_{UK}$, and

$$\begin{aligned}
&\Theta_{k11} = p_{ii_{\hat{k}}} P_{i_{k-k'-\hat{k}}},\ \ \Theta_{k21} = P_{i_{k-k'-\hat{k}}},\ \ \Theta_{k31} = \Upsilon^T_{i_{\hat{k}}} P_{i_{k-k'-\hat{k}}},\\
&\Theta_{k33} = -\mathrm{diag}\left(\Upsilon_{i_{\hat{k}}}\right)\bar{P}_{i_{k-k'-\hat{k}}}, \quad \text{if } i \in \mathscr{I}^i_K\\
&\Upsilon_{i_{\tilde{k}}} = \left[p^{(\tilde{k})}_{i1}\mathrm{I}_n\ \cdots\ p^{(\tilde{k})}_{ii-1}\mathrm{I}_n\ p^{(\tilde{k})}_{ii+1}\mathrm{I}_n\ \cdots\ p^{(\tilde{k})}_{i\sigma}\mathrm{I}_n\right],\\
&\bar{P}_{i_k} = \mathrm{diag}\left(P_{1_k},\ \ldots,\ P_{i-1_k},\ P_{i+1_k},\ \ldots,\ P_{\sigma_k}\right)
\end{aligned}$$

with $\mathrm{diag}(\Upsilon_{i_{\tilde{k}}})\bar{P}_{i_k}$, defined $\forall\, p^{(\tilde{k})}_{ij} \neq 0$, $\forall i, j \in \mathscr{I}^i_K$, $\forall \tilde{k} \in \mathscr{K}(\mathbb{1})$ where $\mathbb{1} = (\overbrace{1,\ldots,1}^{u}, \overbrace{0,\ldots,0}^{\sigma})$, then $K_i = Z_i G_i^{-1}$ is a mode-dependent mean square stabilizing state feedback gain for system (1). Additionally, $\gamma$ is a guaranteed cost for the $\mathscr{H}_\infty$ norm of the closed-loop system.

*Proof*

Consider $P_i(\alpha) = P_i(\alpha)^T = X_i(\alpha)^{-1}$, $A_{cl\,i}(\alpha) = A_i(\alpha) + B_i(\alpha)K_i$, $C_{cl\,i}(\alpha) = C_{z\,i}(\alpha) + D_{z\,i}(\alpha)K_i$ and $\mathrm{He}(X_i(\alpha)A_{cl\,i}(\alpha)) = X_i(\alpha)A_{cl\,i}(\alpha) + A_{cl\,i}(\alpha)^T X_i(\alpha)$. In the case where $i \in \mathscr{I}^i_{UK}$, the conditions (7) and (8) are coupled. First, note that multiplying (7) by $\alpha^k$, summing up $\forall k \in \mathscr{K}(g)$, applying Schur's complement, and a congruence transformation, one obtains

$$\begin{bmatrix} X_j(\alpha) & \star \\ 0 & X_i(\alpha)\end{bmatrix}\begin{bmatrix} -P_j(\alpha) & \star \\ P_i(\alpha) & -P_i(\alpha)\end{bmatrix}\begin{bmatrix} X_j(\alpha) & \star \\ 0 & X_i(\alpha)\end{bmatrix} < 0 \quad \Leftrightarrow \quad X_j(\alpha) - X_i(\alpha) < 0 \qquad (10)$$

Now, multiplying (8) by $\alpha^k$ and summing up $\forall k \in \mathscr{K}(r_i)$, one obtains

$$\underbrace{\begin{bmatrix} \underline{\lambda}^i_K P_i(\alpha) & \star & \star & \star & \star \\ P_i(\alpha) & 0 & \star & \star & \star \\ \theta_i^T P_i(\alpha) & 0 & -\mathrm{diag}\,(\theta_i)\,\hat{P}_i(\alpha) & \star & \star \\ E_i(\alpha)^T & 0 & 0 & -\gamma^2\mathrm{I} & \star \\ 0 & 0 & 0 & E_{z\,i}(\alpha) & -\mathrm{I}\end{bmatrix}}_{Q} + \underbrace{\begin{bmatrix}\xi\mathrm{I}\\ \mathrm{I}\\ 0\\ 0\\ 0\end{bmatrix}}_{V^T} G_i^T \begin{bmatrix} A_{cl_i}(\alpha)\\ -\mathrm{I}\\ 0\\ 0\\ C_{cl_i}\end{bmatrix}^T + \underbrace{\begin{bmatrix} A_{cl_i}(\alpha)\\ -\mathrm{I}\\ 0\\ 0\\ C_{cl_i}\end{bmatrix}}_{U^T} G_i \begin{bmatrix}\xi\mathrm{I}\\ \mathrm{I}\\ 0\\ 0\\ 0\end{bmatrix}^T < 0 \qquad (11)$$

Choosing the following bases for the null space of $U$ and $V$

$$N_U = \begin{bmatrix} \mathrm{I} & 0\ 0 & 0 \\ A_{cl\,i}(\alpha)^T & 0\ 0 & C_{cl\,i}(\alpha)^T \\ 0 & \mathrm{I}\ 0 & 0 \\ 0 & 0\ \mathrm{I} & 0 \\ 0 & 0\ 0 & \mathrm{I}\end{bmatrix} \quad \text{and } N_V = \begin{bmatrix} \mathrm{I} & 0 & 0 & 0 \\ -\xi\mathrm{I} & 0 & 0 & 0 \\ 0 & \mathrm{I} & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 & \mathrm{I}\end{bmatrix} \qquad (12)$$

and using the first equivalence of the Projection Lemma, one obtains

$$N_V^T Q N_V = \begin{bmatrix} \left(\underline{\lambda}^i_K - 2\xi\right) P_i(\alpha) & \star & \star & \star \\ \theta_i^T P_i(\alpha) & -\mathrm{diag}\,(\theta_i)\,\hat{P}_i(\alpha) & \star & \star \\ E_i(\alpha)^T & 0 & -\gamma^2\mathrm{I} & \star \\ 0 & 0 & E_{z\,i}(\alpha) & -\mathrm{I}\end{bmatrix} < 0$$

that provides a lower bound to the scalar parameter, $\xi > \underline{\lambda}_K^i/2$. Since the maximum admissible value for $\underline{\lambda}_K^i$ is zero, one suitable choice for the scalar parameter is $\xi > 0$. Now, using the second inequality of the Projection Lemma, one has

$$N_U^T Q N_U = \begin{bmatrix} A_{cl\,i}(\alpha)P_i(\alpha) + P_i(\alpha)A_{cl\,i}(\alpha)^T + \underline{\lambda}_K^i P_i(\alpha) & \star & \star & \star \\ \theta_i^T P_i(\alpha) & -\mathrm{diag}\,(\theta_i)\,\hat{P}_i(\alpha) & \star & \star \\ E_i(\alpha)^T & 0 & -\gamma^2 \mathrm{I} & \star \\ C_{cl\,i}(\alpha)P_i(\alpha) & 0 & E_{z\,i}(\alpha) & -\mathrm{I} \end{bmatrix} < 0 \tag{13}$$

Defining $\hat{X}_i(\alpha) = \mathrm{diag}(X_j(\alpha), X_{j+1}(\alpha), \ldots) = (\hat{P}_i(\alpha))^{-1}$, $j \in \mathscr{I}_K^i$, $j \neq i$, (13) can be rewritten as

$$\begin{bmatrix} A_{cl\,i}(\alpha)X_i(\alpha)^{-1} + X_i(\alpha)^{-1}A_{cl\,i}(\alpha)^T + \underline{\lambda}_K^i X_i(\alpha)^{-1} & \star & \star & \star \\ \theta_i^T X_i(\alpha)^{-1} & -\mathrm{diag}\,(\theta_i)\,\hat{X}_i(\alpha)^{-1} & \star & \star \\ E_i(\alpha)^T & 0 & -\gamma^2 \mathrm{I} & \star \\ C_{cl\,i}(\alpha)X_i(\alpha)^{-1} & 0 & E_{z\,i}(\alpha) & -\mathrm{I} \end{bmatrix} < 0 \tag{14}$$

multiplying (14) by $S = \mathrm{diag}\,(X_i(\alpha),\ \mathrm{I},\ \mathrm{I},\ \mathrm{I})$ on the right and $S^T$ on the left, and applying the Schur's complement, one obtains

$$\begin{bmatrix} X_i(\alpha)A_{cl\,i}(\alpha) + A_{cl\,i}(\alpha)^T X_i(\alpha) + \sum_{\substack{j \in \mathscr{I}_K^i \\ j \neq i}} \overline{p}_{ij} X_j(\alpha) + \underline{\lambda}_K^i X_i(\alpha) & \star & \star \\ E_i(\alpha)^T X_i(\alpha) & -\gamma^2 \mathrm{I} & \star \\ C_{cl\,i}(\alpha) & E_{z\,i}(\alpha) & -\mathrm{I} \end{bmatrix} < 0$$

which is equivalent to

$$X_i(\alpha)A_{cl\,i}(\alpha) + A_{cl\,i}(\alpha)^T X_i(\alpha) + \sum_{\substack{j \in \mathscr{I}_K^i \\ j \neq i}} \overline{p}_{ij}(\alpha) X_j(\alpha) + \underline{\lambda}_K^i X_i(\alpha) + R(\alpha) < 0 \tag{15}$$

with

$$R(\alpha) = -\left(X_i(\alpha)E_i(\alpha) + C_{cl\,i}(\alpha)^T E_{z\,i}(\alpha)\right)\left(E_{z\,i}(\alpha)^T E_{z\,i}(\alpha) - \gamma^2 \mathrm{I}\right)^{-1} \times \left(E_i(\alpha)^T X_i(\alpha) + E_{z\,i}(\alpha)^T C_{cl\,i}(\alpha)\right) + C_{cl\,i}(\alpha)^T C_{cl\,i}(\alpha)$$

Knowing that

$$\sum_{\substack{j \in \mathscr{I}_K^i \\ j \neq i}} \overline{p}_{ij} X_j(\alpha) \geqslant \sum_{\substack{j \in \mathscr{I}_K^i \\ j \neq i}} p_{ij} X_j(\alpha), \qquad \lambda_K^i \geqslant -\sum_{\substack{j \in \mathscr{I}_K^i \\ j \neq i}} p_{ij}, \qquad \text{and} \qquad \sum_{\substack{j \in \mathscr{I}_{UK}^i \\ j \neq i}} p_{ij} \geqslant 0$$

if (10) and (15) hold, then

$$\mathrm{He}\,(X_i(\alpha)A_{cl\,i}(\alpha)) + \sum_{\substack{j \in \mathscr{I}_K^i \\ j \neq i}} p_{ij} X_j(\alpha) - \sum_{\substack{j \in \mathscr{I}_K^i \\ j \neq i}} p_{ij} X_i(\alpha) + \sum_{\substack{j \in \mathscr{I}_{UK}^i \\ j \neq i}} p_{ij}(X_j(\alpha) - X_i(\alpha)) + R(\alpha) < 0 \tag{16}$$

is also verified. Rearranging the terms of (16), one obtains

$$\mathrm{He}\left(X_i(\alpha)A_{cl\,i}(\alpha)\right) + \sum_{\substack{j\in\mathscr{I}_K^i\\ j\neq i}} p_{ij}X_j(\alpha) + \sum_{\substack{j\in\mathscr{I}_{UK}^i\\ j\neq i}} p_{ij}X_j(\alpha) + \underbrace{\left(-\sum_{\substack{j\in\mathscr{I}_K^i\\ j\neq i}} p_{ij} - \sum_{\substack{j\in\mathscr{I}_{UK}^i\\ j\neq i}} p_{ij}\right)}_{p_{ii}} X_i(\alpha)$$

$$+ R(\alpha) = X_i(\alpha)A_{cl\,i}(\alpha) + A_{cl\,i}(\alpha)^T X_i(\alpha) + \sum_{j=1}^{\sigma} p_{ij}X_j(\alpha) + R(\alpha) < 0 \tag{17}$$

that is equivalent to the bounded real lemma (4) for continuous-time MJLS [10, 26].

On the other hand, if $i \in \mathscr{I}_K^i$, multiplying (9) by $\alpha^k$ and summing up $\forall k \in \mathscr{K}(w_i)$, one obtains a condition equivalent to (11) with

$$Q = \begin{bmatrix} p_{ii}(\alpha)P_i(\alpha) & \star & \star & \star & \star \\ P_i(\alpha) & 0 & \star & \star & \star \\ \Upsilon_i(\alpha)^T P_i(\alpha) & 0 & -\mathrm{diag}\left(\Upsilon_i(\alpha)\right)\bar{P}_i(\alpha) & \star & \star \\ E_i(\alpha)^T & 0 & 0 & -\gamma^2\mathrm{I} & \star \\ 0 & 0 & 0 & E_{z\,i}(\alpha) & -\mathrm{I} \end{bmatrix}$$

Using the inequality $N_V^T Q N_V < 0$ of the Projection Lemma, with $N_V$ given in (12), a lower bound to the scalar parameter is obtained, $\xi > p_{ii}(\alpha)/2$. Again, by choosing $\xi > 0$, this constraint is always fulfilled. Now, multiplying the second condition of the Projection Lemma, $N_U^T Q N_U < 0$, by $S = \mathrm{diag}\left(X_i(\alpha)^{-1},\ \mathrm{I},\ \mathrm{I},\ \mathrm{I}\right)$ on the right and $S^T$ on the left and then applying the Schur's complement, one recovers the bounded real lemma (4) for continuous-time MJLS

$$\begin{bmatrix} X_{pi}(\alpha) + X_i(\alpha)A_{cl\,i}(\alpha) + A_{cl\,i}(\alpha)^T X_i(\alpha) & \star & \star \\ E_i(\alpha)^T X_i(\alpha) & -\gamma^2\mathrm{I} & \star \\ C_{cl\,i}(\alpha) & E_{z\,i}(\alpha) & -\mathrm{I} \end{bmatrix} < 0 \tag{18}$$

where $X_{pi}(\alpha) = \sum_{j=1}^{\sigma} p_{ij}(\alpha)X_j(\alpha) = p_{ii}(\alpha)X_i(\alpha) + \Upsilon_i(\alpha)(\mathrm{diag}(\Upsilon_i(\alpha))\bar{P}_i(\alpha))^{-1}\Upsilon_i(\alpha)^T$. □

The following theorem presents sufficient conditions for the existence of a robust mean square stabilizing state feedback gain assuring an upper bound to the $\mathscr{H}_2$ norm for the closed-loop system (1).

*Theorem 2*

If there exist symmetric positive definite matrices $P_{i_k} \in \mathbb{R}^{n_x \times n_x}$ and $W_{i_k} \in \mathbb{R}^{n_w \times n_w}$, $\forall k \in \mathscr{K}(g)$, $i = 1,\dots,\sigma$, matrices $G_i \in \mathbb{R}^{n_x \times n_x}$, $Z_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1,\dots,\sigma$, partial degrees $g = (g_1, g_2, \dots, g_m) \in \mathbb{N}^m$, $r_i = \max(g,\ \varepsilon_i)$, $w_i = \max(g + \mathbb{1},\ \varepsilon_i)$, and a given scalar parameter $\xi > 0$, such that the following LMIs hold

$$\sum_{i=1}^{\sigma} \mu_i \mathrm{Trace}(W_{i_k}) - \frac{\pi(g)}{\pi(k)}\rho^2 < 0, \quad \forall k \in \mathscr{K}(g) \tag{19}$$

$$\sum_{\substack{k'\in\mathscr{K}(r_i-g)\\ k\geqslant k'}} \frac{\pi(r_i-g)}{\pi(k')}\Theta_{T_k} + \sum_{\substack{\tilde{k}\in\mathscr{K}(r_i-\varepsilon_i)\\ k\geqslant \tilde{k}}} \frac{\pi(r_i-\varepsilon_i)}{\pi(\tilde{k})}\Psi_{T_k} < 0, \quad \forall k \in \mathscr{K}(r_i), \qquad \forall i \in \mathbb{K} \tag{20}$$

and

$$\frac{\pi(w_i)}{\pi(k)}\Phi_{G_k} + \sum_{\substack{k'\in\mathscr{K}(r_i-g)\\ k\geqslant k'}} \frac{\pi(r_i-g)}{\pi(k')}\Theta_{G_k} + \sum_{\substack{\tilde{k}\in\mathscr{K}(w_i-\varepsilon_i)\\ k\geqslant\tilde{k}}} \frac{\pi(r_i-\varepsilon_i)}{\pi(\tilde{k})}\Psi_{G_k} < 0, \quad \forall k\in\mathscr{K}(r_i), \quad \forall i\in\mathscr{I}^i_{UK} \tag{21}$$

$$\frac{\pi(w_i)}{\pi(k)}\Phi_{G_k} + \sum_{\substack{k'\in\mathscr{K}(w_i-g-\mathbb{1})\\ k\geqslant k'}} \sum_{\substack{\hat{k}\in\mathscr{K}(\mathbb{1})\\ k\geqslant k'+\hat{k}}} \frac{\pi(w_i-g-\mathbb{1})}{\pi(k')}\Theta_{G_k} + \sum_{\substack{\tilde{k}\in\mathscr{K}(w_i-\varepsilon_i)\\ k\geqslant\tilde{k}}} \frac{\pi(w_i-\varepsilon_i)}{\pi(\tilde{k})}\Psi_{G_k} < 0 \quad \forall k\in\mathscr{K}(w_i), \quad \forall i\in\mathscr{I}^i_{K} \tag{22}$$

with

$$\Theta_{T_k} = \begin{bmatrix} -W_{i_{k-k'}} & \star \\ 0 & -P_{i_{k-k'}} \end{bmatrix}, \quad \Psi_{T_k} = \begin{bmatrix} 0 & \star \\ E_{i_{k-\tilde{k}}} & 0 \end{bmatrix}$$

$$\Phi_{G_k} = \begin{bmatrix} 0 & \star & \star & \star \\ -\xi G_i & -G_i - G_i^T & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & -\mathrm{I} \end{bmatrix}, \quad \Theta_{G_k} = \begin{bmatrix} \Theta_{k\,11} & \star & \star & \star \\ \Theta_{k\,21} & 0 & \star & \star \\ \Theta_{k\,31} & 0 & \Theta_{k\,33} & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Psi_{G_k} = \begin{bmatrix} \xi\left(A_{i_{k-\tilde{k}}}G_i + B_{i_{k-\tilde{k}}}Z_i + G_i^T A^T_{i_{k-\tilde{k}}} + Z_i^T B^T_{i_{k-\tilde{k}}}\right) & \star & \star & \star \\ \left(A_{i_{k-\tilde{k}}}G_i + B_{i_{k-\tilde{k}}}Z_i\right)^T & 0 & \star & \star \\ 0 & 0 & 0 & \star \\ \xi\left(C_{z i_{k-\tilde{k}}}G_i + D_{z i_{k-\tilde{k}}}Z_i\right) & \left(C_{z i_{k-\tilde{k}}}G_i + D_{z i_{k-\tilde{k}}}Z_i\right)^T & 0 & 0 \end{bmatrix}$$

where

$$\Theta_{k\,11} = \underline{\lambda^i_K} P_{i_{k-\hat{k}}}, \;\; \Theta_{k\,21} = P_{i_{k-\hat{k}}}, \;\; \Theta_{k\,31} = \theta_i^T P_{i_{k-\hat{k}}}, \;\; \Theta_{k\,33} = -\mathrm{diag}\,(\theta_i)\,\hat{P}_{i_{k-\hat{k}}}, \;\; \text{if } i\in\mathscr{I}^i_{UK}$$

$$\theta_i = \left[\overline{p}_{ij}\mathrm{I}_n \;\; \overline{p}_{ij+1}\mathrm{I}_n \;\; \cdots\right], \quad \hat{P}_{i_k} = \mathrm{diag}\left(P_{j_k},\; P_{j+1_k},\; \ldots\right),$$

with $\mathrm{diag}(\theta_i)\hat{P}_{i_k}$, defined $\forall\, p_{ij}\neq 0$, $\forall i\in\mathscr{I}^i_{UK}$, $\forall j\in\mathscr{I}^i_K$, $\underline{\lambda}^i_K = -\sum_{j\in\mathscr{I}^i_K}\underline{p}_{ij}$, $\forall i\in\mathscr{I}^i_{UK}$, and

$$\Theta_{k\,11} = p_{ii_{\hat{k}}} P_{i_{k-k'-\hat{k}}}, \;\; \Theta_{k\,21} = P_{i_{k-k'-\hat{k}}}, \;\; \Theta_{k\,31} = \Upsilon^T_{i_{\hat{k}}} P_{i_{k-k'-\hat{k}}},$$
$$\Theta_{k\,33} = -\mathrm{diag}\left(\Upsilon_{i_{\hat{k}}}\right)\bar{P}_{i_{k-k'-\hat{k}}}, \;\; \text{if } i\in\mathscr{I}^i_K$$

$$\Upsilon_{i_{\tilde{k}}} = \left[p^{(\tilde{k})}_{i1}\mathrm{I}_n \;\cdots\; p^{(\tilde{k})}_{ii-1}\mathrm{I}_n \;\; p^{(\tilde{k})}_{ii+1}\mathrm{I}_n \;\cdots\; p^{(\tilde{k})}_{i\sigma}\mathrm{I}_n\right], \quad \bar{P}_{i_k} = \mathrm{diag}\left(P_{1_k},\; \ldots, P_{i-1_k},\; P_{i+1_k},\; \ldots, P_{\sigma_k}\right)$$

with $\mathrm{diag}(\Upsilon_{i_{\tilde{k}}})\bar{P}_{i_k}$, defined $\forall\, p^{(\tilde{k})}_{ij}\neq 0$, $\forall i,j\in\mathscr{I}^i_K$, $\forall\tilde{k}\in\mathscr{K}(\mathbb{1})$ where $\mathbb{1} = (\overbrace{1,\ldots,1}^{u},\overbrace{0,\ldots,0}^{\sigma})$, then $K_i = Z_i G_i^{-1}$ is a mode-dependent mean square stabilizing state feedback gain for system (1). Additionally, $\rho$ is an upper bound for the $\mathscr{H}_2$ norm of the closed-loop system.

*Proof*
Multiplying (19) and (20) by $\alpha^k$ and summing up, respectively, $\forall k \in \mathscr{K}(g)$ and $\forall k \in \mathscr{K}(r_i)$, one obtains

$$\sum_{j=1}^{\sigma} \mu_i \text{Trace}\left(W_j(\alpha)\right) < \rho^2 \tag{23}$$

$$\begin{bmatrix} -W_i(\alpha) & \star \\ E_i(\alpha) & -P_i(\alpha) \end{bmatrix} < 0 \tag{24}$$

Now, employing the Schur's complement and using the fact that $X_i(\alpha) = P_i(\alpha)^{-1}$, (24) can be rewritten as

$$-W_i(\alpha) + E_i(\alpha)^T X_i(\alpha) E_i(\alpha) < 0$$

recovering the trace condition (5). The proofs for (21) and (22) follow the same lines used in Theorem 1 by eliminating the fourth row and column of (8) and (9). □

Note that $\xi > 0$ represents a degree of freedom to be exploited in the optimization problems of theorems 1 and 2 to determine stabilizing state feedback gains associated to smallest guaranteed costs.

*Remark 1*
Because the complete availability of the Markov modes may be limited by cost, physical constraints, difficulty of measuring, and other factors [31], it can be necessary to synthesize mode-independent gains. To contemplate this assumption, replace $G_i$ and $Z_i$ by $G$, $Z$, for all $i \in \mathbb{K}$, in theorems 1 and 2.

*Remark 2*
The conservativeness of the sufficient conditions presented in theorems 1 and 2 can be progressively reduced by increasing the degrees $g$ of the Lyapunov matrices. Moreover, the corresponding LMI conditions become necessary and sufficient when the system matrices and transition rate matrix are completely known, and the information about the current mode is available. To prove the necessity of theorems 1 and 2, the following steps should be performed: replace $G_i$ by $G_i/\xi$ and $G_i^T$ by $G_i^T/\xi$ in (9) and (22); apply Schur's complement with respect to $-\left(G_i + G_i^T\right)/\xi$; and make $G_i = G_i^T = P_i$. Note that, when $\xi \to \infty$, the resulting inequalities are equivalent, respectively, to the bounded real lemma (4) and the controllability gramian (6) for continuous-time MJLS.

*Remark 3*
The main technical novelty of the design conditions proposed in this paper, when compared with other approaches in the MJLS literature, is the new definition of matrix $X_{pi}(\alpha) = p_{ii}(\alpha)X_i(\alpha) + \Upsilon_i(\alpha)(\text{diag}(\Upsilon_i(\alpha))\bar{P}_i(\alpha))^{-1}\Upsilon_i(\alpha)^T$, which does not exhibit the square roots of the transition rates. The resulting conditions are therefore linear in terms of the transition rates, allowing the use of Lyapunov matrices depending on all uncertain parameters affecting the system to certify the closed-loop mean square stability with $\mathscr{H}_\infty$ and $\mathscr{H}_2$ guaranteed costs for system (1), whenever the bounds for all elements belonging to a row of the transition rate matrix can be inferred.

## 4. EXAMPLES

This section presents numerical comparisons between the proposed approach and other methods from the literature. All routines were implemented in MATLAB (The MathWorks, Inc.), version 7.10 (R2010a) using Yalmip [32] and SeDuMi [33]. Furthermore, as the proposed LMI conditions require the representation of $\Gamma(\alpha)$ in the multi-simplex domain, a routine that automatically generates the vertices of $\Gamma(\alpha)$ is available for download at http://www.dt.fee.unicamp.br/~ricfow/programs/Gamma_Multi_Simplex_Cont.m.

*Example 1*

Consider the numerical example of [18] where an MJLS with four operation modes is presented. The aim is to design mode-dependent $\mathscr{H}_\infty$ state feedback gains regarding three descriptions for the transition rate matrix: the completely known case, the partially known Case I (CI) and Case II (CII). When the transition rate matrix is completely known and all the modes are available, the conditions of [18] and Theorem 1, with $g = (0, 0, 0, 0)$ and a large enough value of $\xi$, for instance $\xi \geqslant 10$, are necessary and sufficient, achieving the same $\mathscr{H}_\infty$ guaranteed cost, $\gamma = 1.2046$.

If the transition rate matrix is partially unknown, all the available LMI conditions are only sufficient, and the proposed methodology becomes less conservative because of the employment of slack variables, scalar parameter search, and to the use of polynomially parameter-dependent Lyapunov matrices. For instance, Theorem 1, with the particular choices $g = \mathbb{1} = (1, 1, 1, 0, 0, 0, 0)$ and $\xi = 10$, provided $\mathscr{H}_\infty$ guaranteed costs of $\gamma = 1.3066$ and $\gamma = 1.3085$ for the cases CI and CII, respectively. These outcomes outperform the $\mathscr{H}_\infty$ performance of [18] that provides $\gamma = 2.0451$ (CI) and $\gamma = 2.1265$ (CII). The behavior of the $\mathscr{H}_\infty$ guaranteed cost with respect to the scalar parameter and the increase of the degrees associated to the Lyapunov matrix is illustrated in Figure 1(a) (CI) and Figure 1(b) (CII).

*Example 2*

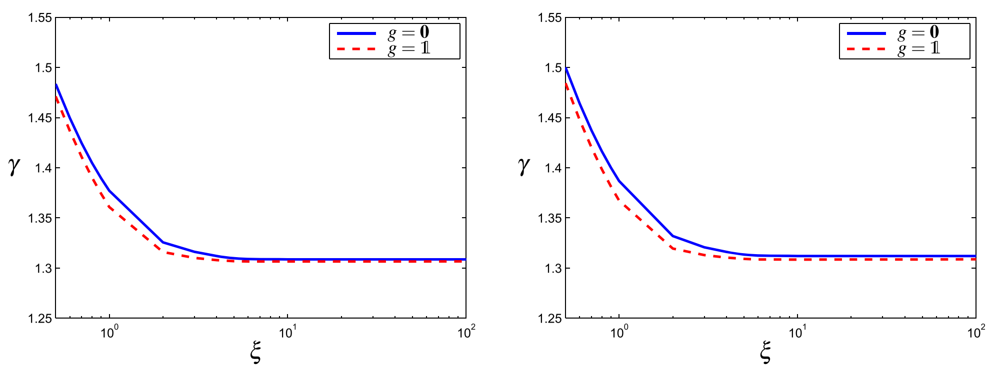
Consider the uncertain MJLS with two polytopic modes and precisely known transition rate matrix

$$A_{11} = \begin{bmatrix} -2.49 & 2.45 \\ 0.04 & -2.17 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -2.52 & 0.45 \\ 2.56 & 1.95 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} -1.22 \\ 0.84 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.43 \\ -1.60 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1.15 & -0.07 \\ -1.22 & 2.74 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2.77 & 1.96 \\ 1.41 & -3.28 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 1.67 \\ -0.42 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} -0.59 \\ -0.06 \end{bmatrix}$$

$$E_{ij} = \begin{bmatrix} -0.3 \\ 0.6 \end{bmatrix}, \quad C_{z ij}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D_{z ij} = 2, \quad \mu_i = 0.5, \quad \forall i, j \in \{1, 2\}, \quad \Gamma = \begin{bmatrix} -1.5 & 1.5 \\ 0.5 & -0.5 \end{bmatrix}$$

A mode-dependent stabilizing state feedback gain that assures an upper bound to the $\mathscr{H}_2$ norm of the closed-loop system is investigated. Three conditions were tested: Theorem 8 (T8) from [22], Lemma 4 (L4) from [22] (which is an $\mathscr{H}_2$ state feedback condition adapted by Theorem 4.1 from [26]), and Theorem 2 (T2) proposed in this paper. The aim is to compare the $\mathscr{H}_2$ performance levels ($\rho$) yielded by each one of those methods. As T8 is based on the reciprocal projection lemma [5], which does not contain the quadratic stability as a particular case, for this specific




(a) Case I (b) Case II

Figure 1. Behavior of the $\mathscr{H}_\infty$ guaranteed cost with respect to the scalar parameter ($\xi$) and the increase of degrees ($g$) of the Lyapunov matrix ($g = \mathbb{1} = (1, 1, 1, 0, 0, 0, 0)$ and $g = \mathbf{0} = (0, 0, 0, 0, 0, 0, 0)$) for the MJLS of the numerical example of [18] for two cases of the transition rate matrix.



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Table I. $\mathscr{H}_2$ guaranteed costs ($\rho$) for Example 2 obtained by Theorem 2 with different values of $\xi$ and $g = (g_1, g_2)$.

| $\xi$ | $g_1$ | $g_2$ | $\rho$ |
|---|---|---|---|
| $10^5$ | 0 | 0 | 6.6061 |
| $10^2$ | 1 | 0 | 6.6037 |
| $10^5$ | 0 | 1 | 6.6034 |
| 10 | 0 | 1 | 5.9167 |
| 10 | 1 | 1 | 5.8672 |

example, no solution has been obtained. On the other hand, T2, with the choices $\xi = 10^5$ and $g = (0, 0)$, produces $\rho = 6.6061$, the same result provided by L4. Theorem 2 can yield lower $\mathscr{H}_2$ guaranteed costs by performing a search of the scalar parameter and increasing the partial degrees of the Lyapunov matrices, as shown in Table I.

*Example 3*

Consider the MJLS presented in [19, Example 2], where the system matrices $\{A_i, B_i, E_i, C_{zi}, D_{zi}\}$ can be found, with transition rate matrix given by

$$\Gamma = \begin{bmatrix} -1.4 & ? & ? & 0.2 \\ 0.3 & -1.8 & 0.3 & 1.2 \\ \beta & ? & ? & ? \\ ? & 0.3 & ? & -0.8 \end{bmatrix} \tag{25}$$

The purpose of this example is to design state feedback gains such that the resulting closed-loop system is MSS with a bound to the $\mathscr{H}_2$ norm.

For the case where all the modes are accessible, the $\mathscr{H}_2$ guaranteed costs obtained by [19, Theorem 3] and T2 for different values of the uncertain rate $\beta$ are shown in Table II. If a mode-independent controller is sought and $\beta \in [0.0,\ 2.0]$, the result attained by [19, Theorem 3] is $\rho = 0.5667$ while the proposed approach, with $\xi = 10$ and $g = (0, 0, 0, 0, 0, 0)$, provides $\rho = 0.5359$. As can be seen, even without performing a search of the scalar parameter or increasing the degree of the Lyapunov matrices, T2 outperforms the results of [19, Theorem 3] for both assumptions on the mode availability.

Now, to deal with the case where the system matrices are polytopic, consider $\Gamma$ given by (25) with $\beta = 0.2$, and assume that all modes are accessible and have two vertices, such that $A_{i1} = A_i$, $A_{i2} = A_i + \delta\mathrm{I}$, $B_{ij} = B_i$, $E_{ij} = E_i$, $C_{zij} = C_{zi}$, and $D_{zij} = D_{zi}$, $\forall i = 1, \ldots, 4$, $\forall j = 1, 2$. The aim is to evaluate the conservativeness of the conditions presented in [19] (adapted for MJLS with polytopic matrices) and T2 for different values of the parameter $\delta \geqslant 0$. As illustrated in Figure 2, the methodology of T2 is less conservative than [19, Theorem 3], because $\delta = 3.77$ is the largest value of $\delta$ for which [19] yields a feasible solution, while the proposed approach can synthesize stabilizing gains until $\delta \cong 3.93$. Figure 2 also shows that, for a fixed value of $\delta$, the approach of T2, with $\xi = 100$ and $g = (0, 0, 0, 0, 0, 0)$, provides smaller attenuation levels than the ones from [19]. Such outcomes can be improved by increasing the degree of the Lyapunov matrices, as indicated in Table III.

Table II. $\mathscr{H}_2$ guaranteed costs ($\rho$) for Example 3 obtained by the conditions in [19] and Theorem 2 with $\xi = 10$ and $g = (0, 0, 0, 0, 0, 0)$.

| | | $0.1 \leqslant \beta \leqslant 0.6$ | $0.15 \leqslant \beta \leqslant 0.55$ | $0.19 \leqslant \beta \leqslant 0.53$ | $\beta = 0.2$ |
|---|---|---|---|---|---|
| $\rho$ | [19, Theorem 3] | 0.2950 | 0.2947 | 0.2946 | 0.2945 |
| | Theorem 2 | 0.2598 | 0.2592 | 0.2591 | 0.2589 |



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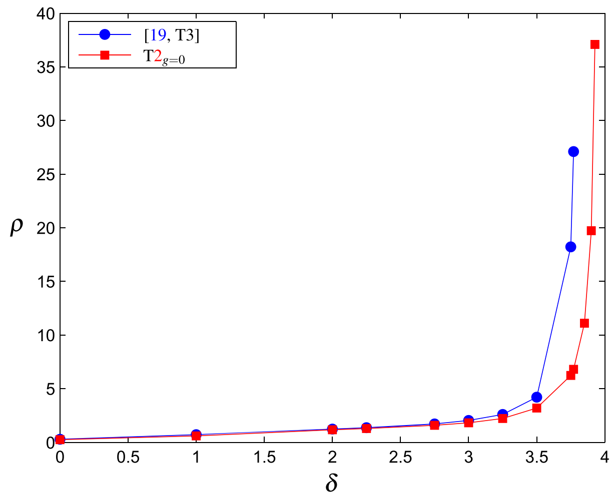


Figure 2. Behavior of the $\mathscr{H}_2$ guaranteed cost obtained by Theorem 2 with $g = (0,0,0,0,0,0)$ (T2$_{g=0}$) and the conditions adapted from [19, Theorem 3] ([19, T3]), with respect to parameter $\delta \geqslant 0$ for Example 3.

Table III. $\mathscr{H}_2$ guaranteed costs ($\rho$) for Example 3 with polytopic system matrices obtained by the conditions in [19] (adapted for Markov jump linear systems with polytopic matrices) and Theorem 2 (T2) with $\xi = 100$ and different degrees $g$.

| | $\delta$ | 2.0 | 3.0 | 3.5 | 3.75 | 3.77 | 3.875 |
|---|---|---|---|---|---|---|---|
| | [19, Theorem 3] (Adapted) | 1.2463 | 2.0482 | 4.2090 | 18.2194 | 27.1025 | – |
| $\rho$ | T2 with $g = (0,0,0,0,0,0)$ | 1.1736 | 1.8291 | 3.1980 | 6.2385 | 6.8065 | 14.1162 |
| | T2 with $g = (1,1,1,1,1,1)$ | 1.1729 | 1.8141 | 3.1556 | 6.0847 | 6.6308 | 13.6864 |

## 5. CONCLUSION

This paper investigated the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ state feedback control design problems for uncertain continuous-time MJLS, whose uncertainties are modeled in a unified representation through the multi-simplex methodology. Differently from the existing approaches, polynomially parameter-dependent Lyapunov matrices are used to assure the closed-loop MSS with $\mathscr{H}_\infty$ or $\mathscr{H}_2$ bounds. For the case of an MJLS free of uncertainties and complete mode availability, the proposed conditions are necessary and sufficient. Numerical examples demonstrated that the use of polynomially parameter-dependent Lyapunov matrices of arbitrary degrees, combined with the search of a scalar parameter, can decrease the conservativeness of the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ guaranteed costs for the considered class of MJLS.

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