Origin of the Geometrical Pattern in the Royal Architecture of Ancient Egypt during the Thinite Age and Old Kingdom

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Abstract. The royal architecture of Ancient Egypt is one of the greatest exponents of that civilization's knowledge. Unfortunately, documentary sources with constructive designs or architectural projects have not been preserved, with the exception of some sketches or details. The only source of information remaining is the archaeology of these buildings themselves. The study of the origins of ancient Egyptian civilization focuses on archaeological, philological and religious studies; however, the origin of the geometrical rationale for its royal architectural designs has not been fully explained. These royal architecture designs, developed over three thousand years of pharaonic culture, would have recalled a certain link between the earthly and divine worlds. This article analyses the design principles pertaining to the main royal buildings during the Thinite Age and Old Kingdom; it thereby aims to propose a geometrical pattern that would explain the architectural design of the ancient Egyptian royal buildings throughout their history.

Keywords: royal, architecture, geometrical, pattern, ancient, Egypt

Introduction

Attempts to discover trends and patterns in the architectural design of royal buildings in ancient Egypt have been the object of considerable study.[1] Notwithstanding this, no study has been able to propose a geometric pattern whose origins in the Thinite Age (First and Second Dynasties) would have been

[1] Corinna Rossi surveys the main proposals made throughout the twentieth century regarding the geometrical patterns in the royal architecture of Ancient Egypt. Rossi compares Viollet-le Duc, Babin and Badawy in a highly convincing manner. However, an architectural design from the Thinite Age to the Late times that covered the whole of ancient Egyptian architecture was not able to be posited. See Rossi 2003: 68-77.

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used throughout pharaonic Egyptian history.[2] This research does not intend to revise the aesthetic development in royal architecture from its beginning; rather it directly engages with the topic of dimensions in the royal architecture of these periods in order to understand how architects were able to achieve such proportions in those times.

It is fundamental in this discussion to recall that the royal Egyptian cubit was the referential unit of length. The best preserved royal cubit-rods are from the New Kingdom. These had a largely ceremonial use and a characteristic design that would range from between 52.3 to 52.5 m in length.[3] The royal cubit was divided into 7 segments (termed fists or palms), which in turn were divided into 4 digits. Generally, there were 28 digits in a royal Egyptian cubit.[4] There are examples of half an Egyptian cubit, in which each segment had 14 digits (Figure 1).[5]

The Egyptian cubit was already in use at the time of the first pharaohs and possibly appeared with writing and the first basic arithmetic operations.[6] The Palermo stone is probably the main example of cubit use during the Thinite Age and Old Kingdom (Third, Fourth, Fifth and Sixth Dynasties). Made in the Fifth Dynasty, the Palermo Stone collected the Nile floods throughout the years of the distinct reigns.[7] Measurements of the floods are referred to, in cubits, at the lower part of each year's inscription (Figure 2).[8]

More than one cubit-rod was used in ancient Egypt.[9] Distinct rods had lengths from 4 to 7 palms.[10] In light of this, it will therefore be necessary to study not only the architectural proportions of the royal buildings themselves, but also to consider their specific measurements in order to rule out coincidences and thereby

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[2] Goyon 2004: 101-103.

[3] Clagett 1999: 7.

[4] Goyon 2004: 385-386.

[5] Figure 1.a, website of the Turin Egyptian Museum about the royal cubit pliable in two pieces, see https://collezioni.museoegizio.it:443/eMP/eMuseumPlus?service=ExternalInterface&module=collection&objectId=103873&viewType=detailView; Goyon 2004: Figure 497. Figure 1.b shows the main representation of a royal cubit rod as a single segment, now exhibited at the Louvre Museum, see https://www.louvre.fr/en/oeuvre-notices/cubit-rod-rule-maya-treasurer-tutankhamun; Monnier / Petit / Tardy 2016: 1-9.

[6] Arnold 1991: 10. The ancient Egyptians already used huge and complex numbers to count large quantities from the beginning of the Thinite Age. One of the most well-known examples is Narmer's mace head where a count is made of 400,000 sheep, 1,422,000 goats and 120,000 prisoners. Narmer's mace head can now be seen in the Ashmolean Museum, Oxford (UK), and is considered an original source of the numbers-and-counting systems in Ancient Egypt. See Wengrow 2007: Figure 2.3.

[7] Shih-Wei 2010: 68-89.

[8] Wilkinson 2000: Figure 1.

[9] Shaw 2012: 157-158.

[10] Clagett 1999: 8-9.

be able to determine the correct type of royal cubit and, finally, to propose a single pattern for royal architectural design in ancient Egypt.

1. Geometrical pattern of architectural design in the royal tombs at Umm el Qa'ab

The first pharaohs were buried at Umm el-Qa'ab, the royal cemetery during the Thinite Age and the necropolis of the ancient city of Tinis.[11] Aha (the Fighter) was the successor of Narmer, the first pharaoh of the First Dynasty (2900 BC, Table 1)[12] and both were buried at Umm el-Qa'ab.[13] W. Kaiser and G. Dreyer[14] excavated the tombs of both Narmer and Aha. Two more royal buildings are connected with Aha; the first of these is a large mastaba at Naqada,[15] initially related to Narmer.[16] It is now associated with Queen Neithhotep who lived at the time as Pharaoh Aha.[17] The second royal building is S3357, at Saqqara, which belongs to the group of large mastabas closely connected to the Memphite elite in the Thinite Age. Figure 3a[18] presents the royal tombs at Umm el-Qa'ab in chronological order, and it is highly possible that these monarchs used a small number of rectangles to define the main proportions of their tombs (Figure 3b).[19]

Figure 3c sets out the plan of one of the three possible tombs of Aha at Umm el-Qa'ab[20] (among others), specifically that which is furthest north (Tomb B10). The width of the three tombs varies from 4.43 m to 4.57 m, whilst their length varies from 7.48 to 7.62 m.[21] This study will proceed on the basis that a royal cubit is 0.523 m.[22] In that respect, 14 royal cubits are equal to 7.32 m, only 16 cm

[11] Wengrow 2007: 292-305.
[12] The present study sets out the Thinite Age and Old Kingdom chronologies in accordance with Erik Hornung, Rolf Krauss and David. A. Warburton's chronology of 2006. See Krauss / Warburton 2006: 490-491.
[13] Wilkinson 1999: 70-71.
[14] Kaiser / Dreyer 1982: 212-218. In this work, Kaiser and Dreyer provide detailed plans of the three tombs of Aha at Umm el-Qa'ab. Additionally, Kaiser publishes research on the tombs of several Thinite kings at Abydos, based on original plans that Petrie had made at the beginning of the twentieth century. See Kaiser 1982: 247-253.
[15] Van Wetering 2012: 91-124.
[16] Kahn et al 2001: 174, Fig 1.
[17] Arnold 2003: 148.
[18] Bestock 2008: 43.
[19] The plans and measurements have been made with AutoCAD 2017 for MAC.
[20] Kaiser / Dreyer 1982: 212-213.
[21] Kaiser / Dreyer 1982: 215.
[22] Lepsius's work is one of the first and principal studies on the Egyptian cubit rod. He determined that the Egyptian cubit was 0.525 m long, although he indicates its measurements as between 0.523 and 0.527 m, see Lepsius 1865: 8. In later works, Alan Gardiner defines the Egyptian cubit as 0.523 m, see Gardiner 1957: 199.

less than Dreyer's 7.48 m. In Figure 3c(1), the rectangle R.α that delimits the excavated burial chamber is 8.66x14 cubits (rectangle R.α, 1x1.62 proportions). Additionally, the burial chambers of Aha's three tombs are surrounded by a wall of 16.5x21 cubits (Figure 3d(1), rectangle R.ß, 1x1.27 proportions). The fact that Kaiser and Dreyer indicate the depth of Aha's tomb as 3.60 m, which means 6.88 cubits (≈ 7 royal cubits = 14/2), might make sense for the designs of Aha's architects. The same proportions of rectangle R.α (1x1.62) were used by the first-dynasty pharaohs Djet, Andjib, Semerkhet and Qa'a (the last pharaoh of the First Dynasty) to define the mortuary chambers or the main proportions of their tombs (Figure 3c). Khasekhemwy, the last pharaoh of the Second Dynasty, would use the same proportions to define his mortuary chamber (Figure 3c).

Figure 3d shows that Djer, the successor of Aha, also defined his mortuary chamber with an R.ß rectangle (1x1.27), as did his antecessor. Djet and Meretneith, from the First Dynasty, used the same proportions to design parts of their tombs, but only Djet mixed rectangles R.α and R.ß again. Meanwhile Peribsen, the predecessor of Khasekhemwy in the Second Dynasty, used the same rectangle R.ß to build the wall surrounding his mortuary chamber.

Meretneith was probably Djet's wife and Den's mother,[23] the fifth king of the First Dynasty. Den lived at the end of the First Dynasty (2814-2772 BC). In addition to bringing in administrative innovations for his kingdom,[24] he also introduced significant ideas into royal funerary architecture. Den added an entry to the death chamber, which made supply and cult easier, with no need for the tomb to be completed.[25]

Figure 3e shows the plan of Den's tomb made by Dreyer.[26] More specifically, Figure 3e defines this mortuary chamber as a rectangle of 1x2 proportions (R.1x2), which would appear to be a further contribution by Den to royal architecture. Den's reign was long-lived.[27] This fact could be the reason that so many mastabas at Saqqara were attributed to him.[28] During the First Dynasty, Den was succeeded by Andjib, Semerkhet and Qa'a. All three used an R.1x2 rectangle to define their mortuary chambers or the main proportions of their tombs (Figure 3e). Finally, Peribsen, in the Second Dynasty, did the same with his mortuary chamber.

It is necessary to point out that Hetepsekhemwy, the first pharaoh of the Second Dynasty, came to the Saqqara area to place his tomb in the form of

[23] Wilkinson 1999: 74-75.
[24] Engel 2008: 33-34.
[25] Wilkinson, 1999: 75.
[26] Dreyer et al 1998: 144.
[27] Wilkinson 1999: 76.
[28] Hendrickx 1999: 62.

underground galleries located 1 km south of the great mastabas, because there appears to have been no break between dynasties.[29] Raneb and Ninetjer, Hetepsekhemwy's successors, are associated with these galleries, but the monarchs of the Second Dynasty were not buried again at Umm el-Qa'ab until the arrival of Peribsen and the last king of the Second Dynasty, Khasekhemwy.[30] Researchers do not agree on the burial location for the remaining kings of the Second Dynasty.[31]

Figure 3f shows the pharaonic tombs of the Thinite Age at Umm el-Qa'ab in chronological order and scaled. This figure represents the proportions set out in Figures 3c, 3d and 3e. Table 2 presents all the tomb measurements and proportions. The lengths of all the rectangles, except for the burial chambers of Den, Semerkhet and Peribsen (with R.1x2 proportions), appear to be multiples of 14 (which is the length of the mortuary chamber in Aha's Tomb). Some of these multiples of 14 are rather sophisticated, although still conforming to basic arithmetic operations. The widths of all the tombs appear to be multiples of 8.66 (when part of an R.α rectangle) and 11 (when part of an R.ß rectangle). The remaining proportions and sections of the tombs appear to be the result of adding the remaining architectural design of each tomb to an R.α, R.ß and R.1x2 rectangles. This conclusion is derived from the fact that, apart from these rectangles, no rectangle of any proportions is repeated in at least three distinct tombs.

2. The funerary enclosures at Abydos

Only six Thinite pharaohs have been identified with their corresponding funerary enclosures at Abydos. The funerary complexes of the monarchs at Abydos consisted of the tombs at Umm el-Qa'ab and the funerary enclosures located 1 km north of this cemetery.[32] Aha was the first pharaoh to build his three enclosures at this place. These three enclosures were, perhaps, an allegorical reference to Aha's three tombs at Umm el-Qa'ab. In their geometrical design, Aha introduced an R.2x3 rectangle to define their proportions (Figure 4b). The specific measurements of the largest of these are 22x33 m[33] or 42x63 cubits (1 cubit = 0.523 m). These specific measurements appear to be direct multiples of 11 and 14, as if Aha had intended to refer to the measurements and proportions of his tombs at Umm el-Qa'ab.

[29] Bard 1999: 125.
[30] Wilkinson 1999: 80-89.
[31] Bard 1999: 35.
[32] Kemp 2006: Figure 30.
[33] Bestock 2008: 49.

Figure 4 shows the main proportions of all enclosures at Abydos. The enclosures of Djer, Djet and Meretneith were reconstructed by the North Abydos Project,[34] although the enclosure remains discovered were in an extremely poor condition. For this reason, Table 3 provides only the specific measurements for the enclosures of Aha, Peribsen and Khasekhemwy. Peribsen and Khasekhemwy were the last two kings of the Thinite Age to build their funerary enclosures at Abydos. Both enclosures can be defined as having a central axis corresponding to an R.1x2 rectangle (Fig. 4.a). The measurements of the outer face of the wall in Peribsen's enclosure are 55x108 m,[35] its thickness being 1.5 m. The central axis of 53.5x106.5 m is therefore equivalent to 101.5x203 cubits.

Bestock gives the approximate measurements of Khasekhemwys's funerary enclosure at Abydos as 65x126 m.[36] These measurements do not seem to correspond to the proportions of an R.1x2 rectangle. It should be observed that, for a rectangle that is 65 m wide, 126 m long and 5.5 m thick, a central axis can be defined, which would be 59.5 m in width and 120.5 m in length. This proportion (59.5 x 120.5 m) corresponds to a close 1x2 proportion (width and length, Figure 4a). The 120.5 m length of the enclosure's perimeter axis can be equated to 231 royal cubits, which amounts to exactly 120.81 m. It is worth observing that that 231 is a multiple of 11 and 14. In particular:

1- $231 = 14 \times \frac{33}{2} = (14 \times 16) + \frac{14}{2} = (14 \times 2 \times 2 \times 2 \times 2) + \frac{14}{2}$
2- $231 = 11 \times 21$

Figure 4 sets out all the mortuary enclosures at Abydos; Table 3 lists this data chronologically.

3. Geometrical pattern in the architectural design (GPAD)[37] of royal buildings that emerged during the Thinite Age

The lengths of the main buildings studied appear to be multiples of 14. As regards width, these are multiples of 8.66 (R. α rectangle) and 11 (R.ß rectangle). A closer understanding of the relationship between 11x14[38] is necessary in order to

[34] Bestock 2008: 42-59.
[35] Bestock 2008: 56.
[36] Bestock 2008: 57.
[37] The Geometrical Pattern in the Architectural Design of royal buildings is hereafter indicated as *GPAD*.
[38] In 1990, Gay Robins proposed a study that analyses the 11x14 proportion in the pyramids' architectural design of the Fourth Dynasty. See Robins 1990: 75-80. Corina Rossi also exposes the importance of the rectangle 11x14 in the pyramidal architecture of the third and fourth dynasties. See

ascertain whether the R. α and R.ß rectangles are directly related. These proportions have a geometrical peculiarity that is entirely accidental. A rectangle of 11x14 cubits defines a circle with a radius of 14 and a perimeter of 87.964 cubits, and a square with a half side of 11 cubits and a perimeter of 88 cubits (Figure 5a). Both Figures have the same perimeter, with a difference of 0.036 cubits. In order to understand how close this geometric solution is to reality, if a square is planned with an 11x14 proportion and its perimeter is the same as 1 cubit, it appears that the perimeter of the circumference is 0.9996 cubits. That is to say, a difference in perimeter of a mere 0.0004 cubits (Figure 5b).

Ancient Egyptians might have been aware of this geometric property between these two numbers, although they had no kind of either mathematical[39] or measurement[40] tools to demonstrate how significant this relation might be. However, such a discovery would have been sufficiently important to be included in their architectural designs. Figure 5c analyses the intersectional measurements between the square having 11 cubits on every half side and the circle having a radius of 14 cubits; its shows that 8.66 cubits is half of the shortest side of the rectangle that appears on the inner intersections of the square and the circle, which defines an R.α rectangle. Figure 5d, in contrast, analyses the intersectional measurements between the square having 11 cubits on every half side and the circle having a radius of 14 cubits, showing that 11 cubits is half of the shortest side of the rectangle that appears on the outer intersections of the square and the circle, which defines an R.ß rectangle.

Ordering all the data and measurements set out so far, the GPAD can basically be defined in the following geometric figures:

- An R.α rectangle
- An R.ß rectangle
- Regular 1x2 and 2x3 rectangles (width x length)

As regards the means by which ancient Egyptians were able measure in centesimal cubits, such as 8.66 cubits, it is plausible to suggest that they never aimed to develop such accuracy in their measurements, given a lack of mathematical and measurement tools that would have facilitated this.

Illustration 1 is a sketch of Aha's tomb construction (Figure 3f.1), made by the author of this article, which shows how the foundation process in royal

Rossi 2003: 215-216. Later works also compare the pyramid geometry of the Old Kingdom, in order to study its astronomical and topographic relations. See Magli 2010: 59-74.

[39] Clagett 1999: 93-95.

[40] Arnold 1991: 251-256.

buildings, such as *stretching the cord*[41] (detail 1, Illustration 1), perfectly matches this type of measurement and architectural design from the Thinite Age. That is to say, in order to obtain the R.α and R.ß rectangles of the GPAD, or rectangles based on their proportions, the priest architects of Aha's Tomb would first begin by stretching a rope to define the longitudinal axis of the temple and its orientation (details 1 and 1* of Illustration 1). They would then draw the square on the ground with a half side of 11, or multiple of 11, and a circumference with a radius of 14 or multiple of 14, both with the same centre (detail 2, Illustration 1). The intersections of the square and circumference would define, in a simple and evident way, an R.α rectangle of the GPAD framing the mortuary chamber (details 3 and 4, Illustration 1). Finally, the outer face of the mud brick wall was possibly designed by an R.ß rectangle with 16.5x21 cubits (details 3 and 4, Illustration 1). An R.ß rectangle does not need to mark the square and circle of the GPAD on the ground and can be defined directly by a rectangle with proportions of 11x14.

As regards the astronomical alignment of the tomb (details 1 and 1*, Illustration 1), there are two points to be made. First, the earliest representation of the *stretching the cord* ritual dates to the times of Khasekhemwy, the earliest mention engraved on the Palermo Stone in the times of Den.[42] Second, the representation of a portable object at the solar temple of Niuserre at Abu Ghorab (detail 1*, Illustration 1) is a clear reference to the item that goddess Seshat wore on her head during the stretching the cord ritual.[43] This symbol, a multi-petalled flower, seems to appear in Narmer's Palette and on his mace head in the First Dynasty.[44] Additionally, the earliest well-preserved representation of Seshat's object is in Snefru's first temple valley.[45] Illustration 1 could therefore be a highly reliable representation of a royal architectural design and construction process during the early Thinite Age. The astronomical alignment, represented in Illustration 1, responds to any astronomical alignment, not to a specific one.

[41] *Stretching the cord* is one of the several steps involved in the ritual ceremony for the foundation of sacred buildings. The king and the goddess Seshat stretched a rope defining the orientation of the sacred building and its four corners. Rossi 2003: 148-153.
[42] Belmonte 2012: Figures 4.7 and 4.8.
[43] Antenmüller 2010: 51.
[44] Achneider 1997: 241.
[45] Belmonte 2012: Figures 4.9 and 9.

4. The GPAD during the Third Dynasty

4.1. The GPAD in the enclosures of the mortuary complexes

Gisr el-Mudir, a mysterious empty enclosure, was built at Saqqara near the tombs of the first pharaohs in the Second Dynasty.[46] This enclosure was defined by a low wall and could be the counterpart to the valley enclosures at Abydos.[47] Figures 6a,[48] 6b[49] and 6c[50] present this enclosure, which seems to be defined with an R.α rectangle. Several attempts have tried to relate the Gisr el-Mudir enclosure design with those of Netjerkhet and Sekhemkhet.[51]

Netjerkhet, the first pharaoh of the Third Dynasty (2592, Tab. 1), built the southern wall of his mortuary complex (Figure 6d)[52] adjoining the entrance of the tomb of Hetepsekhemwy, the first pharaoh of the Second Dynasty.[53] Furthermore, he laid a great ditch around his mortuary complex, which corresponds to the proportions of an R.ß rectangle (Figure 6a). Figures 6a, 6c and 6d define the inside face of Netjerkhet's enclosure wall as a R.1x2 rectangle.

After Netjerkhet, Sekhemkhet was the last king of the Third Dynasty to build a great mortuary complex at Saqqara. Figures 6a, 6c and 6d show this mortuary complex as an R.1x3 rectangle. Figure 6e[54] presents an accurate overview of this enclosure. The architects of Sekhemkhet placed his pyramid, the pyramid temple, its entrance and the south tomb inside an R.α rectangle with a length of 448 cubits ($14x32=14x2^5$). At the same time, this R.α rectangle and an R.1x2 rectangle to the south are framed by another R.1x2 rectangle. Finally, these all define a square to the north (R.1x1 rectangle).

Figure 6 highlights the manner in which the GPDA appears to be have been used by the priest architects of the Thinite Age and the Third Dynasty to design everything from the proportions of royal tombs to enormous royal enclosures. During the Third Dynasty, an R.1x3 rectangle was incorporated into the sacred

[46] Lehner 1997: 82.
[47] Stadelmann 1997: 29-31.
[48] Swelim 1991: 398.
[49] Swelim 1991: 395.
[50] Stadelmann 1997: 30.
[51] Swelim 1991: 389-402.
[52] Stadelmann 1997: 32.
[53] Trigger et al. 1967: 54.
[54] Stadelmann 1997: 73. During the 1990s, Stadelmann developed one of the most complete archaeological works on the Old Kingdom Pyramids. Thirty years earlier, Maragioglio&Rinaldi carried out one of the most exhaustive studies on the architecture and measurements of the Old Kingdom buildings. In their second book, Maragioglio & Rinaldi present Sekhemkhet's mortuary complex plan, which seems to corrrespond to the same proportions as Figure 6.e. See Maragioglio / Rinaldi 1963a: Figure 1.

proportions of the GPAD, although ultimately the R.1x3 rectangle that defines the enclosure wall of the mortuary complex of Sekhemkhet is the sum of an R.1x2 rectangle and a square, or an R.1x1 rectangle.

4.2. The GPAD in the pyramids and pyramid temples of the Third Dynasty

Netjerkhet (Djoser), Khasekhemwy's successor and possibly his son, was the first pharaoh of the Third Dynasty. For no discernible reason, he abandoned the tradition of being buried at Abydos, which had previously been followed by most Thinite kings.[55] Netjerkhet chose the southern zone of the great mastabas at Saqqara to build his mortuary complex among the large mastabas to the north and the tombs of the first pharaohs of the Second Dynasty[56] to the south. Netjerkhet's pyramid plan measures 121x109 m (≈231x209 cubits) with a height of 60 m in its final stage[57] and two specific data directly relate the funerary enclosure of Khasekhemwy at Abydos to his son's step pyramid at Saqqara:

1. The length of the wall's central axis in Khasekhemwy's enclosure is the same as the major base (east-west) in Netjerkhet's step pyramid → 231 royal cubits = $14 \times \frac{33}{2} = (14 \times 2^5) + \frac{14}{2}$
2. The width of the wall's central axis in Khasekhemwy's enclosure is the same as the height of Netjerkhet's step pyramid → 115.5 royal cubits = $\frac{231}{2}$

These data mean that the rectangle with R.1x2 proportions, which defines Khasekhemwy's enclosure of 115.5x231 cubits at Abydos, is exactly the same as the rectangle formed by the major base and the height of Netjerkhet's step pyramid. That is to say, Netjerkhet projected his father's architecture from the horizontal to the vertical plane. It is entirely plausible, therefore that the evolution of the GPAD in Netjerkhet's royal architecture (Figure 7) may act as a *background* to the astronomic symbolism of the royal architecture[58] and the process of solarization[59] in which Egyptian culture was immersed, as presented in Figures 7 and 8a.[60]

An important architectural complex such as that built by Netjerkhet at Saqqara had not been seen since the first temples in Egyptian history, and the social and

[55] Wengrow 2007: 95-98.
[56] Wilkinson 1999: 240-243.
[57] Lehner 1997: 85.
[58] Belmonte 2012: 114-126.
[59] Cervelló-Autuori, 2011: 1125-1149.
[60] Netjerkhet's pyramid plan in its final stage measures 121x109 m. (≈231x209 cubits), see Lehner 1997: 19, 85.

economic revolution that Netjerkhet carried out through his mortuary complex is evident.[61] Until the present, the architectural design of Netjerkhet's step pyramid has been explained by means of the construction of successive mastabas overlying an original, each attempting to make up a pyramidal heap.[62] In his work, for example, Lehner shows that Lauer divided the construction into distinct phases. According to Lauer, Netjerkhet expanded and redesigned his construction throughout his reign in order to build what is now the step pyramid. However, Lehner also mentions Stadelmann's deduction—made on the basis of his excavations at the end of 1980's—that Netjerkhet may have had in mind the final design of the step pyramid from the very outset. Additionally, although other Egyptologists do not defend the hypothesis that Netjerkhet might have intended the final project of the monument from the outset,[63] Stadelmann does maintain that the step pyramid was an intentional project and not merely the overlap of successive mastabas.

If the major base and height of the step pyramid are equal to the length and width of the central axis in the wall of Khasekhemwy's funerary enclosure at Abydos, it is easy to conclude that Netjerkhet had already studied the final design of his pyramid from the outset of constructing the square mastabas, located under the step pyramid. Moreover, the proportions of the step pyramid's north temple (Figure 8b)[64] seem to correspond to the R.ß rectangle in the GPAD.

The most important constructions that are related to the pharaohs of the Third Dynasty are the mortuary complexes of Netjerkhet and Sekhemkhet at Saqqara, and the step pyramid of Khaba (Sek in Zawyet el-Aryan).[65] Khaba was Sekhemkhet's successor[66] and the archaeological evidence relating him to the pyramid is poor, although the consensus to date on this matter is unanimous.[67] The archaeology only allows us to confirm Netjerkhet's pyramid, as it is the only completed step pyramid that has been preserved from the third dynasty[68] (Figure 8a).[69] Its height is 231/2 cubits, which is equivalent to 60.4 m.[70] Additionally, the

[61] Parra 2001: 31-32.
[62] Lehner 1997: 87.
[63] Parra 2008: 64.
[64] Stadelmann 1997: 38.
[65] Wilkinson 1999: 95-99.
[66] Wilkinson 1999: 99.
[67] Parra 2001: 36.
[68] Several provincial step pyramids were built during the end of the third dynasty and the beginning of the fourth dynasty by Huni and Snofru along the bank of the Nile. Probably the best known is the Seila Pyramid in the Fayum Area. See Lehner 1997: 96.
[69] Figure 8 does not pretend to represent the step pyramids of the third dynasty as a smooth side pyramids. In fact, Points 5 and 7 of this paper specify that the step design is crucial to understand the evolutions of the royal building designs.
[70] For Lehner, the height of Netjerkhet's step pyramid is 60 m. See Lehner 1997: 17.

fact that the architects planned six steps to reach the highest point means that the total levels required for this are seven (=14/2). That is to say, the first level would be the ground level itself; the remaining six would be those corresponding to the six steps of the pyramid. Finally, he probably tried to represent the diagonal of an R.ß rectangle through the slopes of the eastern and western faces in his step pyramid (Figures 7 and 8a).

After Netjerkhet, Sekhemkhet devised the first square-based pyramid and added an important innovation to the design: the entrance to the mortuary chamber from outside the base of the pyramid (Figure 8c).[71] Figure 8c is a possible reconstruction of Sekhemkhet's pyramid. He probably defined its base of 231[72] cubits, like his antecessor did, and if this was designed from the GPAD it would imply that its height could have had only two measurements. First, it could have been equal to that of Netjerkhet's (point n in Figure 8a), that is to say, half of the height, that is, 115.5 cubits (point s in Fig 8c). Or second, he could have chosen to completely represent the diagonal of the R.ß rectangle of the GPAD, with a total height of 147 cubits (=14x21/2) and possibly eight pyramid steps (Figure 8c). The same exercise should be performed on Khaba's pyramid. Its base can be considered as 159.5 cubits[73] and is equal to 14.5 (=29/2) x 11. There would therefore be only two possibilities for its height: either half of the pyramid's total base, that is, 159.5/2 cubits, marked as point k in Figure 8d; or a complete representation of the diagonal in the GPAD's R.ß rectangle, that is, multiplying 14 by 29/8 and obtaining a total height of 101.5 cubits. Table 4 presents the main proportions of the royal architecture in the Third Dynasty.

5. The GPAD in the pyramids and main pyramid temples of the Fourth, Fifth and Sixth Dynasties

5.1. The GPAD in the pyramids and their chambers of the Fourth, Fifth and Sixth Dynasties

Pyramidal royal architecture prevailed during the Old Kingdom from Pharaoh Netjerkhet. If the first main evolution of the royal architecture was to project the GPAD from the horizontal to the vertical plane, the second main evolution—once pyramidal architecture had appeared—was possibly the transition from the step pyramid to a smooth-sided pyramid. Snofru (2543-2510 BC, Tab. 1), the first

[71] Stadelmann 1997: 73. In his work, Stadelmann presents a possible reconstruction of Sekhemkhet's pyramid.

[72] Both Lehner and Stadelmann estimated the length of Sekhemkhet's pyramid as c. 120 m. See Lehner 1997: 19; Stadelmann 1997: 71-72.

[73] Stadelmann defines the base of Khaba's pyramid as 84 m, that is to say 160 cubits, and again presents a possible reconstruction of Khaba's pyramid. See Stadelmann 1997: 75-76.

pharaoh of the Fourth Dynasty, built three great pyramids. Two of these were at Dahshur, which scholars have largely attributed to Snofru.[74] The other pyramid was built at Meidum.

Snofru settled in Meidum in the first years of his reign. He may have started the construction of the Meidum pyramid at this time, although this fact does not mean that he was not able to design and build the architectural project at Dahshur in the same period. He moved to Dahshur in the fifteenth year of his reign to found a new necropolis and possibly to be closer to Netjerkhet's mortuary complex, which housed the first pyramid in Ancient Egypt. Snofru then returned to Meidum at the end of his reign to complete his own pyramid.[75] Some scholars defend the hypothesis that Huni initiated the construction of the Meidum pyramid as a step pyramid.[76] However, there is no archaeological evidence for this and it is plausible that this corresponds more to a need to attribute a great pyramid to this pharaoh.[77]

It is particularly difficult to determine whether or not Snofru planned the pyramid of Meidum as a pyramid with smooth sides from its outset and with a square base of 231 cubits.[78] Bearing in mind that step pyramids were the only pyramid construction that had been developed by that period, it is reasonable to assume that this would have been the form chosen by Snofru at Meidum. However, as he clearly thought about the bent pyramid with smooth sides slightly subsequent to the start of his reign, it is plausible that the idea of building a smooth-sided pyramid at Meidum was present from the beginning. Stadelmann, who defends Snofru as the only builder of the Meidum pyramid, marks out the base of the pyramid in 275 royal cubits (=11x25) and defines its inclination as 51º51'.[79]

Both Stadelmann and Lehner measure the height of this pyramid as 92 m. It is significant to observe that neither scholar gives this height in cubits (92 m are equivalent to 175.9 cubits). If the base of the pyramid was designed by the architects of Snofru as 275 cubits, it is relevant to propose that the height would be projected as 175 cubits, equivalent to 91.52 m. The Meidum pyramid presents a rectangular triangle with a base of 275/2 and a height of 175, which is the equal

[74] Stadelmann 1997: 87, 104.

[75] Lehner 1997: 98-101.

[76] Parra 2001: 37-46.

[77] Reader 2015: 214.

[78] Stadelmann gives 230 cubits for the base of the pyramid of Meidum, in its stepped phase. These 230 cubits are equated to 120.75 m as he uses Helck's measurement of 0.525 m. per cubit. Using the length of 0.523 m. per cubit, as applied in this current study for every measurement assessed, the resulting distance is 120.75 m., equivalent to 230.89 cubits (≈231 cubits). See Stadelmann 1997: 84.

[79] Stadelmann 1997: 86.

to 25/2 times a rectangular triangle with a base of 11 and a height of 14. Table 5[80] presents all the rectangle triangles that appear to be the design base of the Old-Kingdom pyramids, starting from the Meidum pyramid.

The priest architects used an R.ß rectangle and basic proportions (2x3, 3x4, 4x5 and 5x6) to define the pyramid slopes of the Old Kingdom. Nevertheless, Tables 6 and 7 show that they did not use R.α and R.ß rectangles to design the main chamber proportions of the Fourth, Fifth and Sixth Dynasty pyramids.[81]

5.2. The GPAD in the main pyramid temples and valley temples of the Fourth, Fifth and Sixth Dynasties

At the beginning of the Fourth Dynasty, Snofru defined the design of the so-called first middle temple of his bent pyramid from the R.ß and R.1x2 rectangles (Figure 9a).[82] After Snofru had built his three great pyramids, Khufu—his successor—possibly attempted to emulate or even outdo him, setting out to implement an ambitious architectonic plan at Giza (Table 5, Figure 9h).[83] Some studies have tried to frame the three great pyramids of Giza within an original architectonic project devised by Khufu.[84]

An ancient harbor complex of the early Fourth Dynasty has recently been excavated at Wadi el-Jarfa.[85] This port was a point of departure to the Sinai Peninsula, where copper and turquoise were obtained. Additionally, translations of the papyri found in this port appear to indicate that an artificial harbor was built on the banks of Menkaure's temple valley in the times of Khufu.[86] Similarly, an artificial canal was discovered by Lehner's work on the Giza Plateau,[87] which allowed access to construction materials through what is now the Menkaure temple valley during the construction of the three great pyramids.[88] Whether or not Khufu initiated a huge architectonic plan at Giza, he did use—as had Snofru—the R.ß and R.1x2 rectangles to define the design of his pyramid temple in this

[80] Table 5 presents Magli exhaustive study of the fourth, fifth and sixth pyramid slopes from Khufu's pyramid. See Magli 2010: 67. Moreover, one of the most current works about the measures of the Old Kingdom Pyramids is the one carried out by Pietro Testa. Table 5 compares Lehner and Testa's measures. See Lehner 1997: 19 and Testa 2009: Volume II.
[81] Lehner 1997: 84-161 and Testa 2009: Volume II.
[82] Arnold 2017: 390.
[83] Digital Giza is one of the most significant works on the digitalization and 3D modelling of all archaeological data related to the Giza Plateau. See http://giza.fas.harvard.edu/3dmodels/71017/allphotos/
[84] Lehner 1985: 109-143.
[85] Tallet / Marouard 2016: 135.
[86] Tallet 2017: 83-95.
[87] Lehner 2014b: 14-23.
[88] Lehner 2014a: 2-7.

case (Figure 9b).[89] His successor, Djedefre, decided to leave this area and built his pyramid at Abu Rowash (Table 5), almost 8.5 km to the northwest of Giza. After Djedefre, Khafre and Menkaure returned to Giza to build their funerary complexes.[90]

Khafre used the longitudinal half of an R.ß rectangle, that is to say, R.ß' (a length of 14 or multiple, and a weight of half of 11 or multiple), to define the main proportions of his pyramid temple (Figures 9c and 9d).[91] This was 111 m in length,[92] with a wall thickness of almost 1 m, which means that the wall of its central axis (rectangle R.ß' in Figures 9c and 9d) has a length of 110 m, or 209.83 cubits (≈210=14x15). The two courtyards in this pyramid temple—one covered, the other uncovered—correspond to rectangles R.1x2 and R.α, respectively. Khafre's valley temple (Figure 9e)[93] was built near the Sphinx temple, which was also constructed by Khafre, its proportions being based on the R.α and R.ß'' rectangles (R.ß'' appears to be the longitudinal third of an R.ß rectangle, that is to say, a length of 14 or multiple, and a weight of 11/3 or multiple). Figure 9f[94] shows how the GPAD defines the whole architectural design of the Sphinx Temple. Finally, on the Giza Plateau, Menkaure used an R.1x2 rectangle to mark off the courtyard of his valley temple (Figure 9g). Table 8 summarizes the proportions of the main temples in the Fourth Dynasty.

Shepseskaf, probably the last king of the Fourth Dynasty (2442-2436, Tab. 1), left Giza and built his large mastaba at South Saqqara. His successor, Userkaf, who was possibly also his son, was the first pharaoh of the Fifth Dynasty. Userkaf was the first monarch to build at Abusir, where he built his solar temple. He chose the area near the northeast corner of Netjerkhet's complex to build his pyramid (Table 5) and used an R.α rectangle for the courtyard design of the pyramid temple (Figure 10a.1).[95] After Userkaf, Sahure applied an R.ß rectangle to define the proportions both of his valley temple (Figure 10b.1) and his pyramid temple courtyard (Figure 10b.2). Neferirkare, in contrast, used an R.α rectangle to design the main proportions of his pyramid temple (Figure 10c.2) and an R.ß rectangle for its courtyard (Figure 10c.1). Niuserre used an R.α rectangle in the same way to define the main proportions of his pyramid temple (Figure 10d.1) and an R.ß rectangle to determine the main proportions of his valley temple (Figure 10d.2).

[89] Maragioglio / Rinaldi 1963b: Figure 10. Moreover, the courtyard of Khufu's pyramid temple is a rectangle R.2x1 defined by 7x14 pillars. See Miatello 2010.
[90] Edwards 1955: 133.
[91] Maragioglio / Rinaldi 1963c: Figures 11 and 13.
[92] Stadelmann 1997: 136.
[93] Maragioglio / Rinaldi 1963c: Figure 14.
[94] Maragioglio / Rinaldi 1963c: Figure 14.
[95] The present study uses Stadelmann's work to reference the plans of Figures 10 and 11. See Stadelmann 1997: 160-197.

After Niuserre, Djedkare defined the main proportions of his pyramid temples with R.α and R.ß rectangles (Figure 10e.1), and the proportions of a queen's pyramid complex with an R.α rectangle (Figure 10e.2). Unas, the last pharaoh of the Fifth Dynasty, applied an R.ß rectangle to determine his pyramid temple proportions (Figure 10f.1) and designed his courtyard with an R.α rectangle (Figure 10f.2), just as the first pharaoh of this dynasty had done. Table 9 summarizes the proportions and measurements of the Fifth Dynasty's main temples.

After the Fifth Dynasty, the kings of the Sixth Dynasty were unable to find new locations in which to build their pyramidal complexes.[96] Teti, the first pharaoh of the Sixth Dynasty, chose the same area as Userkaf, the first pharaoh of the Fifth Dynasty. That is to say, the northeastern zone of Netjerkhet's funerary complex. Figure 11 outlines the main pyramidal complexes of the Sixth Dynasty. As with the pyramidal complexes of the Fifth Dynasty, the pharaohs of the Sixth Dynasty—the last dynasty of the Old Kingdom— continued to use the GPAD in designing the main proportions of the pyramid temples, their courtyards and their valley temples. Table 10 shows the proportions and measurements of the main temples in the Sixth Dynasty.

When the Old Kingdom ended, in approximately 2150 BC, the country of the Two Lands underwent a First Intermediate Period in which the unification of the entire territory was lost. However, following the fall of Heracleopolis to the Theban kings, Montuhotep II succeeded in uniting Upper and Lower Egypt again and the Middle Kingdom then began.[97] Nevertheless, there were still times of strong territorial disputes.[98] In 2009 BC,[99] Montuhotep II chose Deir el-Bahri to build his mortuary temple. This construction is located on the west bank of the Nile, facing the area of Amon at Karnak,[100] constructed on the east bank 2 km north of Ancient Thebes.[101]

Significantly, Montuhotep II did not wish to settle down on any of the lands that the kings of the Old Kingdom had occupied, although he was the pharaoh who had united the whole country. The choice of Deir el-Bahri was probably due to the simple intention to build near Thebes or to taking advantage of the terrain's orography and the mountain that would provide shelter to his temple, as if it were a great pyramid. Until the onset of the Middle Kingdom, the construction of great pyramidal complexes did not reoccur.

[96] Lehner 1997: 156-163.
[97] Moreno / Agut 2016: 161-235.
[98] Diego Espinel 2009: 208-270.
[99] Krauss / Warburton 2006: Table IV.2.
[100] Wilkinson, R. H. 2000: 175.
[101] Bailes / Malek 1991: 84-85.

6. Conclusions

In the origins of Egyptian civilization, a dual sense of existence was one of the bases of their thought,[102] and being able to equate a square perimeter to one of a circumference may very well have been charged with symbolism. It is also probable that the four sides of the square expressed the four cardinal points of Egypt[103] and the living world. The circumference, at a celestial level, expressed the idea of regeneration and eternity,[104] the solar disk and possibly the vault of heaven in which the gods would dwell. The geometry that described by GPAD would appear to have granted a perfect solution to the architects of the time through which to represent this dual conception.

The apparent meaningless of using a measuring tool whose length is divided into 28 segments makes sense in this period, more so still when they developed a decimal mathematical writing system.[105] On the one hand, the royal cubit is divided into 28 digits and, in the GPAD, the value of 28 is the diameter of the circumference with a radius of 14 (Figure 5). On the other hand, assuming that the royal cubit has a length of approximately 0.523 m, measurements in cubits of the royal buildings (since the time of the first pharaohs' early tombs) are multiples of 14 in relation to their lengths, so that the premise of assuming that the royal cubit corresponds to 0.523 m is in consonance with the proposal of the GPAD. Applying GPAD proportions, their main widths appear to be multiples of 8.66 (rectangle R.α) and 11 (rectangle R.ß). These arguments possibly indicate that the ancient Egyptians could have developed their metric tools at the same time as they defined the design of their royal buildings.

The architectural design of the royal tombs of the First and Second Dynasties at Umm el-Qa'ab and the funerary temples at Abydos appear to be based on the GPAD. The most important feature that connect the designs of these royal buildings is the use and combination of the R.α, R.ß, R.1x2 and R.2x3 rectangles. The first evolution of the GPAD took place between Khasekhemwy's temple at Abydos and the step pyramid of Netjerkhet at Saqqara. This first evolution consisted of projecting the GPAD from a horizontal plane (ground level) to a vertical plane (the heavens) and, at the same time, represented an R.ß diagonal rectangle (11x14 proportions). At the beginning of the Fourth Dynasty, the priest architects devised the second evolution of the GPAD through a transition from a step pyramid to a pyramid with smooth sides.

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102 Cervelló-Autuori 1996: 219.
103 Wilkinson 1994: 160. Moreover, the idea of squaring a circle try to match the area of square to the area of a circle not their perimeters.
104 Lull 2016: 43.
105 Sánchez 2000: 143; Clagett 1999: 2.

Tables 5, 8, 9, 10 and Figures 3f, 4, 6, 8, 9, 10, 11 are a representative sample of the pyramid and temple designs during the Thinite Age and Old Kingdom. It seems clear that all of such designs attempted to use the GPAD to define their main proportions, combining these with regular rectangles (Tables 5, 6 and 7), in order to design the pyramid slopes, the main pyramid temple proportions or the courtyard measurements throughout more than seven centuries (Table 1).

The systematic use of the GPAD in all ancient Egyptian royal building designs in the Thinite Age and Old Kingdom can be perceived only when all these royal architectural designs are placed together chronologically. All pharaohs in this period followed this architectural tradition and used the GPAD in their own way, applying it in order to distinguish their own buildings from those of the other pharaohs.

It is relevant to point out that the plans used in this study—obviously—do not refer to the GPAD when such plans were first made. The fact that the GPAD proportions seem to fit in perfectly with all of them upgrades the present research. The findings set out here appear to confirm that a geometrical pattern was established in the Thinite Age and during the Old Kingdom as a means of designing the royal buildings in Ancient Egypt, and was systematically used throughout its history. The geometrical pattern proposed in this article could therefore be a tool available to all researchers to support with their work and help understand in depth the architecture of the ancient buildings involved in such work.

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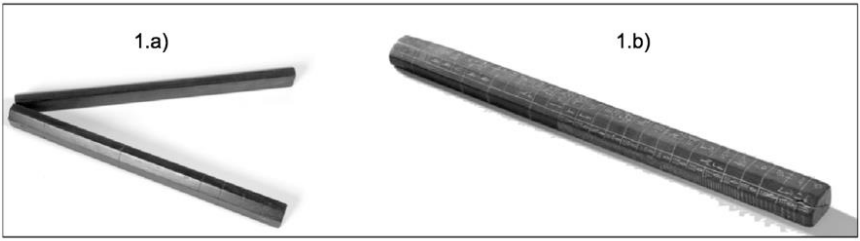


Figure 1a. Royal Egyptian cubit, pliable, in two segments.
Belonging to Kha, architect priest from Amenhotep II to Amenhotep III

Figure 1b. Royal Egyptian cubit, in a single segment.
Belonging to Maya, royal treasurer under Tutankhamun and Horemheb

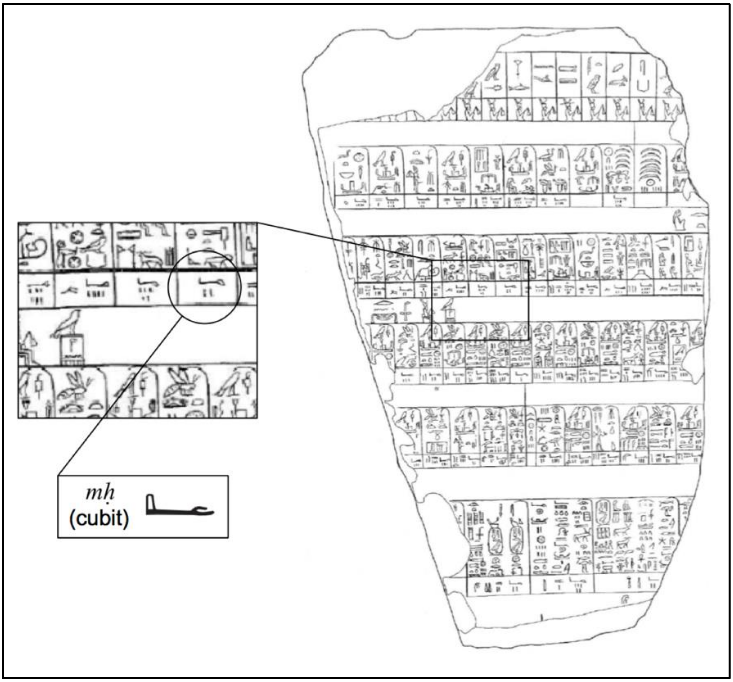
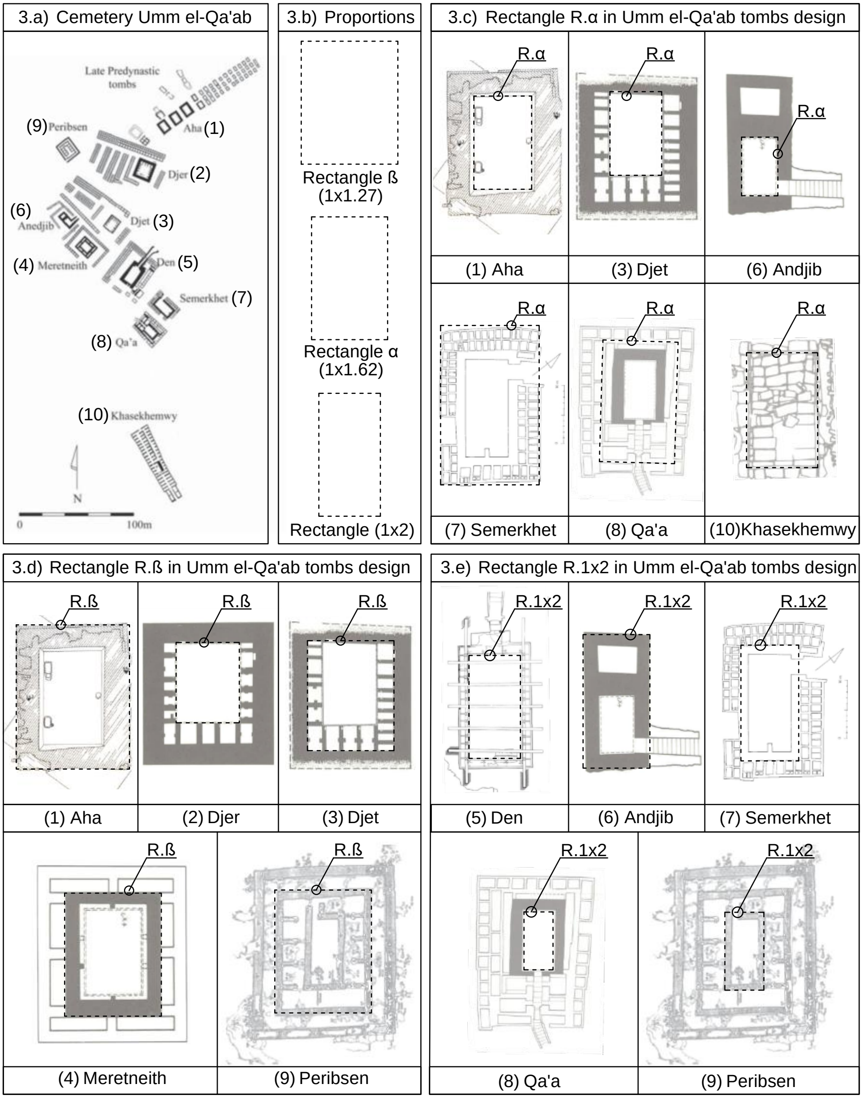


Figure 2. The Palermo Stone



Figures 3a, 3b, 3c, 3d and 3e. Pharaonic tombs of the First and Second Dynasty at Umm el-Qa'ab (without scale)

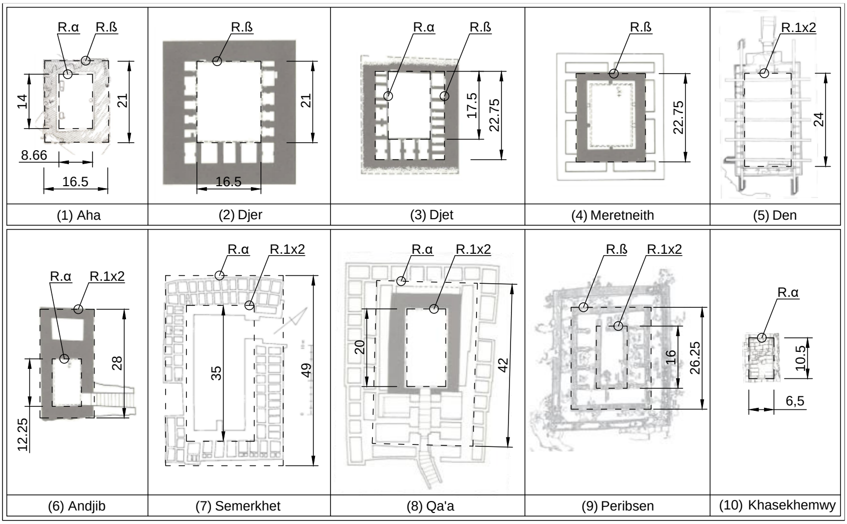


Figure 3f. Pharaonic tombs of the First and Second Dynasty at Umm el-Qa'ab in chronological order and scaled. Measurements in cubits

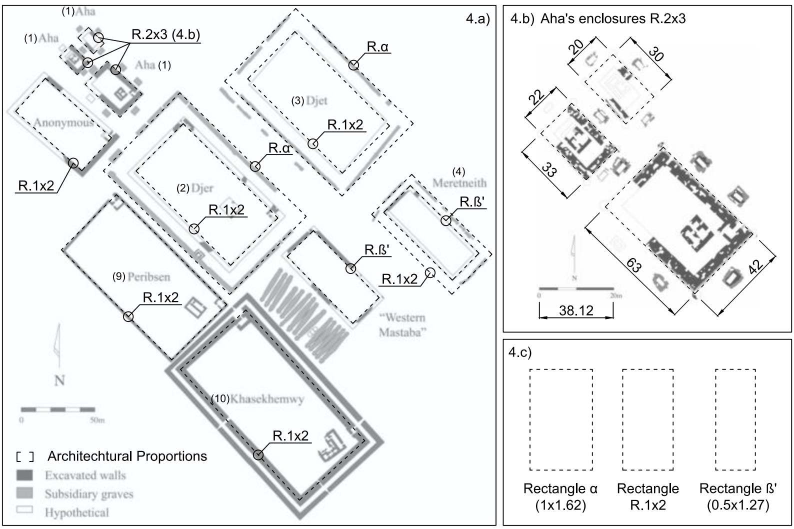


Figure 4. Funerary enclosures at Abydos

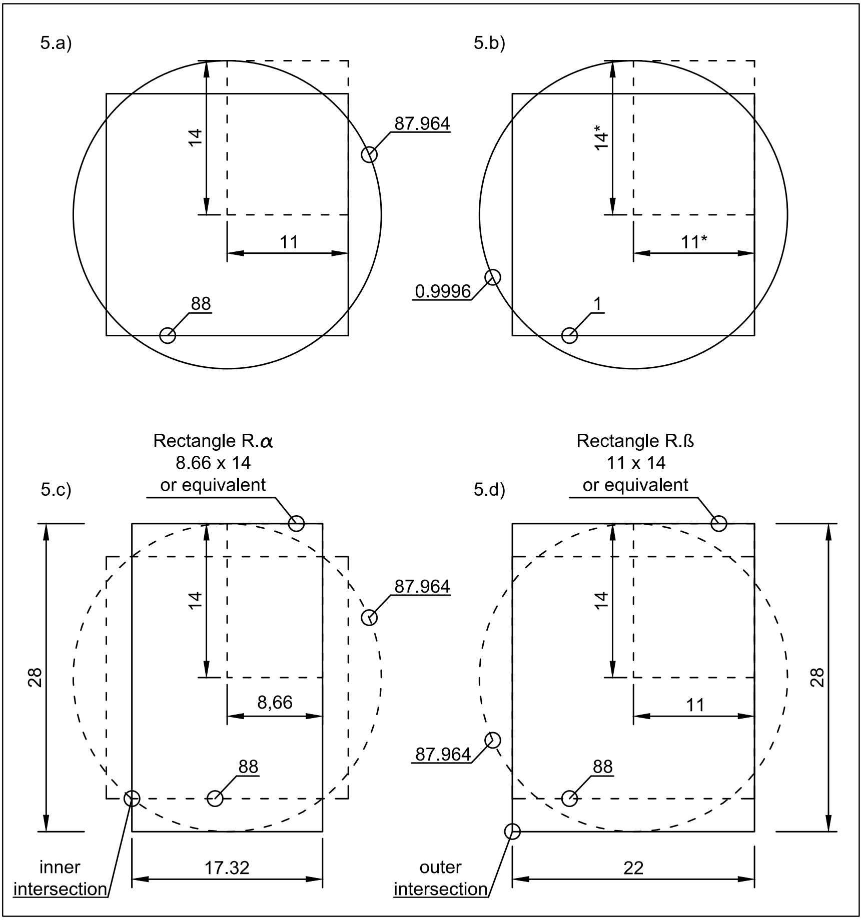


Figure 5. Geometrical pattern of the arch design (GPAD)
in royal buildings that emerged during the Thinite Age

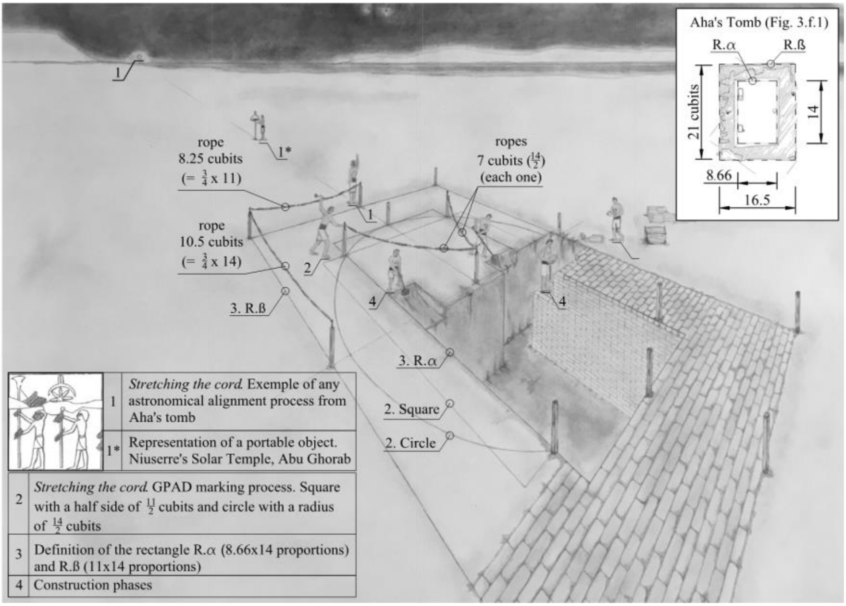


Illustration 1. Representation of the possible construction of Aha's tomb at Umm el-Qa'ab from the GPAD

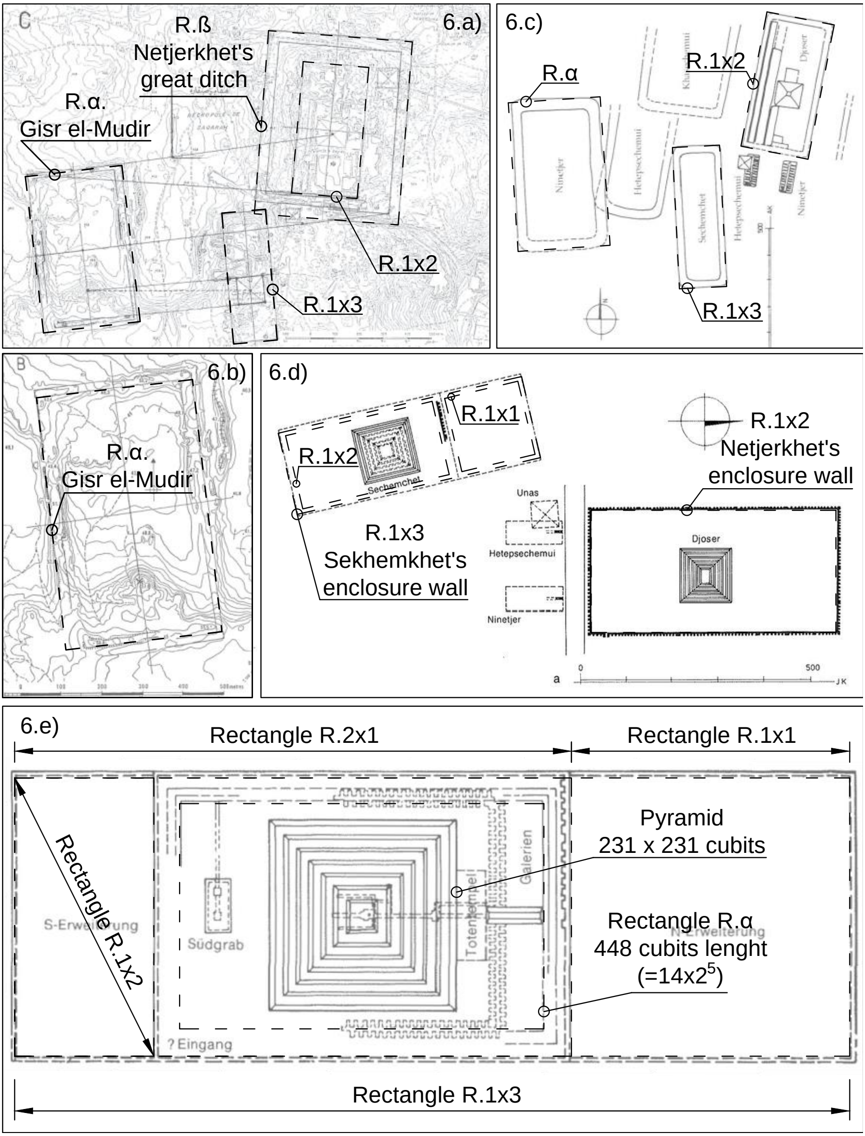


Figure 6. The GPAD in royal enclosures of the Third Dynasty

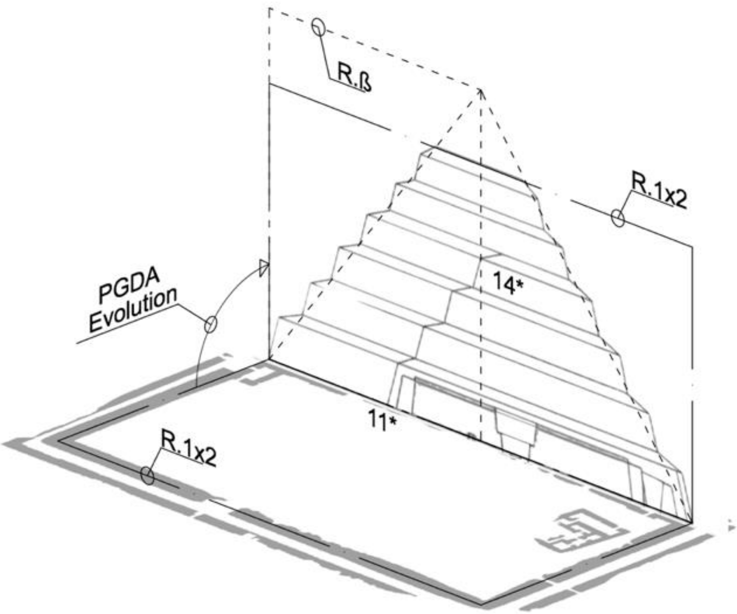


Figure 7. First evolution of the GPAD in the royal architecture of ancient Egypt

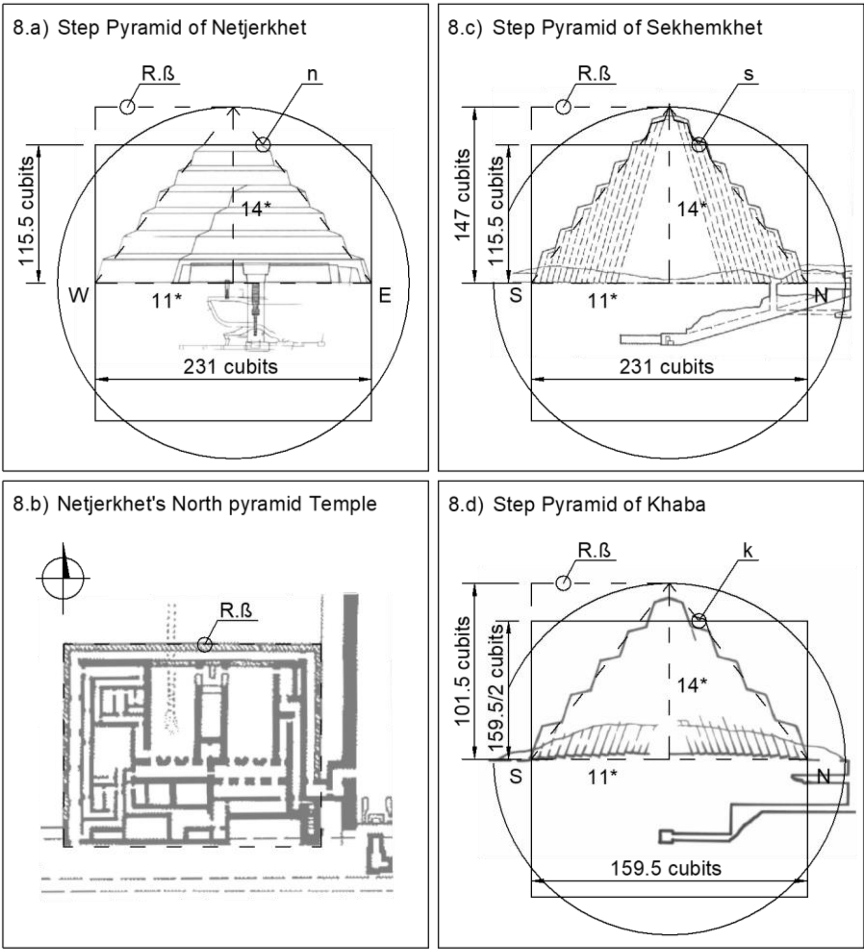
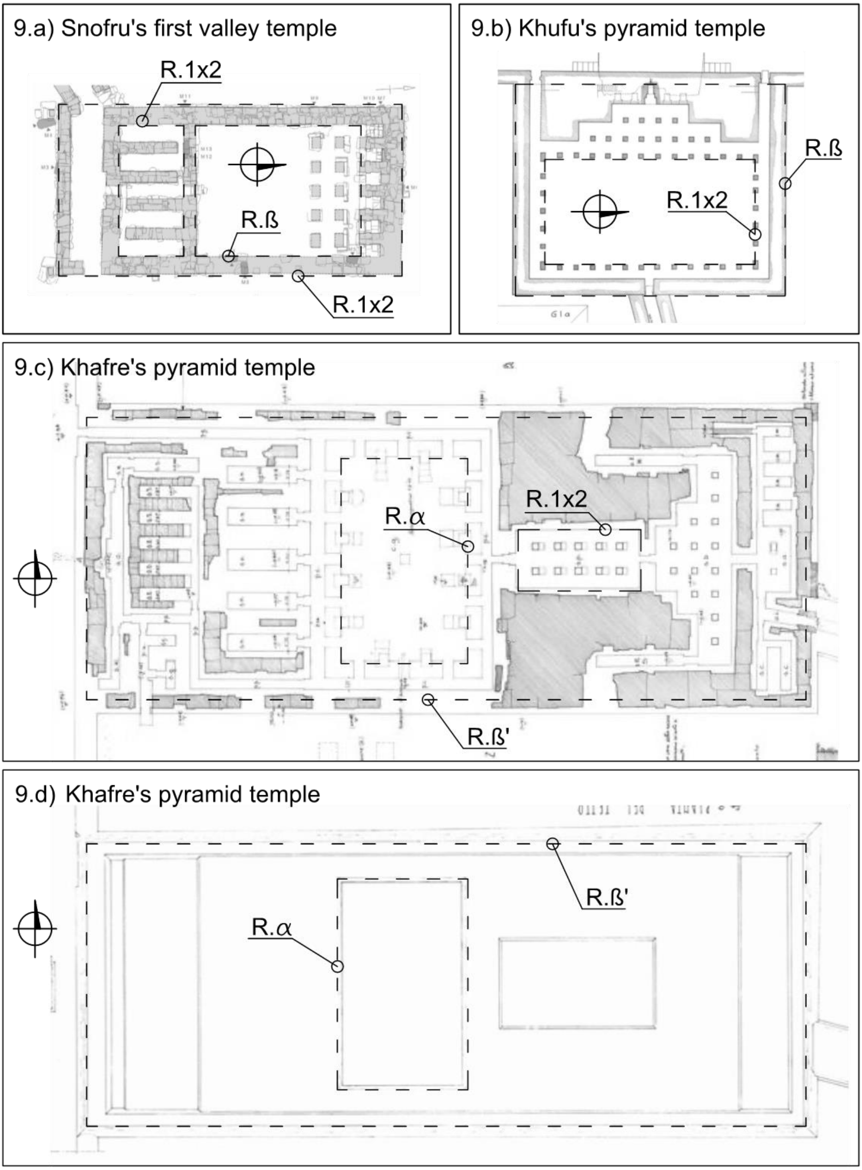
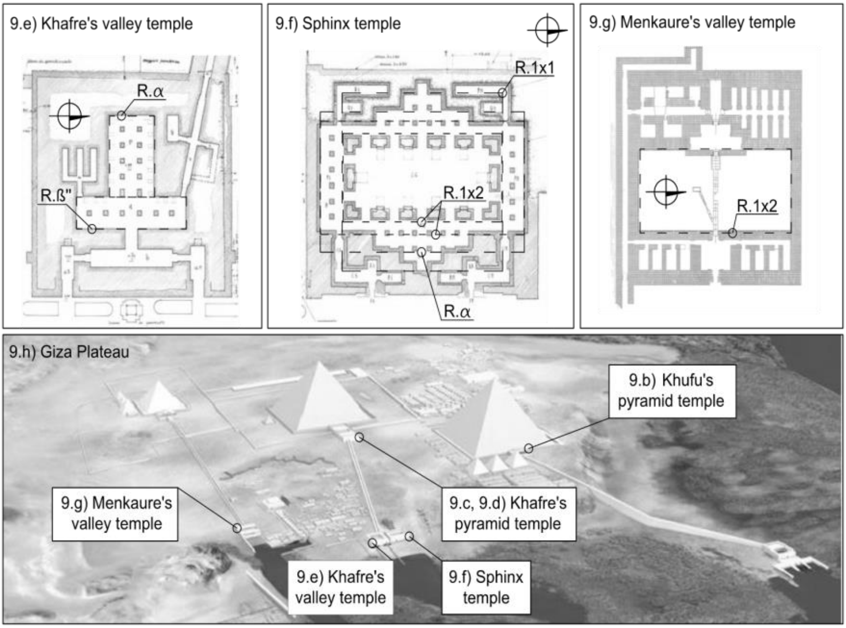


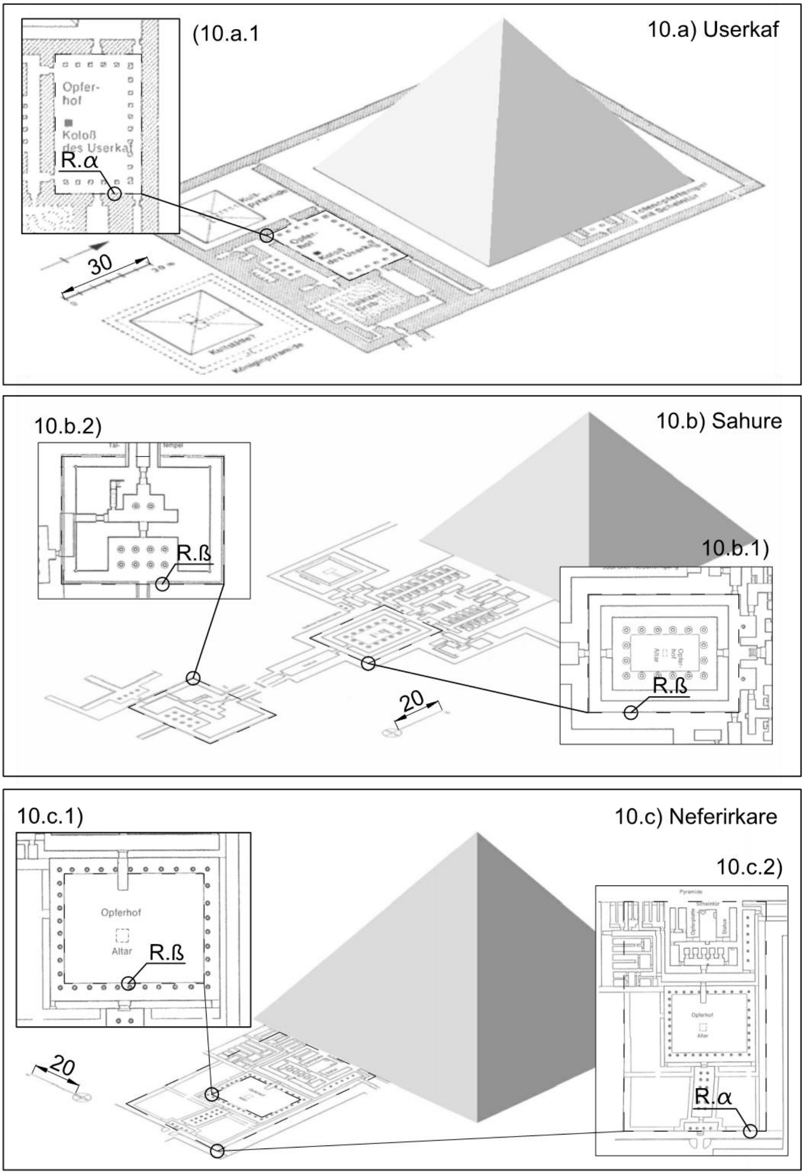
Figure 8. The GPAD in pyramids and temples of the Third Dynasty



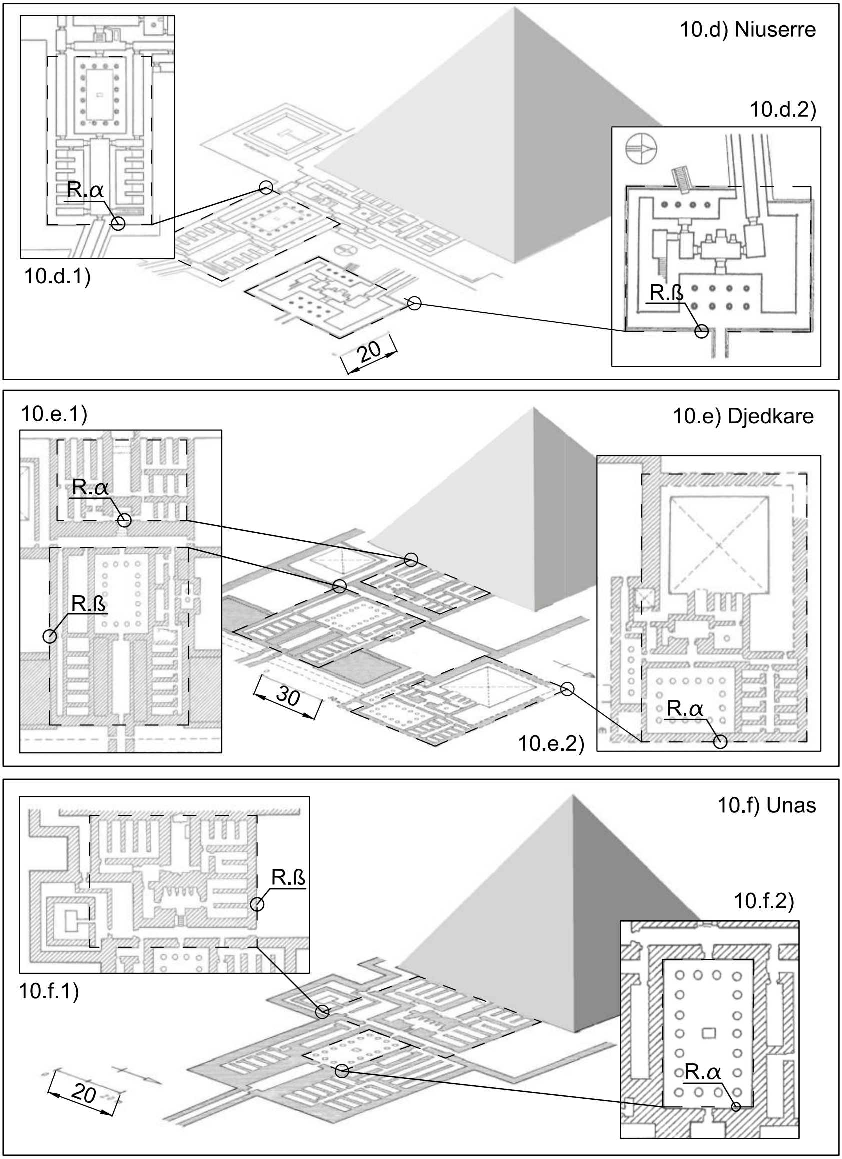
Figures 9a, 9b, 9c and 9d. The GPAD in the main pyramid temples and valley temples of the Fourth Dynasty



Figures 9e, 9f, 9g and 9h. The GPAD in the main pyramid temples and valley temples of the Fourth Dynasty



Figures 10a, 10b and 10c. The GPAD in the main pyramid temples
and valley temples of the Fifth Dynasty (measurements in m)



Figures 10d, 10e and 10f. The GPAD in the main pyramid temples and valley temples of the Fifth Dynasty (measurements in m)

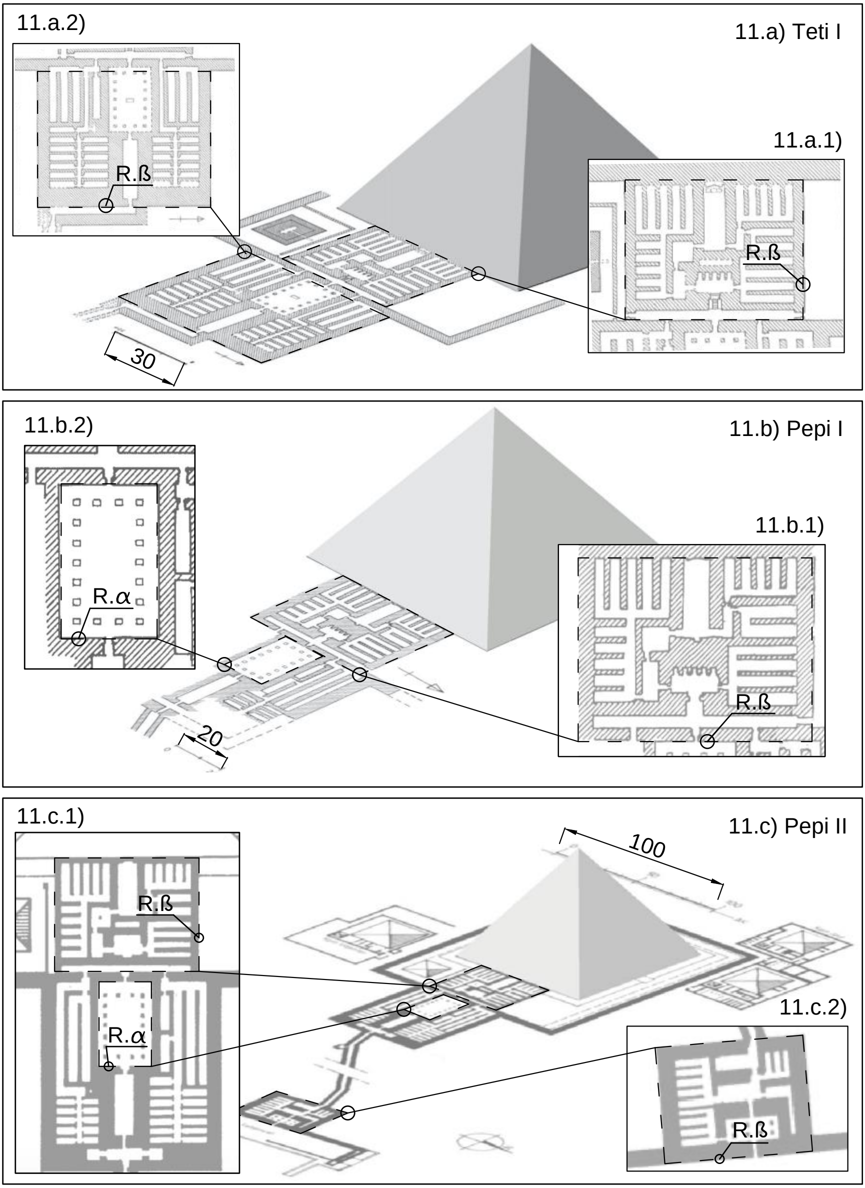


Figure 11. The GPAD in the main pyramid temples
and valley temples of the Sixth Dynasty (measurements in m)

	Regnal years BC (HKW: 2006)
First Dynasty	
Narmer	2900-?$^{+25}$
Aha-Hor	?-2870^{+25}
Djer	2870-2823^{+25}
Djet	2822-2815^{+25}
Meretneith	--
Den	2814-2772^{+25}
Andjib	2771-2764^{+25}
Semerkhet	2763-2756^{+25}
Qa'a	2755-2732^{+25}
Second Dynasty	
Hetepsekhemwy	2730-?$^{+25}$
Raneb	?-2700^{+25}
Ninetjer	2700-2660^{+25}
Wenedj	--
Sekhemib	2650-?$^{+25}$
Senedj	?-2610^{+25}
Peribsen	2660-2650^{+25}
Khasekhemwy	2610-2596^{+25}
Third Dynasty	
Netjerkhet	2592-2566^{+25}
Sekhemkhet	2565-2559^{+25}
Khaba	2559-?$^{+25}$
Sanakht	--
Huni	?-2544^{+25}

	Regnal years BC (HKW: 2006)
Fourth Dynasty	
Snofru	2543-2510^{+25}
Khufu	2509-2483^{+25}
Djedefre	2482-2475^{+25}
Bikheris	2474-2473^{+25}
Khafre	2472-2448^{+25}
Menkaure	2447-2442^{+25}
Shepseskaf	2442-2436^{+25}
Fifth Dynasty	
Userkaf	2435-2429^{+25}
Sahure	2428-2416^{+25}
Neferirkare	2415-2405^{+25}
Neferefre	2404-?$^{+25}$
Shepseskare	2403-?$^{+25}$
Niuserre	2402-2374^{+25}
Menkauhor	2373-2366^{+25}
Djedkare	2365-2322^{+25}
Unas	2321-2306^{+25}
Sixth Dynasty	
Teti	2305-2279^{+25}
Userkare	--
Pepi I	2276-2228^{+25}
Merenre	2227-2217^{+25}
Pepi II	2216-2153^{+25}
Nemtiemsaf II	2152-?$^{+25}$

Table 1. Chronology of the Thinite Age and the Old Kingdom

The royal tombs at Umm el-Qa'ab					
Pharaoh	Architectural Design		Bibliography	Measurements in cubits (1 cubit = 0.523 m) (widthxlength)	Proportions
	Mortuary chamber	Main wall			
Aha	R.α (1x1.62)		Kaiser and Dreyer 1982. Figure 1	8.66x14	**8.66x14**
Djer	R.ß (1x1.27)		Kaiser and Dreyer 1982: Figure 12	16.5x21	(**11**+11/2) x (**14**+14/2)
Djet	R.α (1x1.62)		Kaiser and Dreyer 1982: Figure 12	10.83x17.5	(**8.66**+8.66/4) x (**14**+14/4)
		R.ß (1x1.27)		17.875x22.75	(**11**+11x5/8) x (**14**+14x5/8)
Meretneith	R.ß (1x1.27)		Kaiser and Dreyer. 1982: Figure 12	17.875x22.75	(**11**+11x5/8) x (**14**+14x5/8)
Den	R.1x2		Dreyer and others 1998: 144	12x24	
Andjib		R.1x2	Kaiser and Dreyer. 1982: Figure 12	14x28	**14** x **14**x2
	R.α (1x1.62)		Kaiser and Dreyer. 1982: Figure 12	7.58x12.25	(**8.66**x7/8) x (**14**x7/8)
Semerkhet	R.1x2		Engel. 2008: Tomb U, after Dreyer. 2003	17.5x35	**14**x5/4 x **14**x5/2
		R.α (1x1.62)		30.31x49	(**8.66**x7/2) x (**14**x7/2)
Qa'a	R.1x2		Kaiser and Dreyer. 1982: Figure 12	10x20	
		R.α (1x1.62)		26x42	(**8.66**x3) x (**14**x3)
Peribsen		R.1x2		8x16	
	R.ß (1x1.27)		Dreyer and others. 2000: Figure 26.b.	20.625x26.25	(**11**+11x7/8) x (**14**+14x7/8)
Khasekhemwy	R.α (1x1.62)		Dreyer. 2003: Figure 18.	6.5x10.5	(**8.66**+8.66/4) x (**14**+14/4)

Table 2. The royal tombs at Umm el-Qa'ab

Funerary Temple at Abydos					
	Inner face of the wall	Outer face of the wall	Central axis of the wall	Perimeter wall	Measurements and proportions (in cubits)
Aha		R.2x3			42 x 63 (**14**x3) x (**14**x4.5)
		R.2x3			22 x 33 (**11**x2) x (**11**x3)
		R.2x3			20 x 30 (2x10) x (3x10)
Djer	R.1x2			R.α	
Djet	R.1x2			R.α	
Meretneith		R.ß'		R.1x2	
W. Mastaba	R.ß'				
Anonymous	R.1x2				
Peribsen			R.1x2		101.5 x 203
					(**14**x14+14/2)/2 x (**14**x14+14/2)
Khasekhemwy			R.1x2		115.5 x 231
					(**14**x16/2+14/2)/2 x (**14**x16+14/2)
					(**11**x21)/2 x (**11**x21)

Table 3. Funerary enclosures at Abydos

Pharaoh	Building	Geometric Pattern
Ninetjer	Great empty enclosure (Gisr el-Mudir)	R.α
Netjerkhet	Great empty enclosure	R.ß
	Enclosure wall	R.1x2
	Northern temple	R.ß
	Pyramid	R.ß*
Sekhemkhet	Great enclosure wall	R.1x3
	Pyramid	R.ß*
	Main pyramid enclosure	R.α
Khaba	Pyramid	R.ß*

Table 4. Royal architecture in the Third Dynasty.
R.ß refers to the diagonal of an R.ß rectangle on the vertical plane (Figure 8)

Pyramid (chronological order)	Place	Slope (Magli)	Measurements in cubits (Lehner) (1 cubit = 0.523 m)	Measurements in cubits (Testa)
SNOFRU PYRAMID	Meidum	11 x 14	275 x 175	280x178 (1c=0.515m) Tavola P.1.1
SNOFRU BEND PYRAMID	Dahshur	--	--	360 x 110 (1c=0.523m) Tavola P.2.1
SNOFRU RED PYRAMID	Dahshur	21 x 20	420 x 200	420 x 196 (14·30 x 14·12) (1c=0.523m) Tavola P.3.1
KHUFU	Giza	11 x 14	440 x 280	440 x 280 (1c=0.523m) Tavola P.4.1
DJEDEFRE	Abu Rowash	11 x 14	203.5 x 129.5	200 x 175 (1c= 0.5225 m) Tavola P.5.1
KHAFRE	Giza	3 x 4	411x274	410 x 273 (1c=0.525m) Tavola P.6.1
MENKAURE	Giza	4 x 5	200x125	200x125 (1c=0.528m) Tavola P.7.1
USERKAF	Saqqara	3 x 4	--	140 x 91 (14·10 x 14·13/2) Tavola P.10.1
SAHURE	Abu Sir	5 x 6	--	150 x 90 Tavola P.11.1
NEFERIRKARE	Abu Sir	3 x 4	--	200 x 140 (1c=0,520) Tavola P.12.1
NIUSERRE	Abu Sir	11 x 14	154 x 98	150 x 95 (1c=0.523 m) Tavola P.13.1
DJEDKARE	South. Saqqara	11 x 14	154 x 98	150 x 90 (1c=0.516) Tavola P.14.1
UNAS	Saqqara	2 x 3	--	110 x 88 (1c=0.522) Tavola P.15.1
TETI	Saqqara	3 x 4	150 x 110	150 x 110 (1c=0.522) Tavola P.16.1
PEPI I	South. Saqqara	3 x 4	--	--
MERENRE	South. Saqqara	3 x 4	--	--
PEPI II	South. Saqqara	3 x 4	150 x 100	150 x 110 (1c=0.525) Tavola P.17.1

Table 5. Fourth, Fifth and Sixth Dynasty pyramid slopes and measures

Pharaoh	Building	Burial chamber dimensions (Lehner 1997)				Testa
		Weight x Length (m)	Weight x Length (cubits)	Relation (Length/Weight)	Geometric Pattern correspondence	
Snofru	Meidum Pyramid. Burial Chamber	2.65x5.9	5x11	2.2	R.5x11	2.57x6.09 m Tavola P.1.3
	Bend Pyramid. Lower burial chamber	4.96x6.3	9.5x12	1.27	R.ß	4.97x6.28 m Tavola P.2.6
	Bend Pyramid. Lower burial chamber	5.26x7.97	10x15	1.5	R.2x3	5.24x8.07 m Tavola P.2.3
	Red Pyramid. Antechambers (x2)	3.65 x 8.36	7x16	2.29	R.4x9 = R.(2x2)x(3x3)	-- x 8.36 m Tavola P.3.1
	Red Pyramid. Burial chamber	4.18 x 8.55	8x16	2.04	R.1x2	4.18 x 8.36 m Tavola P.3.1
Khufu	Giza Pyramid. King's chamber	5.2 x 10.5	10x20	2.02	R.1x2	5.23 x 10.46 m Tavola P.4.7
	Giza Pyramid. Queen's chamber	5.3 x 5.8	10x11	1.1	R.10x11	5.23 x 5.75 m Tavola P.4.5
Khafre	Giza Pyramid. Burial chamber	5 x 14.15	10x27	2.83	R.10x27 = R.(11-1) x(14x2-1)	10x27 c (13c hight=14-1) Tavola P.6.3
	Giza Pyramid. Subsidiary chamber	3.12 x 10.41	6x20	3.33	R.3x10	6x20 Tavola P.6.3
Menkaure	Giza Pyramid. Antechamber	3.84 x 14.2	7.5x27	3.6	R.5x18 = R.5x(5x3+3)	3.87x14.2 c Tavola P.7.5
	Giza Pyramid. Antechamber	2.62 x 6.59	5x12	2.4	R.2x5	--
Shepseskaf	Mastaba el-Fara'un	74.4 x 99.6	142x190	1.33	R.3x4	74.44 x 99.26 Tavola P.9.1
	Mastaba el-Fara'un. Burial chamber	3.85 x 7.79	7x15	2.02	R.1x2	4.13 x 7.75 c Tavola P.9.4

Table 6. Measurements of the main chambers in the pyramids of the Fourth Dynasty

Pharaoh	Building	Burial chamber dimensions (Lehner 1997)				Testa
		Weight x Length (m)	Weight x Length (cubits)	Relation (Length/Weight) in meters	Geometric Pattern correspondence	
Userkaf	Pyramid of Saqqara. Antechamber	3.12x4.14	6x8	1.33	R.3x4	3.15x4.2 m Tavola P.10.2
	Pyramid of Saqqara. Burial chamber	3.13x7.87	6x15	2.51	R.2x5	--x7.89 Tavola P.10.2
Sahure	Pyramid of Abusir. Burial chamber	3.15x7.87	6x15	4	R.1x4	6x15 Tavola P11.2 R.2x5 (idem Niusere) Tavola P.13.2
Djedkare-Isesi	Pyramid of Saqqara. Antechamber	3.1x4.02	6x8	1.33	R.3x4	3.1x3.98 Tavola P.14.2
	Pyramid of South Saqqara. Burial chamber	3.1x7.84	6x15	2.52	R.2x5	--
Unas	Pyramid of Saqqara. Antechamber	3.08x3.75	6x7	1.17	R.6x7	6x7 c Tavola P.15.2
	Pyramid of Saqqara. Burial chamber	3.08x7.3	6x14	2.37	R.6x14	6x14 c Tavola P.15.2
Teti	Pyramid of Saqqara. Antechamber	3.12x3.75	6x7	1.17	R.6x7	6x7 c Tavola P.16.2
	Pyramid of Saqqara. Burial chamber	3.45x7.9	6.5x15	2.31	R.2x5	6x13.5 c Tavola P.16.2
Pepi II	Pyramid of Saqqara. Antechamber	3.15x3.69	6x7	1.17	R.6x7	6x7 c Tavola P.17.2
	Pyramid of South Saqqara. Burial chamber	3.15x7.9	6x15	2.5	R.2x5	6x15 c Tavola P.17.2

Table 7. Measurements of the main chambers in the pyramids of the Fifth and Sixth dynasties

Dynasty	Pharaoh	Temple	Architecture	Proportions GPAD	Testa
IV	Snofru	First valley temple	Courtyard	R.ß	40 x 48 Tavola P.2.10
			Group of chambers	R.1x2	18 x 40 Tavola P.2.10
			Wall temple	R.1x2	50 x 104 Tavola P.2.10
	Khufu	Pyramid temple	Courtyard	R.1x2	R.1x2 Tavola P.4.9 (7x14 columns)
			Wall temple	R.3x4	75x100 c Tavola P.4.9 (R.3x4)
	Khafre	Pyramid temple	Courtyard	R.α	-- Tavola P.6.9
			Group of chambers	R.1x2	-- Tavola P.6.9
			Wall temple	R.ß'	209x88 c Tavola P.6.7 (taking in account 1c=0,523) 210x88→R.ß'
		Valley temple	Antechamber	R.ß"	--
			Main chamber	R.α	--x31.5 R.α Tavola P.6.13
		Sphinx Temple	Main chamber	R.1x2	39x78 R.1x2 Tavola P.6.17
			Wall temple design	GPAD	88x88 Tavola P.6.17 14 c hight Tabola P.6.18
	Menkaure	Valley temple	Courtyard	R.1x2	R.1x2 Tavola P.7.9 (7x14 columns)

Table 8. Proportions of the main temples (Fourth Dynasty)

Dynasty	Pharaoh	Temple	Architecture	Proportions GPAD	Length Stadelmann (m)	Length GPAD proportions in cubits	Testa (width x length in cubits)
V	Userkaf	Pyramid temple	10.a.1) Courtyard	R.α	≈ 33	63 = 14x4 + 14/2	66.5 = 14x5 – 14/4 Tavole P.10.7
	Sahure	Pyramid temple	10.b.1) Courtyard	R.ß	≈ 33	63 = 14x4 + 14/2	61 (≈63)
		Valley temple	10.b.2) Wall temple	R.ß	≈ 36.5	70 = 14x5	70 = 14x5 Tavola P.11.14
	Neferirkare	Pyramid temple	10.c.1) Courtyard	R.ß	≈ 23.5	45.5 = 14x3 + 14/4	--
			10.c.2) Wall temple	R.ß	≈ 73	140 = 14x10	--
	Niuserre	Pyramid temple	10.d.1) Wall temple	R.α	≈ 47.5	91 = 14x6 + 14/2	90 (x 56) Tavola P.13.5 and P.13.7
		Valley temple	10.d.2) Wall temple	R.ß	≈ 40	**77** = 14x5 + 14/2	55 x **72** Tavola P.13.15
	Djedkare	Pyramid temple	10.e.1) First pyramid temple	R.α	≈ 40	**77** = 14x5 + 14/2	61.8 x **78.75** (taking into account 1c=0.523) Tabola P.14.5 → R.ß
			10.e.1) Second pyramid temple	R.ß	≈ 55	**105** = 14x7 + 14/2	65.25 x **99.25** (1c=0.516) Tavola P.14.5 → R.2x3
		Queen's pyramid complex	10.e.2) Wall complex	R.α	≈ 66	**126** = 14x9	75.7 x **122.5** (taking into account 1c=0.523) Tavola P.14.5 → R.α
	Unas	Pyramid temple	10.f.1) First pyramid temple	R.ß	≈ 40	**77** = 14x5 + 14/2	59.5 x **75.5**
			10.f.2) Courtyard	R.α	≈ 22	42 = 14x3	25 x 42 → R. α Tavola P.15.4

Table 9. Proportions and measurements of the main temples (Fifth Dynasty)

Dynasty	Pharaoh	Temple	Architecture	Proportions GPAD	Length (m)	Length (cubits&GPAD)	Testa
64	Teti	Pyramid temple	11.a.1) First pyramid temple	R.ß	≈ 44	**84** = 14x4	66 x **86** c → R.ß Tavola P.16.4
			11.a.2) Second pyramid temple	R.ß	≈ 62	**119** = 14x8 + 14/2	118 c → R.ß Tavola P.16.4
	Pepi I	Pyramid temple	11.b.1) First pyramid temple	R.ß	≈ 47.5	91 = 14x6 + 14/2	--
			11.b.2) Courtyard of the second pyramid temple	R.α	≈ 25.5	49 = 14x3 + 14/2	--
	Pepi II	Pyramid temple	11.c.1) First pyramid temple	R.ß	≈ 44	66 x **84** = 14x6	66 x **86** c Tavola P.17.5
			11.c.1) Courtyard of the second pyramid temple	R.α	≈ 25.5	**49** = 14x3 + 14/2	30 x **45** c Tavola P.17.5 R.2x3
		Valley temple	11.c.2) Wall temple	R.ß	≈ 44	**84** = 14x6	63.25 x **80.5** Tavola P.17.8

Table 10. Proportions and measurements of the main temples (Sixth Dynasty)