Mesonic eightfold way from dynamics and confinement in strongly coupled lattice quantum chromodynamics

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We show the existence of all the 36 eightfold way mesons and determine their masses and dispersion curves exactly, from dynamical first principles such as directly from the quark-fluon dynamics. We also give a proof of confinement below the two-meson energy threshold. For this purpose, we consider an imaginary time functional integral representation of a 3+1 dimensional lattice QCD model with Wilson action, $\mathrm{SU}(3)_f$ global and $\mathrm{SU}(3)_c$ local symmetries. We work in the strong coupling regime, such that the hopping parameter $\kappa>0$ is small and much larger than the plaquette coupling $\beta>1/g_0^2\geqslant 0$ $(\beta\ll\kappa\ll 1)$. In the quantum mechanical physical Hilbert space $\mathcal{H}$, a Feynman-Kac type representation for the two-meson correlation and its spectral representation are used to establish an exact rigorous connection between the complex momentum singularities of the two-meson truncated correlation and the energy-momentum spectrum of the model. The total spin operator J and its z-component J_z are defined by using $\pi/2$ rotations about the spatial coordinate axes, and agree with the infinitesimal generators of the continuum for improper zero-momentum meson states. The mesons admit a labelling in terms of the quantum numbers of total isospin I, the third component I_3 of total isospin, the z-component J_z of total spin and quadratic Casimir C_2 for $\mathrm{SU}(3)_f$. With this labelling, the mesons can be organized into two sets of states, distinguished by the total spin J. These two sets are identified with the $\mathrm{SU}(3)_f$ nonet of pseudo-scalar mesons $(J=0)$ and the three nonets of vector mesons $(J=1, J_z=\pm 1, 0)$. Within each nonet a further decomposition can be made using C_2 to obtain the singlet state $(C_2=0)$ and the eight members of the octet $(C_2=3)$. By casting the problem of determination of the meson masses and dispersion curves into the framework of the the anaytic implicit function theorem, all the masses $m(\kappa,\beta)$ are found exactly and are given by convergent expansions in the parameters κ and β. The masses are all of the form $m(\kappa,\beta=0)\equiv m(\kappa)=-2\ln\kappa-3\kappa^2/2+\kappa^4 r(\kappa)$ with $r(0)\neq 0$ and $r(\kappa)$ real analytic; for $\beta>0, m(\kappa,\beta)+2\ln\kappa$ is jointly analytic in κ and β. The masses of the vector mesons are independent of J_z and are all equal within each octet. All isospin singlet masses are also equal for the vector mesons. For each nonet and $\beta=0$, up to and including $\mathcal{O}(\kappa^4)$, the masses of the octet and the singlet are found to be equal. But there is a pseudoscalar-vector meson mass splitting given by $2\kappa^4+\mathcal{O}(\kappa^6)$ and the splitting persists for $\beta>0$. For $\beta=0$, the dispersion curves are all of the form $w(\vec{p})=-2\ln\kappa-3\kappa^2/2+(\frac{1}{4})\kappa^2\Sigma_{j=1}^3 2(1-\cos p^j)+\kappa^4 r(\kappa,\vec{p})$, with $|r(\kappa,\vec{p})|\leqslant$const. For the pseudoscalar mesons, $r(\kappa,\vec{p})$ is jointly analytic in κ and p^j, for $|\kappa|$ and $|\mathrm{Im}\,p^j|$

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small. We use some machinery from constructive field theory, such as the decoupling of hyperplane method, in order to reveal the gauge-invariant eightfold way meson states and a correlation subtraction method to extend our spectral results to all $\mathcal{H}_e$, the subspace of $\mathcal{H}$ generated by vectors with an even number of Grassmann variables, up to near the two-meson energy threshold of $\approx -4 \ln \kappa$. Combining this result with a previously similar result for the baryon sector of the eightfold way, we show that the only spectrum in all $\mathcal{H} \equiv \mathcal{H}_e \oplus \mathcal{H}_o$ ($\mathcal{H}_o$ being the odd subspace) below $\approx -4 \ln \kappa$ is given by the eightfold way mesons and baryons. Hence, we prove confinement up to near this energy threshold. [DOI: 10.1063/1.2903751]

I. INTRODUCTION

A landmark in particle physics was achieved when Gell-Mann and Ne'eman independently proposed the eightfold way scheme to classify the then known hadronic particles (see Refs. 1–5). This model asserts that every hadron is composed of quarks with three flavors—*up* (u), *down* (d), and *strange* (s). The baryons are made of three quarks and the mesons are made of a quark-antiquark pair. Later, a color dynamics was introduced for the quarks via an exchange of gauge vector bosons called gluons, and today the local gauge model of interacting quarks and gluons based on the color symmetry $\mathrm{SU}(3)_c$, which is known as the quantum chromodynamics (QCD), is the best candidate to describe the strong interactions.

The lattice regularization of the continuum theory was introduced by Wilson in Ref. 6. Precisely, in Ref. 6, an imaginary-time functional integral formulation for lattice QCD was developed. In this framework, there are basically two ingredients: quarks, obeying Fermi–Dirac statistics, carrying flavor, color and spin labels, and located in each lattice site, and colored string bits connecting adjacent points on the lattice. Besides preserving gauge invariance and being free of ultraviolet singularities, the lattice formulation is powerful enough to, e.g., obtain the first results on the QCD particle spectrum and to exhibit confinement, i.e., isolated quarks are not observed. Within this construction, one formally recovers the continuum theory in the scaling limit (i.e., the lattice spacing going to zero). Later, it was shown by Osterwalder and Seiler that the lattice regularization of Wilson has the property of reflection positivity (see Refs. 7 and 8 for more details). This property enables the construction of the quantum mechanical Hilbert space of physical states $\mathcal{H}$ and allows us to define a positive self-adjoint energy and self-adjoint momentum operators. Within this framework, a Feynman–Kac (FK) formula is established and the energy-momentum (E-M) spectrum can be investigated.

The low-lying E-M spectrum (one-particle and two-particle bound states) was rigorously determined exactly in Refs. 9–19 for increasingly complex $\mathrm{SU}(3)_c$ lattice QCD models with one and two flavors in the strong coupling regime, i.e., with the hopping parameter κ and plaquette coupling β satisfying $0 < \beta \ll \kappa \ll 1$. In this regime, more recently, the $\mathrm{SU}(3)_f$ scheme for baryons was validated in Refs. 20 and 21 for the $(3+1)$-dimensional lattice QCD with three quark flavors and using the Wilson action. Fifty-six eightfold way baryon states of mass $\approx -3 \ln \kappa$ were obtained from first principles, i.e., directly from the quark-gluon dynamics. By concentrating on the subspace $\mathcal{H}_o \subset \mathcal{H}$ of vectors with an *odd* number of quarks and applying a correlation subtraction method, the eightfold way baryon and antibaryon spectrum was shown to be the only spectrum up to near the meson-baryon energy threshold of $\approx -5 \ln \kappa$. More precisely, up to near the meson-baryon energy threshold, all the E-M spectrum is generated by the 56 eightfold way baryon fields, and the baryon particles are strongly bound, bound states with three quarks. For each baryon, there is a corresponding antibaryon related by charge conjugation and with identical spectral properties. The reason for the restriction $\beta \ll \kappa$ is that in this region of parameters, the hadron spectrum (mesons and baryons of asymptotic masses $-2 \ln \kappa$ and $-3 \ln \kappa$, respectively) is the low-lying spectrum. If, on the other hand, $\beta \gg \kappa$, then the low-lying spectrum consists of only glueballs (of asymptotic mass $-4 \ln \beta$) and their excitations (see Ref. 22).

We point out that even for $\beta=0$ (no plaquette terms in the action), there is still a nontrivial dependence on the gauge field in the quadratic Fermi field hopping term. One manifestation of this dependence is that the meson dispersion curves have a small momentum behavior approximately proportional to $\kappa^2\vec{p}^2$. For free fermions (such as setting the gauge group elements equal to the identity), the behavior is proportional to $\kappa\vec{p}^2$ and the free fermion mass is asymptotically $-\ln\kappa$.

The baryons are detected as complex momentum singularities of the Fourier transform of a two-baryon correlation and are rigorously related to the *E-M* spectrum via a spectral representation to this quantity. The spectral representation for the two-point correlation is derived by using a FK formula.

The 56 baryon states can be grouped into two octets of total spin $J=\frac{1}{2}$ $\left(J_z=\pm\frac{1}{2}\right)$ and four decuplets of total spin $J=\frac{3}{2}$ $\left(J_z=\pm\frac{1}{2},\pm\frac{3}{2}\right)$. On the lattice, there is only the discrete $\pi/2$-rotation group. However, we can adapt the treatment usually employed in solid state physics and use the structure of point groups (see Ref. 23) to rigorously introduce total spin operators J and spin z-component J_z so that for zero-momentum states, there is a partial restoration of rotational symmetry, which means that these operators inherit the same structure as for the continuum. Using this shows that the masses within the octets and within the decuplets are independent of J_z. However, there is a mass splitting between the octets and the decuplets given to leading order of $3\kappa^6/4$ at $\beta=0$. For $\beta\neq 0$ and $m_b(\kappa)$ as the baryon mass, $m_b(\kappa)+3\ln\kappa$ is jointly analytic in κ and β. In particular, the mass splitting between octet and decuplet persists for $\beta\neq 0$.

It is worth remarking that the work of Refs. 20 and 21 is *not* the first publication on the one-particle spectrum of lattice QCD models with three flavors. Specially in the 1980s, many papers were devoted to the existence of baryons, e.g., the work of Refs. 24 and 25. However, these papers do not rely on spectral representations. This can be problematic when tiny splitting among the states is present. Also, the determination of momentum singularities of the two-point function via the zeros of uncontrolled expansion in the denominator of the Fourier transform of approximate propagators leaves the question of the nature and the existence of the supposed singularity unanswered. The same kind of problems may show up in works where the masses are determined by the exponential decay rates of two-point functions, as what is usually done in numerical simulations. We remark that we work with the exact correlation function, its convolution inverse, and their Fourier transforms.

In this work, in the strong coupling regime, and using the decoupling of hyperplane method (see Ref. 26–29), we complete the exact determination of the one-particle *E-M* spectrum initiated in Refs. 20 and 21. For a pedagogical presentation of the basic principles of the hyperplane decoupling method, see Ref. 28. Here, we consider the *even* subspace $\mathcal{H}_e\subset\mathcal{H}$ and show the existence of the 36 eightfold way meson (of asymptotic mass $-2\ln\kappa$) states.

The hyperplane decoupling method has many nice features.

- It enables us to obtain the basic local gauge-invariant excitation fields without any *a priori* guesswork. As will be shown, linear combination of these fields can be identified with the eightfold way particles, namely, the pseudoscalar and vector mesons.
- It gives good control of the global decay properties of the correlation functions involved.
- It enables us to show that the spectrum is generated by isolated dispersion curves, i.e., the upper gap property.
- It permits us to show that the only spectrum in all $\mathcal{H}_e$ is generated by the eightfold way particles.

By using a meson correlation subtraction method, we also show that the spectrum generated by the 36 eightfold way meson states, which are bound states of a quark and an anti-quark, is the only spectrum in the whole $\mathcal{H}_e$ up to near the two-meson threshold of $\approx -4\ln\kappa$. These 36 states can be grouped into four $SU(3)_f$ nonets–one associated with the pseudoscalar mesons ($J=0$) and three with the vector mesons ($J=1$, $J_z=0,\pm 1$). Each nonet admits a further decomposition into a singlet [with quadratic $SU(3)_f$ Casimir $C_2=0$] and an octet ($C_2=3$). The 36 mesons are labeled (and distinguished by this labeling) by J, J_z, C_2, and the quantum numbers of total isospin I (its square), third component of total isospin I_3, and total hypercharge Y.

The meson particles are detected by isolated dispersion curves $w(\vec{p})$ in the energy-momentum spectrum. They are of the form, for $\beta=0$, $w(\vec{p})=-2\ln\kappa-3\kappa^2/2+(\frac{1}{4})\kappa^2\Sigma_{j=1}^{3}2(1-\cos p^j)+\kappa^4 r(\kappa,\vec{p})$ with $|r(\kappa,\vec{p})|\leq$const. For the pseudoscalar mesons, $r(\kappa,\vec{p})$ is jointly analytic in κ and p^j, for $|\kappa|$ and $|\mathrm{Im}\, p^j|$ small. The meson masses are given by $m(\kappa)=-2\ln\kappa-3\kappa^2/2+\kappa^4 r(\kappa)$ with $r(0)\neq 0$ and $r(\kappa)$ real analytic. The nonsingular part of the mass, i.e., $r(\kappa)$, is jointly analytic in κ and β. For a fixed nonet, the masses of all vector mesons are independent of J_z and are all equal within each octet. All singlet masses are also equal. For $\beta=0$, up to and including $\mathcal{O}(\kappa^4)$, for each nonet, the masses of the octet and the singlet are found to be equal. All members of each octet have identical dispersions. Other dispersion curves may differ. Indeed, there is a pseudoscalar vector meson mass splitting (between $J=0$ and $J=1$) given by $2\kappa^4+\mathcal{O}(\kappa^6)$ and, by analyticity, the splitting persists for $\beta\neq 0$. Using a correlation subtraction method, we show that the 36 meson states give the only spectrum in $\mathcal{H}_e$ up to near the two-meson threshold of $\approx -4\ln\kappa$. Up to and including $\mathcal{O}(\kappa^4)$, there is no isospin singlet-isospin octet mass splitting at $\beta=0$. There may be a splitting at higher order in κ and β. For this splitting in the continuum model, see the U(1) problem in Ref. 30.

Combining the present result with the results of Refs. 20 and 21 shows confinement up to the two-meson threshold. We stress that even within the limitation of dealing with only three quark flavors, since not all the eightfold way hadrons have not, up to now, been experimentally observed (specially the heavy decuplet baryons and the scalar mesons), our results still give more strength to the Gell-Mann and Ne'eman quark model.

This paper is organized as follows. In Sec. II, we introduce the Wilson's lattice QCD model that we use. The decoupling of hyperplane method is also used to reveal the form of the basic excitation fields in $\mathcal{H}_e$. To establish a connection with the E-M spectrum, we define a matrix valued two-point meson function $\mathcal{G}$, in terms of the basic excitation fields with individual quark and antiquark spin and isospin labels, and introduce a spectral representation for $\mathcal{G}$. Our main result, stating the existence of the eightfold way mesons, determining their masses, multiplicitiesm, and dispersion curves, is then presented in Theorem 1. To determine the meson masses and dispersion curves, we analyze the convolution inverse Λ of the two-point correlation. We need global bounds on $\mathcal{G}$ and Λ as well as their short distance behavior to the lowest orders in κ. They are given in Theorems 2 and 3, respectively. In Sec. III, we make a basis change from the individual spin and isospin basis to the particle basis. This basis change is implemented by an orthogonal transformation (normalized at $\kappa=0$) and allows us to identify the basic excitation fields with the eightfold way pseudoscalar and vector mesons. As for the baryon case, to carry out this identification, we label the states by their associated quantum numbers—total isospin I, its third component I_3, total hypercharge, and quadratic Casimir C_2—by using the global flavor symmetry. We also need to consider spin and we adopt the same prescription as in Refs. 20 and 21 in such a way that for improper zero-momentum meson states, the total spin operator J and related operators J_x, J_y, and J_z (and also $J_\pm$) agree with the infinitesimal generators of the continuum. Finally, we determine the meson masses (the mass splittings) and the dispersion curves. In Sec. VI, we consider the implementation of total isospin and related operators and total hypercharge operator as operators acting on the quantum mechanical physical Hilbert space $\mathcal{H}$ by following Refs. 20 and 21. In Sec. V, we employ a subtraction method to extend our spectral results from the subspace generated by the eightfold way meson fields to all $\mathcal{H}_e$. In Appendix A, to simplify the proof of Theorem 3, we establish correlation identities using ordinary symmetries as discrete rotations, parity, time reversal, charge conjugation, and coordinate reflections (for further details, we refer the reader to Ref. 10). A new symmetry of time reflection established in Refs. 20, 21, and 31 is also used and a composition with parity and time reversal gives a spin flip symmetry given in Sec. III B. In Appendix B, we derive a very useful formula for calculating contributions to the two-point function $\mathcal{G}(x,y)$ of nonintersecting paths connecting the lattice point x to y. This formula is employed in the proof of Theorem 3.

II. MODEL AND RESULTS

In the first subsection, we introduce the lattice QCD model and the physical quantum mechanical Hilbert space $\mathcal{H}$. In particular, the FK formula and the self-adjoint E-M operators are considered. We also state our main results on the E-M spectrum and the existence of meson particles in all $\mathcal{H}_e$ in Theorem 1. In the second subsection, the decoupling of hyperplane method is introduced and used to obtain the basic excitation states appearing in the two-point function. To determine the E-M spectrum, we need the long distance and short distance behaviors of the two-point function and its convolution inverse, which are presented in Theorems 2 and 3.

A. Model and the physical quantum mechanical Hilbert space

We use the same model presented in Refs. 20 and 21. The model is the $\mathrm{SU}(3)_f$ lattice QCD with the partition function formally given by $Z=\int e^{-\mathcal{S}(\psi,\bar{\psi},g)}d\psi d\bar{\psi}d\mu(g)$, and for a function $F(\psi,\bar{\psi},g)$, the normalized correlations are denoted by

$$\langle F\rangle = \frac{1}{Z}\int F(\psi,\bar{\psi},g)e^{-\mathcal{S}(\psi,\bar{\psi},g)}d\psi d\bar{\psi}d\mu(g). \tag{1}$$

The action $\mathcal{S}\equiv\mathcal{S}(\psi,\bar{\psi},g)$ is given by

$$\mathcal{S}(\psi,\bar{\psi},g) = \frac{\kappa}{2}\sum \bar{\psi}_{a,\alpha,f}(u)\Gamma^{\epsilon e^\rho}_{\alpha\beta}U(g_{u,u+\epsilon e^\rho})_{ab}\psi_{b,\beta,f}(u+\epsilon e^\rho) + \sum_{u\in\mathbb{Z}^4_o}\bar{\psi}_{a,\alpha,f}(u)M_{\alpha\beta}\psi_{a,\beta,f}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p), \tag{2}$$

where the first sum is over $u\in\mathbb{Z}^4_o$, $\epsilon=\pm 1$ and $\rho=0,1,2,3$ and over repeated indices. By denoting 0 as the temporal direction, the lattice is given by $\mathbb{Z}^4_o$, where $u=(u^0,\vec{u})=(u^0,u^1,u^2,u^3)\in\mathbb{Z}^4_o \equiv\mathbb{Z}_{1/2}\times\mathbb{Z}^3$, where $\mathbb{Z}_{1/2}=\{\pm\frac{1}{2},\pm\frac{3}{2},\ldots\}$. For each site $u\in\mathbb{Z}^4_o$, there are fermionic fields represented by Grassmann variables $\psi_{a,\alpha,f}(u)$, associated with a quark, and $\bar{\psi}_{a,\alpha,f}(u)$, associated with an antiquark, which carry a Dirac spin $\alpha=1,2,3,4$, a color $a=1,2,3$, and flavor or "isospin" $f=u,d,s=1,2,3$ index. Γ is related to the Dirac matrices by $\Gamma^{\pm e^\rho}=-1\pm\gamma^\rho$. The γ^ρ are the Dirac 4×4 matrices

$$\gamma^0=\begin{pmatrix} I_2 & 0\\ 0 & -I_2\end{pmatrix} \quad\text{and}\quad \gamma^j=\begin{pmatrix} 0 & i\sigma^j\\ -i\sigma^j & 0\end{pmatrix},$$

where σ^j, $j=1,2,3$, denote the Pauli 2×2 matrices and I_2 is the 2×2 identity matrix. For each oriented bound on the lattice $\langle u,u\pm e^\rho\rangle$, there is a matrix $U(g_{u,u\pm e^\rho})\in\mathrm{SU}(3)$ parametrized by the gauge group element $U(g_{u,u\pm e^\rho})$ satisfying $U(g_{u,u+e^\rho})^{-1}=U(g_{u+e^\rho,u})$. The parameter κ is the hopping parameter and $\beta\equiv 1/g_0^2$ is the plaquette coupling. The measure $d\mu(g)$ is the product measure over nonoriented bonds of normalized $\mathrm{SU}(3)_c$ Haar measures (see Ref. 32). There is only one integration variable per bond, so that g_{uv} and g^{-1}_{vu} are not treated as distinct integration variables. The integrals over Grassmann fields are defined according to Ref. 33. For a polynomial in the Grassmann variables with coefficients depending on the gauge variables, the fermionic integral is defined as the coefficient of the monomial of maximum degree, i.e., of $\Pi_{u,k}\psi_k(u)\bar{\psi}_k(u)$, $k\equiv(a,\alpha,f)$. In Eq. (1), $d\psi d\bar{\psi}$ means $\Pi_{u,k}d\psi_k(u)d\bar{\psi}_k(u)$ such that with a normalization $\mathcal{N}=\langle 1\rangle$, we have $\langle\psi_{k_1}(x)\bar{\psi}_{k_2}(y)\rangle=(1/\mathcal{N})\int\psi_{k_1}(x)\bar{\psi}_{k_2}(y)e^{-\Sigma_{u,k_3,k_4}\bar{\psi}_{k_3}(u)O_{k_3k_4}\psi_{k_4}(u)}d\psi d\bar{\psi}=O^{-1}_{\alpha_1\alpha_2}\delta_{a_1a_2}\delta_{f_1f_2}\delta(x-y)$ with a Kronecker delta for space-time coordinates, where O is diagonal in the color and isospin indices.

By polymer expansion methods (see Refs. 7, 27, and 28), the thermodynamic limit of correlations exists and truncated correlations have an exponential tree decay. The limiting correlation functions are lattice translational invariant. Furthermore, the correlation functions extend to analytic functions in the global coupling parameters κ and $\beta \equiv 1/g_0^2$ and also in any finite number of local coupling parameters. For the formal hopping parameter expansion, see Refs. 34–36.

Associated with the $\mathrm{SU}(3)_f$ model, there is an underlying physical Hilbert space which we denote by $\mathcal{H}$. Starting from gauge-invariant correlations with support restricted to $u^0=\frac{1}{2}$ and letting $T_0^{x^0}$, $T_i^{x^i}$, $i=1,2,3$, denoting translation of the functions of Grassmann and gauge variables (is used to denote Hilbert space operators) by $x^0 \geqslant 0$, $\vec{x}=(x^1,x^2,x^3) \in \mathbb{Z}^3$, there is the FK formula for F and G only depending on coordinates with $u^0=\frac{1}{2}$ given by

$$(G, \check{T}_0^{x^0} \check{T}_1^{x^1} \check{T}_2^{x^2} \check{T}_3^{x^3} F)_{\mathcal{H}} = \langle [T_0^{x^0} \vec{T}^{\vec{x}} F] \Theta G \rangle, \tag{3}$$

where $\vec{T}^{\vec{x}}=T_1^{x^1}T_2^{x^2}T_3^{x^3}$ and Θ is an antilinear operator that involves time reflection. Following Ref. 10, the action of Θ on single fields is given by

$$\Theta \bar{\psi}_{a,\alpha,f}(u) = (\gamma^0)_{\alpha\beta} \psi_{a,\beta,f}(u_t),$$

$$\Theta \psi_{a,\alpha,f}(u) = \bar{\psi}_{a,\beta,f}(u_t)(\gamma^0)_{\beta\alpha},$$

where $u_t=(-u^0,\vec{u})$ for A and B monomials, $\Theta(AB)=\Theta(B)\Theta(A)$, and $\Theta f(\{g_{uv}\})=f^*(\{g_{u_t v_t}\})$, $u,v \in Z_o^4$, for a function of the gauge fields where $*$ means complex conjugate. Θ antilinearly extends to the algebra. For simplicity, we do not distinguish between Grassmann and gauge variables and their associated Hilbert space vectors in our notation. As linear operators in $\mathcal{H}$, $\check{T}_\rho$, $\rho=0,1,2,3$, are mutually commuting; $\check{T}_0$ is self-adjoint with $-1 \leqslant \check{T}_0 \leqslant 1$ and $\check{T}_{j=1,2,3}$ are unitary. So, we write $\check{T}_j = e^{iP^j}$ and $\vec{P}=(P^1,P^2,P^3)$ is the self-adjoint momentum operator with spectral points $\vec{p} \in \mathbf{T}^3 \equiv (-\pi,\pi]^3$. Since $\check{T}_0^2 \geqslant 0$, the energy operator $H>0$ can be defined by $\check{T}_0^2=e^{-2H}$. We call a point in the E-M spectrum associated with spatial momentum $\vec{p}=\vec{0}$ a mass and, to be used below, we let $\mathcal{E}(\lambda_0,\vec{\lambda})$ be the product of the spectral families of $\check{T}_0$, P^1, P^2, and P^3. By the spectral theorem (see Ref. 37), we have

$$\check{T}_0 = \int_{-\pi}^{\pi} \lambda^0 dE_0(\lambda^0), \quad \check{T}_{j=1,2,3} = \int_{-\pi}^{\pi} e^{i\lambda^j} dF_j(\lambda^j),$$

so that $\mathcal{E}(\lambda_0,\vec{\lambda})=E_0(\lambda_0)\Pi_{j=1}^3 F_j(\lambda^j)$. The positivity condition $\langle F\Theta F\rangle \geqslant 0$ is established in Ref. 8, but there may be nonzero F's such that $\langle F\Theta F\rangle=0$. If the collection of such F's is denoted by $\mathcal{N}$, a pre-Hilbert space $\mathcal{H}'$ can be constructed from the inner product $\langle G\Theta F\rangle$ and the physical Hilbert space $\mathcal{H}$ is the completion of the quotient space $\mathcal{H}'/\mathcal{N}$, including also the Cartesian product of the inner space sectors, the color space $\mathbb{C}^3$, the spin space $\mathbb{C}^4$, and the isospin space $\mathbb{C}^3$.

By considering the parameters κ, β, it is to be understood that the following conditions hold in the sequel: there exist $\kappa_0>0$, $\beta_0>0$, and $\beta_0/\kappa_0>0$ sufficiently small. Under this condition, our results hold for all physical values of κ and β such that $\kappa<\kappa_0$, $\beta<\beta_0$, and $\beta/\kappa \leqslant \beta_0/\kappa_0 \ll 1$. The main result of this paper is summarized in Theorem 1 below.

Theorem 1: *The low-lying energy-momentum spectrum of the lattice QCD model given by the action of Eq. (2) in the strong coupling regime, in the even subspace $\mathcal{H}_e \subset \mathcal{H}$, and up to near the two-meson threshold of $\approx -4 \ln \kappa$ is generated by 36 states, which are bound states of a quark and an anti-quark. These 36 states are labelled by the $\mathrm{SU}(3)_f$ quantum numbers I, I_3, Y, and C_2. Also, for zero-momentum states, a spin labeling can be introduced. The meson states can be distinguished and grouped into three $\mathrm{SU}(3)_f$ nonets associated with the vector mesons $(J=1, J_z=0, \pm 1)$ and one nonet associated with the pseudoscalar mesons $(J=0)$. Each nonet admits a further decomposition into a $\mathrm{SU}(3)_f$ singlet $(C_2=0)$ and an octet $(C_2=3)$. The particles are detected by isolated dispersion curves $w(\vec{p})$ in the energy-momentum spectrum. The 36 dispersion curves are all of the form, for $\beta=0$,*

$$w(\vec{p}) = -2\ln\kappa - 3\kappa^2/2 + \left(\tfrac{1}{4}\right)\kappa^2\sum_{j=1}^{3} 2(1-\cos p^j) + \kappa^4 r(\kappa,\vec{p}),$$

with $|r(\kappa,\vec{p})| \le \text{const}$. *For the pseudoscalar mesons, we can show that* $r(\kappa,\vec{p})$ *is jointly analytic in* κ *and* p^j, *for* $|\kappa|$ *and* $|\text{Im}\, p^j|$ *small. The meson masses are of the form*

$$m(\kappa) = -2\ln\kappa - 3\kappa^2/2 + \kappa^4 r(\kappa),$$

with $r(0)\neq 0$ *and* $r(\kappa)$ *real analytic. The* $m(\kappa)+2\ln\kappa$ *is jointly analytic in* κ *and* β. *For a fixed nonet, the masses of all vector mesons are independent of* J_z *and are all equal within each octet. All singlet masses are also equal for the vector mesons. For* $\beta=0$, *up to and including* $\mathcal{O}(\kappa^4)$, *for each nonet, the masses of the octet and the singlet are found to be equal. All members of each octet have identical dispersion curves. Other dispersion curves may differ. Indeed, there is a pseudoscalar vector meson mass splitting (between* $J=0$ *and* $J=1$) *given by* $2\kappa^4+\mathcal{O}(\kappa^6)$; *the splitting persists for* $\beta\neq 0$. *Finally, by, combining the above results with similar results for the eightfold way baryons, i.e., the eightfold way baryons are the only spectrum in all* $\mathcal{H}_o$ *up to near the meson-baryon threshold* ($\approx -5\ln\kappa$), *we also show that up to near the two-meson threshold, the one-hadron spectrum in* $\mathcal{H}$ *of our lattice QCD model solely consists of the eightfold way gauge-invariant baryon and meson states, and confinement is thus proven up to near this threshold.*

□

Proof. The proof of Theorem 1 follows from Theorems 2 and 3 stating global bounds and the short-distance behavior in κ, respectively, of the two-point function and its convolution inverse. The fact that the masses depend only on the total spin follows from the analysis of Sec. III B. In Sec. III D, by using Theorems 2 and 3, the pseudoscalar and vector meson masses, dispersion curves, and their multiplicities are exactly determined. By using a meson correlation subtraction method, in Sec. V, we extend our spectral results to all $\mathcal{H}_e$, up to near the two-meson threshold. ■

B. One-meson spectrum

We start by introducing a spectral representation for the two-point subtracted correlation. To obtain the spectral representation, we use the FK formula and observe that for $M,L\in\mathcal{H}_e$ and with support at time $u^0=\frac{1}{2}$, denoting them by $M\left(\frac{1}{2}\right)\equiv M\left(\frac{1}{2},\vec{0}\right)$ and $L\left(\frac{1}{2}\right)\equiv L\left(\frac{1}{2},\vec{0}\right)$,

$$\left((1-P_\Omega)M\left(\tfrac{1}{2}\right),\check{T}_0^{|v^0-u^0|-1}\check{\vec{T}}^{\vec{v}-\vec{u}}(1-P_\Omega)L\left(\tfrac{1}{2}\right)\right)_{\mathcal{H}} = \left\langle [T_0^{|v^0-u^0|-1}\vec{T}^{\vec{v}-\vec{u}}L\left(\tfrac{1}{2}\right)](\Theta M)\left(-\tfrac{1}{2}\right)\right\rangle_T. \tag{4}$$

Note that P_Ω is the projection onto the vacuum state $\Omega\equiv 1$ since we are interested in the spectrum generated by vectors orthogonal to the vacuum. From the relation obtained above, we have for $v^0>u^0$,

$$\begin{aligned}\left\langle\left[T_0^{v^0-u^0-1}\vec{T}^{\vec{v}-\vec{u}}L\left(\tfrac{1}{2}\right)\right](\Theta M)\left(-\tfrac{1}{2}\right)\right\rangle_T &= \left\langle\left[T_0^{v^0-1/2}\vec{T}^{\vec{v}}L\left(\tfrac{1}{2}\right)\right]\left[T_0^{u^0+1/2}\vec{T}^{\vec{u}}(\Theta M)\left(-\tfrac{1}{2}\right)\right]\right\rangle_T\\ &= \langle\Theta M(u^0,\vec{u})L(v^0,\vec{v})\rangle_T,\end{aligned}$$

where we have defined $L(v^0,\vec{v})\equiv T_0^{v^0-1/2}\vec{T}^{\vec{v}}L\left(\frac{1}{2}\right)$ and $\Theta M(u^0,\vec{u})=T_0^{u^0+1/2}\vec{T}^{\vec{u}}(\Theta M)\left(-\frac{1}{2}\right)$. Similarly, for $v^0<u^0$, by moving the energy and momentum operators to the left-hand (LHS) and taking the complex conjugate, we have

$$\left((1-P_\Omega)L\left(\tfrac{1}{2}\right),\check{T}_0^{u^0-v^0-1}\check{\vec{T}}^{\vec{u}-\vec{v}}(1-P_\Omega)M\left(\tfrac{1}{2}\right)\right)^*_{\mathcal{H}} = \left\langle [T_0^{u^0-v^0-1}\vec{T}^{\vec{u}-\vec{v}}M\left(\tfrac{1}{2}\right)]\Theta L(-1/2)\right\rangle^*_T. \tag{5}$$

From the relation obtained above, we have $\left\langle\left[T_0^{u^0-v^0-1}M\left(\frac{1}{2},\vec{u}\right)\right]\Theta L\left(-\frac{1}{2},\vec{v}\right)\right\rangle^*_T = \langle M(u^0,\vec{u})\Theta L(v^0,\vec{v})\rangle^*_T$.

With this, we define the general two-point function below, with M,L in $\mathcal{H}_e$, and using the $u^0 < v^0$ value to extend the correlation values to $u^0 = v^0$,

$$\mathcal{G}_{ML}(u,v) \equiv \begin{cases} \langle \Theta M(u) L(v) \rangle_T, & u^0 \leqslant v^0 \\ \langle M(u) \Theta L(v) \rangle_T^*, & u^0 > v^0. \end{cases} \tag{6}$$

The correlation in Eq. (6) admits the spectral representation, for $x^0 \neq 0$, $x = (x^0, \vec{x}) = v - u$, $\mathcal{G}_{ML}(x) \equiv \mathcal{G}_{ML}(0, v-u) = \mathcal{G}_{ML}(u,v)$,

$$\mathcal{G}_{ML}(x) = \int_{-1}^{1} \int_{T^d} \lambda_0^{|x^0|-1} e^{i\vec{\lambda}\cdot\vec{x}} d((1-P_\Omega)M(1/2), \mathcal{E}(\lambda_0, \vec{\lambda})(1-P_\Omega)L(1/2))_{\mathcal{H}}.$$

Our starting point to obtain the eightfold way mesons, as mentioned before, is the product structure obtained by using the decoupling of hyperplane method. Before we introduce the method, we use a convenient form to represent the subtracted two-point function, namely, the duplicate of variable representation. By this, we mean to replace the two-point function by an equivalent expression depending on two independent variables $\hat{\psi}$, $\hat{\bar{\psi}}$, g, g' to obtain (we take $M = \Theta L$ in what follows)

$$\langle M(u) L(v) \rangle_T = \frac{1}{2Z^2} \int [M(u) - M'(u)][L(v) - L'(v)] \exp[-\mathcal{S}(\psi, \bar{\psi}, g) - \mathcal{S}(\psi', \bar{\psi}', g')] d\psi d\bar{\psi} d\mu(g) d\psi' d\bar{\psi}' d\mu(g') \equiv \langle\langle [M(u) - M'(u)][L(v) - L'(v)] \rangle\rangle, \tag{7}$$

where

$$\langle M(u) L(v) \rangle_T = \langle M(u) L(v) \rangle - \langle M(u) \rangle \langle L(v) \rangle \tag{8}$$

is the truncated two-point function.

The decoupling of hyperplane method consists in replacing the hopping parameter κ by $\kappa_p \in \mathbb{C}$ for bonds p connecting $u^0 + \frac{1}{2} \leqslant p \leqslant v^0 - \frac{1}{2}$ and β by $\beta_p \in \mathbb{C}$ for plaquettes connecting the hyperplanes $u^0 + \frac{1}{2} \leqslant p \leqslant v^0 - \frac{1}{2}$. Actually, the procedure should start in a finite volume, and then to reach the infinite volume, we use standard consequences of the polymer expansion. This must be understood in what follows. For ease of presentation, we refer to the time direction as the vertical direction. We expand in the two variables κ_p and β_p, around $\kappa_p, \beta_p = 0$, the numerator $\mathcal{N}$ and denominator $\mathcal{D}$ of the general two-point function of Eq. (6) and we use the notation $\mathcal{N}^{(m,n)}(u,v)$ $[\mathcal{D}^{(m,n)}(u,v)]$ with $m,n \geqslant 0$, which means the coefficient of $\kappa_p^m \beta_p^n$ in the expansion of the numerator (denominator). First, $\mathcal{N}^{(0,0)} = 0$ since $M(u)$ decouples from $L(v)$ under the expectation $\langle \cdot \rangle$ in the two-point subtracted function. For κ_p^0, the first nonvanishing coefficient is β_p^4, arising from four vertical plaquettes composing the four vertical sides of a cube. Next, $\mathcal{N}^{(2m+1,n)} = 0 = \mathcal{D}^{(2m+1,n)}$, recalling that each expectation factorizes and each factor has an odd number of fermion fields giving zero. The term $\mathcal{N}^{(0,1)} = 0 = \mathcal{D}^{(0,1)}$ is trivially zero due to gauge integration. The term $\mathcal{N}^{(0,2)}$, corresponding to two superposed vertical plaquettes with opposite orientation, is zero. More precisely, by using the gauge integral $\mathcal{I}_2$ [see Eq. (B2)] and the fact that, as in $\mathcal{N}^{(0,0)} = 0$, the fields decouple, we get the desired result. $\mathcal{N}^{(0,3)}$ is zero, corresponding to three superposed vertical plaquettes with the same orientation and using the gauge integral $\mathcal{I}_3$ [see Eq. (B3)] for the vertical sides of the superposed plaquettes. Therefore by collecting our results for $m+n \leqslant 3$, we get

$$\mathcal{G}_{ML}(u,v) = \mathcal{G}_{ML}^{(2,0)}(u,v) \kappa_p^2 + \mathcal{G}_{ML}^{(2,1)}(u,v) \kappa_p^2 \beta_p + (\text{higher order terms}), \tag{9}$$

where we have used

$$\mathcal{G}^{(0,0)}_{ML}(u,v) = \mathcal{G}^{(1,0)}_{ML}(u,v) = \mathcal{G}^{(0,1)}_{ML}(u,v) = \mathcal{G}^{(1,1)}_{ML}(u,v) = \mathcal{G}^{(0,2)}_{ML}(u,v) = \mathcal{G}^{(3,0)}_{ML}(u,v) = \mathcal{G}^{(1,2)}_{ML}(u,v) = \mathcal{G}^{(0,3)}_{ML}(u,v) = 0.$$

Considering the second κ_p derivative of $\mathcal{G}$, i.e., $\mathcal{G}^{(2,0)}_{ML}$, and for the time ordering $u^0 \leq p < v^0$,

$$\begin{aligned}\langle M(u)L(v)\rangle^{(2,0)}_T &= \sum_{\vec{w}} \langle\langle [M(u) - M'(u)]\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle\rangle^{(0,0)} \langle\langle \mathcal{M}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})[L(v) - L'(v)]\rangle\rangle^{(0,0)} \\ &= \sum_{\vec{w}} \langle [M(u) - \langle M(u)\rangle]\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle^{(0,0)} \langle \mathcal{M}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})[L(v) - \langle L(v)\rangle]\rangle^{(0,0)},\end{aligned} \tag{10}$$

which we write schematically as

$$\mathcal{G}^{(2,0)}_{LL}(u,v) = [\mathcal{G}^{(0,0)}_{L\bar{\mathcal{M}}} \circ \mathcal{G}^{(0,0)}_{\bar{\mathcal{M}}L}](u,v).$$

Similarly, for the time ordering $u^0 > p \geq v^0$, we have for $L = \Theta M$,

$$\begin{aligned}\langle M(u)L(v)\rangle^{(2,0)}_T &= \sum_{\vec{w}} \langle\langle [M(u) - M'(u)]\mathcal{M}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\rangle\rangle^{(0,0)} \langle\langle \bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(p,\vec{w})[L(v) - L'(v)]\rangle\rangle^{(0,0)} \\ &= \sum_{\vec{w}} \langle [M(u) - \langle M(u)\rangle]\mathcal{M}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\rangle^{(0,0)} \langle \bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(p,\vec{w})[L(v) - \langle L(v)\rangle]\rangle^{(0,0)},\end{aligned} \tag{11}$$

written also as, after taking the complex conjugate of Eq. (11),

$$\mathcal{G}^{(2,0)}_{MM}(u,v) = [\mathcal{G}^{(0,0)}_{M\bar{\mathcal{M}}} \circ \mathcal{G}^{(0,0)}_{\bar{\mathcal{M}}M}](u,v).$$

In Eq. (10), we have defined

$$\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}} = \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\gamma_\ell,g_1}\psi_{a,\gamma_u,g_2}, \tag{12}$$

and, similarly, we define the $\mathcal{M}$ fields

$$\mathcal{M}_{\vec{\gamma}\vec{g}} = \tfrac{1}{\sqrt{3}}\psi_{a,\gamma_\ell,g_1}\bar{\psi}_{a,\gamma_u,g_2}, \tag{13}$$

i.e., making the change $\psi \rightarrow \bar{\psi}$ and $\bar{\psi} \rightarrow \psi$ and preserving the color ($a=1,2,3$), spin ($\alpha_u, \beta_u, \gamma_u = 1,2$, $\alpha_\ell, \beta_\ell, \gamma_\ell = 3,4$), and flavor or isospin index ($f,g,h=u,d,s\equiv 1,2,3$). In Eqs. (12) and (13) and in the sequel, we use the indices $\vec{f}$, $\vec{g}$, $\vec{h}$ and $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ to denote (f_1,f_2), (g_1,g_2), (h_1,h_2) and (α_ℓ,α_u), (β_ℓ,β_u), (γ_ℓ,γ_u), respectively. Also, by considering Eq. (12) [Eq. (13)], the spin index of $\bar{\psi}$ (ψ) is always a lower one, i.e., $\gamma_\ell = 3,4$, in contrast to ψ ($\bar{\psi}$) for which it is an upper one, i.e., $\gamma_u = 1,2$. The normalization factor $\frac{1}{\sqrt{3}}$ is such that at coincident points, $f_1 = f_2$, and for $\kappa = 0$, the two-point function of Eq. (16) below is the 4×4 identity matrix I_4. We refer to the quark-antiquark fields in Eq. (12) as the fundamental excitation fields in the individual basis since each isospin and spin index of the fermion fields individually appears.

The next term in the expansion of Eq. (9) is $\mathcal{G}^{(2,1)}_{ML}$, but because of our parameter restriction, i.e., $\beta_p \ll \kappa_p$, this is a subdominant term. This term is associated with a vertical plaquette with two bonds, coming from the quark field dependent part of the action, which is superposed the two vertical sides of the plaquette and in opposite orientation. We note that $\kappa_p^2\beta_p \ll \kappa_p^3$ from which we get the restriction $\beta_p/\kappa_p \ll 1$.

Remark 1: Note that the fields in Eqs. (12) and (13) are local composite fermion fields and gauge invariant (colorless).

We now obtain a fundamental property called as the *product structure*, which plays a major role in our analysis. In what follows, for simplicity, we drop the superscript notation $(m,0)$ and simply write (m), which means the coefficient of κ_p^m at $\kappa_p=\beta_p=0$. For closure, which means that correlations on the LHS and right-hand side (RHS) of Eq. (10) are the same, we take the fields $M=\mathcal{M}_{\vec{\alpha}\vec{f}}$ and $L=\bar{\mathcal{M}}_{\vec{\beta}\vec{h}}$ in Eq. (10) to obtain, for $u^0\leqslant p<v^0$,

$$\langle \mathcal{M}_{\vec{\alpha}\vec{f}}(u)\bar{\mathcal{M}}_{\vec{\beta}\vec{h}}(v)\rangle_T^{(2)} = \sum_{\vec{w}} \langle \mathcal{M}_{\vec{\alpha}\vec{f}}(u)\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(p,\vec{w})\rangle_T^{(0)}\langle \mathcal{M}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\bar{\mathcal{M}}_{\vec{\beta}\vec{h}}(v)\rangle_T^{(0)}, \tag{14}$$

i.e., the aforesaid *product structure*. Similarly to Eq. (14), we have for $v^0\leqslant p<u^0$, taking $M=\bar{\mathcal{M}}_{\vec{\alpha}\vec{f}}$ and $L=\mathcal{M}_{\vec{\beta}\vec{h}}$ in Eq. (11),

$$\langle \bar{\mathcal{M}}_{\vec{\alpha}\vec{f}}(u)\mathcal{M}_{\vec{\beta}\vec{h}}(v)\rangle_T^{(2)} = \sum_{\vec{w}} \langle \bar{\mathcal{M}}_{\vec{\alpha}\vec{f}}(u)\mathcal{M}_{\vec{\gamma}\vec{g}}(p+1,\vec{w})\rangle_T^{(0)}\langle \bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(p,\vec{w})\mathcal{M}_{\vec{\beta}\vec{h}}(v)\rangle_T^{(0)}. \tag{15}$$

Equations (14) and (15) can be put together by schematically writing

$$\mathcal{G}^{(2)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(u,v) = [\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}} \circ \mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}](u,v).$$

This property is our guide to define the two-point function of Eq. (16) and to perform our analysis, i.e., to obtain the mesonic eightfold way from a dynamical perspective.

In agreement with the general definition of Eq. (6), the two-point function for the basic excitation fields is defined by, writing $\mathcal{G}_{\bar{M}_\ell \bar{M}_{\ell'}}(u,v)\equiv \mathcal{G}_{\ell\ell'}(u,v)=\mathcal{G}_{\ell\ell'}(x=u-v)$,

$$\begin{aligned}\mathcal{G}_{\ell\ell'}(x) &= \langle \mathcal{M}_\ell(u)\bar{\mathcal{M}}_{\ell'}(v)\rangle_T \chi_{u^0\leqslant v^0} + \langle \bar{\mathcal{M}}_\ell(u)\mathcal{M}_{\ell'}(v)\rangle_T^* \chi_{u^0>v^0} \\ &= \langle \mathcal{M}_\ell(u)\bar{\mathcal{M}}_{\ell'}(v)\rangle \chi_{u^0\leqslant v^0} + \langle \bar{\mathcal{M}}_\ell(u)\mathcal{M}_{\ell'}(v)\rangle^* \chi_{u^0>v^0},\end{aligned} \tag{16}$$

where the subscripts $\ell=(\vec{\alpha},\vec{f})$ and $\ell'=(\vec{\beta},\vec{h})$ are collective indices. Note that in Eq. (16), the subtraction $\langle \mathcal{M}_\ell(u)\rangle=\langle \bar{\mathcal{M}}_\ell(u)\rangle=0$ is zero by using parity symmetry (to be defined ahead). This must be understood whenever we refer to the meson two-point function. For the dimension of the two-point matrix regarding (spin $\times$ isospin), we have to consider $(2\times 3)^2=36$.

By considering Eq. (16), we have the spectral representation, for $x^0\neq 0$,

$$\mathcal{G}_{\ell\ell'}(x) = \int_{-1}^{1}\int_{T^d} \lambda_0^{|x^0|-1} e^{i\vec{\lambda}\cdot\vec{x}} d(\bar{\mathcal{M}}_\ell(1/2),\mathcal{E}(\lambda_0,\vec{\lambda})\bar{\mathcal{M}}_{\ell'}(1/2))_{\mathcal{H}}, \tag{17}$$

which is an even function of $\vec{x}$ by parity symmetry (more details will be given in Appendix A).

Remark 2: We note that from the spectral representation of Eq. (17), the fields $\bar{\mathcal{M}}$ create particles in contrast to the fields $\mathcal{M}$, which are auxiliary fields entering the definition of the two-point function of Eq. (16).

By taking the Fourier transform of Eq. (17), i.e., $\tilde{\mathcal{G}}_{\ell\ell'}(p)=\Sigma_{x\in\mathbb{Z}_0^4}\mathcal{G}_{\ell\ell'}(x)e^{-ipx}$ and after separating the equal time contribution, we obtain

$$\tilde{\mathcal{G}}_{\ell\ell'}(p) = \tilde{\mathcal{G}}_{\ell\ell'}(\vec{p}) + (2\pi)^3\int_{-1}^{1} f(p^0,\lambda^0)d_{\lambda^0}d\alpha_{\vec{p},\ell\ell'}(\lambda^0), \tag{18}$$

where

$$d\alpha_{\vec{p},\ell\ell'}(\lambda^0) = \int_{\mathbb{T}^3} \delta(\vec{p}-\vec{\lambda})d_{\lambda^0}d_{\vec{\lambda}}(\bar{\mathcal{M}}_\ell,\mathcal{E}(\lambda^0,\vec{\lambda})\bar{\mathcal{M}}_{\ell'})_{\mathcal{H}}, \tag{19}$$

with $f(x,y)=(e^{ix}-y)^{-1}+(e^{-ix}-y)^{-1}$, and we set $\tilde{\mathcal{G}}(\vec{p})=\Sigma_{\vec{x}}e^{-i\vec{p}\cdot\vec{x}}\mathcal{G}(x^0=0,\vec{x})$.

Points in the E-M spectrum are detected as singularities of $\tilde{\mathcal{G}}_{\ell\ell'}(p)$ on Im p^0, which are given by the $w(\vec{p})$ solutions of Eq. (21). To determine the singularities, we define by a Neumann series the convolution inverse to the two-point function in the individual spin and isospin basis, namely, $\Lambda \equiv \mathcal{G}^{-1}$,

$$\Lambda = (1 + \mathcal{G}_d^{-1}\mathcal{G}_n)^{-1}\mathcal{G}_d^{-1} = \sum_{i=0}^{\infty} (-1)^i [\mathcal{G}_d^{-1}\mathcal{G}_n]^i \mathcal{G}_d^{-1}, \tag{20}$$

where $\mathcal{G}_d$ is given by $\mathcal{G}_{d,\ell\ell'}(u,v) = \mathcal{G}_{d,\ell\ell'}(u,u)\delta_{\ell\ell'}\delta_{u,v}$ with $\mathcal{G} = \mathcal{G}_d + \mathcal{G}_n$.

More precisely, the reason behind the introduction of Λ is that it decays faster than $\mathcal{G}$. Thus,its Fourier transform $\tilde{\Lambda}(p)$, $\tilde{\mathcal{G}}(p)\tilde{\Lambda}(p) = 1$ has a larger analyticity domain in p^0 than $\tilde{\mathcal{G}}(p)$, which turns out to be the strip $|\text{Im } p^0| < -(4-\epsilon)\ln \kappa$ (see Ref. 38). Then,

$$\tilde{\Lambda}^{-1}(p) = [\text{cof } \tilde{\Lambda}(p)]^t / \det[\tilde{\Lambda}(p)]$$

provides a meromorphic extension of $\tilde{\mathcal{G}}(p)$. The singularities of $\tilde{\Lambda}^{-1}(p)$ are solutions $w(\vec{p})$ of the equation

$$\det[\tilde{\Lambda}(p^0 = iw(\vec{p}), \vec{p})] = 0. \tag{21}$$

The solutions $w(\vec{p})$ will be shown to be the meson dispersion curves and the masses correspond to $w(\vec{p} = \vec{0})$.

Remark 3: The meson dispersion relations $w(\vec{p})$ *are given by the zeros of* $\det[\tilde{\Lambda}(p^0 = iw(\vec{p}), \vec{p})] = 0$. *Note that due to the determinant, we are free to take any new basis related to the individual spin and isospin basis by an orthogonal transformation.*

As we will see in the next section, the mesonic eightfold way particles are related to the basic excitation fields of Eq. (12) by a real orthogonal transformation implying that both have identical spectral properties.

To determine the meson masses (dispersion relations) up to and including $\mathcal{O}(\kappa^4)$ ($\mathcal{O}(\kappa^2)$), we need the global bounds for long distances and the low order in κ short distance behavior of $\mathcal{G}$ and Λ. In the sequel, we consider the two-point function of Eq. (16) and its inverse given by Eq. (20), where the spin and isospin indices are in the individual basis. Before we state the theorems concerning the expansion in κ of the two-point function and its convolution inverse, we make some remarks. In the κ, expansion of $\mathcal{G}$, we first observe that each link of the path is composed by an even number of bonds, with two by two in opposite orientation. We call intersecting paths those either with links intercepting in one point or superposed links connecting two points on the lattice. For example, in the first case, a path starting at zero and going around a square is an intersecting path at zero; for the second case, $0 \to x_1 \to x_1 + x_2 \to x_1$, with $0, x_1, x_2$ lattice points, has an overlapping link connecting x_1 and x_2. In our case, for a path with one link, we need intersecting paths corresponding to four and six overlapping bonds, with two by two in opposite orientation, connecting the points separated by distance one. For paths with two or more links, as we will determine mass splitting between pseudoscalar and vector mesons to leading order, which turns out to be $\mathcal{O}(\kappa^4)$, our calculation shows that we only need nonintersecting paths. Intersecting paths would give a contribution of $\mathcal{O}(\kappa^6)$ or higher to the mass and dispersion curves. For this purpose, we have derived a general formula for calculating nonintersecting paths [see Eq. (B6) in Appendix B]. A direct application of this formula to obtain the low order in κ short distance behavior of the two-point function of Eq. (16) shows that the labeling ℓ and ℓ' in $\mathcal{G}_{\ell\ell'}$ is such that $\vec{f} = \vec{h}$ and is independent of $\vec{f}$ up to and including $\mathcal{O}(\kappa^4)$.

Remark 4: Using the symmetries of Sec. III A, we can decompose each nonet into an octet and flavor singlet. The octet states have the same mass and, up to and including $\mathcal{O}(\kappa^4)$ *at* $\beta = 0$, *their masses agree with the singlet of isospin. We point out that we cannot guarantee, solely based on symmetry considerations, that the equality of masses between octet and singlet states holds to* *higher orders in κ and β. For this splitting in the continuum model, see the* U(1) *problem* (*see Ref. 30*).

In the sequel, we use the convenient notation $(\mathcal{G}_{\alpha\beta})$ to denote $(\mathcal{G}_{\ell\ell'})$ with ℓ, ℓ' as in Eq. (16) with fixed $\vec{f}=\vec{h}$ and independent of $\vec{f}$ the 4×4 matrix in the individual spin basis. In a similar way,we introduce the matrix $(\Lambda_{\alpha\beta})$. In the sequel, we use the ordering $\alpha=1=(3,1)$, $\alpha=2=(4,2)$, $\alpha=3=(4,1)$, and $\alpha=4=(3,2)$. We also let $|\vec{x}|\equiv\Sigma_{i=1}^{3}|x^i|$ and c as an arbitrary constant, which may differ from place to place. By considering the behavior of $\mathcal{G}$ and Λ, we have the two theorems given below.

Theorem 2: *$\mathcal{G}$ and Λ are jointly analytic in κ and β and satisfy the following global bounds*:

(1)

$$|\mathcal{G}_{\ell\ell'}(x)| \leqslant c|\kappa|^{2|x^0|+2|\vec{x}|}. \tag{22}$$

(2)

$$|\Lambda_{\ell\ell'}(x)| \leqslant \begin{cases} c|\kappa|^2|\kappa|^{4(|x^0|-1)+2|\vec{x}|}, & |x^0| \geqslant 1 \\ c|\kappa|^{2|\vec{x}|}, & |x^0| = 0, \end{cases} \tag{23}$$

where c is independent of ℓ and ℓ'.

□

Proof: The proof of items (1) and (2) are direct applications of the hyperplane decoupling method. To compute the hyperplane derivatives of Λ we use Leibniz formula, i.e., $\Lambda^{(n)}= -\Sigma_{m=0}^{n}\Lambda^{(0)}\mathcal{G}^{(n-m)}\Lambda^{(m)}$ where, we recall that the superscript (m) stands for the derivative of order m. We warn the reader about the two time orderings: for the time ordering $u^0\geqslant p > v^0$ we must take complex conjugation of the coefficients in the Taylor expansion in the complex parameter κ_p. For more details, we refer the reader to Ref. 11. ■

The short distance behavior of $\mathcal{G}$ and Λ needed in the proof of Theorem 1 is summarized in the next theorem.

Theorem 3: *Let $\rho,\sigma=0,1,2,3$, e^0 the unitary vector in the temporal direction, e^i, e^j $(i,j=1,2,3)$ the unitary vectors in the spatial directions, and $\epsilon,\epsilon'=\pm 1$. The following properties, independent of isospin labeling, hold for $(\mathcal{G}_{\alpha\beta})$ and $(\Lambda_{\alpha\beta})$, at $\beta=0$:*

(1)

$$\mathcal{G}_{\alpha\beta}(x) = \begin{cases} \delta_{\alpha\beta} + \delta_{\alpha\beta}\mathcal{O}(\kappa^8), & x=0 \\ \delta_{\alpha\beta}\kappa^2 + 2c_2c_6\kappa^6 + \mathcal{O}(\kappa^8), & x=\epsilon e^0 \\ c_2\delta_{\alpha\beta}\kappa^2 + 2c_6\kappa^6 + \mathcal{O}(\kappa^8), & x=\epsilon e^j \\ \delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8), & x=2\epsilon e^0 \\ c_2\delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8), & x=2\epsilon e^j \\ 2c_2\kappa^4 + \mathcal{O}(\kappa^8), & x=\epsilon e^0 + \epsilon' e^\sigma \\ c_2\delta_{\vec{\alpha}\vec{\gamma}_u}\delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8), & x=\epsilon e^1 + \epsilon' e^2 \\ [2c_2^2\delta_{\alpha\beta} + (\delta_{\vec{\alpha}\vec{\gamma}_u} - \delta_{\vec{\alpha}\vec{\gamma}_\ell})(1-\delta_{\alpha\beta})]\kappa^4 + \mathcal{O}(\kappa^8), & x=\epsilon e^1 + \epsilon' e^3 \\ [2c_2^2\delta_{\alpha\beta} + (\delta_{\vec{\alpha}\vec{\gamma}_u} + \delta_{\vec{\alpha}\vec{\gamma}_\ell})(1-\delta_{\alpha\beta})]\kappa^4 + \mathcal{O}(\kappa^8), & x=\epsilon e^2 + \epsilon' e^3 \\ c_6^2\delta_{\alpha\beta}\kappa^6 + \mathcal{O}(\kappa^8), & x=\epsilon e^0 + 2\epsilon' e^j \\ c_6\delta_{\alpha\beta}\kappa^6 + \mathcal{O}(\kappa^8), & x=2\epsilon e^0 + \epsilon' e^j \\ (2c_2\delta_{\vec{\alpha}\vec{\gamma}_u} + 2c_2^2)\delta_{\alpha\beta}\kappa^6 + \mathcal{O}(\kappa^8), & x=\epsilon e^0 + \epsilon' e^1 + \epsilon'' e^2 \\ [6c_2^2\delta_{\alpha\beta} + c_2(\delta_{\vec{\alpha}\vec{\gamma}_u} - \delta_{\vec{\alpha}\vec{\gamma}_\ell})(1-\delta_{\alpha\beta})]\kappa^6 + \mathcal{O}(\kappa^8), & x=\epsilon e^0 + \epsilon' e^1 + \epsilon'' e^3 \\ [6c_2^2\delta_{\alpha\beta} + c_2(\delta_{\vec{\alpha}\vec{\gamma}_u} + \delta_{\vec{\alpha}\vec{\gamma}_\ell})(1-\delta_{\alpha\beta})]\kappa^6 + \mathcal{O}(\kappa^8), & x=\epsilon e^0 + \epsilon' e^2 + \epsilon'' e^3 \\ \delta_{\alpha\beta}\kappa^6 + \mathcal{O}(\kappa^8), & x=3\epsilon e^0, \end{cases} \tag{24}$$

where the κ independent constants are given by $c_2=\frac{1}{4}$, $c_6=\frac{3}{4}$, and we have defined $\delta_{\vec{\alpha}\vec{\gamma}_c} = \delta_{\alpha\alpha_c}\delta_{\beta\beta_c}$, $c=u,\ell$ for $\vec{\alpha}=(\alpha,\beta)$, and $\vec{\gamma}_c=(\alpha_c,\beta_c)$.

(2)

$$\Lambda_{\alpha\beta}(x) = \begin{cases} \delta_{\alpha\beta} + (2+6c_2^2)\delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8), & x=0 \\ -\delta_{\alpha\beta}\kappa^2 - \delta_{\alpha\beta}\kappa^6 + \mathcal{O}(\kappa^8), & x=\epsilon e^0 \\ -c_2\delta_{\alpha\beta}\kappa^2 + \mathcal{O}(\kappa^6), & x=\epsilon e^j \\ \mathcal{O}(\kappa^{10}), & x=2\epsilon e^0 \\ (-c_2+c_2^2)\delta_{\alpha\beta}\kappa^4 + \mathcal{O}(\kappa^8), & x=2\epsilon e^j \\ \mathcal{O}(\kappa^8), & x=\epsilon e^0 + \epsilon' e^j \\ (-c_2\delta_{\vec{\alpha}\vec{\gamma}_u}\delta_{\alpha\beta} + 2c_2^2\delta_{\alpha\beta})\kappa^4 + \mathcal{O}(\kappa^8), & x=\epsilon e^1 + \epsilon' e^2 \\ (\delta_{\vec{\alpha}\vec{\gamma}_u} - \delta_{\vec{\alpha}\vec{\gamma}_\ell})(1-\delta_{\alpha\beta})\kappa^4 + \mathcal{O}(\kappa^8), & x=\epsilon e^1 + \epsilon' e^3 \\ (\delta_{\vec{\alpha}\vec{\gamma}_u} + \delta_{\vec{\alpha}\vec{\gamma}_\ell})(1-\delta_{\alpha\beta})\kappa^4 + \mathcal{O}(\kappa^8), & x=\epsilon e^2 + \epsilon' e^3 \\ \mathcal{O}(\kappa^8), & x=\epsilon e^0 + 2\epsilon' e^j \\ \mathcal{O}(\kappa^{10}), & x=2\epsilon e^0 + \epsilon' e^j \\ \mathcal{O}(\kappa^8), & x=\epsilon e^0 + \epsilon' e^i + \epsilon'' e^j \\ \mathcal{O}(\kappa^{12}), & x=3\epsilon e^0. \end{cases} \tag{25}$$

□

Proof: The proof is given in Appendix B. The second item of Theorem 3 uses the Neumann series of Eq. (20) by using the nonintersecting paths of Eq. (B6). ■

Remark 5: The action of charge conjugation $\mathcal{C}$ (see Appendix A) leaves the subspace $\mathcal{H}_{\bar{M}}$, i.e., the subspace of $\mathcal{H}_e$ generated by the fields of Eq. (12), stable in the sense that particle and antiparticle states are linearly dependent (l.d.) and, hence, have the same spectral representation. More explicitly, $\mathcal{C}\bar{\mathcal{M}}_{31,f_1f_2}=\bar{\mathcal{M}}_{42,f_2f_1}$, $\mathcal{C}\bar{\mathcal{M}}_{42,f_1f_2}=\bar{\mathcal{M}}_{31,f_2f_1}$, $\mathcal{C}\bar{\mathcal{M}}_{41,f_1f_2}=\bar{\mathcal{M}}_{41,f_2f_1}$, $\mathcal{C}\bar{\mathcal{M}}_{32,f_1f_2}=\bar{\mathcal{M}}_{32,f_2f_1}$, and so considering det $\tilde{\Lambda}(p^0=iw(\vec{p}),\vec{p})=0$, *they have the same dispersion curves.*

Remark 6: Considering Theorem 3 above, we note that for $x=2\epsilon e^0$, $x=\epsilon e^0+\epsilon' e^j$, $x=\epsilon e^0+2\epsilon' e^j$, $x=2\epsilon e^0+\epsilon' e^j$, $x=\epsilon e^0+\epsilon' e^i+\epsilon'' e^j$, $x=3\epsilon e^0$, we would expect the contributions $\mathcal{O}(\kappa^6)$, $\mathcal{O}(\kappa^4)$, $\mathcal{O}(\kappa^6)$, $\mathcal{O}(\kappa^8)$, $\mathcal{O}(\kappa^6)$, $\mathcal{O}(\kappa^{10})$, respectively, taking into account the global bound $|\Lambda_{\ell\ell'}(x)|\leqslant c|\kappa|^{2+4(|x^0|-1)+2|\vec{x}|}$ from Theorem 2. The absence of lower order terms in Λ for $|x^0|=1,2,3$ is related to explicit cancellations in the Neumann series and improves the global bounds obtained by the decoupling of the hyperplane method.

In the next section, starting from the basic excitation fields of Eq. (12), we introduce a new basis, which is the total basis, related to the individual spin and isospin basis by a real orthogonal transformation and we show how to make the conventional identification with the pseudoscalar and vector mesons. We also determine the meson masses, dispersion curves, and their multiplicities.

III. PARTICLE BASIS: THE PSEUDOSCALAR AND VECTOR MESONS FIELDS

This section is divided into four subsections. In the first subsection, we introduce the symmetries at the level of correlations associated with $SU(3)_f$ such as total isospin, third component of total isospin, and total hypercharge. The fact that these symmetries can be implemented as unitary operators in $\mathcal{H}$ is devoted to the next section. The use of orthogonality relations at the level of correlations enables us to show that the two-point function reduces to a block diagonal form. In the second subsection, we introduce spin operators and other symmetries, namely, $\mathcal{G}$-parity ($\mathcal{G}_p$) and spin flip ($\mathcal{F}_s$), used to reduce the two-point function to a simpler diagonal form. In the third subsection, we establish the conventional connection with the eightfold way meson states. The final subsection is devoted to the determination of the mesons masses, dispersion curves, and their multiplicities.

A. Flavor symmetry considerations

Here, we define as linear operators on the Grassmann algebra the total isospin, total hypercharge, and other operators associated with $\mathrm{SU}(3)_f$ symmetry. For fixed $\vec{\alpha}$, $\{\mathcal{M}_{\vec{\alpha}\vec{f}}\}$ form a basis for the $3\otimes\bar{3}$ representations of $\mathrm{SU}(3)_f$. We define, with F as a function of Grassmann fields suppressing the gauge field dependence,

$$\mathcal{W}(U)F = F(\{U\bar{\psi}\},\{\bar{U}\psi\}), \quad \bar{U}\psi = \psi U^{\dagger}. \tag{26}$$

In Sec. VI, we show how to implement this operator as a unitary operator in the physical Hilbert space $\mathcal{H}$.

Letting $U=e^{i\theta F_j}$, we define the operators A_j, $j=1,2,\ldots 8$, by

$$A_j F = \lim_{\theta\searrow 0} \frac{(\mathcal{W}(U)-1)}{i\theta}F, \tag{27}$$

where $F_j=\lambda_j/2$ are the traceless self-adjoint Gell-Mann matrices given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

For $F=\bar{M}_{\vec{\alpha}\vec{f}}$, $A_j=I_j$, $j=1,2,3$, in Eq. (27), we define the components of Isospin (where "$-$" means complex conjugation) as

$$I_j = \mathrm{i}_j \times 1 - 1 \times \bar{\mathrm{i}}_j, \tag{28}$$

where $\mathrm{i}_j=F_j$ $(j=1,2,3)$. Similarly, we have for total hypercharge, $U=e^{i\theta F_8}$,

$$YF = \frac{2}{\sqrt{3}} \lim_{\theta\searrow 0} \frac{(\mathcal{W}(U)-1)}{i\theta}F,$$

from which we get

$$Y = y \times 1 - 1 \times \bar{y}, \tag{29}$$

with $y=\frac{2}{\sqrt{3}}F_8$. Also, we define the isospin raising and lowering operators by letting $\mathrm{i}_{\pm}=\mathrm{i}_1 \pm i\mathrm{i}_2$ as

$$I_{\pm} = i_{\pm} \times 1 - 1 \times (\bar{\mathrm{i}}_1 \pm \bar{\mathrm{i}}_2),$$

$$= i_{\pm} \times 1 - 1 \times i_{\mp}, \tag{30}$$

the total isospin squared as

$$\vec{I}^2 = \sum_{j=1}^{3} I_j^2 = \frac{1}{2}(I_+ I_- + I_- I_+) + I_3^2, \tag{31}$$

raising and lowering operators for $\mathrm{SU}(3)_f$ as

$$U_\pm = u_\pm \times 1 - 1 \times u_\mp, \quad u_\pm = F_6 \pm iF_7, \tag{32}$$

$$V_\pm = v_\pm \times 1 - 1 \times v_\mp, \quad v_\pm = F_4 \pm iF_5, \tag{33}$$

and the quadratic Casimir as

$$C_2 = \sum_{j=1}^{8} A_j^2. \tag{34}$$

The operators above obey the commutation relations as follows:

$$[I_+, I_-] = 2I_3, \quad [I_3, I_\pm] = \pm I_\pm, \quad [I_j, \vec{I}^2] = 0,$$

$$[Y, I_\pm] = 0, \quad [Y, I_3] = 0,$$

$$[I_3, U_\pm] = \mp \frac{U_\pm}{2}, \quad [Y, U_\pm] = \pm U_\pm,$$

$$[I_3, V_\pm] = \pm \frac{V_\pm}{2}, \quad [Y, V_\pm] = \pm V_\pm,$$

$$[C_2, A_j] = 0.$$

Here, we follow the convention of Ref. 3.

From the commutation relations above, we can see that $I_\pm$ changes I_3 by ± 1 but does not change Y, $U_\pm$ changes I_3 by $\mp\frac{1}{2}$ and changes Y by ± 1, and $V_\pm$ changes I_3 by $\pm\frac{1}{2}$ and changes Y by ± 1.

In Sec. IV, we show how to lift $\vec{I}^2$, I_3, Y, and C_2 to operators in $\mathcal{H}$ (we recall the notation $\check{A}$ for an operator in $\mathcal{H}$ if A is the Grassmann algebra operator). We use the eigenvectors of the commuting set $\{\check{\vec{I}}^2, \check{I}_3, \check{Y}, \check{C}_2\}$ to form a new basis for each fixed $\vec{\alpha}$. The vectors are denoted by

$$\bar{\mathcal{M}}_{\vec{\alpha}\mathcal{L}}, \tag{35}$$

where $\mathcal{L} = (I, I_3, Y, C_2)$ denotes the eigenvalues of $\check{\vec{I}}^2$, $\check{I}_3$, $\check{Y}$, and $\check{C}_2$; really, $I(I+1)$ is the eigenvalue of $\check{\vec{I}}^2$, but we drop this from the notation and, for simplicity, we write I.

Explicitly, following the usual procedure to pass from one vector to another inside the nonet, we can start, for example, with $\bar{\psi}_{a,\alpha_1,u}\psi_{a,\alpha_2,d}$, and apply the operators $I_\pm$, $U_\pm$, $V_\pm$ to generate eight basis vectors. These eight vectors all have the same value of C_2, which is conveniently calculated on the vector $U_+\bar{\psi}_{a,\alpha_1,u}\psi_{a,\alpha_2,d}$. For this vector, $C_2 = \vec{I}^2 + V_- V_+ + V_3 + U_- U_+ + U_3 + \frac{3}{4}Y^2$ reduces to $C_2 = \vec{I}^2 + \frac{3}{2}Y + \frac{3}{4}Y^2$ since V_+ and U_+ are zero acting on this vector and V_3 and U_3 add to give $3Y/2$. Thus, C_2 takes the value 3.

Furthermore, for the vector

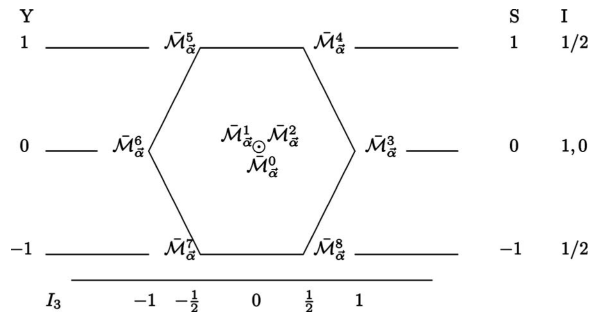

FIG. 1. Graphical representation of the tensor product decomposition $3\otimes\bar{3}=8\oplus 1$. $\bar{\mathcal{M}}^0_{\vec{\alpha}}$ is the singlet state and the remaining fields $\{\bar{\mathcal{M}}^k_{\vec{\alpha}}\}_{k=1}^8$ are members of the octet.

$$\bar{\mathcal{M}}^0_{\vec{\alpha}} = \tfrac{1}{3}(\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,u} + \bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,d} + \bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,s}),$$

it is seen that $\vec{I}^2$, U_+, V_+, Y acting on it give zero, so that C_2 has the eigenvalue 0.

In this way, for fixed $\vec{\alpha}=(\alpha_\ell,\alpha_u)$, we decompose the basis into the direct sum of irreducible representations of $\mathrm{SU}(3)_f$: a one-dimensional flavor singlet (denoted by $\bar{\mathcal{M}}^0_{\vec{\alpha}}$ with $C_2=0$) and an eight-dimensional octet ($\{\bar{\mathcal{M}}^k_{\vec{\alpha}}\}_{k=1}^8$ with $C_2=3$) and the labeling distinguishes between them. These states with their quantum numbers are displayed in Fig. 1 where for simplicity we have labeled them by $\bar{\mathcal{M}}^k_{\vec{\alpha}}$ $(k=0,1,\ldots,8)$. We list $\{\bar{\mathcal{M}}^k_{\vec{\alpha}}\}$ as follows:

$$\begin{aligned}
\bar{\mathcal{M}}^0_{\vec{\alpha}} &= \tfrac{1}{3}(\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,u} + \bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,d} + \bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,s}),\\
\bar{\mathcal{M}}^1_{\vec{\alpha}} &= \tfrac{1}{3\sqrt{2}}(\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,u} + \bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,d} - 2\bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,s}),\\
\bar{\mathcal{M}}^2_{\vec{\alpha}} &= \tfrac{1}{\sqrt{6}}(\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,u} - \bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,d}),\\
\bar{\mathcal{M}}^3_{\vec{\alpha}} &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,d},\\
\bar{\mathcal{M}}^4_{\vec{\alpha}} &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,u}\psi_{a,\alpha_u,s},\\
\bar{\mathcal{M}}^5_{\vec{\alpha}} &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,s},\\
\bar{\mathcal{M}}^6_{\vec{\alpha}} &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,d}\psi_{a,\alpha_u,u},\\
\bar{\mathcal{M}}^7_{\vec{\alpha}} &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,u},\\
\bar{\mathcal{M}}^8_{\vec{\alpha}} &= \tfrac{1}{\sqrt{3}}\bar{\psi}_{a,\alpha_\ell,s}\psi_{a,\alpha_u,d}.
\end{aligned} \tag{36}$$

The set $\{\bar{\mathcal{M}}^k_{\vec{\alpha}}\}$ is related to $\{\bar{\mathcal{M}}_{\vec{\alpha}\vec{f}}\}$ by a real orthogonal transformation B. Explicitly, by using the $\bar{\mathcal{M}}_{\vec{\alpha}\vec{f}}$ ordering $\vec{f}=(u,u)$, (d,d), (s,s), (u,d), (u,s), (d,s), (d,u), (s,u), (s,d) with fixed $\vec{\alpha}$, B is given by

$$B = B_3 \oplus I_6, \tag{37}$$

with

$$B_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{38}$$

From the above, by recalling the normalization $\langle \mathcal{M}_{\vec{\alpha}\vec{f}} \bar{\mathcal{M}}_{\vec{\alpha}\vec{f}} \rangle^{(0)} = 1$, we have $\langle \mathcal{M}^k_{\vec{\alpha}} \bar{\mathcal{M}}^k_{\vec{\alpha}} \rangle^{(0)} = 1$.

The graphical representation of the tensor product decomposition is $3 \otimes \bar{3} = 8 \oplus 1$. $\bar{\mathcal{M}}^0_{\vec{\alpha}}$ is the singlet state and the remaining fields $\{\bar{\mathcal{M}}^k_{\vec{\alpha}}\}_{k=1}^8$ are members of the octet.

The two-point function is given by the following equation, where we recall the convenient notation $\mathcal{L} = (I, I_3, Y, C_2)$ labeling the quantum numbers of each member of the nonet:

$$\mathcal{G}_{\vec{\alpha}\mathcal{L},\vec{\beta}\mathcal{L}'}(u,v) = \langle \mathcal{M}_{\vec{\alpha}\mathcal{L}}(u) \bar{\mathcal{M}}_{\vec{\beta}\mathcal{L}'}(v) \rangle \chi_{u^0 \leq v^0} + \langle \bar{\mathcal{M}}_{\vec{\alpha}\mathcal{L}}(u) \mathcal{M}_{\vec{\beta}\mathcal{L}'}(v) \rangle^* \chi_{u^0 > v^0}, \tag{39}$$

which decomposes into eight identical (4×4) blocks and one (4×4) block associated with the octet and flavor singlet, respectively. We remark that for the two-point function of Eq. (39), the product structure still holds because the states $\{\mathcal{M}_{\vec{\alpha}\mathcal{L}}\}$ are related to $\{\mathcal{M}_{\vec{\alpha}\vec{f}}\}$ by an orthogonal transformation [see Eqs. (37) and (38)], and we still have the faster decay in κ for the convolution inverse (see Theorem 2).

B. Block diagonalization of the two-point function: Spin operators and other symmetries

Now, we want to further reduce these (4×4) blocks of $\mathcal{G}$ by using additional symmetries. Then, we will make the conventional identification with meson particles. Here, we list the symmetries and leave details for Appendix A. The time reversal $\mathcal{T}$ and parity $\mathcal{P}$ symmetries are used to show that the two-point matrix function for fixed u and v is self-adjoint.

We introduce a generalized $\mathcal{G}_p$ symmetry, which is a composition of charge conjugation and discrete $\mathrm{SU}(3)_f$ symmetry, namely, permutations of flavor indices. For F as a function of the field algebra, we define the linear operator $\mathcal{G}_p$ by following equation, suppressing the gauge fields:

$$\mathcal{G}_p F_c(\bar{\psi}, \psi) = F_c(P\bar{\psi}, \psi P), \tag{40}$$

where $F_c(\bar{\psi}, \psi) = \mathcal{C}F(\bar{\psi}, \psi)$, $\mathcal{C}$ is the charge conjugation linear operator, and $P \in \mathrm{SU}(3)_f$ is one of the six permutations of the flavor indices, i.e., of u, d, s. For example, if P permutes u and d, we take

$$P \equiv P_{ud} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{41}$$

$\mathcal{G}_p$ is implemented by a unitary operator on $\mathcal{H}$ (see Sec. IV). We use this symmetry to further decompose the space of excitations, which will be seen to correspond to the decomposition into pseudoscalar and vector excitations. Taking P to be the permutation of (f_1, f_2), $\mathcal{G}_p$ acts on $\bar{\mathcal{M}}_{\vec{\alpha}\vec{f}}$ as

$$\bar{\mathcal{M}}_{31,f_1f_2} \to \bar{\mathcal{M}}_{42,f_1f_2},$$

$$\bar{\mathcal{M}}_{42,f_1f_2} \to \bar{\mathcal{M}}_{31,f_1f_2}, \tag{42}$$

$$\bar{\mathcal{M}}_{41,f_1f_2} \to \bar{\mathcal{M}}_{41,f_1f_2},$$

$$\bar{\mathcal{M}}_{32,f_1f_2} \to \bar{\mathcal{M}}_{32,f_1f_2}.$$

For fixed $\mathcal{L}$, we decompose the space $\bar{\mathcal{M}}_{\vec{\alpha}\mathcal{L}}$ into eigenvectors of $\mathcal{G}_p$ given by (suppressing all but the spin index)

$$\tfrac{1}{\sqrt{2}}(\bar{\mathcal{M}}_{31}+\bar{\mathcal{M}}_{42}) \quad \text{with eigenvalue } 1,$$

$$\bar{\mathcal{M}}_{32}, \tfrac{1}{\sqrt{2}}(\bar{\mathcal{M}}_{31}-\bar{\mathcal{M}}_{42}), \bar{\mathcal{M}}_{41} \quad \text{with eigenvalue } -1. \tag{43}$$

We denote the vectors in Eq. (43) by $\bar{\mathcal{M}}_{\mathcal{J}}$, which are related to $\bar{\mathcal{M}}_{\vec{\alpha}}$ by a real orthogonal transformation explicitly given by

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{44}$$

recalling the ordering for $\vec{\alpha}$ such as (3,1), (4,2), (4,1), and (3,2).

We now turn to the definition of the spin operator. In the continuum, we identify the components of the total angular momentum with the generators of infinitesimal rotations about the coordinate axis. Starting from a state $\Phi(x)$ created by a local single or composite field, we consider a zero spatial momentum improper state $\Phi_0=\int\Phi(x)d\vec{x}$. Φ_0 is expected to have a zero spatial angular momentum, i.e., only spin angular momentum, and the rotation operator reduces to a rotation in spin space.

On the lattice, a $\pi/2$ rotation about any one of the coordinate axes x, y, z is a symmetry giving rise to a unitary operator

$$\check{\mathcal{Y}}(U) = \int_{-\pi}^{\pi} e^{i\lambda} dE(\lambda), \tag{45}$$

which is a lift from the linear transformation

$$\mathcal{Y}(U)F = F(\{U\psi\}(x_r),\{\bar{U}\bar{\psi}\}(x_r)), \tag{46}$$

with x_r denoting the coordinates of the rotated point and $U=U_2\oplus U_2 \equiv e^{i\theta j}$, where $U_2=e^{i(\theta/2)\sigma_{x,y,z}}$, $\theta=\pi/2$. By the spectral theorem, we can define $M_{x,y,z}$, which are the components of a lattice total angular momentum, by

$$\check{\mathcal{M}} = \frac{2}{i\pi}\ln \check{\mathcal{Y}}(U) = \frac{2}{i\pi}\int_{-\pi}^{\pi}\lambda dE(\lambda). \tag{47}$$

If we consider the zero spatial momentum improper state $\Phi_0=\Sigma_{\vec{x}}\Phi(\vec{x})$ (expected to have only spin angular momentum) for the special case, omitting all indices,

$$\Phi(x) = \bar{\mathcal{M}}(x) = \bar{\psi}\psi(x),$$

then

$$\mathcal{Y}(U)\Phi_0=\sum_{\vec{x}}(U\bar{\psi})(\bar{U}\psi)(\vec{x}),$$

such that only the spin space is transformed and the total angular momentum is expected to reduce to spin angular momentum only. We define the components $J_{x,y,z}$ of the total spin J, acting on Φ_0 by

$$J\Phi_0=\frac{2}{i\pi}\ln\check{\mathcal{Y}}(U)\Phi_0=\sum_{\vec{x}}[(j\bar{\psi})\psi-\bar{\psi}(\bar{j}\psi)](x),$$

where we have used the spectral theorem with $U=e^{i(\pi/2)j}=\Sigma e^{i(\pi/2)\lambda}P_\lambda$, and for a function $f(w)=\Sigma_n a_n w^n$,

$$f(\mathcal{Y}(U))\Phi_0=\sum_{\vec{x}}\sum_{\lambda_1,\lambda_2}f(e^{i(\pi/2)(\lambda_1-\lambda_2)})P_{\lambda_1}\bar{\psi}P_{\lambda_2}\psi.$$

The argument of ln is well defined for $(\max|J_z|)\pi/2<\pi$, which includes the meson states, i.e., $|J_z|=0,1$. On the other hand, $J\Phi_0$ is precisely what we obtain from the continuous rotation limit

$$\lim_{\theta\searrow 0}\frac{(\mathcal{Z}(U)-1)}{i\theta}\Phi_0,$$

where $\mathcal{Z}(U)F=F(\{U\bar{\psi}\},\{\bar{U}\psi\})$.

$J_{x,y,z}$ obey the usual angular momentum algebra. Below, we will show that the correlations for different spin states of zero-momentum states within a member of the nonet, are related by the usual raising and lowering operations, which implies that the masses of the $(J,J_z)=(1,1)$ and $(J,J_z)=(1,0)$ spin states are equal.

The vectors in Eq. (43) are eigenvectors of $\vec{J}^2$ and J_z with eigenvalues (0,0), (1,1), (1,0), $(1,-1)$. It turns out that these eigenvalues and eigenvectors in the total spin basis correspond to those of $\mathcal{G}_p$. It should be noted that $\mathcal{G}_p$ does not distinguish among the states $(J,J_z)=(1,1),(1,0),(1,-1)$. We will label the vectors in Eq. (43) by $\bar{\mathcal{M}}_{\mathcal{J}}$ with this total spin notation $\mathcal{J}=(J,J_z)$ in the order given above. The associated two-point function in the total spin basis is denoted by $G_{\mathcal{J}\mathcal{J}'}(x)$ and has the structure

$$G_{\mathcal{J}\mathcal{J}'}(x)=\begin{pmatrix} a & 0 & 0 & 0\\ 0 & b & c & d\\ 0 & c^* & e & f\\ 0 & d^* & f^* & g\end{pmatrix},$$

where a,b,e,g are real (see Lemma 1 in Appendix A).

We obtain further relations among the elements of $G_{\mathcal{J}\mathcal{J}'}(x)$ by exploiting a new local antilinear symmetry that we call spin flip, which is denoted by $\mathcal{F}_s$. This symmetry is the composition

$$\mathcal{F}_s=-i\mathcal{TC}T, \tag{48}$$

where $\mathcal{T}$ is time reversal, $\mathcal{C}$ is the charge conjugation, and T is the time reflection (see Refs. 20, 21, and 31) symmetries. $\mathcal{F}_s$ can be implemented as a antiunitary transformation in $\mathcal{H}$ (see Refs. 20 and 21). These symmetries are discussed in more detail in Appendix A. By using this symmetry, we obtain the structure

$$G_{\mathcal{J}\mathcal{J}'}(x) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & d \\ 0 & c^* & e & c \\ 0 & d^* & c^* & b \end{pmatrix}. \tag{49}$$

It is interesting to observe the symmetry of $G_{\mathcal{J}\mathcal{J}'}$ about the secondary diagonal in the lower right (3×3) block. The convolution inverse, which is denoted by $\Gamma_{\mathcal{J}\mathcal{J}'}(x)$, has the same structure of Eq. (49).

There is a partial restoration of continuous rotational symmetry. In particular, fixing a nonet member, we show that the zero-momentum two-point correlation diagonal elements are identical for fixed J and all J_z. Of course, in the continuum, for the model with rotational symmetry fixing the energy, this is true for all spatial momentum. We already know from previous symmetry considerations, i.e., the spin flip symmetry, that they only depend on $|J_z|$.

To see the lattice result, we write $\bar{\psi}\psi$ for a typical term of a meson creating field. By the $\pi/2$ spatial rotation symmetry, we have, with $u=(u^0,\vec{0})$,

$$\langle \psi\bar{\psi}(u)(U\bar{\psi})(\bar{U}\psi)(v)\rangle = \langle(\psi U)(\bar{\psi}\bar{U})(u)\bar{\psi}\psi(v_r)\rangle,$$

where $v_r=(v^0,\vec{v}_r)$ is the rotated point. By summing over $\vec{v}_r$, $\vec{v}_r$ can be replaced by $\vec{v}$ and the identity holds for powers of U and for linear combinations of correlations. By using the spectral theorem for $U=\Sigma_\lambda e^{i\pi/2}P_\lambda$ as before and for a function $f(z)=\Sigma_n a_n z^n$, we obtain

$$\sum_{\vec{v}}\left\langle \psi\bar{\psi}(u)\sum_{\lambda_1,\lambda_2} f(e^{i\pi/2(\lambda_1-\lambda_2)})(U\bar{\psi})(\bar{U}\psi)(v)\right\rangle = \sum_{\vec{v}}\left\langle \sum_{\lambda_1,\lambda_2}(\psi U)(\bar{\psi}\bar{U})(u)f(e^{i\pi/2(\lambda_1-\lambda_2)})\bar{\psi}\psi(v)\right\rangle.$$

For the function $f(x)=2/(i\pi)\ln x$, we get

$$\sum_{\vec{v}}\langle \psi\bar{\psi}(u)(J\bar{\psi}\psi)_{34}(v)\rangle = \sum_{\vec{v}}\langle(\psi\bar{\psi}J)_{12}(u)\bar{\psi}\psi(v)\rangle,$$

where $J=j\times 1-1\times\bar{j}$, $j=j_{x,y,z}$.

Multiplying by $\bar{w}_{34}$, v_{12} and summing over components, we have

$$(Jw,Rv)=(w,RJv),$$

where we have used the usual complex Hilbert space notation and that j is self-adjoint. In the above, $(Jw,Rv)=\overline{(Jw)}_{\vec{\alpha}}R_{\vec{\beta}\vec{\alpha}}v_{\vec{\beta}}$, where $R_{\vec{\beta}\vec{\alpha}}=\Sigma_{\vec{v}}\langle\mathcal{M}_{\vec{\beta}}(u)\bar{\mathcal{M}}_{\vec{\alpha}}(v)\rangle$. Thus, letting $J_\pm=J_x\pm iJ_y$, we have

$$(J_+w,Rv)=(w,RJ_-v),$$

and taking $v=\chi_{J,J_z}$, $w=\chi_{J,J_z-1}$, where χ_{Jm} is the normalized eigenfunction of total spin J and z-component m, we get

$$(\chi_{J,J_z-1},R\chi_{J,J_z-1})=(\chi_{J,J_z},R\chi_{J,J_z}),$$

so that for diagonal elements of the nonets $\Sigma_{\vec{x}}G_{J,J_z}(x^0,\vec{x})=\Sigma_{\vec{x}}G_{J,J_z-1}(x^0,\vec{x})$,

$$\tilde{G}_{J,J_z}(p^0=\vec{p}=\vec{0})=\tilde{G}_{J,J_z-1}(p^0=\vec{p}=\vec{0})$$

and the same for $\tilde{\Gamma}_{J,J_z-1}(p^0=\vec{p}=\vec{0})$.

For $\vec{p}\neq\vec{0}$, the determinant of $\tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}(p^0=i\chi,\vec{p})$ factorizes using the formula for the roots of a cubic equation. For $\vec{p}=\vec{0}$, $\tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}(p^0=i\chi,\vec{p}=\vec{0})$ is diagonal by using additional symmetries of $\pi/2$ rotations about e_3. We can obtain further relations among the diagonal elements in $\tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}(p^0=i\chi,\vec{p}=\vec{0})$, i.e., for $\tilde{\Gamma}_{\mathcal{J}}=\tilde{\Gamma}_{\mathcal{J}\mathcal{J}}$, $\tilde{\Gamma}_{(1,1)}=\tilde{\Gamma}_{(1,-1)}$, by using the symmetry of reflection in e^1 (for more details, see Appendix A). It is interesting to observe that to obtain these properties for $\vec{p}=\vec{0}$, we do not need the spin flip symmetry $\mathcal{F}_s$. We use the auxiliary function method in Sec. III D to obtain convergent expansions for the masses $M=M(\kappa)$ in $\det \tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}(p^0=iM,\vec{p}=\vec{0})=0$.

C. Particle identification and basic excitation states

We make some remarks about the basis for the octet and flavor singlet and the identification of particles. Applying $I_-(I_+)$ to $\bar{\psi}_u\psi_d(\bar{\psi}_d\psi_u)$, which is identified with the $\bar{\mathcal{M}}^3$ $(\bar{\mathcal{M}}^6)$ state of Fig. 1 where we suppress spin and gauge indices for simplicity, we obtain $\bar{\psi}_u\psi_u-\bar{\psi}_d\psi_d$; by applying U_+, U_-, V_+, V_- to the outer four octet members (i.e., to $\bar{\mathcal{M}}^8$, $\bar{\mathcal{M}}^5$, $\bar{\mathcal{M}}^7$, $\bar{\mathcal{M}}^4$ in Fig. 1, respectively) on the outer rim, we generate the vectors $\bar{\psi}_s\psi_s-\bar{\psi}_d\psi_d$ and $\bar{\psi}_s\psi_s-\bar{\psi}_u\psi_u$. These three vectors are l.d. and we can take the linear combinations

$$w_1=\bar{\psi}_u\psi_u-\bar{\psi}_d\psi_d,$$

$$w_2=2\bar{\psi}_s\psi_s-\bar{\psi}_u\psi_u-\bar{\psi}_d\psi_d,$$

which are identified with $\bar{\mathcal{M}}^2$ and $\bar{\mathcal{M}}^1$, respectively, together with the six outer rim vectors to form a basis for the eight-dimensional representation of $\mathrm{SU}(3)_f$ in the decomposition $3\otimes\bar{3}=8\oplus 1$. The quantum numbers of w_1,w_2 are

$$w_1\colon I=1,\quad I_3=0,\quad Y=0,\quad C_2=3,$$

$$w_2\colon I=0,\quad I_3=0,\quad Y=0,\quad C_2=3.$$

On the other hand, the basis vector for the one-dimensional representation is the flavor singlet $(\bar{\mathcal{M}}^0)$,

$$w_0=\bar{\psi}_u\psi_u+\bar{\psi}_d\psi_d+\bar{\psi}_s\psi_s,$$

with quantum numbers

$$w_0\colon I=0,\quad I_3=0,\quad Y=0,\quad C_2=0,$$

so that the three vectors w_0,w_1,w_2 have distinct quantum numbers.

We now consider the identification of physical particles for broken $\mathrm{SU}(3)_f$. The outer rim vectors are identified with particles as well as $w_0=\eta'$, $w_1=\pi^0$, and $w_2=\eta$ for pseudoscalar mesons; for vector mesons, w_1 is identified with ρ^0 and the ψ and ω seem to be best described as strong mixtures of w_0 and w_2. In order to make the conventional identification of states presented in Fig. 1 with particles, the pseudoscalar and vector mesons along with the associated quantum numbers are depicted in Figs. 2 and 3. Referring to Fig. 1, we see that $\bar{\mathcal{M}}^0_{\tilde{\alpha}}\to\eta'$, $\bar{\mathcal{M}}^1_{\tilde{\alpha}}\to\eta$, $\bar{\mathcal{M}}^2_{\tilde{\alpha}}\to\pi^0$, $\bar{\mathcal{M}}^3_{\tilde{\alpha}}\to\pi^+$, and so on. Note that since the baryon number for the mesons is zero, we get $S=Y-B=Y$. The charges verify the Gell-Mann and Nishijima relation $Q=I_3+Y/2$.

Note that η', η, and π^0 are invariant under charge conjugation. Also, $\mathcal{C}\pi^\pm=\pi^\mp$, $\mathcal{C}K^\pm=K^\mp$, and $\mathcal{C}K^0=\bar{K}^0$. Hence, charge conjugation $\mathcal{C}$ changes the sign of the hypercharge Y and the third component of total isospin I_3 of a pseudoscalar state.

D. Mesonic eightfold way masses and dispersion curves

We now turn to the exact determination of the mesonic eightfold way dispersion curves and masses. In what follows, in our determination of meson masses and dispersion curves due the restriction of $\beta\ll\kappa$, the $\beta=0$ behavior dominates and we will explicitly carry out the analysis for $\beta=0$. At the end, we show what modifications are needed for $\beta\neq 0$. We explicitly determine the

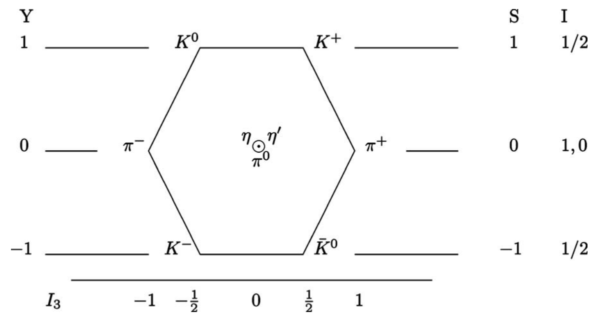

FIG. 2. The pseudoscalar mesons ($J=0$). The hypercharge (Y), strangeness (S), total isospin (I), and third component of total isospin (I_3) are indicated.

masses (dispersion curves) up to and including $\mathcal{O}(\kappa^4)$ ($\mathcal{O}(\kappa^2)$) at $\beta=0$; for this, we need the short distance behavior of $\mathcal{G}$ and Λ of Theorem 3. Fixing a member of the octet or singlet, the mass-determining equation is $\det \tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}(p^0=iM,\vec{p}=\vec{0})=0$. $\tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}$ is seen to be diagonal by using the symmetry of $\pi/2$ rotations about the e^3 axis, as shown in Appendix A, and is seen to have the diagonal structure of Eq. (49) by using e^1 reflections. We have one equation for each factor $\tilde{\Gamma}_{\mathcal{J}\mathcal{J}'}=0$, and for notational simplicity, we write $\tilde{\Gamma}_{\mathcal{J}}$. We note that the same global bounds of Theorem 2 hold for $\Gamma_{\mathcal{J}}(x)$. The short distance behavior of $\Gamma_{\mathcal{J}}(x)$ is related to the behavior in the individual spin basis given in Theorem 3 by the similarity transformation with the orthogonal transformation B given in Eqs. (37) and (38).

The solution of $\det \tilde{\Gamma}_{\mathcal{J}}=0$ for all $\vec{p}$ approaches infinity as κ goes to zero. To find the solutions of $\det \tilde{\Gamma}_{\mathcal{J}}=0$ without approximation, we make a nonlinear transformation from p^0 to an auxiliary variable w and introduce an auxiliary matrix function $H_{\mathcal{J}\mathcal{J}'}(w,\kappa,\vec{p})$ (for $\vec{p}=\vec{0}$ we have the mass) to bring the solution for the nonsingular part $w(\vec{p})+2\ln\kappa$ of the dispersion curves from infinity to close to $w=0$ for small κ. With this function, we can cast the problem of determining dispersion curves and masses into the framework of the analytic implicit function theorem. To this end, we introduce the new variable with $c_2(\vec{p})\equiv c_2\Sigma_{j=1}^{3}2\cos p^j$ and we recall from Theorem 3 that $c_2=\frac{1}{4}$,

$$w = 1 - c_2(\vec{p})\kappa^2 - \kappa^2 e^{-ip^0}, \tag{50}$$

and the auxiliary function $H_{\mathcal{J}}(w,\kappa,\vec{p})$ such that

$$\tilde{\Gamma}_{\mathcal{J}}(p^0,\vec{p}) = H_{\mathcal{J}}(w = 1 - c_2(\vec{p})\kappa^2 - \kappa^2 e^{-ip^0},\kappa,\vec{p}),$$

where $H_{\mathcal{J}}(w,\kappa,\vec{p})$ is defined by the following using $\Gamma(x^0,\vec{x})=\Gamma(-x^0,\vec{x})$ [see Lemma 1, item (2), in appendix A]:

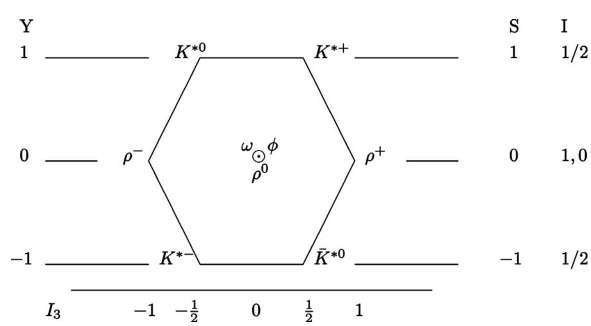

FIG. 3. The vector mesons ($J=1$).

$$H_{\mathcal{J}\mathcal{J}'}(w,\kappa,\vec{p}) = \sum_{\vec{x}} \Gamma_{\mathcal{J}\mathcal{J}'}(x^0=0,\vec{x})e^{-i\vec{p}\cdot\vec{x}} + \sum_{n\geq 1,\vec{x}} \Gamma_{\mathcal{J}\mathcal{J}'}(x^0=n,\vec{x})e^{-i\vec{p}\cdot\vec{x}}\left[\left(\frac{1-w-c_2(\vec{p})\kappa^2}{\kappa^2}\right)^n + \left(\frac{\kappa^2}{1-w-c_2(\vec{p})\kappa^2}\right)^n\right], \tag{51}$$

with $\mathcal{J}=\mathcal{J}'$. By the global bounds of Theorem 2, H is jointly analytic in w and κ for $|w|$ and $|\kappa|$ small.

The mass-determining equation becomes

$$H_{\mathcal{J}}(w,\kappa) \equiv H_{\mathcal{J}\mathcal{J}}(w,\kappa,\vec{p}=\vec{0}) = 0. \tag{52}$$

In the sequel, we determine the masses up to and including $\mathcal{O}(\kappa^4)$.

Remark 7: We have not found any symmetry that allows us to show that the flavor singlet masses are the same as the octet masses. Our calculation shows that their masses are the same up to and including $\mathcal{O}(\kappa^4)$.

For convenience, we separate the time zero and one and the remaining contributions to $H_{\mathcal{J}} = H_{\mathcal{J}}(w,\kappa)$, and we use the short distance behavior of $\Gamma_{\mathcal{J}}$ in the total spin basis [see Eq. (49)]. The contributions are

$$n=0{:}(1+c_0\kappa^4) - c_2(\vec{0})\kappa^2 + c_2(\vec{0})c_4\kappa^4 + a_{\mathcal{J}}\kappa^4 + \mathcal{O}(\kappa^6),$$

where $c_0=2+6c_2^2$, $c_4=c_2-1$ and $a_{\mathcal{J}}\kappa^4$ are the κ^4 contributions from all points of the form $x=\epsilon e^i + \epsilon' e^j$, $i,j=1,2,3$, which are called angle contributions,

$$n=1{:}\ -\kappa^4 - (1-w-c_2(\vec{0})\kappa^2) - \frac{\kappa^4}{(1-w-c_2(\vec{0})\kappa^2)} + \mathcal{O}(\kappa^6),$$

$$n\geq 2{:}\ \mathcal{O}(\kappa^6).$$

Thus, we can write $H_{\mathcal{J}}(w,\kappa)$ in the form, with $b_{\mathcal{J}}=-1+c_0+c_2(\vec{0})c_4$,

$$H_{\mathcal{J}} = w + b_{\mathcal{J}}\kappa^4 - \frac{\kappa^4}{1-w-c_2(\vec{0})\kappa^2} + a_{\mathcal{J}}\kappa^4 + \kappa^6 r_{\mathcal{J}}(w,\kappa),$$

where $r_{\mathcal{J}}(w,\kappa)$ is jointly analytic in w and κ. We see that $H_{\mathcal{J}}(0,0)=0$ and $(\partial H_{\mathcal{J}}/\partial w)(0,0)=1$ so that the analytic implicit function theorem applies and yields the analytic function $w_{\mathcal{J}}(\kappa)$ such that

$$H_{\mathcal{J}}(w_{\mathcal{J}}(\kappa),\kappa) = 0. \tag{53}$$

The solution $w_{\mathcal{J}}(\kappa)$ has the form

$$w_{\mathcal{J}}(\kappa) = \kappa^4 - (a_{\mathcal{J}} + b_{\mathcal{J}})\kappa^4 + \mathcal{O}(\kappa^6).$$

By returning to Eq. (50), the mass is given by

$$M_{\mathcal{J}} = \ln e^{-i(p^0=iM_{\mathcal{J}})} = -2\ln\kappa + \ln(1-w_{\mathcal{J}} - c_2(\vec{0})\kappa^2) = -2\ln\kappa + (-1+c_0+6c_2c_4)\kappa^4 - 6c_2\kappa^2 - \tfrac{1}{2}6^2c_2^2\kappa^4 + a_{\mathcal{J}}\kappa^4 + \mathcal{O}(\kappa^6). \tag{54}$$

Suppressing in what follows the subscript $\mathcal{J}$ from the notation, the implicitly defined $w_{\mathcal{J}}(\kappa)$ for each $\mathcal{J}$ has an explicit representation in terms of the Cauchy integral,

$$w(\kappa) = \frac{1}{2\pi i}\oint_{|w|<\alpha} \frac{w}{H(w,\kappa)} \frac{\partial H}{\partial w}(w,\kappa) dw, \tag{55}$$

where $\alpha > 0$ is sufficiently small (see Ref. 39). From this representation, we see that $w(\kappa)$ is analytic in κ. We note that the integral formula of Eq. (55) permits us to deduce an explicit formula for $w_n = (1/n!) d^n w(0)/d\kappa^n$, which is the nth Taylor coefficient of the analytic function $w(\kappa)$, implicitly defined by Eq. (53). For the general procedure to obtain w_n from Eq. (55), we refer the reader to Ref. 40.

By noting that $a_{\mathcal{J}} = 4[\Gamma_{\mathcal{J}}(e^1+e^2) + \Gamma_{\mathcal{J}}(e^1+e^3) + \Gamma_{\mathcal{J}}(e^2+e^3)]$, we find

$$a_{\mathcal{J}} = \begin{cases} (-3 + 4! c_2^2)\kappa^4 = -3\kappa^4/2, & J=0 \\ (-1 + 4! c_2^2)\kappa^4 = \kappa^4/2, & J=1, \quad J_z = 0, -1, 1. \end{cases} \tag{56}$$

Thus, we see that there is a mass splitting between the total spin one and total spin zero states given by

$$M_{(1,J_z)} - M_{(0,0)} = 2\kappa^4 + \mathcal{O}(\kappa^6).$$

For $\beta \neq 0$, $\beta \ll \kappa$, the arguments above hold and the nonsingular contribution to the mass is jointly analytic in κ and β. In particular, the mass splitting persists for $\beta \neq 0$. The implicit function $w(\kappa, \beta)$ is given by the above integral representation of Eq. (55) making the obvious replacements $H(w(\kappa))$ by $H(w(\kappa,\beta))$ in the integrant.

We now turn to the determination of dispersion curves. We recall the block decomposition of $G_{\mathcal{J}\mathcal{J}'}$ in Eq. (49), which is the same as $\Gamma_{\mathcal{J}\mathcal{J}'}$. We write $\Gamma_{\mathcal{J}\mathcal{J}'} = D_1 \oplus D_3$ (where D_n a $n \times n$ matrix), which implies that for the pseudoscalar meson, i.e., the number D_1, we can still apply the auxiliary function method to determine the dispersion curves $w_p(\vec{p})$. They are given by

$$w_p(\vec{p}) = -2 \ln \kappa - 6c_2\kappa^2 + c_2\kappa^2 \sum_{j=1}^{3} 2(1 - \cos p^j) + \kappa^4 r_p(\kappa, \vec{p}), \tag{57}$$

with $r_p(\kappa, \vec{p})$ jointly analytic in κ, $\mathrm{Im}\, p^j$ for $|\kappa|$, $|\mathrm{Im}\, p^j|$ small.

By considering the (3×3) block, for fixed κ and $\vec{p}$, we can apply a Rouché's theorem argument to the analytic function

$$f(w) \equiv \det H(w,\kappa) = \det(wI_3) + [\det H(w,\kappa) - \det(wI_3)] = g(w) + h(w).$$

In the disk $|w| \leqslant c\kappa^4$, $c \gg 1$, $g(w) > h(w)$, so that the equations $f(w)=0$ and $g(w)=0$ have the same number of solutions. Since $g(w) = w^3 = 0$ has three solutions, we see that $g(w)=0$ has exactly three not necessarily distinct solutions for the dispersion curves.

From the relation between w and p^0 of Eq. (50), the dispersion curves are given by

$$w(\vec{p}) = -2 \ln \kappa - 6c_2\kappa^2 + c_2\kappa^2 \sum_{j=1}^{3} 2(1 - \cos p^j) + \mathcal{O}(\kappa^4). \tag{58}$$

We still do not know if the dispersion curves are the same for the singlet flavor and octet.

By recalling Eq. (49), we note that for the vector mesons, we can use Cardano's formula for the roots of a cubic equation to obtain factorization of the 3×3 determinant. Each factor is a sum of terms with square and cubic roots of polynomials of maximum degree 6 in the matrix elements of $\tilde{\Gamma}$. The expressions are lengthy and we will not present them here but they can be viewed by using standard mathematical softwares such as MAPLE. The presence of the square of cubic roots and the lack of knowledge of the order in κ where $\tilde{\Gamma}_{00} - \tilde{\Gamma}_{11}$ is nonzero prevent us from solving the equations for the dispersion curves using the auxiliary function method.

IV. ISOSPIN, HYPERCHARGE, SPIN FLIP, AND $\mathcal{G}_p$ OPERATORS IN $\mathcal{H}$

In this section, by following Refs. 20 and 21 and inspired by the treatment of point groups given in Ref. 23, we construct the total isospin, total hypercharge, $\mathcal{G}_p$, and spin flip defined in Eqs. (28), (29), (40), and (48), respectively, as self-adjoint operators acting on the physical Hilbert space $\mathcal{H}$. The spin flip symmetry and its implementation by an antiunitary operator is treated in Refs. 20 and 21. The spin operators are defined on the field algebra but not in the physical Hilbert space. For a function F on the field algebra and $U\in \mathrm{SU}(3)$, we define a linear operator $\mathcal{W}(U)$ by using Eq. (26). By using the FK formula and for functions F and G of the basic fields of finite support, we define the Hilbert space operator $\check{\mathcal{W}}(U)$ by the sesquilinear form

$$(G,\check{\mathcal{W}}(U)F)_{\mathcal{H}} = \langle [\mathcal{W}(U)F]\Theta G\rangle,$$

so that by using Eq. (26), we have

$$\begin{aligned}(G,\check{\mathcal{W}}(U)F)_{\mathcal{H}} &= \langle F(\{\psi U^\dagger\},\{U\bar{\psi}\})\Theta G(\{\psi\},\{\bar{\psi}\})\rangle = \langle F(\{\psi\},\{\bar{\psi}\})\Theta G(\{\psi U\},\{U^\dagger\bar{\psi}\})\rangle = \langle F\Theta\mathcal{W}(U^\dagger)G\rangle \\ &= (\check{\mathcal{W}}(U^\dagger)G,F)_{\mathcal{H}},\end{aligned} \tag{59}$$

where $\mathcal{W}(U)^\dagger=\mathcal{W}(U^\dagger)$ and where we have used the $\mathrm{SU}(3)_f$ symmetry on the correlation functions in the RHS. Furthermore, we have

$$(\check{\mathcal{W}}(U)G,\check{\mathcal{W}}(U)F)_{\mathcal{H}} = \langle [W(U)F]\Theta[W(U)G]\rangle = \langle F\Theta G\rangle = (G,F)_{\mathcal{H}}.$$

Hence, $\check{\mathcal{W}}(U)$ is an isometry, i.e., $\check{\mathcal{W}}(U)^\dagger\check{\mathcal{W}}(U)=1$. Interchanging U and $U^\dagger$, we also have $\check{\mathcal{W}}(U)\check{\mathcal{W}}(U)^\dagger=1$. Then, $\check{\mathcal{W}}(U)^\dagger$ is also isometric, which implies that $\check{\mathcal{W}}$ is unitary. The isometry property of $\check{\mathcal{W}}(U)$ is seen by first considering F and G monomials and then extending to F and G elements of $\mathcal{H}$ by continuity. That $\check{\mathcal{W}}$ in Eq. (59) is well defined follows from the fact that taking $F=G\in\mathcal{N}$ (recall that $\mathcal{N}$ denotes the set of nonzero F such that $\langle F\Theta F\rangle=0$) and then if $F\in\mathcal{N}$, $\check{\mathcal{W}}(U)F$ is also in $\mathcal{N}$. We note that $\mathcal{G}_p$ is a composition of unitary operators (charge conjugation and discrete flavor permutations) and can lift to a unitary operator in $\mathcal{H}$. Generators $\grave{A}_j$ associated with the eight one parameter subgroups of $\mathrm{SU}(3)_f$ are self-adjoint by Stone's theorem.[37]

We see that $\check{\mathcal{W}}(U)$ commutes with time evolution $\check{T}_0^{x^0}$ by noting that $\mathcal{W}(U)T_0^{x^0}F=T_0^{x^0}\mathcal{W}(U)F$. Thus, the $\mathrm{SU}(3)_f$ generators defined as operators in $\mathcal{H}$ also commute with $\check{T}_0^{x^0}$.

We now turn to the spin flip symmetry defined in Refs. 20, 21, and 31. From a consideration of the composition of the symmetries of time reversal $\mathcal{T}$, charge conjugation $\mathcal{C}$, and time reflection T given in Appendix A, the local spin flip operator is defined by $\mathcal{F}_s=-i\mathcal{T}\mathcal{C}T$, which acts on single Fermi fields by $\psi_\alpha(x)\to A_{\alpha\beta}\psi_\beta(x)$, $\bar{\psi}_\alpha(x)\to\bar{\psi}_\beta(x)B_{\beta\alpha}$, where

$$A = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$$

is real antisymmetric and $B=A^{-1}=-A$. For functions of the gauge fields, $f(g_{xy})\to\bar{f}(g_{xy}^*)$, where $*$ denotes the complex conjugate. More explicitly, $\tilde{\psi}_1\to\tilde{\psi}_2$, $\tilde{\psi}_2\to-\tilde{\psi}_2$, $\tilde{\psi}_3\to\tilde{\psi}_4$ and $\tilde{\psi}_4\to-\tilde{\psi}_3$.

In more detail, if F is a polynomial (not necessarily local) in the gauge and Fermi fields, suppressing the lattice site arguments,

$$F = \sum a_{\ell mn} g^\ell \bar{\psi}^m \psi^n,$$

then extend $\mathcal{F}_s$ by

$$\mathcal{F}_s F = \sum a^*_{\ell mn} g^\ell (\bar{\psi} B)^m (A\psi)^n$$

and take $\mathcal{F}_s$ to be order preserving. Note that $\mathcal{F}_s$ is antilinear but not in the same sense as Θ (no complex conjugation of g for $\mathcal{F}_s$).

With these definitions, the action of Eq. (2) is termwise invariant and the symmetry operation is a symmetry of the system satisfying $\langle F\rangle = \langle \mathcal{F}_s F\rangle^*$. For the (4×4) block of G, applying $\mathcal{F}_s$ gives the structure presented in Appendix A. For the implementation of $\mathcal{F}_s$ as an antiunitary operator in $\mathcal{H}$, we refer the reader to Refs. 20 and 21.

V. UPPER GAP PROPERTY AND EXTENSION OF THE SPECTRAL RESULTS FROM $\mathcal{H}_{\bar{\mathcal{M}}}$ TO ALL $\mathcal{H}_e$

Up to now, we have considered the spectrum generated by vectors in $\mathcal{H}_{\bar{\mathcal{M}}} \subset \mathcal{H}_e$. As in Refs. 10 and 11, we use a correlation subtraction method (see Ref. 22) to show that the eightfold way meson spectrum is the only spectrum in all $\mathcal{H}_e$, up to near the two-meson threshold of $\approx -4 \ln \kappa$. For $L \in \mathcal{H}_e$, we have the spectral representation and FK formula (with as P_Ω the projection onto the vacuum state $\Omega \equiv 1$)

$$((1-P_\Omega)L, \check{T}_0^{|v^0-u^0|-1} \check{\vec{T}}^{\vec{v}-\vec{u}} (1-P_\Omega)L)_{\mathcal{H}} = \mathcal{G}(u,v), \quad u^0 \neq v^0,$$

where, with $M = (1-P_\Omega)L$,

$$\mathcal{G}(u,v) = \mathcal{G}_{MM}(u,v)\chi_{u^0\leqslant v^0} + \mathcal{G}_{MM}(u_t,v_t)\chi_{u^0>v^0} = \mathcal{G}_{MM}(u,v)\chi_{u^0\leqslant v^0} + \mathcal{G}^*_{MM}(u,v)\chi_{u^0>v^0},$$

and we have used the notation $z_t = (-z^0, \vec{z})$ if $z = (z^0, \vec{z})$.

M may have contributions to the energy spectrum in the interval $(0, -(4-\epsilon)\ln \kappa)$ that arise from states not in $\mathcal{H}_{\bar{M}}$. We show that this is not the case by considering the decay of the subtracted function

$$\mathcal{F} = \mathcal{G} - \mathcal{P}\Lambda\mathcal{Q}, \tag{60}$$

where the kernels of $\mathcal{P}$, Λ, and $\mathcal{Q}$ are given by

$$\mathcal{P}(u,w) = \mathcal{G}_{M\bar{\mathcal{M}}}(u,w)\chi_{u^0\leqslant w^0} + \mathcal{G}_{M\bar{\mathcal{M}}}(u_t,w_t)\chi_{u^0>w^0} = \mathcal{G}_{M\bar{\mathcal{M}}}(u,w)\chi_{u^0\leqslant w^0} + \mathcal{G}^*_{M\bar{\mathcal{M}}}(u,w)\chi_{u^0>w^0},$$

$$\mathcal{Q}(z,v) = \mathcal{G}_{\bar{\mathcal{M}}M}(z,v)\chi_{z^0\leqslant v^0} + \mathcal{G}_{\bar{\mathcal{M}}M}(z_t,v_t)\chi_{z^0>v^0} = \mathcal{G}_{\bar{\mathcal{M}}M}(z,v)\chi_{z^0\leqslant v^0} + \mathcal{G}^*_{\bar{\mathcal{M}}M}(z,v)\chi_{z^0>v^0},$$

$$\mathcal{J}(w,z) = \mathcal{G}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(w,z)\chi_{w^0\leqslant z^0} + \mathcal{G}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(w_t,z_t)\chi_{w^0>z^0} = \mathcal{G}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(w,z)\chi_{w^0\leqslant z^0} + \mathcal{G}^*_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(w,z)\chi_{w^0>z^0},$$

with $\Lambda(w,z) = \mathcal{J}^{-1}(w,z)$. The identities above are obtained by using time reversal, which gives

$$\mathcal{G}_{MM}(u_t,v_t) = \mathcal{G}^*_{MM}(u,v), \quad \mathcal{G}_{M\bar{\mathcal{M}}}(u_t,w_t) = \mathcal{G}^*_{M\bar{\mathcal{M}}}(u,w),$$

$$\mathcal{G}_{\bar{\mathcal{M}}M}(z_t,v_t) = \mathcal{G}^*_{\bar{\mathcal{M}}M}(z,v), \quad \mathcal{G}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(w_t,z_t) = \mathcal{G}^*_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(w,z).$$

The motivation for the definitions of the kernels of $\mathcal{G}$ $\mathcal{P}$, Λ, and $\mathcal{Q}$ is such that time reflected points give the same value for the $u^0 < v^0$ and $u^0 > v^0$ definitions.

The kernels of $\mathcal{P}$ and $\mathcal{Q}$ also have spectral representations for noncoincident temporal points given by

$$(M, \check{T}_0^{|v^0-u^0|-1} \check{\vec{T}}^{\vec{v}-\vec{u}} \bar{\mathcal{M}})_{\mathcal{H}} = \mathcal{P}(u,v), \quad u^0 \neq v^0,$$

$$(\bar{\mathcal{M}},\check{T}_0^{|v^0-u^0|-1}\check{\vec{T}}^{\vec{v}-\vec{u}}M)_{\mathcal{H}}=\mathcal{Q}(u,v),\quad u^0\neq v^0.$$

We remark that in the two equations above, we made use of $\langle\bar{\mathcal{M}}(u)\rangle=\langle\mathcal{M}(u)\rangle=0$ by parity symmetry. Using the hyperplane decoupling method, we show below that $\mathcal{F}^{(r)}(u,v)=0$, $r=0,1,2,3$ for $|u^0-v^0|>2$, which implies that $\tilde{F}(p)$ is analytic in p^0 in the strip $|\mathrm{Im}\,p^0|\leqslant-(4-\epsilon)\ln\kappa$. Again, only the expansion in κ_p is needed because of our restriction $\beta\ll\kappa$. However, $\tilde{F}(p)=\tilde{G}(p)-\tilde{P}(p)\tilde{\Lambda}(p)\tilde{Q}(p)$ so that possible singularities of $\tilde{G}(p)$ in the strip are canceled by those in the term $\tilde{P}(p)\tilde{\Lambda}(p)\tilde{Q}(p)$. From their spectral representations, it is seen that $\tilde{P}(p)$ and $\tilde{Q}(p)$ only have singularities at the one-meson particle spectrum and the same holds for $\tilde{P}(p)\tilde{\Lambda}(p)\tilde{Q}(p)$ since $\tilde{\Lambda}(p)$ is analytic in the strip. Thus, the singularities of $\tilde{G}(p)$ and the spectrum generated by L in the interval $(0,-(4-\epsilon)\ln\kappa)$ are contained in the one-meson spectrum.

By expanding $\mathcal{F}$ in Eq. (60) in powers of κ_p, we get

$$\begin{aligned}\mathcal{F}=\mathcal{F}^{(0)}\kappa_p^0+\mathcal{F}^{(1)}\kappa_p+\mathcal{F}^{(2)}\kappa_p^2+\mathcal{O}(\kappa_p^3)&=(\mathcal{G}^{(0)}-\mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(0)})\kappa_p^0+(\mathcal{G}^{(1)}-\mathcal{P}^{(1)}\Lambda^{(0)}\mathcal{Q}^{(0)}-\mathcal{P}^{(0)}\Lambda^{(1)}\mathcal{Q}^{(0)}\\&\quad-\mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(1)})\kappa_p+(\mathcal{G}^{(2)}-\mathcal{P}^{(2)}\Lambda^{(0)}\mathcal{Q}^{(0)}-\mathcal{P}^{(0)}\Lambda^{(2)}\mathcal{Q}^{(0)}-\mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(2)}-\mathcal{P}^{(1)}\Lambda^{(1)}\mathcal{Q}^{(0)}\\&\quad-\mathcal{P}^{(1)}\Lambda^{(0)}\mathcal{Q}^{(1)}-\mathcal{P}^{(0)}\Lambda^{(1)}\mathcal{Q}^{(1)})\kappa_p^2+\mathcal{O}(\kappa_p^3).\end{aligned}$$

That $\mathcal{F}^{(r)}(u,v)=0$ $(r=0,1,3)$ follows from gauge integration and imbalance of fermion fields appearing in the expectations. The second derivative of $\mathcal{F}(u,v)$ for the time ordering $u^0\leqslant p<v^0$ is

$$\mathcal{F}^{(2)}=\mathcal{G}^{(2)}-\mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(2)}-\mathcal{P}^{(0)}\Lambda^{(2)}\mathcal{Q}^{(0)}-\mathcal{P}^{(2)}\Lambda^{(0)}\mathcal{Q}^{(0)}=A_1+A_2+A_3+A_4.$$

We will use in the sequel for $r^0\leqslant p<s^0$ the following special cases of Eq. (14):

$$\mathcal{G}^{(2)}_{MM}(r,s)=[\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}M}](r,s),\quad\mathcal{G}^{(2)}_{M\bar{\mathcal{M}}}(r,s)=[\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}](r,s),$$

$$\mathcal{G}^{(2)}_{\bar{\mathcal{M}}M}(r,s)=[\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}M}](r,s),\quad\mathcal{G}^{(2)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(r,s)=[\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}](r,s).$$

For the A_2 term, we have

$$\begin{aligned}A_2&=-\sum_{w^0,z^0\leqslant p}\mathcal{P}^{(0)}(u,w)\Lambda^{(0)}(w,z)[\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}M}](z,v)=-\sum_{\vec{w}}\mathcal{P}^{(0)}(u,(p,\vec{w}))\mathcal{G}^{(0)}_{\bar{\mathcal{M}}M}((p+1,\vec{w}),v)\\&=-[\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}M}](u,v)=-A_1,\end{aligned}$$

where in the equation above we have extended the sum to all z by using the support properties of $\Lambda^{(0)}$ and $\mathcal{G}^{(0)}$.

For the A_4 term, we similarly get $A_4=A_2$. Now, we consider the A_3 term,

$$A_3=-\sum_{w^0\leqslant p,z^0\geqslant p+1}\mathcal{P}^{(0)}(u,w)\Lambda^{(2)}(w,z)\mathcal{Q}^{(0)}(z,v).$$

However, for $w^0\leqslant p$, $z^0\geqslant p+1$, $\Lambda^{(2)}(w,z)=-[\Lambda^{(0)}\mathcal{J}^{(2)}\Lambda^{(0)}](w,z)$, which is obtained by taking the second derivative of the relation $\Lambda\mathcal{J}=1$ and observing that $[\Lambda^{(0)}\mathcal{J}^{(1)}\Lambda^{(1)}](w,z)=[\Lambda^{(1)}\mathcal{J}^{(1)}\Lambda^{(0)}]\times(w,z)=0$ for $w^0\leqslant p$, $z^0\geqslant p+1$. With these restrictions on sums, we get

$$\mathcal{J}^{(2)}(x,y)=\mathcal{G}^{(2)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(x,y)=[\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}](x,y),$$

so that

$$A_3 = \sum_{\substack{w^0,x^0\leqslant p,\\ z^0,y^0\geqslant p+1}} \mathcal{P}^{(0)}(u,w)\Lambda^{(0)}(w,x)[\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}](x,y)\Lambda^{(0)}(y,z)\mathcal{Q}^{(0)}(z,v).$$

By extending the sum to all x^0 and y^0, we get

$$A_3 = \sum_{\vec{w}} \mathcal{P}^{(0)}(u,(p,\vec{w}))\mathcal{Q}^{(0)}((p+1,\vec{w}),v) = [\mathcal{G}^{(0)}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)}_{\bar{\mathcal{M}}M}](u,v) = A_1.$$

By collecting the results above, we have $\mathcal{F}^{(2)}(u,v)=0$ for $u^0\leqslant p<v^0$.

The treatment for the other time ordering is more intricate but similar. For $u^0\geqslant p>v^0$, we have

$$\mathcal{F}^{(2)} = \mathcal{G}^{(2)} - \mathcal{P}^{(0)}\Lambda^{(0)}\mathcal{Q}^{(2)} - \mathcal{P}^{(0)}\Lambda^{(2)}\mathcal{Q}^{(0)} - \mathcal{P}^{(2)}\Lambda^{(0)}\mathcal{Q}^{(0)} = A_1' + A_2' + A_3' + A_4'. \tag{61}$$

By considering the A_2' term in the expression above, we get

$$A_2' = -\sum_{w^0,z^0>p} \mathcal{P}^{(0)}(u,w)\Lambda^{(0)}(w,z)[\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}\circ\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}M}](z,v) = -\sum_{w^0,z^0>p} \mathcal{P}^{(0)}(u,w)\Lambda^{(0)}(w,z)\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}$$
$$\times(z,(p+1,\vec{w}))\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}M}((p,\vec{w}),v). \tag{62}$$

We write

$$\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w})) = \mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w}))\chi_{z^0>p+1} + \mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w}))\chi_{z^0\leqslant p+1}$$
$$+ \mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w}))\delta_{z^0,p+1} - \mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w}))\delta_{z^0,p+1} = A_5' + A_6' + A_7' + A_8' \tag{63}$$

under the w^0,z^0 summations. The idea behind this decomposition is that, as we will show below, $A_2'=-A_1'$.

Recall that

$$\mathcal{J}(z,(p+1,\vec{w})) = \mathcal{G}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w}))\chi_{z^0\leqslant p+1} + \mathcal{G}^{*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w}))\chi_{z^0>p+1}.$$

For the first two terms in Eq. (63), going back to Eq. (62), we have

$$\begin{aligned}
&-\mathcal{P}^{(0)}(u,x)\Lambda^{(0)}(x,y)(A_5'+A_6')(y,z)\mathcal{G}^{*}_{\bar{\mathcal{M}}M}(z,v)\\
&\quad= -\sum_{\vec{w}} \mathcal{P}^{(0)}(u,w)\Lambda^{(0)}(w,z)\mathcal{J}^{(0)}(z,(p+1,\vec{w}))\mathcal{G}^{*}_{\bar{\mathcal{M}}M}((p,\vec{w}),v)\\
&\quad= -\sum_{\vec{w}} \mathcal{P}^{(0)}(u,w)(\delta_{w^0,p+1})\mathcal{G}^{*}_{\bar{\mathcal{M}}M}((p,\vec{w}),v)\\
&\quad= -\sum_{\vec{w}} \mathcal{G}^{(0)*}_{M\bar{\mathcal{M}}}(u,(p+1,\vec{w}))\mathcal{G}^{*}_{\bar{\mathcal{M}}M}((p,\vec{w}),v)\\
&\quad= -\sum_{\vec{w}} \mathcal{G}^{(0)*}_{M\bar{\mathcal{M}}}\circ\mathcal{G}^{*}_{\bar{\mathcal{M}}M}(u,v) = -A_1',
\end{aligned}$$

for $u^0>p+1$. For the remaining terms, we get

$$A_7' + A_8' = \mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w}))\delta_{z^0,p+1} - \mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(z,(p+1,\vec{w}))\delta_{z^0,p+1},$$

which is zero by time reversal symmetry $\mathcal{T}$.

In the sequel, we show that $A_3'=A_1'$. For the A_3', term we have

$$A_3' = -\sum_{w^0>p,z^0\geqslant p} \mathcal{P}^{(0)}(u,w)\Lambda^{(2)}(w,z)\mathcal{Q}^{(0)}(z,v).$$

With this w_0,z_0 restrictions, we have

$$\Lambda^{(2)}(w,z) = -\sum_{x^0>p,y^0\geqslant p} \Lambda^{(0)}(w,x)\mathcal{J}^{(2)}(x,y)\Lambda^{(0)}(y,z).$$

With the x_0,y_0 restrictions, we have

$$\mathcal{J}^{(2)}(x,y) = \mathcal{G}^{(2)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(x,y) = [\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}} \circ \mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}](x,y).$$

Thus,

$$A_3' = \sum_{\substack{w^0,x^0>p,\\ z^0,y^0\leqslant p}} \mathcal{P}^{(0)}(u,w)\Lambda^{(0)}(w,x)[\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}} \circ \mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}](x,y)\Lambda^{(0)}(y,z)\mathcal{Q}^{(0)}(z,v).$$

By considering x,y sums in the second and fifth factors above, we get, $r^0=p+1$,

$$A_3' = \sum_{x^0>p,y^0\leqslant p} \Lambda^{(0)}(w,x)[\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(x,r)\chi_{x^0>p+1} + \mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(x,r)\chi_{x^0\leqslant p+1} + \mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(x,r)\delta_{x^0,p+1}$$
$$- \mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(x,r)\delta_{x^0,p+1}]\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(r,y)\Lambda^{(0)}(y,z).$$

The x sum for $x^0=p+1$, which upon using Eq. (63), gives

$$\delta_{w^0,p+1} + \sum_{\vec{x}} \Lambda^{(0)}(w,x)[\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(x,(p+1,\vec{r})) - \mathcal{G}^{(0)}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(x,(p+1,\vec{r}))]$$

and using time reversal, the term in $[\cdot]$ is zero. Similarly, the sum in y gives δ_{p,z^0}.

By returning to A_3', we have for $u^0>p+1$, $v^0<p$,

$$A_3' = \sum_{\vec{w}} \mathcal{P}^{(0)}(u,(p+1,\vec{w}))\mathcal{Q}^{(0)*}((p,\vec{w}),v) = \sum_{\vec{w}} \mathcal{G}^{(0)*}_{M\bar{\mathcal{M}}}(u,(p+1,\vec{w}))\mathcal{G}^{(0)*}_{\bar{\mathcal{M}}M}((p,\vec{w}),v)$$
$$= [\mathcal{G}^{(0)*}_{M\bar{\mathcal{M}}} \circ \mathcal{G}^{(0)*}_{\bar{\mathcal{M}}M}](u,v) = A_1'.$$

Similarly, we have $A_4'=A_2'$. Thus, $\mathcal{F}^{(2)}(u,v)=0$ for $u^0>p+1$ or $v^0<p$ so we have a minimum separation of $|u^0-v^0|>2$ to get that $\mathcal{F}^{(2)}(u,v)=0$ and we are done.

VI. CONCLUDING REMARKS

We completed the exact determination of the one-particle E-M spectrum associated with the 3+1 Wilson's lattice QCD model with three quark flavors initiated in Refs. 20 and 21 for the odd sector $\mathcal{H}_o$ of the physical Hilbert space, where the baryons (of asymptotic mass $-3\ln\kappa$) lie. Here, we analyzed the even sector $\mathcal{H}_e$ and obtained the eightfold way mesons (of asymptotic mass $-2\ln\kappa$) from first principles, i.e., directly from the quark-gluon dynamics. We obtain a spectral representation for the two-point correlation. The nonsingular parts of the eightfold way meson masses are given by a function jointly analytic in κ and β. In this way, in particular, we control the expansion of the mass to all orders in κ and β. We obtain a pseudoscalar vector meson mass splitting given by $2\kappa^4+\mathcal{O}(\kappa^6)$ at $\beta=0$ and, by analyticity, the splitting persists for $\beta>0$, $\beta\ll\kappa$. Up to and including $\mathcal{O}(\kappa^4)$ at $\beta=0$, there is no isospin singlet-octet mass splitting. The splitting may occur at higher orders of κ and β or may take place by breaking the $\mathrm{SU}(3)_f$ symmetry with a heavier strange quark mass. For the splitting in the continuum model, see the U(1) problem (see Ref. 30). A correlation subtraction method is used to guarantee that there is no other spectrum in all $\mathcal{H}_e$ except that generated by the eightfold way mesons up to near the two-meson threshold ($\approx -4 \ln \kappa$). By combining this result with a similar one for baryons in Refs. 20 and 21, the one-hadron E-M spectrum in all $\mathcal{H} \equiv \mathcal{H}_e \oplus \mathcal{H}_o$, up to near the two-meson threshold, is the one generated by the Gell-Mann and Ne'eman eightfold gauge-invariant meson and baryon fields. Thus, confinement is proved up to near the two-meson threshold.

The determination of the one-meson spectrum is an essential step toward the analysis of the existence of two-hadron bound states as we previously did in simpler QCD models. Hence, our present work opens the way to attack interesting open questions such as the existence of tetraquark and pentaquark states, for example, meson-meson and meson-baryon bound states.

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APPENDIX A: SYMMETRY CONSIDERATIONS

Now, we list several symmetries used to obtain the general structure of Eq. (49) and the relations among $\mathcal{G}$'s for distinct lattice points. Those properties are used to simplify the proof of Theorem 3 devoted to Appendix B. Before we list the symmetries and determine the properties of the two-point function of Eq. (16), we remark that the use of $\mathcal{G}_p$ plays a fundamental role in our analysis. The (4×4) matrix in the total spin basis $(G_{\mathcal{J}\mathcal{J}'})$ can be reduced to an even more diagonal form. More precisely, it breaks into a direct sum with one block (1×1) associated with the labeling total spin zero (the pseudoscalar meson) and a (3×3) block related to total spin 1 (the vector meson). Before we prove the lemma, we recall various symmetry results of Theorem 4 in Ref. 10. The required symmetries are summarized as follows, omitting color and flavor indices.

- Time reversal $\mathcal{T}$: $\psi_\alpha(x) \to \bar\psi_\beta(x_t) A_{\beta\alpha}$, $\bar\psi_\alpha(x) \to B_{\alpha\beta}\psi_\beta(x_t)$, $A=B=B^{-1}=\gamma^0$, $f(g_{xy}) \to f^*(g_{x_t y_t})$, with $z_t=(-z^0,\vec{z})$.
- Parity $\mathcal{P}$: $\psi_\alpha(x) \to A_{\alpha\beta}\psi_\beta(x_p)$, $\bar\psi_\alpha(x) \to \bar\psi_\beta(x_p) B_{\beta\alpha}$, $A=B=B^{-1}=\gamma^0$, $f(g_{xy}) \to f(g_{x_p y_p})$, with $z_p=(z^0,-\vec{z})$.
- Charge conjugation $\mathcal{C}$: $\psi_\alpha(x) \to \bar\psi_\beta(x) A_{\beta\alpha}$, $\bar\psi_\alpha(x) \to B_{\alpha\beta}\psi_\beta(x)$,

$$A = -B = B^{-1} = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix},$$

  $f(g_{xy}) \to f(g^*_{xy})$.
- Time Reflection T: $\psi_\alpha(x) \to A_{\alpha\beta}\psi_\beta(x)$, $\bar\psi_\alpha(x_t) \to \bar\psi_\beta(x_t) B_{\beta\alpha}$, where

$$B = A^{-1} = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix},$$

  $f(g_{xy}) \to f(g_{x_t y_t})$; recall that $z_t=(-z^0,\vec{z})$.
- Rotation r_3 by $\pi/2$ about e^3: $\psi_\alpha(x) \to A_{\alpha\beta}\psi_\beta(x_r)$, $\bar\psi_\alpha \to \bar\psi_\beta(x_r) B_{\alpha\beta}$, where $A=B^{-1}=\mathrm{diag}(\mathrm{e}^{-\mathrm{i}\theta},\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta},\mathrm{e}^{\mathrm{i}\theta})$, $f(g_{xy}) \to f(g_{x_r y_r})$, with $z_r=(z^0,-z^2,z^1,z^3)$ and $\theta=\pi/4$.
- Reflection in e^1: $\psi_\alpha(x) \to A_{\alpha\beta}\psi_\beta(x_{\bar r})$, $\bar\psi_\alpha \to \bar\psi_\beta(x_{\bar r}) B_{\alpha\beta}$, where

$$A = A^{-1} = B = \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix},$$

  $f(g_{xy}) \to f(g_{x_{\bar r} y_{\bar r}})$, with $z_{\bar r}=(z^0,-z^1,z^2,z^3)$.

The above symmetries are defined on single fields, extended linearly to polynomials, and taken to be order preserving, except for $\mathcal{T}$, which is antilinear, and $\mathcal{C}$, both of which are order reversing. For all of them, the action is invariant and the transformed fields equal the field average, except for time reversal where the transformed field equals the complex conjugate of the field average.

We are now ready to state the following lemma.

Lemma 1: The following properties of symmetry holds for $\mathcal{G}$, Λ, G, *and* Γ.

(1) $\mathcal{G}_{\alpha\beta}(x)=\mathcal{G}^{*}_{\beta\alpha}(x)$.

(2) $\mathcal{G}_{\alpha\beta}(x_t)=\mathcal{G}^{*}_{\beta\alpha}(x)=\mathcal{G}_{\alpha\beta}(x)$.

(3) $\mathcal{G}_{\alpha\alpha}(x)=\mathcal{G}_{\alpha\alpha}(x_t)=\mathcal{G}^{*}_{\alpha\alpha}(x)$, *and the same for* G, Λ, *and* Γ.

(4) Using the ordering $\alpha=1=(3,1)$, $\alpha=2=(4,2)$, $\alpha=3=(4,1)$, *and* $\alpha=4=(3,2)$, *for fixed isospin and hypercharge,* $(\mathcal{G}_{\alpha\beta})$ *has the following structure, with* $a,d\in\mathbb{R}$, *and* $b,c,e\in\mathbb{C}$:

$$(\mathcal{G}_{\alpha\beta})=\begin{pmatrix} a & b & c & c^* \\ b^* & a & -c & -c^* \\ c^* & -c^* & d & e \\ c & -c & e^* & d \end{pmatrix}. \tag{A1}$$

Hence the (4×4) *matrix* $(G_{\mathcal{J}\mathcal{J}'})$ *in the total spin basis has the following structure*:

$$(G_{\mathcal{J}\mathcal{J}'})=\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & d \\ 0 & c^* & e & c \\ 0 & d^* & c^* & b \end{pmatrix}, \tag{A2}$$

and the same structure holds for $(\Gamma_{\mathcal{J}\mathcal{J}'})$.

(5) For $\chi\in\mathbb{R}$, *let* $p^0=i\chi$. *We obtain* $\tilde{G}_{\mathcal{J}\mathcal{J}'}(i\chi,\vec{p})=\tilde{G}^{*}_{\mathcal{J}'\mathcal{J}}(i\chi,\vec{p})$.

(6) When $\vec{p}=\vec{0}$, $\tilde{G}_{\mathcal{J}\mathcal{J}'}(p^0,\vec{p}=\vec{0})=\mathrm{diag}(\tilde{a},\tilde{b},\tilde{e},\tilde{b})$, $\tilde{a},\tilde{b},\tilde{e}\in\mathbb{R}$, *and the same for the matrix* Γ.

□

Proof: Items (1), (2), and (3) directly follow by applying parity and time reversal symmetries. To prove item (4), we apply the spin flip symmetry $\mathcal{F}_s=-i\mathcal{TCT}$ and use the fact that the matrix $(\mathcal{G}_{\alpha\beta})$ is self-adjoint, as follows from previous items. Hence, in the individual spin basis, we get the structure

$$(\mathcal{G}_{\alpha\beta})=\begin{pmatrix} a & d & c & e \\ d^* & a & -e^* & -c^* \\ c^* & -e & b & f \\ e^* & -c & f^* & b \end{pmatrix}.$$

Next, we use charge conjugation followed by permutation of isospin indices. Explicitly, for the block associated with $\bar{M}^3_{\tilde{\alpha}}$ or $\bar{M}^6_{\tilde{\alpha}}$, we consider the permutation matrix acting on the isospin degree of freedom, $U\equiv P_{ud}\in\mathrm{SU}(3)_f$, which interchanges $(u\leftrightarrows d)$ and is given by Eq. (41). By the global flavor symmetry, we get, suppressing all but the isospin index,

$$\langle (U^\dagger\psi)_{f_1}(\bar{\psi}U)_{f_2}(\bar{\psi}U)_{f_3}(U^\dagger\psi)_{f_4}\rangle=\langle\psi_{f_1}\bar{\psi}_{f_2}\bar{\psi}_{f_3}\psi_{f_4}\rangle, \tag{A3}$$

from which $c=e^*$ and, hence, Eq. (A1) follows. The treatment for the other blocks is similar, except by the flavor permutation matrix, which, according to the isospin indices of each block, interchanges $(d\leftrightarrows s)$ and $(u\leftrightarrows s)$. We now pass to the total spin basis, which is related to the individual spin basis by the matrix A of Eq. (44), i.e., $(G_{\mathcal{J}\mathcal{J}'})=A(\mathcal{G}_{\alpha\beta})A^T$ and we get Eq. (A2).

To prove item (5) recalling that

$$\tilde{G}_{\mathcal{J}\mathcal{J}'}(i\chi,\vec{p}) = \sum_{x^0,\vec{x}} e^{\chi x^0} e^{-i\vec{p}.\vec{x}} G_{\mathcal{J}\mathcal{J}'}(x^0,\vec{x}),$$

we obtain, by the first item of this lemma,

$$\tilde{G}_{\mathcal{J}\mathcal{J}'}(i\chi,\vec{p}) = \sum_{x^0,\vec{x}} e^{\chi x^0} e^{-i\vec{p}.\vec{x}} G^*_{\mathcal{J}'\mathcal{J}}(x^0,\vec{x}) = \left(\sum_{x^0,\vec{x}} e^{\chi x^0} e^{i\vec{p}.\vec{x}} G_{\mathcal{J}\mathcal{J}'}(x^0,\vec{x})\right)^* = \tilde{G}^*_{\mathcal{J}'\mathcal{J}}(i\chi,-\vec{p}),$$

and using parity symmetry, $G_{\mathcal{J}\mathcal{J}'}(x^0,\vec{x})=G_{\mathcal{J}\mathcal{J}'}(x^0,-\vec{x})$, which implies that $\tilde{G}_{\mathcal{J}\mathcal{J}'}(p^0,\vec{p})=\tilde{G}_{\mathcal{J}\mathcal{J}'}(p^0,-\vec{p})$, and the proof of the fifth item follows.

Finally, the proof of item (6) follows by using $\pi/2$ rotations about e^3. ■

By using Lemmas 1 and 2 below, we only need to prove Theorem 3 for $x^0>0$ and $\epsilon,\epsilon',\epsilon''=1$, for $ij=12,13$.

Lemma 2: For $\rho,\sigma=0,1$ *and* $\epsilon,\epsilon'\in\{-1,+1\}$, *the following relations are verified:*

(1) $\mathcal{G}_{\alpha\alpha}(0,\rho e^0+\epsilon e^i+\sigma\epsilon' e^j)=\mathcal{G}_{\alpha\alpha}(0,\rho e^0+e^i+\sigma e^j)$.
(2) $\mathcal{G}_{\alpha\alpha}(0,\rho e^0+e^1+\sigma e^3)=\mathcal{G}_{\alpha\alpha}(0,\rho e^0+e^2+\sigma e^3)$.
(3) $\mathcal{G}_{12}(0,\rho e^0+e^1+\sigma e^3)=-\mathcal{G}_{34}(0,\rho e^0+e^2+\sigma e^3)$. □

Proof: Items (1)–(3) all follow by using rotation of $\pi/2$ about e^3, reflections about e^3, and parity. ■

APPENDIX B: SMALL DISTANCE BEHAVIOR OF $\mathcal{G}$ AND Λ

In this appendix, we consider the contributions obtained by expanding $\mathcal{G}(0,x)$ in powers of κ, but all we need are nonintersecting paths. In particular, we develop a general formula with applications to the determination of the small distance behavior of $\mathcal{G}$ and Λ. Recall that Λ is defined as a Neumann series according to Eq. (25), and we will need the short distance behavior of Λ until and including $\mathcal{O}(\mathcal{G}_n^5)$,

$$\Lambda = \sum_{i=0}^{5}(-1)^i[\mathcal{G}_n]^i + \mathcal{O}(\mathcal{G}_n^6), \tag{B1}$$

where we made use of $\mathcal{G}_d^{-1}=1+\mathcal{O}(\kappa^8)$. When $x\neq 0$, $\mathcal{G}(0,x)=\mathcal{G}_n(0,x)$ and Eq. (B6) furnishes us with a formula to determine $\mathcal{G}_n(0,x)$ in Eq. (B1) for nonintersecting paths.

In Theorems 2 and 3, we will use gauge integrals with two overlapping bonds of opposite orientation given by

$$\mathcal{I}_2 = \int U_{a_1b_1}(g)U^{-1}_{a_2b_2}(g)d\mu(g) = \int g_{a_1b_1}g^{-1}_{a_2b_2}d\mu(g) = \tfrac{1}{3}\delta_{a_1b_2}\delta_{a_2b_1}. \tag{B2}$$

Although we will not use the gauge integral for three bonds with the same orientation, namely, $\mathcal{I}_3$, in Theorems 2 and 3, we present it here since $\mathcal{I}_3$ was used in the determination of the coefficient $\mathcal{G}_{LL}^{(0,3)}$ in Eq. (9),

$$\mathcal{I}_3 = \int g_{a_1b_1}g_{a_2b_2}g_{a_3b_3}d\mu(g) = \tfrac{1}{6}\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}. \tag{B3}$$

Also, we will need gauge integrals with four and six overlapping bonds [in which case we use the convenient notation $(123)\equiv((a_1b_1),(a_2b_2),(a_3b_3))$, $(132)\equiv((a_1b_1),(a_3b_3),(a_2b_2))$, etc.],

$$\begin{aligned}\mathcal{I}_4 = \int g_{a_1b_1}g^{-1}_{a_2b_2}g_{a_3b_3}g^{-1}_{a_4b_4}d\mu(g) &= \tfrac{1}{8}[\delta_{a_1b_2}\delta_{a_3b_4}\delta_{b_1a_2}\delta_{b_3a_4} + (a_2 \leftrightarrows a_4; b_2 \leftrightarrows b_4)]\\ &\quad - \tfrac{1}{24}[\delta_{a_1b_2}\delta_{a_3b_4}\delta_{b_1a_4}\delta_{b_3a_2} + (a_2 \leftrightarrows a_4; b_2 \leftrightarrows b_4)],\end{aligned} \tag{B4}$$

$$\mathcal{I}_6 = \int g_{a_1b_1}g_{a_2b_2}g_{a_3b_3}g^{-1}_{a_4b_4}g^{-1}_{a_5b_5}g^{-1}_{a_6b_6}d\mu(g) = \tfrac{1}{90}\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}\epsilon_{a_4a_5a_6}\epsilon_{b_4b_5b_6}$$
$$+ \tfrac{1}{30}\{\delta_{a_1b_6}\delta_{a_2b_5}\delta_{a_3b_4}\delta_{a_4b_3}\delta_{a_5b_2}\delta_{a_6b_1} + [(123)\to(132)] + [(123)\to(213)] + [(123)\to(231)]$$
$$+ [(123)\to(312)] + [(123)\to(321)]\} + \tfrac{1}{180}\{\delta_{a_1b_6}\delta_{a_2b_5}\delta_{a_3b_4}(\delta_{a_4b_1}\delta_{a_5b_3}\delta_{a_6b_2} + \delta_{a_4b_2}\delta_{a_5b_1}\delta_{a_6b_3})$$
$$+ [(123)\to(132)] + [(123)\to(213)] + [(123)\to(231)] + [(123)\to(312)] + [(123)$$
$$\to(321)]\}. \tag{B5}$$

In nonintersecting paths, only $\mathcal{I}_2$ occurs. The general formula for nonintersecting path is given by

$$\langle \mathcal{M}_{\vec{\alpha}\vec{f}}(0)\bar{\mathcal{M}}_{\vec{\beta}\vec{f}'}(x)\rangle =_p \left(\frac{\kappa}{2}\right)^{2L}\delta_{\vec{f}\vec{f}'}\Gamma^p_{\alpha_\ell\beta_\ell}\Gamma^{-p}_{\beta_u\alpha_u}, \tag{B6}$$

where we recall that $\vec{\alpha}=(\alpha_\ell,\alpha_u)$, $\vec{\beta}=(\beta_\ell,\beta_u)$, $\vec{f}=(f_1,f_2)$, $\vec{f}'=(f_3,f_4)$, and L is the length of the path. The subscript p in Eq. (B6) above means that we take only the contribution coming from nonintersecting paths, with any consecutive points of the path linked by two overlapping bonds of opposite orientation.

The notation $\Gamma^p_{\alpha\beta}$ ($\Gamma^{-p}_{\alpha\beta}$) means that the $\alpha\beta$ element of the ordered product of Γ matrices along the path that connects 0 to x (x to 0). For example, if $x=e^0+e^1+e^2$ and the path is chosen such that $0\to e^0\to e^0+e^1\to e^0+e^1+e^2$, hence, $\Gamma^p=\Gamma^0\Gamma^1\Gamma^2$ where we have used the notation $\Gamma^{\epsilon e^\rho}\equiv\Gamma^{\epsilon\rho}$. On the one hand, for the reversing path, the product of the Γ matrices is in the opposite order, i.e., $\Gamma^{-p}=\Gamma^{-2}\Gamma^{-1}\Gamma^{-0}$. In general, if the path goes as $0\to x_1\to x_2\to\cdots\to x_n\to x$, then $\Gamma^p\equiv\Gamma^{0\to x}=\Gamma^{x_1}\Gamma^{x_2-x_1}\cdots\Gamma^{x_n-x_{n-1}}\Gamma^{x-x_n}$ and $L=n+1$. On the other hand, we have $\Gamma^{-p}\equiv\Gamma^{x\to 0}=\Gamma^{-(x-x_n)}\Gamma^{-(x_n-x_{n-1})}\cdots\Gamma^{-(x_2-x_1)}\Gamma^{-x_1}$.

We give a brief deduction of the general formula (B6). Expanding the exponential of the action in the numerator of $\langle\mathcal{M}_{\vec{\alpha}\vec{f}}(0)\bar{M}_{\vec{\beta}\vec{f}}(x)\rangle$, we pick up two overlapping bonds with opposite orientation for each bond of the path. Using $\mathcal{I}_2$ and carrying out the Fermi integration over the intermediate fields, we arrive at two kinds of products of Γ matrices—one of them is zero using the *come and go* property, i.e., $\Gamma^p\Gamma^{-p}=0$. For the remaining product, we get

$$\langle \mathcal{M}_{\vec{\alpha}\vec{f}}(0)\bar{\mathcal{M}}_{\vec{\beta}\vec{f}'}(x)\rangle =_p \frac{1}{9}\left(\frac{\kappa}{2}\right)^{2L}\langle\psi_{a,\alpha_\ell,f_1}\bar{\psi}_{a,\alpha_u,f_2}\bar{\psi}_{a_1,\alpha_1,g_1}\psi_{a_1,\beta_2,g_2}(0)\rangle^{(0)}$$
$$\times\langle\psi_{a_2,\beta_1,g_1}\bar{\psi}_{a_2,\alpha_2,g_2}\bar{\psi}_{b,\beta_\ell,f_3}\psi_{b,\beta_u,f_4}(x)\rangle^{(0)}\Gamma^p_{\alpha_1\beta_1}\Gamma^{-p}_{\alpha_2\beta_2}, \tag{B7}$$

where $\langle\cdot\rangle^{(0)}$ is the expectation with the hopping parameter κ set to zero in the action $\mathcal{S}$. Note that the expectations in Eq. (B7) can be easily calculated and the result is

$$\langle\psi_{a,\alpha_\ell,f_1}\bar{\psi}_{a,\alpha_u,f_2}\bar{\psi}_{a_1,\alpha_1,g_1}\psi_{a_1,\beta_2,g_2}(0)\rangle^{(0)}/3 = (\delta_{\alpha_\ell\alpha_1}\delta_{\beta_2\alpha_u} - \delta_{\alpha_\ell\alpha_u}\delta_{\beta_2\alpha_1})\delta_{\vec{f}\vec{g}} \equiv \det(\delta_{\vec{\alpha}\vec{\gamma}})\delta_{\vec{f}\vec{g}} \tag{B8}$$

and

$$\langle\psi_{a_2,\beta_1,g_1}\bar{\psi}_{a_2,\alpha_2,g_2}\bar{\psi}_{b,\beta_\ell,f_3}\psi_{b,\beta_u,f_4}(0)\rangle^{(0)}/3 = (\delta_{\beta_1\beta_\ell}\delta_{\beta_u\alpha_2} - \delta_{\beta_1\alpha_2}\delta_{\beta_u\beta_\ell})\delta_{\vec{g}\vec{f}'} \equiv \det(\delta_{\vec{\gamma}'\vec{\beta}})\delta_{\vec{g}\vec{f}'}, \tag{B9}$$

where we made use of the following notation, for the matrices $(\delta_{\vec{\alpha}\vec{\gamma}})$ and $(\delta_{\vec{\gamma}'\vec{\beta}})$:

$$(\delta_{\vec{\alpha}\vec{\gamma}}) = \begin{pmatrix}\delta_{\alpha_\ell\alpha_1} & \delta_{\alpha_\ell\alpha_u}\\ \delta_{\beta_2\alpha_1} & \delta_{\beta_2\alpha_u}\end{pmatrix}, \quad (\delta_{\vec{\gamma}'\vec{\beta}}) = \begin{pmatrix}\delta_{\beta_1\beta_\ell} & \delta_{\beta_1\alpha_2}\\ \delta_{\beta_u\beta_\ell} & \delta_{\beta_u\alpha_2}\end{pmatrix}. \tag{B10}$$

From Eq. (B7)–(B9), and by carrying out the sum over $\vec{g}$, we get

$$\langle \mathcal{M}_{\vec{\alpha}\vec{f}}(0)\bar{\mathcal{M}}_{\vec{\beta}\vec{f}'}(x)\rangle =_p \left(\frac{\kappa}{2}\right)^{2L} \delta_{\vec{f}\vec{f}'}[\det(\delta_{\vec{\alpha}\vec{\gamma}})][\det(\delta_{\vec{\gamma}'\vec{\beta}})]\Gamma^{p}_{\alpha_1\beta_1}\Gamma^{-p}_{\alpha_2\beta_2} \tag{B11}$$

or, more explicitly,

$$\langle \mathcal{M}_{\vec{\alpha}\vec{f}}(0)\bar{\mathcal{M}}_{\vec{\beta}\vec{f}'}(x)\rangle =_p \left(\frac{\kappa}{2}\right)^{2L} \delta_{\vec{f}\vec{f}'}(\Gamma^{p}_{\alpha_\ell\beta_\ell}\Gamma^{-p}_{\beta_u\alpha_u} - \delta_{\beta_u\beta_\ell}\Gamma^{p}_{\alpha_\ell\beta_1}\Gamma^{-p}_{\beta_1\alpha_u} - \delta_{\alpha_\ell\alpha_u}\Gamma^{p}_{\alpha_1\beta_\ell}\Gamma^{-p}_{\beta_u\alpha_1} + \delta_{\alpha_\ell\alpha_u}\delta_{\beta_u\beta_\ell}\Gamma^{p}_{\alpha_1\beta_1}\Gamma^{-p}_{\beta_1\alpha_1}). \tag{B12}$$

Formula (B6) follows from Eq. (B12) by observing that $\delta_{\alpha_\ell\alpha_u}=\delta_{\beta_u\beta_\ell}=0$ since $\{\alpha_\ell,\beta_\ell\}$ and $\{\alpha_u,\beta_u\}$ are lower and upper spin index sets, respectively.

Before we prove Theorem 3, we note that many possible configurations are shown to be zero by using the come and go property of the Γ matrices ($\rho=0,1,2,3$, $\epsilon=\pm 1$),

$$\Gamma^{\epsilon e^\rho}\Gamma^{-\epsilon e^\rho}=0. \tag{B13}$$

For example, if a path doubles back upon itself at an isolated point, then at this point, the come and go property of Eq. (B13) holds to give a zero contribution. Also, possible contributions can give zero due to an imbalance of the number of fermions or the number of fermion components at a site.

Also, other useful properties involving Γ matrices ($\rho=0,1,2,3$, $\epsilon=\pm 1$) used in evaluating possible contributions are

$$\Gamma^{\epsilon e^\rho}\Gamma^{\epsilon e^\rho}=-2\Gamma^{\epsilon e^\rho}, \tag{B14}$$

$$\Gamma^{\epsilon e^\rho}\Gamma^{\epsilon' e^\sigma}=2I_4-\Gamma^{-\epsilon' e^\sigma}\Gamma^{-\epsilon e^\rho}. \tag{B15}$$

Especially, the property in Eq. (B14) shows that lattice bond segments in a straight line similarly behave. Finally, the property of Eq. (B15) is useful to sum over different orders in a path with fixed endpoints. Due to Lemma 2 of Appendix A, we only need to consider points $x=0$, e^ρ, $2e^\rho$, $3e^0$, e^0+e^1, e^1+e^σ, $e^0+e^1+e^\sigma$, e^0+2e^1, $2e^0+e^1$ with $\rho=0,1$ and $\sigma=2,3$ in the proof below.

In what follows, whenever we write $x\to y$ connecting two distinct points on the lattice x and y, $\to$ means a link of the path, i.e., two opposite oriented bonds connecting points x and y.

We now turn to the proof of Theorem 3.

Proof of Theorem 3 item (1): The proof of the short distance behavior of $\mathcal{G}$ directly follows from the nonintersecting path formula in Eq. (B6). We give the following details.

- We begin by considering $x=0$. Using the symmetry of $\pi/2$ rotations about e^3 shows that the off-diagonal elements are zero. The gauge integral and Eq. (B13) show that the first nonvanishing contribution occurs at κ^8 and consists of two paths which go around a square in opposite directions. The square has one vertex at zero.
- Now, take $x=e^0,e^1$. The κ^2 contribution is a direct application of the path formula of Eq. (B6). The κ^4 and κ^6 contributions of the type $0\to e^0\to 0$ and $0\to e^0\to 0\to e^0$, respectively, are zero by the come and go property of Eq. (B13). For $x=e^0$, parity symmetry at the level of correlations can also be used to show that the κ^4 contribution is zero. The nonvanishing κ^6 contribution comes from paths of the type $0\to e^j\to e^0+e^j\to e^0$, which we call U's, which means two oppositely oriented bonds on the three sides of the path $0\to e^j\to e^0+e^j\to e^0$. A direct application of Eq. (B6) gives $\mathcal{G}(e^0)=3\kappa^6/8$ and $G(e^1)=3\kappa^6/2$.
- For $x=e^1+e^0$, we have a straightforward application of the path formula. The same also holds for $x=2e^0$, $x=2e^1$, $x=e^1+e^2$, $x=e^1+e^3$, $x=e^0+2e^1$, $x=2e^0+e^1$, $x=e^0+e^1+e^2$, $x=e^0+e^1+e^3$, $x=3e^0$. There are vertical (temporally oriented) U. The vertical contributions are given by the path formula as well as the U contributions. Possible contributions which are vertical backtracking paths, i.e., the path $0\to e^0\to 2e^0\to e^0\to 2e^0\to 3e^0$, are zero by using imbalance of fermion components at e^0 or at $2e^0$. For the points $x=\epsilon e^\rho+\epsilon' e^\sigma$, the contribution of $\mathcal{O}(\kappa^6)$ is zero by using the come and go property of Eq. (B13).

Proof of Theorem 3 item (2): To prove this item, we use the simplified formula for Λ, as follows from Eq. (B1).

- For $x=0$, we get by using Eq. (B1) that $\Lambda(0)=\mathcal{G}_d(0)-\mathcal{G}_n^2(0)+\mathcal{O}(\kappa^8)$ with $\mathcal{G}_d(0)=\mathcal{O}(1)$ and $\mathcal{G}_n^2(0)=\mathcal{O}(\kappa^4)$. By noting that $\mathcal{G}_n^2(0)=\mathcal{G}_n(0,x)\mathcal{G}_n(x,0)$ and taking into account contributions coming from $x=0,\epsilon e^0,\epsilon' e^j$, the result follows by using Eq. (B6).
- For $x=e^1$, we get by using Eq. (B1) that $\Lambda(x)=\mathcal{G}_n(x)+\mathcal{O}(\kappa^6)$ with $\mathcal{G}_n(x)=\mathcal{O}(\kappa^2)$. A direct application of Eq. (B6) gives the result.
- For $x=e^0$, the κ^2 contributions directly comes from Eq. (B6). The κ^4 contribution is zero due to the property of Eq. (B13). Finally, the κ^6 contribution is related to vertical paths, such as those connecting $0\to e^0$, i.e., (a) $0\to e^0\to 0\to e^0$, (b) $0\to e^0\to 2e^0\to e^0$, and (c) $0\to -e^0\to 0\to e^0$ contributing to $\Lambda(e^0)$ as

(a) $\mathcal{G}_n^3(e^0)=\mathcal{G}_n(0,e^0)\mathcal{G}_n(e^0,0)\mathcal{G}_n(0,e^0)$,
(b) $\mathcal{G}_n^2(e^0)=\mathcal{G}_n(0,2e^0)\mathcal{G}_n(2e^0,e^0)$, $\mathcal{G}_n^3(e^0)=\mathcal{G}_n(0,e^0)\mathcal{G}_n(e^0,2e^0)\mathcal{G}_n(2e^0,e^0)$,
(c) $\mathcal{G}_n^2(e^0)=\mathcal{G}_n(0,-e^0)\mathcal{G}_n(-e^0,e^0)$, $\mathcal{G}_n^3(e^0)=\mathcal{G}_n(0,-e^0)\mathcal{G}_n(-e^0,0)\mathcal{G}_n(0,e^0)$,

respectively. They sum up to give the final result $-\kappa^6$. We note that U-type contributions are canceled out in the Neumann series of Eq. (B1).

- For $x=2e^0$, the contribution of $\mathcal{O}(\kappa^4)$ gives zero since $\mathcal{G}_n(2e^0)=\mathcal{G}_n^2(2e^0)$ and $\mathcal{G}_n(2e^0),\mathcal{G}_n^2(2e^0)=\mathcal{O}(\kappa^4)$. The contribution of $\mathcal{O}(\kappa^6)$ gives zero by the come and go property of Eq. (B13). Next, there are two types of contributions of order $\mathcal{O}(\kappa^8)$ to the series of Eq. (B1). One of them is a U-type contribution coming from, for example, $0\to e^j\to e^0+e^j\to 2e^0+e^j\to 2e^0$, their sum in Eq. (B1) giving zero. There are also vertical backtracking contributions that also contribute to zero in the Neumann series. More explicitly, in the Neumann series, we have, for example, the $\mathcal{O}(\kappa^8)$ path, $0\to e^0\to 2e^0\to 3e^0\to 2e^0$, with

$$\Lambda(2e^0)=\mathcal{G}_n^2(2e^0)+\mathcal{G}_n^3(2e^0)+\mathcal{G}_n^4(2e^0),$$

with

$$\mathcal{G}_n^2(2e^0)=\mathcal{G}_n(0,3e^0)\mathcal{G}_n(3e^0,2e^0),$$

$$\mathcal{G}_n^3(2e^0)=\mathcal{G}_n(0,2e^0)\mathcal{G}_n(2e^0,3e^0)\mathcal{G}_n(3e^0,2e^0),$$

$$\mathcal{G}_n^3(2e^0)=\mathcal{G}_n(0,e^0)\mathcal{G}_n(e^0,3e^0)\mathcal{G}_n(3e^0,2e^0),$$

$$\mathcal{G}_n^4(2e^0)=\mathcal{G}_n(0,e^0)\mathcal{G}_n(e^0,2e^0)\mathcal{G}_n(2e^0,3e^0)\mathcal{G}_n(3e^0,2e^0),$$

and $\mathcal{G}_n^2(2e^0)=\mathcal{G}_n^3(2e^0)=\mathcal{G}_n^4(2e^0)$. We also need to consider $0\to e^0\to 2e^0\to e^0\to 2e^0$, $0\to e^0\to 0\to e^0\to 2e^0$ and $0\to -e^0\to 0\to e^0\to 2e^0$. We are left with the contribution of $\mathcal{O}(\kappa^{10})$.
- For $x=e^1+e^2$, the first nonzero contribution comes from $\Lambda(x)=\mathcal{G}_n(x)-\mathcal{G}_n^2(x)+\mathcal{O}(\kappa^8)$ with $\mathcal{G}_n(x),\mathcal{G}_n^2(x)=\mathcal{O}(\kappa^4)$. Note that $\mathcal{G}_n^2(x)=\mathcal{G}_n(0,y)\mathcal{G}_n(y,x)$ and we must take into account contributions coming from $y=e^1,e^2$; next, the use of Eq. (B6) gives the result. The same procedure can be applied to $x=e^1+e^3$.
- For $x=e^0+e^1$, $\mathcal{G}_n^2(x)=\mathcal{G}_n(0,y)\mathcal{G}_n(y,x)$, where $y=e^0,e^1$, and $\mathcal{G}_n(x)=\mathcal{O}(\kappa^4)$, $\mathcal{G}_n(0,y),\mathcal{G}_n(y,x)=\mathcal{O}(\kappa^2)$. The contribution of $\mathcal{O}(\kappa^6)$ is zero by the imbalance of fermion fields. We are left with $\mathcal{O}(\kappa^8)$.
- For $x=e^0+e^1+e^2$, we will show that $\Lambda(x=e^0+e^1+e^2)=\mathcal{O}(\kappa^8)$. This result improves the bounds coming from hyperplane decoupling method calculations, which in this case gives $|\Lambda(x=e^0+e^1+e^2)|\leqslant c|\kappa|^2|\kappa|^{4\times(1-1)+2\times 2}=|\kappa|^6$. Other points, such as $x=\epsilon e^0+2\epsilon' e^j$, $x=2\epsilon e^0+\epsilon' e^j$, and $x=3\epsilon e^0$, in calculating Λ present similar cancellations in the Neumann series and, hence, our calculation improves the global bounds of Theorem 3. The details in those cases

are left aside but can be reproduced by following the steps below. The first contribution to $\Lambda(x=e^0+e^1+e^2)$ that we need to consider is $\mathcal{O}(\kappa^6)$ and comes from, recalling Eq. (B1) (in what follows we take $x=e^0+e^1+e^2$),

$$\mathcal{G}_n(0,x) = (2c_2\delta_{\vec{\alpha}\vec{\gamma}_u} + 2c_2^2)\delta_{\alpha\beta}\kappa^6$$

$$\mathcal{G}_n(0,y)\mathcal{G}_n(y,x) = \begin{cases} c_2\delta_{\vec{\alpha}\vec{\gamma}_u}\delta_{\alpha\beta}\kappa^6, & y=e^0,e^1+e^2 \\ 2c_2^2\kappa^6, & y=e^1,e^2,e^0+e^1,e^0+e^2, \end{cases}$$

$$\mathcal{G}_n(0,y)\mathcal{G}_n(y,z)\mathcal{G}_n(z,x) = c_2^2\kappa^6, \quad y=e^\rho, \quad z=e^\sigma, \quad \rho,\sigma \in \{0,1,2\}, \quad \rho \neq \sigma,$$

where we made use of the nonintersecting formula (B6) to calculate $\mathcal{G}_n(u,v)$. Note that $\mathcal{G}_n^i = \mathcal{O}(\kappa^8)$, $i \geqslant 4$, does not contribute to $\mathcal{O}(\kappa^6)$. By summing up, we get

$$(\Lambda_{\alpha\beta}(x)) = [-(2c_2\delta_{\vec{\alpha}\vec{\gamma}_u} + 2c_2^2)\delta_{\alpha\beta} + 2c_2\delta_{\vec{\alpha}\vec{\gamma}_u}\delta_{\alpha\beta} + 8c_2^2\delta_{\alpha\beta} - 6c_2^2\delta_{\alpha\beta}]\kappa^6 = 0.$$

- For $x=3e^0$, similar to the case $x=2e^0$, we have cancellations in the Neumann series until and including $\mathcal{O}(\kappa^8)$. For contributions $\mathcal{O}(\kappa^{10})$, we also have U-type and vertical paths. U-type path is given by, for example, $0 \to e^0 \to e^0+e^1 \to 2e^0+e^1 \to 2e^0 \to 3e^0$, etc., by summing up to zero in Eq. (B1). Vertical contributions, as for $x=2e^0$, up to and including $\mathcal{G}_n^5$, sum up to zero in Eq. (B1). ■

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