Title: On absolute continuity of invariant measures associated with a piecewise-deterministic Markov process with random switching between flows

Author: Dawid Czapla, Katarzyna Horbacz, Hanna Wojewódka-Ściążko

Citation style: Czapla Dawid, Horbacz Katarzyna, Wojewódka-Ściążko Hanna. (2021). On absolute continuity of invariant measures associated with a piecewise-deterministic Markov process with random switching between flows."Nonlinear Analysis" (2021, Vol. 213, art. no. 112522), doi 10.1016/j.na.2021.112522

Nonlinear Analysis 213 (2021) 112522

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On absolute continuity of invariant measures associated with a piecewise-deterministic Markov process with random switching between flows

Dawid Czapla [a,*], Katarzyna Horbacz [a], Hanna Wojewódka-Ściążko [a,b]

[a] *Institute of Mathematics, University of Silesia in Katowice, Bankowa 14, Katowice, 40-007, Poland*
[b] *Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, Bałtycka 5, Gliwice, 44-100, Poland*

A R T I C L E   I N F O

Article history:
Received 5 April 2020
Accepted 21 July 2021
Communicated by Dejan Slepcev

MSC:
60J25
60G30
60J35
60J05

Keywords:
Piecewise-deterministic Markov process
Invariant measure
Ergodic measure
Absolute continuity
Singularity
Switching semiflows

A B S T R A C T

We are concerned with the absolute continuity of stationary distributions corresponding to some piecewise deterministic Markov process, being typically encountered in biological models. The process under investigation involves a deterministic motion punctuated by random jumps, occurring at the jump times of a Poisson process. The post-jump locations are obtained via random transformations of the pre-jump states. Between the jumps, the motion is governed by continuous semiflows, which are switched directly after the jumps. The main goal of this paper is to provide a set of verifiable conditions implying that any invariant distribution of the process under consideration that corresponds to an ergodic invariant measure of the Markov chain given by its post-jump locations has a density with respect to the Lebesgue measure.

0. Introduction

The object of our study is a subclass of piecewise-deterministic Markov processes (PDMPs), somewhat similar to that considered in [2–4,10,22,24], which plays an important role in biology, providing a mathematical framework for the analysis of gene expression dynamics (cf. [25,30]). Recall that a Markov process may be regarded as belonging to the class of PDMPs whenever, roughly speaking, its randomness stems only from the jump mechanism and, in particular, it admits no diffusive dynamics. This huge class of processes has been introduced by Davis [16], and arises naturally in many applied areas, such as population dynamics [5,9], neuronal activity [27], excitable membranes [29], storage modelling [8] or internet traffic [19].

* Corresponding author.
E-mail addresses: dawid.czapla@us.edu.pl (D. Czapla), katarzyna.horbacz@us.edu.pl (K. Horbacz), hanna.wojewodka@us.edu.pl (H. Wojewódka-Ściążko).

https://doi.org/10.1016/j.na.2021.112522

The process considered in this paper is an instance of that introduced in [11], and further examined in [15] (cf. also [12–14]). More specifically, we study a Markov process $\{(Y(t),\xi(t))\}_{t\geq 0}$ evolving on $X:=Y\times I$, where Y is a closed subset of $\mathbb{R}^d$ (but not necessarily bounded, in contrast to e.g. [4]), and I is a finite set. It is assumed that the process involves a deterministic motion punctuated by random jumps, appearing at random moments $\tau_1<\tau_2<\cdots$, coinciding with the jump times of a homogeneous Poisson process. The underlying random dynamical system can be described in terms of a finite collection $\{S_i:\, i\in I\}$ of semiflows, acting from $[0,\infty)\times Y$ to Y, and an arbitrary family $\{w_\theta:\,\theta\in\Theta\}$ of transformations from Y into itself. In the main part of the paper, we assume that Θ is either an interval in $\mathbb{R}$ or a finite set. Between any two consecutive jumps, the evolution of the first coordinate $Y(\cdot)$ is driven by a semiflow S_i, where i is the value of the second coordinate $\xi(\cdot)$. The latter is constant on each time interval between jumps and it is randomly changed right after the jump, depending on the current states of both coordinates. Moreover, the post-jump location of the first coordinate after the nth jump, i.e. $Y(\tau_n)$, is obtained as a result of transforming the pre-jump state $Y(\tau_n-)$, using a map w_θ, where the index θ is randomly drawn from Θ, depending on this state. It is worth noting here that such transformations are not present e.g. in the models discussed in [2–4,10], where the jumps are only related to the semiflow changes. Consequently, the first coordinate of the process can be shortly expressed as

$$Y(t)=\begin{cases}S_{\xi(t)}(t-\tau_n,Y(\tau_n)) & \text{for } t\in[\tau_n,\tau_{n+1}),\\ w_{\eta_{n+1}}(Y(\tau_{n+1}-)) & \text{for } t=\tau_{n+1},\end{cases}$$

where $n\in\mathbb{N}\cup\{0\}$, $\tau_0:=0$, and $\{\eta_n\}_{n\in\mathbb{N}}$ is an appropriate sequence of random variables with values in Θ. In our study, a significant role will be also played by the discrete-time Markov chain $\{(Y_n,\xi_n)\}_{n\in\mathbb{N}\cup\{0\}}$ defined by

$$Y_n:=Y(\tau_n),\quad \xi_n:=\xi(\tau_n)\quad \text{for}\quad n\in\mathbb{N}\cup\{0\},$$

to which we will further refer as to the chain given by the post-jump locations.

In [11, Theorem 4.1] (cf. also [15]), we have provided a set of tractable conditions implying that the chain $\{(Y_n,\xi_n)\}_n$ is geometrically ergodic in the Fortet–Mourier metric (also known as the dual-bounded Lipschitz distance; see [20]), which induces the topology of weak convergence of probability measures (see [18]). This means that the chain possesses a unique, and thus ergodic, stationary distribution, and, for any initial state, the distribution of the chain (at consecutive time points) converges weakly to the stationary one at a geometric rate with respect to the above-mentioned distance. Moreover, we have established a one-to-one correspondence between invariant distributions of that chain and those of the process $\{(Y(t),\xi(t))\}_t$ (see [11, Theorem 4.4]). This has led us to the conclusion that the aforementioned conditions guarantee the existence and uniqueness of a stationary distribution for the PDMP as well. Although not relevant here, it is worth mentioning that the aforesaid results are valid in a more general setting than the one given above; namely, it is enough to require that Y is a Polish metric space, and Θ is an arbitrary measurable topological space endowed with a finite measure.

The main goal of the present paper is to provide certain verifiable conditions that would imply the absolute continuity of all the stationary distributions of the PDMP $\{(Y(t),\xi(t))\}_t$ which correspond to ergodic stationary distributions of the associated chain $\{(Y_n,\xi_n)\}_n$ (see Theorem 3.2). The absolute continuity is understood here to hold with respect to the product measure $\bar{\ell}_d$ of the d-dimensional Lebesgue measure and the counting measure on I. As we shall see in Theorem 3.1(ii), the problem reduces, in fact, to examining the invariant distributions of the Markov chain given by the post-jump locations.

Simultaneously, it should be emphasized that the hypotheses of the above-mentioned [11, Theorem 4.4] do not ensure that the unique (and thus ergodic) stationary distribution of the chain $\{(Y_n,\xi_n)\}_n$ (or that of the continuous-time process) is absolutely continuous. The simplest example illustrating this claim is a system including only one transformation $w_1\equiv 0$, for which the Dirac measure at 0 is a unique stationary distribution.

On the other hand, it is well known and not hard to prove that, whenever the transition operator of a Markov chain preserves the absolute continuity of measures, then any ergodic stationary distribution of the chain (or, in other words, any ergodic invariant probability measure of the transition operator) must be either singular or absolutely continuous (see [21, Lemma 2.2 with Remark 2.1] and cf. [2, Theorem 6]). As will be clarified later (in Lemma 3.1), this is the case for the chain $\{(Y_n, \xi_n)\}_n$ if, for instance, all the transformations w_θ and $S_i(t, \cdot)$ are non-singular with respect to the Lebesgue measure. Yet, as shown in Example 5.2, even under this assumption, the conditions imposed in [11] do not guarantee that a unique invariant distribution of the chain and, thus, that of the PDMP, is absolutely continuous. It should be also stressed that, in general, the singularity of some of the transformations w_θ does not necessarily exclude the absolute continuity of invariant measures as well (see e.g. [24]).

Obviously, the above-mentioned absolute continuity/singularity dichotomy significantly simplifies the analysis, since, in such a setting, we only need to guarantee that the continuous part of a given ergodic invariant distribution of $\{(Y_n, \xi_n)\}_n$, say μ_*, is non-trivial. One way to achieve this is to provide the existence of an open $\bar{\ell}_d$-small set (in the sense of [26]) that is uniformly accessible from some measurable subset of X with positive measure μ_* in a specified number of steps (see Proposition 3.1).

Following ideas of [4], we show (in Lemma 3.3) that the existence of an open small set, containing a given point (y_0, j_0), can be accomplished by assuming that, for some $n \geq d$ and certain "admissible" paths $(j_1, \ldots, j_{n-1}) \in I^{n-1}$, $(\theta_1, \ldots, \theta_n) \in \Theta^n$, the composition

$$(0, \infty)^n \ni (t_1, \ldots, t_n) \mapsto w_{\theta_n}(S_{j_{n-1}}(t_n, \ldots, w_{\theta_1}(S_{j_0}(t_1, y_0)) \ldots))$$

has at least one regular point (at which it is a submersion). This requirement is similar in nature to that employed e.g. in [2,4,30], involving the so-called cumulative flows, which can be usually checked by using a Hörmander's type condition (see [2, Theorems 4 and 5]). Furthermore, if the chain is asymptotically stable, i.e., it admits a unique invariant probability measure to which the distribution of the chain converges weakly, independently of the initial state (which is the case, e.g., under the hypotheses employed in [11]), and (y_0, j_0) belongs to the support of μ_*, then the Portmanteau theorem ([6, Theorem 2.1]) ensures that every open neighbourhood of (y_0, j_0) is uniformly accessible from some other (sufficiently small) neighbourhood of this point with positive measure μ_* in a given number of steps (cf. Corollary 3.1). In general, the latter may, however, be difficult to verify directly, and the argument works only if the chain is asymptotically stable. Therefore, we also propose a more practical condition ensuring the accessibility (cf. Lemma 3.4), which concerns the above-specified compositions of w_θ and S_j.

Finally, let us draw attention to the special case where $S_i(t, y) := y$ for every $i \in I$ (which is, however, out of the scope of this paper). In this case, we have $Y_{n+1} = w_{\theta_{n+1}}(Y_n)$ for every $n \in \mathbb{N} \cup \{0\}$, and thus $\{Y_n\}_n$ can be viewed as the Markov chain arising from an iterated function system (IFS in short) with place-dependent probabilities (also called a learning system; cf. [20,23,32]). The results in [31] (cf. also [21]) show that for most (in the sense of Baire category) such systems the corresponding invariant measures are singular, at least in the case where Θ is finite and Y is a compact convex subset of $\mathbb{R}^d$. More precisely, it has been proved that asymptotically stable IFSs with singular invariant measures constitute a residual subset of the family of all Lipschitzian IFSs enjoying some additional property, which somehow links the Lipschitz constants of w_θ with the associated probabilities.

The outline of the paper is as follows. In Section 1, we introduce the notations and basic definitions regarding Markov operators acting on measures, as well as we give a proof of the aforementioned result on the absolute continuity/singularity dichotomy for their ergodic invariant measures. Section 2 provides a detailed description of the model under study. The main results are established in Section 3, which is divided into two parts. Section 3.1 contains an interpretation of the dichotomy criterion in the given framework and a significant conclusion on the mutual dependence between the absolute continuity of stationary distributions of the chain given by the post-jump locations and the corresponding invariant distributions of the PDMP. Here we also state a general key observation, linking the absolute continuity of the ergodic invariant distributions of $\{(Y_n, \xi_n)\}_n$ with the existence of a suitable open $\bar{\ell}_d$-small set. Further, in Section 3.2, we provide some testable conditions implying the existence of such a set and, therefore, guaranteeing the absolute continuity of the invariant measures under consideration. Section 4 contains the statement of [11, Theorem 4.1], providing the exponential ergodicity of the chain $\{(Y_n, \xi_n)\}_n$ (and hence the existence and uniqueness of a stationary distribution for the PDMP). Some remarks and examples related to our main result are given in Section 5.

1. Preliminaries

Let (E, ρ) be an arbitrary separable metric space, endowed with the Borel σ-field $\mathcal{B}(E)$. Further, let $\mathcal{M}_{fin}(E)$ be the set of all finite non-negative Borel measures on E, and let $\mathcal{M}_{prob}(E)$ stand for the subset of $\mathcal{M}_{fin}(E)$ consisting of all probability measures. Moreover, by $\mathcal{M}^1_{prob}(E)$ we will denote the set of all measures $\mu \in \mathcal{M}_{prob}(E)$ with finite first moment, i.e. satisfying

$$\int_E \rho(x, x^*)\,\mu(dx) < \infty \quad \text{for some} \quad x^* \in E.$$

Now, suppose that we are given a σ-finite non-negative Borel measure m on E. Then, a σ-finite Borel measure μ on E is called *absolutely continuous* with respect to m, which is denoted by $\mu \ll m$, whenever

$$\mu(A) = 0 \quad \text{for every} \quad A \in \mathcal{B}(E) \quad \text{such that} \quad m(A) = 0.$$

By the Radon–Nikodym theorem, $\mu \ll m$ can be equivalently characterized by saying that there is a unique (modulo sets of m − measure 0) Borel measurable function $f^\mu : E \to [0, \infty)$, usually denoted by $d\mu/dm$, such that

$$\mu(A) = \int_A f^\mu(x)\,m(dx) \quad \text{for ever} \quad A \in \mathcal{B}(E).$$

Obviously, if $\mu \in \mathcal{M}_{fin}(E)$, then f^μ is a member of $\mathcal{L}^1(E, m)$, i.e. the space of all Borel measurable and m-integrable functions from E to $\mathbb{R}$ (identified, as usual, with the corresponding quotient space under the relation of m-almost everywhere equality).

The measure μ is said to be *singular* with respect to m, which is denoted by $\mu \perp m$, if there exists a set $F \in \mathcal{B}(E)$ such that

$$\mu(F) = 0 \quad \text{and} \quad m(E \backslash F) = 0.$$

It is well-known that, due to the Lebesgue decomposition theorem, any σ-finite Borel measure μ can be uniquely decomposed as

$$\mu = \mu_{ac} + \mu_{sin}, \quad \text{so that} \quad \mu_{ac} \ll m \quad \text{and} \quad \mu_{sin} \perp m.$$

With regard to the definitions given above, we will use the following notation:

$$\mathcal{M}_{ac}(E, m) := \{\mu \in \mathcal{M}_{fin}(E) :\ \mu \ll m\},$$
$$\mathcal{M}_{sin}(E, m) := \{\mu \in \mathcal{M}_{fin}(E) :\ \mu \perp m\}.$$

Let us now briefly recall the concept of *Frobenius–Perron operator*, which will be used in the analysis that follows. For this aim, suppose that we are given a Borel measurable transformation $S : E \to E$ that is *non-singular* with respect to m, i.e.

$$m(S^{-1}(A)) = 0 \quad \text{for every} \quad A \in \mathcal{B}(E) \quad \text{satisfying} \quad m(A) = 0.$$

The non-singularity condition assures that, if $\mu \in \mathcal{M}_{ac}(E,m)$, and μ_S is defined by

$$\mu_S(A) := \mu(S^{-1}(A)) \quad \text{for} \quad A \in \mathcal{B}(E),$$

then $\mu_S \in \mathcal{M}_{ac}(E,m)$. This allows one to define a linear operator $\mathcal{P}_S : \mathcal{L}^1(E,m) \to \mathcal{L}^1(E,m)$ in such a way that

$$\mathcal{P}_S\left(\frac{d\mu}{dm}\right) = \frac{d\mu_S}{dm} \quad \text{for every} \quad \mu \in \mathcal{M}_{ac}(E,m),$$

which, in other words, means that

$$\int_A \mathcal{P}_S f(x)\, m(dx) = \int_{S^{-1}(A)} f(x)\, m(dx) \quad \text{for all} \quad A \in \mathcal{B}(E),\ f \in \mathcal{L}^1(E,m). \tag{1.1}$$

Such an operator $\mathcal{P}_S$ is commonly known as the Frobenius–Perron operator.

Now, we shall recall several basic definitions from the theory of Markov operators, which will be used throughout the paper. A function $P : E \times \mathcal{B}(E) \to [0,1]$ is called a *stochastic kernel* if for each $A \in \mathcal{B}(E)$, $x \mapsto P(x,A)$ is a measurable map on E, and for each $x \in E$, $A \mapsto P(x,A)$ is a probability Borel measure on $\mathcal{B}(E)$. Given a stochastic kernel P, we can consider the corresponding operator $P : \mathcal{M}_{fin}(E) \to \mathcal{M}_{fin}(E)$, acting on measures, given by

$$P\mu(A) = \int_E P(x,A)\, \mu(dx) \quad \text{for} \quad \mu \in \mathcal{M}_{fin}(E),\ A \in \mathcal{B}(E). \tag{1.2}$$

Such an operator is usually called *a regular Markov operator*. For notational simplicity, we use here the same symbol for the stochastic kernel and the corresponding Markov operator. This slight abuse of notation will not, however, lead to any confusion.

We say that the Markov operator P is *Feller* (or that it enjoys the *Feller property*) whenever the map $x \mapsto \int_E f(y) P(x,dy)$ is continuous for every bounded continuous function $f : E \to \mathbb{R}$.

A measure $\mu_* \in \mathcal{M}_{fin}(E)$ is called *invariant* for the Markov operator P (or, simply, P-invariant) if $P\mu_* = \mu_*$. If there exists a unique P-invariant measure $\mu_* \in \mathcal{M}_{prob}(E)$ such that, for every $\mu \in \mathcal{M}_{prob}(E)$, the sequence $\{P^n\mu\}_{n\in\mathbb{N}}$ is weakly convergent to μ_*, then the operator P is said to be *asymptotically stable*. Let us recall here that a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{fin}(E)$ is said to be *weakly convergent* to $\mu \in \mathcal{M}_{fin}(E)$ whenever

$$\int_E f\, d\mu_n \to \int_E f\, d\mu, \ \text{ as } \ n \to \infty,$$

for each bounded continuous function $f : E \to \mathbb{R}$.

Remark 1.1. Suppose that P is a regular Markov–Feller operator, and that there exists a measure $\mu_* \in \mathcal{M}_{prob}(E)$ such that $\{P^n\delta_x\}_{n\in\mathbb{N}}$ is weakly convergent to μ_* for every $x \in E$. Then P is asymptotically stable.

Indeed, note that, due to the Feller property, $P : \mathcal{M}_{prob}(E) \to \mathcal{M}_{prob}(E)$ is continuous in the topology of weak convergence of measures. Taking this into account, we infer that

$$P\mu_* = P(\lim_{n\to\infty} P^n\delta_x) = \lim_{n\to\infty} P^{n+1}\delta_x = \mu_* \quad (\text{weakly, with any } x \in E),$$

which shows that μ_* is P-invariant. Moreover, using the assumption of the weak convergence of $\{P^n\delta_x\}_{n\in\mathbb{N}}$ (for each $x \in E$) and the Lebesgue's dominated convergence theorem we can simply conclude that $\{P^n\mu\}_{n\in\mathbb{N}}$ converges weakly to μ_* for every $\mu \in \mathcal{M}_{prob}(E)$. This also proves that μ_* is a unique invariant probability measure for P.

An invariant probability measure $\mu_* \in \mathcal{M}_{prob}(E)$ is said to be *ergodic* with respect to P (or P-ergodic) whenever $\mu_*(A) \in \{0,1\}$ for every $A \in \mathcal{B}(E)$ satisfying

$$P(x,A) = 1 \quad \text{for} \quad \mu_* \text{ - almost every } \; x \in A.$$

It is well-known (see e.g. [17, Corollary 7.17]) that, if μ_* is a unique invariant probability measure for P, then it must be ergodic. Moreover, according to [1, Theorem 19.25], the P-ergodic measures are precisely the extreme points of the set of all P-invariant probability measures. These observations lead to the following simple, but extremely useful, conclusion regarding the dichotomy between absolute continuity and singularity of P-ergodic measures, which can be found e.g. in [21, Lemma 2.2, Remark 2.1], and whose proof is also given in the Appendix.

Lemma 1.1. *Let m be an arbitrary σ-finite non-negative Borel measure on E, and suppose that $P : \mathcal{M}_{fin}(E) \to \mathcal{M}_{fin}(E)$ is a regular Markov operator which preserves absolute continuity of measures, i.e. $P(\mathcal{M}_{ac}(E,m)) \subset \mathcal{M}_{ac}(E,m)$. Then, every ergodic invariant probability measure of P is either absolutely continuous or singular with respect to m.*

For any given E-valued time-homogeneous Markov chain $\{\Phi_n\}_{n\in\mathbb{N}\cup\{0\}}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the stochastic kernel $P(\cdot,*)$ satisfying

$$P(x,A) = \mathbb{P}(\Phi_{n+1} \in A \mid \Phi_n = x) \quad \text{for all} \quad x \in E,\ A \in \mathcal{B}(E),\ n \in \mathbb{N}\cup\{0\}$$

is called a *one-step transition law* of this process, and the distribution of Φ_0 is said to be its *initial distribution*. Obviously, in this case, the Markov operator defined by (1.2) describes the evolution of the distributions $\mu_n(\cdot) := \mathbb{P}(\Phi_n \in \cdot)$, $n \in \mathbb{N}\cup\{0\}$, that is, $\mu_n = P\mu_{n-1}$ for each $n \in \mathbb{N}$. In this connection, an invariant probability measure of P is called a stationary distribution of the chain. Moreover, it is well known (see e.g. [28]) that, for any given stochastic kernel $P(\cdot,*)$ on $E \times \mathcal{B}(E)$ and $\mu \in \mathcal{M}_{prob}(E)$, on some probability space, there exists a time-homogeneous Markov chain for which $P(\cdot,*)$ serves as a description of the one-step transition law, and μ is the initial distribution.

A family of regular Markov operators $\{P_t\}_{t\geq 0}$ on $\mathcal{M}_{fin}(E)$, generated according to (1.2), is called a regular Markov semigroup whenever it constitutes a semigroup under composition with $P_0 = \mathrm{id}$ as the unity element. A measure $\nu_* \in \mathcal{M}_{fin}(E)$ is said to be invariant for such a semigroup if $P_t\,\nu_* = \nu_*$ for every $t \geq 0$.

Analogously to the discrete-time case, by the transition law (or a transition semigroup) of a homogeneous continuous-time Markov process $\{\Phi(t)\}_{t\geq 0}$ we mean the family $\{P_t(\cdot,*)\}_{t\geq 0}$ of stochastic kernels satisfying

$$P_t(x,A) = \mathbb{P}(\Phi(t+s) \in A \mid \Phi(s) = x) \quad \text{for all} \quad x \in E,\ A \in \mathcal{B}(E),\ s,t \geq 0.$$

Since $\{P_t(\cdot,*)\}_{t\geq 0}$ satisfies the Chapman–Kolmogorov equation, the family $\{P_t\}_{t\geq 0}$ of Markov operators generated by such kernels is a Markov semigroup (under the composition), which describes the evolution of the distributions $\mu(t)(\cdot) := \mathbb{P}(\Phi(t) \in \cdot)$, $t \geq 0$, i.e. $\mu(s+t) = P_t\mu(s)$ for all $s,t \geq 0$. In this context, an invariant probability measure of $\{P_t\}_{t\geq 0}$ is called a stationary distribution of the process $\{\Phi(t)\}_{t\geq 0}$.

2. Description of the model

Let us now present a formal description of the investigated model (originating from [11]), which has already been briefly discussed in the introduction. Recall that such a system can be viewed as a PDMP evolving through random jumps, which arrive one by one (at random time points τ_n) in exponentially distributed time intervals. The parameter of the exponential distribution, determining the jump rate, will be denoted by λ. The deterministic evolution of the process will be governed by a finite number of continuous semiflows, randomly switched at the jump times.

Let Y be a Polish metric space, endowed with the Borel σ-field $\mathcal{B}(Y)$, $\mathbb{R}_+ := [0,\infty)$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Further, suppose that we are given a finite collection $\{S_i : i \in I\}$ of continuous semiflows, where $I = \{1,\dots,N\}$ and $S_i : \mathbb{R}_+ \times Y \to Y$ for every $i \in I$. In what follows, we will assume that I is equipped with the discrete topology. The semiflows will be switched at the jump times according to a matrix $\{\pi_{ij} : i,j \in I\}$ of place-dependent continuous probabilities $\pi_{ij} : Y \to [0,1]$, satisfying

$$\sum_{j\in I} \pi_{ij}(y) = 1 \quad \text{for all} \quad i \in I,\, y \in Y.$$

Moreover, let Θ be an arbitrary separable topological space equipped with a finite Borel measure ϑ, and let $\{w_\theta : \theta \in \Theta\}$ be an arbitrary family of transformations from Y to itself, such that the map $(y,\theta) \mapsto w_\theta(y)$ is continuous. These transformations will be related to the post-jump locations of the process; more specifically, if the system is in the state y just before a jump, then its position directly after the jump should be $w_\theta(y)$ with some randomly selected $\theta \in \Theta$. The choice of θ will depend on the current state y and is determined by a probability density function $\theta \mapsto p_\theta(y)$ such that $(\theta, y) \mapsto p_\theta(y)$ is continuous.

We shall first introduce a discrete-time model. For this aim, define $Z := Y \times I \times \Theta \times \mathbb{R}_+$ (endowed with the product topology) and a stochastic kernel $\bar{P} : Z \times \mathcal{B}(Z) \to [0,1]$ by setting

$$\bar{P}(z,C) = \sum_{j\in I} \int_0^\infty \int_\Theta \lambda e^{-\lambda t} \mathbb{1}_C(w_\theta(S_i(t,y)), j, \theta, t_0 + t)\pi_{ij}(w_\theta(S_i(t,y)))p_\theta(S_i(t,y))\, \vartheta(d\theta)\, dt$$

for $z = (y,i,\theta_0,t_0) \in Z$ and $C \in \mathcal{B}(Z)$. Further, for an arbitrarily given $\bar{\mu} \in \mathcal{M}_{prob}(Z)$, consider a Z-valued time-homogeneous Markov chain $\{(Y_n,\xi_n,\eta_n,\tau_n)\}_{n\in\mathbb{N}_0}$ (wherein Y_n, ξ_n, η_n and τ_n take values in Y, I, Θ and $\mathbb{R}_+$, respectively) with transition law $\bar{P}$ and initial distribution $\bar{\mu}$, constructed on some suitable probability space, equipped with a probability measure $\mathbb{P}$. It is then easy to check that

$$Y_n = w_{\eta_n}(S_{\xi_{n-1}}(\Delta\tau_n, Y_{n-1})) \;\; \mathbb{P}-a.s. \quad \text{for every} \quad n \in \mathbb{N}, \quad \text{where} \quad \Delta\tau_n := \tau_n - \tau_{n-1}, \tag{2.1}$$

and that the following laws hold:

$$\begin{gathered}
\mathbb{P}(\xi_n = j \mid \xi_{n-1} = i,\, Y_n = y) = \pi_{ij}(y) \quad \text{for} \quad y \in Y,\; i,j \in I, \\
\mathbb{P}(\eta_n \in D \mid S_{\xi_{n-1}}(\Delta\tau_n, Y_{n-1}) = y) = \int_D p_\theta(y)\, \vartheta(d\theta) \quad \text{for} \quad y \in Y,\; D \in \mathcal{B}(\Theta), \\
\mathbb{P}(\Delta\tau_n \le t \mid \tau_{n-1} = t_0) = \left(1 - e^{-\lambda(t-t_0)}\right) \mathbb{1}_{[t_0,\infty)(t)} \quad \text{for} \quad t_0, t \in \mathbb{R}_+.
\end{gathered}$$

Moreover, note that the increments $\Delta\tau_n$, $n \in \mathbb{N}$, form a sequence of independent and exponentially distributed random variables with the same rate parameter λ, and thus $\tau_n \uparrow \infty$ $\mathbb{P}$-almost everywhere (a.e.), as $n \to \infty$ (due to the strong law of large numbers). These observations confirm that such a model coincides with our description of the jump mechanism.

Let $X := Y \times I$ and suppose that $\bar{\mu}$ has the form $\bar{\mu} = \mu \otimes \hat{\delta}_{\theta_0} \otimes \delta_0$, where $\mu \in \mathcal{M}_{prob}(X)$, θ_0 is an arbitrarily fixed element of Θ, and $\hat{\delta}_{\theta_0}$, δ_0 are the Dirac measures on $\mathcal{B}(\Theta)$, $\mathcal{B}(\mathbb{R}_+)$, respectively. In what follows we will focus on the sequence $\{\Phi_n\}_{n\in\mathbb{N}_0}$ given by

$$\Phi_n = (Y_n, \xi_n) \quad \text{for} \quad n \in \mathbb{N}_0,$$

which can be viewed as an X-valued Markov chain with initial distribution μ and transition law of the form

$$\begin{aligned}
P((y,i),A) &:= \bar{P}((y,i,\theta_0,0), A \times \Theta \times \mathbb{R}_+) \\
&= \sum_{j\in I} \int_0^\infty \int_\Theta \lambda e^{-\lambda t} \mathbb{1}_A(w_\theta(S_i(t,y)), j)\pi_{ij}(w_\theta(S_i(t,y)))\, p_\theta(S_i(t,y))\, \vartheta(d\theta)\, dt.
\end{aligned} \tag{2.2}$$

for $(y,i) \in X$ and $A \in \mathcal{B}(X)$.

We can now define an interpolation $\{\Phi(t)\}_{t\geq 0}$ of the chain $\{\Phi_n\}_{n\in\mathbb{N}_0}$ as follows:

$$\Phi(t) := (Y(t), \xi(t)) \quad \text{for} \quad t \geq 0,$$

where

$$Y(t) := S_{\xi_n}(t-\tau_n, Y_n) \;\text{ and }\; \xi(t) := \xi_n, \;\text{ whenever }\; t \in [\tau_n, \tau_{n+1}) \;\text{ for }\; n \in \mathbb{N}_0. \tag{2.3}$$

It is easily seen that $\{\Phi(t)\}_{t\geq 0}$ is a time-homogeneous Markov process, and that $\Phi(\tau_n) = \Phi_n$ for every $n \in \mathbb{N}_0$, which means that $\{\Phi_n\}_{n\in\mathbb{N}}$ describes the post-jump locations of this process. By $\{P_t\}_{t\geq 0}$ we will denote the transition semigroup of $\{\Phi(t)\}_{t\geq 0}$, i.e.

$$P_t((y,i),A) = \mathbb{P}_\mu(\Phi(s+t) \in A \mid \Phi(s) = (y,i)) \quad \text{for} \quad (y,i) \in X,\; A \in \mathcal{B}(X),\; s,t \geq 0. \tag{2.4}$$

Referring to P and $\{P_t\}_{t\geq 0}$ in our further discussion, we will always mean the Markov operator generated by the transition law of $\{\Phi_n\}_{n\in\mathbb{N}_0}$, given by (2.2), and the Markov semigroup induced by the transition law of the process $\{\Phi(t)\}_{t\geq 0}$, satisfying (2.4), respectively. It is worth noting here that, by continuity of functions S_i, w_θ, p_θ and π_{ij}, both the operator P and the semigroup $\{P_t\}_{t\geq 0}$ are Feller (cf. [11, Lemma 6.3]).

As mentioned in the introduction, a set of directly testable conditions for the existence and uniqueness of P-invariant probability measures (which, simultaneously, guarantee a form of geometric ergodicity for P) has already been provided in [11, Theorem 4.1]. This theorem will be quoted in Section 4. The main results of the paper, concerning the absolute continuity of ergodic invariant measures for P, presented in Section 3, will be derived by assuming a priori that such measures exist.

Let us also recall that, by virtue of [11, Theorem 4.4], there is a one-to-one correspondence between invariant probability measures of the operator P and those of the semigroup $\{P_t\}_{t\geq 0}$. Moreover, such a correspondence can be expressed explicitly by using the Markov operators G and W induced by the stochastic kernels of the form

$$G((y,i),A) = \int_0^\infty \lambda e^{-\lambda t} \mathbb{1}_A(S_i(t,y),i)\, dt, \tag{2.5}$$

$$W((y,i),A) = \sum_{j\in I} \int_\Theta \mathbb{1}_A(w_\theta(y),j)\pi_{ij}(w_\theta(y))p_\theta(y)\, \vartheta(d\theta) \tag{2.6}$$

for $(y,i) \in X$ and $A \in \mathcal{B}(X)$. More precisely, the following holds:

Theorem 2.1 (*[11, Theorem 4.4]*)**.** *Let P and $\{P_t\}_{t\geq 0}$ denote the Markov operator and the Markov semigroup induced by (2.2) and (2.4), respectively.*

(i) If $\mu_* \in \mathcal{M}_{prob}(X)$ is an invariant measure of P, then $G\mu_*$ is an invariant measure of $\{P_t\}_{t\geq 0}$ and $WG\mu_* = \mu_*$.

(ii) If $\nu_* \in \mathcal{M}_{prob}(X)$ is an invariant measure of $\{P_t\}_{t\geq 0}$, then $W\nu_*$ is an invariant measure of P and $GW\nu_* = \nu_*$.

3. Main results

Throughout the remainder of the paper, we assume that Y is a closed subset of $\mathbb{R}^d$ (endowed with the Euclidean norm $\|\cdot\|$) such that $\operatorname{int} Y \neq \emptyset$, and we write ℓ_d for the d-dimensional Lebesgue measure restricted to $\mathcal{B}(Y)$. Moreover, by $\bar{\ell}_d$ we denote the product measure $\ell_d \otimes m_c$ on $X = Y \times I$, where m_c is the counting measure on I. The latter can be therefore expressed as $m_c(J) = \sum_{j\in I} \delta_j(J)$ for $J \subset I$, where δ_j stands for the Dirac measure at j. Our aim is to find conditions ensuring the absolute continuity (with respect to $\bar{\ell}_d$) of ergodic invariant probability measures of the operator P, if any exist, and the corresponding invariant measures of the semigroup $\{P_t\}_{t\geq 0}$.

3.1. Singularity/absolute continuity dichotomy of ergodic P-invariant measures

We begin our analysis with simple observations regarding the case where the Markov operator P, induced by (2.2), as well as the operators G and W, corresponding to (2.5) and (2.6), respectively, preserve the absolute continuity of measures.

Lemma 3.1. *Suppose that, for all $\theta \in \Theta$, $k \in I$ and $t \geq 0$, the transformations w_θ and $S_k(t, \cdot)$ are non-singular with respect to ℓ_d. Then the Markov operator P induced by (2.2) satisfies*

$$P\left(\mathcal{M}_{ac}(X, \bar{\ell}_d)\right) \subset \mathcal{M}_{ac}\left(X, \bar{\ell}_d\right).$$

Lemma 3.2. *Suppose that the assumption of Lemma 3.1 is fulfilled. Then the Markov operators G and W, generated by (2.5) and (2.6), respectively, satisfy*

$$G\left(\mathcal{M}_{ac}(X, \bar{\ell}_d)\right) \subset \mathcal{M}_{ac}\left(X, \bar{\ell}_d\right) \quad \text{and} \quad W\left(\mathcal{M}_{ac}(X, \bar{\ell}_d)\right) \subset \mathcal{M}_{ac}\left(X, \bar{\ell}_d\right).$$

The idea underlying the proofs of these lemmas is to construct the densities of $P\mu$, $G\mu$ and $W\mu$ by using the density of $\mu \in \mathcal{M}_{ac}(X, \bar{\ell}_d)$ and the Frobenius–Perron operators $\mathcal{P}_{\theta,j,t}$, $\mathcal{P}_t$ and $\mathcal{P}_{\theta,j}$ associated with the non-singular transformations mapping (y, i) to $(w_\theta(S_i(t, y)), j)$, $(S_i(t, y), i)$ and $(w_\theta(y), j)$, respectively. The details are rather technical, and thus the rigorous proofs are postponed to the Appendix.

Collecting all the results obtained so far, we can state the following theorem:

Theorem 3.1. *Let P, G, W be the Markov operators generated by (2.2), (2.5) and (2.6), respectively, and let $\{P_t\}_{t\geq 0}$ be the Markov semigroup corresponding to (2.4). Further, suppose that, for all $\theta \in \Theta$, $k \in I$ and $t \geq 0$, the transformations w_θ and $S_k(t, \cdot)$ are non-singular with respect to the Lebesgue measure ℓ_d. Then*

(i) Every ergodic invariant probability measure of P is either absolutely continuous or singular with respect to $\bar{\ell}_d$.

(ii) If $\mu_*, \nu_* \in \mathcal{M}_{prob}(X)$ are invariant probability measures for P and $\{P_t\}_{t\geq 0}$, respectively, which correspond to each other in the manner of Theorem 2.1, that is,

$$\nu_* = G\mu_* \;\; \text{or, equivalently,} \;\; \mu_* = W\nu_*,$$

then the measure μ_* is absolutely continuous with respect to $\bar{\ell}_d$ if and only if so is ν_*.

(iii) If $\mu_*, \nu_* \in \mathcal{M}_{prob}(X)$ are the unique invariant probability measures for P and $\{P_t\}_{t\geq 0}$, respectively, then μ_* is absolutely continuous with respect to $\bar{\ell}_d$ if and only if so is ν_*.

Proof. The first statement of the theorem follows immediately from Lemmas 1.1 and 3.1. The second one is just a summary of Theorem 2.1 and Lemma 3.2. Finally, the last assertion is a straightforward consequence of the second one. □

For a given ergodic P-invariant probability measure μ_*, Theorem 3.1 enables us to restrict our enquiry about the absolute continuity of both μ_* and $G\mu_*$ to the one about the non-triviality of the continuous part of μ_*. Certain general conditions providing the positive answer to this question are given in the result below. These conditions should be viewed as a starting point for the forthcoming discussion regarding possible restrictions on the component functions of the model that would guarantee the desired absolute continuity.

Proposition 3.1. *Let μ_* be any invariant probability measure of the Markov operator P, corresponding to (2.2). Suppose that there exist non-empty open subsets U, V of Y and an index $i \in I$ such that, for some $n \in \mathbb{N}$ and some $\bar{c} > 0$,*

$$P^n(x, B \times \{j\}) \geq \bar{c}\, \ell_d(B \cap V) \quad \text{for all} \quad x \in U \times \{i\},\; j \in I \;\; \text{and} \;\; B \in \mathcal{B}(Y). \tag{3.1}$$

Furthermore, assume that there exist a set $\widetilde{X} \in \mathcal{B}(X)$ *with* $\mu_*(\widetilde{X}) > 0$, $m \in \mathbb{N}$ *and* $\delta > 0$ *such that*

$$P^m(x, U \times \{i\}) \geq \delta \quad \textit{for every} \quad x \in \widetilde{X}. \tag{3.2}$$

Then the absolutely continuous part of μ_* *with respect to* $\bar{\ell}_d$ *is non-trivial. If, additionally,* μ_* *is ergodic and the assumption of Theorem 3.1 is fulfilled, then both* μ_* *and* $G\mu_*$ *are absolutely continuous with respect to* $\bar{\ell}_d$.

Proof. Let $B \in \mathcal{B}(Y)$ and $j \in I$. Taking into account the invariance of μ_* and condition (3.1), we can write

$$\begin{aligned}\mu_*(B \times \{j\}) = P^n\mu_*(B \times \{j\}) &= \int_X P^n(x, B \times \{j\})\, \mu_*(dx) \\ &\geq \int_{U \times \{i\}} P^n(x, B \times \{j\})\, \mu_*(dx) \geq \bar{c}\,\ell_d(B \cap V)\mu_*(U \times \{i\}).\end{aligned}$$

Using again the invariance of μ_*, we get

$$\mu_*(B \times \{j\}) \geq \bar{c}\,\ell_d(B \cap V)P^m\mu_*(U \times \{i\}) \geq \bar{c}\,\ell_d(B \cap V)\int_{\widetilde{X}} P^m(x, U \times \{i\})\, \mu_*(dx).$$

Finally, applying hypothesis (3.2) gives

$$\mu_*(B \times \{j\}) \geq \bar{c}\delta\mu_*(\widetilde{X})\ell_d(B \cap V),$$

which shows that μ_* indeed has a non-trivial absolutely continuous part. The second part of the assertion follows immediately from Theorem 3.1. $\square$

The assumptions of Proposition 3.1, referring to an open set $U \times \{i\}$, may be interpreted as follows. Condition (3.1) says that this set is $(n, \ell_d|_V)$-*small* in the sense of [26]. According to (3.2), it is also *uniformly accessible* from some subset of X with positive measure μ_* in some specified number of steps.

3.2. A criterion on absolute continuity of ergodic invariant measures associated with the model

In this section, as well as in the rest of the paper, we require that $\Theta \subset \mathbb{R}$ is either a finite set, viewed as a subset of $\mathbb{R}$ with the discrete topology (in which it is open) or an arbitrary interval, considered as a subset of $\mathbb{R}$ with the usual (Euclidean) topology. Let us note that, in the first case, we just have $\operatorname{int}\Theta = \Theta$, while in the second one, $\operatorname{int}\Theta$ means Θ without its endpoints.

If Θ is finite, we assume it is equipped with the counting measure $\vartheta = \sum_{\theta\in\Theta}\delta_\theta$, while in the case where it is an interval, we require that ϑ is a non-atomic Borel measure that is positive on every non-empty open subset of Θ (e.g., if Θ is bounded, we can just take $\vartheta = \ell_1|_{\mathcal{B}(\Theta)}$).

As mentioned in the introduction, our main goal is to provide a set of tractable conditions for the components of the model, which are sufficient for the absolute continuity of the unique invariant probability measures associated with the Markov operator P and the semigroup $\{P_t\}_{t\geq 0}$, induced by (2.2) and (2.4), respectively. To do this, we shall need an explicit form of the nth-step kernel $(x, A) \mapsto P^n(x, A)$. For this reason, it is convenient to introduce the following piece of notation.

For each $k \in \mathbb{N}$, let $\mathbf{j}_k$, $\mathbf{t}_k$, $\boldsymbol{\theta}_k$ denote $(j_1, \ldots, j_k) \in I^k$, $(t_1, \ldots, t_k) \in \mathbb{R}_+^k$, $(\theta_1, \ldots, \theta_k) \in \Theta^k$, respectively. Further, given any $i, j \in I$, we employ the following convention:

$$(i, \mathbf{j}_k) := (i, j_1, \ldots, j_k) \quad \text{and} \quad (i, \mathbf{j}_k, j) := (i, j_1, \ldots, j_k, j).$$

Here, for notational consistency, we additionally put $(i, \mathbf{j}_0) := i$ and $(i, \mathbf{j}_0, j) := (i, j)$ if $k = 0$. In some places, we will write $\hat{\mathbf{j}}_k, \hat{\mathbf{t}}_k, \hat{\boldsymbol{\theta}}_k$ or $\bar{\mathbf{j}}_k, \bar{\mathbf{t}}_k, \bar{\boldsymbol{\theta}}_k$ (instead of $\mathbf{j}_k, \mathbf{t}_k, \boldsymbol{\theta}_k$, respectively), using the same convention.

With this notation, for any $n \in \mathbb{N}$, $y \in Y$, $i \in I$ and $(\mathbf{j}_n, \mathbf{t}_n, \boldsymbol{\theta}_n) \in I^n \times \mathbb{R}_+^n \times \Theta^n$, we may define

$$\begin{aligned}
\mathcal{W}_1(y,i,t_1,\theta_1) &:= w_{\theta_1}(S_i(t_1,y)),\\
\mathcal{W}_n(y,\,(i,\mathbf{j}_{n-1}),\mathbf{t}_n,\boldsymbol{\theta}_n) &:= w_{\theta_n}(S_{j_{n-1}}(t_n,\mathcal{W}_{n-1}(y,(i,\mathbf{j}_{n-2}),\mathbf{t}_{n-1},\boldsymbol{\theta}_{n-1})));\\
\Pi_1(y,(i,j_1),t_1,\theta_1) &:= \pi_{ij_1}(w_{\theta_1}(S_i(t_1,y))),\\
\Pi_n(y,(i,\mathbf{j}_n),\mathbf{t}_n,\boldsymbol{\theta}_n) &:= \Pi_{n-1}(y,(i,\mathbf{j}_{n-1}),\mathbf{t}_{n-1},\boldsymbol{\theta}_{n-1})\pi_{j_{n-1}j_n}(\mathcal{W}_n(y,(i,\mathbf{j}_{n-1}),\mathbf{t}_n,\boldsymbol{\theta}_n));\\
\mathcal{P}_1(y,i,t_1,\theta_1) &:= p_{\theta_1}(S_i(t_1,y)),\\
\mathcal{P}_n(y,(i,\mathbf{j}_{n-1}),\mathbf{t}_n,\boldsymbol{\theta}_n) &:= \mathcal{P}_{n-1}(y,(i,\mathbf{j}_{n-2}),\mathbf{t}_{n-1},\boldsymbol{\theta}_{n-1})\\
&\quad\times p_{\theta_n}(S_{j_{n-1}}(t_n,\mathcal{W}_{n-1}(y,(i,\mathbf{j}_{n-2}),\mathbf{t}_{n-1},\boldsymbol{\theta}_{n-1}))).
\end{aligned}$$

The nth-step transition law of the chain $\{\Phi_k\}_{k\in\mathbb{N}_0}$ can be now expressed as

$$\begin{aligned}
P^n((y,i),A) = \sum_{\mathbf{j}_n\in I^n}\int_{\Theta^n}\int_{\mathbb{R}_+^n} &\lambda^n e^{-\lambda(t_1+\cdots+t_n)}\mathbb{1}_A(\mathcal{W}_n(y,(i,\mathbf{j}_{n-1}),\mathbf{t}_n,\boldsymbol{\theta}_n),j_n)\\
&\times \Pi_n(y,(i,\mathbf{j}_n),\mathbf{t}_n,\boldsymbol{\theta}_n)\mathcal{P}_n(y,(i,\mathbf{j}_{n-1}),\mathbf{t}_n,\boldsymbol{\theta}_n)\,d\mathbf{t}_n\,\vartheta^{\otimes n}(d\boldsymbol{\theta}_n)
\end{aligned} \tag{3.3}$$

for every $(y,i) \in X = Y \times I$ and each $A \in \mathcal{B}(X)$, where the symbols $d\mathbf{t}_n$ and $\vartheta^{\otimes n}(d\boldsymbol{\theta}_n)$ represent $\ell_n(dt_1,\ldots,dt_n)$ and $(\vartheta\otimes\cdots\otimes\vartheta)(d\theta_1,\ldots,d\theta_n)$, respectively.

In what follows, we shall assume that, for every $\theta \in \Theta$ and each $i \in I$, the maps

$$Y \ni (y_1,\ldots,y_d) =: y \mapsto w_\theta(y) \quad\text{and}\quad (0,\infty)\times Y \ni (t,y) \mapsto S_i(t,y)$$

are continuously differentiable with respect to each of the variables y_k, $k = 1,\ldots,d$, and t. In the case where Θ is an interval, we additionally require that the map $\operatorname{int}\Theta \ni \theta \mapsto w_\theta(y)$ is continuously differentiable for all $y \in Y$.

Let $\hat{y} := (\hat{y}_1,\ldots,\hat{y}_d) \in Y$, $i \in I$, $n \in \mathbb{N}$, $\hat{\boldsymbol{\theta}}_n := (\hat{\theta}_1,\ldots,\hat{\theta}_n) \in \Theta^n$, $\hat{\mathbf{t}}_n := (\hat{t}_1,\ldots,\hat{t}_n) \in (0,\infty)^n$, and, if $n > 1$, also $\hat{\mathbf{j}}_{n-1} := (\hat{j}_1,\ldots,\hat{j}_{n-1}) \in I^{n-1}$. For every $m \le n$ and any pairwise different indices $k_1,\ldots,k_m \in \{1,\ldots,n\}$, the Jacobi matrix of the map

$$(t_{k_1},\ldots,t_{k_m}) \mapsto \mathcal{W}_n(\hat{y},(i,\hat{\mathbf{j}}_{n-1}),\mathbf{t}_n,\hat{\boldsymbol{\theta}}_n) \;\text{ with fixed }\; t_r = \hat{t}_r \;\text{ for }\; r \in \{1,\ldots,n\}\backslash\{k_1,\ldots,k_m\}$$

at point $(\hat{t}_{k_1},\ldots,\hat{t}_{k_m})$ will be denoted by $\partial_{(t_{k_1},\ldots,t_{k_m})}\mathcal{W}_n(\hat{y},(i,\hat{\mathbf{j}}_{n-1}),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n)$. More precisely, assuming that $\mathcal{W}_n = (\mathcal{W}_n^{(1)},\ldots,\mathcal{W}_n^{(d)})$, where $\mathcal{W}_n^{(l)}$ takes values in $\mathbb{R}$ for each $l \in \{1,\ldots,d\}$, we put

$$\partial_{(t_{k_1},\ldots,t_{k_m})}\mathcal{W}_n(\hat{y},(i,\hat{\mathbf{j}}_{n-1}),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n) := \left[\frac{\partial\mathcal{W}_n^{(l)}}{\partial t_{k_r}}(\hat{y},(i,\hat{\mathbf{j}}_{n-1}),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n)\right]_{\substack{l\in\{1,\ldots,d\}\\ r\in\{1,\ldots,m\}}}.$$

Analogously, for any pairwise different $l_1,\ldots,l_m \in \{1,\ldots,n\}$ and $r_1,\ldots,r_m \in \{1,\ldots,d\}$, we can define the matrices

$$\partial_{(\theta_{l_1},\ldots,\theta_{l_m})}\mathcal{W}_n(\hat{y},(i,\hat{\mathbf{j}}_{n-1}),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n) \quad\text{and}\quad \partial_{(y_{r_1},\ldots,y_{r_m})}\mathcal{W}_n(\hat{y},(i,\hat{\mathbf{j}}_{n-1}),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n).$$

A key role in our discussion will be played by the following lemma, which provides a tractable condition under which the law P verifies the first hypothesis of Proposition 3.1, expressed in (3.1). The proof of this result is based upon ideas found in [4] (cf. the proofs of [4, Lemmas 6.2 and 6.3]).

Lemma 3.3. *Let $(\hat{y},i) \in \operatorname{int} Y \times I$, and suppose that, for some integer $n \ge d$, there exist sequences $\hat{\mathbf{t}}_n \in (0,\infty)^n$, $\hat{\boldsymbol{\theta}}_n \in (\operatorname{int}\Theta)^n$ and, in the case of $n > 1$, also $\hat{\mathbf{j}}_{n-1} \in I^{n-1}$, such that*

$$\mathcal{P}_n(\hat{y},(i,\hat{\mathbf{j}}_{n-1}),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n)\,\Pi_n(\hat{y},(i,\hat{\mathbf{j}}_{n-1},j),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n) > 0 \quad \textit{for every} \quad j \in I, \tag{3.4}$$

and

$$\operatorname{rank} \partial_{\mathbf{t}_n} \mathcal{W}_n(\hat{y}, (i, \hat{\mathbf{j}}_{n-1}), \hat{\mathbf{t}}_n, \hat{\boldsymbol{\theta}}_n) = d. \tag{3.5}$$

Then there is an open neighbourhood $U_{\hat{y}} \subset Y$ *of* $\hat{y}$ *and an open neighbourhood* $U_{\hat{w}} \subset Y$ *of the point*

$$\hat{w} := \mathcal{W}_n(\hat{y}, (i, \hat{\mathbf{j}}_{n-1}), \hat{\mathbf{t}}_n, \hat{\boldsymbol{\theta}}_n) \tag{3.6}$$

such that, for some constant $\bar{c} > 0$, *we have*

$$P^n(x, B \times \{j\}) \geq \bar{c}\, \ell_d(B \cap U_{\hat{w}}) \quad \textit{for all} \quad x \in U_{\hat{y}} \times \{i\},\, B \in \mathcal{B}(Y),\; j \in I. \tag{3.7}$$

Proof. According to (3.5), there exist $k_1, \ldots, k_d \in \{1, \ldots, n\}$ such that

$$\det \partial_{(t_{k_1}, \ldots, t_{k_d})} \mathcal{W}_n(\hat{y},\, (i, \hat{\mathbf{j}}_{n-1}), \hat{\mathbf{t}}_n, \hat{\boldsymbol{\theta}}_n) \neq 0.$$

Without loss of generality, we may further assume that $(k_1, \ldots, k_d) = (1, \ldots, d)$, i.e.

$$\det \partial_{\mathbf{t}_d} \mathcal{W}_n(\hat{y},\, (i, \hat{\mathbf{j}}_{n-1}), \hat{\mathbf{t}}_n, \hat{\boldsymbol{\theta}}_n) \neq 0. \tag{3.8}$$

In the analysis that follows, given $\mathbf{t}_n = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$, we shall write $\mathbf{t}^{n-d}$ to denote $(t_{d+1}, \ldots, t_n)$, so that $\mathbf{t}_n = (\mathbf{t}_d, \mathbf{t}^{n-d})$.

Case I: Consider first the case where Θ is finite. For each $y \in Y$, let us introduce the map $\mathcal{R}_y : (0, \infty)^n \to Y \times \mathbb{R}_+^{n-d} \subset \mathbb{R}^n$ given by

$$\mathcal{R}_y(\mathbf{t}_n) := (\mathcal{W}_n(y, (i, \hat{\mathbf{j}}_{n-1}), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n), \mathbf{t}^{n-d}) \quad \text{for} \quad \mathbf{t}_n \in (0, \infty)^n.$$

We can then easily observe that

$$\partial_{\mathbf{t}_n} \mathcal{R}_y(\mathbf{t}_n) = \left[\begin{array}{c|c} \partial_{\mathbf{t}_d} \mathcal{W}_n & \partial_{\mathbf{t}^{n-d}} \mathcal{W}_n \\ \hline \mathbf{0}_{n-d,\,d} & I_{n-d} \end{array}\right] (y, (i, \hat{\mathbf{j}}_{n-1}), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n),$$

where $\mathbf{0}_{n-d,\,d}$ and I_{n-d} are the zero matrix of size $(n-d) \times d$ and the identity matrix of order $n-d$, respectively. This yields that

$$\det \partial_{\mathbf{t}_n} \mathcal{R}_y(\mathbf{t}_n) = \det\, \partial_{\mathbf{t}_d} \mathcal{W}_n(y, (i, \hat{\mathbf{j}}_{n-1}), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n) \quad \text{for all} \quad \mathbf{t}_n \in (0, \infty)^n,\; y \in Y. \tag{3.9}$$

Further, let us define $\mathcal{H} : (0, \infty)^n \times \operatorname{int} Y \to (Y \times \mathbb{R}_+^{n-d}) \times Y$, acting from an open subset of $\mathbb{R}^{n+d}$ into itself, by

$$\mathcal{H}(\mathbf{t}_n, y) := (\mathcal{R}_y(\mathbf{t}_n), y) \quad \text{for all} \quad \mathbf{t}_n \in (0, \infty)^n,\; y \in \operatorname{int} Y.$$

Since the Jacobi matrix of $\mathcal{H}$ can also be written in a block form, namely

$$\partial_{(\mathbf{t}_n, y)} \mathcal{H}(\mathbf{t}_n, y) = \left[\begin{array}{c|c} \partial_{\mathbf{t}_d} \mathcal{W}_n & \partial_{\mathbf{t}^{n-d}} \mathcal{W}_n \;\; \partial_y \mathcal{W}_n \\ \hline \mathbf{0}_{n,\,d} & I_n \end{array}\right] (y, (i, \hat{\mathbf{j}}_{n-1}), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n),$$

it follows, due to (3.8), that

$$\det \partial_{(\mathbf{t}_n, y)} \mathcal{H}(\hat{\mathbf{t}}_n, \hat{y}) = \det\, \partial_{\mathbf{t}_d} \mathcal{W}_n(\hat{y},\, (i, \hat{\mathbf{j}}_{n-1}), \hat{\mathbf{t}}_n, \hat{\boldsymbol{\theta}}_n) \neq 0. \tag{3.10}$$

Consequently, by virtue of the local inversion theorem, we can choose an open neighbourhood $\widehat{V}_{(\hat{\mathbf{t}}_n, \hat{y})} \subset (0, \infty)^n \times \operatorname{int} Y$ of $(\hat{\mathbf{t}}_n, \hat{y})$ so that $\mathcal{H}|_{\widehat{V}_{(\hat{\mathbf{t}}_n, \hat{y})}} : \widehat{V}_{(\hat{\mathbf{t}}_n, \hat{y})} \to \mathcal{H}(\widehat{V}_{(\hat{\mathbf{t}}_n, \hat{y})})$ is a diffeomorphism. Obviously, $\mathcal{H}(\widehat{V}_{(\hat{\mathbf{t}}_n, \hat{y})}) \subset (Y \times \mathbb{R}_+^{n-d}) \times Y$.

If we now define

$$\mathcal{T}_n(y, (i, \hat{\mathbf{j}}_{n-1}, j), \mathbf{t}_n, \boldsymbol{\theta}_n) := \lambda^n e^{-\lambda(t_1 + \cdots + t_n)} \mathcal{P}_n(y, (i, \hat{\mathbf{j}}_{n-1}), \mathbf{t}_n, \boldsymbol{\theta}_n) \varPi_n(y, (i, \hat{\mathbf{j}}_{n-1}, j), \mathbf{t}_n, \boldsymbol{\theta}_n), \tag{3.11}$$

then, using (3.4) and (3.10), together with continuity of the component functions of the model and the map $\widehat{V}_{(\hat{\mathbf{t}}_n,\hat{y})} \ni (\mathbf{t}_n, y) \mapsto \det \partial_{(\mathbf{t}_n,y)}\mathcal{H}(\mathbf{t}_n, y)$, we may find an open neighbourhood $\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})} \subset \widehat{V}_{(\hat{\mathbf{t}}_n,\hat{y})}$ of $(\hat{\mathbf{t}}_n, \hat{y})$ such that, for some constant $\tilde{c} > 0$,

$$\left|\det \partial_{(\mathbf{t}_n,y)}\mathcal{H}(\mathbf{t}_n, y)\right|^{-1} \mathcal{T}_n(y, (i, \hat{\mathbf{j}}_{n-1}, j), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n) \geq \tilde{c} \quad \text{for all} \quad (\mathbf{t}_n, y) \in \widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})},\ j \in I. \tag{3.12}$$

Taking into account that, due to (3.9),

$$\det \partial_{(\mathbf{t}_n,y)}\mathcal{H}(\mathbf{t}_n, y) = \det\, \partial_{\mathbf{t}_d}\mathcal{W}_n(y, (i, \hat{\mathbf{j}}_{n-1}), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n) = \det \partial_{\mathbf{t}_n}\mathcal{R}_y(\mathbf{t}_n),$$

we then obtain

$$|\det \partial_{\mathbf{t}_n}\mathcal{R}_y(\mathbf{t}_n)|^{-1}\, \mathcal{T}_n(y, (i, \hat{\mathbf{j}}_{n-1}, j), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n) \geq \tilde{c} \quad \text{for all} \quad (\mathbf{t}_n, y) \in \widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})},\ j \in I. \tag{3.13}$$

Clearly, $\mathcal{H}|_{\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})}} : \widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})} \to \mathcal{H}(\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})})$ is also a diffeomorphism, and thus, in particular, the set $\mathcal{H}(\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})})$ is open. Since $((\hat{w}, \hat{\mathbf{t}}^{n-d}), \hat{y}) \in \mathcal{H}(\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})})$, where $\hat{w}$ is given by (3.6), there exist open bounded neighbourhoods $U_{(\hat{w},\, \hat{\mathbf{t}}^{n-d})} \subset Y \times \mathbb{R}_+^{n-d}$ and $U_{\hat{y}} \subset Y$ of the points $(\hat{w}, \hat{\mathbf{t}}^{n-d})$ and $\hat{y}$, respectively, with the property that $U_{(\hat{w},\, \hat{\mathbf{t}}^{n-d})} \times U_{\hat{y}} \subset \mathcal{H}(\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y})})$. Let

$$V_{(\hat{\mathbf{t}}_n,\hat{y})} := \mathcal{H}^{-1}(U_{(\hat{w},\, \hat{\mathbf{t}}^{n-d})} \times U_{\hat{y}}),$$

and, for any $((w, \mathbf{t}^{n-d}), y) \in U_{(\hat{w},\, \hat{\mathbf{t}}^{n-d})} \times U_{\hat{y}}$, write $\mathcal{H}^{-1}((w, \mathbf{t}^{n-d}), y) = (\overline{\mathcal{R}}_y(w, \mathbf{t}^{n-d}), y)$. Then, it follows immediately that $\mathcal{R}_y\left(\overline{\mathcal{R}}_y(w, \mathbf{t}^{n-d})\right) = (w, \mathbf{t}^{n-d})$, whence $\overline{\mathcal{R}}_y$ is the continuously differentiable inverse of an appropriate restriction of $\mathcal{R}_y$. More specifically, introducing

$$W(y) := \{\mathbf{t}_n \in (0, \infty)^n : (\mathbf{t}_n, y) \in V_{(\hat{\mathbf{t}}_n,\hat{y})}\} \quad \text{for every} \quad y \in U_{\hat{y}},$$

we see that each of these sets is open, and that $\mathcal{R}_y|_{W(y)} : W(y) \to U_{(\hat{w},\, \hat{\mathbf{t}}^{n-d})}$ is a diffeomorphism for every $y \in U_{\hat{y}}$. Obviously, by the definition of $W(y)$, we have

$$(\mathbf{t}_n, y) \in V_{(\hat{\mathbf{t}}_n,\hat{y})} \quad \text{whenever} \quad \mathbf{t}_n \in W(y),\ y \in U_{\hat{y}}. \tag{3.14}$$

In view of the above, we can choose (independently of y) open neighbourhoods $U_{\hat{w}} \subset Y$ and $U_{\hat{\mathbf{t}}^{n-d}} \subset \mathbb{R}_+^{n-d}$ of $\hat{w}$ and $\hat{\mathbf{t}}^{n-d}$, respectively, in such a way that

$$U_{\hat{w}} \times U_{\hat{\mathbf{t}}^{n-d}} \subset U_{(\hat{w},\, \hat{\mathbf{t}}^{n-d})} = \mathcal{R}_y(W(y)) \quad \text{for every} \quad y \in U_{\hat{y}}. \tag{3.15}$$

Now, keeping in mind (3.3), (3.13) and (3.14), for any $B \in \mathcal{B}(Y)$, $j \in I$ and $y \in U_{\hat{y}}$, we can write

$$\begin{aligned}
P^n((y, i), B \times \{j\}) &\geq \int_{\mathbb{R}_+^n} \mathbb{1}_B(\mathcal{W}_n(y,\, (i, \hat{\mathbf{j}}_{n-1}), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n))\mathcal{T}_n(y, (i, \hat{\mathbf{j}}_{n-1}, j), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n) d\mathbf{t}_n \\
&\geq \int_{W(y)} \mathbb{1}_{B \times \mathbb{R}^{n-d}}(\mathcal{R}_y(\mathbf{t}_n))\mathcal{T}_n(y, (i, \hat{\mathbf{j}}_{n-1}, j), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n) d\mathbf{t}_n \\
&= \int_{W(y)} \mathbb{1}_{B \times \mathbb{R}^{n-d}}(\mathcal{R}_y(\mathbf{t}_n))|\det \partial_{\mathbf{t}_n}\mathcal{R}_y(\mathbf{t}_n)| \cdot |\det \partial_{\mathbf{t}_n}\mathcal{R}_y(\mathbf{t}_n)|^{-1}\mathcal{T}_n(y, (i, \hat{\mathbf{j}}_{n-1}, j), \mathbf{t}_n, \hat{\boldsymbol{\theta}}_n) d\mathbf{t}_n \\
&\geq \tilde{c} \int_{W(y)} \mathbb{1}_{B \times \mathbb{R}^{n-d}}(\mathcal{R}_y(\mathbf{t}_n))|\det \partial_{\mathbf{t}_n}\mathcal{R}_y(\mathbf{t}_n)|\, d\mathbf{t}_n.
\end{aligned}$$

If we now change variables by setting $\mathbf{s}_n = \mathcal{R}_y(\mathbf{t}_n)$ and, further, apply (3.15), then we can conclude that

$$\begin{aligned}P^n((y,i), B\times\{j\}) &\geq \tilde{c}\int_{\mathcal{R}_y(W(y))} \mathbb{1}_{B\times\mathbb{R}^{n-d}}(\mathbf{s}_n)\, d\mathbf{s}_n \geq \tilde{c}\int_{U_{\hat{w}}\times U_{\hat{\mathbf{t}}^{n-d}}} \mathbb{1}_{B\times\mathbb{R}^{n-d}}(\mathbf{s}_n)\, d\mathbf{s}_n\\ &= \tilde{c}\,\ell_n((B\times\mathbb{R}^{n-d})\cap(U_{\hat{w}}\times U_{\hat{\mathbf{t}}^{n-d}})) = \tilde{c}\,\ell_{n-d}(U_{\hat{\mathbf{t}}^{n-d}})\ell_d(B\cap U_{\hat{w}}),\end{aligned}$$

Finally, we see that (3.7) holds with $\bar{c} := \tilde{c}\,\ell_{n-d}(U_{\hat{\mathbf{t}}^{n-d}}) > 0$.

Case II: Let us now assume that Θ is an interval in $\mathbb{R}$. The proof in this case is similar to the previous one. This time, however, we need to consider a family $\{\mathcal{R}_{y,\boldsymbol{\theta}_n} : y\in Y,\ \boldsymbol{\theta}_n\in\Theta^n\}$ of maps from $(0,\infty)^n$ into $Y\times\mathbb{R}_+^{n-d}\subset\mathbb{R}^n$, wherein $\mathcal{R}_{y,\boldsymbol{\theta}_n}$ is defined by

$$\mathcal{R}_{y,\boldsymbol{\theta}_n}(\mathbf{t}_n) := (\mathcal{W}_n(y,(i,\hat{\mathbf{j}}_{n-1}),\mathbf{t}_n,\boldsymbol{\theta}_n),\mathbf{t}^{n-d}) \quad \text{for} \quad \mathbf{t}_n\in(0,\infty)^n.$$

Furthermore, $\mathcal{H}$ will now stand for the map

$$\mathcal{H} : (0,\infty)^n \times \operatorname{int} Y \times (\operatorname{int}\Theta)^n \to (Y\times\mathbb{R}_+^{n-d})\times Y\times\Theta^n$$

(acting from an open subset of $\mathbb{R}^{2n+d}$ into itself) given by

$$\mathcal{H}(\mathbf{t}_n,y,\boldsymbol{\theta}_n) := (\mathcal{R}_{y,\boldsymbol{\theta}_n}(\mathbf{t}_n),y,\boldsymbol{\theta}_n) \quad \text{for} \quad \mathbf{t}_n\in(0,\infty)^n,\ y\in\operatorname{int}Y,\ \boldsymbol{\theta}_n\in(\operatorname{int}\Theta)^n.$$

Since the Jacobi matrix of $\mathcal{H}$ is of the form

$$\partial_{(\mathbf{t}_n,y,\boldsymbol{\theta}_n)}\mathcal{H}(\mathbf{t}_n,y,\boldsymbol{\theta}_n) = \left[\begin{array}{c|c} \partial_{\mathbf{t}_d}\mathcal{W}_n & \partial_{\mathbf{t}^{n-d}}\mathcal{W}_n \quad \partial_y\mathcal{W}_n \quad \partial_{\boldsymbol{\theta}_n}\mathcal{W}_n \\ \hline \mathbf{0}_{2n,\,d} & I_{2n}\end{array}\right](y,(i,\hat{\mathbf{j}}_{n-1}),\mathbf{t}_n,\boldsymbol{\theta}_n),$$

similarly as in the previous case, we obtain

$$\det\partial_{(\mathbf{t}_n,y,\boldsymbol{\theta}_n)}\mathcal{H}(\hat{\mathbf{t}}_n,\hat{y},\hat{\boldsymbol{\theta}}_n) = \det\partial_{\mathbf{t}_d}\mathcal{W}_n(\hat{y},\,(i,\hat{\mathbf{j}}_{n-1}),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n)\neq 0. \tag{3.16}$$

This enables us to choose an open neighbourhood $\widehat{V}_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)}\subset(0,\infty)^n\times\operatorname{int}Y\times(\operatorname{int}\Theta)^n$ of the point $(\hat{\mathbf{t}}_n,\hat{y},\hat{\boldsymbol{\theta}}_n)$ so that $\mathcal{H}|_{\widehat{V}_{(\hat{\mathbf{t}}_n,\hat{y},\hat{\boldsymbol{\theta}}_n)}}$ is a diffeomorphism from $\widehat{V}_{(\hat{\mathbf{t}}_n,\hat{y},\hat{\boldsymbol{\theta}}_n)}$ onto $\mathcal{H}(\widehat{V}_{(\hat{\mathbf{t}}_n,\hat{y},\hat{\boldsymbol{\theta}}_n)})$.

Appealing to (3.4), (3.16) and the continuity of the map

$$\widehat{V}_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)}\ni(\mathbf{t}_n,y,\boldsymbol{\theta}_n)\mapsto\det\partial_{(\mathbf{t}_n,y,\boldsymbol{\theta}_n)}\mathcal{H}(\mathbf{t}_n,y,\boldsymbol{\theta}_n),$$

we may find an open neighbourhood $\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)}\subset\widehat{V}_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)}$ of the point $(\hat{\mathbf{t}}_n,\hat{y},\hat{\boldsymbol{\theta}}_n)$ such that, for some constant $\tilde{c}>0$,

$$\left|\det\partial_{(\mathbf{t}_n,y,\,\boldsymbol{\theta}_n)}\mathcal{H}(\mathbf{t}_n,y,\boldsymbol{\theta}_n)\right|^{-1}\mathcal{T}_n(y,(i,\hat{\mathbf{j}}_{n-1},j),\mathbf{t}_n,\boldsymbol{\theta}_n)\geq\tilde{c} \quad \text{for} \quad (\mathbf{t}_n,y,\boldsymbol{\theta}_n)\in\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)},\ j\in I,$$

where $\mathcal{T}_n$ is defined by (3.11). This obviously yields

$$|\det\partial_{\mathbf{t}_n}\mathcal{R}_{y,\boldsymbol{\theta}_n}(\mathbf{t}_n)|^{-1}\mathcal{T}_n(y,(i,\hat{\mathbf{j}}_{n-1},j),\mathbf{t}_n,\boldsymbol{\theta}_n)\geq\tilde{c} \quad \text{for} \quad (\mathbf{t}_n,y,\boldsymbol{\theta}_n)\in\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)},\ j\in I. \tag{3.17}$$

Since $\mathcal{H}(\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y},\hat{\boldsymbol{\theta}}_n)})$ is open and $((\hat{w},\hat{\mathbf{t}}^{n-d}),\hat{y},\hat{\boldsymbol{\theta}}_n)\in\mathcal{H}(\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y},\hat{\boldsymbol{\theta}}_n)})$, it follows that there exist open bounded neighbourhoods $U_{(\hat{w},\,\hat{\mathbf{t}}^{n-d})}\subset Y\times\mathbb{R}_+^{n-d}$, $U_{\hat{y}}\subset Y$ and $U_{\hat{\boldsymbol{\theta}}_n}\subset\Theta^n$ of the points $(\hat{w},\hat{\mathbf{t}}^{n-d})$, $\hat{y}$ and $\hat{\boldsymbol{\theta}}_n$, respectively, such that $U_{(\hat{w},\,\hat{\mathbf{t}}^{n-d})}\times U_{\hat{y}}\times U_{\hat{\boldsymbol{\theta}}_n}\subset\mathcal{H}(\widetilde{V}_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)})$. Define

$$\begin{aligned}V_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)} &:= \mathcal{H}^{-1}(U_{(\hat{w},\,\hat{\mathbf{t}}^{n-d})}\times U_{\hat{y}}\times U_{\hat{\boldsymbol{\theta}}_n}),\\ W(y,\boldsymbol{\theta}_n) &:= \{\mathbf{t}_n\in(0,\infty)^n : (\mathbf{t}_n,y,\boldsymbol{\theta}_n)\in V_{(\hat{\mathbf{t}}_n,\hat{y},\,\hat{\boldsymbol{\theta}}_n)}\} \quad \text{for} \quad (y,\boldsymbol{\theta}_n)\in U_{\hat{y}}\times U_{\hat{\boldsymbol{\theta}}_n}.\end{aligned}$$

Then, arguing analogously as in Case I, we can conclude that all the sets $W(y,\boldsymbol{\theta}_n)$ are open, and that $\mathcal{R}_{y,\boldsymbol{\theta}_n}|_{W(y,\boldsymbol{\theta}_n)}$ is a diffeomorphism from $W(y,\boldsymbol{\theta}_n)$ onto $U_{(\hat{w},\,\hat{\mathbf{t}}^{n-d})}$ for every $(y,\boldsymbol{\theta}_n)\in U_{\hat{y}}\times U_{\hat{\boldsymbol{\theta}}_n}$. This observation, as before, enables us to choose (independently of y and $\boldsymbol{\theta}_n$) open neighbourhoods $U_{\hat{w}}\subset Y$ and $U_{\hat{\mathbf{t}}^{n-d}}\subset\mathbb{R}_+^{n-d}$ of the points $\hat{w}$ and $\hat{\mathbf{t}}^{n-d}$, respectively, so that

$$U_{\hat{w}}\times U_{\hat{\mathbf{t}}^{n-d}}\subset U_{(\hat{w},\,\hat{\mathbf{t}}^{n-d})}=\mathcal{R}_{y,\boldsymbol{\theta}_n}(W(y,\boldsymbol{\theta}_n))\quad\text{for all}\quad (y,\boldsymbol{\theta}_n)\in U_{\hat{y}}\times U_{\hat{\boldsymbol{\theta}}_n}. \tag{3.18}$$

Proceeding similarly as in the first part of the proof, from (3.3) and (3.17) we may now deduce that, for any $B\in\mathcal{B}(Y)$, $j\in I$ and $y\in U_{\hat{y}}$,

$$\begin{aligned}
P^n((y,i),B\times\{j\}) &\geq \int_{\Theta^n}\int_{\mathbb{R}_+^n}\mathbb{1}_B(\mathcal{W}_n(y,\,(i,\hat{\mathbf{j}}_{n-1}),\mathbf{t}_n,\boldsymbol{\theta}_n))\mathcal{T}_n(y,(i,\hat{\mathbf{j}}_{n-1},j),\mathbf{t}_n,\boldsymbol{\theta}_n)d\mathbf{t}_n\vartheta^{\otimes n}(d\boldsymbol{\theta}_n)\\
&\geq \int_{U_{\hat{\boldsymbol{\theta}}_n}}\int_{W(y,\boldsymbol{\theta}_n)}\mathbb{1}_{B\times\mathbb{R}^{n-d}}(\mathcal{R}_{y,\boldsymbol{\theta}_n}(\mathbf{t}_n))\mathcal{T}_n(y,(i,\hat{\mathbf{j}}_{n-1},j),\mathbf{t}_n,\boldsymbol{\theta}_n)d\mathbf{t}_n\,\vartheta^{\otimes n}(d\boldsymbol{\theta}_n)\\
&\geq \tilde{c}\int_{U_{\hat{\boldsymbol{\theta}}_n}}\int_{W(y,\boldsymbol{\theta}_n)}\mathbb{1}_{B\times\mathbb{R}^{n-d}}(\mathcal{R}_{y,\boldsymbol{\theta}_n}(\mathbf{t}_n))|\det\partial_{\mathbf{t}_n}\mathcal{R}_{y,\boldsymbol{\theta}_n}(\mathbf{t}_n)|\,d\mathbf{t}_n\,\vartheta^{\otimes n}(d\boldsymbol{\theta}_n).
\end{aligned}$$

Finally, substituting $\mathbf{s}_n=\mathcal{R}_{y,\boldsymbol{\theta}_n}(\mathbf{t}_n)$ (for every fixed $(y,\boldsymbol{\theta}_n)$ separately) and applying (3.18), gives

$$\begin{aligned}
P^n((y,i),B\times\{j\}) &\geq \tilde{c}\int_{U_{\hat{\boldsymbol{\theta}}_n}}\int_{\mathcal{R}_{y,\boldsymbol{\theta}_n}(W(y,\boldsymbol{\theta}_n))}\mathbb{1}_{B\times\mathbb{R}^{n-d}}(\mathbf{s}_n)\,d\mathbf{s}_n\,\vartheta^{\otimes n}(d\boldsymbol{\theta}_n)\\
&\geq \tilde{c}\int_{U_{\hat{\boldsymbol{\theta}}_n}}\int_{U_{\hat{w}}\times U_{\hat{\mathbf{t}}^{n-d}}}\mathbb{1}_{B\times\mathbb{R}^{n-d}}(\mathbf{s}_n)\,d\mathbf{s}_n\,\vartheta^{\otimes n}(d\boldsymbol{\theta}_n)\\
&= \tilde{c}\,\vartheta^{\otimes n}(U_{\hat{\boldsymbol{\theta}}_n})\ell_{n-d}(U_{\hat{\mathbf{t}}^{n-d}})\ell_d(B\cap U_{\hat{w}}),
\end{aligned}$$

which shows that (3.7) holds with $\bar{c}:=\tilde{c}\,\vartheta^{\otimes n}(U_{\hat{\boldsymbol{\theta}}_n})\ell_{n-d}(U_{\hat{\mathbf{t}}^{n-d}})>0$ and, therefore, completes the proof. □

Remark 3.1. Note that, in the case where $d=1$, condition (3.5) can be expressed in the following simple form:

$$\sum_{r=1}^{n}\left(\frac{\partial\mathcal{W}_n}{\partial t_r}(\hat{y},(i,\hat{\mathbf{j}}_{n-1}),\hat{\mathbf{t}}_n,\hat{\boldsymbol{\theta}}_n)\right)^2>0.$$

Assuming that conditions (3.4) and (3.5) hold with some point $(\hat{y},i)\in\operatorname{int}Y\times I$, we intend to apply Proposition 3.1 with $U=U_{\hat{y}}$ and $V=U_{\hat{w}}$, where $U_{\hat{y}}$ and $U_{\hat{w}}$ are the open sets guaranteed by Lemma 3.3. To do this, we need to know that, for any given P-ergodic invariant measure μ_*, the set $U_{\hat{y}}\times\{i\}$ is uniformly accessible from an open set $\widetilde{X}\subset X$, satisfying $\mu_*(\widetilde{X})>0$, in some given number of steps, i.e. condition (3.2) holds with $U=U_{\hat{y}}$ and the given i for some $m\in\mathbb{N}$. This is the case, for example, if the operator P is asymptotically stable, and the point $(\hat{y},i)$, verifying the desired properties, belongs to the support of the unique P-invariant measure.

Corollary 3.1. *Let P and $\{P_t\}_{t\geq 0}$ stand for the Markov operator and the Markov semigroup induced by (2.2) and (2.4), respectively. Further, suppose that, for some $\mu_*\in\mathcal{M}_{prob}(X)$, and for every $x\in X$, the sequence $\{P^n\delta_x\}_{n\in\mathbb{N}}$ is weakly convergent to μ_* (which, by Remark 1.1, is equivalent to say that P is asymptotically stable). Moreover, assume that all the transformations w_θ and $S_k(t,\cdot)$ are non-singular with respect to ℓ_d, and that there exists a point $(\hat{y},i)\in(\operatorname{int}Y\times I)\cap\operatorname{supp}\mu_*$, for which the assumptions of Lemma 3.3 are fulfilled. Then both μ_* and $G\mu_*$, which are then unique invariant measures for P and $\{P_t\}_{t\geq 0}$, respectively, are absolutely continuous with respect to $\bar{\ell}_d$.*

Proof. In the light of Proposition 3.1 and Lemma 3.3, it suffices to show that (3.2) holds for $U = U_{\hat{y}}$ and the given i. Since $\hat{x} := (\hat{y}, i) \in \operatorname{supp} \mu_*$, it follows that $\delta_* := \mu_*(U \times \{i\}) > 0$. Taking into account that $\{P^n(x, \cdot)\}_{n\in\mathbb{N}}$ converges weakly to μ_* for every $x \in X$, we can apply the Portmanteau theorem ([6, Theorem 2.1]) to deduce that

$$\liminf_{n\to\infty} P^n(x, U \times \{i\}) \geq \delta_* \quad \text{for every} \quad x \in X.$$

In particular, we therefore get $P^m(\hat{x}, U \times \{i\}) > \delta_*/2$ for some $m \in \mathbb{N}$. Since the operator P is Feller, the map $X \ni x \mapsto P^m(x, U \times \{i\})$ is lower-semicontinuous, and thus there exists an open neighbourhood of $\hat{x}$, say $\widetilde{X}$, such that $P^m(x, U \times \{i\}) > \delta_*/3$ for every $x \in \widetilde{X}$. Moreover, $\mu_*(\widetilde{X}) > 0$, since $\hat{x} \in \operatorname{supp} \mu_*$. This shows that (3.2) is indeed satisfied (with $\delta = \delta_*/3$) and completes the proof. □

The requirement $(\hat{y}, i) \in \operatorname{supp} \mu_*$ is rather implicit and difficult to verify without any additional information regarding the measure μ_*. Moreover, the above-stated results are limited by the assumption that the underlying operator is asymptotically stable. In the remainder of the paper, we therefore derive a somewhat more practical result, which does not require the stability, and enables one to establish the uniform accessibility of $U_{\hat{y}} \times \{i\}$ in the sense of (3.2), using a more intuitive argument, which refers directly to the component functions of the model. More precisely, given $(\hat{y}, i) \in X$, we shall use the following condition:

(A) For every open neighbourhood $V_{\hat{y}}$ of $\hat{y}$ and each $(y, j) \in X$, there exist $n \in \mathbb{N}$, $\mathbf{t}_n \in \mathbb{R}_+^n$, $\boldsymbol{\theta}_n \in \Theta^n$ and, whenever $n > 1$, also $\mathbf{j}_{n-1} \in I^{n-1}$, such that

$$\mathcal{W}_n(y, (j, \mathbf{j}_{n-1}), \mathbf{t}_n, \boldsymbol{\theta}_n) \in V_{\hat{y}} \quad \text{and} \quad \mathcal{P}_n(y, (j, \mathbf{j}_{n-1}), \mathbf{t}_n, \boldsymbol{\theta}_n) \Pi_n(y, (j, \mathbf{j}_{n-1}, i), \mathbf{t}_n, \boldsymbol{\theta}_n) > 0.$$

The following lemma, which is essentially based on [4, Lemma 3.16], should be treated as an intermediate result on the way to the above-mentioned implication (A) ⇒ (3.2).

Lemma 3.4. *Let $\mu \in \mathcal{M}_{fin}(X)$ be an arbitrary non-zero measure. Further, suppose that condition (A) holds for some $(\hat{y}, i) \in \operatorname{int} Y \times I$, and let $U_{\hat{y}} \subset Y$ be an arbitrary open neighbourhood of $\hat{y}$. Then, there exist constants $\varepsilon > 0$, $\beta > 0$, $m \in \mathbb{N}$, sequences $\bar{\mathbf{j}}_{m-1} \in I^{m-1}$, $\bar{\mathbf{t}}_m \in \mathbb{R}_+^m$, $\bar{\boldsymbol{\theta}}_m \in \Theta^m$ (the former only if $m > 1$) and an open set $\widetilde{X} \subset X$ with $\mu(\widetilde{X}) > 0$ such that*

$$\begin{gathered}\mathcal{W}_m(y, (j, \bar{\mathbf{j}}_{m-1}), \mathbf{t}_m, \boldsymbol{\theta}_m) \in U_{\hat{y}}, \\ \mathcal{P}_m(y, (j, \bar{\mathbf{j}}_{m-1}), \mathbf{t}_m, \boldsymbol{\theta}_m) \Pi_m(y, (j, \bar{\mathbf{j}}_{m-1}, i), \mathbf{t}_m, \boldsymbol{\theta}_m) > \beta,\end{gathered} \tag{3.19}$$

whenever $(y, j) \in \widetilde{X}$, $\mathbf{t}_m \in B_{\mathbb{R}_+}(\bar{\mathbf{t}}_m, \varepsilon)$ and $\boldsymbol{\theta}_m \in B_\Theta(\bar{\boldsymbol{\theta}}_m, \varepsilon)$, where

$$B_{\mathbb{R}_+}(\bar{\mathbf{t}}_m, \varepsilon) := \{\mathbf{t}_m \in \mathbb{R}_+^m : \left\|\mathbf{t}_m - \bar{\mathbf{t}}_m\right\|_m < \varepsilon\},$$

$$B_\Theta(\bar{\boldsymbol{\theta}}_m, \varepsilon) := \begin{cases} \{\boldsymbol{\theta}_m \in \Theta^m : \ \|\boldsymbol{\theta}_m - \bar{\boldsymbol{\theta}}_m\|_m < \varepsilon\} & \text{if } \Theta \text{ is an interval,} \\ \{\bar{\boldsymbol{\theta}}_m\} & \text{if } \Theta \text{ is finite,} \end{cases}$$

and $\|\cdot\|_m$ stands for the Euclidean norm in $\mathbb{R}^m$.

Proof. For each $k \in \mathbb{N}$, let $\mathcal{A}_k$ denote the set of all $(\mathbf{j}_{k-1}, \mathbf{t}_k, \boldsymbol{\theta}_k, \beta')$, where $\mathbf{j}_{k-1} \in I^{k-1}$, $\mathbf{t}_k \in \mathbb{R}_+^k$, $\boldsymbol{\theta}_k \in \Theta^k$ and $\beta' > 0$ (excluding the first member whenever $k = 1$). Further, let $V_{\hat{y}}$ be a bounded open neighbourhood of $\hat{y}$ such that $\operatorname{cl} V_{\hat{y}} \subset U_{\hat{y}}$, and define

$$O(\mathbf{j}_{k-1}, \mathbf{t}_k, \boldsymbol{\theta}_k, \beta') := \{(y, j) \in X : \mathcal{W}_k(y, (j, \mathbf{j}_{k-1}), \mathbf{t}_k, \boldsymbol{\theta}_k) \in V_{\hat{y}}, \ \mathcal{Q}_k(y, (j, \mathbf{j}_{k-1}, i), \mathbf{t}_k, \boldsymbol{\theta}_k) > \beta'\},$$

with

$$\mathcal{Q}_k(y, (j, \mathbf{j}_{k-1}, i), \mathbf{t}_k, \boldsymbol{\theta}_k) := \mathcal{P}_k(y, (j, \mathbf{j}_{k-1}), \mathbf{t}_k, \boldsymbol{\theta}_k) \Pi_k(y, (j, \mathbf{j}_{k-1}, i), \mathbf{t}_k, \boldsymbol{\theta}_k),$$

for $k \in \mathbb{N}$ and $(\mathbf{j}_{k-1}, \mathbf{t}_k, \boldsymbol{\theta}_k, \beta') \in \mathcal{A}_k$.

Obviously, by continuity of the functions underlying the model, all the sets $\mathcal{O}(\cdot)$ are open. Hence, from hypothesis (A) it follows that

$$\mathcal{V} := \{O(\mathbf{j}_{k-1}, \mathbf{t}_k, \boldsymbol{\theta}_k, \beta') :\ k \in \mathbb{N},\ (\mathbf{j}_{k-1}, \mathbf{t}_k, \boldsymbol{\theta}_k, \beta') \in \mathcal{A}_k\}.$$

is an open cover of X.

Since X is a Lindelöf space (as a separable metric space) there exists a countable subcover of $\mathcal{V}$. Consequently, we can choose sequences $\{k_r\}_{r\in\mathbb{N}} \subset \mathbb{N}$ and $\left\{\left(\mathbf{j}^{(r)}_{k_r-1}, \mathbf{t}^{(r)}_{k_r}, \boldsymbol{\theta}^{(r)}_{k_r}, \beta_r\right)\right\}_{r\in\mathbb{N}}$, wherein $\left(\mathbf{j}^{(r)}_{k_r-1}, \mathbf{t}^{(r)}_{k_r}, \boldsymbol{\theta}^{(r)}_{k_r}, \beta_r\right) \in \mathcal{A}_{k_r}$ for every $r \in \mathbb{N}$, so that

$$X = \bigcup_{r\in\mathbb{N}} O\left(\mathbf{j}^{(r)}_{k_r-1}, \mathbf{t}^{(r)}_{k_r}, \boldsymbol{\theta}^{(r)}_{k_r}, \beta_r\right).$$

Now, taking into account that $\mu(X) > 0$, we may find $p \in \mathbb{N}$ such that

$$\mu\left(O\left(\mathbf{j}^{(p)}_{k_p-1}, \mathbf{t}^{(p)}_{k_p}, \boldsymbol{\theta}^{(p)}_{k_p}, \beta_p\right)\right) > 0.$$

Define

$$m := k_p,\quad (\bar{\mathbf{j}}_{m-1}, \bar{\mathbf{t}}_m, \bar{\boldsymbol{\theta}}_m, \bar{\beta}) := \left(\mathbf{j}^{(p)}_{k_p-1}, \mathbf{t}^{(p)}_{k_p}, \boldsymbol{\theta}^{(p)}_{k_p}, \beta_p\right),\quad \widetilde{X} := O\left(\bar{\mathbf{j}}_{m-1}, \bar{\mathbf{t}}_m, \bar{\boldsymbol{\theta}}_m, \bar{\beta}\right).$$

Clearly, we then have

$$\mathcal{W}_m(y, (j, \bar{\mathbf{j}}_{m-1}), \bar{\mathbf{t}}_m, \bar{\boldsymbol{\theta}}_m) \in V_{\hat{y}} \quad\text{and}\quad \mathcal{Q}_m(y, (j, \bar{\mathbf{j}}_{m-1}, i), \bar{\mathbf{t}}_m, \bar{\boldsymbol{\theta}}_m) > \bar{\beta} \quad\text{for every}\quad (y, j) \in \widetilde{X}.$$

Since $\operatorname{cl} V_{\hat{y}} \cap U^c_{\hat{y}} = \emptyset$ and $\operatorname{cl} V_{\hat{y}}$ is compact, the distance between $V_{\hat{y}}$ and $U^c_{\hat{y}}$ is positive. This, together with continuity of $\mathcal{W}_m$ and $\mathcal{Q}_m$ with respect to y, $\mathbf{t}_m$ and (if Θ is an interval) $\boldsymbol{\theta}_m$, enables one to choose $\varepsilon > 0$ so small that

$$\mathcal{W}_m(y, (j, \bar{\mathbf{j}}_{m-1}), \mathbf{t}_m, \boldsymbol{\theta}_m) \in U_{\hat{y}} \quad\text{and}\quad \mathcal{Q}_m(y, (j, \bar{\mathbf{j}}_{m-1}, i), \mathbf{t}_m, \boldsymbol{\theta}_m) > \bar{\beta}/2,$$

whenever $(y, j) \in \widetilde{X}$, $\mathbf{t}_m \in B_{\mathbb{R}_+}(\bar{\mathbf{t}}_m, \varepsilon)$ and $\boldsymbol{\theta}_m \in B_\Theta(\bar{\boldsymbol{\theta}}_m, \varepsilon)$. The proof is now complete. □

Remark 3.2. It is worth noting here that, in the proof of [4, Lemma 3.16], the authors choose a finite cover of a compact space M (that plays the role of Y) consisting of the sets $\mathcal{O}((j, \mathbf{j}_{n-1}, i), \mathbf{t}_n, \beta)$ (defined similarly to our sets $O(\cdot)$) with common $n \in \mathbb{N}$ and $\beta > 0$. This enables them to derive a condition resembling (3.19), but valid for all initial states (y, j). Such a result, in turn, lead them to [4, Propositions 3.13 and 3.14], which guarantee that the analogue of our neighbourhood $U_{\hat{y}} \times \{i\}$ is uniformly accessible from the whole state space. The aforementioned argument obviously fails within our framework, due to the lack of compactness of Y. What is more, even while assuming that Y is compact, the union of the sets $O(\cdot)$ need not be increasing with respect to the length of multi-indices (through the presence of jumps $y \mapsto w_\theta(y)$), which is the case in [4]. That is the reason why condition (3.2) states the accessibility only from a subset of X with positive measure μ_* (in contrast to that obtained in the above-mentioned propositions in [4]). Thanks to such a configuration, (3.2) can be derived from (A) by using the assertion of Lemma 3.4, which is weaker than that of [4, Lemma 3.16].

We are now in a position to establish the main result of this paper, which provides conditions sufficient for the absolute continuity of invariant measures for both the operator P and the semigroup $\{P_t\}_{t\geq 0}$.

Theorem 3.2. *Suppose that the transformations w_θ, $\theta \in \Theta$, and $S_k(t, \cdot)$, $k \in I$, $t \geq 0$, are non-singular with respect to ℓ_d. Further, assume that there exists a point $(\hat{y}, i) \in \operatorname{int} Y \times I$ with property (A), for which (3.4) and (3.5) hold with some integer $n \geq d$ and some $(\hat{\mathbf{j}}_{n-1}, \hat{\mathbf{t}}_n, \hat{\boldsymbol{\theta}}_n) \in I^{n-1} \times (0, \infty)^n \times (\operatorname{int} \Theta)^n$ (excluding $\hat{\mathbf{j}}_0$ in the case of $n = 1$). Then every ergodic invariant measure $\mu_* \in \mathcal{M}_{prob}(X)$ of the Markov operator P, induced by (2.2), as well as the corresponding invariant measure $G\mu_*$ of the semigroup $\{P_t\}_{t\geq 0}$, generated by (2.4), is absolutely continuous with respect to $\bar{\ell}_d$.*

Proof. Let $\mu_* \in \mathcal{M}_{prob}(X)$ be an ergodic invariant probability measure of P. By virtue of Lemma 3.3 we can choose an open neighbourhood $U_{\hat{y}} \subset Y$ of $\hat{y}$, an open set $U_{\hat{w}} \subset Y$ and a constant $\bar{c} > 0$ so that (3.1) holds with $U = U_{\hat{y}}$, $V = U_{\hat{w}}$ and the given i, i.e.

$$P^n(x, B \times \{j\}) \geq \bar{c}\ell_d(B \cap U_{\hat{w}}) \quad \text{for all} \quad x \in U_{\hat{y}} \times \{i\},\ j \in I \ \text{ and } \ B \in \mathcal{B}(Y).$$

On the other hand, in view of Lemma 3.4, we may find $\varepsilon > 0$, $\beta > 0$, $m \in \mathbb{N}$, sequences $\bar{\mathbf{j}}_{m-1} \in I^{m-1}$ (if $m > 1$), $\bar{\mathbf{t}}_m \in \mathbb{R}_+^m$, $\bar{\boldsymbol{\theta}}_m \in \Theta^m$ and an open set $\widetilde{X} \subset X$ with $\mu_*(\widetilde{X}) > 0$ such that conditions (3.19) hold for any $z = (y, j) \in \widetilde{X}$, $\mathbf{t}_m \in B_{\mathbb{R}_+}(\bar{\mathbf{t}}_m, \varepsilon)$ and $\boldsymbol{\theta}_m \in B_\Theta(\bar{\boldsymbol{\theta}}_m, \varepsilon)$. Hence, appealing to (3.3), we see that

$$P^m(z, U_{\hat{y}} \times \{i\}) \geq \beta\, \vartheta^{\otimes m}(B_\Theta(\bar{\boldsymbol{\theta}}_m, \varepsilon)) \int_{B(\bar{\mathbf{t}}_m, \varepsilon)} \lambda^m e^{-\lambda(t_1 + \cdots + t_m)}\, d\mathbf{t}_m := \delta > 0 \quad \text{for all} \quad z \in \widetilde{X},$$

which exactly means that condition (3.2) holds for $U = U_{\hat{y}}$ and the given i. The desired absolute continuity of μ_* and $G\mu_*$ now follows from Proposition 3.1. $\square$

Finally, as a straightforward consequence of Theorems 3.2 and 3.1(iii), we obtain the following conclusion:

Corollary 3.2. *Suppose that there exists a unique invariant probability measure for the operator P or, equivalently, for the semigroup $\{P_t\}_{t\geq 0}$. Then, under the hypotheses of Theorem 3.2, both of the invariant measures, that for P, and that for $\{P_t\}_{t\geq 0}$, are absolutely continuous with respect to $\bar{\ell}_d$.*

4. The existence and uniqueness of invariant measures

It is clear that to ensure the existence and uniqueness of an invariant probability measure for the Markov operator P (and therefore for the Markov semigroup $\{P_t\}_{t\geq 0}$), some additional restrictions should be imposed on the functions composing the model under consideration.

In what follows, we quote [11, Theorem 4.1] (cf. also [15, Theorem 4.1]), which, apart from the existence of a unique P-invariant measure, also assures the geometric ergodicity of P in the Fortet–Mourier distance on $\mathcal{M}_{prob}(X)$ (see e.g. [20] or [18] for the equivalent Dudley metric).

Assuming that $X = Y \times I$ is equipped with the metric of the form

$$\rho_c((u, i), (v, j)) = \|u - v\| + c\, \mathbf{d}(i, j) \quad \text{for} \quad (u, i), (v, j) \in X, \tag{4.1}$$

where c is a given positive constant, the Forter–Mourier distance can be defined by

$$d_{FM}(\mu, \nu) := \sup\left\{ \left| \int_X f\, d(\mu - \nu) \right| :\ f \in \mathcal{F}_{FM}(X) \right\} \quad \text{for} \quad \mu, \nu \in \mathcal{M}_{prob}(X), \tag{4.2}$$

where

$$\mathcal{F}_{FM}(X) := \left\{ f : X \to [0, 1] :\ \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho_c(x, y)} \leq 1 \right\}.$$

It is well-known (see e.g. [7, Theorem 8.3.2]) that the topology induced on $\mathcal{M}_{prob}(X)$ by d_{FM} is equal to the topology of weak convergence of probability measures (whenever X is a Polish space, which is the case here).

Before we formulate the above-mentioned stability result, let us emphasize that it holds with a sufficiently large constant c, whose magnitude depends on the quantities occurring in the hypotheses to be imposed on the component functions of the model (see [11, Section 6]). Let us also note that, although we have assumed that the metric on Y is induced by a norm, just to stay with the framework introduced in Section 3 (wherein Y is a closed subset of $\mathbb{R}^d$), the result remains valid for any Polish metric space (cf. [15]).

Theorem 4.1 ([11, Theorem 4.1]). *Suppose that there exist $\alpha \in \mathbb{R}$, $L > 0$ and $L_w > 0$ satisfying*

$$LL_w + \frac{\alpha}{\lambda} < 1, \tag{4.3}$$

as well as constants L_p, L_π, c_π, $c_p > 0$, a point $y^* \in Y$ and two Borel measurable functions: $\mathcal{L} : Y \to \mathbb{R}_+$, which is bounded on bounded sets, and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\int_{\mathbb{R}_+} \varphi(t) e^{-\lambda t}\, dt < \infty,$$

such that, for any $u, v \in Y$, the following conditions hold:

$$\|S_i(t,u) - S_i(t,v)\| \le L e^{\alpha t} \|u - v\| \quad \textit{for all} \quad i \in I,\ t \ge 0; \tag{4.4}$$

$$\|S_i(t,u) - S_j(t,u)\| \le \varphi(t)\mathcal{L}(u) \quad \textit{for all} \quad i,j \in I,\ t \ge 0; \tag{4.5}$$

$$\sup_{y \in Y} \int_\Theta \int_0^\infty e^{-\lambda t} \|w_\theta(S_i(t,y^*)) - y^*\|\, p_\theta(S_i(t,y))\, dt\, \vartheta(d\theta) < \infty \quad \textit{for every} \quad i \in I; \tag{4.6}$$

$$\int_\Theta \|w_\theta(u) - w_\theta(v)\|\, p_\theta(u)\, \vartheta(d\theta) \le L_w \|u - v\|\,; \tag{4.7}$$

$$\int_\Theta |p_\theta(u) - p_\theta(v)|\, \vartheta(d\theta) \le L_p \|u - v\|\,; \tag{4.8}$$

$$\sum_{k \in I} \min\{\pi_{ik}(u), \pi_{jk}(u)\} \ge c_\pi \quad \textit{for} \ \ i,j \in I, \quad \textit{and} \quad \int_{\Theta(u,v)} \min\{p_\theta(u), p_\theta(v)\}\, \vartheta(d\theta) \ge c_p, \tag{4.9}$$

where

$$\Theta(u,v) := \{\theta \in \Theta : \|w_\theta(u) - w_\theta(v)\| \le L_w \|u - v\|\}.$$

Then the Markov operator P generated by (2.2) admits a unique invariant distribution μ_* such that $\mu_* \in \mathcal{M}^1_{prob}(X)$. Moreover, there exists $\beta \in (0,1)$ such that, for each $\mu \in \mathcal{M}^1_{prob}(X)$ and some constant $C(\mu) \in \mathbb{R}$, we have

$$d_{FM}(P^n \mu, \mu_*) \le C(\mu)\beta^n \quad \textit{for every} \quad n \in \mathbb{N}. \tag{4.10}$$

In particular, P is then also asymptotically stable (cf. Remark 1.1).

Obviously, due to Theorem 2.1, the hypotheses of Theorem 4.1 also guarantee the existence and uniqueness of an invariant probability measure for the semigroup $\{P_t\}_{t \ge 0}$, generated by (2.4).

Remark 4.1. In paper [11], the above-stated theorem is proved under the assumption that (4.4) holds with $\varphi(t) = t$. It is, however, easy to check that the same proof works without any significant changes if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an arbitrary function such that $t \mapsto \varphi(t)\exp(-\lambda t)$ is integrable over $\mathbb{R}_+$.

Remark 4.2. It is easy to verify (cf. [11, Corollary 3.4]) that, if Θ is compact, and there exist a Borel measurable function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ and $y^* \in Y$ such that

$$\int_{\mathbb{R}_+} \psi(t) e^{-\lambda t}\, dt < \infty \quad \text{and} \quad \|S_j(t,y^*) - y^*\| \le \psi(t) \quad \text{for all} \quad t \ge 0,\ j \in I,$$

then (4.6) holds under each of the following two conditions:

(i) The map $y \mapsto p_\theta(y)$ is constant for every $\theta \in \Theta$ and (4.7) is fulfilled.
(ii) There exists $L_w > 0$ such that all w_θ, $\theta \in \Theta$, are Lipschitz continuous with the same constant L_w.

5. Examples

In this section, we shall illustrate the applicability of Theorem 3.2 by analysing a simple example, inspired by [4, Example 5.2], wherein also the hypotheses of Theorem 4.1 are fulfilled. Furthermore, we will provide two examples showing the necessity of some of the conditions imposed in Theorem 3.2. However, before that, let us discuss some special cases wherein condition (A), introduced prior to Lemma 3.4, is fulfilled for some identifiable point of X.

Proposition 5.1. *Suppose that there exist $\bar{\theta} \in \Theta$, $z \in Y$ and $i \in I$ such that the following statements are fulfilled:*

(i) $w_{\bar{\theta}}$ is a contraction satisfying $w_{\bar{\theta}}(z) = z$;
(ii) $p_{\bar{\theta}}(y) > 0$ for all $y \in Y$;
(iii) for every $n \in \mathbb{N}$, there is $(j_1, \dots, j_n) \in I^n$ with $j_n = i$ such that

$$\pi_{j_{k-1}j_k}(y) > 0 \quad \text{for all} \quad k \in \{1, \dots, n\} \quad \text{and} \quad y \in w_{\bar{\theta}}(Y) \quad \text{with each} \quad j_0 \in I. \tag{5.1}$$

Then condition (A) holds with $\hat{y} = z$ and the given i.

Proof. Fix $(y, j) \in X$ and $\varepsilon > 0$. Letting $K < 1$ denote a Lipschitz constant of $w_{\bar{\theta}}$, we can choose $n \in \mathbb{N}$, $n > 1$, so that $K^n \|y - z\| < \varepsilon$. According to (iii), for this n, we may find $(j_1, \dots, j_n) \in I^n$ with $j_n = i$ such that (5.1) is satisfied. Taking $\mathbf{j}_{n-1} := (j_1, \dots, j_{n-1})$, $\mathbf{0} := (0, \dots, 0) \in \mathbb{R}^n_+$ and $\boldsymbol{\theta}_n := (\bar{\theta}, \dots, \bar{\theta}) \in \Theta^n$, we now see that

$$\|\mathcal{W}_n(y, (j, \mathbf{j}_{n-1}), \mathbf{0}, \boldsymbol{\theta}_n) - z\| = \left\| w^n_{\bar{\theta}}(y) - w^n_{\bar{\theta}}(z) \right\| \le K^n \|y - z\| < \varepsilon,$$

and $\mathcal{P}_n(y, (j, \mathbf{j}_{n-1}), \mathbf{0}, \boldsymbol{\theta}_n) \Pi_n(y, (j, \mathbf{j}_{n-1}, i), \mathbf{0}, \boldsymbol{\theta}_n) > 0$, due to (ii) and (5.1). $\square$

Proposition 5.2. *Suppose that condition (4.4) holds with $\alpha < 0$, and that (4.7) is satisfied. Further, assume that there exist $z \in Y$, $k \in I$, $\bar{\theta} \in \Theta$ and $i \in I$ such that the following statements are fulfilled:*

(i) $S_k(t, z) = z$ for all $t \ge 0$;
(ii) $w_{\bar{\theta}}$ is Lipschitz continuous;
(iii) $p_{\bar{\theta}}(y) > 0$ for all $y \in Y$;
(iv) $\pi_{jk}(y)\pi_{ki}(y) > 0$ for every $j \in I$ and each $y \in w_{\bar{\theta}}(Y)$.

Then condition (A) holds with $\hat{y} = w_{\bar{\theta}}(z)$ and the given i.

Proof. Let $(y, j) \in X$ and $\varepsilon > 0$. Further, choose $t > 0$ so that $L_{\bar{\theta}} L e^{\alpha t} \|w_{\bar{\theta}}(y) - z\| < \varepsilon$, where $L_{\bar{\theta}}$ stands for a Lipschitz constant of $w_{\bar{\theta}}$. Now, keeping in mind that $S_j(0, u) = u$ for all $u \in Y$, and applying (ii), (i), (4.4), sequentially, we infer that

$$\begin{aligned} \left\| \mathcal{W}_2(y, (j, k), (0, t), (\bar{\theta}, \bar{\theta})) - w_{\bar{\theta}}(z) \right\| &= \|w_{\bar{\theta}}(S_k(t, w_{\bar{\theta}}(y))) - w_{\bar{\theta}}(z)\| \le L_{\bar{\theta}} \|S_k(t, w_{\bar{\theta}}(y)) - z\| \\ &= L_{\bar{\theta}} \|S_k(t, w_{\bar{\theta}}(y)) - S_k(t, z)\| \\ &\le L_{\bar{\theta}} L e^{\alpha t} \|w_{\bar{\theta}}(y) - z\| < \varepsilon. \end{aligned}$$

Moreover, from (iii) and (iv) it follows that

$$\mathcal{P}_2(y, (j, k), (0, t), (\bar{\theta}, \bar{\theta})) = p_{\bar{\theta}}(y) p_{\bar{\theta}}(S_k(t, w_{\bar{\theta}}(y))) > 0,$$

$$\Pi_2(y, (j, k, i), (0, t), (\bar{\theta}, \bar{\theta})) = \pi_{jk}(w_{\bar{\theta}}(y)) \pi_{ki}(w_{\bar{\theta}}(S_k(t, w_{\bar{\theta}}(y)))) > 0. \quad \square$$

Remark 5.1. Note, that in the case where $\varTheta$ is finite (and ϑ is the counting measure), condition (ii) of Proposition 5.2 can be guaranteed by assuming condition (4.7) and a strengthened version of (iii), namely $p := \inf_{y\in Y} p_{\bar\theta}(y) > 0$. Under these settings, $w_{\bar\theta}$ is Lipschitz continuous with $L_{\bar\theta} = p^{-1}L_w$. To see this, it suffices to write

$$p\,\|w_{\bar\theta}(u) - w_{\bar\theta}(v)\| \le \sum_{\theta\in\varTheta} \|w_\theta(u) - w_\theta(v)\|\, p_\theta(u) \le L_w\,\|u-v\| \quad \text{for} \quad u,v\in Y.$$

Proposition 5.2, together with the observation recorded in Remark 5.1, prove to be useful in analysing the example given below.

Example 5.1. Let $\alpha < 0$ and $a \in \mathbb{R}\backslash\{0\}$. Consider an instance of the dynamical system introduced in Section 2, with $\varTheta$ satisfying the assumptions of Section 3.2, $Y := \mathbb{R}$, $I := \{1,2\}$, and two semiflows S_1, S_2 induced by the initial value problems (in $\mathbb{R}_+$) associated with the equations $u' = \alpha u$ and $u' = \alpha(u-a)$, respectively. Clearly, the semiflows are of the form

$$S_1(t,y) := e^{\alpha t}y \quad \text{and} \quad S_2(t,y) := e^{\alpha t}(y-a)+a \quad \text{for} \quad t\ge 0,\ y\in\mathbb{R}.$$

Furthermore, assume that conditions (4.6)–(4.8) hold for the transformations w_θ, $\theta\in\varTheta$, and the densities $\theta\mapsto p_\theta(y)$, $y\in\mathbb{R}$, with $L_w = 1$ and some $L_p > 0$, as well as that

$$\inf_{y\in\mathbb{R}} \pi_{ij}(y) > 0 \quad \text{and} \quad \inf_{y\in\mathbb{R}} p_\theta(y) > 0 \quad \text{for all} \quad i,j\in I,\ \theta\in\varTheta. \tag{5.2}$$

Obviously, the foregoing requirement is just a strengthened form of condition (4.9). It is also worth noting that (4.6) holds, for example, if $\varTheta$ is compact, and at least one of conditions (i) or (ii) from Remark 4.2 is satisfied.

Clearly, the semiflows S_1, S_2 satisfy conditions (4.4), (4.5) with $\alpha < 0$, $L = 1$, $\mathcal{L}\equiv 1$, $\varphi(t) = |a|(1-e^{\alpha t})$, and inequality (4.3) is then trivially fulfilled as well. Hence, due to Theorem 4.1, the Markov operator P, corresponding to the chain given by the post-jump locations, possesses a unique invariant probability measure μ_*. What is more, due to Theorem 2.1, $\nu_* := G\mu_*$ is the unique invariant probability measure of the transition semigroup $\{P_t\}_{t\ge 0}$, associated with the corresponding PDMP.

Suppose now that all the transformations $y\mapsto w_\theta(y)$, $\theta\in\varTheta$, and, if $\varTheta$ is an interval, also $\theta\mapsto w_\theta(y)$, $y\in\mathbb{R}$, are continuously differentiable and non-singular with respect to ℓ_1. Furthermore, assume that, for at least one $\bar\theta\in\varTheta$, $w_{\bar\theta}(a)w'_{\bar\theta}(w_{\bar\theta}(a)) \ne 0$, and that the transformation $w_{\bar\theta}$ is Lipschitz continuous. Plainly, in the case where $\varTheta$ is finite, assuming the latter is unnecessary, since the Lipschitz continuity is assured by (4.7) and (5.2) (due to Remark 5.1). Under the aforesaid conditions, both the invariant measures μ_* and ν_* are absolutely continuous with respect to $\bar\ell_1$. To see this, first observe that $S_2(t,a) = a$ for all $t\ge 0$. Then, due to Proposition 5.2, condition (A) holds for $(\hat y, i) := (w_{\bar\theta}(a), 1)$. Moreover, we have

$$\frac{\partial}{\partial t}\mathcal{W}_1(\hat y, i, t, \bar\theta) = \frac{d}{dt} w_{\bar\theta}(S_1(t, w_{\bar\theta}(a))) = \frac{d}{dt} w_{\bar\theta}\left(e^{\alpha t} w_{\bar\theta}(a)\right) = \alpha e^{\alpha t} w_{\bar\theta}(a)\, w'_{\bar\theta}(e^{\alpha t} w_{\bar\theta}(a)) \ne 0$$

for small enough $t > 0$, which ensures that (3.5) is satisfied with $n = 1$, $\hat y = w_{\bar\theta}(a)$, $i = 1$, $\hat\theta_1 = \bar\theta$ and some sufficiently small $\hat t_1 > 0$. Obviously, (3.4) is also fulfilled, due to (5.2). Consequently, in view of Corollary 3.2, the measures μ_* and ν_* are absolutely continuous with respect to $\bar\ell_1$.

It is worth noting that the assumptions of non-singularity of the transformations $S_k(t,\cdot)$, w_θ, and the existence of a point $(\hat y, i)$ for which (A) holds are not yet sufficient for the absolute continuity of the unique P-invariant measure, even though the hypotheses of Theorem 4.1 are fulfilled. In other words, conditions (3.5) and (3.4) in Theorem 3.2 cannot be omitted. The following simple example justifies this assertion:

Example 5.2. Let $Y := \mathbb{R}$, $I := \{1\}$, $\Theta := \{1\}$, and suppose that $S_1(t,y) := e^{-t}y$, $w_1(y) := y$ for $t \geq 0$ and $y \in \mathbb{R}$. In such a case, the state space $X = \mathbb{R} \times \{1\}$ of our dynamical system can be identified with $\mathbb{R}$, and the transition law of $\{\Phi_n\}_{n\in\mathbb{N}_0}$, given by (2.2), takes the form

$$P(y,A) = \int_0^\infty \lambda e^{-\lambda t} \mathbb{1}_A(ye^{-t})\, dt \quad \text{for} \quad y \in \mathbb{R},\ A \in \mathcal{B}(\mathbb{R}).$$

Obviously, conditions (4.3)–(4.9) hold in this setup ((4.6) follows directly from Remark 4.2), and thus, due to Theorem 4.1, there exists a unique invariant measure for P. Moreover, note that $S_1(t,\cdot)$, $t \geq 0$, and w_1 are non-singular with respect to ℓ_1, and that condition (A) is fulfilled for $(\hat{y}, i) := (0,1)$, since $S_1(t,0) = 0$ and $w_1(0) = 0$ (cf. Proposition 5.2). On the other hand, it is easily seen that the unique P-invariant measure is δ_0, which is singular with respect to ℓ_1.

The last example demonstrates that, under the assumptions of Theorem 3.2, there may exist a singular invariant probability measure for P. This means that the assertion of our result is not valid for non-ergodic invariant measures, and, simultaneously, shows that the conditions of Theorem 3.2 do not guarantee the uniqueness of invariant distributions.

Example 5.3. Let $Y := \mathbb{R}_+$, $I := \{1,2\}$, $X := Y \times I$ and $\Theta := \{1\}$. Consider the semiflows S_1, S_2 generated by the initial value problems (in $\mathbb{R}_+$) associated with $u' = u$ and $u' = u(1-u)$, respectively, that is

$$S_1(t,y) := e^t y \quad \text{and} \quad S_2(t,y) := \frac{e^t y}{1 + (e^t - 1)y} \quad \text{for} \quad t, y \in \mathbb{R}_+.$$

Further, put $w_1(y) := y$, and take $\pi_{ij}(y) := 1/2$ for all $y \in \mathbb{R}_+$, $i,j \in \{1,2\}$. Under this setting, the transition law P of $\{\Phi_n\}_{n\in\mathbb{N}_0}$ is given by

$$P((y,k),A) = \frac{1}{2}\int_0^\infty \lambda e^{-\lambda t} \left(\mathbb{1}_A(S_k(t,y),1) + \mathbb{1}_A(S_k(t,y),2)\right) dt \quad \text{for} \quad (y,k) \in X,\ A \in \mathcal{B}(X).$$

Obviously, all the transformations $S_1(t,\cdot)$, $S_2(t,\cdot)$, $t \geq 0$, and w_1 are non-singular with respect to ℓ_1. Moreover, since $S_2(t,1) = 1$ for every $t \geq 0$, and $\pi_{j2}(y)\pi_{21}(y) > 0$ for all $y \in \mathbb{R}_+$, $j \in \{1,2\}$, it follows from Proposition 5.2 that condition (A) is fulfilled with $(\hat{y}, i) := (1,1)$, and we also get

$$\frac{\partial}{\partial t}\mathcal{W}_1(\hat{y}, i, t, 1) = \frac{d}{dt} S_1(t,1) = e^t \neq 0 \quad \text{for every} \quad t > 0,$$

which shows that (3.5) holds (with $n = 1$) as well. Condition (3.4) is trivially satisfied, since $\pi_{ij}(y) > 0$ for all $y \in \mathbb{R}_+$ and $i,j \in \{1,2\}$. Hence, all the hypotheses of Theorem 3.2 are fulfilled.

On the other hand, we see that $S_1(t,0) = S_2(t,0) = 0$ for every $t \geq 0$, which implies that

$$\frac{1}{2}\left(\delta_{(0,1)} + \delta_{(0,2)}\right)$$

is an invariant probability measure for the operator P. According to Theorem 3.2 such a measure cannot be ergodic (since it is singular with respect to $\bar{\ell}_d$), and, in turn, it cannot be the unique P-invariant measure too.

Funding

The work of Hanna Wojewódka-Ściążko has been partly supported by the National Science Centre of Poland, Grant Number 2018/02/X/ST1/01518.

Appendix

As announced in Section 1, for the self-containedness of the paper, we give here a proof of Lemma 1.1. However, prior to that, note a simple consequence of [1, Theorem 19.25], which says that P-ergodic measures are the extreme points of the set of P-invariant probability measures.

If $\mu_* \in \mathcal{M}_{prob}(E)$ is an ergodic invariant measure of a Markov operator $P : \mathcal{M}_{fin}(E) \to \mathcal{M}_{fin}(E)$, then it cannot be a sum of two distinct non-zero P-invariant measures. To see this, suppose that $\mu_* = \mu_1 + \mu_2$ for certain non-trivial invariant measures $\mu_1, \mu_2 \in \mathcal{M}_{fin}(E)$, and let $\alpha_i := \mu_i(E)$ for $i = 1, 2$. Then $\alpha_1 + \alpha_2 = 1$, and $\widetilde{\mu}_i := \mu_i/\alpha_i$, $i = 1, 2$, are invariant probability measures for P. Since $\mu_* = \alpha_1\widetilde{\mu}_1 + \alpha_2\widetilde{\mu}_2$, and μ_* is an extreme point of the set of P-invariant probability measures, it follows that $\mu_1 = \mu_2$.

Proof of Lemma 1.1. Let $\mu_* \in \mathcal{M}_{prob}(E)$ be an ergodic P-invariant measure. By virtue of the Lebesgue decomposition theorem we can write

$$\mu_* = \mu_{ac} + \mu_{sin},$$

where $\mu_{ac} \in \mathcal{M}_{ac}(E, m)$ and $\mu_{sin} \in \mathcal{M}_{sin}(E, m)$ are uniquely determined by μ_*. Consequently, it now follows that

$$P\mu_* = P\mu_{ac} + P\mu_{sin}.$$

From the principal assumption of the lemma we know that $P\mu_{ac} \in \mathcal{M}_{ac}(E, m)$. Further, using the invariance of μ_*, we also get $\mu_* = P\mu_{ac} + P\mu_{sin}$. If we now take the absolutely continuous part of each side of this equality, then we get

$$\mu_{ac} = P\mu_{ac} + (P\mu_{sin})_{ac},$$

which, in particular, implies that

$$\mu_{ac}(E) = \mu_{ac}(E) + (P\mu_{sin})_{ac}(E).$$

Hence $(P\mu_{sin})_{ac} \equiv 0$, and thus $P\mu_{sin} \in \mathcal{M}_{sin}(E, m)$. From the identity

$$\mu_{ac} + \mu_{sin} = \mu_* = P\mu_{ac} + P\mu_{sin}$$

and the uniqueness of the Lebesgue decomposition it now follows that both measures μ_{ac} and μ_{sin} are invariant for the operator P. Finally, taking into account the aforementioned consequence of [1, Theorem 19.25] and the fact that $\mu_{ac} \neq \mu_{sin}$, we can apply the above remark to conclude that at least one of the measures μ_{ac}, μ_{sin} must be trivial, which gives the desired conclusion. □

In the remainder of this section, we provide the proofs of Lemmas 3.1 and 3.2, referring to the Markov operators P, G and W, induced by the kernels (2.2), (2.5) and (2.6), respectively.

Proof of Lemma 3.1. For any $\theta \in \Theta$, $j \in I$ and $t \geq 0$, let us define $T_{\theta,j,t} : X \to X$ by

$$T_{\theta,j,t}(y, i) := (w_\theta(S_i(t, y)), j) \quad \text{for} \quad (y, i) \in X.$$

Obviously, each of the transformations $T_{\theta,j,t}$ is then Borel measurable and non-singular with respect to $\bar{\ell}_d$. Consequently, for any $\theta \in \Theta$, $j \in I$ and $t \geq 0$, we can consider the Frobenius–Perron operator associated with $T_{\theta,j,t}$, say $\mathcal{P}_{\theta,j,t}$, which satisfies

$$\int_X \mathbb{1}_A(T_{\theta,j,t}(y, i)) f(y, i)\, \bar{\ell}_d(dy, di) = \int_A \mathcal{P}_{\theta,j,t} f(y, i)\, \bar{\ell}_d(dy, di) \;\text{ for }\; A \in \mathcal{B}(X),\; f \in \mathcal{L}^1(X, \bar{\ell}_d). \tag{A.1}$$

Let $\mu \in \mathcal{M}_{ac}(X, \bar{\ell}_d)$, and, for each $(\theta, j, t) \in \Theta \times I \times \mathbb{R}_+$, define

$$f^{\mu}_{\theta,j,t}(y,i) := \pi_{ij}(w_\theta(S_i(t,y)))p_\theta(S_i(t,y))\frac{d\mu}{d\bar{\ell}_d}(y,i), \quad (y,i) \in X.$$

Note that $f^{\mu}_{\theta,j,t} \in \mathcal{L}^1(X, \bar{\ell}_d)$ for all $j \in I$ and $\vartheta \otimes \ell_1$ - a.e. $(\theta, t) \in \mathbb{R}_+ \times \Theta$. To justify this, observe that, for every $i \in I$, the map $(\theta, t, y) \mapsto e^{-\lambda t} p_\theta(S_i(t,y))$ is measurable with respect to $\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(Y)$, since the functions $(\theta, y) \mapsto p_\theta(y)$ and $(t, y) \mapsto S_i(t, y)$ are continuous. Moreover, since $\theta \mapsto p_\theta(y)$ is a density for each $y \in Y$, we have

$$\int_Y \int_0^\infty \int_\Theta e^{-\lambda t} p_\theta(S_i(t,y))\, \vartheta(d\theta)\, dt\, \mu(dy \times \{i\}) = \lambda^{-1}\mu(Y \times \{i\}) < \infty,$$

which (by [7, Theorem 3.4.5]) guarantees that the function $(\theta, t, y) \mapsto e^{-\lambda t} p_\theta(S_i(t,y))$ is $\vartheta \otimes \ell_1 \otimes \mu(\cdot \times \{i\})$-integrable for every $i \in I$. From Fubini's theorem ([7, Theorem 3.4.4]) it now follows that, for each $i \in I$, the map $y \mapsto e^{-\lambda t} p_\theta(S_i(t,y))$ is $\mu(\cdot \times \{i\})$-integrable for $\vartheta \otimes \ell_1$-a.e. $(\theta, t) \in \Theta \times \mathbb{R}_+$, and so is $y \mapsto p_\theta(S_i(t,y))$. This obviously implies that such a function is also μ-integrable for $\vartheta \otimes \ell_1$-a.e. $(\theta, t) \in \Theta \times \mathbb{R}_+$, which finally gives

$$\int_X f^{\mu}_{\theta,j,t}(y,i)\, \bar{\ell}_d(dy, di) \le \int_X p_\theta(S_i(t,y))\mu(dy, di) < \infty \;\text{ for }\; \vartheta \otimes \ell_1 - \text{a.e. } (\theta, t) \in \Theta \times \mathbb{R}_+,\; j \in I.$$

In view of the above, there exists a set $N \in \mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbb{R}_+)$, with $(\vartheta \otimes \ell_1)(N) = 0$, such that $f^{\mu}_{\theta,j,t} \in \mathcal{L}^1(X, \bar{\ell}_d)$ for all $(\theta, t) \in (\Theta \times \mathbb{R}_+) \backslash N$ and $j \in I$. Hence we can now use (A.1) to conclude that, for every $A \in \mathcal{B}(X)$,

$$\begin{aligned}
P\mu(A) &= \sum_{j\in I} \int_0^\infty \int_\Theta \lambda e^{-\lambda t} \int_X \mathbb{1}_A(w_\theta(S_i(t,y)), j)\pi_{ij}(w_\theta(S_i(t,y)))\, p_\theta(S_i(t,y))\mu(dy, di)\vartheta(d\theta)\, dt \\
&= \sum_{j\in I} \int_{(\Theta\times\mathbb{R}_+)\backslash N} \lambda e^{-\lambda t} \left(\int_X \mathbb{1}_A(T_{\theta,j,t}(y,i)) f^{\mu}_{\theta,j,t}(y,i)\, \bar{\ell}_d(dy, di) \right) (\vartheta \otimes \ell_1)(d\theta \times dt) \\
&= \sum_{j\in I} \int_{(\Theta\times\mathbb{R}_+)\backslash N} \lambda e^{-\lambda t} \left(\int_A \mathcal{P}_{\theta,j,t}\left(f^{\mu}_{\theta,j,t}\right)(y,i)\, \bar{\ell}_d(dy, di) \right) (\vartheta \otimes \ell_1)(d\theta \times dt) \\
&= \int_A \left(\sum_{j\in I} \int_{(\Theta\times\mathbb{R}_+)\backslash N} \lambda e^{-\lambda t} \mathcal{P}_{\theta,j,t}\left(f^{\mu}_{\theta,j,t}\right)(y,i)\, (\vartheta \otimes \ell_1)(d\theta \times dt) \right) \bar{\ell}_d(dy, di).
\end{aligned}$$

We have therefore shown that the map

$$X \ni (y,i) \mapsto \sum_{j\in I} \int_{(\Theta\times\mathbb{R}_+)\backslash N} \lambda e^{-\lambda t} \mathcal{P}_{\theta,j,t}\left(f^{\mu}_{\theta,j,t}\right)(y,i)\, (\vartheta \otimes \ell_1)(d\theta \times dt)$$

is a Radon–Nikodym derivative of $P\mu$ with respect to $\bar{\ell}_d$, whence $P\mu \in \mathcal{M}_{ac}(X, \bar{\ell}_d)$ and the proof is complete. □

Proof of Lemma 3.2. To prove the first inclusion, for every $t \ge 0$, we define $H_t : X \to X$ by setting

$$H_t(y,i) := (S_i(t,y), i) \quad \text{for} \quad (y,i) \in X.$$

Such a transformation is then Borel measurable and non-singular with respect to $\bar{\ell}_d$. Hence, we can consider the Frobenius–Perron operator associated with H_t, say $\mathcal{P}_t$, which satisfies

$$\int_X \mathbb{1}_A(H_t(y,i)) f(y,i)\, \bar{\ell}_d(dy, di) = \int_A \mathcal{P}_t f(y,i)\, \bar{\ell}_d(dy, di) \;\text{ for all }\; A \in \mathcal{B}(X),\; f \in \mathcal{L}^1(X, \bar{\ell}_d).$$

Letting $\mu \in \mathcal{M}_{ac}(X,\bar{\ell}_d)$ and putting $h^\mu := d\mu/d\bar{\ell}_d \in \mathcal{L}^1(X,\bar{\ell}_d)$, we then see that, for any $A \in \mathcal{B}(X)$,

$$\begin{aligned} G\mu(A) &= \int_0^\infty \int_X \lambda e^{-\lambda t} \mathbb{1}_A(S_i(t,y),i)\,\mu(dy,di)\,dt \\ &= \int_0^\infty \lambda e^{-\lambda t} \left(\int_X \mathbb{1}_A(H_t(y,i))\,h^\mu(y,i)\,\bar{\ell}_d(dy,di) \right) dt \\ &= \int_0^\infty \lambda e^{-\lambda t} \left(\int_A \mathcal{P}_t h^\mu(y,i)\,\bar{\ell}_d(dy,di) \right) dt = \int_A \left(\int_0^\infty \lambda e^{-\lambda t} \mathcal{P}_t h^\mu(y,i)\,dt \right) \bar{\ell}_d(dy,di). \end{aligned}$$

This shows that

$$(y,i) \mapsto \int_0^\infty \lambda e^{-\lambda t} \mathcal{P}_t h^\mu(y,i)\,dt$$

is a Radon–Nikodym derivative of $G\mu$ with respect to $\bar{\ell}_d$, which means that $G\mu \in \mathcal{M}_{ac}(X,\bar{\ell}_d)$ and, therefore, shows the first inclusion in the assertion of the lemma.

The proof of the second inclusion goes similarly. In this case, for every $\theta \in \Theta$ and each $j \in I$, we consider $R_{\theta,j} : X \to X$ given by

$$R_{\theta,j}(y,i) = (w_\theta(y), j) \quad \text{for} \quad (y,i) \in X.$$

Obviously, all the transformations $R_{\theta,j}$ are Borel measurable and non-singular with respect to $\bar{\ell}_d$. This observation, as before, enables us to introduce the Frobenius–Perron operator associated with $R_{\theta,j}$, say $\mathcal{P}_{\theta,j}$, which satisfies

$$\int_X \mathbb{1}_A(R_{\theta,j}(y,i)) f(y,i)\,\bar{\ell}_d(dy,di) = \int_A \mathcal{P}_{\theta,j} f(y,i)\,\bar{\ell}_d(dy,di) \;\text{ for }\; A \in \mathcal{B}(X),\; f \in \mathcal{L}^1(X,\bar{\ell}_d). \tag{A.2}$$

Let $\mu \in \mathcal{M}_{ac}(X,\bar{\ell}_d)$ and define

$$r^\mu_{\theta,j}(y,i) := \pi_{ij}(w_\theta(y)) p_\theta(y) \frac{d\mu}{d\bar{\ell}_d}(y,i) \quad \text{for} \quad (y,i) \in X.$$

Proceeding analogously as in the proof of Lemma 3.1, one can show that there exists $N \in \mathcal{B}(\theta)$ satisfying $\vartheta(N) = 0$ such that $r^\mu_{\theta,j} \in \mathcal{L}^1(X,\bar{\ell}_d)$ for all $\theta \in \Theta \backslash N$ and $j \in I$. Hence, taking into account (A.2), we infer that, for every $A \in \mathcal{B}(X)$,

$$\begin{aligned} W\mu(A) &= \sum_{j\in I} \int_\Theta \int_X \mathbb{1}_A(w_\theta(y),j) \pi_{ij}(w_\theta(y)) p_\theta(y)\,\mu(dy,di)\,\vartheta(d\theta) \\ &= \sum_{j\in I} \int_\Theta \left(\int_X \mathbb{1}_A(w_\theta(y),j) \pi_{ij}(w_\theta(y)) p_\theta(y)\,\mu(dy,di) \right) \vartheta(d\theta) \\ &= \sum_{j\in I} \int_{\Theta\backslash N} \left(\int_X \mathbb{1}_A(R_{\theta,j}(y,i)) r^\mu_{\theta,j}(y,i)\,\bar{\ell}_d(dy,di) \right) \vartheta(d\theta) \\ &= \sum_{j\in I} \int_{\Theta\backslash N} \left(\int_A \mathcal{P}_{\theta,j}(r^\mu_{\theta,j})(y,i)\,\bar{\ell}_d(dy,di) \right) \vartheta(d\theta) \\ &= \int_A \left(\sum_{j\in I} \int_{\Theta\backslash N} \mathcal{P}_{\theta,j}(r^\mu_{\theta,j})(y,i)\,\vartheta(d\theta) \right) \bar{\ell}_d(dy,di). \end{aligned}$$

Consequently, we now see that the map

$$X \ni (y,i) \mapsto \sum_{j\in I} \int_{\Theta\backslash N} \mathcal{P}_{\theta,j}(r^\mu_{\theta,j})(y,i)\,\vartheta(d\theta)$$

is a Radon–Nikodym derivative of $W\mu$ with respect to $\bar{\ell}_d$, which, in turn, yields that $W\mu \in \mathcal{M}_{ac}(X,\bar{\ell}_d)$ and completes the proof of the lemma. $\square$

References

[1] C.D. Aliprantis, K.C. Border, Infinite Dimensional Analysis. a Hitchhiker's Guide, third ed., Springer-Verlag, Berlin, 2006.

[2] Y. Bakhtin, T. Hurth, Invariant densities for dynamical systems with random switching, Nonlinearity 25 (10) (2012) 2937–2952, http://dx.doi.org/10.1088/0951-7715/25/10/2937.

[3] M. Benaïm, S. Le Borgne, F. Malrieu, P.-A. Zitt, Quantitative ergodicity for some switched dynamical systems, Electron. Commun. Probab. 17 (2012) 1–14, http://dx.doi.org/10.1214/ecp.v17-1932.

[4] M. Benaïm, S. Le Borgne, F. Malrieu, P.-A. Zitt, Qualitative properties of certain piecewise deterministic Markov processes, Ann. Inst. Henri Poincaré Probab. 51 (3) (2015) 1040–1075, http://dx.doi.org/10.1214/14-aihp619.

[5] M. Benaïm, C. Lobry, Lotka–Volterra with randomly fluctuating environments or "how switching between beneficial environments can make survival harder", Ann. Appl. Probab. 26 (6) (2016) 3754–3785, http://dx.doi.org/10.1214/16-aap1192.

[6] P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1999.

[7] V. Bogachev, Measure Theory, Vol. I, II, Springer-Verlag, Berlin, 2007.

[8] O. Boxma, H. Kaspi, O. Kella, D. Perry, On/off storage systems with state-dependent inpout, outpout and switching rates, Probab. Engrg. Inform. Sci. 19 (1) (2005) 1–14, http://dx.doi.org/10.1017/s0269964805050011.

[9] B. Cloez, R. Dessalles, A. Genadot, F. Malrieu, A. Marguet, R. Yvinec, Probabilistic and piecewise deterministic models in biology, ESAIM: Proc. Surv. 60 (2017) 225–245, http://dx.doi.org/10.1051/proc/201760225.

[10] O. Costa, F. Dufour, Stability and ergodicity of piecewise deterministic Markov processes, SIAM J. Control Optim. 47 (2) (2008) 1053–1077.

[11] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, Ergodic properties of some piecewise-deterministic Markov process with application to gene expression modelling, Stochastic Process. Appl. 130 (5) (2020) 2851–2885, http://dx.doi.org/10.1016/j.spa.2019.08.006.

[12] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, A useful version of the central limit theorem for a general class of Markov chains, J. Math. Anal. Appl. 484 (1) (2020) 123725, http://dx.doi.org/10.1016/j.jmaa.2019.123725.

[13] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, The law of the iterated logarithm for a piecewise deterministic Markov process assured by the properties of the Markov chain given by its post-jump locations, Stoch. Anal. Appl. 39 (2) (2021) 357–379, http://dx.doi.org/10.1080/07362994.2020.1798252.

[14] D. Czapla, K. Horbacz, H. Wojewódka-Ściążko, The strassen invariance principle for certain non-stationary Markov-Feller chains, Asymptot. Anal. 121 (1) (2021) 1–34, http://dx.doi.org/10.3233/ASY-191592.

[15] D. Czapla, J. Kubieniec, Exponential ergodicity of some Markov dynamical systems with application to a Poisson-driven stochastic differential equation, Dyn. Syst. - Int. J. 34 (1) (2019) 130–156, http://dx.doi.org/10.1080/14689367.2018.1485879.

[16] M. Davis, Piecewise-deterministic Markov processes: a general class of nondiffusion stochastic models, J. R. Statist. Soc. Ser. B 46 (3) (1984) 353–388.

[17] D. Douc, E. Moulinesand, D. Stoffer, Nonlinear Time Series: Theory, Methods and Applications with R Examples, Chapman and Hall/CRC, New York, 2014.

[18] R. Dudley, Convergence of baire measures, Studia Math. 27 (1966) 251–268, http://dx.doi.org/10.4064/sm-27-3-251-268.

[19] C. Graham, P. Robert, Interacting multi-class transmissions in large stochastic networks, Ann. Appl. Probab. 19 (6) (2009) 2334–2361, http://dx.doi.org/10.1214/09-aap614.

[20] A. Lasota, From fractals to stochastic differential equations, in: Chaos – The Interplay Between Stochastic and Deterministic Behaviour, in: Lecture Notes in Phys., vol. 457, (Springer Verlag), 1995, pp. 235–255.

[21] A. Lasota, J. Myjak, Generic properties of fractal measures, Bull. Pol. Acad. Sci. Math. 42 (4) (1994) 283–296.

[22] A. Lasota, J. Traple, Invariant measures related with Poisson driven stochastic differential equation, Stochastic Process. Appl. 106 (1) (2003) 81–93, http://dx.doi.org/10.1016/s0304-4149(03)00017-6.

[23] A. Lasota, J. Yorke, Lower bound technique for Markov operators and iterated function systems, Random Comput. Dyn. 2 (1) (1994) 41–77.

[24] E. Löcherbach, Absolute continuity of the invariant measure in piecewise deterministic Markov processes having degenerate jumps, Stochastic Process. Appl. 128 (6) (2018) 1797–1829, http://dx.doi.org/10.1016/j.spa.2017.08.011.

[25] M. Mackey, M. Tyran-Kamińska, R. Yvinec, Dynamic behavior of stochastic gene expression models in the presence of bursting, SIAM J. Appl. Math. 73 (5) (2013) 1830–1852.

[26] S. Meyn, R. Tweedie, Markov Chains and Stochastic Stability, Springer-Verlag, London, 1993.

[27] K. Pakdaman, M. Thieullen, G. Wainrib, Fluid limit theorems for stochastic hybrid systems with application to neuron models, Adv. Appl. Probab. 42 (3) (2010) 761–794, http://dx.doi.org/10.1239/aap/1282924062.

[28] D. Revuz, Markov Chains, Elsevier Science Publishers B.V., North-Holland, Amsterdam, 1984.

[29] M. Riedler, M. Thieullen, G. Wainrib, Limit theorems for infinite-dimensional piecewise deterministic Markov processes. applications to stochastic excitable membrane models, Electron. J. Probab. 17 (2012) http://dx.doi.org/10.1214/ejp.v17-1946.

[30] R. Rudnicki, M. Tyran-Kamińska, Piecewise Deterministic Processes in Biological Models, Springer, 2017, http://dx.doi.org/10.1007/978-3-319-61295-9.

[31] T. Szarek, Generic properties of learning systems, Ann. Polon. Math. 73 (2) (2000) 93–103, http://dx.doi.org/10.4064/ap-73-2-93-103.

[32] T. Szarek, Invariant measures for Markov operators with application to function systems, Studia Math. 154 (3) (2003) 207–222, URL: https://doi.org/10.4064/sm154-3-2.