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A K-CONTACT SIMPLY CONNECTED 5-MANIFOLD WITH NO SEMI-REGULAR SASAKIAN STRUCTURE

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Abstract: We construct the first example of a 5-dimensional simply connected compact manifold that admits a K-contact structure but does not admit any semi-regular Sasakian structure. For this, we need two ingredients: (a) to construct a suitable simply connected symplectic 4-manifold with disjoint symplectic surfaces spanning the homology, all of them of genus 1 except for one of genus $g > 1$; (b) to prove a bound on the second Betti number b_2 of an algebraic surface with $b_1 = 0$ and having disjoint complex curves spanning the homology, all of them of genus 1 except for one of genus $g > 1$.

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1. Introduction

In geometry, it is a central question to determine when a given manifold admits an specific geometric structure. Complex geometry provides numerous examples of compact manifolds with rich topology, and there is a number of topological properties that have to be satisfied by complex manifolds. For instance, compact Kähler manifolds satisfy strong topological properties like the hard Lefschetz property, the formality of its rational homotopy type [**10**], or restrictions on the fundamental group [**1**]. A natural approach is to weaken the given structure and to ask to what extent a manifold having the weaker structure may admit the stronger one. In the case of Kähler manifolds, if we forget about the integrability of the complex structure, then we are dealing with symplectic manifolds. There has been enormous interest in the construction of (compact) symplectic manifolds that do not admit Kähler structures and in determining its topological properties [**29**]. In dimension 4, when we deal with complex surfaces, we have the Enriques–Kodaira classification [**4**] that helps in the understanding of this question.

In odd dimension, Sasakian and K-contact manifolds are analogues of Kähler and symplectic manifolds, respectively. Sasakian geometry has become an important and active subject, especially after the appearance of the fundamental treatise of Boyer and Galicki [**6**]. Chapter 7 of this book contains an extended discussion of topological problems of Sasakian and K-contact manifolds.

The precise definition of the structures that we are dealing with in this paper is as follows. Let M be a smooth manifold. A *K-contact structure* on M consists of tensors (η, J) such that η is a contact form $\eta \in \Omega^1(M)$, i.e. $\eta \wedge (d\eta)^n > 0$ everywhere, and J is an endomorphism of TM such that:

- $J^2 = -\operatorname{Id} + \xi \otimes \eta$, where ξ is the Reeb vector field of η, $i_\xi \eta = 1$, $i_\xi(d\eta) = 0$,
- $d\eta(JX, JY) = d\eta(X, Y)$ for all vector fields X, Y and $d\eta(JX, X) > 0$ for all nonzero $X \in \ker \eta$, and
- the Reeb field ξ is Killing with respect to the Riemannian metric $g(X, Y) = d\eta(JX, Y) + \eta(X)\eta(Y)$.

We may denote (η, J) or (η, J, g, ξ) a K-contact structure on M, since g and ξ are in fact determined by η and J. Note that the endomorphism J defines a complex structure on $\mathcal{D} = \ker \eta$ compatible with $d\eta$, hence J is orthogonal with respect to the metric $g|_{\mathcal{D}}$. By definition, the Reeb vector field ξ is orthogonal to $\mathcal{D}$. Finally, a *K-contact manifold* is (M, η, J, g, ξ), a manifold M endowed with a K-contact structure. For a K-contact manifold M, the condition that the Reeb vector field be Killing with respect to the metric g is very rigid and it imposes strong constraints on the topology. In particular, it is not difficult to find manifolds that admit contact but do not admit K-contact structures in any odd dimension, for instance the odd dimensional tori; see [**6**, Corollary 7.4.2]. For simply-connected 5-manifolds (i.e. Smale–Barden manifolds), one can also find infinitely many of them admitting contact but not K-contact structures; see [**6**, Theorem 10.2.9 and Corollary 10.2.11].

Just as for almost complex structures, there is the notion of integrability of a K-contact structure. More precisely, a K-contact structure (η, J, g, ξ) is called *normal* if the Nijenhuis tensor N_J associated to the tensor field J, defined by

$$N_J(X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY],$$

satisfies the equation $N_J = -d\eta \otimes \xi$. A *Sasakian structure* on M is a normal K-contact structure (η, J, g, ξ) and we call (M, η, J, g, ξ) a *Sasakian manifold*.

Let (M, η, J, g, ξ) be a K-contact manifold. Consider the cone as the Riemannian manifold $C(M) = (M \times \mathbb{R}_+, t^2 g + dt^2)$. One defines an almost complex structure I on $C(M)$ by:

- $I(X) = J(X)$ on $\ker \eta$, and
- $I(\xi) = t\frac{\partial}{\partial t}$, $I\big(t\frac{\partial}{\partial t}\big) = -\xi$, for the Reeb vector field ξ of η.

Then (M, η, J, g, ξ) is Sasakian if and only if I is integrable, that is, if $(C(M), I, t^2 g + dt^2)$ is a Kähler manifold; see [**6**, Definition 6.5.15].

Slightly abusing notation, if we are given a smooth manifold M with no specified contact structure, we will say that M is K-contact (Sasakian) if it admits some K-contact (Sasakian) structure. In this paper we will mainly be concerned with geography questions, i.e. which smooth manifolds admit K-contact or Sasakian structures.

In dimension 3, every K-contact manifold admits a Sasakian structure [**17**]. For dimension greater than 3, there is much interest on constructing K-contact manifolds which do not admit Sasakian structures. The odd Betti numbers up to degree n of Sasakian $(2n+1)$-manifolds must be even. The parity of b_1 was used to produce the first examples of K-contact manifolds with no Sasakian structure [**6**, Example 7.4.16]. In the case of even Betti numbers, more refined tools are needed to distinguish K-contact from Sasakian manifolds. The cohomology algebra of a Sasakian manifold satisfies a hard Lefschetz property [**9**]. Using it examples of K-contact non-Sasakian manifolds are produced in [**8**] in dimensions 5 and 7. These examples are nilmanifolds with even Betti numbers, so in particular they are not simply connected.

When one moves to simply connected manifolds, K-contact non-Sasakian examples of any dimension ≥ 9 were constructed in [**16**] using the evenness of b_3 of a compact Sasakian manifold. Alternatively, using the hard Lefschetz property for Sasakian manifolds there are examples [**20**] of simply connected K-contact non-Sasakian manifolds of any dimension ≥ 9. In [**5, 28**] the rational homotopy type of Sasakian manifolds is studied. All higher order Massey products for simply connected Sasakian manifolds vanish, although there are Sasakian manifolds with non-vanishing triple Massey products [**5**]. This yields examples of simply connected K-contact non-Sasakian manifolds in dimensions ≥ 17. However, Massey products are not suitable for the analysis of lower dimensional manifolds.

The problem of the existence of simply connected K-contact non-Sasakian compact manifolds (Open Problem 7.4.1 in [**6**]) is still open in dimension 5. It was solved for dimensions ≥ 9 in [**9, 8, 16**] and for dimension 7 in [**22**] by a combination of various techniques based on the homotopy theory and symplectic geometry. In the least possible dimension the problem appears to be much more difficult. A simply connected compact oriented 5-manifold is called a *Smale–Barden manifold.* These manifolds are classified topologically by $H_2(M,\mathbb{Z})$ and the second Stiefel–Whitney class; see [**3, 26**] for the classification by Smale and Barden. Chapter 10 of the book [**6**] by Boyer and Galicki is devoted to a description of some Smale–Barden manifolds which carry Sasakian structures. The following problem is still open [**6**, Open Problem 10.2.1].

Do there exist Smale–Barden manifolds which carry K-contact but do not carry Sasakian structures?

In the pioneering work [**21**] a first step towards a positive answer to the question is taken. A homology Smale–Barden manifold is a compact 5-dimensional manifold with $H_1(M,\mathbb{Z}) = 0$. A Sasakian structure is *regular* if the leaves of the Reeb flow are a foliation by circles with the structure of a circle bundle over a smooth manifold. The Sasakian structure is *quasi-regular* if the foliation is a Seifert circle bundle over a (cyclic) orbifold, and it is *semi-regular* if the base orbifold has only locus of non-trivial isotropy of codimension 2, i.e. its underlying space is a topological manifold. Recall that the isotropy locus of an orbifold is the subset of points with non-trivial isotropy group. It is a remarkable result, although not difficult to prove, that any manifold admitting a Sasakian structure has also a quasi-regular Sasakian structure (in any odd dimension). Therefore, a Sasakian manifold is a Seifert bundle over a cyclic Kähler orbifold [**21**].

Correspondingly, for K-contact manifolds we also define regular, quasi-regular, and semi-regular K-contact structures with the same conditions. Any K-contact manifold admits a quasi-regular K-contact structure by [**6**, Theorem 7.1.10] and [**25**]. Hence, a K-contact manifold is a Seifert bundle over a cyclic symplectic orbifold. Such orbifold has a isotropy locus which is a (stratified) collection of symplectic suborbifolds. The K-contact structure is semi-regular if the symplectic orbifold has isotropy locus of codimension 2. The main result of [**21**] is:

Theorem 1 ([**21**])**.** *There exists a homology Smale–Barden manifold which admits a semi-regular K-contact structure but which does not carry any semi-regular Sasakian structure.*

The construction of [**21**] relies upon subtle obstructions to admit Sasakian structures in dimension 5 found by Kollár [**18**]. If a 5-dimensional manifold M has a Sasakian structure, then it is a Seifert bundle over a Kähler orbifold X with isotropy locus a collection of complex curves D_i with isotropy (multiplicity) m_i. We have the following topological characterization of the homology of M in terms of that of X.

Theorem 2 ([**21**, Theorem 16]). *Suppose that $\pi\colon M \to X$ is a semi-regular Seifert bundle with isotropy surfaces D_i with multiplicities m_i. Then $H_1(M,\mathbb{Z}) = 0$ if and only if*

(1) *$H_1(X,\mathbb{Z}) = 0$,*
(2) *the map $H^2(X,\mathbb{Z}) \to \oplus_i H^2(D_i,\mathbb{Z}/m_i)$ induced by the inclusions $D_i \subset X$, is surjective, and*
(3) *the Chern class $c_1(M/e^{2\pi i/\mu}) \in H^2(X,\mathbb{Z})$ of the circle bundle $M/e^{2\pi i/\mu}$ is a primitive element, where μ is the lcm of all m_i.*

Moreover, $H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus (\mathbb{Z}/m_i)^{2g_i}$, g_i =genus of D_i, $k+1 = b_2(X)$.

Recall that an element x of a $\mathbb{Z}$-module is called *primitive* if it is not of the form $x = ny$ for some integer $n > 1$.

Corollary 3 ([**21**, Corollary 18]). *Suppose that M is a 5-manifold with $H_1(M,\mathbb{Z}) = 0$ and $H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus_{i=1}^{k+1} (\mathbb{Z}/p^i)^{2g_i}$, $k \geq 0$, p a prime, and $g_i \geq 1$. If $M \to X$ is a semi-regular Seifert bundle, then $H_1(X,\mathbb{Z}) = 0$, $H_2(X,\mathbb{Z}) = \mathbb{Z}^{k+1}$, and the ramification locus has $k+1$ disjoint surfaces D_i linearly independent in rational homology, and of genus $g(D_i) = g_i$.*

In [**21**, Theorem 23] the authors construct a symplectic 4-dimensional orbifold with disjoint symplectic surfaces spanning the second homology. This is the first example of such phenomenon and has $b_2 = 36$. The genera of the isotropy surfaces satisfy $1 \leq g_i \leq 3$, with several of them having genus 3. Using this symplectic orbifold X, we obtain a semi-regular K-contact 5-manifold M with

$$H_1(M,\mathbb{Z}) = 0, \quad H_2(M,\mathbb{Z}) = \mathbb{Z}^{35} \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p^i)^{2g_i}. \tag{1}$$

For understanding the Sasakian side, the following result is proved in [**21**]:

Theorem 4 ([**21**, Theorem 32]). *Let S be a smooth Kähler surface with $H_1(S,\mathbb{Q}) = 0$ and containing $D_1, \ldots, D_b$, $b = b_2(S)$, smooth disjoint complex curves with $g(D_i) = g_i > 0$, and spanning $H_2(S,\mathbb{Q})$. Assume that:*

(1) *at least two g_i are > 1, and*
(2) *$1 \leq g_i \leq 3$.*

Then $b \leq 2\max\{g_i\} + 3$.

As a corollary [**21**, Proposition 31], there is no Sasakian semi-regular 5-dimensional manifold with homology given by (1). So M is K-contact, but does not admit any semi-regular Sasakian structure, proving Theorem 1.

Theorem 4 is a result in accordance with the following conjecture from [**21**]:

Conjecture 5. *There does not exist a Kähler manifold or a Kähler orbifold X with $b_1 = 0$ and with $b_2 \geq 2$ having disjoint complex curves spanning $H_2(X, \mathbb{Q})$, all of genus $g \geq 1$.*

The present work enhances the main result from [**21**] given in Theorem 1, to achieve a 5-manifold that it is furthermore simply connected. Our main result is the following:

Theorem 6. *There exists a (simply connected) Smale–Barden manifold which admits a semi-regular K-contact structure but which does not carry any semi-regular Sasakian structure.*

On the one hand, we provide a new construction of a symplectic 4-manifold X with $b_1 = 0$ and $b_2 = b > 1$, having a collection of disjoint symplectic surfaces $C_1, \ldots, C_b$ spanning $H_2(X, \mathbb{Q})$, and all with genus $g_i \geq 1$. This is based on the following phenomenon which can be performed in the symplectic setting but not in the algebro-geometric situation.

Start with the complex projective plane $\mathbb{C}P^2$ and two generic (smooth) complex cubic curves C_1, C_2. Note C_1 and C_2 have genus 1 by the genus-degree formula, and they intersect in nine points $P_1, \ldots, P_9$. A third complex cubic curve passing through $P_1, \ldots, P_8$ has to go necessarily through P_9. This is a purely algebraic phenomenon. However, it is possible to construct a third symplectic cubic C_3 going through $P_1, \ldots, P_8$, but intersecting C_1 at another point P_{10}, and C_2 at a different point P_{11}. Note that each C_i misses exactly one of the eleven points $P_1, \ldots, P_{11}$. Looking at this more symmetrically, we aim to have a collection of eleven points $\Delta = \{P_1, \ldots, P_{11}\}$ and eleven cubic complex curves $C_1, \ldots, C_{11}$ such that C_i passes through the points of $\Delta - \{P_i\}$, $i = 1, \ldots, 11$. In this way, the intersections are $C_i \cap C_j = \Delta - \{P_i, P_j\}$ and no more points. Blowing up at all points of Δ, we get the (symplectic) 4-manifold $X = \mathbb{C}P^2 \# 11 \overline{\mathbb{C}P}^2$, with eleven disjoint complex curves of genus 1. An extra (complex) curve can be obtained by taking a singular complex curve G of degree 10 with ordinary triple points at the points of Δ. Note that G has genus 3 by the Plücker formulas. Moreover, as $G \cdot C_i = 30$ equals the geometric intersection, that is, three times for each of the ten triple points in $G \cap C_i = \Delta - \{P_i\}$, we would not have more intersections.

This curve is of genus $g_G = 3$ and it becomes a smooth genus 3 curve in the blow-up, that is disjoint from the others. This heuristic argument has to be carried out in a slightly different guise, by making a symplectic construction in a tubular neighbourhood of a cubic curve and a complex line and gluing it in symplectically (see Section 2).

Theorem 7. *Let $P_1, \dots, P_{11}$ be eleven points in $\mathbb{C}P^2$. Then there exist symplectic surfaces*

$$C_1, C_2, \dots, C_{11}, G \subset \mathbb{C}P^2$$

such that:

(1) *C_i is a genus 1 smooth surface and $P_j \in C_i$ for $j \neq i$, $P_i \notin C_i$.*
(2) *The surfaces C_i, C_j, $i \neq j$, intersect exactly at $\{P_1, \dots, P_{11}\} - \{P_i, P_j\}$, positively and transversely.*
(3) *G is a genus 3 singular symplectic surface whose only singularities are eleven triple points at P_i (with different branches intersecting positively). Moreover G intersects each C_i only at the points P_j, $j \neq i$, and all the intersections of C_i with the branches of G are positive and transverse.*

Using this, we construct our K-contact 5-manifold. First we blow up $\mathbb{C}P^2$ at the eleven points $P_1, \dots, P_{11}$, to obtain a symplectic manifold, which topologically is $X = \mathbb{C}P^2 \# 11 \overline{\mathbb{C}P}^2$. The proper transforms of $C_1, \dots, C_{11}, G$ are symplectic surfaces in X, via the method in [**21**, Section 5.2]. The proper transform of G becomes a smooth genus 3 symplectic surface. Therefore $b_2(X) = 12$ and it has twelve disjoint symplectic surfaces, eleven of them of genus $g_i = 1$ and one of genus $g_{12} = 3$. Take numbers m_i. Using [**21**, Proposition 7], we make X into an orbifold X' whose isotropy locus is C_i with multiplicity m_i and G with multiplicity m_{12}. Then we can take a Seifert bundle $M \to X'$ with primitive Chern class $c_1(M/e^{2\pi i/\mu}) = [\omega]$ after a small perturbation of the symplectic form, as in [**21**, Lemma 20]. The manifold M is K-contact and has

$$H_1(M, \mathbb{Z}) = 0, \quad H_2(M, \mathbb{Z}) = \mathbb{Z}^{11} \oplus \bigoplus_{i=1}^{12} (\mathbb{Z}/m_i)^{2g_i}. \tag{2}$$

We choose a prime p and $m_i = p^i$, so that all m_i are distinct and pairwise non-coprime.

Given a Seifert bundle $M \to X'$, the fundamental group of M is directly related to the *orbifold fundamental group* of X' by the long exact sequence

$$\cdots \longrightarrow \pi_1(S^1) = \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1^{\mathrm{orb}}(X') \longrightarrow 1.$$

When $\pi_1^{\mathrm{orb}}(X') = 1$, we have that $\pi_1(M)$ is abelian, and hence if $H_1(M,\mathbb{Z}) = 0$, then M is simply connected. We prove the following in Section 4.

Theorem 8. *For the orbifold X' constructed above, $\pi_1^{\mathrm{orb}}(X') = 1$. Hence M is a Smale–Barden manifold.*

On the other hand, we have to prove that M cannot admit a semi-regular Sasakian structure. If this were the case, then there would be a Seifert bundle $M \to Y$, where Y is a Kähler orbifold. By [**21**, Proposition 10], this orbifold Y is a complex manifold, and as the Sasakian structure is semi-regular, Y is smooth. As the homology of M is given by (2), then Corollary 3 guarantees that Y has $b_1 = 0$, $b_2 = 12$, and contains twelve disjoint smooth complex curves $C'_1, \ldots, C'_{11}, G'$, where $g(C'_i) = 1$ and $g(G') = 3$. We prove the corresponding instance of Conjecture 5. Note that this is not covered by Theorem 4.

Theorem 9. *Let S be a smooth complex surface with $H_1(S,\mathbb{Q}) = 0$ and containing $D_1, \ldots, D_b$, $b = b_2(S)$, smooth disjoint complex curves with genus $g(D_i) = g_i > 0$, and spanning $H_2(S,\mathbb{Q})$. Assume that $g_i = 1$, for $1 \leq i \leq b-1$. Then $b \leq 2g_b^2 - 4g_b + 3$.*

In particular, the case $b_2 = 12$, $g_i = 1$, for $1 \leq i \leq 11$ and $g_{12} = 3$, cannot happen.

Corollary 10. *Let M be a 5-dimensional manifold with $H_1(M,\mathbb{Z}) = 0$ and*

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^{11} \oplus \bigoplus_{i=1}^{12} (\mathbb{Z}/p^i)^{2g_i},$$

where $g_i = 1$ for $1 \leq i \leq 11$, $g_{12} = 3$, and p is a prime number. Then M does not admit a semi-regular Sasakian structure.

This proves Theorem 6. It remains to see Theorems 7, 8, and 9. We prove Theorem 7 in Section 3, Theorem 8 in Section 4, and Theorem 9 in Section 5.

The manifold M in Corollary 10 is spin if $p = 2$, and can be chosen to be spin or non-spin if $p > 2$.

Both here and in [**21**] we have provided the first examples of symplectic 4-manifolds containing symplectic surfaces of positive genus and spanning the homology. Whereas the example of [**21**] is a symplectic 4-manifold that does not admit a complex structure (see Remark 32), the manifold constructed here, $X = \mathbb{C}P^2 \# 11 \overline{\mathbb{C}P}^2$ does admit a Kähler structure. So X is symplectic deformation equivalent to a Kähler manifold, but the twelve symplectic surfaces inside it cannot be deformed to complex curves in the way. We thank Roger Casals from prompting this question to us.

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2. Symplectic plumbing

The specific aim of this section is to give suitable local models for a small neighborhood of a union of two positively intersecting symplectic surfaces inside a 4-manifold. See references [**12, 13**] for related content.

2.1. Definition of symplectic plumbing. Let (S,ω) be a compact symplectic surface and $\pi\colon E\to S$ be a complex line bundle. Topologically, E is determined by the Chern class $d=c_1(E)$ which is the self-intersection of S inside E, $d=[S]^2$. We put a hermitian structure in E, so we can define a neighbourhood via a disc bundle of some fixed radius $c>0$, denoted by $B_c(S)\subset E$. We construct a symplectic form on $B_c(S)$ next. First, we write $V'\Subset V$ if V' is an open subset such that its closure $\overline{V'}\subset V$.

Lemma 11. *For small enough $c>0$, $B_c(S)$ admits a symplectic form ω_E which is compatible with the complex structure of the fibers of the complex line bundle, and such that the inclusion $(S,\omega)\hookrightarrow(B_c(S),\omega_E)$ is symplectic. If $V\subset S$ is a trivializing open set, $E|_V\cong V\times\mathbb{C}$, and $V'\Subset V$, we can arrange that $\omega_E|_{B_c(S)\cap E|_{V'}}$ is the symplectic product structure on $B_c(S)\cap E|_{V'}\cong V'\times B_c(0)$, with $B_c(0)\subset\mathbb{C}$ a ball centered at 0.*

Proof: Take $S=\bigcup_\alpha U_\alpha$ a cover of S, with each U_α symplectomorphic to a ball, and trivializations $E|_{U_\alpha}\cong U_\alpha\times\mathbb{C}$. In the fiber $\mathbb{C}$ we put coordinates $u+iv$ and consider the standard symplectic form $\omega_0=du\wedge dv=d(udv)=d\eta$. Denote $\varpi_\alpha\colon U_\alpha\times\mathbb{C}\to\mathbb{C}$ the projection over the second factor, and take ρ_α a smooth partition of unity subordinated to the cover U_α of S. Define

$$\omega_E=\pi^*\omega_S+\sum_\alpha d((\pi^*\rho_\alpha)\cdot(\varpi_\alpha^*\eta)).$$

For $x\in S$, we have $\omega_E|_{E_x}=\sum_\alpha\rho_\alpha(x)\omega_0=\omega_0$, using that the changes of trivializations preserve ω_0. Then using the decomposition $T_xE=T_xS\oplus E_x$, we have that $(\omega_E)^2(x)=\omega_S(x)\wedge\omega_0>0$. Therefore ω_E is symplectic on the zero section $S \subset E$. Since this is an open condition, it holds in some neighborhood $B_c(S)$ of the zero section.

For the last part, just take an open cover U_α of $S - V'$ together with V in the construction above. □

The submanifold $S \subset E$ and any fiber $E_x \subset E$ are symplectic, and they are symplectically orthogonal.

Now we move to the definition of plumbing as a symplectic neighbourhood of the union of two intersecting symplectic surfaces S_1, S_2. Take points $P_1, \ldots, P_m \in S_1$ and $Q_1, \ldots, Q_m \in S_2$. We define

$$S = S_1 \sqcup S_2 / P_i \sim Q_i, \quad i = 1, \ldots, m,$$

and we can write $S = S_1 \cup S_2$. Let now $E_1 \to S_1$ and $E_2 \to S_2$ be two complex line bundles, where $d_i = c_1(E_i)$ is the self-intersection of S_i inside E_i, $d_i = S_i^2$. Take hermitian metrics on the line bundles, so that $B_c(S_1) \subset E_1$ and $B_c(S_2) \subset E_2$ are symplectic manifolds for $c > 0$ by using Lemma 11.

For each $i = 1, \ldots, m$, take small neighbourhoods $B(P_i) \subset S_1$, symplectomorphic to the ball $B_c(0)$ via $f_{1i} \colon B(P_i) \to B_c(0)$. Take a trivialization $\varphi_{1i} \colon E_1|_{B(P_i)} \xrightarrow{\cong} B(P_i) \times \mathbb{C}$. Therefore we have

$$(f_{1i} \times \mathrm{Id}) \circ \varphi_{1i} \colon B_c(S_1) \cap E_1|_{B(P_i)} \xrightarrow{\cong} B_c(0) \times B_c(0). \tag{3}$$

Using Lemma 11, we endow E_1 with a 2-form ω_{E_1} such that $(B_c(S_1), \omega_{E_1})$ is symplectic and the symplectic form is a product on $B_c(S_1) \cap E_1|_{B(P_i)}$. This means that (3) is a symplectomorphism. We do the same for $Q_i \in S_2$, obtaining a symplectomorphism $f_{2i} \colon B(Q_i) \to B_c(0)$, a trivialization $\varphi_{2i} \colon E_2|_{B(Q_i)} \xrightarrow{\cong} B(Q_i) \times \mathbb{C}$, a symplectic form ω_{E_2} on $B_c(S_2)$, and a symplectomorphism

$$(f_{2i} \times \mathrm{Id}) \circ \varphi_{2i} \colon B_c(S_2) \cap E_2|_{B(Q_i)} \xrightarrow{\cong} B_c(0) \times B_c(0).$$

Let $R \colon B_c(0) \times B_c(0) \to B_c(0) \times B_c(0)$, $R(z_1, z_2) = (z_2, z_1)$, be the map reversal of coordinates, which is a symplectomorphism swapping horizontal and vertical directions. Then we take the gluing map

$$\Phi_i = ((f_{2i} \times \mathrm{Id}) \circ \varphi_{2i})^{-1} \circ R \circ ((f_{1i} \times \mathrm{Id}) \circ \varphi_{1i}) \colon \\ B_c(S_1) \cap E_1|_{B(P_i)} \to B_c(S_2) \cap E_2|_{B(Q_i)}.$$

Definition 12. We define the *symplectic plumbing* $P_c(S_1 \cup S_2)$ of $S = S_1 \cup S_2$ as the symplectic manifold

$$X = (B_c(S_1) \sqcup B_c(S_2))/x \sim \Phi_i(x), \quad x \in B_c(S_1) \cap E_1|_{B(P_i)}, \quad i = 1, \ldots, m.$$

Note that $S_1 \cup S_2 \subset P_c(S_1 \cup S_2)$ are symplectic submanifolds and they intersect transversely.

2.2. Symplectic tubular neighbourhood. We need a symplectic tubular neighbourhood theorem for two intersecting surfaces $S_1 \cup S_2$. We start with the case of a single submanifold. We include the proof since our result is a minor modification of the one appearing in the literature.

Proposition 13 (Symplectic tubular neighborhood)**.** *Suppose that (X, ω) and (X', ω') are two symplectic 4-manifolds (maybe open) with compact symplectic surfaces $S \subset X$ and $S' \subset X'$. Suppose that S and S' are symplectomorphic as symplectic manifolds via $f \colon S \to S'$, and assume also that their normal bundles are smoothly isomorphic.*

Let V, V' be tubular neigbourhoods of S and S' with projections $\pi \colon V \to S$, $\pi' \colon V' \to S'$, and let $g \colon V \to V'$ be a diffeomorphism of tubular neighbourhoods of S and S' with $g|_S = f$. Let $W \subset S$, $W' \subset S'$ be such that $g|_{\pi^{-1}(W)} \colon \pi^{-1}(W) \to \pi'^{-1}(W')$ is a symplectomorphism. Suppose that $H^1(W) = 0$, and let $\hat{W} \Subset W$. Then there are tubular neighborhoods $S \subset U \subset X$ and $S' \subset U' \subset X'$ which are symplectomorphic via $\varphi \colon U \to U'$, where $\varphi|_S = f$ and $\varphi|_{U \cap \pi^{-1}(\hat{W})} = g$.

Proof: This is an extension of the symplectic tubular neighbourhood theorem [**7**], which is the case where W is empty. Let $g \colon V \to V'$ be the diffeomorphism of tubular neighbourhoods where $g|_S = f$. We start by isotopying g so that $d_x g \colon T_x X \to T_{g(x)} X'$ is a linear symplectic map for all $x \in S$. We do this without modifying g on $\pi^{-1}(\hat{W})$, since g is symplectic there. Then the symplectic orthogonal to $T_x S \subset T_x V$ is sent to the symplectic orthogonal to $T_{f(x)} S' \subset T_{f(x)} V'$.

We take $\omega_0 = \omega$ and $\omega_1 = g^* \omega'$ and note that $i^*(\omega_1 - \omega_0) = 0$, where $i \colon S \to V$ is the inclusion map. As $i^* \colon H^2(V) \to H^2(S)$ is an isomorphism, we have that $[\omega_1 - \omega_0] = 0$, hence there exists a 1-form $\mu \in \Omega^1(V)$ such that $d\mu = \omega_1 - \omega_0$. We can suppose that $i^*\mu = 0$, since otherwise we would consider the form $\mu - \pi^* i^* \mu$.

Take an open set $\tilde{W}$ such that $\hat{W} \Subset \tilde{W} \Subset W$. We can also suppose that $\mu|_{\pi^{-1}(\tilde{W})} = 0$. As $\omega_1 - \omega_0 = 0$ on $\pi^{-1}(W)$, $d\mu = 0$ on $\pi^{-1}(W)$, and hence $\mu = df$ for some function $f \in C^\infty(\pi^{-1}(W))$, since we are assuming that $H^1(W) = 0$. As $i^*\mu = 0$ we can change f by $f - \pi^* i^* f$, so that $df = \mu$ and $i^* f = 0$. Let ρ be a step function on S such that $\rho|_{\tilde{W}} \equiv 1$ and $\rho \equiv 0$ outside W. Then we can substitute μ by $\mu - d((\pi^*\rho) f)$.

We can also suppose that the restriction $\mu|_S = 0$. In local coordinates (x_1, x_2, y_1, y_2) where $S = \{(x_1, x_2, 0, 0)\}$, we have $\mu = \sum a_j(x_1, x_2)\, dy_j + O(y)$. We cover S with balls B_α and then $(\mu|_S)|_{B_\alpha} = \sum a_j^\alpha\, dy_j^\alpha$. The balls are chosen so that they are inside $S - \hat{W}$ or inside $\tilde{W}$. Take a partition of unity $\{\rho_\alpha\}$ subordinated to it. We define $k_\alpha = \sum a_j^\alpha y_j^\alpha$ and $k = \sum \rho_\alpha k_\alpha$. For those $B_\alpha \subset \tilde{W}$, we can take $k_\alpha = 0$. Then $dk|_S = \mu|_S$, and we can substitute μ by $\mu - dk$. Note that $k = 0$ on $\pi^{-1}(\hat{W})$, so we keep $\mu|_{\pi^{-1}(\hat{W})} = 0$.

Now consider the form $\omega_t = t\omega_1 + (1-t)\omega_0 = \omega_0 + t\, d\mu$ for $0 \leq t \leq 1$. Since $d_x g$ is a symplectomorphism for all $x \in S$, we have $\omega_1|_S = \omega_0|_S$ and hence $\omega_t|_S = \omega_0|_S$ is symplectic over all points of S. So, reducing V if necessary, ω_t is symplectic on some neighborhood V of S. The equation $\iota_{X_t}\omega_t = -\mu$ admits a unique solution X_t which is a vector field on V. By the above, $X_t|_S = 0$ and $X_t|_{\hat{W}} = 0$. Take the flow φ_t of the family of vector fields X_t. There is some $U \subset V$ such that $\varphi_t(U) \subset V$ for all $t \in [0,1]$. Moreover $\varphi_0 = \mathrm{Id}_U$, $\varphi_t|_S = \mathrm{Id}_S$, and $\varphi_t|_{\hat{W}} = \mathrm{Id}_{\hat{W}}$. We compute

$$\begin{aligned} \frac{d}{dt}\Big|_{t=s} \varphi_t^*\omega_t &= \varphi_s^*(L_{X_s}\omega_s) + \varphi_s^*(d\mu) = \varphi_s^*(d(\iota_{X_s}\omega_s) + \iota_{X_s} d\omega_s) + \varphi_s^*\, d\mu \\ &= -\varphi_s^*(d\mu) + \varphi_s^*(d\mu) = 0. \end{aligned}$$

This implies that $\omega_0 = \varphi_0^*\omega_0 = \varphi_1^*\omega_1$. So $\varphi_1 \colon (U,\omega) \to (V, g^*\omega')$ is a symplectomorphism. The composition $\varphi = g \circ \varphi_1 \colon (U,\omega) \to (V', \omega')$ is a symplectomorphism of U onto $U' = \varphi(U) \subset V'$. □

2.3. Symplectic tubular neighbourhood of two intersecting submanifolds. Now we move to the case of the union of two intersecting symplectic submanifolds.

Definition 14. Let (X,ω) be a symplectic 4-manifold. We say that two symplectic surfaces $S_1, S_2 \subset X$ intersect ω-orthogonally if for every $\phi \in S_1 \cap S_2$ there are (complex) Darboux coordinates (z_1, z_2) such that $S_1 = \{z_2 = 0\}$ and $S_2 = \{z_1 = 0\}$ around p.

By definition, S_1 and S_2 intersect ω-orthogonally in the symplectic plumbing $P_c(S_1 \cup S_2)$.

Lemma 15 ([**21**, Lemma 6])**.** *Let (X,ω) be a symplectic 4-manifold and suppose that $S_1, S_2 \subset X$ are symplectic surfaces intersecting transversely and positively. Then we can perturb S_1 to get another surface S_1' in such a way that:*

(1) *The perturbed surface S_1' is symplectic.*
(2) *The perturbation is small in the C^0-sense and only changes S_1 near the intersection points with S_2, leaving these points fixed, i.e. $S_1 \cap S_2 = S_1' \cap S_2$.*
(3) *S_1' and S_2 intersect ω-orthogonally.*

Let $S = S_1 \cup S_2 \subset X$ be a union of two intersecting symplectic submanifolds of a symplectic manifold X. We use the expression *tubular neighborhood* of S to refer to a small neighborhood U of S in X such that S is a deformation retract of U.

Theorem 16 (Symplectic tubular neighborhood)**.** *Suppose that (X, ω) and (X', ω') are two symplectic 4-manifolds (maybe open) with compact symplectic surfaces $S_1, S_2 \subset X$ and $S_1', S_2' \subset X'$. Assume that S_1 and S_2 intersect symplectically orthogonally, and similarly for S_1' and S_2'. Suppose that there is a map $f\colon S = S_1 \cup S_2 \to S' = S_1' \cup S_2'$ which is a symplectomorphism $f\colon S_1 \to S_1'$ and a symplectomorphism $f\colon S_2 \to S_2'$. Assume also that the normal bundles satisfy $\nu_{S_1} \cong \nu_{S_1'}$ and $\nu_{S_2} \cong \nu_{S_2'}$. Then, there are tubular neighborhoods $S \subset U \subset X$ and $S' \subset U' \subset X'$ which are symplectomorphic via $\varphi\colon U \to U'$, with $\varphi|_S = f$.*

Proof: Take a point $P_i \in S_1 \cap S_2$. Let $\varphi_i\colon B_i \to B_\epsilon(0) \subset \mathbb{C}^2$ be Darboux coordinates so that $S_1 = \{z_2 = 0\}$ and $S_2 = \{z_1 = 0\}$, $\varphi_i(P_i) = 0$. For $f(P_i) \in S_1' \cap S_2'$ we also take $\varphi_i'\colon B_i' \to B_\epsilon(0) \subset \mathbb{C}^2$ Darboux coordinates so that $S_1' = \{z_2' = 0\}$ and $S_2' = \{z_1' = 0\}$, $\varphi_i'(f(P_i)) = 0$. The composite $(\varphi_i')^{-1} \circ \varphi_i\colon B_i \to B_i'$ may not coincide with f on $B_i \cap (S_1 \cup S_2)$. To arrange this, take

$$h_1 = \varphi_i' \circ (f|_{B_i \cap S_1}) \circ \varphi_i^{-1}\colon B_{\epsilon'}(0) \times \{0\} \longrightarrow B_\epsilon(0) \times \{0\},$$
$$h_2 = \varphi_i' \circ (f|_{B_i \cap S_2}) \circ \varphi_i^{-1}\colon \{0\} \times B_{\epsilon'}(0) \longrightarrow \{0\} \times B_\epsilon(0),$$

which are symplectomorphisms onto their image. Then $h = h_1 \times h_2$ is a symplectomorphism of $\mathbb{C}^2$ on a neighbourhood of the origin. So consider the symplectomorphism

$$\psi_i = (\varphi_i')^{-1} \circ h \circ \varphi_i\colon W_i \longrightarrow W_i'$$

defined on a neighbourhood $W_i \subset B_i$. It satisfies

$$\psi_i|_{B_i \cap (S_1 \cup S_2)} = f|_{B_i \cap (S_1 \cup S_2)}.$$

Fix also $\hat{W}_i \Subset W_i$, and denote $W = \bigcup W_i$, $\hat{W} = \bigcup \hat{W}_i$, $W_i' = \psi_i(W_i)$, $W' = \bigcup W_i'$, $\hat{W}_i' = \psi_i(\hat{W}_i)$, $\hat{W}' = \bigcup \hat{W}_i'$, and $\psi\colon W \to W'$ the map which is ψ_i on each W_i.

Now take small tubular neighbourhoods U_1, U_2 of S_1, S_2 respectively. Then $U_1 \cap U_2$ is a neighbourhood of the intersection $S_1 \cap S_2$ and can be made as small as we want. We require that $U_1 \cap U_2 \subset \hat{W}$. We also take neighbourhoods U_1', U_2' of S_1', S_2' respectively such that $U_1' \cap U_2' \subset \hat{W}'$. We can define diffeomorphisms $g_j\colon U_j \to U_j'$ with $g_j|_{S_j} = f|_{S_j}$ and $g_j|_{\tilde{W} \cap U_j} = \psi|_{\tilde{W} \cap U_j}$ for some $\hat{W} \Subset \tilde{W} \Subset W$, for $j = 1, 2$. Apply Proposition 13 to g_j, to obtain symplectomorphisms $\varphi_j\colon V_j \to V_j'$, where

$S_j \subset V_j \subset U_j$ and $S'_j \subset V'_j \subset U'_j$, such that $\varphi_j|_{S_j} = f|_{S_j}$ and $\varphi_j|_{\hat{W}\cap V_j} = \psi|_{\hat{W}\cap V_j}$. As $V_1 \cap V_2 \subset U_1 \cap U_2 \subset \hat{W}$, we have that φ_1, φ_2 coincide in the overlap region, defining thus a symplectomorphism

$$\varphi\colon V_1 \cup V_2 \longrightarrow V'_1 \cup V'_2$$

with $\varphi|_S = f|_S$. □

Corollary 17. *Let (X,ω) be a symplectic 4-manifold and $S_1, S_2 \subset X$ two compact symplectic surfaces intersecting symplectically orthogonally. Then there is a neighbourhood U of $S = S_1 \cup S_2$ which is symplectomorphic to a symplectic plumbing $P_c(S)$.*

Proof: Let $i_j\colon S_j \hookrightarrow S$ be the inclusion map, and denote $\{P_1,\ldots,P_m\} = i_1^{-1}(S_1 \cap S_2) \subset S_1$ and $\{Q_1,\ldots,Q_m\} = i_2^{-1}(S_1 \cap S_2) \subset S_2$. Take complex line bundles $E_j \to S_j$ with $c_1(E_j) = d_j = [S_j]^2$, and define a symplectic plumbing $P_c(S_1 \cup S_2)$ with these data. Now apply Theorem 16 to $S \subset X$ and $S \subset P_c(S)$. □

Corollary 18. *Let (S_1,ω_1), (S_2,ω_2) and (S'_1,ω'_1), (S'_2,ω'_2) be compact symplectic surfaces. Consider a symplectic plumbing $P_c(S_1 \cup S_2)$ with $\#S_1 \cap S_2 = m$ and $d_j = [S_j]^2$, $j = 1,2$, and another symplectic plumbing $P_c(S'_1 \cup S'_2)$ with $\#S'_1 \cap S'_2 = m'$ and $d'_j = [S'_j]^2$, $j = 1,2$. If $m = m'$, $\langle[\omega_j],[S_j]\rangle = \langle[\omega'_j],[S'_j]\rangle$, and $d_j = d'_j$, $j = 1,2$, then there are neighbourhoods $S_1 \cup S_2 \subset U \subset P_c(S_1 \cup S_2)$ and $S'_1 \cup S'_2 \subset U' \subset P_c(S'_1 \cup S'_2)$ which are symplectomorphic.*

Proof: Note that two compact surfaces Σ, Σ' are symplectomorphic if and only if they have the same area $\langle[\omega],[\Sigma]\rangle = \langle[\omega'],[\Sigma']\rangle$. Moreover the symplectomorphism can be chosen so that it sends some finite collection of m points of Σ to another collection of m points of Σ'. Applying this to S_j, S'_j, we get a symplectomorphism $f_j\colon S_j \to S'_j$ with $f_j|_{S_1\cap S_2}\colon S_1 \cap S_2 \to S'_1 \cap S'_2$ sending the intersection points in the required order, $j = 1,2$. Therefore $f_1|_{S_1\cap S_2} = f_2|_{S_1\cap S_2}$, thus defining a map $f\colon S_1 \cup S_2 \to S'_1 \cup S'_2$. As the intersections are symplectically orthogonal, we can apply Theorem 16 to get the stated result. □

This gives uniqueness of symplectic plumbings. In particular, they do not depend on the choices of symplectomorphisms of the surfaces, or the choice of Darboux coordinates at the intersection points.

Remark 19. Theorem 16 holds for a symplectic manifold X of any dimension and symplectic submanifolds $S_1, S_2 \subset X$ of complementary dimension intersecting symplectically orthogonally.

The plumbing can be defined for symplectic manifolds S_1, S_2 of any dimension $2n$, and $P_c(S_1 \cup S_2)$ will have dimension $4n$.

3. A configuration of symplectic surfaces in $\mathbb{C}P^2\#11\overline{\mathbb{C}P}^2$

3.1. Homology of $\mathbb{C}P^2\#11\overline{\mathbb{C}P}^2$. Let $X=\mathbb{C}P^2\#11\overline{\mathbb{C}P}^2$ be the symplectic manifold obtained by blowing up the projective plane $\mathbb{C}P^2$ at eleven points $\Delta=\{P_1,\ldots,P_{11}\}$. We call $h\in H_2(X)$ the homology class of the line, and e_i, $1\le i\le 11$, the homology classes of the exceptional divisors, so that $H_2(X)=\langle h,e_1,\ldots,e_{11}\rangle$. Moreover, the intersection form of X is diagonal with respect to the basis $\{h,e_1,\ldots,e_{11}\}$. Now consider the collection of homology classes in $H_2(X)$ given by:

$$c_k=3h-\sum_{i\neq k}^{11}e_i,\quad 1\le k\le 11,$$

$$d=10h-\sum_{i=1}^{11}3e_i.$$

Proposition 20. *The homology classes $\{c_1,\ldots,c_{11},d\}$ form a basis of $H_2(X)$. The intersection form is diagonal with respect to this basis, and the self-intersections are $c_k^2=-1$, for $1\le k\le 11$, and $d^2=1$.*

Proof: The second sentence follows from $e_i\cdot h=0$, $e_i^2=-1$, for all i, and $h^2=1$. This implies that the determinant of the intersection form with respect to this basis is -1, hence it is a basis over $\mathbb{Z}$. □

Our focus is to prove that the basis $\{c_1,\ldots,c_{11},d\}$ of $H_2(X)$ can be realized by symplectic surfaces. For this, we need the following configuration of symplectic surfaces in $\mathbb{C}P^2$:

- Eleven symplectic surfaces $C_1,\ldots,C_{11}$ such that their homology classes are $[C_i]=3h$ in $\mathbb{C}P^2$. These surfaces C_i, being cubics, must have $g=1$ by the symplectic adjunction formula. The surface C_i is required to pass through the ten points in $\Delta-\{P_i\}$, but not through P_i. Therefore, the proper transform $\tilde{C}_i$ of C_i in the blow-up $X=\mathbb{C}P^2\#11\overline{\mathbb{C}P}^2$ of $\mathbb{C}P^2$ at S has homology class $[\tilde{C}_i]=c_i$.
- The intersection $C_i\cap C_j$ contains the nine points $\Delta-\{P_i,P_j\}$, for $i\neq j$. Note that the algebraic intersection is $C_i\cdot C_j=9$. If these intersections are transverse and positive (e.g. if the C_i are holomorphic around the intersection points) and if there are no more intersections, then the proper transforms $\tilde{C}_i$, $\tilde{C}_j$ are disjoint.

- One singular symplectic surface G such that $[G]=10h$ and G has eleven ordinary triple points at the points of Δ. By the adjunction formula the genus of G is
$$g=\frac{1}{2}(10-1)(10-2)-11\frac{3\cdot 2}{2}=36-33=3.$$
If G is holomorphic at a neighbourhood of the triple points and the branches intersect transversely (and hence also positively), then the proper transform $\tilde{G}$ of G in the blow-up $X=\mathbb{C}P^2\#11\overline{\mathbb{C}P}^2$ of $\mathbb{C}P^2$ at S has homology class $[\tilde{G}]=10h-3(e_1+\cdots+e_{11})=d$. Moreover, if there are no more singularities, then $\tilde{G}$ is a smooth symplectic surface in X.
- The intersections $C_i\cap G$ contain the ten points $\Delta-\{P_i\}$. Note that the algebraic intersection is $C_i\cdot G=30$. If the intersections with each of the three branches at each intersection point are transverse and positive (e.g. if the C_i and G are holomorphic around the intersection points), and if there are no more intersections, then these account for all intersections. In the blow-up X, the proper transforms $\tilde{C}_i$, $\tilde{G}$ are disjoint.

Our aim now is to construct these surfaces in $\mathbb{C}P^2$. For this, we will make the construction in a local model and then we will transplant it to $\mathbb{C}P^2$.

3.2. Construction of a local model. Now we are going to construct the required eleven surfaces of genus 1 and the singular surface of genus 3 in a local model. The local model is as follows: take a genus 1 complex curve C and a rational complex curve $L\cong\mathbb{C}P^1$. Take three points $Q_1,Q_2,Q_3\in C$ and another three $Q_1',Q_2',Q_3'\in L$. Take a line bundle $E\to C$ of degree 9 and a line bundle $E'\to L$ of degree 1, and perform the plumbing as given in Subsection 2.1. This produces a symplectic manifold $P_c(C\cup L)$, which contains $C\cup L$.

Proposition 21. *Let $C'\subset\mathbb{C}P^2$ and $L'\subset\mathbb{C}P^2$ be a smooth cubic and a line in the complex plane, intersecting transversely. Then $P_c(C\cup L)$ can be symplectically embedded in a neighbourhood of $C'\cup L'$, where C is sent to C' and L is sent to a C^0-small perturbation of L', preserving the intersection points.*

Proof: We start by modifying L' to L'' using Lemma 15, so that C' and L'' intersect symplectically orthogonally. By Corollary 17, a small neighbourhood of $C'\cup L''$ is symplectomorphic to a small neighbourhood of the plumbing of $C\cup L$, that is, some $P_c(C\cup L)$, for $c>0$ small, and the symplectomorphism sends C to C' and L to L''. □

Therefore, to prove Theorem 7, it is enough to prove the following:

Theorem 22. *There are eleven points* $P_1, \dots, P_{11}$ *in* $P_c(C \cup L)$ *and symplectic surfaces* $C_1, C_2, \dots, C_{11}, G \subset P_c(C \cup L)$ *such that:*

(1) C_i *is a section of a complex line bundle* $E \to C$ *of degree* 3, *and* $P_j \in C_i$ *for* $j \neq i$, $P_i \notin C_i$. *In particular, they have genus* 1.
(2) *The surfaces* C_i, C_j, $i \neq j$, *intersect exactly at* $\{P_1, \dots, P_{11}\} - \{P_i, P_j\}$, *positively and transversely.*
(3) G *is a genus* 3 *singular symplectic surface whose only singularities are eleven triple points at* P_i *(with different branches intersecting positively). Moreover,* G *intersects each* C_i *only at the points* P_j, $j \neq i$, *and all the intersections of* C_i *with the branches of* G *are positive and transverse.*

To be more concrete, we proceed as follows. We fix a complex structure on C and a degree 9 complex line bundle $E \to C$. This is going to be as follows: take a complex disc $D \subset C$, which we assume as the radius 1 disc $D = D(0,1) \subset \mathbb{C}$. Let $V = C - \bar{D}(0, 1/2)$, and consider the change of trivialization given by the function $g(z) = z^9$ with $D(0,1) - \bar{D}(0,1/2)$. This means that E is formed by gluing $E|_D = D \times \mathbb{C}$ with $E|_V = V \times \mathbb{C}$ via $(z, y) \sim (z, z^9 y)$. We endow E with an auxiliary hermitian metric which is of the form $h(z) = 1$ on the trivialization $E|_D$. We will choose the points $Q_1, Q_2, Q_3 \in D \subset C$. We also take a complex line bundle $E' \to L$ of degree 1, for which we fix a hermitian structure. Fixing three points $Q'_1, Q'_2, Q'_3 \in L$, we perform the plumbing $P_c(C \cup L)$.

3.3. Construction of the genus 1 surfaces. The genus 1 symplectic surfaces will be constructed as sections of the line bundle $E \to C$. Consider the previous cover $C = V \cup D$ and trivializations $E|_V \cong V \times \mathbb{C}$ and $E|_D \cong D \times \mathbb{C}$. Fix distinct numbers $z_1, \dots, z_{10}, z_{11} \in D$ with $z_{11} = 0$ the origin. Take $\lambda > 0$ a small positive real number to be fixed later. Take the points

$$P_j = (\lambda z_j, 0), \quad j = 1, \dots, 10, \text{ and } P_{11} = (0, 1) \tag{4}$$

in E, in the given trivialization $E|_D = D \times \mathbb{C}$. We define eleven holomorphic local sections in the chart $D \subset C$ as

$$\sigma_j(z) = \prod_{i \neq j}^{10} \left(1 - \frac{z}{\lambda z_i}\right), \quad j = 1, \dots, 10,$$

and $\sigma_{11}(z) = 0$.

Clearly $\sigma_j(z) = \sigma_{11}(z) = 0$ at the nine points $\lambda z_1, \dots, \widehat{\lambda z_j}, \dots, \lambda z_{10}$. Also, for $1 \leq j < k \leq 10$, we have that $\sigma_j(z) = \sigma_k(z)$ at the nine points given by

$$\lambda z_1, \dots, \widehat{\lambda z_j}, \dots, \widehat{\lambda z_k}, \dots, \lambda z_{10}, z_{11} = 0. \tag{5}$$

All the intersections of the graphs are transverse and positive since the points λz_i are simple roots and σ_j are holomorphic sections. By construction, the graph $\Gamma(\sigma_j)$ of the local section σ_j in the trivialization $E|_D \cong D \times \mathbb{C}$ contains the set of points $\{P_1, \dots, P_{11}\} - \{P_j\}$, as desired.

Now we move to the trivialization $E|_V$. Let us see that we can extend the sections σ_j to all of V without introducing any new intersection points between their graphs. For $z \in D \cap V$, the sections σ_j become, for $|z| \geq 1/2$, in the trivialization of $E|_V \cong V \times \mathbb{C}$,

$$\tilde{\sigma}_j = z^{-9} \prod_{i \neq j}^{10} \left(1 - \frac{z}{\lambda z_i}\right) = \lambda^{-9} A z_j \prod_{i \neq j}^{10} \left(1 - \frac{\lambda z_i}{z}\right), \quad A = -(z_1 \cdots z_{10})^{-1},$$

and $\tilde{\sigma}_{11} = 0$. Then $\tilde{\sigma}_j$ has the form

$$\tilde{\sigma}_j = A \lambda^{-9} z_j (1 + \lambda f_j(z, \lambda)),$$

where

$$f_j(z, \lambda) = \frac{1}{\lambda} \left(\prod_{i \neq j}^{10} \left(1 - \frac{\lambda z_i}{z}\right) - 1 \right)$$

is a holomorphic function of z depending on the parameter λ such that

$$|f_j(z, \lambda)| \leq M_0, \text{ for } \lambda \leq \tfrac{1}{4}, \ |z| \geq \tfrac{1}{2},$$

being M_0 a constant depending only on $z_1, \dots, z_{11}$.

Let ρ be the smooth non-increasing function with $\rho(r) = 0$ for $r \geq 3/4$ and $\rho(r) = 1$ for $r \leq 2/3$. Here $r = |z|$ is the radius in the disc D. Now we modify the local sections $\tilde{\sigma}_j$ to sections $\hat{\sigma}_j$ that can be extended to global sections in $E \to C$. We define for $z \in U$, $|z| \geq 1/2$,

$$\hat{\sigma}_j(z) = \rho(|z|)\tilde{\sigma}_j(z) + (1 - \rho(|z|))\lambda^{-9} A z_j = \lambda^{-9} A z_j (1 + \lambda \rho(|z|) f_j(z, \lambda)). \tag{6}$$

We also put $\hat{\sigma}_{11} = 0$.

We have that $\hat{\sigma}_j = \tilde{\sigma}_j$ in $\{1/2 \leq |z| \leq 2/3\}$, so $\hat{\sigma}_j$ extends to the trivialization $E|_D$ as σ_j in $\{|z| \leq 1/2\} \subset D$. Moreover, $\hat{\sigma}_j(z) = \lambda^{-9} A z_j$ is constant for $|z| \geq 3/4$, so $\hat{\sigma}_j$ extends to all of V, hence they give global sections in the line bundle $E \to C$. We call $\hat{\sigma}_j$ these global sections and $\Gamma(\hat{\sigma}_j)$ their graphs.

Now let us check that no undesired intersection points are introduced between any pair of surfaces C_j, $1 \le j \le 11$. On $|z| \le 1/2$, $\hat\sigma_j = \tilde\sigma_j$, so $\tilde\sigma_j$, $\tilde\sigma_k$, $j \neq k$, have nine intersection points given by (5), which are the set $\{P_1, \dots, P_{11}\} - \{P_j, P_k\}$. As $\tilde\sigma_j$ and $\tilde\sigma_k$ are holomorphic there, and the roots are simple, the intersections are positive and transverse.

For $|z| \ge 3/4$, $\hat\sigma_j = \lambda^{-9} A z_j$, $j = 1, \dots, 10$, and $\hat\sigma_{11} = 0$. Therefore the sections do not intersect since the $\{z_j\}$ are distinct points. Now assume that $1/2 \le |z| \le 3/4$. If $\hat\sigma_j(z) = \hat\sigma_k(z)$ with $k \neq j \le 10$, then

$$z_j + z_j \lambda \rho(|z|) f_j(z, \lambda) = z_k + z_k \lambda \rho(|z|) f_k(z, \lambda).$$

Taking $\lambda > 0$ small enough, the discs $B(z_j, M_0|z_j|\lambda)$ and $B(z_k, M_0|z_k|\lambda)$ are all pairwise disjoint, so the above equality does not happen. Analogously, if $\hat\sigma_j(z) = \hat\sigma_{11}(z) = 0$ for $1/2 \le |z| \le 3/4$, we have a contradiction as long as λ is small enough so that the discs $B(z_j, \lambda|z_j|M_0)$ do not contain the origin.

Finally, considering $\hat\sigma_j^\epsilon = \epsilon\hat\sigma_j$, being $\epsilon > 0$ small enough, the intersections of the graphs remain the same except that P_{11} is changed to the point $(0, \epsilon)$. This ensures that the graphs are all contained in the given neighbourhood $B_c(C)$, for any $c > 0$ given beforehand. Moreover the graphs become C^1-close to the zero section $C \subset E$, in particular the graphs are symplectic surfaces of $B_c(E)$.

3.4. Construction of the genus 3 surface. The genus 3 surface will be constructed inside the neighbourhood $P_c(C \cup L)$ of $C \cup L$, where C is the genus 1 surface and L the genus 0 surface, both intersecting at three points. We will take three sections of the bundle $E \to C$, all of them passing through the eleven points $P_1, \dots, P_{11}$. In this way we get the eleven triple points. Then we add the line L, and glue the three sections with L around the intersection points of L and C. Let us give the details.

As before, take the previous cover $C = V \cup D$, $D = D(0,1)$, $V = C - \bar D(0, 1/2)$, and trivializations $E|_D \cong D \times \mathbb{C}$ and $L|_V \cong V \times \mathbb{C}$, with change of trivialization $g(z) = z^9$. We have fixed $z_1, \dots, z_{10}, z_{11} = 0 \in D$ and the points

$$P_j = (\lambda z_j, 0), \quad j = 1, \dots, 10, \text{ and } P_{11} = (0, 1)$$

in $E|_D = D \times \mathbb{C}$, where $0 < \lambda \le 1/4$ is some small number as arranged in Subsection 3.3.

We choose another three distinct values $w_1, w_2, w_3 \in D$, different to $z_1, \dots, z_{11}$. We take the points

$$Q_1 = (\lambda w_1, 0), \quad Q_2 = (\lambda w_2, 0), \quad Q_3 = (\lambda w_3, 0), \tag{7}$$

in the trivialization $E|_D = D \times \mathbb{C}$. Consider (meromorphic) sections τ_k, defined in $D - \{Q_k\}$ by the formula

$$\tau_k(z) = \frac{\prod_{i=1}^{10}\left(1 - \frac{z}{\lambda z_i}\right)}{1 - \frac{z}{\lambda w_k}},$$

for $k = 1, 2, 3$. The graph of τ_k passes through all eleven points (4).

Let us see that we can extend the sections τ_k to the trivialization $E|_V$, giving thus sections over $C - \{Q_k\}$. For $z \in D \cap V$, i.e. $|z| \geq 1/2$, we express τ_k in the trivialization $L|_V$, which is given by $\tilde{\tau}_k(z) = z^{-9}\tau_k(z)$.

$$\begin{aligned}\tilde{\tau}_k(z) &= z^{-9}\frac{\prod_{i=1}^{10}\left(1 - \frac{z}{\lambda z_i}\right)}{1 - \frac{z}{\lambda w_k}} \\ &= \lambda^{-9}Aw_k\frac{\prod_{i=1}^{10}\left(1 - \frac{\lambda z_i}{z}\right)}{1 - \frac{\lambda w_k}{z}} = \lambda^{-9}Aw_k(1 + \lambda g_k(z, \lambda)),\end{aligned}$$

where $A = -(z_1 \cdots z_{10})^{-1}$ as before, and

$$g_k(z, \lambda) = \frac{1}{\lambda}\left(\frac{\prod_{i=1}^{10}\left(1 - \frac{\lambda z_i}{z}\right)}{1 - \frac{\lambda w_k}{z}} - 1\right)$$

are bounded functions for $|z| \geq 1/2$ and $0 < \lambda \leq 1/4$, say $|g_k(\lambda, z)| \leq M$, for $M > 0$ a constant.

Let $\rho\colon [0, \infty) \to \mathbb{R}$ be a non-increasing smooth function such that $\rho(r) = 1$ for $r \leq 1/2$ and $\rho(r) = 0$ for $r \geq 3/4$. Now we modify $\tilde{\tau}_k(z)$ for $z \in D \cap V$, i.e. $|z| \geq 1/2$. Consider

$$\hat{\tau}_k(z) = \lambda^{-9}Aw_k(1 + \rho(|z|)\lambda g_k(\lambda, z)).$$

Clearly, $\hat{\tau}_k(z) = \tilde{\tau}_k(z)$ for $|z| \leq 1/2$, so $\hat{\tau}_k$ extends to the trivialization $E|_D$. Also, for $|z| \geq 3/4$ we have $\hat{\tau}_k(z) = \lambda^{-9}Aw_k$ is constant so $\hat{\tau}_k$ extends to all the trivialization $L|_V$. This yields a global section defined in $C - \{Q_k\}$, given by τ_k in $\{|z| \leq 1/2\} \subset D$, and by $\hat{\tau}_k$ in V. We call from now on $\hat{\tau}_k$ this global section. Let us denote

$$\Theta_k = \Gamma(\hat{\tau}_k) = \{(z, \hat{\tau}_k(z)) \mid z \in C - \{Q_k\}\}$$

the graph of $\hat{\tau}_k$.

Let us see that the graphs Θ_1, Θ_2, Θ_3 only intersect at the points $P_1, \ldots, P_{10}, P_{11}$, i.e. that the sections only coincide for the values $\lambda z_1, \ldots, \lambda z_{10}, z_{11} = 0$. Let $j \neq k$. On $|z| \leq 1/2$, $z \neq \lambda w_j, \lambda w_k$, if $\hat{\tau}_j(z) = \hat{\tau}_k(z)$, then

$$\frac{\prod_{i=1}^{10}\left(1 - \frac{z}{\lambda z_i}\right)}{1 - \frac{z}{\lambda w_j}} = \frac{\prod_{i=1}^{10}\left(1 - \frac{z}{\lambda z_i}\right)}{1 - \frac{z}{\lambda w_k}}.$$

Hence either $z = \lambda z_i$ for some $1 \le i \le 10$ or $\frac{z}{\lambda w_j} = \frac{z}{\lambda w_k}$. The latter implies $z = 0 = z_{11}$.

For $|z| \ge 3/4$, if $\hat{\tau}_j(z) = \hat{\tau}_k(z)$, then $\lambda^{-9} A w_j = \lambda^{-9} A w_k$, which is false since $w_j \neq w_k$. Finally, for $1/2 \le |z| \le 3/4$, if $\hat{\tau}_j(z) = \hat{\tau}_k(z)$, then

$$w_j + w_j \rho(|z|)\lambda g_j(\lambda, z)) = w_k + w_k \rho(|z|)\lambda g_k(\lambda, z)).$$

Choosing λ small enough, we have that the discs $D(w_j, \lambda|w_j|M)$ and $D(w_k, \lambda|w_k|M)$ do not intersect. So the above equality does not happen.

Finally, let us check the intersections of Θ_k with $\Gamma(\hat{\sigma}_j)$. Take $|z| \le 1/2$. Suppose that $\tau_k(z) = \sigma_j(z)$. This implies that

$$\sigma_j(z) \frac{1 - \frac{z}{\lambda z_j}}{1 - \frac{z}{\lambda w_k}} = \sigma_j(z),$$

hence either $\sigma_j(z) = 0$ or $\frac{z}{\lambda z_j} = \frac{z}{\lambda w_k}$. In the first case we have that $z = \lambda z_i$ for some $i \neq j$. In the second case we have that either $z = 0 = z_{11}$, or $\lambda z_j = \lambda w_k$ which is not possible because the points w_k are different from the points z_j.

Suppose now that $1/2 \le |z| \le 3/4$ and $\sigma_j(z) = \tau_k(z)$. Then

$$z_j + z_j \rho(|z|)\lambda f_j(z, \lambda) = w_k + w_k \rho(|z|)\lambda g_k(\lambda, z).$$

If we take λ small, the discs $D(z_j, M_0|z_j|\lambda)$ and $D(w_k, M|w_k|\lambda)$ are disjoint, so the above equality is impossible. Finally, if $|z| \ge 3/4$ and $\tau_k(z) = \sigma_j(z)$, then $\lambda^{-9} A z_j = \lambda^{-9} A w_k$, and this is false.

3.5. Gluing the transversal in the plumbing. The plumbing $P_c(C \cup L)$ is defined only for $c > 0$ small enough. Let us arrange that our sections lie inside it suitably. For this, let $N > 0$ be an upper bound of all $|\hat{\sigma}_j|$, $j = 1, \dots, 11$, such that $(|\hat{\tau}_k|)^{-1}([N, \infty)) \subset B(Q_k)$, where $B(Q_k) \subset D(0, 1/2)$ are small balls around Q_k, $k = 1, 2, 3$. Recall that $\hat{\tau}_k = \tau_k$ is holomorphic on $B(Q_k) - \{Q_k\}$. As τ_k has a simple pole at Q_k, we have that

$$z' = z'_k = h_k(z) = \frac{1}{\tau_k(z)}$$

is a biholomorphism from a neighbourhood of Q_k (that we keep calling $B(Q_k)$) to a ball $B_c(0)$. We take the coordinate z'_k on $B(Q_k)$. We need to modify the symplectic form so that z'_k is also a Darboux coordinate.

Lemma 23. *Consider the disc $D = D(0, 1)$. We can perturb the standard symplectic form ω_D to a nearby symplectic form ω'_D such that, maybe after reducing the balls $B(Q_k)$, the coordinates z'_k are Darboux. The perturbation is made only on a (slightly larger) ball around Q_k, and keeping the total area.*

Proof: We write $z' = z'_k = x' + iy'$. The standard symplectic form ω_D on the coordinates z is clearly Kähler, therefore it is also Kähler for the holomorphic coordinate z'. In particular, it has a Kähler potential $\phi(x', y')$, with $\omega_D = \partial\bar\partial\phi(x', y')$. We can assume that ϕ has no linear part, so $\phi(x', y') = \phi_2(x', y') + \phi_3(x', y')$, where $\phi_2(x', y')$ is quadratic and $|\phi_3| = O(|(x', y')|^3)$. Then take some bump function ρ that vanishes on a neighbourhood $B_\eta(Q_k)$ of Q_k (the size measured with respect to the radial coordinate $r' = |z'|$), and $\rho \equiv 1$ on a slightly larger neighbourhood $B_{2\eta}(Q_k)$, $|d\rho| = O(\eta^{-1})$, and $|\nabla d\rho| = O(\eta^{-2})$. Set $\omega'_D = \partial\bar\partial(\phi_2 + \rho\phi_3)$. Then $|\omega'_D - \omega_D| = O(\eta)$ and ω'_D is standard on $(B_\eta(Q_k), z')$. As the difference $\omega'_D - \omega_D = \partial\bar\partial((\rho - 1)\phi_3) = d(\bar\partial(\rho - 1)\phi_3)$ is exact and compactly supported, the total area remains the same. □

Remark 24. Lemma 23 also holds in higher dimension. More concretely, let (Z, ω, J) be a Kähler manifold of real dimension $2n$, $p \in Z$, and $\varphi\colon U \to B \subset \mathbb{C}^n$ holomorphic coordinates around p. Then there exists a symplectic form ω' on Z so that (Z, ω', J) is Kähler, $\omega' = \omega$ in $Z - U$, and ω' is a linear symplectic form near p on the coordinates φ, in some smaller neighborhood $V \subset U$. Moreover, the cohomology classes $[\omega] = [\omega']$.

Over $E|_{B(Q_k)} \cong B_c(0) \times \mathbb{C}$, the section τ_k is given by $v = 1/z'$ (making $c > 0$ smaller if needed), writing $z' = z'_k$ for brevity. For $Q'_1, Q'_2, Q'_3 \in L$, take holomorphic balls $B(Q'_j) \cong B_c(0)$, and arrange the symplectic structure on L to be standard over them. Finally, take symplectic structures on the total spaces of the complex line bundles $\pi\colon E \to C$ and $\pi'\colon E' \to L$ so that they are product symplectic structures on $B_c(C) \cap E|_{B(Q_k)} \cong B(Q_k) \times B_c(0)$ and $B_c(L) \cap E'|_{B(Q'_k)} \cong B(Q'_k) \times B_c(0)$, respectively. The plumbing $P_c(C \cup L)$ is done by gluing $B_c(C)$ and $B_c(L)$ along $R\colon B(Q_k) \times B_c(0) \to B(Q'_k) \times B_c(0)$, the map reversal of coordinates. Note that the uniqueness result of Corollary 18 allows to do the plumbing with these choices. We only have to take care of keeping the total areas $\langle [C], [\omega_E] \rangle$ and $\langle [L], [\omega_{E'}] \rangle$ fixed.

Now take $\epsilon > 0$ small enough so that:

- The graphs of the sections $\sigma_j^\epsilon = \epsilon\sigma_j$ are inside $B_c(C)$. For this $\epsilon N < c$ is enough.
- The graphs of the sections σ_j^ϵ are C^1-close to the zero section. This implies that these graphs are symplectic (a submanifold C^1-close to a symplectic one is symplectic).
- All sections $\tau_k^\epsilon = \epsilon\tau_k$ satisfy $|\tau_k^\epsilon| < c$ on $C - B(Q_k)$, so the graph Θ_k^ϵ of τ_k^ϵ satisfies that $\Theta_k^\epsilon \cap \pi^{-1}(C - B(Q_k)) \subset B_c(C)$. For this it is enough that $\epsilon N < c$ again.

- The graphs of the sections τ_k^ϵ are C^1-close to the zero section on $C-B(Q_k)$, so they are symplectic.

Now we look at the graph $\Theta_k^\epsilon \cap (E|_{B(Q_k)})$. We have

$$\Theta_k^\epsilon \cap (E|_{B(Q_k)}) \cong \left\{ (z',v) \in B_c(0) \times \mathbb{C} \,\middle|\, v = \frac{\epsilon}{z'} \right\}$$
$$= \left\{ (z',v) \in B_c(0) \times \mathbb{C} \,\middle|\, |v| \geq \epsilon c^{-1},\, z' = \frac{\epsilon}{v} \right\}.$$

Make $\epsilon > 0$ smaller if necessary, so that $\epsilon c^{-1} \leq c/2$. Take $\rho(r)$ a smooth non-increasing function so that $\rho(r)=1$ for $r \leq 1/2$ and $\rho(r)=0$ for $r \geq 3/4$. Define

$$\hat{\Theta}_k^\epsilon = \left\{ (z',v) \in B_c(0) \times \mathbb{C} \,\middle|\, \epsilon c^{-1} \leq |v| \leq c,\, z' = \epsilon \rho(|v|/c) \frac{1}{v} \right\}.$$

This can be smoothly glued to $\bar{\Theta}_k^\epsilon = \Theta_k^\epsilon \cap \pi^{-1}(C - B(Q_k))$. On the part of the plumbing corresponding to $E' \to L$, this has the form $z' = \epsilon \rho(|v|/c)\frac{1}{v}$ on $B(Q_k') \times B_c(0)$, where v is the coordinate for $B(Q_k')$ and z' is the vertical coordinate. Note that this can be extended as $z' = 0$ in the bundle $E' \to L$, over $L - (B(Q_1') \cup B(Q_2') \cup B(Q_3'))$. The resulting smooth manifold is

$$G = \bigcup_{k=1,2,3} (\bar{\Theta}_k^\epsilon \cup \hat{\Theta}_k^\epsilon) \cup (L - (B(Q_1') \cup B(Q_2') \cup B(Q_3'))). \tag{8}$$

Clearly, as $|v| \geq \epsilon c^{-1}$ for the points of $\hat{\Theta}_k^\epsilon$, there are no new intersections with the graphs $\Gamma(\sigma_j^\epsilon)$ or Θ_l^ϵ, $l \neq k$, since they are bounded by ϵN, and we can take $c < N^{-1}$ to start with.

The graphs $\hat{\Theta}_k^\epsilon$ are symplectic, since the graphs of $z' = \epsilon \rho(|v|/c)\frac{1}{v}$ are symplectic over $|v| \geq c/2$, by taking $\epsilon > 0$ small enough so that it is C^1-close to the zero section $z' = 0$ of the bundle $E' \to L$. On $\epsilon c^{-1} \leq |v| \leq c/2$, the graph coincides with $z' = \epsilon \frac{1}{v}$, which is holomorphic hence symplectic.

Remark 25. The homology class of the graph $\Gamma(\hat{\sigma}_j)$ in $P_c(C \cup L)$ is equal to $[C]$, since they are sections of $E \to C$. The manifold G of (8) can be retracted to $3[C] + [L]$ in $P_c(C \cup L)$ by making $\epsilon \to 0$.

When we embed $P_c(C \cup L) \hookrightarrow \mathbb{C}P^2$, the class $[C] \mapsto 3h$, and $[L] \mapsto h$, where h is the class of the line in $\mathbb{C}P^2$. Hence $[G] \mapsto 10h$, so G has degree 10.

The genus of G is 3 since topologically it is the gluing (connected sum) of three punctured surfaces of genus 1 (given by the graphs of the sections Θ_k, $k = 1, 2, 3$), with a sphere with three holes given by $L - (B(Q_1') \cup B(Q_2') \cup B(Q_3'))$.

4. Fundamental group of the K-contact 5-manifold

Let X be the symplectic manifold constructed as the symplectic blow-up of $\mathbb{C}P^2$ at the eleven points $P_1, \dots, P_{11}$. The underlying smooth manifold is $X = \mathbb{C}P^2 \# 11\overline{\mathbb{C}P^2}$ with $b_2 = 12$. It has eleven surfaces of genus 1, named $\tilde{C}_1, \dots, \tilde{C}_{11}$, and a genus 3 surface $\tilde{G}$ all of them disjoint. We set the isotropy of $\tilde{C}_i$ to be $\mathbb{Z}/(p^i)$, $i = 1, \dots, 11$, and that of $\tilde{G}$ to be $\mathbb{Z}/(p^{12})$, for a fixed prime p. This determines a symplectic orbifold X' uniquely by [**21**, Proposition 7].

We start by computing the *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(X')$ of X'. The reader can find alternative definitions in [**27**, Chapter 13] and in [**6**, Definition 4.3.6]. We only need a presentation of $\pi_1^{\mathrm{orb}}(X')$, which follows from [**15**, Théorème A.1.4]. For this, fix a base point $p_0 \in X'$. Take loops from p_0 to a point near $\tilde{C}_i$, followed by a loop δ_i around $\tilde{C}_i$, and going back to p_0, $i = 1, \dots, 11$. In the same vein, we add another loop δ_{12} around $\tilde{G}$. Then

$$\pi_1^{\mathrm{orb}}(X') = \frac{\pi_1(X - (\tilde{C}_1 \cup \cdots \cup \tilde{C}_{11} \cup \tilde{G}))}{\langle \delta_1^p, \dots, \delta_{11}^{p^{11}}, \delta_{12}^{p^{12}} \rangle}.$$

Let us see that $\pi_1^{\mathrm{orb}}(X')$ is trivial. It suffices to see that $\pi_1(X - (\tilde{C}_1 \cup \cdots \cup \tilde{C}_{11} \cup \tilde{G}))$ is trivial. We start with a lemma.

Lemma 26. *We can arrange a complex cubic curve and a complex line $C', L' \subset \mathbb{C}P^2$ intersecting transversally, such that a small neighborhood $B_\epsilon(C' \cup L')$ of $C' \cup L'$ satisfies the following: there are generators of $\pi_1(C')$ represented by loops α, β away from $B_\epsilon(L')$, that can be homotoped (outside $B_\epsilon(L')$) to loops $\hat{\alpha}$, $\hat{\beta}$ in $\partial B_\epsilon(C')$. The loops $\hat{\alpha}$, $\hat{\beta}$ are contractible in $\mathbb{C}P^2 - B_\epsilon(C' \cup L')$.*

Proof: We consider a particular family of complex cubics in $\mathbb{C}P^2$ given by the affine equations $C_r = \{y^2 = x^3 - r^2 x\}$, with $r > 0$ small. As $r \to 0$, the cubic C_r collapses to a cuspidal rational curve $C_0 = \{y^2 = x^3\}$, which has trivial first homology group. It is known [**19**] that the vanishing cycles generate the homology $H_1(C_r)$. Here we give an explicit description, as the loops

$$\alpha_r = \{(x, y) \mid x \in [-r, 0],\, y \in \mathbb{R},\, y^2 = x^3 - r^2 x\},$$
$$\beta_r = \{(x, y) \mid x = -x' \in [0, r],\, y = iy' \in i\mathbb{R},\, (y')^2 = (x')^3 - r^2 x'\}.$$

Note that α_r, β_r intersect transversally at one point, hence they generate $\pi_1(C_r) \cong \mathbb{Z}^2$. The homotopies given by α_t, $t \in [0, r]$, and β_t, $t \in [0, r]$ (with base-point at $(0, 0)$), produce discs that contract α_r, β_r. These discs do not intersect C_r. Now fix some $C' = C_r$ and take a tubular neighbourhood $B_\epsilon(C')$ by considering all C_s with $|s-r|<\epsilon$. Then we can homotop the loops $\alpha=\alpha_r$, $\beta=\beta_r$ to $\hat{\alpha}=\alpha_{r-\epsilon}$, $\hat{\beta}=\beta_{r-\epsilon}$ which lie at the boundary, and can be contracted outside $B_\epsilon(C')$.

Finally, take a complex line $L'\subset\mathbb{C}P^2$ intersecting transversally C', but well away from the loops α_r, β_r, and the homotopies above (e.g. a small perturbation of the line at infinity). Therefore all previous statement happen outside $B_\epsilon(L')$. □

Proposition 27. *We have that the fundamental group* $\pi_1(X-(\tilde{C}_1\cup\cdots\cup\tilde{C}_{11}\cup\tilde{G}))=1$. *In particular,* $\pi_1^{\mathrm{orb}}(X')=1$.

Proof: We constructed $C_1,\ldots,C_{11},G$ inside a plumbing $\mathbf{P}=P_c(C\cup L)$, and then we have transferred it to a neighbourhood $\mathbf{P}'=P_c(C'\cup L'')$ of a cubic C' and a perturbation L'' of a line L' in $\mathbb{C}P^2$. Note that $C'\cup L''$ is smoothly isotopic to $C'\cup L'$. Then we blew-up at the eleven points $P_1,\ldots,P_{11}$ which lie inside $\mathbf{P}'$, and took the proper transforms $\tilde{C}_1,\ldots,\tilde{C}_{11},\tilde{G}\subset\tilde{\mathbf{P}}'$, where $\tilde{\mathbf{P}}'$ is the blow-up of $\mathbf{P}'$. Let $B_\epsilon(\tilde{C}_i),B_\epsilon(\tilde{G})\subset\tilde{\mathbf{P}}'$ be small and disjoint tubular neighbourhoods of $\tilde{C}_i$, $\tilde{G}$, $i=1,\ldots,11$, respectively.

Put $X=W\cup W'$, with

$$W=\bigcup_i B_{2\epsilon}(\tilde{C}_i)\cup B_{2\epsilon}(\tilde{G})\cup T_0,\quad W'=X-\left(\bigcup_i B_\epsilon(\tilde{C}_i)\cup B_\epsilon(\tilde{G})\right),$$

where T_0 denotes an open contractible set constructed by fattening paths joining the base point with the tubular neighbourhoods $B_{2\epsilon}(\tilde{C}_i)$, $B_{2\epsilon}(\tilde{G})$. As $\pi_1(X)$ is trivial, the Seifert Van-Kampen Theorem shows that the map

$$\pi_1(W\cap W')\longrightarrow\pi_1(W')\cong\pi_1(X-(\tilde{C}_1\cup\cdots\cup\tilde{C}_{11}\cup\tilde{G}))$$

is surjective. Note that $W\cap W'$ is homotopy equivalent to the wedge sum $Y_1\vee\cdots\vee Y_{11}\vee Y_{12}$, where $Y_i=\partial B_\epsilon(\tilde{C}_i)$ is the boundary of a small tubular neighbourhood of C_i, and $Y_{12}=\partial B_\epsilon(\tilde{G})$. Hence it is enough to see that every loop in Y_i for $1\le i\le 11$ and every loop in Y_{12} are contractible in $\pi_1(X-(\tilde{C}_1\cup\cdots\cup\tilde{C}_{11}\cup\tilde{G}))$.

Take the plumbing $\mathbf{P}=P_c(C\cup L)$ and the curves $C_1,\ldots,C_{11},G$. We have decomposed $C=D\cup V$, where D is a disc, so we may take α, β inside $C-D$. For each of the cubics C_i, $\pi_1(C_i)$ is generated by loops α_i,β_i which can be taken by lifting α, β via the sections $\hat{\sigma}_i$, $i=1,\ldots,11$, of the complex line bundle $E\to C$. For the curve $G\subset\mathbf{P}$ of genus 3, we have generators $\alpha^{(1)}$, $\beta^{(1)}$, $\alpha^{(2)}$, $\beta^{(2)}$, $\alpha^{(3)}$, $\beta^{(3)}$ of the fundamental group $\pi_1(G)$ with $\prod_{j=1}^3[\alpha^{(j)},\beta^{(j)}]=1$. These can be taken by lifting the loops α, β via the sections $\hat{\tau}_j$, $j=1,2,3$. The base point is also chosen outside the disc D. In $\mathbf{P}-(C_1\cup\cdots\cup C_{11}\cup G)$, we can move vertically (along the fiberwise directions of the bundle $E \to C$) all the loops α_i, β_i, $\alpha^{(j)}$, $\beta^{(j)}$ without touching the other curves. Once we reach the boundary of $\mathbf{P} \cong \mathbf{P}'$, these can be contracted in the complement $\mathbb{C}P^2 - \mathbf{P}'$ by Lemma 26 above.

Now we blow-up inside $\mathbf{P}$ the eleven points $P_1, \ldots, P_{11}$ to obtain $\tilde{\mathbf{P}}$ and the proper transforms $\tilde{C}_1, \ldots, \tilde{C}_{11}, \tilde{G}$. Consider $Y_i = \partial B_\epsilon(\tilde{C}_i)$ as before. This is a circle bundle $S^1 \to Y_i = \partial B_\epsilon(\tilde{C}_i) \to \tilde{C}_i$ with Chern class $c_1(Y_i) = [\tilde{C}_i]^2 = -1$. We have a short exact sequence

$$0 \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(Y_i) \longrightarrow \pi_1(\tilde{C}_i) \longrightarrow 0.$$

Since we are away from the blow-up locus we call the generators of $\pi_1(\tilde{C}_i)$ again α_i, β_i. The loop $[\alpha_i, \beta_i]$ can be homotoped in $B_\epsilon(\tilde{C}_i)$ to the base point through a homotopy transversal to $\tilde{C}_i$. This homotopy intersects $\tilde{C}_i$ in $\tilde{C}_i^2 = -1$ points counted with signs. Via the retraction $B_e(\tilde{C}_i) - \tilde{C}_i \to Y_i$, this gives a homotopy in Y_i between the lifting of $[\alpha_i, \beta_i]$ and γ_i^{-1}, where γ_i is the loop going along the fiber S^1. We conclude that

$$\pi_1(Y_i) = \langle \alpha_i, \beta_i, \gamma_i \mid [\alpha_i, \beta_i] = \gamma_i^{-1},\ \gamma_i \text{ central} \rangle.$$

Note that α_i, β_i can be moved to Y_i without touching the other cubics $\tilde{C}_j$ and then contracted in $\mathbb{C}P^2 - \mathbf{P}$ via the blow-up map. The conclusion is that α_i and β_i can be contracted to a point through a homotopy in $X - (\tilde{C}_1 \cup \cdots \cup \tilde{C}_{11} \cup \tilde{G})$. Therefore the same happens to γ_i.

Analogously, $Y_{12} = \partial B_\epsilon(\tilde{G})$ is a circle bundle $S^1 \to Y_{12} = \partial B_\epsilon(\tilde{G}) \to \tilde{G}$ with Chern class $c_1(Y_{12}) = [\tilde{G}]^2 = 1$. Denoting by γ_{12} the loop along the fiber S^1, we have that

$$\pi_1(Y_{12}) = \Big\langle \alpha^{(1)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(2)}, \alpha^{(3)}, \beta^{(3)}, \gamma_{12} \ \Big| \ \prod_{j=1}^{3} [\alpha^{(j)}, \beta^{(j)}] = \gamma_{12},\ \gamma_{12} \text{ central} \Big\rangle.$$

The loops $\alpha^{(j)}$, $\beta^{(j)}$ can be moved to the boundary Y_{12} and then contracted in $\mathbb{C}P^2 - \mathbf{P}$ via the blow-up map. Thus the same happens to γ_{12}. So all generators of $\pi_1(\partial B_\epsilon(\tilde{C}_i))$, $i = 1, \ldots, 11$, and of $\pi_1(\partial B_\epsilon(\tilde{G}))$ become trivial in $\pi_1(X - (\tilde{C}_1 \cup \cdots \cup \tilde{C}_{11} \cup \tilde{G}))$. This concludes the proof. □

Once we have the symplectic orbifold X', we construct a Seifert bundle $M \to X'$ with primitive Chern class $c_1(M/e^{2\pi i/\mu}) = [\omega]$. This is a K-contact manifold, which is simply-connected.

Theorem 28. *The 5-manifold M is simply-connected, hence it is a Smale–Barden manifold.*

Proof: By [**6**, Theorem 4.3.18], we have an exact sequence $\pi_1(S^1) = \mathbb{Z} \to \pi_1(M) \to \pi_1^{\mathrm{orb}}(X') = 1$. In particular, $\pi_1(M)$ is abelian. Therefore $\pi_1(M) = H_1(M, \mathbb{Z}) = 0$, by Theorem 2. □

5. Non-existence of an algebraic surface with the given pattern of curves

In this section we show that it is not possible to construct an algebraic surface with the same configuration of complex curves as the manifold we constructed in Section 3, that is, twelve disjoint complex curves spanning $H_2(S, \mathbb{Q})$, one of genus 3 and all the others of genus 1. More concretely, we prove Theorem 9.

Theorem 29. *Suppose S is a complex surface with $b_1 = 0$ and disjoint smooth complex curves spanning $H_2(S, \mathbb{Q})$, one of them of genus $g \geq 1$ and all the others elliptic (and thus of genus 1). Then $b_2 \leq 2g^2 - 4g + 3$.*

Proof: Let S be a complex surface with $b_1 = 0$, containing disjoint complex curves spanning $H_2(S, \mathbb{Q})$, one of them, say D_1, of genus g and the other curves $D_2, \ldots, D_{b_2}$ all of genus 1.

The Poincaré duals $[D_1], \ldots, [D_{b_2}]$ are a basis of $H^2(S, \mathbb{Q})$, since $\{D_1, \ldots, D_{b_2}\}$ is a basis of $H_2(S, \mathbb{Q})$. Furthermore, these classes are all of type $(1,1)$, so we have that $h^{1,1} = b_2$ and the geometric genus is $p_g = h^{2,0} = 0$. The irregularity is $q = h^{1,0} = 0$ since $b_1 = 0$. In particular, S is an algebraic surface [**4**]. The holomorphic Euler characteristic is

$$\chi(\mathcal{O}_S) = 1 - q + p_g = 1. \tag{9}$$

By the Riemann–Hodge relations, the signature of $H^{1,1}(S)$ is $(1, b_2 - 1)$. Therefore, the self-intersection of one of the D_i's is positive and it is negative for the others.

Case 1. Assume for the moment that $g = g(D_1) \geq 2$. We show first that $D_1^2 > 0$. Suppose otherwise that $D_i^2 > 0$ for one of the genus 1 curves. After reordering, we can suppose this is true for D_2. By the adjunction formula, $K_S \cdot D_2 + D_2^2 = 2g(D_2) - 2 = 0$, so $K_S \cdot D_2 = -D_2^2$. And, by Riemann–Roch's theorem, we have,

$$\chi(D_2) = \chi(\mathcal{O}_S) + \frac{D_2^2 - K_S \cdot D_2}{2} = 1 + D_2^2.$$

Hence using Serre duality,

$$h^0(D_2) + h^0(K_S - D_2) = h^0(D_2) + h^2(D_2) \geq \chi(D_2) = 1 + D_2^2 \geq 2.$$

Also, from the exact short sequence $0 \to \mathcal{O}_S(K_S - D_2) \to \mathcal{O}_S(K_S) \to \mathcal{O}_{D_2}(K_S|_{D_2}) \to 0$ we deduce that $h^0(K_S - D_2) = 0$, since $h^0(K_S) = h^{2,0}(S) = 0$. Thus $h^0(D_2) \geq 2$ and we can consider a pencil $\mathbb{P}^1 \leq |D_2|$.

This gives a rational map $S \dashrightarrow \mathbb{C}P^1$ and, after blowing-up the base points of the pencil, an elliptic fibration $\tilde{S} \to \mathbb{C}P^1$, with the proper transform of D_2 as a smooth fiber. However, since they are pairwise disjoint, all the proper transforms of D_i, $i \neq 2$, have to lie in fibers. To see this, consider the projections of the proper transforms of D_i by the elliptic fibration $\tilde{S} \to \mathbb{C}P^1$. These projections must be Zariski-closed, connected subsets of $\mathbb{C}P^1$. As they are not all $\mathbb{C}P^1$ since they do not intersect the proper transform of D_2, they should be points. In particular, since all the fibers are connected, the arithmetic genus of each irreducible component of a fiber has to be at most 1, which gives rise to a contradiction as $g(D_1) = g > 1$. Therefore, $D_1^2 > 0$ and $D_2^2, \ldots, D_{b_2}^2 < 0$.

Denote $m_1 = D_1^2$ and $m_i = -D_i^2$, $i = 2, \ldots, b_2$. All of the m_i's are positive integers. Write $K_S \equiv \sum_{i=1}^{b_2} \lambda_i D_i$ for its homology class in $H_2(S, \mathbb{Q})$, with $\lambda_i \in \mathbb{Q}$. Notice that $K_S \cdot D_i = \lambda_i D_i^2$, from where $\lambda_i = \frac{K_S \cdot D_i}{D_i^2}$. By the adjunction formula,

$$\begin{aligned} K_S \cdot D_1 &= 2g(D_1) - 2 - D_1^2 = 2g - 2 - m_1, \\ K_S \cdot D_i &= 2g(D_i) - 2 - D_i^2 = m_i, \quad i \geq 2. \end{aligned} \tag{10}$$

Therefore

$$K_S \equiv \frac{2g - 2 - m_1}{m_1} D_1 - \sum_{i=2}^{b_2} D_i,$$

and we get

$$K_S^2 = \frac{(2g - 2 - m_1)^2}{m_1} - \sum_{i=2}^{b_2} m_i. \tag{11}$$

Consider the following short exact sequence of sheaves,

$$0 \longrightarrow \mathcal{O}(K_S) \longrightarrow \mathcal{O}(K_S + D_1) \longrightarrow \mathcal{O}_{D_1}(K_{D_1}) \longrightarrow 0,$$

where $K_{D_1} = (K_S + D_1)|_{D_1}$, by adjunction. This gives a long exact sequence in cohomology,

$$0 \longrightarrow H^0(K_S) \longrightarrow H^0(K_S + D_1) \longrightarrow H^0(K_{D_1}) \longrightarrow H^1(K_S) \longrightarrow \cdots,$$

where $H^0(K_S) = H^{2,0}(S) = 0$ and $H^1(K_S) = H^1(\mathcal{O}_S) = H^{0,1}(S) = 0$. So we have an isomorphism $H^0(K_S + D_1) \cong H^0(K_{D_1})$ and we deduce that $h^0(K_S + D_1) = h^0(K_{D_1}) = g$. In particular, the linear system $|K_S + D_1|$ is not empty and it has dimension $g - 1 \geq 1$. Let $Z = Z(|K_S + D_1|)$ be the fixed part of $|K_S + D_1|$ (that is, the largest effective divisor such that $D \geq Z$ for all $D \in |K_S + D_1|$). Notice that $Z \cdot D_1 = 0$, since the restriction of the linear system to D_1, $|(K_S + D_1)|_{D_1}| = |K_{D_1}|$, has no fixed points as $g \geq 2$.

Write now Z as an effective divisor $Z = \sum_{i=1}^{b_2} \alpha_i D_i + T$, where α_i are non-negative integers and T is an effective divisor not containing any of the D_i's. Notice that the latter implies $T \cdot D_i \geq 0$ for all i. Since $0 = Z \cdot D_1 = \alpha_1 m_1 + T \cdot D_1$, we have $\alpha_1 = 0$ and $T \cdot D_1 = 0$. So we can write $Z = \sum_{i=2}^{b_2} \alpha_i D_i + T$ and T does not intersect D_1.

Let us see that $T = 0$. Write $T \equiv \sum_{i=1}^{b_2} \mu_i D_i$ for its homology class in $H_2(S, \mathbb{Q})$, with $\mu_i \in \mathbb{Q}$. First note that, since $T \cdot D_1 = 0$, we have $\mu_1 = 0$, so $T \equiv \sum_{i=2}^{b_2} \mu_i D_i$. For $i \geq 2$, $0 \leq T \cdot D_i = -\mu_i m_i$, hence $\mu_i \leq 0$. Let $n \geq 1$ be an integer such that $n\mu_i \in \mathbb{Z}$ for all i. Hence nT is effective and $-nT = \sum(-n\mu_i) D_i$ is also effective. This implies that $nT = 0$ and thus $T = 0$. This means that the fixed part is $Z = \sum_{i=2}^{b_2} \alpha_i D_i$.

Write $|K_S + D_1| = Z + |F|$, where F is the free part, which is a fully movable divisor. We look now at the self-intersection $F^2 = (K_S + D_1 - Z)^2 \geq 0$. Recall that the self-intersection of a fully movable divisor is $F^2 \geq 0$, since if we take a different $F' \equiv F$ such that F and F' do not share components, then $F^2 = F \cdot F' \geq 0$.

Let $j \geq 2$ and suppose both that $m_j = 1$ and $D_j \nleq Z$. In this case, the restriction of an effective divisor $C \in |K_S + D_1|$ to D_j is an effective degree 1 divisor, since $(K_S + D_1) \cdot D_j = K_S \cdot D_j = m_j = 1$, by (10). So $C \cap D_j$ is a point P_j. Furthermore, since D_j is not a rational curve, any pair of linearly equivalent points are actually equal. Therefore $P_j \in D_j$ is a fixed point of $|K_S + D_1|$ and, since $Z \cap D_j = \emptyset$, $P_j \in F$. So P_j is counted in the self-intersection $(K_S + D_1 - Z)^2$.

There are at most $\sum_{i=2}^{b_2} m_i - (b_2 - 1)$ curves among $D_2, \ldots, D_{b_2}$ with $m_i > 1$. So there are at least $2(b_2 - 1) - \sum_{i=2}^{b_2} m_i$ curves with self-intersection -1. Hence there are at least $2(b_2 - 1) - \sum_{i=2}^{b_2} m_i - r$ fixed points $P_j \in D_j$ of some $|(K_S + D_1 - Z)|_{D_j}|$, where $r = \#\{\alpha_i > 0\}$, and thus

$$(K_S + D_1 - Z)^2 \geq 2(b_2 - 1) - \sum_{i=2}^{b_2} m_i - r. \tag{12}$$

Note that in case we have $2(b_2 - 1) - \sum_{1=2}^{b_2} m_i - r \leq 0$ we cannot assure the existence of any fixed point but the inequality still holds since $(K_S + D_1 - Z)^2 \geq 0$.

We now compute $(K_S + D_1 - Z)^2$,

$$\begin{aligned}
(K_S + D_1)^2 &= K_S^2 + 2K_S \cdot D_1 + D_1^2 = K_S^2 + 2(2g - 2 - m_1) + m_1 \\
&= K_S^2 + 4g - 4 - m_1, \\
(K_S + D_1 - Z)^2 &= (K_S + D_1)^2 - 2(K_S + D_1) \cdot Z + Z^2 \\
&= K_S^2 + 4g - 4 - m_1 - 2\sum_{i=2}^{b_2} \alpha_i m_i - \sum_{i=2}^{b_2} \alpha_i^2 m_i.
\end{aligned}$$

Thus (12) gives

$$K_S^2 + 4g - 4 - m_1 - 2\sum_{i=2}^{b_2} \alpha_i m_i - \sum_{i=2}^{b_2} \alpha_i^2 m_i \geq 2(b_2 - 1) - \sum_{i=2}^{b_2} m_i - r,$$

from which

$$\begin{aligned} 2b_2 &\leq 4g - 2 - m_1 + K_S^2 + \sum_{i=2}^{b_2} m_i + r - 2\sum_{i=2}^{b_2} \alpha_i m_i - \sum_{i=2}^{b_2} \alpha_i^2 m_i \\ &\leq 4g - 2 - m_1 + K_S^2 + \sum_{i=2}^{b_2} m_i + r - 3r \leq 4g - 2 - m_1 + K_S^2 + \sum_{i=2}^{b_2} m_i. \end{aligned}$$

Using (11), we have

$$2b_2 \leq 4g - 2 - m_1 + \frac{(2g - 2 - m_1)^2}{m_1}.$$

The expression on the right is a decreasing function on m_1. Therefore, we can bound it by its value in $m_1 = 1$, that is

$$2b_2 \leq 4g - 3 + (2g - 3)^2 = 4g^2 - 8g + 6.$$

Hence $b_2 \leq 2g^2 - 4g + 3$, as required.

Case 2. Suppose now $g = 1$. Using the adjunction formula, we get $K_S \equiv -\sum_{i=1}^{b_2} D_i$. And using the same argument as above, we have that $h^0(K_S + D_1) = h^0(K_{D_1}) = g = 1$. Thus, there is an effective divisor in S linearly equivalent to $K_S + D_1 = -\sum\limits_{i=2}^{b_2} D_i$ which is clearly anti-effective if $b_2 \geq 2$. Therefore, $b_2 \leq 1$. □

Let us end up by giving a different proof of the non-existence of a Kähler surface S with $b_1 = 0$ and $b_2 = 12$, containing disjoint smooth complex curves spanning $H_2(S, \mathbb{Q})$, one of them of genus $g = 3$, all the others of genus $g_i = 1$. It makes very specific use of the numbers at hand.

We follow the notations in the proof of Theorem 29. We have the curves $D_1, D_2, \ldots, D_b$, $b = 12$, with $D_1^2 = m_1$, $D_i^2 = -m_i$, $2 \leq i \leq b$, and all the m_i's are positive integers. The curve D_1 has genus $g = 3$ and D_i have genus 1, $2 \leq i \leq b$. By (9) and Noether's formula [**4**] we have that

$$\frac{1}{12}(K_S^2 + c_2(S)) = \chi(\mathcal{O}_S) = 1 - q + p_g = 1.$$

Note that $c_2(S) = \chi(S) = 2 + b$, where $b = b_2 = 12$ and $b_1 = b_3 = 0$. Therefore $K_S^2 = 10 - b = -2$. Now (11) says that

$$-2 = K_S^2 = \frac{(4-m_1)^2}{m_1} - m_2 - \cdots - m_b \leq \frac{(4-m_1)^2}{m_1} - 11,$$

using that $g = 3$. Therefore $(4 - m_1)^2 \geq 9m_1$, which is rewritten as $(m_1 - 16)(m_1 - 1) \geq 0$.

If $m_1 \geq 16$, then the curve D_1 of genus $g = 3$ has self-intersection $D_1^2 \geq 2g + 1$. The argument of [**21**, Theorem 32] concludes that $b \leq 2g + 3$. This is a contradiction since $g = 3$ and $b = 12$.

Therefore we have that $m_1 = 1$. So

$$K_S = 3D_1 - D_2 - \cdots - D_b$$

and $K_S^2 = -2 = 9 - m_2 - \cdots - m_b \leq 9 - 11 = -2$. Therefore there must be equality and $m_2 = \cdots = m_b = 1$. The basis $\{D_1, D_2, \ldots, D_b\}$ is a diagonal basis of $H_2(S, \mathbb{Z})$. Now we try to reconstruct S in "reverse". Let $H, E_2, \ldots, E_b \in H_2(S, \mathbb{Z})$ be defined by the equalities:

$$\begin{aligned} D_1 &= 10H - 3E_2 - \cdots - 3E_b, \\ D_j &= (3H - E_2 - \cdots - E_b) + E_j, \quad j = 2, \ldots, b. \end{aligned}$$

This is solved as:

$$\begin{aligned} H &= 10D_1 - 3D_2 - \cdots - 3D_b, \\ E_j &= 3D_1 + D_j - \sum_{k=2}^{b} D_k, \quad j = 2, \ldots, b, \\ K_S &= -3H + \sum_{k=2}^{b} E_k. \end{aligned}$$

The following self-intersections are easily computed:

$$\begin{array}{lll} H^2 = 1, & H \cdot E_j = 0, & j = 2, \ldots, b, \\ E_j^2 = -1, & E_j \cdot E_k = 0, & j \neq k, \\ K_S \cdot H = -3, & K_S \cdot E_j = -1, & j = 2, \ldots, b, \\ D_j \cdot E_j = 0, & D_j \cdot E_k = 1, & j \neq k. \end{array}$$

Now let us prove that the classes $E_2, \ldots, E_b$ are defined by effective divisors. Note that $\chi(E_j) = 1 + \frac{1}{2}(E_j^2 - K_S \cdot E_j) = 1$. Also $h^0(K_S - E_j) = 0$ since $(K_S - E_j) = -D_j < 0$. Therefore $h^0(E_j) \geq 1$ and E_j is effective.

Next note that $K_S + D_j = E_j$. Consider the long exact sequence in cohomology associated to the exact sequence

$$0 \longrightarrow \mathcal{O}(K_S) \longrightarrow \mathcal{O}(K_S + D_j) \longrightarrow \mathcal{O}_{D_j}(K_{D_j}) \longrightarrow 0.$$

As $H^0(K_S) = H^1(K_S) = 0$, we have that $h^0(E_j) = h^0(K_S + D_j) = h^0(\mathcal{O}_{D_j}(K_{D_j})) = 1$, since D_j is an elliptic curve. Also $h^2(E_j) = h^0(K_S - E_j) = 0$, and hence $h^1(E_j) = 0$ since $\chi(E_j) = 1$.

Consider now the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(D_2 + \cdots + D_b) \longrightarrow \bigoplus_{j=2}^{b} \mathcal{O}_{D_j}(D_j) \longrightarrow 0, \tag{13}$$

which holds since the D_j are disjoint. As $D_j^2 = -m_j = -1$ and D_j is an elliptic curve, we have $h^0(\mathcal{O}_{D_j}(D_j)) = 0$ and $h^1(\mathcal{O}_{D_j}(D_j)) = 1$. Therefore (13) and the fact that $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$ imply that $h^0(D_2 + \cdots + D_b) = 1$ and $h^1(D_2 + \cdots + D_b) = b - 1 = 11$. Using that $3D_1 \equiv D_2 + \cdots + D_{b-1} + E_b$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(E_b) \longrightarrow \mathcal{O}(3D_1) \longrightarrow \bigoplus_{j=2}^{b-1} \mathcal{O}_{D_j}(E_b) \longrightarrow 0.$$

As $D_j \cdot E_b = 1$ and D_j is elliptic, we have $h^0(\mathcal{O}_{D_j}(E_b)) = 1$. Then $h^0(3D_1) = b - 1 = 11$, using $h^0(E_b) = 1$ and $h^1(E_b) = 0$ computed before.

Now we compute $h^0(3D_1)$ in a different way. We have exact sequences:

$$\begin{aligned}
&0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(D_1) \longrightarrow \mathcal{O}_{D_1}(D_1) \longrightarrow 0,\\
&0 \longrightarrow \mathcal{O}(D_1) \longrightarrow \mathcal{O}(2D_1) \longrightarrow \mathcal{O}_{D_1}(2D_1) \longrightarrow 0, \text{ and}\\
&0 \longrightarrow \mathcal{O}(2D_1) \longrightarrow \mathcal{O}(3D_1) \longrightarrow \mathcal{O}_{D_1}(3D_1) \longrightarrow 0
\end{aligned}$$

so

$$\begin{aligned}
h^0(3D_1) &\leq h^0(2D_1) + h^0(\mathcal{O}_{D_1}(3D_1))\\
&\leq h^0(D_1) + h^0(\mathcal{O}_{D_1}(2D_1)) + h^0(\mathcal{O}_{D_1}(3D_1))\\
&\leq 1 + h^0(\mathcal{O}_{D_1}(D_1)) + h^0(\mathcal{O}_{D_1}(2D_1)) + h^0(\mathcal{O}_{D_1}(3D_1)).
\end{aligned} \tag{14}$$

We use Clifford's theorem [**2**, p. 107] that says that for a curve of genus $g \geq 1$ and a divisor D of degree $0 \leq d \leq 2g - 2$, we have $h^0(D) \leq \left[\frac{d}{2}\right] + 1$. Applying this to the curve D_1, we have $h^0(\mathcal{O}_{D_1}(D_1)) \leq 1$, $h^0(\mathcal{O}_{D_1}(2D_1)) \leq 2$, and $h^0(\mathcal{O}_{D_1}(3D_1)) \leq 2$, recalling that $D_1^2 = 1$. Therefore (14) implies $h^0(3D_1) \leq 1 + 1 + 2 + 2 = 6$. This is a contradiction with the previous computation of $h^0(3D_1)$.

6. The second Stiefel–Whitney class of the Smale–Barden manifold

We close with a proof of the last comments in the introduction.

We start by computing the Stiefel–Whitney class $w_2(M)$ of the 5-manifold M constructed in Corollary 10. This manifold is a Seifert circle bundle $\pi\colon M \to X'$ over the cyclic 4-orbifold constructed with the manifold $X = \mathbb{C}P^2 \# 11\overline{\mathbb{C}P}^2$ of Section 3, ramified over the curves $\tilde{C}_1, \dots, \tilde{C}_{11}$, $\tilde{C}_{12} = \tilde{G}$, with multiplicities $m_i = p^i$, $i = 1, \dots, 12$, and p a fixed prime.

Remark 30. We could use other powers of p for the m_i's, as long as they are distinct. Also we can use $m_i = p^i \tilde{m}_i$, with $\gcd(\tilde{m}_i, p) = 1$. However, for the computations below we stick to our choice $m_i = p^i$.

The Seifert bundle $\pi\colon M \to X'$ is determined by the Chern class

$$c_1(M/X') = c_1(B) + \sum \frac{b_i}{m_i}[\tilde{C}_i],$$

where b_i are called the orbit invariants (they should satisfy $\gcd(b_i, m_i) = 1$), and B is a suitable line bundle over X. By Theorem 2, we have to impose the condition that the cohomology class

$$c_1(M/e^{2\pi i \mu}) = \mu\, c_1(B) + \sum b_i \frac{\mu}{m_i}[\tilde{C}_i] = p^{12} c_1(B) + \sum b_i p^{12-i}[\tilde{C}_i]$$

is primitive and represented by some orbifold symplectic form $[\hat{\omega}]$, being $\mu = \operatorname{lcm}(m_i) = p^{12}$. The proof of [**21**, Lemma 20] shows that in order to ensure this we can take $b_i = 1$ and a class $a = c_1(B) \in H^2(X, \mathbb{Z})$ with $a = \sum a_i [\tilde{C}_i]$ and $\gcd(pa_1 + 1, p^2 a_2 + 1) = 1$. Certainly, with this condition on the class a and the numbers b_i, the Chern class is

$$c_1(M/e^{2\pi i \mu}) = \sum p^{12-i}(p^i a_i + 1)[\tilde{C}_i],$$

so it is primitive.

Let us see that we can ensure that $\gcd(pa_1 + 1, p^2 a_2 + 1) = 1$. Take any $a_2 \in \mathbb{Z}$ and write

$$p^2 a_2 + 1 = \prod_{j=1}^{l} q_j^{m_j}$$

with q_j different primes. The condition $\gcd(pa_1 + 1, p^2 a_2 + 1) = 1$ is equivalent to the condition that q_j does not divide $pa_1 + 1$ for all j. Since q_j and p are coprime, by Bezout's identity we have $1 + \alpha p = \beta q_j$ for some $\alpha, \beta \in \mathbb{Z}$. In fact, all the numbers α, β satisfying that condition are of the form $(\alpha, \beta) = (\alpha_j + tq, \beta_j + tp)$, $t \in \mathbb{Z}$. Applying this to $q_1, \dots, q_l$ we get that the condition is that a_1 does not belong to the set

$$A = \{\alpha_1 + tq_1 \mid t \in \mathbb{Z}\} \cup \dots \cup \{\alpha_l + tq_l \mid t \in \mathbb{Z}\}. \tag{15}$$

The set A above is a union of arithmetic progressions of ratios $q_1, \dots, q_l$ which are different primes. By the Chinese remainder theorem there is $\alpha \pmod{\prod q_i}$ so that $\alpha \equiv \alpha_i \pmod{q_i}$ for all i. Therefore the set $\mathbb{Z} - A$ modulo $\prod q_i$ contains $\phi\big(\prod q_i\big) = \prod (q_i - 1)$ elements. In particular it is infinite and of positive density.

The conclusion is that possible choices of $a_1, \dots, a_{12}$ consist of choosing freely $a_2, \dots, a_{12} \in \mathbb{Z}$, and then choose $a_1 \in \mathbb{Z} - A$.

Proposition 31. *If $p = 2$, then M is spin. If $p > 2$, then we can arrange $c_1(B)$ and b_i so that M is spin or non-spin.*

Proof: If $p = 2$, then [**23**, Proposition 13] says that the map $\pi^*\colon H^2(X, \mathbb{Z}_2) \to H^2(M, \mathbb{Z}_2)$ is zero, since the $[\tilde{C}_i]$ are in $\ker \pi^*$ and they span the cohomology. By formula (3) in [**23**], we have $w_2(M) = \pi^*(w_2(X) + \sum b_i[\tilde{C}_i] + c_1(B)) = 0$.

If $p > 2$, then [**23**, Proposition 13] says that the map $\pi^*\colon H^2(X, \mathbb{Z}_2) \to H^2(M, \mathbb{Z}_2)$ has one-dimensional kernel spanned by $c_1(B) + \sum b_i[\tilde{C}_i]$. Formula (2) in [**23**] says that $w_2(M) = \pi^*(w_2(X))$. As $X = \mathbb{C}P^2 \# 11 \overline{\mathbb{C}P}^2$, $w_2(X) = H + E_1 + \dots + E_{11} = \tilde{C}_1 + \dots + \tilde{C}_{12} \pmod 2$. The manifold M is spin or non-spin according to whether $c_1(B) + \sum b_i[\tilde{C}_i]$ is proportional to $\sum [\tilde{C}_i]$ or not (mod 2). We can arrange that M is non-spin by taking, say, a_2 odd. To get M spin we need to take all $a_2, \dots, a_{12}$ even, and then choose $a_1 \in \mathbb{Z} - A'$, where A' is defined as (15) but including also $q_0 = 2$, $\alpha_0 = 1$ (note that all q_j are odd in this case). So a_1 is also even. $\square$

Remark 32. We end with the proof that the symplectic manifold produced in [**21**] does not admit a complex structure. That manifold Z is constructed as follows: take the 4-torus $\mathbb{T}^4$ with coordinates (x_1, x_2, x_3, x_4) and take 2-tori T_{12}, T_{13}, T_{14} along the directions (x_1, x_2), (x_1, x_3), (x_1, x_4), respectively. We arrange these 2-tori to be disjoint and make them symplectic. Now perform Gompf connected sums [**14**] with 3-copies of the rational elliptic surface $E(1)$ along a fiber F. This produces

$$Z' = \mathbb{T}^4 \#_{T_{12}=F} E(1) \#_{T_{13}=F} E(1) \#_{T_{14}=F} E(1).$$

Now we blow-up twice to get $Z = Z' \# 2\overline{\mathbb{C}P}^2$. Then Z is simply connected and it has $b_2(Z) = 36$ and contains thirty-six disjoint surfaces, of which thirty-one of genus 1 and negative self-intersection, two of genus 2, and three of genus 3, all of positive self-intersection (see Theorem 23 in [**21**]). Then $b_2^+(Z) = 5$ and $b_2^-(Z) = 31$.

Suppose that Z admits a complex structure. First, for a complex manifold $b_2^+ = 1 + p_g$ and $b_2^- = h^{1,1} + p_g - 1$, thus $b_2^- \geq b_2^+ - 1$, as $h^{1,1} \geq 1$.

This implies that the orientation of Z has to be the same as a complex manifold. Also $p_g = 4$ and $h^{1,1} = 28$. Using Noether's formula,

$$\frac{K_Z^2 + c_2}{12} = \chi(\mathcal{O}_Z) = 1 - q + p_g = 5.$$

As $c_2 = \chi(Z) = 38$, we get $K_Z^2 = 22$. As $K_Z^2 > 9$, the Enriques classification ([**4**, p. 188]) implies that Z is of general type.

Next we use the Seiberg–Witten invariants of Z. For a minimal surface X of general type, the only Seiberg–Witten basic classes are $\pm K_X$ (see Proposition 2.2 in [**11**]). If Z is the blow-up of X at s points, then the basic classes of Z are $\kappa_Z = \pm K_X \pm E_1 \pm \cdots \pm E_s$, where $E_1, \ldots, E_s$ are the exceptional divisors. Note that $\kappa_Z^2 = K_X^2 - s = K_Z^2 = 22$.

Now we compute the Seiberg–Witten basic classes of Z. The only Seiberg–Witten basic class of $\mathbb{T}^4$ is $\kappa = 0$. The Seiberg–Witten basic classes of a Gompf connected sum along a torus can be found in [**24**, Corollary 15]. Using the relative Seiberg–Witten invariant of $E(1)$ in [**24**, Theorem 18], we have that the Seiberg–Witten invariants satisfy

$$SW_{X\#_F E(1)} = SW_X \cdot (e^F + e^{-F}),$$

for a 4-manifold X. Therefore the basic classes of Z' are only $\kappa' = \pm T_{12} \pm T_{13} \pm T_{14}$. When blowing-up, the basic classes of Z are $\kappa_Z = \pm T_{12} \pm T_{13} \pm T_{14} \pm E_1 \pm E_2$. Then $\kappa_Z^2 = -2$, which is a contradiction.

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