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GLOBAL EXISTENCE FOR VECTOR VALUED FRACTIONAL REACTION-DIFFUSION EQUATIONS

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Abstract: In this paper we study the initial value problem for infinite dimensional fractional non-autonomous reaction-diffusion equations. Applying general time-splitting methods, we prove the existence of solutions globally defined in time using convex sets as invariant regions. We expose examples where biological and pattern formation systems, under suitable assumptions, achieve global existence. We also analyze the asymptotic behavior of solutions.

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1. Introduction

In this paper we prove global existence of solutions for vector valued fractional non-autonomous reaction-diffusion equations. We study the non-autonomous system

$$\partial_t u + \sigma(-\Delta)^\beta u = F(t,u), \tag{1.1}$$

where $u(t,x) \in Z$ for $x \in \mathbb{R}^n$, $t > 0$, $\sigma \geq 0$, and $0 < \beta \leq 1$, $F\colon \mathbb{R}\times Z \to Z$ is a continuous map, and Z is a Banach space. We consider the initial value problem $u(x,0) = u_0(x)$.

The aim of this paper is to develop a new method to obtain behavioral results on the fractional reaction-diffusion equation, using recent numerical splitting techniques [**6, 14**] introduced for other purposes. The main goal is to obtain general conditions for global well-posedness of the fractional reaction-diffusion equation in Banach spaces.

Fractional reaction-diffusion equations are commonly used in many applications such as biological models, population dynamics models, nuclear reactor models, just to name a few (for references to examples see [**4**]). The difference between classical and fractional diffusion is that the classical Laplacian term associated with classical diffusion implies a Gaussian dispersal kernel in the corresponding equation, which does not represent all possible models in practice. The fractional model captures the faster spreading rates and power law invasion profiles observed in many applications. This behavior is given by the fractional Laplacian, that is described by standard theories of fractional calculus (for a complete survey see [**24**]). There are many different equivalent definitions of the fractional Laplacian and its properties are well understood (see [**7, 15, 21, 23, 31, 28, 22**]).

The non-autonomous nonlinear reaction-diffusion equation dynamics were studied, among others, by the authors of [**29**], analyzing the stability and evolution of the problem. Global existence in reaction-diffusion equations in bounded sets was studied in the book by Smoller [**33**] and in [**12**] where the authors consider the n-dimensional case with classical diffusion and the intersection of half-spaces as invariant regions in $\mathbb{R}^n$, where the equation evolves. The case of adopting a convex set as an invariant set is considered only when the diffusion coefficients $\sigma_{ij} \in \mathbb{R}^{n\times n}$ form the identity matrix (see Corollary 14.8 (b) in [**33**]). Morgan ([**26**]) considered a similar case where the σ_{ij} do not form the identity matrix, but other conditions are needed for the system to achieve the result. These techniques have been used frequently to obtain local and global well-posedness for different classical diffusion problems (see [**1, 5, 25, 32**]). In our case, we study global existence of fractional reaction-diffusion equations using a completely different approach. We use time splitting methods in Banach spaces using closed convex sets as invariant regions. We remark that, any difficulties that the nonlocal operator could add are avoided by the splitting method, requiring only to prove results about the linear equation.

As an example, we explore the scalar system where the nonlinearity is given by $F(u) = (1+ia)u-(1+ib)|u|^2u$ with $a, b \in \mathbb{R}$ (see [**12, 35, 10**]). For particular nonlinearities exact solutions are known, for instance, in [**20**] the existence of scalar traveling waves for the quadratic, cubic, and quartic cases was studied using the tanh method. We also explore a FitzHugh–Nagumo pattern formation system in $\mathbb{R}^2$ and a population dynamic system in a Banach space. In each case, we found an appropriate invariant region that allows us to prove global existence. Finally, we also analyze the asymptotic behavior of solutions in $\mathbb{R}$.

This paper is organized as follows:

In Section 2 we introduce the notation and prove preliminary results concerning local well-posedness of the linear and nonlinear components of the fractional reaction-diffusion equation. In Section 3 we introduce the propagators, allowing us to construct a splitting reaction-diffusion equation. This is important to build up the linear part. In Section 4 we obtain several results to finally prove the convergence of the “splitting” equation to the “original” equation. This allows us to study, separately, the linear and nonlinear parts in order to obtain interesting results about the original equation. In Section 5 we prove global well-posedness for invariant closed convex sets of a Banach space. We show that the linear and nonlinear parts of the splitting equation maintain independently the solution inside the convex set. We expose examples, such us the Ginzburg–Landau equation and the Fisher–Kolmogorov equation. In Subsection 5.1 we describe an interesting example, a population dynamics model, with a trait variable in a Banach space. This shows the importance of extending the results to Banach spaces. In Subsection 5.2 we extend the results from Section 5 (Proposition 5.7 and Theorem 5.8) to products of Banach spaces. In Section 6 we analyze a completely different problem, namely the asymptotic behavior of a solution. The strategy is, again, to split the linear and nonlinear parts, analyze them separately, and use the results from Sections 4 and 5.

2. Notations and preliminaries

We are interested in continuous functions with vectorial values, that is to say, Banach space valued functions. The main reason for this is to analyze well-posedness of population dynamics problems with discrete or continuous traits that distinguish the population components (see Subsection 5.1).

Let Z be a Banach space. We define $C_{\mathrm{u}}(\mathbb{R}^d, Z)$ as the set of uniformly continuous and bounded functions on $\mathbb{R}^d$ with values in Z. Defining the norm

$$\|u\|_{\infty,Z} = \sup_{x\in\mathbb{R}^d} |u(x)|_Z,$$

$C_{\mathrm{u}}(\mathbb{R}^d, Z)$ is a Banach space. It is easy to see that, if $g \in L^1(\mathbb{R}^d)$ and $u \in C_{\mathrm{u}}(\mathbb{R}^d, Z)$, the Bochner integral is defined in the following way:

$$(g * u)(x) = \int_{\mathbb{R}^d} g(y)u(x - y)\, dy.$$

This defines an element of $C_{\mathrm{u}}(\mathbb{R}^d, Z)$ and the linear operator $u \mapsto g * u$ is continuous (see [**11**]).

The following results show that the operator $-(-\Delta)^{\beta}$ defines a continuous contraction semigroup in the Banach space $C_{\mathrm{u}}(\mathbb{R}^d, Z)$. The following lemma is a consequence of the Lévy–Khintchine formula for infinitely divisible distributions and properties of the Fourier transform.

Lemma 2.1. *Let* $0<\beta\le 1$ *and* $g_\beta\in C_0(\mathbb{R}^d)$ *such that* $\hat g_\beta(\xi)=e^{-|\xi|^{2\beta}}$. *Then, the function* g_β *is positive, invariant under rotations of* $\mathbb{R}^d$*, integrable, and*

$$\int_{\mathbb{R}^d} g_\beta(x)\,dx=1.$$

Proof: The first statement follows from Theorem 14.14 of [**30**], and the remaining claims are an easy consequence of the definition of $\hat g_\beta$. □

Based on the previous lemma, we study Green's function related to the linear operator $\partial_t+\sigma(-\Delta)^\beta$.

Proposition 2.2. *Let* $\sigma>0$ *and* $0<\beta\le 1$*. The function* $G_{\sigma,\beta}$*, given by*

$$G_{\sigma,\beta}(t,x)=(\sigma t)^{-\frac{d}{2\beta}}g_\beta((\sigma t)^{-\frac{1}{2\beta}}x),$$

verifies

(i) $G_{\sigma,\beta}(\cdot,t)>0$.

(ii) $G_{\sigma,\beta}(\cdot,t)\in L^1(\mathbb{R}^d)$ *and*

$$\int_{\mathbb{R}^d} G_{\sigma,\beta}(t,x)\,dx=1.$$

(iii) $G_{\sigma,\beta}(\cdot,t)*G_{\sigma,\beta}(\cdot,t')=G_{\sigma,\beta}(\cdot,t+t')$ *for* $t,t'>0$.

(iv) $\partial_t G_{\sigma,\beta}+\sigma(-\Delta)^\beta G_{\sigma,\beta}=0$ *for* $t>0$.

Proof: The first and second statements are a consequence of the definition of $\hat g_\beta$. The third and fourth statements are immediate applying Fourier transform. □

If X is Banach space, then we denote the space of bounded linear operators from X to X as $\mathcal{B}(X)$. In our case we consider $X=C_{\mathrm{u}}(\mathbb{R}^d,Z)$. In the following proposition we show that the linear operator $-\sigma(-\Delta)^\beta$ defines a continuous contraction semigroup in $C_{\mathrm{u}}(\mathbb{R}^d,Z)$.

Proposition 2.3. *If* $u\in C_{\mathrm{u}}(\mathbb{R}^d,Z)$*, then for any* $\sigma>0$ *and* $0<\beta\le 1$*, the map* $\mathsf{S}\colon\mathbb{R}_+\to\mathcal{B}(C_{\mathrm{u}}(\mathbb{R}^d,Z))$ *defined by* $\mathsf{S}(t)u=G_{\sigma,\beta}(\cdot,t)*u$ *is a continuous contraction semigroup.*

Proof: We first prove the semigroup property, which is deduced from (iii) of Proposition 2.2:

$$\begin{aligned}\mathsf{S}(t)\mathsf{S}(t')u&=G_{\sigma,\beta}(\cdot,t)*(G_{\sigma,\beta}(\cdot,t')*u)\\&=(G_{\sigma,\beta}(\cdot,t)*G_{\sigma,\beta}(\cdot,t'))*u=(G_{\sigma,\beta}(\cdot,t+t'))*u=\mathsf{S}(t+t')u.\end{aligned}$$

We show that $\mathsf{S}(t)u$ converges to u for all $u \in C_{\mathrm{u}}(\mathbb{R}^d, Z)$ when $t \to 0$. Indeed, we have, for $\delta > 0$,

$$\begin{aligned}|(\mathsf{S}(t)u)(x) - u(x)|_Z &\leq \int_{\mathbb{R}^n} G_{\sigma,\beta}(y,t)|u(x-y) - u(x)|_Z\,dy \\ &= \int_{|y|<\delta} G_{\sigma,\beta}(y,t)|u(x-y) - u(x)|_Z\,dy \\ &\quad + \int_{|y|\geq\delta} G_{\sigma,\beta}(y,t)|u(x-y) - u(x)|_Z\,dy.\end{aligned}$$

The first integral on the right hand side of the equality can be estimated as follows:

$$\begin{aligned}&\int_{|y|<\delta} G_{\sigma,\beta}(y,t)|u(x-y) - u(x)|_Z\,dy \\ &\qquad\leq \int_{\mathbb{R}^n} G_{\sigma,\beta}(y,t) \max_{|y|<\delta} |u(x-y) - u(x)|_Z\,dy \\ &\qquad= \max_{|y|<\delta} |u(x-y) - u(x)|_Z.\end{aligned}$$

This can be small enough because $|y| < \delta$ and u is uniformly continuous. For the second term we proceed in the following way,

$$\begin{aligned}&\int_{|y|\geq\delta} G_{\sigma,\beta}(y,t)|u(x-y) - u(x)|_Z\,dy \\ &\qquad\leq 2\|u\|_{\infty,Z}(\sigma t)^{-\frac{d}{2\beta}} \int_{|y|\geq\delta} g_\beta((\sigma t)^{-\frac{1}{2\beta}}y)\,dy \\ &\qquad= 2\|u\|_{\infty,Z} \int_{|y|\geq\delta(\sigma t)^{-1/(2\beta)}} g_\beta(y)\,dy.\end{aligned}$$

Since $\delta(\sigma t)^{-1/(2\beta)} \to \infty$ when $t \to 0^+$ and $g_\beta \in L^1(\mathbb{R}^d)$, the right hand side of the previous equality tends to 0. The next property proves that S is well defined, that is, $\mathsf{S}u \in C_{\mathrm{u}}(\mathbb{R}^d, Z)$.

$$\begin{aligned}|(\mathsf{S}(t)u)(x_1) - (\mathsf{S}(t)u)(x_2)|_Z &\leq \int_{\mathbb{R}^n} G_{\sigma,\beta}(y,t)|u(x_1-y) - u(x_2-y)|_Z\,dy \\ &\leq \varepsilon \int_{\mathbb{R}^n} G_{\sigma,\beta}(y,t)\,dy = \varepsilon.\end{aligned}$$

In the last inequality we used that u is uniformly continuous. Finally, we prove the contraction semigroup property:

$$|(\mathsf{S}(t)u)(x)|_Z \leq \int_{\mathbb{R}^n} G_{\sigma,\beta}(y,t)|u(x-y)|_Z\,dy \leq \|u\|_{\infty,Z}. \qquad \square$$

Remark 2.4. If $u \in C_{\mathrm{u}}(\mathbb{R}^d, Z)$ is a constant, then $\mathsf{S}(t)u = u$.

We consider local in time integral solutions of problem (1.1). We say that $u \in C([0,T], C_{\mathrm{u}}(\mathbb{R}^d, Z))$ is a mild solution of (1.1) iff u verifies

$$u(t) = \mathsf{S}(t)u_0 + \int_0^t \mathsf{S}(t-t')F(t', u(t'))\,dt'. \tag{2.1}$$

Since our method to build solutions of (2.1) is based on the Lie–Trotter method, it is necessary to study the nonlinear problem related to F. We remark that some regularity condition is necessary for convergence, as it is shown in the counterexample given in [**9**].

Let $F\colon \mathbb{R}_+ \times Z \to Z$ be a continuous map. We say that F is locally Lipschitz in the second variable if, given $R, T > 0$, there exists $L = L(R,T) > 0$ such that, if $t \in [0,T]$ and $z, \tilde{z} \in Z$ with $|z|_Z, |\tilde{z}|_Z \le R$, then

$$|F(t,z) - F(t,\tilde{z})|_Z \le L|z - \tilde{z}|_Z.$$

In this case, for any $z_0 \in Z$ there exists a unique local in time (maximal) solution of the Cauchy problem

$$z(t) = z_0 + \int_{t_0}^t F(t', z(t'))\,dt' \tag{2.2}$$

defined on $[t_0, t_0 + T^*(t_0, z_0))$, with $T^*(t_0, z_0)$ the maximal time of existence. It is easy to see that there exists a nonincreasing function $\mathcal{T}\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ such that

$$\mathcal{T}(T,R) \le \inf\{T^*(t_0, z_0) : 0 \le t_0 \le T,\ |z_0|_Z \le R\}.$$

Also, one of the following alternatives holds:

- $T^*(t_0, z_0) = \infty$;
- $T^*(t_0, z_0) < \infty$ and $|z(t)|_Z \to \infty$ when $t \uparrow t_0 + T^*(t_0, z_0)$.

We can see that $F\colon \mathbb{R}_+ \times C_{\mathrm{u}}(\mathbb{R}^d, Z) \to C_{\mathrm{u}}(\mathbb{R}^d, Z)$, given by $F(t,u)(x) = F(t, u(x))$, is continuous and locally Lipschitz in the second variable. Therefore, we can consider problem (2.2) in $C_{\mathrm{u}}(\mathbb{R}^d, Z)$. We denote by $\mathsf{N}\colon \mathbb{R} \times \mathbb{R} \times C_{\mathrm{u}}(\mathbb{R}^d, Z) \to C_{\mathrm{u}}(\mathbb{R}^d, Z)$ the flow generated by the integral equation (2.2) as $u(t) = \mathsf{N}(t, t_0, u_0)$, defined for $t_0 \le t < t_0 + T^*(t_0, u_0)$.

The following result relates the solutions of (2.2) with problem (2.1) in the case where the initial data is constant.

Proposition 2.5. *If u_0 is a constant function, then $u(t) = \mathsf{N}(t, t_0, u_0)$ is a solution of* (2.1).

Proof: Since u_0 is a constant function, from the uniqueness of problem (2.2) we have that $u(t)$ is a constant function for any $t > 0$ where the solution is defined. Therefore,

$$u(t) = u_0 + \int_0^t F(t', u(t'))\,dt' = \mathsf{S}(t)u_0 + \int_0^t \mathsf{S}(t-t')F(t', u(t'))\,dt',$$

which proves our assertion. □

Theorem 2.6. *There exists a function $T^*\colon C_{\mathrm{u}}(\mathbb{R}^d, Z) \to \mathbb{R}_+$ such that, for $u_0 \in C_{\mathrm{u}}(\mathbb{R}^d, Z)$, there is a unique $u \in C([0, T^*(u_0)), C_{\mathrm{u}}(\mathbb{R}^d, Z))$, which is a mild solution of* (1.1) *with $u(0) = u_0$. Moreover, one of the following alternatives holds:*

(i) *$T^*(u_0) = \infty$;*
(ii) *$T^*(u_0) < \infty$ and $\lim_{t\uparrow T^*(u_0)} \|u(t)\|_{\infty,Z} = \infty$.*

Proof: See Theorem 4.3.4 in [**11**]. □

Proposition 2.7. *Under the conditions of the above theorem, we have*

(i) *$T^*\colon C_{\mathrm{u}}(\mathbb{R}^d, Z) \to \mathbb{R}_+$ is lower semi-continuous.*
(ii) *If $u_{0,n} \to u_0$ in $C_{\mathrm{u}}(\mathbb{R}^d, Z)$ and $0 < T < T^*(u_0)$, then $u_n \to u$ in the Banach space $C([0, T], C_{\mathrm{u}}(\mathbb{R}^d, Z))$.*

Proof: See Proposition 4.3.7 in [**11**]. □

3. Propagators

To build the approximate solutions, we decompose the time variable in regular intervals and consider the evolution, in an alternate form, of the linear and nonlinear problem. To achieve this, we *turn on and off* each term of the equation. The first step is to consider the abstract linear problem

$$\partial_t u - \alpha(t)Au = 0,$$
$$u(s) = u_0,$$

where $\alpha(t) \geq 0$ and A is the infinitesimal generator of S, a strongly continuous semigroup of operators defined on the Banach space X. The mild solution of the non-autonomous problem can be written as $u(t) = \mathsf{S}_\alpha(t, s)u_0 = \mathsf{S}(\tau(t, s))u_0$, where τ is defined by

$$\tau(t, s) = \int_s^t \alpha(t')\, dt'.$$

Formally, we have $\partial_t u = \partial_t \mathsf{S}(\tau(t, s))u_0 = \partial_t \tau(t, s) A\mathsf{S}(\tau(t, s))u_0$. To analyze the Lie–Trotter method, we define a periodic function $\alpha\colon \mathbb{R} \to \mathbb{R}$ of period 1 as:

$$\alpha(t) = \begin{cases} 2 & \text{if } k \leq t < k + 1/2, \\ 0 & \text{if } k - 1/2 \leq t < k, \end{cases} \tag{3.1}$$

for $k \in \mathbb{Z}$. Given $h > 0$, we define the function $\alpha_h\colon \mathbb{R} \to \mathbb{R}$ as $\alpha_h(t) = \alpha(t/h)$. Clearly $0 \leq \alpha_h \leq 2$, α_h is h-periodic, and its mean value is 1.

We consider $\tau_h\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$\tau_h(t,t') = \int_{t'}^{t} \alpha_h(t'')\,dt''.$$

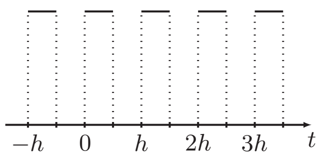

Figure 1. Graph of $\alpha_h(t)$.

The following results show that S_{α_h} defines a propagator in X. We also obtain some estimates that we will use in the following section concerning the convergence.

Lemma 3.1. *The map τ_h is continuous in $\mathbb{R}^2$ and satisfies*

(i) $0 \le \tau_h(t,t') \le 2(t-t')$ *if* $t' \le t$.
(ii) $\tau_h(t,t') + \tau_h(t',t'') = \tau_h(t,t'')$ *if* $t'' < t' < t$.
(iii) $\tau_h(t+kh, t'+kh) = \tau_h(t,t')$ *for* $k \in \mathbb{Z}$.
(iv) $\tau_h(t'+kh, t') = kh$ *for* $k \in \mathbb{Z}$.
(v) $|(t-t') - \tau_h(t,t')| \le h$.

Proof: The first statement is a consequence of the inequality $0 \le \alpha_h \le 2$. The additivity property is immediate from the definition. The third statement is a consequence of the periodicity of α_h. As the mean value of α_h is 1, we have $\tau_h(t'+h,t') = h$, and using the additivity property (ii), we obtain

$$\tau_h(t'+kh,t') = \sum_{j=1}^{k} \tau_h(t'+jh, t'+(j-1)h) = kh.$$

For the last claim, we consider $t = t'+kh+sh$, with $k \in \mathbb{Z}$ and $0 \le s < 1$. As $|1-\alpha_h(t)| \le 1$, we get

$$\begin{aligned} |(t-t')-\tau_h(t,t')| &= |kh+sh-\tau_h(t'+kh+sh,t')| \\ &= |(kh+sh)-\tau_h(t'+kh+sh,t'+kh)-\tau_h(t'+kh,t')| \\ &= |sh-\tau_h(t'+kh+sh,t'+kh)| \\ &= \left|\int_{t'+kh}^{t'+kh+sh} (1-\alpha_h(t''))\,dt''\right| \\ &\le \int_{t'+kh}^{t'+kh+sh} |1-\alpha_h(t'')|\,dt'' \le h, \end{aligned}$$

and this proves the last assertion. □

We define $\Omega = \{(t,t') \in \mathbb{R}^2 : 0 \le t' \le t\}$ and the application $\mathsf{S}_h \colon \Omega \to \mathcal{B}(X)$ by $\mathsf{S}_h(t,t') = \mathsf{S}(\tau_h(t,t'))$. From the previous lemma we have:

Corollary 3.2. *Let* $\mathsf{S} \colon \mathbb{R}_+ \to \mathcal{B}(X)$ *a strongly continuous one-parameter semigroup of operators. We have that* S_h *satisfies:*

(i) $\mathsf{S}_h(t,t) = \mathbb{1}$.

(ii) $\mathsf{S}_h(t,t'') = \mathsf{S}_h(t,t')\mathsf{S}_h(t',t'')$ *if* $0 \le t'' \le t' \le t$.

(iii) *There exist constants* $M \ge 1$ *and* $\omega \in \mathbb{R}$ *such that* $\|\mathsf{S}_h(t,t')\|_{\mathcal{B}(X)} \le Me^{2\omega(t-t')}$ *for* $(t,t') \in \Omega$.

(iv) *If* $u \in X$*, the map* $(t,t') \mapsto \mathsf{S}_h(t,t')u$ *is continuous.*

(v) *If* $u \in D = \mathrm{Dom}(A)$ *and* $t' \le t \ne kh/2$ *with* $k \in \mathbb{Z}$*, then the map* $t \mapsto \mathsf{S}_h(t,t')u$ *is differentiable and we have*

$$\partial_t \mathsf{S}_h(t,t')u = \begin{cases} 2A\mathsf{S}_h(t,t')u & \text{if } kh < t < (k+1/2)h, \\ 0 & \text{if } (k-1/2)h < t < kh. \end{cases}$$

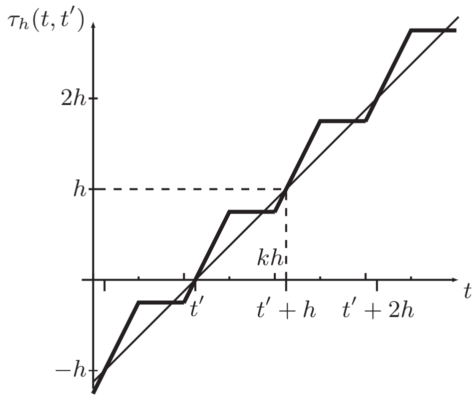

FIGURE 2. Graph of $\tau_h(t,t')$. The steps are in the half integer multiples of h.

4. Approximate solutions

In this section we develop the basic tools (Proposition 4.3 and Theorem 4.4) that allow us to obtain some properties of the solutions of problem (2.1) from the approximations obtained with the Lie–Trotter method. Theorem 4.4 is an extension of Theorem 3.9 in [**14**] to the non-autonomous case. We define the system

$$\begin{cases} \partial_t u_h - \alpha_h(t)Au_h = (2-\alpha_h(t))F(t,u_h), \\ u_h = u_{h,0}, \end{cases} \tag{4.1}$$

where $\alpha_h(t)$ is as in (3), $u_h \in X$, $t > 0$, $F\colon \mathbb{R}_+ \times X \to X$ is a continuous function, and X is a Banach space. Similarly with define the integral equation:

$$u_h(t) = \mathsf{S}_h(t,0)u_{h,0} + \int_0^t (2-\alpha_h(t'))\mathsf{S}_h(t,t')F(t',u_h(t'))\,dt'. \tag{4.2}$$

Proposition 4.1. *Let $u_{h,0} \in C([0,T],\mathrm{Dom}(A)) \cap \mathrm{Lip}([0,T],X)$. If u_h is solution of system* (4.1)*, then u_h is solution of* (4.2) *for $t \in [0,T]$.*

Proof: The procedure is similar to [**11**, Lemma 4.1.1]. □

Theorem 4.2. *There exists a function $T^*\colon X \to \mathbb{R}_+$ such that, for $u_{h,0} \in X$, there is a unique $u_h \in C([0,T^*(u_{h,0})),X)$, which is a solution of* (4.2) *with $u_h(0) = u_{h,0}$. Moreover, one of the following alternatives holds:*

(i) $T^*(u_{h,0}) = \infty$;
(ii) $T^*(u_{h,0}) < \infty$ *and* $\lim_{t\uparrow T^*(u_{h,0})} \|u_h(t)\|_X = \infty$.

Proof: The proof is similar to Theorem 4.3.4 in [**11**]. □

In the following proposition we show that the solution of the integral problem (4.2) corresponds to the approximations obtained with the Lie–Trotter method.

Proposition 4.3. *Let u_h be the solution of* (4.2)*. If $U_{h,k} = u_h(kh)$ and $V_{h,k} = u_h(kh-h/2)$, then*

$$V_{h,k+1} = \mathsf{S}(h)U_{h,k}, \tag{4.3a}$$

$$U_{h,k+1} = \mathsf{N}(kh+h, kh+h/2, V_{h,k+1}), \tag{4.3b}$$

where N is the flux associated to $2F$, that is, $w(t) = \mathsf{N}(t,t_0,w_0)$ where w is the solution of

$$\begin{cases} \dot{w} = 2F(t,w(t)), \\ w(t_0) = w_0. \end{cases} \tag{4.4}$$

Proof: For $t_1 \in (0,t)$

$$u_h(t) = \mathsf{S}_h(t,t_1)u_h(t_1) + \int_{t_1}^t (2-\alpha_h(t'))\mathsf{S}_h(t,t')F(t',u_h(t'))\,dt'$$

is verified. If we consider the time interval $[kh, kh+h/2]$, then with $t_1 = kh$ and $t = kh+h/2$, we have

$$\begin{aligned} V_{h,k+1} &= \mathsf{S}_h(kh+h/2,kh)U_{h,k} \\ &\quad + \int_{kh}^{kh+h/2} (2-\alpha_h(t'))\mathsf{S}_h(kh+h/2,t')F(t',u_h(t'))\,dt'. \end{aligned}$$

Given that $\alpha_h(t) = 2$ for $t \in [kh, kh+h/2)$, we have $\tau_h(kh+h/2, kh) = h$ and therefore (4.3a). Similarly, $\alpha_h(t) = 0$ for $t \in [kh+h/2, kh+h)$, then $\tau_h(t, kh+h/2) = 0$ and therefore

$$u_h(t) = V_{h,k+1} + 2\int_{kh+h/2}^{t} F(t', u_h(t'))\,dt',$$

evaluating in $t = kh + h$, we obtain (4.3b). □

Theorem 4.4. *Let $u \in C([0,T^*), X)$ be the solution of the integral problem* (2.1)

$$u(t) = \mathsf{S}(t)u_0 + \int_0^t \mathsf{S}(t-t')F(t', u(t'))\,dt',$$

$T \in (0, T^*)$, and $\varepsilon > 0$. There exists $h^* > 0$ such that, if $0 < h < h^*$, then the solution u_h of (4.2) *with $u_{h,0} = u_0$ is defined in $[0,T]$ and verifies $\|u(t) - u_h(t)\|_X \leq \varepsilon$ for $t \in [0,T]$.*

To prove the theorem, we need two lemmas. We follow the procedure from Theorem 3.9 in [**14**] (see also [**6**]).

Lemma 4.5. *Let $f \in C([0,T], X)$. If*

$$I_h(t,t') = (\mathsf{S}(t-t') - \mathsf{S}_h(t,t'))f(t'),$$

then $\lim_{h\to 0^+} \sup_{(t,t')\in\Omega_T} \|I_h(t,t')\|_X = 0$, where $\Omega_T = \{(t,t') \in \mathbb{R}^2 : 0 \leq t' \leq t \leq T\}$.

Proof: Given $\varepsilon > 0$, there exists $g \in C([0,T], X)$ such that $g(t) \in \mathrm{Dom}(A)$ for $t \in [0,T]$, $Ag \in C([0,T], X)$, and $\max_{t\in[0,T]} \|f(t) - g(t)\|_X < \varepsilon$. We have

$$\begin{aligned} \|(\mathsf{S}(t-t') - \mathsf{S}_h(t,t')(f(t') - g(t'))\|_X &\leq 2Me^{2\omega T} \max_{t\in[0,T]} \|f(t) - g(t)\|_X \\ &\leq 2Me^{2\omega T}\varepsilon. \end{aligned} \tag{4.5}$$

On the other hand,

$$\mathsf{S}(t-t')g(t') = g(t') + \int_0^{t-t'} \mathsf{S}(\xi)Ag(t')\,d\xi,$$

$$\mathsf{S}_h(t,t')g(t') = g(t') + \int_0^{\tau_h(t,t')} \mathsf{S}(\xi)Ag(t')\,d\xi.$$

Subtracting both equations we obtain

$$(\mathsf{S}(t-t') - \mathsf{S}_h(t,t'))g(t') = \pm \int_{J(t,t')} \mathsf{S}(\xi)Ag(t')\,d\xi,$$

where $J(t,t')$ is the interval $J(t,t') = [\min\{(t-t'), \tau_h(t,t')\}, \max\{(t-t'), \tau_h(t,t')\}]$. Then

$$
\begin{aligned}
\|(\mathsf{S}(t-t')-\mathsf{S}_h(t,t'))g(t')\|_X &\le M e^{2\omega T}|(t-t')-\tau_h(t,t')| \max_{t\in[0,T]}\|Ag(t)\|_X \\
&\le M e^{2\omega T} h \max_{t\in[0,T]}\|Ag(t)\|_X.
\end{aligned}
\tag{4.6}
$$

From equations (4.5) and (4.6) we obtain the result. □

Lemma 4.6. *Let $f \in C(\Omega_T, X)$, with Ω_T as in the previous lemma. If*

$$I_h(t) = \int_0^t (\alpha_h(t') - 1) f(t,t')\, dt',$$

then $\lim_{h\to 0^+} \sup_{t\in[0,T]} \|I_h(t)\|_X = 0$.

Proof: From the uniform continuity of f, we can see that there exists $\delta > 0$ such that, if $0 \le t', t'' \le t \le T$ and $|t' - t''| < \delta$, then $\|f(t,t') - f(t,t'')\|_X < \varepsilon$. Let $k = \lfloor t/h \rfloor$. We can write

$$I_h(t) = \sum_{j=1}^{k} \int_{(j-1)h}^{jh} (\alpha_h(t') - 1) f(t,t')\, dt' + \int_{kh}^{t} (\alpha_h(t') - 1) f(t,t')\, dt'.$$

As the mean value of α_h is 1 in intervals of length h, for $f_j \in X$ we have

$$0 = \int_{(j-1)h}^{jh} (\alpha_h(t') - 1) f_j\, dt'. \tag{4.7}$$

Therefore

$$\int_{(j-1)h}^{jh} (\alpha_h(t') - 1) f(t,t')\, dt' = \int_{(j-1)h}^{jh} (\alpha_h(t') - 1)(f(t,t') - f_j)\, dt'.$$

If $h < \delta$ and $f_j = f(t, jh)$, then $\|f(t,t') - f_j\|_X < \varepsilon$ for $t' \in [(j-1)h, jh]$ and therefore

$$\left\| \int_{(j-1)h}^{jh} (\alpha_h(t') - 1)(f(t,t') - f_j)\, dt' \right\|_X \le \varepsilon h. \tag{4.8}$$

If $M = \max_{(t,t')\in\Omega_T} \|f(t,t')\|_X$, then we have

$$\left\| \int_{kh}^{t} (\alpha_h(t') - 1) f(t,t')\, dt' \right\|_X \le \int_{kh}^{t} \|f(t,t')\|_X\, dt' \le Mh. \tag{4.9}$$

From (4.8), (4.7), and (4.9), we obtain

$$\|I_h(t)\|_X \le \sum_{j=1}^{k} \varepsilon h + Mh \le T\varepsilon + Mh,$$

from where we get the result. □

Proof of Theorem 4.4: If $[0, T_h^*)$ is the interval of existence of the integral equation (4.2), for $0 \le t < \min\{T, T_h^*\}$ the subtraction $u(t) - u_h(t)$ satisfies

$$u(t)-u_h(t)=(\mathsf{S}(t) - \mathsf{S}_h(t,0))u_0 + \int_0^t \mathsf{S}(t-t')F(t',u(t'))\,dt' - \int_0^t (2-\alpha_h(t'))\mathsf{S}_h(t,t')F(t',u_h(t'))\,dt'.$$

If we define

$$\begin{aligned} I_{1,h}(t) &= (\mathsf{S}(t) - \mathsf{S}_h(t,0))u_0, \\ I_{2,h}(t) &= \int_0^t (2-\alpha_h(t'))(\mathsf{S}(t-t') - \mathsf{S}_h(t,t'))F(t',u(t'))\,dt', \\ I_{3,h}(t) &= \int_0^t (\alpha_h(t')-1)\mathsf{S}(t-t')F(t',u(t'))\,dt', \end{aligned}$$

then

$$\begin{aligned} u(t)-u_h(t) &= I_{1,h}(t) + I_{2,h}(t) + I_{3,h}(t) \\ &\quad + \int_0^t (2-\alpha_h(t'))\mathsf{S}_h(t,t')(F(t',u(t'))-F(t',u_h(t')))\,dt'. \end{aligned} \tag{4.10}$$

Using Lemma 4.5, with $f(t)=u_0$, we can see that $\lim_{h\to 0} \sup_{t\in[0,T]} \|I_{1,h}(t)\|_X = 0$. Given that

$$\begin{aligned} \|I_{2,h}(t)\|_X &\le 2\int_0^t \|(\mathsf{S}(t-t') - \mathsf{S}_h(t,t'))F(t',u(t'))\|_X\,dt' \\ &\le 2T \sup_{(t,t')\in\Omega_T} \|(\mathsf{S}(t-t') - \mathsf{S}_h(t,t'))F(t',u(t'))\|_X, \end{aligned}$$

and using once again Lemma 4.5 with $f(t) = F(t, u(t))$, we obtain

$$\lim_{h\to 0} \sup_{t\in[0,T]} \|I_{2,h}(t)\|_X = 0.$$

The map $f(t,t') = \mathsf{S}(t-t')F(t',u(t'))$ is continuous in Ω_T, and therefore from Lemma 4.6 we can deduce $\lim_{h\to 0} \sup_{t\in[0,T]} \|I_{3,h}(t)\|_X = 0$.

We consider $R = \max_{t\in[0,T]} \|u(t)\|_X + \varepsilon$ and L the Lipschitz constant of F for $B_R(0) \subset X$. If we define

$$J_\varepsilon = \{0 \le t < \min\{T, T_h^*\} : u_h(t')\|_X < R + \varepsilon,\ 0 \le t' \le t\},$$

from estimate (4.10) we obtain, for $t \in J_\varepsilon$,

$$\begin{aligned}\|u(t) - u_h(t)\|_X \le \|I_{1,h}(t)\|_X + \|I_{2,h}(t)\|_X + \|I_{3,h}(t)\|_X \\ + 2Me^{2\omega T} L \int_0^t \|u(t') - u_h(t')\|_X \, dt',\end{aligned}$$

and, from Gronwall's lemma

$$\begin{aligned}&\|u(t) - u_h(t)\|_X \\ &\quad \le e^{CT}\Big(\sup_{t\in[0,T]} \|I_{1,h}(t)\|_X + \sup_{t\in[0,T]} \|I_{2,h}(t)\|_X + \sup_{t\in[0,T]} \|I_{3,h}(t)\|_X\Big),\end{aligned}$$

where $C = 2Me^{2\omega T}L$. Taking $h^* > 0$ small enough, we have $\|u(t) - u_h(t)\|_X < \varepsilon/2$ for $t \in J_\varepsilon$ and $0 < h < h^*$. Therefore $\sup J_\varepsilon = \min\{T, T_h^*\}$ but, as $\|u_h(t)\| \le R + \varepsilon < \infty$, the inequality $T < T_h^*$ is verified. This proves the theorem. □

Remark 4.7. Convergence in Theorem 4.4 can be proved for $F\colon \mathbb{R}_+ \times \mathbb{R}^d \times X \to X$ without modifications in the proof, using an auxiliary map $\mathcal{F}\colon \mathbb{R}_+ \times X \to X$, such that $\mathcal{F}(t, u)(x) = F(t, x, u)$.

5. Global well-posedness of the Cauchy problem

In this section we analyze the well-posedness of problem (2.2) for different interesting cases. The local case can be analyzed using standard methods, so we refer the reader to the bibliography. We address the global problem, $t \in [0, \infty)$, by the notion of positively invariant convex families. For classical diffusion ($\beta = 1$), similar ideas can be found in Chapter 14 of [**33**]. But this method presents two problems in order to use a maximum principle: the operator must be differential elliptic and $u(x)$ belongs to a finite dimensional space. To obtain global well-posedness for the lineal equation, we use the main characteristics of the nonlocal operator, through its semigroup, mentioned in Section 2. Both difficulties are overcome considering the Lie–Trotter approximations and then taking the limit. We take advantage of this to study the evolution of a population model, where individuals have a characteristic trait that differentiates them. In [**2**] the existence of stationary solutions is studied for a scalar characteristic trait. In order not to limit a priori the possibilities of modeling this problem we consider the abstract case, where the characteristic trait is an element of a measure space.

Definition 5.1. Let $\{K(t)\}_{t\in\mathbb{R}_+}$ be a family of closed sets of Z. We say that $\{K(t)\}_{t\in\mathbb{R}_+}$ is positively F-invariant if for any $t_0 \in \mathbb{R}_+$, $z_0 \in K(t_0)$, the solution z of (2.2) verifies $z(t) \in K(t)$ for $t \in [t_0, t_0 + T^*(t_0, z_0))$. The family $\{K(t)\}_{t\in\mathbb{R}_+}$ is increasing if $K(t') \subseteq K(t)$ for $0 \le t' \le t$.

Example 5.2. Let $a, b \in C(\mathbb{R}_+)$ be positive continuous functions defined on $\mathbb{R}_+$ such that $|F(t,z)|_Z \le a(t) + b(t)|z|_Z$ for $(t,z) \in \mathbb{R}_+ \times Z$. We claim that the family of closed balls given by $B(t) = \{z \in Z : |z|_Z \le \lambda(t)\}$, with

$$\lambda(t) = \left(\lambda_0 + \int_0^t a(t')\,dt'\right) \exp\left(\int_0^t b(t')\,dt'\right),$$

is an increasing and positively F-invariant family of (convex) closed sets. Indeed, since $\lambda(t)$ is an increasing function, it is clear that $\{B(t)\}_{t\in\mathbb{R}_+}$ is an increasing family. Let $z_0 \in B(t_0)$. From (2.2) we obtain

$$|z(t)|_Z \le |z_0|_Z + \int_{t_0}^t |F(t', z(t'))|_Z\,dt' \le \lambda(t_0) + \int_{t_0}^t \left(a(t') + b(t')|z|_Z\right)\,dt'.$$

From Gronwall's lemma, we have

$$|z(t)|_Z \le \left(\lambda(t_0) + \int_{t_0}^t a(t')\,dt'\right) \exp\left(\int_{t_0}^t b(t')|z|_Z\right) \le \lambda(t),$$

which implies $z(t) \in B(t)$.

Lemma 5.3. *Let $\{K(t)\}_{t\in\mathbb{R}_+}$ be a family of closed sets of Z. If $\{K(t)\}_{t\in\mathbb{R}_+}$ is positively F-invariant and F is autonomous, then for any $z_0 \in K(t_0)$ and $0 < h < T^*(z_0)$, the solution z of* (4.4) *with initial condition $z(t_0 + h/2) = z_0$ verifies $z(t_0 + h) \in K(t_0 + h)$.*

Proof: Let $w(t) = z((t + t_0 + h)/2)$, we have

$$\begin{aligned} w(t_0 + h) = z(t_0 + h) &= z_0 + \int_{t_0+h/2}^{t_0+h} 2F(z(t'))\,dt' \\ &= z_0 + \int_{t_0}^{t_0+h} F(z((t + t_0 + h)/2))\,dt = z_0 + \int_{t_0}^{t_0+h} F(w(t))\,dt. \end{aligned}$$

Using that $\{K(t)\}_{t\in\mathbb{R}_+}$ is positively F-invariant, we have $w(t_0 + h) \in K(t_0 + h)$. ☐

Lemma 5.4. *Let $\{K(t)\}_{t\in\mathbb{R}_+}$ be an increasing family of closed sets of Z such that $\{K(t)\}_{t\in\mathbb{R}_+}$ is positively $2F$-invariant. Then, for any $z_0 \in K(t_0)$ and $0 < h < T^*(z_0)$, the solution z of* (4.4) *with initial condition $z(t_0 + h/2) = z_0$ verifies $z(t_0 + h) \in K(t_0 + h)$.*

Proof: Since $z_0 \in K(t_0) \subseteq K(t_0 + h/2)$ and $\{K(t)\}_{t\in\mathbb{R}_+}$ is positively $2F$-invariant, the result follows. ☐

Corollary 5.5. *Let $F\colon \mathbb{R}_+ \times Z \to Z$ be a continuous map locally Lipschitz in the second variable and let $\{K(t)\}_{t\in\mathbb{R}_+}$ be a family of closed sets of Z. If one of the following conditions holds:*

(i) *F is autonomous and $\{K(t)\}_{t\in\mathbb{R}_+}$ is positively F-invariant;*
(ii) *$\{K(t)\}_{t\in\mathbb{R}_+}$ is increasing and positively $2F$-invariant,*

then for any $u_0 \in C_{\mathrm{u}}(\mathbb{R}^d, K(t_0))$ and $0 < h < T^*(t_0, u_0)$, we have

$$\mathsf{N}(t_0 + h, t_0 + h/2, u_0) \in C_{\mathrm{u}}(\mathbb{R}^d, K(t_0 + h)).$$

Lemma 5.6. *Let $\sigma \geq 0$, $0 < \beta \leq 1$, and let K be a closed convex set of Z. Then $\mathsf{S}(t)u \in C_{\mathrm{u}}(\mathbb{R}^d, K)$ for any $t > 0$ and $u \in C_{\mathrm{u}}(\mathbb{R}^d, K)$.*

Proof: Suppose, contrary to our claim, that the assertion of the lemma is false. Then, there exists $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ such that $v = (\mathsf{S}(t)u)(x) \notin K$. Using the Hahn–Banach Separation Theorem, we take a separating hyperplane, i.e., $\omega \in Z^*$ and $\lambda \in \mathbb{R}$ verifying $\langle \omega, z\rangle \leq \lambda$ for any $z \in K$ and $\langle \omega, v\rangle > \lambda$. But then

$$\langle \omega, v\rangle = \int_{\mathbb{R}^d} G_{\sigma,\beta}(x - y, t)\langle \omega, u(y)\rangle\, dy \leq \lambda \int_{\mathbb{R}^d} G_{\sigma,\beta}(x - y, t)\, dy = \lambda,$$

a contradiction. □

Proposition 5.7. *Let F and $\{K(t)\}_{t\in\mathbb{R}_+}$ be as in Corollary 5.5. If $K(t)$ is convex for $t \geq 0$, then $u(t) \in C_{\mathrm{u}}(\mathbb{R}^d, K(t))$ for any $u_0 \in C_{\mathrm{u}}(\mathbb{R}^d, K(0))$ and $t \in (0, T^*(u_0))$, where u is the solution of (2.1).*

Proof: For $t \in [0, T^*(u_0))$ and $n \in \mathbb{N}$, let $h = t/n$ and $\{U_{h,k}\}_{0\leq k\leq n}$, $\{V_{h,k}\}_{1\leq k\leq n}$ be the sequences given by $U_{h,0} = u_0$,

(5.1a) $$V_{h,k+1} = \mathsf{S}(h)U_{h,k},$$

(5.1b) $$U_{h,k+1} = \mathsf{N}(kh + h, kh + h/2, V_{h,k+1}), \quad k = 0, \dots, n - 1.$$

From Proposition 4.3 and Theorem 4.4, it may be concluded that $U_{h,k}$ is defined and $\|u(t) - U_{h,n}\|_{\infty,Z} \to 0$ when $n \to \infty$. Since $K(t)$ is a closed set, it suffices to prove that $U_{h,n} \in K(t)$. We claim that $U_{h,k} \in C_{\mathrm{u}}(\mathbb{R}^d, K(kh))$. The proof is by induction on k. If $U_{h,k} \in C_{\mathrm{u}}(\mathbb{R}^d, K(kh))$, as $K(kh)$ is convex, Lemma 5.6 implies $V_{h,k+1} \in C_{\mathrm{u}}(\mathbb{R}^d, K(kh))$. From Corollary 5.5, $U_{h,k+1} \in C_{\mathrm{u}}(\mathbb{R}^d, K((k + 1)h))$ and our claim follows. □

Theorem 5.8. *Let $\{K(t)\}_{t\in\mathbb{R}_+}$ be a family of bounded convex closed sets of Z. Suppose that F and, $\{K(t)\}_{t\in\mathbb{R}_+}$ satisfy the hypothesis of Corollary 5.5 and for any $T > 0$, we have $M(T) = \sup\{|z|_Z : z \in K(t),\ t \in [0, T]\} < \infty$. Then $T^*(u_0) = \infty$ for any $u_0 \in C_{\mathrm{u}}(\mathbb{R}^d, K(0))$ and $u(t) \in C_{\mathrm{u}}(\mathbb{R}^d, K(t))$ for $t > 0$.*

Proof: From Proposition 5.7, we have $u(t) \in C_{\rm u}(\mathbb{R}^d, K(t))$ for $t \in (0, T^*(u_0))$. Suppose $T^*(u_0) < \infty$. Then $\lim_{t\to T^*(u_0)} \|u(t)\|_{\infty,Z} = \infty$. But $\|u(t)\|_{\infty,Z} \le M(T^*(u_0)) < \infty$, a contradiction. □

Example 5.9 (Ginzburg–Landau equation)**.** The Ginzburg–Landau equation is given by (1.1), where $\beta = 1$, $\sigma > 0$, and $F(u) = (1 + ia)u - (1 + ib)|u|^2 u$ with $a, b \in \mathbb{R}$ (see [**10, 12, 35**]). In general, we consider $F(u) = f_{\rm R}(|u|^2)u + i f_{\rm I}(|u|^2)u$, where $f_{\rm R}, f_{\rm I} \colon \mathbb{R}_+ \to \mathbb{R}$ are smooth functions. If $f_{\rm R}(\eta) \le 0$ for $\eta > 0$, then $K = B(0, \eta)$ is a bounded convex positively F-invariant set of $\mathbb{C}$. For $0 < \beta \le 1$, from Theorem 5.8 we obtain that the fractional Ginzburg–Landau equation is globally well-posed for $u_0 \in C_{\rm u}(\mathbb{R}^d, K)$.

Example 5.10 (Fisher–Kolmogorov equation)**.** Fisher ([**17**]) and Kolmogorov et al. ([**19**]) introduced a classical model to describe the propagation of an advantageous gene in a one-dimensional habitat. We consider the generalized nonlinear reaction-diffusion equation

$$\partial_t u + \sigma(-\Delta)^\beta u = \chi u(1 - u),$$

where u is the chemical concentration, σ is the diffusion coefficient, and the positive constant χ represents the growth rate of the chemical reaction. Since then, a great deal of work has been carried out to extend their model to take into account other biological, chemical, and physical factors. This equation is also used in flame propagation [**18**], nuclear reactor theory [**8**], autocatalytic chemical reactions [**13, 16**], logistic growth models [**27**], and neurophysiology [**34**]. Consider $b_0 > 1$ and $K(t) = [0, b(t)]$, with

$$b(t) = \frac{b_0 e^{\chi t}}{1 + b_0(e^{\chi t} - 1)}.$$

We can see that $\{K(t)\}_{t\in\mathbb{R}_+}$ is a family of compact intervals, positively F-invariant for $F(z) = \chi z(1 - z)$. In particular, for any $u_0 \in C_{\rm u}(\mathbb{R})$ with $u_0(x) \ge 0$, taking $b_0 = \sup_{x\in\mathbb{R}^d} u(x)$, we can see that $T^*(u_0) = \infty$ and $\limsup_{t\to\infty} |u(t,x)| \le 1$ for any $x \in \mathbb{R}^d$. In the case $0 < a_0 = \inf_{x\in\mathbb{R}^d} u(x) < 1$, we have that $K(t) = [a(t), b(t)]$, with

$$a(t) = \frac{a_0 e^{\chi t}}{1 + a_0(e^{\chi t} - 1)},$$

is F-positive. Therefore, $\lim_{t\to\infty} \|u(t) - 1\|_\infty = 0$.

5.1. Population dynamics with a continuous trait. In [**2**], Arnold et al. consider a model of population dynamics in which the population is structured with respect to the space variable x and a trait variable denoted by θ in a measure space. The distribution function $u(t,x,\theta) \geq 0$ denotes the density of individuals at time $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^d$, and whose trait is $\theta \in \Theta$. The evolution of u is governed by an integro-PDE model of reaction-diffusion type in infinite (continuous) dimensions in which selection, mutations, competition, and migrations are taken into account. The modeling assumptions are the following: migration is described by a (normal or anomalous) diffusion operator $-\sigma(-\Delta)^\beta$; mutations are described by a linear kernel $M(\theta,\vartheta)$ which is related to the probability that individuals with trait ϑ have offsprings with trait θ; selection is implemented in the model, thanks to a fitness function k which may depend on trait θ; finally, a logistic term involving a kernel $C(\theta,\vartheta)$ models the competition (felt by individuals of trait θ) due to individuals of trait ϑ. Under those assumptions, the evolution of the population is governed by the following integro-PDE:

$$\partial_t u + \sigma(-\Delta_x)^\beta u = F(t,u(t)) \tag{5.2}$$

with initial condition $u(0) = u_0$. The map F is given by

$$F(t,z)(\theta) = k(t,\theta)z(\theta) + \int_\Theta M(t,\theta,\vartheta)z(\vartheta)\,d\mu(\vartheta) - \left(\int_\Theta C(t,\theta,\vartheta)z(\vartheta)\,d\mu(\vartheta)\right) z(\theta).$$

Let Θ be a compact Hausdorff space, $\mathcal{B}$ the Borel algebra, and μ a regular Borel probability. We set the problem on $C_{\mathrm{u}}(\mathbb{R}^d, Z)$, with $Z = L^1(\Theta,\mathcal{B},\mu)$. Following [**2**], we assume $k \in C(\mathbb{R}_+ \times \Theta)$, $M, C \in C(\mathbb{R}_+ \times \Theta \times \Theta)$ verifying $M \geq 0$ and $C > 0$. For any $T > 0$, we define

$$\begin{aligned}
\|k\|_{T,\infty} &= \max\{|k(t,\theta)| : (t,\theta) \in [0,T] \times \Theta\},\\
\|M\|_{T,\infty} &= \max\{M(t,\theta,\vartheta) : (t,\theta,\vartheta) \in [0,T] \times \Theta \times \Theta\},\\
\|C\|_{T,\infty} &= \max\{C(t,\theta,\vartheta) : (t,\theta,\vartheta) \in [0,T] \times \Theta \times \Theta\}.
\end{aligned}$$

Also, we need

$$k_+(t) = \max\left\{k(t',\theta) + \int_\Theta M(t',\vartheta,\theta)\,d\mu(\vartheta) : (t',\theta) \in [0,t] \times \Theta\right\}, \tag{5.3a}$$

$$c_-(t) = \min\{C(t',\theta,\vartheta) : (t',\theta,\vartheta) \in [0,t] \times \Theta \times \Theta\}. \tag{5.3b}$$

We assume that $c_-(t) > 0$ for $t > 0$, the lower bound for C means that all individuals are in competition. To obtain well-posedness of (5.2), we need to prove some results.

Lemma 5.11. *The map $F\colon \mathbb{R}_+ \times Z \to Z$ is continuous and locally Lipschitz in the second variable.*

Proof: Let $R, T > 0$ and $z, \tilde{z} \in Z$ with $|z|_Z, |\tilde{z}|_Z \le R$. Then

$$\begin{aligned}|F(t,z) - F(t,\tilde{z})|_Z \le & \int_\Theta |k(t,\theta)||z(\theta) - \tilde{z}(\theta)|\, d\mu(\theta) \\ & + \int_{\Theta\times\Theta} M(t,\theta,\vartheta)|z(\vartheta) - \tilde{z}(\vartheta)|\, d\mu(\vartheta)\, d\mu(\theta) \\ & + \int_{\Theta\times\Theta} C(t,\theta,\vartheta)|z(\vartheta)||z(\theta) - \tilde{z}(\theta)|\, d\mu(\vartheta)\, d\mu(\theta) \\ & + \int_{\Theta\times\Theta} C(t,\theta,\vartheta)|\tilde{z}(\theta)||z(\vartheta) - \tilde{z}(\vartheta)|\, d\mu(\vartheta)\, d\mu(\theta).\end{aligned}$$

Since k, M, C are bounded for $t \in [0,T]$ and $\theta, \vartheta \in \Theta$, we get

$$|F(t,z) - F(t,\tilde{z})|_Z \le (\|k\|_{T,\infty} + \|M\|_{T,\infty} + 2\|C\|_{T,\infty} R)|z - \tilde{z}|_Z.$$

Let $(t_n, z_n) \to (t,z) \in [0,T] \times \Theta$. We see that

$$\begin{aligned}|F(t,z) - F(t_n,z_n)|_Z &\le |F(t,z) - F(t_n,z)|_Z + |F(t_n,z) - F(t_n,z_n)|_Z \\ &\le |F(t,z) - F(t_n,z)|_Z + L(R,T)|z - z_n|_Z.\end{aligned}$$

Using that

$$\begin{aligned}|F(t,z) - F(t_n,z)|_Z \le & \int_\Theta |k(t,\theta) - k(t_n,\theta)||z(\theta)|\, d\mu(\theta) \\ & + \int_{\Theta\times\Theta} |M(t,\theta,\vartheta) - M(t_n,\theta,\vartheta)||z(\vartheta)|\, d\mu(\vartheta)\, d\mu(\theta) \\ & + \int_{\Theta\times\Theta} |C(t,\theta,\vartheta) - C(t_n,\theta,\vartheta)||z(\vartheta)||z(\theta)|\, d\mu(\vartheta)\, d\mu(\theta),\end{aligned}$$

and uniform continuity of k, M, C, we obtain $F(t_n, z) \to F(t,z)$ in Z, which completes the proof. □

We have the same result for continuous functions:

Lemma 5.12. *The map $F\colon \mathbb{R}_+ \times C(\Theta) \to C(\Theta)$ is continuous and locally Lipschitz in the second variable.*

Proof: The proof is similar to the lemma above. □

The non-negativity of density $z(t,\theta)$ is established by the next proposition (and corollary below).

Proposition 5.13. *Let z be the solution of* (2.2) *with* $z(t_0) = z_0 \in C(\Theta)$. *If* $z(t_0) > 0$, *then* $z(t) > 0$ *for any* $t \in [t_0, t_0 + T^*(t_0, z_0))$.

Proof: Let $0 < T < T^*(t_0, z_0)$. For any $(t, \theta) \in [t_0, t_0 + T] \times \Theta$, we define

$$g(t,\theta) = \int_\Theta M(t,\theta,\vartheta) z(t,\vartheta)\, d\mu(\vartheta),$$

$$a(t,\theta) = \int_\Theta C(t,\theta,\vartheta) z(t,\vartheta)\, d\mu(\vartheta).$$

Then $g(\cdot,\theta)$, $a(\cdot,\theta)$ are continuous, the solution verifies $z(\cdot,\theta) \in C^1([t_0, t_0 + T^*(t_0, z_0)))$, and

$$\begin{cases} \partial_t z(t,\theta) = (k(t,\theta) - a(t,\theta)) z(t,\theta) + g(t,\theta), \\ z(t_0,\theta) = z_0(\theta). \end{cases}$$

Then

$$z(t,\theta) = e^{A(t,t_0,\theta)} z_0(\theta) + \int_{t_0}^{t} e^{A(t,t',\theta)} g(t',\theta)\, dt', \tag{5.4}$$

where

$$A(t,t',\theta) = \int_{t'}^{t} k(t'',\theta) - a(t'',\theta)\, dt''.$$

Let $t_* = \sup\{t \in [t_0, t_0 + T] : \min_{[t_0,t]\times\Theta} z(t,\theta) > 0\}$. Suppose $t_* < t_0 + T$. Then there exists $\theta_* \in \Theta$ with $z(\theta_*, t_*) = 0$. But from (5.4), we have

$$z(t_*,\theta_*) = e^{A(t_*,t_0,\theta_*)} z_0(\theta_*) + \int_{t_0}^{t_*} e^{A(t_*,t',\theta_*)} g(t',\theta_*)\, dt' > 0,$$

a contradiction. Since T is arbitrary, we obtain the result. □

Corollary 5.14. *Let z be the solution of* (2.2) *with* $z(t_0) = z_0 \in C(\Theta)$. *If* $z_0 \geq 0$, *then* $z(t) \geq 0$ *for any* $t \in [t_0, t_0 + T^*(t_0, z_0))$.

Proof: Consider $z_{0,n} = z_0 + 1/n$. For any $0 < T < T^*(t_0, z_0)$, there exists $n_0 \in \mathbb{N}$ such that $T < T^*(t_0, z_{0,n})$ if $n \geq n_0$. Since $z_{0,n} > 0$, using Proposition 5.13 we have $z_n(t) > 0$ for $t \in [t_0, t_0 + T]$. As z_n converges to z in $C(\Theta \times [t_0, t_0 + T])$, we see that $z \geq 0$. Since T is arbitrary, we obtain the result. □

We now show global well-posedness in $C(\Theta)$ for $z_0 \geq 0$.

Proposition 5.15. *If* $z_0 \in C(\Theta)$ *with* $z_0 \geq 0$, *then* $T^*(t_0, z_0) = \infty$.

Proof: Let $0 < T < T^*(t_0, z_0)$. From Corollary 5.14 we obtain that $a(t,\theta), g(t,\theta) \geq 0$ and then $A(t,t',\theta) \leq \|k\|_{T,\infty}(t-t')$. Integrating (5.4) on Θ we get, for $t \in [t_0, t_0+T]$

$$\int_\Theta z(t,\theta)\,d\mu(\theta) \leq \exp(\|k\|_{T,\infty}(t-t_0))\int_\Theta z_0(\theta)\,d\mu(\theta)$$
$$+\int_{t_0}^t\int_{\Theta\times\Theta}\exp(\|k\|_{T,\infty}(t-t'))M(t',\theta,\vartheta)z(t',\vartheta)\,d\mu(\vartheta)\,d\mu(\theta)\,dt'$$
$$\leq \exp(\|k\|_{T,\infty}(t-t_0))$$
$$\times\left(\int_\Theta z_0(\theta)\,d\mu(\theta)+\|M\|_{T,\infty}\int_0^t\int_\Theta \exp(-\|k\|_{T,\infty}(t'-t_0))z(t',\vartheta)\,d\mu(\vartheta)\,dt'\right).$$

Using Gronwall's lemma, we obtain

$$\int_\Theta z(t,\theta)\,d\mu(\theta) \leq \exp((\|k\|_{T,\infty}+\|M\|_{T,\infty})(t-t_0))\int_\Theta z_0(\theta)\,d\mu(\theta)$$
$$\leq \exp((\|k\|_{T,\infty}+\|M\|_{T,\infty})(t-t_0))\|z_0\|_{\infty,Z},$$

which implies

$$0 \leq g(t,\theta) \leq \|M\|_{T,\infty}\exp((\|k\|_{T,\infty}+\|M\|_{T,\infty})(t-t_0))\|z_0\|_{\infty,Z}.$$

From (5.4), we get

$$\|z\|_{T,\infty} \leq \exp((\|k\|_{T,\infty}+\|M\|_{T,\infty})T)\|z_0\|_{\infty,Z}.$$

And finally we have $T^*(z_0) = \infty$. □

Now we construct a positive F-invariant convex set of Z.

Lemma 5.16. *Let $w \in C^1([t_0, t_0+T])$, $w \geq 0$, be such that $\dot{w} \leq kw - cw^2$, with $k, c > 0$. If $\lambda \geq k/c$ and $0 \leq w(t_0) \leq \lambda$, then $0 \leq w(t) \leq \lambda$ for $t \in [t_0, t_0+T]$.*

Proof: Suppose $w(t_+) > \lambda$ with $t_0 < t_+ \leq t_0 + T$. Consider $t_- = \sup\{t \in [t_0, t_+] : w(t) \leq \lambda\}$. Using the Mean Value Theorem, there exists $t_1 \in (t_-, t_+)$ such that

$$w(t_+) - w(t_-) = \dot{w}(t_1)(t_+ - t_-),$$

and then $\dot{w}(t_1) > 0$. But $w(t_1) > \lambda$, which implies $kw(t_1) - cw^2(t_1) < 0$, a contradiction. □

Proposition 5.17. *Let $z_0 \in C(\Theta)$, $z_0 \geq 0$. If $\lambda(t) \geq \max\{k_+(t)/c_-(t), |z_0|_Z\}$, then the solution $z \in C([t_0,\infty), C(\Theta))$ of (2.2) verifies $z(t) \geq 0$ and $|z(t)|_Z \leq \lambda(t)$ for any $t \geq t_0$.*

Proof: From Corollary 5.14, we see that $z(t) \geq 0$. Let $t>0$. For any $t' \in [t_0, t]$ we have

$$\begin{aligned}\frac{d}{dt}\int_\Theta z(t',\theta)\,d\mu(\theta) &= \int_\Theta k(t',\theta)z(t',\theta)\,d\mu(\theta)\\ &\quad+\int_{\Theta\times\Theta} M(t',\theta,\vartheta)z(t',\vartheta)\,d\mu(\vartheta)\,d\mu(\theta)\\ &\quad-\int_{\Theta\times\Theta} C(t',\theta,\vartheta)z(t',\vartheta)z(t',\theta)\,d\mu(\vartheta)\,d\mu(\theta)\\ &=\int_\Theta\left(k(t',\theta)+\int_\Theta M(t',\vartheta,\theta)\,d\mu(\vartheta)\right)z(t',\theta)\,d\mu(\theta)\\ &\quad-\int_{\Theta\times\Theta} C(t',\theta,\vartheta)z(t',\vartheta)z(t',\theta)\,d\mu(\vartheta)\,d\mu(\theta).\end{aligned}$$

From (5.3), we have

$$\frac{d}{dt}\int_\Theta z(t',\theta)\,d\mu(\theta) \leq k_+(t)\int_\Theta z(t',\theta)\,d\mu(\theta) - c_-(t)\left(\int_\Theta z(t',\theta)\,d\mu(\theta)\right)^2.$$

Using Lemma 5.16, we obtain $|z(t)|_Z \leq \lambda(t)$. □

Proposition 5.18. *Let $\lambda \in C(\mathbb{R}_+)$ be an increasing function such that $\lambda(t) \geq k_+(t)/c_-(t)$. Then, the family of bounded convex closed set $\{K(t)\}_{t\in\mathbb{R}_+}$ given by $K(t) = \{z \in Z : z \geq 0$ a.e., $|z|_Z \leq \lambda(t)\}$ is increasing and positive F-invariant.*

Proof: Let $z_0 \in K(t_0)$. Taking $\{z_{0,n}\}_{n\in\mathbb{N}} \subset C(\Theta) \cap K(t_0)$ such that $|z_0 - z_{0,n}|_Z \to 0$, we see from Proposition 5.17 that $T^*(z_{0,n}) = \infty$ and $z_n(t) \in K(t)$, for $t \geq t_0$. Using continuous dependence on initial data, we get that $|z(t) - z_n(t)|_Z \to 0$ for any $t \in [t_0, t_0 + T^*(t_0, z_0))$, and since $K(t)$ is closed, we obtain $z(t) \in K(t)$. □

Remark 5.19. Also, the family $\{K(t)\}_{t\in\mathbb{R}_+}$ is positive $2F$-invariant.

Theorem 5.20. *Let $u_0 \in C_{\mathrm{u}}(\mathbb{R}^d, Z)$, with $u_0(x) \geq 0$ a.e. in Θ. Then the mild solution of equation* (5.2) *is globally well-posed and verifies* $\|u(t)\|_{\infty,Z} \leq \max\{\|u_0\|_{\infty,Z}, k_+(t)/c_-(t)\}$.

Proof: The result is an immediate consequence of Theorem 5.8 and Proposition 5.18 using $\lambda(t) = \max\{\|u_0\|_{\infty,Z}, k_+(t)/c_-(t)\}$. □

5.2. Global existence for products of Banach spaces. We generalize the previous results by proving global existence for products of Banach spaces. Lemma 5.21 proves that the semigroup operator maintains the solution inside the invariant region. Next, Theorem 5.22 proves that if u_0 is inside the invariant region, then $u(t)$ remains in it for all $t > 0$. Let $\{Z_j\}_{1\leq j\leq m}$ be Banach spaces and $Z = Z_1 \times \cdots \times Z_m$ with the usual norm. We denote $\pi_j \colon Z \to Z_j$ the projection map. If $\sigma_j > 0$, $0 < \beta_j \leq 1$, and $\mathsf{S}_j(t)u = G_{\sigma_j,\beta j}(\cdot,t) * u$ for $u \in C_{\mathrm{u}}(\mathbb{R}^d, Z_j)$, then $\mathsf{S} \colon \mathbb{R}_+ \to \mathcal{B}(C_{\mathrm{u}}(\mathbb{R}^d, Z))$ given by

$$\mathsf{S}(t)u = (\mathsf{S}_1(t)\pi_1 u, \ldots, \mathsf{S}_m(t)\pi_m u)$$

is a continuous contraction semigroup.

Lemma 5.21. *Let $K_j \subset Z_j$ be a closed convex set and $K = K_1 \times \cdots \times K_m \subset Z$. If $u \in C_{\mathrm{u}}(\mathbb{R}^d, K)$, then $\mathsf{S}(t)u \in C_{\mathrm{u}}(\mathbb{R}^d, K)$ for any $t > 0$.*

Proof: The proof is a consequence of the definition above and Lemma 5.6. □

Theorem 5.22. *Let $K_j(t) \subset Z_j$ be bounded closed convex sets. If $K(t) = K_1(t) \times \cdots \times K_m(t)$ and F satisfy the hypotheses of Corollary 5.5 and Theorem 5.8, then $T^*(u_0) = \infty$ for any $u_0 \in C_{\mathrm{u}}(\mathbb{R}^d, K(0))$ and $u(t) \in C_{\mathrm{u}}(\mathbb{R}^d, K(t))$ for $t > 0$.*

Proof: Let $u_0 \in C_{\mathrm{u}}(\mathbb{R}^d, K(0))$ and $T^*(u_0)$ the maximal time of existence of the solution u of (2.1). Let $t \in (0, T^*(u_0))$, $h = t/n$, $n \in \mathbb{N}$, $\{V_{h,k}\}_{1\leq k\leq n}$, and $\{U_{h,k}\}_{0\leq k\leq n}$, defined as in Proposition 5.7. Suppose that $U_{h,k} \in C_{\mathrm{u}}(\mathbb{R}^d, K(kh))$. Lemma 5.21 implies that $V_{h,k+1} \in C_{\mathrm{u}}(\mathbb{R}^d, K(kh))$. Using that $K(t) = K_1(t) \times \cdots \times K_m(t)$ satisfies the hypothesis of Corollary 5.5, we have that $U_{h,k+1} \in C_{\mathrm{u}}(\mathbb{R}^d, K((k+1)h))$. Using the same argument as in Proposition 5.7, we have that $U_{h,k} \to u(kh)$ in $C_{\mathrm{u}}(\mathbb{R}^d, Z)$ when $n \to \infty$ and $u(t) \in C_{\mathrm{u}}(\mathbb{R}^d, K(t))$. Following the same arguments as in Theorem 5.8, we obtain the result. □

Example 5.23. We expose an example, where we construct an invariant convex set that consists of a product of intervals in which we can apply the above results. In [**3**] a FHN Model for pattern formation is presented:

$$\begin{cases} \partial_t u = \sigma_u \Delta u + (a-u)(u-1)u - v, \\ \partial_t v = \sigma_v \Delta v + e(bu - v), \end{cases} \tag{5.5}$$

with $0 < a < 1$, $e > 0$, and $b \geq 0$. A similar example is analyzed in [**33**]. To apply Theorem 5.22, we need to find positive F-invariant rectangle $K = K_1 \times K_2$, $K_j = [-R_j, R_j]$, where F is given by

$$F(u,v) = (au^2 - u^3 - au + u^2 - v, e(bu - v)).$$

Let $R_1 > \max\{4, \sqrt{2b}\}$ and $2bR_1 < 2R_2 < R_1^3$. We see that the rectangle with R_1 and R_2 is F-invariant:

$$\begin{aligned} F_1(R_1, v) &\leq a(R_1^2 - R_1) - R_1^3 + R_1^2 + |v| \\ &\leq a(R_1^2 - R_1) - R_1^3 + R_1^2 + R_2 < 0, \\ F_1(-R_1, v) &\geq a(R_1^2 + R_1) + R_1^3 + R_1^2 - |v| \\ &\geq a(R_1^2 + R_1) + R_1^3 + R_1^2 - R_2 > 0, \\ F_2(u, R_2) &\leq e(b|u| - R_2) \leq e(bR_1 - R_2) < 0, \\ F_2(u, -R_2) &\geq e(-b|u| + R_2) \geq e(-bR_1 + R_2) > 0. \end{aligned}$$

Then the field evaluated at the border of K points inward. By Theorem 5.22, equation (5.5) is globally well-posed.

6. Asymptotic behavior

We analyze the situation in which u_0 has a horizontal asymptote at z_0. Then, using the introduced splitting methods, we prove that $u(t)$ approaches asymptotically to the time evolution of z_0. We consider the 1-dimensional real case. We first show in Lemma 6.2 that if u_0 has a horizontal asymptote at z_0, then $\mathsf{S}(t)u_0$ remains with the same horizontal asymptote. Next, we prove in Lemma 6.3 that $\mathsf{N}(t, t_0, u_0)(x)$ has a time dependent horizontal asymptote, which is the solution of equation (2.2) with z_0 as an initial condition. Finally, we combine both results and a continuous dependence argument in Lemma 6.4 to achieve Proposition 6.1, which tells us that the solution $u(t)$ of (1.1) maintains a similar asymptotic behavior as $z(t)$.

These results can be applied, for example, to the Fisher–Kolmogorov equation. Specifically, in [**19**], solutions with the mentioned asymptotic behavior are analyzed.

Proposition 6.1. *Let* $u_0 \in C_{\mathrm{u}}(\mathbb{R}, Z)$ *such that* $\lim_{x\to\pm\infty} u_0(x) = z_0^{\pm} \in Z$. *If* $u(t)$ *is the solution of* (2.1) *with* F *as in Corollary 5.5, then* $\lim_{x\to\pm\infty} u(t,x) = z^{\pm}(t)$, *where* $z^{\pm}$ *is the solution of* (2.2) *with* $z^{\pm}(0) = z_0^{\pm}$.

Lemma 6.2. *Let* $u_0 \in C_{\mathrm{u}}(\mathbb{R}, Z)$ *be such that* $\lim_{x\to\pm\infty} u_0(x) = z_0^{\pm} \in Z$. *If* $u(t) = \mathsf{S}(t)u_0$, *then* $\lim_{x\to\pm\infty} u(t,x) = z_0^{\pm}$.

Proof: We only prove the conclusion for z_0^+, the z_0^- case being similar. Let $\varepsilon > 0$. There exists $x_*^+ > 0$ such that $|u_0(x) - z_0^+|_Z < \varepsilon$ for $x > x_*^+$. Before computing the limit, we need an estimate of $g_\beta(\xi)$. Taking $r > 0$ large enough, we have

$$\int_{|\xi|>(\sigma t)^{-1/(2\beta)} r} g_\beta(\xi)\, d\xi < \varepsilon/(2\|u_0\|_{\infty,Z}). \tag{6.1}$$

Next, to study the asymptotic convergence, we analyze two cases. If $x > x_* + r$, then

$$|u(t,x) - z_0^+| \leq \int_{\mathbb{R}} G_{\sigma,\beta}(t, x-y)|u_0(y) - z_0^+|\, dy$$
$$= \int_{y>x-r} G_{\sigma,\beta}(t, x-y)|u_0(y) - z_0^+|\, dy$$
$$+ \int_{y<x-r} G_{\sigma,\beta}(t, x-y)|u_0(y) - z_0^+|\, dy = I_1 + I_2.$$

Since $y > x - r > x_*^+$, we have $|u_0(y) - z_0^+| < \varepsilon$ and therefore we can bound the first integral:

$$I_1 \leq \varepsilon \int_{\mathbb{R}} G_{\sigma,\beta}(t, x-y)\, dy = \varepsilon.$$

For the second integral, we use estimate (6.1) and the norm of the initial condition u_0:

$$I_2 \leq 2\|u_0\|_{\infty,Z} \int_{y<x-r} G_{\sigma,\beta}(t, x-y)\, dy = 2\|u_0\|_{\infty,Z} \int_{\xi>r} G_{\sigma,\beta}(t,\xi)\, d\xi$$
$$= 2\|u_0\|_{\infty,Z} \int_{|\xi'|>(\sigma t)^{-1/(2\beta)} r} g_\beta(\xi')\, d\xi' < \varepsilon.$$

Bounding both integrals we prove the result. □

Lemma 6.3. *Let $u_0 \in C_{\mathrm{u}}(\mathbb{R}, Z)$ be such that $\lim_{x\to\pm\infty} u_0(x) = z_0^\pm \in Z$. If $u(t) = \mathsf{N}(t, t_0, u_0)$, then $\lim_{x\to\pm\infty} u(t,x) = z^\pm(t)$, where $z^\pm(t)$ is the solution* (2.2) *with $z^\pm(0) = z_0^\pm$.*

Proof: We again consider only the z^+ case. From continuous dependence of the initial data, for $\varepsilon > 0$, there exists $\delta > 0$ such that if $|z_0^+ - z_0|_Z < \delta$, then $|z^+(t) - z(t)|_Z < \varepsilon$. Let $x_*^+ \in \mathbb{R}$ be such that if $x > x_*^+$, then $|u_0(x) - z_0^+|_Z < \delta$. Therefore $|u(t,x) - z^+(t)|_Z < \varepsilon$. □

Lemma 6.4. *Let $\{u_n\}_{n\in\mathbb{N}} \subset C_{\mathrm{u}}(\mathbb{R}^d, Z)$ be such that $u_n \to u$ in $C_{\mathrm{u}}(\mathbb{R}^d, Z)$. If $\lim_{x\to\pm} u_n(x) = z^\pm$ for $n \in \mathbb{N}$, then $\lim_{x\to\pm} u(x) = z^\pm$.*

Proof: Let $\varepsilon > 0$. We can take $n \in \mathbb{N}$ such that $\|u - u_n\|_{\infty,Z} < \varepsilon/2$. Then there exists $x_*^+ \in \mathbb{R}$ such that $|u_n(x) - z^+|_Z < \varepsilon/2$ if $x > x_*^+$. Therefore,

$$|u(x) - z^+|_Z \leq |u(x) - u_n(x)|_Z + |u_n(x) - z^+|_Z < \varepsilon. \qquad \square$$

Proof of Proposition 6.1: Let $n \in \mathbb{N}$, $h = t/n$, and consider the sequences $\{U_{h,k}\}_{0\leq k\leq n}$, $\{V_{h,k}\}_{1\leq k\leq n}$ defined by (5.1). We claim that $\lim_{x\to\pm\infty} U_{h,k}(x) = z^\pm(kh)$ for $k = 0, \ldots, n$. Clearly, the assertion is true for $k=0$. If $\lim_{x\to\pm\infty} U_{h,k}(x)=z^{\pm}(kh)$, then $\lim_{x\to\pm\infty} V_{h,k+1}=z^{\pm}(kh)$ from Lemma 6.2, and using Lemma 6.3 we obtain $\lim_{x\to\pm\infty} U_{h,k+1}(x) = z^{\pm}((k+1)h)$. We conclude $z^{\pm}(t) = z^{\pm}(nh) = \lim_{x\to\pm\infty} U_{h,n}(x)$ and, since by Proposition 5.7 and Theorem 5.8 we have that $U_{h,n} \to u(t)$, Lemma 6.4 implies the result. □

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