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STRUCTURE MONOIDS OF SET-THEORETIC SOLUTIONS OF THE YANG–BAXTER EQUATION

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Abstract: Given a set-theoretic solution (X,r) of the Yang–Baxter equation, we denote by $M=M(X,r)$ the structure monoid and by $A=A(X,r)$, respectively $A'=A'(X,r)$, the left, respectively right, derived structure monoid of (X,r). It is shown that there exist a left action of M on A and a right action of M on A' and 1-cocycles π and π' of M with coefficients in A and in A' with respect to these actions, respectively. We investigate when the 1-cocycles are injective, surjective, or bijective. In case X is finite, it turns out that π is bijective if and only if (X,r) is left non-degenerate, and π' is bijective if and only if (X,r) is right non-degenerate. In case (X,r) is left non-degenerate, in particular π is bijective, we define a semi-truss structure on $M(X,r)$ and then we show that this naturally induces a set-theoretic solution $(\overline{M},\overline{r})$ on the least cancellative image $\overline{M}=M(X,r)/\eta$ of $M(X,r)$. In case X is naturally embedded in $M(X,r)/\eta$, for example when (X,r) is irretractable, then $\overline{r}$ is an extension of r. It is also shown that non-degenerate irretractable solutions necessarily are bijective.

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1. Introduction

Let V be a vector space over a field K. Solutions $R\colon V\otimes V\to V\otimes V$ of the linear braid or Yang–Baxter equation (abbreviated YBE)

$$(R\otimes \mathrm{id}_V)\circ(\mathrm{id}_V\otimes R)\circ(R\otimes \mathrm{id}_V)=(\mathrm{id}_V\otimes R)\circ(R\otimes \mathrm{id}_V)\circ(\mathrm{id}_V\otimes R)$$

on the vector space $V\otimes V\otimes V$ have led to several algebraic structures, including some classes of bialgebras, quantum groups, and Hopf algebras. Because the variety of solutions remains elusive, Drinfeld ([**11**]) in 1992 proposed to consider solutions that are linearizations of solutions on a basis of V. These are the so called set-theoretic solutions of the YBE.

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Thus a pair (X,r), where X is a non-empty set and $r\colon X\times X\to X\times X$ is a map, is called a set-theoretic solution of the YBE if

$$(r\times \mathrm{id}_X)\circ(\mathrm{id}_X\times r)\circ(r\times \mathrm{id}_X)=(\mathrm{id}_X\times r)\circ(r\times \mathrm{id}_X)\circ(\mathrm{id}_X\times r).$$

For $x,y\in X$, write $r(x,y)=(\sigma_x(y),\gamma_y(x))$. The solution (X,r) is said to be left (resp. right) non-degenerate if each map σ_x (resp. γ_y) is bijective. A left and right non-degenerate solution is simply called a non-degenerate solution. The solution (X,r) is said to be involutive if $r^2=\mathrm{id}_{X\times X}$, and in particular such a solution is bijective.

This study started in the seminal papers of Etingof, Schedler, and Soloviev [**12**] and Gateva-Ivanova and Van den Bergh [**17**]. Since then, different aspects of this combinatorial problem have been developed [**14, 17, 25, 26, 31**] and several interesting connections have been found, such as braid and Garside groups [**7, 10**], (semi)groups of I-type [**17, 21**], matched pairs of groups [**25, 32**], Artin–Schelter regular algebras [**13**], Jacobson radical rings and generalizations [**5, 27**], regular subgroups and Hopf–Galois extensions [**30**], affine manifolds [**28**], orderability [**3, 8**], and factorizable groups [**33**].

It is now well-known that all non-degenerate involutive set-theoretic solutions (X,r) are restrictions of a set-theoretic solution on the structure monoid

$$M(X,r)=\langle x\in X\mid xy=\sigma_x(y)\gamma_y(x)\text{ for all }x,y\in X\rangle.$$

Furthermore, in this case, the structure group

$$G(X,r)=\mathrm{gr}(x\in X\mid xy=\sigma_x(y)\gamma_y(x)\text{ for all }x,y\in X)$$

and the permutation group $\mathcal{G}(X,r)=\mathrm{gr}(\sigma_x\mid x\in X)$ have a brace structure, an algebraic structure introduced by Rump in [**27**]. Moreover, in [**2**], it is shown that all finite non-degenerate involutive set-theoretic solutions with a given permutation group, as a brace, can be explicitly constructed. For this case of finite solutions (X,r), Etingof, Schedler, and Soloviev ([**12**]) proved that $G(X,r)$ is a finitely generated, solvable abelian-by-finite group and independently Gateva-Ivanova and Van den Bergh ([**17**]) have shown that $G(X,r)$ is a Bieberbach group, i.e. $G(X,r)$ is an abelian-by-finite group, torsion-free, and finitely generated. To deal with arbitrary finite bijective non-degenerate solutions Guarnieri and Vendramin ([**18**]) introduced the algebraic structure called a skew brace. Bachiller ([**1**]) then also showed that all such solutions can be described from finite skew braces. Lu, Yan, and Zhu ([**25**]) and Soloviev ([**31**]) showed that for such solutions the structure group $G(X,r)$ is abelian-by-finite (see also Lebed and Vendramin – [**24**] – for another proof), and Jespers, Kubat, and Van Antwerpen ([**19**]) showed that the structure monoid $M(X,r)$ is also abelian-by-finite. Note that, if (X,r) is not involutive, then the canonical map $i\colon X \to G(X,r)$ is not necessarily injective and thus one cannot recover r from the associated solution on $G(X,r)$. However, it can be recovered from the solution associated to $M(X,r)$.

The associated structure algebras, i.e. the monoid algebra $KM(X,r)$ and the group algebra $KG(X,r)$, where K is any field, have also been studied by Jespers and Okniński [**21**], Gateva-Ivanova and Van den Bergh [**17**], and Jespers, Kubat, and Van Antwerpen [**19**]. In the latter it is shown that if (X,r) is a left non-degenerate bijective finite set-theoretic solution, then the algebra $KM(X,r)$ (and $KG(X,r)$) is a module-finite normal extension of a commutative affine subalgebra. In particular, these algebras are Noetherian PI-algebras of finite Gelfand–Kirillov dimension. Furthermore, it was shown that many properties, such as being a domain or prime, of the algebra $KM(X,r)$ are equivalent with the solution (X,r) being involutive.

A crucial fact to prove the above results is (see [**12, 19, 25**]) that if (X,r) is a left non-degenerate solution, then the structure monoid $M(X,r)$ is a regular submonoid of the semidirect product

$$A(X,r) \rtimes \mathcal{G}(X,r),$$

where

$$A(X,r) = \langle x \in X \mid x\sigma_x(y) = \sigma_x(y)\sigma_{\sigma_x(y)}(\gamma_y(x)) \text{ for all } x,y \in X\rangle,$$

i.e. for any element $a \in A(X,r)$ there is a unique $\phi(a) \in \mathcal{G}(X,r)$ such that $(a,\phi(a)) \in M(X,r)$. In particular, one has a bijective 1-cocycle $M(X,r) \to A(X,r)$, determined by the natural action of $\mathcal{G}(X,r)$ on $A(X,r)$. Here, the derived monoid $A(X,r)$ "encodes" the relations determined by the map $r^2\colon X^2 \to X^2$. If, furthermore, the left non-degenerate solution (X,r) is bijective, then the monoid $A = A(X,r)$ is such that $aA = Aa$ for all $a \in A$. So $A(X,r)$ consists of normal elements and thus A has a much richer structure than $M(X,r)$. For example, if (X,r) is involutive, then A is a free abelian monoid of rank $|X|$. It is this "richer structure" that has been exploited in several papers to obtain information on the structure monoid $M(X,r)$ and the structure algebra $KM(X,r)$.

In this paper we continue these investigations for arbitrary set-theoretic solutions (X,r). So, r is not necessarily bijective and X is any set. In the first section we recall the important result of Gateva-Ivanova and Majid [**16**]: there exists a unique set-theoretic solution (M,r_M)

associated to the structure monoid $M = M(X,r)$ such that the restriction of r_M to X^2 equals r. In the second section we introduce two derived monoids $A(X,r)$ and $A'(X,r)$ and we prove that there is a unique 1-cocycle $\pi\colon M(X,r) \to A(X,r)$, with respect to the natural left action $\lambda'\colon M(X,r) \to \mathrm{End}(A(X,r))$, such that $\pi(x) = x$, and a unique 1-cocycle $\pi'\colon M(X,r) \to A'(X,r)$, with respect to the natural right action $\rho'\colon M(X,r) \to \mathrm{End}(A'(X,r))$ such that $\pi'(x) = x$. Hence one gets a monoid homomorphism $f\colon M(X,r) \to A(X,r) \rtimes \mathrm{Im}(\lambda')\colon a \mapsto (\pi(a), \lambda'_a)$ and a monoid anti-homomorphism $f'\colon M(X,r) \to A'(X,r)^{\mathrm{op}} \rtimes \mathrm{Im}(\rho')\colon a \mapsto (\pi'(a), \rho'_a)$, where $\lambda'_x(y) = \sigma_x(y)$ and $\rho'_x(y) = \gamma_x(y)$ for all $x, y \in X$. In general these 1-cocycles are not bijective. But we investigate when they are injective, respectively surjective. In case (X,r) is finite, the bijectiveness of π (respectively π') is equivalent to the solution being left (respectively right) non-degenerate. In Section 4 we prove the surprising result that any non-degenerate irretractable solution is necessarily bijective. In Section 5 we link the algebraic structure of $M(X,r)$ to that of semi-trusses as introduced by Brzeziński [**4**]. We determine the left cancellative (additive) congruence η on $M(X,r)$ for (X,r) a left non-degenerate solution, and we show that we obtain a solution $(M/\eta, \bar{r})$ determined by a semi-truss structure on M/η.

2. Solution associated with the structure monoid

In this section we recall a result of Gateva-Ivanova and Majid in [**16**, Section 3.2], stating that any set-theoretic solution (X,r) of the YBE can be extended to a set-theoretic solution on its structure monoid $M(X,r)$. The result in [**16**] is stated for bijective solutions but the proof remains valid without this assumption.

We recall this construction. Let (X,r) be a set-theoretic solution of the YBE which is not necessarily bijective. We write $r(x,y) = (\sigma_x(y), \gamma_y(x))$ for all $x, y \in X$. It is known that (X,r) is a set-theoretic solution of the YBE if and only if the following conditions hold:

$$\sigma_x \sigma_y = \sigma_{\sigma_x(y)} \sigma_{\gamma_y(x)}, \tag{1}$$

$$\sigma_{\gamma_{\sigma_x(y)}(z)}(\gamma_y(x)) = \gamma_{\sigma_{\gamma_x(z)}(y)}(\sigma_z(x)), \tag{2}$$

$$\gamma_x \gamma_y = \gamma_{\gamma_x(y)} \gamma_{\sigma_y(x)}, \tag{3}$$

for all $x, y, z \in X$.

Let $M = M(X,r)$ be the structure monoid of (X,r), that is, the multiplicative monoid with operation $\circ$ and with presentation

$$M(X,r) = \langle X \mid x \circ y = \sigma_x(y) \circ \gamma_y(x) \text{ for all } x, y \in X \rangle.$$

One defines the following "left action" on M:

$$\lambda\colon M \longrightarrow \mathrm{Map}(M,M)\colon a \longmapsto \lambda_a,$$

with $\lambda_1 = \mathrm{id}_M$, and for $x_1,\dots,x_m,y_1,\dots,y_n \in X$ and $n>1$, $\lambda_{x_1}(1)=1$,

$$\lambda_{x_1}(y_1)=\sigma_{x_1}(y_1), \quad \lambda_{x_1}(y_1\circ\cdots\circ y_n)=\sigma_{x_1}(y_1)\circ\lambda_{\gamma_{y_1}(x_1)}(y_2\circ\cdots\circ y_n), \tag{4}$$

and for $m>1$,

$$\lambda_{x_1\circ\cdots\circ x_m} = \lambda_{x_1}\circ\cdots\circ\lambda_{x_m}. \tag{5}$$

One also defines a "right action" on M:

$$\rho\colon M \longrightarrow \mathrm{Map}(M,M)\colon a \longmapsto \rho_a,$$

with $\rho_1 = \mathrm{id}_M$, and for $x_1,\dots,x_m,y_1,\dots,y_n \in X$ and $n>1$,

$$\rho_{x_1}(y_1)=\gamma_{x_1}(y_1), \quad \rho_{x_1}(y_1\circ\cdots\circ y_n)=\rho_{\sigma_{y_n}(x_1)}(y_1\circ\cdots\circ y_{n-1})\circ\gamma_{x_1}(y_n), \tag{6}$$

and for $m>1$,

$$\rho_{x_1\circ\cdots\circ x_m} = \rho_{x_m}\circ\cdots\circ\rho_{x_1}. \tag{7}$$

In [**16**] it is proved that λ and ρ are well defined. Furthermore, it is then shown that every set-theoretic solution (X,r) of the YBE is the restriction of a set-theoretic solution defined on the structure monoid $M(X,r)$.

Theorem 2.1 (Gateva-Ivanova and Majid [**16**, Theorem 3.6])**.** *Let (X,r) be a set-theoretic solution of the YBE. Then the mapping λ is a monoid homomorphism and the mapping ρ is monoid anti-homomorphism such that*

$$\rho_b(c\circ a) = \rho_{\lambda_a(b)}(c)\circ\rho_b(a), \tag{8}$$

$$\lambda_b(a\circ c) = \lambda_b(a)\circ\lambda_{\rho_a(b)}(c), \tag{9}$$

for all $a,b,c\in M$. Furthermore, for $a,b\in M = M(X,r)$,

$$a\circ b = \lambda_a(b)\circ\rho_b(a). \tag{10}$$

Let $r_M\colon M\times M \to M\times M$ be defined by $r_M(a,b) = (\lambda_a(b),\rho_b(a))$ for all $a,b\in M$. Then, (M,r_M) is a set-theoretic solution of the YBE. Obviously, r_M extends the solution r.

Note that if the solution (X,r) is bijective and left and right non-degenerate, i.e. all σ_x and γ_x are bijective maps, then as in the proof of the above result one can show that the mappings σ_x and γ_x induce actually left and right actions on $G = G(X,r)$, say $\lambda^e\colon G\to \mathrm{Sym}(G)$ and $\rho^e\colon G\to\mathrm{Sym}(G)$; this is Theorem 4 in [**25**]. Furthermore, the mapping $r_G(a,b) = (\lambda^e_a(b),\rho^e_b(a))$, for $a,b\in G$, defines a set-theoretic solution on G. Note that, in general, the natural map $i\colon X\to G$ is not injective. One obtains that r_G is an extension of the induced set-theoretic solution $(i(X), r_{i(X)^2}) = (i(X), (r_G)_{i(X)^2})$.

A natural question is whether one can extend a solution (X, r), via the actions induced from σ_x and γ_y, to a solution on the structure group. This however is not possible in general as shown by the following example. Consider the set-theoretic solution (X, id_{X^2}) on a set X with more than one element. Obviously, each σ_x and γ_x is constant with image $\{x\}$. Hence, $M = M(X, \mathrm{id}_{X^2})$ is the free monoid on the set X and $G = G(X, \mathrm{id}_{X^2})$ is the free group on X. However, because the maps σ_x are not injective one cannot extend the maps σ_x to a monoid homomorphism $\lambda\colon G \to \mathrm{Map}(G, G)$ with $\lambda_x(y) = \sigma_x(y)$ for $y \in G$.

A remarkable fact shown by Lu, Yan, and Zhu in [**25**] is that if one can extend the mappings σ_x and γ_x to left and right actions on the structure group, then the induced set-theoretic solution is bijective.

3. Derived monoids

Let (X, r) be a set-theoretic solution of the YBE. Write $r(x, y) = (\sigma_x(y), \gamma_y(x))$ for all $x, y \in X$. If (X, r) is left non-degenerate, then Soloviev defined in [**31**] its derived solution (X, r') by

$$r'(x, y) = (y, \sigma_y \gamma_{\sigma_x^{-1}(y)}(x))$$

for all $x, y \in X$. For general solutions one cannot define such a derived solution. But in [**19**] one defines the derived monoids of (X, r) as

$$A(X, r) = \langle X \mid x + \sigma_x(y) = \sigma_x(y) + \sigma_{\sigma_x(y)}\gamma_y(x) \text{ for all } x, y \in X\rangle$$

and

$$A'(X, r) = \langle X \mid \gamma_y(x) \oplus y = \gamma_{\gamma_y(x)}\sigma_x(y) \oplus \gamma_y(x) \text{ for all } x, y \in X\rangle.$$

The zero element of $A(X, r)$ is denoted 0 and the zero element of $A'(X, r)$ is denoted $0'$. We will say that $A(X, r)$ is the *left derived* structure monoid of (X, r) and $A'(X, r)$ is the *right derived* structure monoid of (X, r).

Note that $X \subseteq M(X, r)$, $X \subseteq A(X, r)$, and $X \subseteq A'(X, r)$, because the defining relations of these three monoids are homogeneous of degree 2.

Proposition 3.1. *Let (X, r) be a set-theoretic solution of the YBE, where $r(x, y) = (\sigma_x(y), \gamma_y(x))$ for all $x, y \in X$. Then there exists a unique monoid homomorphism $\lambda'\colon M(X, r) \to \mathrm{End}(A(X, r))$ such that $\lambda'(x)(y) = \sigma_x(y)$ for all $x, y \in X$ and there exists a unique anti-homomorphism $\rho'\colon M(X, r) \to \mathrm{End}(A'(X, r))$ such that $\rho'(x)(y) = \gamma_x(y)$ for all $x, y \in X$. Furthermore, if (X, r) is left (right) non-degenerate, then $\mathrm{Im}(\lambda') \subseteq \mathrm{Aut}(A(X, r))$ ($\mathrm{Im}(\rho') \subseteq \mathrm{Aut}(A'(X, r))$).*

Proof: We will write $\lambda'(a)=\lambda'_a$ and $\rho'(a)=\rho'_a$ for all $a\in M(X,r)$.

Let $x_1,\dots,x_m,y_1,\dots,y_n\in X$. We denote by 1, 0, $0'$ the identity elements of the monoids $M(X,r)$, $A(X,r)$, $A'(X,r)$, respectively. We define $\lambda'_1=\mathrm{id}_{A(X,r)}$, $\rho'_1=\mathrm{id}_{A'(X,r)}$, $\lambda'_a(0)=0$, $\rho'_a(0')=0'$, for all $a\in M(X,r)$, and

$$\lambda'_{x_1\circ\cdots\circ x_m}(y_1+\cdots+y_n)=\sigma_{x_1}\dots\sigma_{x_m}(y_1)+\cdots+\sigma_{x_1}\dots\sigma_{x_m}(y_n),$$

and

$$\rho'_{x_1\circ\cdots\circ x_m}(y_1\oplus\cdots\oplus y_n)=\gamma_{x_m}\dots\gamma_{x_1}(y_1)\oplus\cdots\oplus\gamma_{x_m}\dots\gamma_{x_1}(y_n).$$

First we shall prove that λ' and ρ' are well-defined. To do so it is enough to prove that the following equalities hold:

$$\lambda'_{x_1\circ x_2}(y_1+\cdots+y_n)=\lambda'_{\sigma_{x_1}(x_2)\circ\gamma_{x_2}(x_1)}(y_1+\cdots+y_n),\tag{11}$$

$$\lambda'_{x_1\circ\cdots\circ x_m}(y_1+\sigma_{y_1}(y_2))=\lambda'_{x_1\circ\cdots\circ x_m}(\sigma_{y_1}(y_2)+\sigma_{\sigma_{y_1}(y_2)}(\gamma_{y_2}(y_1))),\tag{12}$$

$$\rho'_{x_1\circ x_2}(y_1\oplus\cdots\oplus y_n)=\rho'_{\sigma_{x_1}(x_2)\circ\gamma_{x_2}(x_1)}(y_1\oplus\cdots\oplus y_n),\tag{13}$$

$$\rho'_{x_1\circ\cdots\circ x_m}(\gamma_{y_2}(y_1)\oplus y_2)=\rho'_{x_1\circ\cdots\circ x_m}(\gamma_{\gamma_{y_2}(y_1)}(\sigma_{y_1}(y_2))\oplus\gamma_{y_2}(y_1)).\tag{14}$$

Using relations (1) and (3), equations (11) and (13) are easily checked:

$$\begin{aligned}\lambda'_{x_1\circ x_2}(y_1+\cdots+y_n)&=\sigma_{x_1}\sigma_{x_2}(y_1)+\cdots+\sigma_{x_1}\sigma_{x_2}(y_n)\\&=\sigma_{\sigma_{x_1}(x_2)}\sigma_{\gamma_{x_2}(x_1)}(y_1)+\cdots+\sigma_{\sigma_{x_1}(x_2)}\sigma_{\gamma_{x_2}(x_1)}(y_n)\\&=\lambda'_{\sigma_{x_1}(x_2)\circ\gamma_{x_2}(x_1)}(y_1+\cdots+y_n),\\\rho'_{x_1\circ x_2}(y_1\oplus\cdots\oplus y_n)&=\gamma_{x_2}\gamma_{x_1}(y_1)\oplus\cdots\oplus\gamma_{x_2}\gamma_{x_1}(y_n)\\&=\gamma_{\gamma_{x_2}(x_1)}\gamma_{\sigma_{x_1}(x_2)}(y_1)\oplus\cdots\oplus\gamma_{\gamma_{x_2}(x_1)}\gamma_{\sigma_{x_1}(x_2)}(y_n)\\&=\rho'_{\sigma_{x_1}(x_2)\circ\gamma_{x_2}(x_1)}(y_1\oplus\cdots\oplus y_n).\end{aligned}$$

Using relations (1), (2), and (3) we shall prove equations (12) and (14) by induction on m. For $m=0$, (12) and (14) follows by the defining relations of $A(X,r)$ and $A'(X,r)$. Suppose that $m>0$. Assume that $\lambda'_{x_1\circ\cdots\circ x_k}(y_1+\sigma_{y_1}(y_2))=\lambda'_{x_1\circ\cdots\circ x_k}(\sigma_{y_1}(y_2)+\sigma_{\sigma_{y_1}(y_2)}(\gamma_{y_2}(y_1)))$ and $\rho'_{x_1\circ\cdots\circ x_k}(\gamma_{y_2}(y_1)\oplus y_2)=\rho'_{x_1\circ\cdots\circ x_k}(\gamma_{\gamma_{y_2}(y_1)}(\sigma_{y_1}(y_2))\oplus\gamma_{y_2}(y_1))$, for $k<m$, then

$$\begin{aligned}
&\lambda'_{x_1\circ\cdots\circ x_m}(y_1+\sigma_{y_1}(y_2))\\
&\quad=\sigma_{x_1}\cdots\sigma_{x_m}(y_1)+\sigma_{x_1}\cdots\sigma_{x_m}(\sigma_{y_1}(y_2))\\
&\quad=\lambda'_{x_1\circ\cdots\circ x_{m-1}}(\sigma_{x_m}(y_1)+\sigma_{x_m}(\sigma_{y_1}(y_2)))\\
&\quad=\lambda'_{x_1\circ\cdots\circ x_{m-1}}(\sigma_{x_m}(y_1)+\sigma_{\sigma_{x_m}(y_1)}(\sigma_{\gamma_{y_1}(x_m)}(y_2)))\\
&\quad=\lambda'_{x_1\circ\cdots\circ x_{m-1}}(\sigma_{\sigma_{x_m}(y_1)}(\sigma_{\gamma_{y_1}(x_m)}(y_2))\\
&\qquad\qquad+\sigma_{\sigma_{\sigma_{x_m}(y_1)}(\sigma_{\gamma_{y_1}(x_m)}(y_2))}(\gamma_{\sigma_{\gamma_{y_1}(x_m)}(y_2)}(\sigma_{x_m}(y_1))))\\
&\quad=\lambda'_{x_1\circ\cdots\circ x_{m-1}}(\sigma_{x_m}(\sigma_{y_1}(y_2))+\sigma_{\sigma_{x_m}(\sigma_{y_1}(y_2))}(\gamma_{\sigma_{\gamma_{y_1}(x_m)}(y_2)}(\sigma_{x_m}(y_1))))\\
&\quad=\lambda'_{x_1\circ\cdots\circ x_{m-1}}(\sigma_{x_m}(\sigma_{y_1}(y_2))+\sigma_{\sigma_{x_m}(\sigma_{y_1}(y_2))}(\sigma_{\gamma_{\sigma_{y_1}(y_2)}(x_m)}(\gamma_{y_2}(y_1))))\\
&\quad=\lambda'_{x_1\circ\cdots\circ x_{m-1}}(\sigma_{x_m}(\sigma_{y_1}(y_2))+\sigma_{x_m}(\sigma_{\sigma_{y_1}(y_2)}(\gamma_{y_2}(y_1))))\\
&\quad=\sigma_{x_1}\cdots\sigma_{x_m}(\sigma_{y_1}(y_2))+\sigma_{x_1}\cdots\sigma_{x_m}(\sigma_{\sigma_{y_1}(y_2)}(\gamma_{y_2}(y_1)))\\
&\quad=\lambda'_{x_1\circ\cdots\circ x_m}(\sigma_{y_1}(y_2)+\sigma_{\sigma_{y_1}(y_2)}(\gamma_{y_2}(y_1))),
\end{aligned}$$

and

$$\begin{aligned}
&\rho'_{x_1\circ\cdots\circ x_m}(\gamma_{y_2}(y_1)\oplus y_2)\\
&\quad=\rho'_{x_2\circ\cdots\circ x_m}(\gamma_{x_1}(\gamma_{y_2}(y_1))\oplus\gamma_{x_1}(y_2))\\
&\quad=\rho'_{x_2\circ\cdots\circ x_m}(\gamma_{\gamma_{x_1}(y_2)}(\gamma_{\sigma_{y_2}(x_1)}(y_1))\oplus\gamma_{x_1}(y_2))\\
&\quad=\rho'_{x_2\circ\cdots\circ x_m}(\gamma_{\gamma_{\gamma_{x_1}(y_2)}(\gamma_{\sigma_{y_2}(x_1)}(y_1))}(\sigma_{\gamma_{\sigma_{y_2}(x_1)}(y_1)}(\gamma_{x_1}(y_2)))\\
&\qquad\qquad\oplus\gamma_{\gamma_{x_1}(y_2)}(\gamma_{\sigma_{y_2}(x_1)}(y_1)))\\
&\quad=\rho'_{x_2\circ\cdots\circ x_m}(\gamma_{\gamma_{x_1}(\gamma_{y_2}(y_1))}(\sigma_{\gamma_{\sigma_{y_2}(x_1)}(y_1)}(\gamma_{x_1}(y_2)))\oplus\gamma_{x_1}(\gamma_{y_2}(y_1)))\\
&\quad=\rho'_{x_2\circ\cdots\circ x_m}(\gamma_{\gamma_{x_1}(\gamma_{y_2}(y_1))}(\gamma_{\sigma_{\gamma_{y_2}(y_1)}(x_1)}(\sigma_{y_1}(y_2)))\oplus\gamma_{x_1}(\gamma_{y_2}(y_1)))\\
&\quad=\rho'_{x_2\circ\cdots\circ x_m}(\gamma_{x_1}(\gamma_{\gamma_{y_2}(y_1)}(\sigma_{y_1}(y_2)))\oplus\gamma_{x_1}(\gamma_{y_2}(y_1)))\\
&\quad=\gamma_{x_m}\cdots\gamma_{x_1}(\gamma_{\gamma_{y_2}(y_1)}(\sigma_{y_1}(y_2)))\oplus\gamma_{x_m}\cdots\gamma_{x_1}(\gamma_{y_2}(y_1))\\
&\quad=\rho'_{x_1\circ\cdots\circ x_m}(\gamma_{\gamma_{y_2}(y_1)}(\sigma_{y_1}(y_2))\oplus\gamma_{y_2}(y_1)).
\end{aligned}$$

This proves that λ'_a and ρ'_a are well-defined and clearly $\lambda'_a\in\mathrm{End}(A(X,r))$ and $\rho'_a\in\mathrm{End}(A'(X,r))$ for all $a\in M(X,r)$. Thus λ' and ρ' are well-defined. It is clear that λ' is a monoid homomorphism and that it is unique with respect to the condition $\lambda'_x(y)=\sigma_x(y)$ for all $x,y\in X$. It is also clear that ρ' is a monoid anti-homomorphism and that it is unique for the condition $\rho'_x(y)=\gamma_x(y)$ for all $x,y\in X$.

Assume now that (X,r) is left non-degenerate. Let $x, y_1, \dots, y_n \in X$. We define $f_x \in \mathrm{End}(A(X,r))$ by

$$f_x(y_1 + \dots + y_n) = \sigma_x^{-1}(y_1) + \dots + \sigma_x^{-1}(y_n).$$

To see that f_x is well-defined it is enough to prove that

$$f_x(y_1 + \sigma_{y_1}(y_2)) = f_x(\sigma_{y_1}(y_2) + \sigma_{\sigma_{y_1}(y_2)}(\gamma_{y_2}(y_1))).$$

Note that, from (1),

$$\sigma_x^{-1}\sigma_{y_1}(y_2) = \sigma_{\sigma_x^{-1}(y_1)}\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2) \tag{15}$$

and thus, also using (2), we get that

$$\begin{aligned}
&\sigma_{\gamma_{\sigma_x^{-1}\sigma_{y_1}(y_2)}(x)}\gamma_{\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2)}(\sigma_x^{-1}(y_1)) \\
&= \sigma_{\gamma_{\sigma_{\sigma_x^{-1}(y_1)}\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2)}(x)}\gamma_{\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2)}(\sigma_x^{-1}(y_1)) \\
&= \gamma_{\sigma_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2))}\sigma_x(\sigma_x^{-1}(y_1)) \\
&= \gamma_{y_2}(y_1).
\end{aligned} \tag{16}$$

We have that

$$\begin{aligned}
f_x(y_1+\sigma_{y_1}(y_2)) &= \sigma_x^{-1}(y_1) + \sigma_x^{-1}(\sigma_{y_1}(y_2)) \\
&\overset{(15)}{=} \sigma_x^{-1}(y_1) + \sigma_{\sigma_x^{-1}(y_1)}(\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2)) \\
&= \sigma_{\sigma_x^{-1}(y_1)}(\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2)) \\
&\quad + \sigma_{\sigma_{\sigma_x^{-1}(y_1)}(\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2))}(\gamma_{\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2)}(\sigma_x^{-1}(y_1))) \\
&\overset{(15)}{=} \sigma_x^{-1}(\sigma_{y_1}(y_2)) + \sigma_{\sigma_x^{-1}(\sigma_{y_1}(y_2))}(\gamma_{\sigma^{-1}_{\gamma_{\sigma_x^{-1}(y_1)}(x)}(y_2)}(\sigma_x^{-1}(y_1))) \\
&\overset{(16)}{=} \sigma_x^{-1}(\sigma_{y_1}(y_2)) + \sigma_{\sigma_x^{-1}(\sigma_{y_1}(y_2))}(\sigma^{-1}_{\gamma_{\sigma_x^{-1}\sigma_{y_1}(y_2)}(x)}(\gamma_{y_2}(y_1))) \\
&\overset{(15)}{=} \sigma_x^{-1}(\sigma_{y_1}(y_2)) + \sigma_x^{-1}(\sigma_{\sigma_{y_1}(y_2)}(\gamma_{y_2}(y_1))) \\
&= f_x(\sigma_{y_1}(y_2) + \sigma_{\sigma_{y_1}(y_2)}(\gamma_{y_2}(y_1))),
\end{aligned}$$

where the third equality follows from the defining relations in $A(X,r)$. Hence f_x is well-defined. Note that $f_x\lambda'_x = \lambda'_x f_x = \mathrm{id}$. Thus $\lambda'_x \in \mathrm{Aut}(A(X,r))$ for all $x \in X$. Therefore $\mathrm{Im}(\lambda') \subseteq \mathrm{Aut}(A(X,r))$.

Similarly one can prove that if (X,r) is right non-degenerate, then $\mathrm{Im}(\rho') \subseteq \mathrm{Aut}(A'(X,r))$. □

Proposition 3.2. *Let (X,r) be a set-theoretic solution of the YBE. Then*

(i) *There is a unique 1-cocycle $\pi\colon M(X,r)\to A(X,r)$ with respect to the left action λ' such that $\pi(x)=x$ for all $x\in X$.*
(ii) *There is a unique 1-cocycle $\pi'\colon M(X,r)\to A'(X,r)$ with respect to the right action ρ' such that $\pi'(x)=x$ for all $x\in X$.*

Furthermore, the mapping

$$f\colon M(X,r)\longrightarrow A(X,r)\rtimes \mathrm{Im}(\lambda')\colon a\longmapsto (\pi(a),\lambda'_a)$$

is a monoid homomorphism and the mapping

$$f'\colon M(X,r)\longrightarrow A'(X,r)^{\mathrm{op}}\rtimes \mathrm{Im}(\rho')\colon a\longmapsto (\pi'(a),\rho'_a)$$

is a monoid anti-homomorphism.

Proof: We define for $x_1,\dots,x_m\in X$,

$$\begin{aligned}\pi(1)&=0,\\ \pi(x_1)&=x_1,\qquad \text{and for } m>1,\\ \pi(x_1\circ\cdots\circ x_m)&=x_1+\lambda'_{x_1}(\pi(x_2\circ\cdots\circ x_m)),\\ \pi'(1)&=0',\\ \pi'(x_1)&=x_1,\qquad \text{and for } m>1,\\ \pi'(x_1\circ\cdots\circ x_m)&=\rho'_{x_m}(\pi'(x_1\circ\cdots\circ x_{m-1}))\oplus x_m.\end{aligned}$$

We prove that $\pi(x_1\circ\cdots\circ x_m)$ and $\pi'(x_1\circ\cdots\circ x_m)$ are well-defined by induction on m. For $m=1$ it is clear. Suppose that $m>1$ and that $\pi(x_1\circ\cdots\circ x_{m-1})$ and $\pi'(x_1\circ\cdots\circ x_{m-1})$ are well-defined.

By the induction hypothesis, it is enough to show that

$$x_1+\lambda'_{x_1}(\pi(x_2\circ\cdots\circ x_m))=\sigma_{x_1}(x_2)+\lambda'_{\sigma_{x_1}(x_2)}(\pi(\gamma_{x_2}(x_1)\circ x_3\circ\cdots\circ x_m)) \tag{17}$$

and

$$\begin{aligned}&\rho'_{x_m}(\pi'(x_1\circ\cdots\circ x_{m-1}))\oplus x_m\\ &\quad=\rho'_{\gamma_{x_m}(x_{m-1})}(\pi'(x_1\circ\cdots\circ x_{m-2}\circ\sigma_{x_{m-1}}(x_m)))\oplus\gamma_{x_m}(x_{m-1}).\end{aligned} \tag{18}$$

By (11) and (13) we get that

$$\begin{aligned}
&\sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)}(\pi(\gamma_{x_2}(x_1) \circ x_3 \circ \cdots \circ x_m))\\
&\quad = \sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)}(\gamma_{x_2}(x_1) + \lambda'_{\gamma_{x_2}(x_1)}(\pi(x_3 \circ \cdots \circ x_m)))\\
&\quad = \sigma_{x_1}(x_2) + \sigma_{\sigma_{x_1}(x_2)}(\gamma_{x_2}(x_1)) + \lambda'_{\sigma_{x_1}(x_2)}(\lambda'_{\gamma_{x_2}(x_1)}(\pi(x_3 \circ \cdots \circ x_m)))\\
&\quad = x_1 + \sigma_{x_1}(x_2) + \lambda'_{\sigma_{x_1}(x_2)\circ\gamma_{x_2}(x_1)}(\pi(x_3 \circ \cdots \circ x_m))\\
&\quad = x_1 + \sigma_{x_1}(x_2) + \lambda'_{x_1 \circ x_2}(\pi(x_3 \circ \cdots \circ x_m))\\
&\quad = x_1 + \sigma_{x_1}(x_2) + \lambda'_{x_1}(\lambda'_{x_2}(\pi(x_3 \circ \cdots \circ x_m)))\\
&\quad = x_1 + \lambda'_{x_1}(x_2 + \lambda'_{x_2}(\pi(x_3 \circ \cdots \circ x_m)))\\
&\quad = x_1 + \lambda'_{x_1}(\pi(x_2 \circ \cdots \circ x_m))
\end{aligned}$$

and

$$\begin{aligned}
&\rho'_{\gamma_{x_m}(x_{m-1})}(\pi'(x_1 \circ \cdots \circ x_{m-2} \circ \sigma_{x_{m-1}}(x_m))) \oplus \gamma_{x_m}(x_{m-1})\\
&\quad = \rho'_{\gamma_{x_m}(x_{m-1})}(\rho'_{\sigma_{x_{m-1}}(x_m)}(\pi'(x_1 \circ \cdots \circ x_{m-2})) \oplus \sigma_{x_{m-1}}(x_m)) \oplus \gamma_{x_m}(x_{m-1})\\
&\quad = \rho'_{\gamma_{x_m}(x_{m-1})}(\rho'_{\sigma_{x_{m-1}}(x_m)}(\pi'(x_1 \circ \cdots \circ x_{m-2})))\\
&\qquad \oplus \gamma_{\gamma_{x_m}(x_{m-1})}(\sigma_{x_{m-1}}(x_m)) \oplus \gamma_{x_m}(x_{m-1})\\
&\quad = \rho'_{\sigma_{x_{m-1}}(x_m)\circ\gamma_{x_m}(x_{m-1})}(\pi'(x_1 \circ \cdots \circ x_{m-2})) \oplus \gamma_{x_m}(x_{m-1}) \oplus x_m\\
&\quad = \rho'_{x_{m-1}\circ x_m}(\pi'(x_1 \circ \cdots \circ x_{m-2})) \oplus \gamma_{x_m}(x_{m-1}) \oplus x_m\\
&\quad = \rho'_{x_m}(\rho'_{x_{m-1}}(\pi'(x_1 \circ \cdots \circ x_{m-2}))) \oplus \gamma_{x_m}(x_{m-1}) \oplus x_m\\
&\quad = \rho'_{x_m}(\rho'_{x_{m-1}}(\pi'(x_1 \circ \cdots \circ x_{m-2})) \oplus x_{m-1}) \oplus x_m\\
&\quad = \rho'_{x_m}(\pi'(x_1 \circ \cdots \circ x_{m-1})) \oplus x_m.
\end{aligned}$$

Thus, indeed, π and π' are well-defined.

For all $a, b \in M(X, r)$, we shall prove by induction on $\deg(a) + \deg(b)$ that

$$\pi(a \circ b) = \pi(a) + \lambda'_a(\pi(b)) \tag{19}$$

and

$$\pi'(a \circ b) = \rho'_b(\pi'(a)) \oplus \pi'(b). \tag{20}$$

If $\deg(a) = \deg(b) = 1$, then (19) and (20) follow by definition. Hence, we may suppose that $\deg(a) + \deg(b) > 2$ and that $\pi(a' \circ b') = \pi(a') + \lambda'_{a'}(\pi(b'))$ and $\pi'(a' \circ b') = \rho'_{b'}(\pi'(a')) \oplus \pi'(b')$ for all $a', b' \in M(X, r)$ such that $\deg(a') + \deg(b') < \deg(a) + \deg(b)$.

Write $a = x \circ a'$ and $b = b' \circ y$ for some $x, y \in X$ and $a', b' \in M(X, r)$. By the induction hypothesis we have

$$\begin{aligned}\pi(a \circ b) &= \pi(x \circ a' \circ b)\\ &= x + \lambda'_x(\pi(a' \circ b))\\ &= x + \lambda'_x(\pi(a') + \lambda'_{a'}(\pi(b)))\\ &= x + \lambda'_x(\pi(a')) + \lambda'_x(\lambda'_{a'}(\pi(b)))\\ &= \pi(x \circ a') + \lambda'_{x\circ a'}(\pi(b))\\ &= \pi(a) + \lambda'_a(\pi(b))\end{aligned}$$

and

$$\begin{aligned}\pi'(a \circ b) &= \pi'(a \circ b' \circ y)\\ &= \rho'_y(\pi'(a \circ b')) \oplus y\\ &= \rho'_y(\rho'_{b'}(\pi'(a)) \oplus \pi'(b')) \oplus y\\ &= \rho'_y(\rho'_{b'}(\pi'(a))) \oplus \rho'_y(\pi'(b')) \oplus y\\ &= \rho'_{b'\circ y}(\pi'(a)) \oplus \pi'(b' \circ y)\\ &= \rho'_b(\pi'(a)) \oplus \pi'(b).\end{aligned}$$

Thus (19) and (20) follow by induction. It is clear that π and π' are the unique 1-cocycles satisfying the hypothesis. Therefore the result follows. □

A natural question is the following.

Question 3.3. *When are the* 1*-cocycles* π *and* π' *bijective?*

In general, these 1-cocycles are not bijective. We provide two examples. The first one is an example where π is injective but not surjective, and the second one where π and π' are neither injective nor surjective.

Example 3.4. Let (X, r) be a set-theoretic solution of the YBE, where X is set of cardinality greater than 1 and $r\colon X \times X \to X \times X$ is a map defined by $r(x, y) = (x, x)$ for all $x, y \in X$. The associated monoids are

$$\begin{aligned}M(X, r) &= \langle X \mid x \circ y = x \circ x \text{ for all } x, y \in X\rangle,\\ A(X, r) &= \langle X \mid x + x = x + x \text{ for all } x, y \in X\rangle,\\ A'(X, r) &= \langle X \mid x \oplus y = x \oplus x \text{ for all } x, y \in X\rangle.\end{aligned}$$

The 1-cocycle π' is bijective, but it is clear that the 1-cocycle π is not. The latter is not surjective. For example, the element $x + y$, where $x \neq y \in X$ is not in the image of π. Note that π is still injective. Similarly, (X, r) with $r\colon X \times X \to X \times X$ defined by $r(x, y) = (y, y)$ is an example of a set-theoretic solution of the YBE where π' is injective but not surjective.

Example 3.5. Let $S = \{0, 1, 2\}$ and define the skew lattice $(S, \wedge, \vee)$ by

$\wedge$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	1
2	2	2	2

The skew lattice $(S, \wedge, \vee)$ is an example of a distributive and cancellative skew lattice that is not a co-strongly distributive skew lattice; see ([**9**, Example 4.4]). By [**9**, Theorem 5.7], $(S, \wedge, \vee)$ is a left distributive solution, i.e. (S, r) is a set-theoretic solution of the YBE, where $r\colon S \times S \to S \times S$ is defined by $r(x, y) = (x \wedge y, y \vee x)$ for all $x, y \in S$. The associated monoids are

$$M(X,r) = \langle 0,1,2 \mid 1 \circ 0 = 0 \circ 1, 2 \circ 0 = 0 \circ 2, 1 \circ 2 = 2 \circ 2, 2 \circ 1 = 1 \circ 1 \rangle,$$
$$A(X,r) = \langle 0,1,2 \mid 1+0=0+0, 2+0=0+0, 1+2=2+2, 2+1=1+1 \rangle,$$
$$A'(X,r) = \langle 0,1,2 \mid 1 \oplus 0 = 1 \oplus 1, 2 \oplus 0 = 2 \oplus 2 \rangle.$$

Both π and π' are not injective, as $\pi(1 \circ 0) = 1 + 0 = 0 + 0 = \pi(0 \circ 0)$ and $\pi'(1 \circ 0) = 1 \oplus 0 = 1 \oplus 1 = \pi'(1 \circ 1)$, but $1 \circ 0 \neq 0 \circ 0$ and $1 \circ 0 \neq 1 \circ 1$ in $M(X, r)$. Both π and π' are not surjective as $0+1$ (resp. $0 \oplus 1$) is not in the image of π (resp. π').

Proposition 3.6. *Let (X, r) be a set-theoretic solution of the YBE. Write $r(x, y) = (\sigma_x(y), \gamma_y(x))$. Let $\pi\colon M(X, r) \to A(X, r)$ and $\pi'\colon M(X, r) \to A'(X, r)$ be the 1-cocycles of Proposition 3.2. Then*

(i) *π is surjective if and only if σ_x is surjective for all $x \in X$.*
(ii) *π' is surjective if and only if γ_x is surjective for all $x \in X$.*

Proof: Suppose that σ_x is surjective for all $x \in X$. First, we claim that σ_x being surjective implies that λ'_x is surjective. Take n an arbitrary positive integer. Let $x_1, \ldots, x_n \in X$ such that $x_1 + \cdots + x_n \in A(X, r)$. As σ_x is surjective, there exist $y_1, \ldots, y_n \in X$ such that $\sigma_x(y_i) = x_i$ for all $i \in \{1, \ldots, n\}$. Then, $\lambda'_x(y_1 + \cdots + y_n) = \sigma_x(y_1) + \cdots + \sigma_x(y_n) = x_1 + \cdots + x_n$, which proves that λ'_x is surjective.

Next, we prove that π is surjective by induction on the length of the elements in $A(X, r)$. As $\pi(x) = x$ for all $x \in X$, π is surjective on elements of length 1. Assume now that for a fixed positive integer n and for any $x_1, \ldots, x_n \in X$, there exist $y_1, \ldots, y_n \in X$ such that $\pi(y_1 \circ \cdots \circ y_n) = x_1 + \cdots + x_n$. Take $x_1, \ldots, x_{n+1} \in X$. Since λ'_{x_1} is surjective, there exist $z_2, \ldots, z_{n+1} \in X$ such that $\lambda'_{x_1}(z_2 + \cdots + z_{n+1}) = x_2 + \cdots + x_{n+1}$.

Using the induction hypothesis, there exist $y_2, \dots, y_{n+1} \in X$ such that $\pi(y_2 \circ \cdots \circ y_{n+1}) = z_2 + \cdots + z_{n+1}$. Thus, we obtain

$$\begin{aligned} x_1 + \cdots + x_{n+1} &= x_1 + \lambda'_{x_1}(z_2 + \cdots + z_{n+1}) \\ &= x_1 + \lambda'_{x_1}(\pi(y_2 \circ \cdots \circ y_{n+1})) \\ &= \pi(x_1 \circ y_2 \circ \cdots \circ y_{n+1}), \end{aligned}$$

and π is surjective.

Suppose now that π is surjective. Let $x, y \in X$ and consider $x + y \in A(X,r)$. Since π is surjective (and it preserves the degree), there exist $z, t \in X$ such that $\pi(z \circ t) = x + y$. Thus $z + \sigma_z(t) = x + y$ in $A(X,r)$. By the defining relations of $A(X,r)$, this equality implies that there exists $y' \in X$ such that $\sigma_x(y') = y$. Therefore σ_x is surjective for all $x \in X$.

The proof for π' is similar. □

Proposition 3.7. *Let (X,r) be a set-theoretic solution of the YBE. Write $r(x,y) = (\sigma_x(y), \gamma_y(x))$. Let $\pi\colon M(X,r) \to A(X,r)$ and $\pi'\colon M(X,r) \to A'(X,r)$ be the 1-cocycles of Proposition 3.2.*

(i) *If σ_x is injective for all $x \in X$, then π is injective.*
(ii) *If γ_x is injective for all $x \in X$, then π' is injective.*

Proof: We shall prove (i). The proof of (ii) is similar. Let $\mathrm{FM}(X)$ be the (multiplicative) free monoid on X. Suppose that σ_x is injective for all $x \in X$. Since $\pi(x) = x$ for all $x \in X$, the restriction of π to elements of degree one in $M(X,r)$ is injective. Let n be an integer greater than 1. Let $x_1, \dots, x_n, y_1, \dots, y_n \in X$ be elements such that $\pi(x_1 \circ \cdots \circ x_n) = \pi(y_1 \circ \cdots \circ y_n)$. Thus, in $A(X,r)$, we have that

$$x_1 + \sigma_{x_1}(x_2) + \cdots + \sigma_{x_1} \cdots \sigma_{x_{n-1}}(x_n) = y_1 + \sigma_{y_1}(y_2) + \cdots + \sigma_{y_1} \cdots \sigma_{y_{n-1}}(y_n).$$

Let $w_1, w_2 \in \mathrm{FM}(X)$ be two elements of degree n. Suppose that $w_1 = z_1 \cdots z_n$ and $w_2 = t_1 \cdots t_n$, for some $z_i, t_i \in X$. We say that $w_1 \sim w_2$ if there exist $1 \le i \le n-1$ and $z \in X$ such that $z_j = t_j$ for all $j \in \{1, 2, \dots, n\} \setminus \{i, i+1\}$ and, either $z_{i+1} = \sigma_{z_i}(z) = t_i$ and $t_{i+1} = \sigma_{t_i}\gamma_z(z_i)$, or $t_{i+1} = \sigma_{t_i}(z) = z_i$ and $z_{i+1} = \sigma_{z_i}\gamma_z(t_i)$. Note that $z_1 + \cdots + z_n = t_1 + \cdots + t_n$ in $A(X,r)$ if and only if there exist $w'_1, \dots, w'_m \in \mathrm{FM}(X)$ of degree n such that

$$w_1 = w'_1 \sim w'_2 \sim \cdots \sim w'_m = w_2.$$

Hence, to prove that $x_1 \circ \cdots \circ x_n = y_1 \circ \cdots \circ y_n$, we may assume that there exist $1 \le i \le n-1$ and $z \in X$ such that $\sigma_{x_1} \cdots \sigma_{x_{j-1}}(x_j) = \sigma_{y_1} \cdots \sigma_{y_{j-1}}(y_j)$ for all $j \in \{1, 2, \dots, n\} \setminus \{i, i+1\}$, and also $\sigma_{x_1} \cdots \sigma_{x_i}(x_{i+1}) = \sigma_{\sigma_{x_1} \cdots \sigma_{x_{i-1}}(x_i)}(z) = \sigma_{y_1} \cdots \sigma_{y_{i-1}}(y_i)$, as well as $\sigma_{y_1} \cdots \sigma_{y_i}(y_{i+1}) = \sigma_{\sigma_{y_1} \cdots \sigma_{y_{i-1}}(y_i)}\gamma_z(\sigma_{x_1} \cdots \sigma_{x_{i-1}}(x_i))$.

Since $\sigma_{x_1}\cdots\sigma_{x_{j-1}}(x_j) = \sigma_{y_1}\cdots\sigma_{y_{j-1}}(y_j)$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i, i+1\}$, and σ_x is injective for all $x \in X$, we have that $x_j = y_j$ for all $j \in \{1, \ldots, i-1\}$. Hence, since $\sigma_{x_1}\cdots\sigma_{x_i}(x_{i+1}) = \sigma_{y_1}\cdots\sigma_{y_{i-1}}(y_i)$, and σ_x is injective for all $x \in X$, we have that $y_i = \sigma_{x_i}(x_{i+1})$. Now we have that

$$\begin{aligned}\sigma_{\sigma_{x_1}\cdots\sigma_{x_{i-1}}(x_i)}(z) &= \sigma_{x_1}\cdots\sigma_{x_i}(x_{i+1})\\ &= \lambda_{x_1\circ\cdots\circ x_{i-1}}\lambda_{x_i}(x_{i+1})\\ &= \lambda_{\lambda_{x_1\circ\cdots\circ x_{i-1}}(x_i)}\lambda_{\rho_{x_i}(x_1\circ\cdots\circ x_{i-1})}(x_{i+1})\\ &= \sigma_{\sigma_{x_1}\cdots\sigma_{x_{i-1}}(x_i)}\lambda_{\rho_{x_i}(x_1\circ\cdots\circ x_{i-1})}(x_{i+1}),\end{aligned}$$

where the third equality follows Theorem 2.1.

Hence, since σ_x is injective for all $x \in X$, we have that

$$z = \lambda_{\rho_{x_i}(x_1\circ\cdots\circ x_{i-1})}(x_{i+1}).$$

By Theorem 2.1,

$$\begin{aligned}\sigma_{y_1}\cdots\sigma_{y_i}(y_{i+1}) &= \sigma_{\sigma_{y_1}\cdots\sigma_{y_{i-1}}(y_i)}\gamma_z(\sigma_{x_1}\cdots\sigma_{x_{i-1}}(x_i))\\ &= \sigma_{\sigma_{x_1}\cdots\sigma_{x_{i-1}}(\sigma_{x_i}(x_{i+1}))}\gamma_z(\sigma_{x_1}\cdots\sigma_{x_{i-1}}(x_i))\\ &= \lambda_{\lambda_{x_1\circ\cdots\circ x_{i-1}}(\lambda_{x_i}(x_{i+1}))}\rho_{\lambda_{\rho_{x_i}(x_1\circ\cdots\circ x_{i-1})}(x_{i+1})}(\lambda_{x_1\circ\cdots\circ x_{i-1}}(x_i))\\ &= \lambda_{\lambda_{x_1\circ\cdots\circ x_{i-1}}(\lambda_{x_i}(x_{i+1}))}\lambda_{\rho_{\lambda_{x_i}(x_{i+1})}(x_1\circ\cdots\circ x_{i-1})}(\rho_{x_{i+1}}(x_i))\\ &= \lambda_{x_1\circ\cdots\circ x_{i-1}}\lambda_{\lambda_{x_i}(x_{i+1})}(\rho_{x_{i+1}}(x_i))\\ &= \lambda_{y_1\circ\cdots\circ y_{i-1}}\lambda_{y_i}(\rho_{x_{i+1}}(x_i))\\ &= \sigma_{y_1}\cdots\sigma_{y_{i-1}}\sigma_{y_i}(\gamma_{x_{i+1}}(x_i)).\end{aligned}$$

Since σ_x is injective for all $x \in X$, we have that $y_{i+1} = \gamma_{x_{i+1}}(x_i)$. Thus,

$$y_i \circ y_{i+1} = \sigma_{x_i}(x_{i+1}) \circ \gamma_{x_{i+1}}(x_i) = x_i \circ x_{i+1}.$$

Since $\sigma_{x_1}\cdots\sigma_{x_{j-1}}(x_j) = \sigma_{y_1}\cdots\sigma_{y_{j-1}}(y_j)$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i, i+1\}$ and σ_x is injective for all $x \in X$, we have that $x_j = y_j$ for all $j \in \{i+2, \ldots, n\}$. Hence $x_1 \circ \cdots \circ x_n = y_1 \circ \cdots \circ y_n$, and therefore π is injective. □

Remark 3.8. Note that in the set-theoretic solution of the YBE of Example 3.4, $\sigma_x(y) = x$ for all $x, y \in X$, so σ_x is not injective. But π is injective. Similarly, (X, r) with $r\colon X \times X \to X \times X$ defined by $r(x, y) = (y, y)$ is a set-theoretic solution of the YBE where π' is injective (see Example 3.4) but $\gamma_y(x) = y$ for all $x, y \in X$. So γ_y is not injective.

If π (resp. π') is injective, then it is clear that the map f (resp. f') defined in Proposition 3.2 is an embedding. The latter was proved in [**19**] under the assumption that (X, r) is a left non-degenerate solution. In this case π is bijective and $M(X, r)$ is a regular submonoid of the semidirect product $A(X, r) \rtimes \operatorname{gr}(\sigma_x \mid x \in X)$.

The following result answers Question 3.3 for finite solutions.

Corollary 3.9 (Jespers, Kubat, and Van Antwerpen [**19**]). *Let (X, r) be a set-theoretic solution of the YBE, λ' (resp. ρ') the left (resp. right) action as defined before, π (resp. π') the unique* 1*-cocycle with respect to λ' (resp. ρ'). Then, π (resp. π') is bijective if (X, r) is left non-degenerate (resp. right non-degenerate). The converse holds if X is finite.*

Proof: Assume first that (X, r) is a left non-degenerate set-theoretic solution of the YBE. Then, by Propositions 3.6 and 3.7, π is bijective. Similarly, one can prove that (X, r) being a right non-degenerate solution implies that π' is bijective.

Assume now that $\pi\colon M(X, r) \to A(X, r)$ is bijective and X is finite. By Proposition 3.6, σ_x is surjective for all $x \in X$. Since X is finite, σ_x is bijective for all $x \in X$, that is, (X, r) is left non-degenerate. □

The next example shows the difficulty of Question 3.3 for infinite solutions.

Example 3.10. Consider the set $\mathbb{N}$ of the non-negative integers. Let $r\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be the map defined by $r(x, y) = (\xi(y), \xi(x))$ for all $x, y \in \mathbb{N}$, where $\xi(x) = \max\{0, x - 1\}$ for all $x \in \mathbb{N}$. Then $(\mathbb{N}, r)$ is a set-theoretic solution of the YBE, such that the associated 1-cocycles π and π' are bijective but, for every $x \in \mathbb{N}$, $\sigma_x = \gamma_x = \xi$ is not injective because $\xi(0) = \xi(1)$.

Proof: It is easy to check that $(\mathbb{N}, r)$ is a set-theoretic solution of the YBE. Note that, for every $x \in \mathbb{N}$, $\xi^x(x) = 0$. Hence

$$M(\mathbb{N}, r) = \langle \mathbb{N} \mid x \circ y = 0 \circ 0 \rangle,$$

$$A(\mathbb{N}, r) = \langle \mathbb{N} \mid x + y = 0 + 0 \rangle,$$

and

$$A'(\mathbb{N}, r) = \langle \mathbb{N} \mid x \oplus y = 0 \oplus 0 \rangle.$$

Therefore, for every integer $n > 1$, the monoids $M(\mathbb{N}, r)$, $A(\mathbb{N}, r)$, and $A'(\mathbb{N}, r)$ have only one element of degree n. Since π and π' preserve the degree and $\pi(x) = x$ and $\pi'(x) = x$, for all $x \in \mathbb{N}$, we have that π and π' are bijective. Thus the result follows. □

4. Non-degenerate irretractable solutions

In [**26**, Theorem 2] (and independently in [**20**, Corollary 2.3]) it is proven that any finite involutive left non-degenerate set-theoretic solution of the YBE also is right non-degenerate. In the infinite case, the latter is no longer true. The following example from [**26**] shows this.

Example 4.1. Let X be the set of the integers, and define $r\colon X^2 \to X^2$ by

$$r(x,y) = (\lambda_x(y), \lambda^{-1}_{\lambda_x(y)}(x)),$$

where $\lambda_x(y) = y + \min(x,0)$ for all $x, y \in X$. Note that λ_x is bijective and $\lambda_x^{-1}(y) = y - \min(x,0)$. It is easy to check that (X,r) is an involutive solution. Note that it is not right non-degenerate. In fact, if $a < 0$, we have that

$$\rho_a(b) = \lambda^{-1}_{\lambda_b(a)}(b) = b - \min(a + \min(b,0), 0) = b - (a+b) = -a$$

for all $b < 0$. Hence ρ_a is not bijective if $a < 0$.

It is unclear whether the above holds for arbitrary bijective solutions. Hence the following question is pertinent.

Question 4.2. *Is any finite bijective left non-degenerate set-theoretic solution of the YBE right non-degenerate?*

A natural question is the converse:

Question 4.3. *Are non-degenerate solutions of the YBE always bijective?*

We will give a positive answer to this question in case the solution (X,r) is irretractable, i.e. $\sigma_x = \sigma_y$ implies $x = y$ for all $x, y \in X$. Note that Example 4.1 is a retractable involutive solution. To our knowledge it is unknown whether there exist infinite involutive irretractable solutions that are left but not right non-degenerate. Note that irretractability has been defined with respect to the maps σ_x. One could equally well define retractability with respect to the maps γ_x. However, this makes no difference since any solution $r(x,y) = (\sigma_x(y), \gamma_y(x))$ has a dual solution $r'(y,x) = (\gamma_y(x), \sigma_x(y))$. Clearly r is (bijective) non-degenerate if and only if r' is (bijective) non-degenerate.

To prove the result we will make use of the following result of Rump [**29**, Proposition 1]: Let X be a non-empty set and let $r\colon X \times X \to X \times X$ be a map, with $r(x,y) = (\sigma_x(y), \gamma_y(x))$, such that $\gamma_y\colon X \to X$ is bijective for all $y \in X$. Then (X,r) is a solution of the YBE if and only if the following conditions hold for all $x, y, z \in X$:

(R1) $(x \cdot y) \cdot (x \cdot z) = (y : x) \cdot (y \cdot z)$,
(R2) $(x : y) : (x : z) = (y \cdot x) : (y : z)$,
(R3) $(x \cdot y) : (x \cdot z) = (y : x) \cdot (y : z)$,
where $x \cdot y = \gamma_x^{-1}(y)$, $x : y = \sigma_{\gamma_y^{-1}(x)}(y)$. Furthermore, r is a bijective solution if the map $X \to X$ defined by $z \mapsto x : z$ is bijective. The use of this result has been proposed by the referee to avoid the arboresque sub- and superscripts in the original proof.

We also will make use of a lemma that was proved by Lebed and Vendramin in [**24**] for finite non-degenerate bijective solutions.

Lemma 4.4. *Let (X, r) be a non-degenerate set-theoretic solution of the YBE. Let $h\colon X \to X$ be the map defined by $h(x) = \sigma_x^{-1}(x)$ for all $x \in X$. If (X, r) is irretractable, then h is bijective and $h^{-1}(x) = \gamma_x^{-1}(x)$ for all $x \in X$.*

Proof: As commented above, we may assume that (X, r) is a non-degenerate set-theoretic solution of the YBE such that $\gamma_x = \gamma_y$ implies that $x = y$. Thus conditions (R1), (R2), (R3) hold. Then, by (R1),

$$\gamma_{x\cdot x}^{-1}(x \cdot z) = \gamma_{x:x}^{-1}(x \cdot z)$$

for all $x, z \in X$. Hence,

$$x \cdot x = x : x.$$

Now $x : x = \sigma_{x\cdot x}(x)$ and thus $\sigma_{x\cdot x}^{-1}(x \cdot x) = x$. This shows that the map $x \mapsto x \cdot x = \gamma_x^{-1}(x)$ is injective. For $x, y \in X$, put

$$\sigma_x^{-1}(y) = x * y.$$

For $y = x * x$, we have $x = \sigma_x(y) = \sigma_{\gamma_y^{-1}(\gamma_y(x))}(y) = \gamma_y(x) : y$. Hence, by (R1),

$$x \cdot (\gamma_y(x) \cdot z) = (\gamma_y(x) : y) \cdot (\gamma_y(x) \cdot z) = (y \cdot \gamma_y(x)) \cdot (y \cdot z) = x \cdot (y \cdot z).$$

Therefore $\gamma_y(x) \cdot z = y \cdot z$, which yields $\gamma_y(x) = y$. Hence

$$x = y \cdot y = (x * x) \cdot (x * x), \tag{21}$$

which shows that the map $x \mapsto x \cdot x = \gamma_x^{-1}(x)$ is bijective. Furthermore the inverse of this map is the map $x \mapsto x * x = \sigma_x^{-1}(x) = h(x)$. □

Theorem 4.5. *Let (X, r) be an irretractable non-degenerate set-theoretic solution of the YBE. Then r is bijective.*

Proof: Again we may assume that (X, r) is a non-degenerate set-theoretic solution of the YBE such that $\gamma_x = \gamma_y$ implies that $x = y$. From (R3) we get that

$$x : (z \cdot z) = (z \cdot \gamma_z(x)) : (z \cdot z) = (\gamma_z(x) : z) \cdot (\gamma_z(x) : z) = \sigma_x(z) \cdot \sigma_x(z).$$

From Lemma 4.4 we then get that (see equation (21))

$$x : z = \sigma_x(z * z) \cdot \sigma_x(z * z).$$

Since $x * x = \sigma_x^{-1}(x) = h(x)$, we get from Lemma 4.4 that the map $z \mapsto x : z = \sigma_{\gamma_z^{-1}(x)}(z)$ is bijective. Hence, by Rump's earlier mentioned result, r is bijective. □

Note that if (X, r) is an irretractable non-degenerate solution, then for every $x \in X$ there is a unique $y \in X$ such that $r(x, y) = (x, y)$ and there is a unique $z \in X$ such that $r(z, x) = (z, x)$. Because (X, r) is left non-degenerate, to prove the former, it is sufficient to show that $\sigma_x(y) = x$ implies $\gamma_y(x) = y$. Now, because (X, r) is a solution we obtain from (1) that $\sigma_x\sigma_y = \sigma_{\sigma_x(y)}\sigma_{\gamma_y(x)} = \sigma_x\sigma_{\gamma_y(x)}$ and thus $\sigma_y = \sigma_{\gamma_y(x)}$. The irretractable assumption yields that $y = \gamma_y(x)$, as claimed. Similarly one proves the other claim. Hence there are at least $\binom{|X|}{2}$ defining relations for the structure monoid. Furthermore, there are precisely $\binom{|X|}{2}$ defining relations if r also is involutive and thus, in this case, $M(X, r)$ is a monoid with a presentation of the type $\langle x_1, \ldots, x_n \mid R\rangle$, where R is a set consisting of $\binom{n}{2}$ relations of the type $x_ix_j = x_kx_l$ with $(x_i, x_j) \neq (x_k, x_l)$ and every word x_ix_j appears in at most one relation. Note that such a presentation has associated a map $r\colon X \times X \to X \times X$, where $X = \{x_1, \ldots x_n\}$, $r^2 = \mathrm{id}_{X^2}$, and $r(x_i, x_j) = (x_k, x_l)$ if and only if either $x_ix_j = x_kx_l$ is one of the relations in R or x_ix_j does not appear in any relation in R and $(x_k, x_l) = (x_i, x_j)$ in this case. Monoids with this type of presentation and their algebras have a rich algebraic structure when r is non-degenerate, even if (X, r) is not a solution of the YBE. Such monoids are said to be of quadratic type, and if x_ix_i does not appear in any defining relation, then they are said to be of skew type. We refer the reader to [**6, 15, 23, 22**]. In [**23**] it has been shown that for such a monoid r is a non-degenerate solution of the YBE if and only if the monoid is cancellative and r is non-degenerate and satisfies the cyclic condition, i.e. if for every $x_1, y \in X$ there exist $x_2, y_1, y_2, z_1, z_2 \in X$ such that $x_1y = y_1z_1$ and $x_2y_1 = y_2z_2$ with $r(x_2, x_1) = (x_2, x_1)$ and $r(z_2, z_1) = (z_2, z_1)$. The latter monoids were first investigated by Gateva-Ivanova and Van den Bergh in [**17**].

5. The structure left semi-truss

Braces and skew braces were introduced to deal with bijective non-degenerate solutions (X, r) of the YBE. In order to translate such solutions to associative structures the structure group $G(X, r)$ and the structure monoid $M(X, r)$ were introduced. The group $G(X, r)$ turns out to be a skew brace, however a structure monoid does not fit in this context. Recently, Brzeziński introduced the algebraic notion of a semi-truss which is built on two semigroup structures on a given set. We show that structure monoids of left non-degenerate solutions of the YBE fit in this context: they turn out to be left semi-trusses with additive structure that is close to being a normal monoid. We then show that also the least left cancellative epimorphic image of $M(X,r)$ inherits a left non-degenerate solution of the YBE that restricts to the original solution r for some interesting classes, in particular if (X,r) is irretractable.

We first recall the definition of a left semi-truss.

Definition 5.1 ([**4**])**.** A left semi-truss is a quadruple $(A,+,\circ,\phi)$ such that $(A,+)$ and $(A,\circ)$ are non-empty semigroups and $\phi\colon A\times A\to A$ is a function such that

$$a\circ(b+c)=(a\circ b)+\phi(a,c)$$

for all $a,b,c\in A$.

Example 5.2. Let (X,r) be a left non-degenerate set-theoretic solution of the YBE (not necessarily bijective). As stated in Section 3, and with the same notation, the map $r'(x,y)=(y,\sigma_y\gamma_{\sigma_x^{-1}(y)}(x))$ defines the left derived solution on X. Let $M=M(X,r)$ and $M'=A(X,r)=M(X,r')$ be the structure monoids of the solutions (X,r) and (X,r') respectively. From Corollary 3.9 and Proposition 3.1 we obtain a left action $\lambda'\colon(M,\circ)\to\operatorname{Aut}(M',+)$ and a bijective 1-cocycle $\pi\colon M\to M'$ with respect to λ' satisfying $\lambda'(x)(y)=\sigma_x(y)$ and $\pi(x)=x$ for all $x,y\in X$. We identify M and M' via π, that is, $a=\pi(a)$ for all $a\in M$. With this identification, we obtain the operation $+$ on M, and $a\circ b=a+\lambda'_a(b)$ for all $a,b\in M$. Put $\phi(a,b)=\lambda'_a(b)$ for all $a,b\in M$. Then

$$a\circ(b+c)=a+\lambda'_a(b+c)=a+\lambda'_a(b)+\lambda'_a(c)=(a\circ b)+\phi(a,c).$$

Furthermore $M+a\subseteq a+M$ for all $a\in M$. Hence $(M,+,\circ,\phi)$ is a left semi-truss. Note that, if r is furthermore bijective, then it can easily be verified that (X,r') is a right non-degenerate solution and thus $M+a=a+M$ for all $a\in M$; that is, $(M,+)$ consists of normal elements. As shown in [**19**], this property is fundamental in the study of the associated structure algebra $KM(X,r)$.

In the remainder of this section we show that if $(M,+,\circ,\phi)$ is a left semi-truss such that for every $a,b\in M$ there exists a unique $c(a,b)\in M$ such that $a+b=b+c(a,b)$, then there exists a set-theoretical solution of the YBE on M, say (M,r'). In the case that $M=M(X,r)/\eta$, the least cancellative epimorphic image of $M(X,r)$, it follows that r' is the (unique) extension of r to M.

Lemma 5.3. *Let* $(A,+)$ *be a non-empty semigroup such that, for each* $(a,b)\in A\times A$ *there exists a unique* $c(a,b)\in A$ *such that*

$$a+b=b+c(a,b).$$

Then (A,r')*, where*

$$r'(a,b)=(b,c(a,b)),$$

for all $a,b\in A$*, is a set-theoretic solution of the YBE.*

Proof: Let $(a,b,d)\in A^3$. We have

$$\begin{aligned} a+b+d &= b+c(a,b)+d \\ &= b+d+c(c(a,b),d) \end{aligned}$$

and also

$$\begin{aligned} a+b+d &= a+d+c(b,d) \\ &= d+c(a,d)+c(b,d) \\ &= d+c(b,d)+c(c(a,d),c(b,d)) \\ &= b+d+c(c(a,d),c(b,d)). \end{aligned}$$

Hence, by the uniqueness assumption,

$$c(a,b+d)=c(c(a,b),d)=c(c(a,d),c(b,d)). \tag{22}$$

Now we have

$$\begin{aligned} r'_1r'_2r'_1(a,b,d) &= r'_1r'_2(b,c(a,b),d)=r'_1(b,d,c(c(a,b),d)) \\ &= (d,c(b,d),c(c(a,b),d)) \end{aligned}$$

and

$$\begin{aligned} r'_2r'_1r'_2(a,b,d) &= r'_2r'_1(a,d,c(b,d))=r'_2(d,c(a,d),c(b,d)) \\ &= (d,c(b,d),c(c(a,d),c(b,d))). \end{aligned}$$

Therefore, by (22), $r'_1r'_2r'_1=r'_2r'_1r'_2$, and the result follows. □

Proposition 5.4. *Let* $(A,+)$ *and* $(A,\circ)$ *be non-empty semigroups. Let* $\lambda\colon(A,\circ)\to\mathrm{Aut}(A,+)$ *be a homomorphism such that* $a\circ b=a+\lambda_a(b)$ *for all* $a,b\in A$*, where* $\lambda(a)=\lambda_a$*. In particular,* $(A,+,\circ,\phi)$ *is a left semi-truss with* $\phi(a,b)=\lambda_a(b)$ *for all* $a,b\in A$*. Suppose that for each* $(a,b)\in A\times A$ *there exists a unique* $c(a,b)\in A$ *such that*

$$a+b=b+c(a,b).$$

Then (A,r)*, where*

$$r(a,b)=(\lambda_a(b),\lambda^{-1}_{\lambda_a(b)}(c(a,\lambda_a(b)))),$$

for all $a,b\in A$*, is a left non-degenerate set-theoretic solution of the YBE.*

Proof: Let $J\colon A^3 \to A^3$ be the map defined by $J(a,b,d)=(a,\lambda_a(b),\lambda_a\lambda_b(d))$. Clearly J is bijective and $J^{-1}(a,b,d) = (a, \lambda_a^{-1}(b), \lambda^{-1}_{\lambda_a^{-1}(b)}\lambda_a^{-1}(d))$ for all $a,b,d \in A$. We have

$$\begin{aligned} J^{-1}r'_1J(a,b,d) &= J^{-1}r'_1(a,\lambda_a(b),\lambda_a\lambda_b(d)) \\ &= J^{-1}(\lambda_a(b), c(a,\lambda_a(b)), \lambda_a\lambda_b(d)) \\ &= (\lambda_a(b), \lambda^{-1}_{\lambda_a(b)}(c(a,\lambda_a(b))), \lambda^{-1}_{\lambda^{-1}_{\lambda_a(b)}(c(a,\lambda_a(b)))}\lambda^{-1}_{\lambda_a(b)}\lambda_a\lambda_b(d)), \end{aligned}$$

where r' is defined as in Lemma 5.3. Since $a \circ b = a + \lambda_a(b) = \lambda_a(b) + c(a,\lambda_a(b)) = \lambda_a(b) \circ \lambda^{-1}_{\lambda_a(b)}(c(a,\lambda_a(b)))$, it follows that $J^{-1}r'_1J = r_1$. Similarly

$$\begin{aligned} J^{-1}r'_2J(a,b,d) &= J^{-1}r'_2(a,\lambda_a(b),\lambda_a\lambda_b(d)) \\ &= J^{-1}(a,\lambda_a\lambda_b(d), c(\lambda_a(b),\lambda_a\lambda_b(d))) \\ &= (a,\lambda_b(d), \lambda^{-1}_{\lambda_b(d)}\lambda_a^{-1}(c(\lambda_a(b),\lambda_a\lambda_b(d)))). \end{aligned}$$

Note that

$$\lambda_a^{-1}(d) + \lambda_a^{-1}(c(b,d)) = \lambda_a^{-1}(d + c(b,d)) = \lambda_a^{-1}(b+d) = \lambda_a^{-1}(b) + \lambda_a^{-1}(d)$$

for all $a,b,d \in A$. Hence, by the uniqueness assumption, $\lambda_a^{-1}(c(b,d)) = c(\lambda_a^{-1}(b), \lambda_a^{-1}(d))$. Since each λ_a is bijective it follows that

$$\begin{aligned} J^{-1}r'_2J(a,b,d) &= (a,\lambda_b(d), \lambda^{-1}_{\lambda_b(d)}\lambda_a^{-1}(c(\lambda_a(b),\lambda_a\lambda_b(d)))) \\ &= (a,\lambda_b(d), \lambda^{-1}_{\lambda_b(d)}(c(b,\lambda_b(d)))). \end{aligned}$$

Thus $J^{-1}r'_2J = r_2$. By Lemma 5.3, (A,r') is a set-theoretic solution of the YBE. Therefore also (A,r) is a set-theoretic solution of the YBE, and the result follows. □

Let (X,r) be a left non-degenerate set-theoretic solution of the YBE. We will write $r(x,y) = (\sigma_x(y), \gamma_y(x))$ for all $x,y \in X$. Thus the σ_x are bijective maps. The derived solution of (X,r) is (X,r'), where

$$r'(x,y) = (y, \sigma_y(\gamma_{\sigma_x^{-1}(y)}(x)))$$

for all $x,y \in X$. We will use the notation of Example 5.2. Thus we have $M = M(X,r)$ and the left semi-truss $(M,+,\circ,\phi)$, where $\phi(a,b) = \lambda'_a(b)$ for all $a,b \in M$. Recall that $\lambda'\colon (M,\circ) \to \operatorname{Aut}(M,+)$ is an homomorphism, that is, an action of $(M,\circ)$ on $(M,+)$, and $\mathrm{id}\colon M \to M$ is a bijective 1-cocycle with respect to λ' (because $a \circ b = a + \lambda'_a(b)$).

Let η be the left cancellative congruence on $(M,+)$, that is, η is the smallest congruence such that $\overline{M}=(M,+)/\eta$ is a left cancellative monoid.

We shall see a description of the elements in η. Let

$$\eta_0=\{(a,b)\in M^2 \mid \exists c\in M \text{ such that } c+a=c+b\}.$$

Note that η_0 is a reflexive and symmetric binary relation on M. Let η_1 be its transitive closure, that is,

$$\eta_1=\{(a,b)\in M^2 \mid \exists a_1,\dots,a_n\in M \text{ such that } (a,a_1),(a_1,a_2),\dots,(a_n,b)\in\eta_0\}.$$

Thus η_1 is an equivalence relation on M. Let

$$\eta_2=\{(c+a,c+b)\in M^2 \mid c\in M \text{ such that } (a,b)\in\eta_1\} \cup \{(a,b)\in M^2 \mid \exists c\in M \text{ such that } (c+a,c+b)\in\eta_1\},$$

and for every $m\geq 1$ we define

$$\eta_{2m+1}=\{(a,b)\in M^2 \mid \exists a_1,\dots,a_n\in M \text{ such that } (a,a_1),(a_1,a_2),\dots,(a_n,b)\in\eta_{2m}\}$$

and

$$\eta_{2m+2}=\{(c+a,c+b)\in M^2 \mid c\in M \text{ such that } (a,b)\in\eta_{2m+1}\} \cup \{(a,b)\in M^2 \mid \exists c\in M \text{ such that } (c+a,c+b)\in\eta_{2m+1}\}.$$

Note that $\eta_n\subseteq\eta_{n+1}\subseteq\eta$ for all $n\geq 0$. Let $\eta'=\cup_{n=0}^{\infty}\eta_n$.

Lemma 5.5. *With the above notation we have $\eta'=\eta$ and $\lambda'_a=\lambda'_b$ for all $(a,b)\in\eta$. Furthermore, for all $z\in M$,*

$$\eta=\{(\lambda'_z(a),\lambda'_z(b)) \mid (a,b)\in\eta\}=\{((\lambda'_z)^{-1}(a),(\lambda'_z)^{-1}(b)) \mid (a,b)\in\eta\},$$

and η also is a congruence on $(M,\circ)$.

Proof: First we shall prove that η' is a congruence on $(M,+)$. Clearly η' is reflexive and symmetric because so is each η_n. Let $a,b,c\in M$ such that $(a,b),(b,c)\in\eta'$. There exists a positive integer m such that $(a,b),(b,c)\in\eta_{2m+1}$. Since η_{2m+1} is the transitive closure of η_{2m}, we have $(a,c)\in\eta_{2m+1}\subseteq\eta'$. Hence η' is an equivalence relation. Note that every η_n satisfies $(x+z,y+z)\in\eta_n$ for all $(x,y)\in\eta_n$. Hence $(a+c,b+c)\in\eta_{2m+1}\subseteq\eta'$. Since $(a,b)\in\eta_{2m+1}$, we have $(c+a,c+b)\in\eta_{2m+2}\subseteq\eta'$. Therefore η' is a congruence.

Let $a, b, c \in M$ be elements such that $(c+a, c+b) \in \eta'$. There exists a positive integer t such that $(c+a, c+b) \in \eta_{2t+1}$. Thus $(a, b) \in \eta_{2t+2} \subseteq \eta'$. Hence $(M, +)/\eta'$ is a left cancellative monoid. Since $\eta' \subseteq \eta$, we have $\eta' = \eta$ by the definition of η.

Let $(a, b) \in \eta_0$. Then there exists $c \in M$ such that $c + a = c + b$. Let $z \in M$. We have

$$(\lambda'_z)^\varepsilon(c) + (\lambda'_z)^\varepsilon(a) = (\lambda'_z)^\varepsilon(c+a) = (\lambda'_z)^\varepsilon(c+b) = (\lambda'_z)^\varepsilon(c) + (\lambda'_z)^\varepsilon(b),$$

for $\varepsilon = \pm 1$. Therefore $\eta_0 = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_0\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta_0\}$. Thus, clearly

$$\eta_1 = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_1\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta_1\}.$$

Let $(a, b) \in \eta_2$. Then, either there exist $c, a', b' \in M$ such that $(a', b') \in \eta_1$ and $(a, b) = (c+a', c+b')$, or there exists $d \in M$ such that $(d+a, d+b) \in \eta_1$. In the first case, we have

$$((\lambda'_z)^\varepsilon(a), (\lambda'_z)^\varepsilon(b)) = ((\lambda'_z)^\varepsilon(c) + (\lambda'_z)^\varepsilon(a'), (\lambda'_z)^\varepsilon(c) + (\lambda'_z)^\varepsilon(b'))$$

for $\varepsilon = \pm 1$. Since $((\lambda'_z)^\varepsilon(a'), (\lambda'_z)^\varepsilon(b')) \in \eta_1$, we get that $((\lambda'_z)^\varepsilon(a), (\lambda'_z)^\varepsilon(b)) \in \eta_2$, in this case. In the second case, since $(d+a, d+b) \in \eta_1$, we have $((\lambda'_z)^\varepsilon(d) + (\lambda'_z)^\varepsilon(a), (\lambda'_z)^\varepsilon(d) + (\lambda'_z)^\varepsilon(b)) \in \eta_1$. Thus also in this case we have $((\lambda'_z)^\varepsilon(a), (\lambda'_z)^\varepsilon(b)) \in \eta_2$. Therefore

$$\eta_2 = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_2\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta_2\}.$$

Now it is easy to show by induction on n that

$$\eta_n = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_n\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta_n\},$$

for all non-negative integer n. Hence

$$\eta = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta\} = \{((\lambda'_z)^{-1}(a), (\lambda'_z)^{-1}(b)) \mid (a, b) \in \eta\}.$$

Let $(a, b) \in \eta_0$. Then there exists $c \in M$ such that $c + a = c + b$. Hence $c \circ (\lambda'_c)^{-1}(a) = c + a = c + b = c \circ (\lambda'_c)^{-1}(b)$. Hence,

$$\lambda'_c \lambda'_{(\lambda'_c)^{-1}(a)} = \lambda'_{c \circ (\lambda'_c)^{-1}(a)} = \lambda'_{c \circ (\lambda'_c)^{-1}(b)} = \lambda'_c \lambda'_{(\lambda'_c)^{-1}(b)}$$

and thus

$$\lambda'_{(\lambda'_c)^{-1}(a)} = \lambda'_{(\lambda'_c)^{-1}(b)}.$$

Since $\eta_0 = \{(\lambda'_c(a), \lambda'_c(b)) \mid (a, b) \in \eta_0\}$, we have $\lambda'_a = \lambda'_b$ for all $(a, b) \in \eta_0$. Because

$$\eta_n = \{(\lambda'_z(a), \lambda'_z(b)) \mid (a, b) \in \eta_n\},$$

for all non-negative integers n, it is easy to prove, by induction on n, that $\lambda'_a = \lambda'_b$ for all $(a, b) \in \eta_n$. Hence $\lambda'_a = \lambda'_b$ for all $(a, b) \in \eta$.

Let $(a,b) \in \eta$. Then $(\lambda'_c(a), \lambda'_c(b)) \in \eta$. Thus $(c \circ a, c \circ b) = (c + \lambda'_c(a), c + \lambda'_c(b)) \in \eta$. Since $\lambda'_a = \lambda'_b$, we have

$$(a \circ c, b \circ c) = (a + \lambda'_a(c), b + \lambda'_b(c)) = (a + \lambda'_a(c), b + \lambda'_a(c)) \in \eta.$$

Hence η is a congruence on $(M, \circ)$, and the result follows. □

With the assumptions and notations as in Example 5.2, $\overline{M} = M/\eta$. Let $M \to \overline{M} \colon a \mapsto \overline{a}$ be the natural projection. Let $\overline{\lambda} \colon (\overline{M}, \circ) \to \operatorname{Aut}(\overline{M}, +)$ be the map defined by $\overline{\lambda}(\overline{a}) = \overline{\lambda}_{\overline{a}}$ and $\overline{\lambda}_{\overline{a}}(\overline{b}) = \overline{\lambda'_a(b)}$ for all $a, b \in M$.

Note that $\overline{\lambda}$ is well-defined, because if $\overline{c} = \overline{a}$ and $\overline{d} = \overline{b}$, then, by Lemma 5.5, $\overline{\lambda'_a(b)} = \overline{\lambda'_a(d)}$ and $\lambda'_a = \lambda'_c$, and

$$\overline{\lambda'_a(b)} = \overline{\lambda'_a(d)} = \overline{\lambda'_c(d)}.$$

Now it is easy to check that $\overline{\lambda}_{\overline{a}} \in \operatorname{Aut}(\overline{M}, +)$ and that $\overline{\lambda}$ is a homomorphism such that $\overline{a} \circ \overline{b} = \overline{a} + \overline{\lambda}_{\overline{a}}(\overline{b})$ for all $a, b \in M$.

Remark 5.6. If, furthermore, the left non-degenerate set-theoretic solution (X, r) is finite and bijective then one can say more. To do so, it is convenient to keep the notation $M = M(X, r)$ and $A = A(X, r)$. So $M \subseteq A \rtimes \operatorname{Im} \lambda'$. Jespers, Kubat, and Van Antwerpen ([**19**, Proposition 2.9]) proved that there exists $t \geq 1$ and a central element $(z, 1) \in M$, with $z \in Z(A)$ and $g(z) = z$ for all $g \in \operatorname{Im}(\lambda')$, such that the least cancellative congruence on $(A, +)$ is

$$\begin{aligned} \eta &= \{(a,b) \in A \times A \mid a + \underbrace{z + \cdots + z}_{\text{i times}} = b + \underbrace{z + \cdots + z}_{\text{i times}}, \ \text{ for all } i \geq t\} \\ &= \{(a,b) \in A \times A \mid c + a = c + b \text{ for some } c \in A\} \\ &= \eta_0. \end{aligned}$$

Note that $(a, b) \in \eta$ implies that $\lambda'_a = \lambda'_b$. Hence, it follows from Proposition 4.2 in [**19**] that the (least) cancellative congruence on $(M, \circ)$ is

$$\eta_M = \{((a, \lambda'_a), (b, \lambda'_b)) \mid (a, b) \in \eta\}.$$

It follows that the natural map

$$M/\eta_M \longrightarrow (A/\eta) \rtimes \operatorname{Im}(\lambda'),$$

i.e. $\overline{(a, \lambda'_a)} \mapsto (\overline{a}, \lambda'_a)$, is an injective monoid homomorphism and M/η_M is a regular submonoid of $(A/\eta) \rtimes \operatorname{Im}(\lambda')$. So we obtain a bijective 1-cocycle $(M/\eta_M, \circ) \to (A/\eta, +)$, with respect to $\overline{\lambda}$, that extends the mapping $\overline{(a, \lambda'_a)} \mapsto \overline{a}$. Because r is bijective we know (see explanation in Example 5.2) that $(A, +)$ consists of normal elements and thus $(A/\eta, +)$ is a left and right Ore monoid and also $(M/\eta_M, \circ)$ is a left and right Ore monoid. Hence they both have a group of fractions, denoted $\mathrm{gr}(A/\eta)$ and $\mathrm{gr}(M/\eta_M)$ respectively. It is easily verified that $\mathrm{gr}(M/\eta_M) = G(X,r)$, the structure group of (X,r), $\mathrm{gr}(A/\eta) = G(X,r')$, the structure group of the derived solution (X,r'), and $\mathrm{gr}(M/\eta_M) \subseteq \mathrm{gr}(A/\eta) \rtimes \mathrm{Im}(\lambda')$ where, by abuse of notation, $\lambda'\colon \mathrm{gr}(M/\eta_M) \to \mathrm{Aut}(A/\eta)$ is the natural extension of the mapping $\overline{\lambda}$ and also $\mathrm{gr}(M/\eta_M)$ is a regular subgroup of $\mathrm{gr}(A/\eta)\rtimes\mathrm{Im}(\lambda')$. The latter was proven by Lebed and Vendramin in [**24**, Theorem 3.4.] in case (X,r) is bijective, (left and right) non-degenerate, and finite.

Question 5.7. *If (X,r) is a left non-degenerate solution of the YBE, does there exist a bijective 1-cocycle $(M/\eta_M,\circ) \to (A/\eta,+)$, with respect to $\overline{\lambda}$, that extends the mapping $\overline{(a,\lambda'_a)} \mapsto \overline{a}$? In other words, can one avoid the bijective assumption in Remark 5.6?*

Let $\overline{\phi}\colon \overline{M}\times\overline{M} \to \overline{M}$ be the map defined by $\overline{\phi}(\overline{a},\overline{b}) = \overline{\lambda}_{\overline{a}}(\overline{b})$ for all $a,b \in M$. Then $(\overline{M},+,\circ,\overline{\phi})$ is a left semi-truss.

Lemma 5.8. *Let $a,b \in M = M(X,r)$. Then there exists $c \in M$ such that $a+b = b+c$.*

Proof: There exist non-negative integers n, m, and $x_1,\dots,x_n,y_1,\dots,y_m \in X$ such that $a = x_1+\dots+x_n$ and $b = y_1+\dots+y_m$. Clearly we may assume that n, m are positive integers. We shall prove the result by induction on $n+m$. If $n=m=1$, then $x_1+y_1 = y_1+\sigma_{y_1}(\gamma_{\sigma_{x_1}^{-1}(y_1)}(x_1))$, by the defining relations of $(M,+)$. Suppose that $m+n>2$, and that the result is true for $m+n-1$. If $n>1$, then by the induction hypothesis there exists $c' \in M$ such that $a+b = x_1+b+c'$, and by the induction hypothesis again there exists $c'' \in M$ such that $x_1+b = b+c''$. Hence $a+b = b+c''+c'$, in this case. Suppose that $n=1$. In this case $m>1$ and

$$a+b = x_1+b = y_1+\sigma_{y_1}(\gamma_{\sigma_{x_1}^{-1}(y_1)}(x_1))+y_2+\dots+y_m.$$

Hence, by the induction hypothesis, there exists $c \in M$ such that

$$\sigma_{y_1}(\gamma_{\sigma_{x_1}^{-1}(y_1)}(x_1))+y_2+\dots+y_m = y_2+\dots+y_m+c.$$

Thus $a+b = b+c$ in this case. Therefore the result follows by induction. □

By Lemma 5.8, the left cancellative monoid $(\overline{M},+)$ satisfies that, for all $\overline{a},\overline{b} \in \overline{M}$, there exists a unique $\overline{c} \in \overline{M}$ such that $\overline{a}+\overline{b} = \overline{b}+\overline{c}$. So, the multiplicative monoid $(\overline{M},\circ)$ is left cancellative. Hence, we have the following corollary.

Corollary 5.9. *Let (X, r) be a left non-degenerate set-theoretic solution of the YBE. Let η be the left cancellative congruence on $(M(X, r'), +)$. Then $(\overline{M}, +, \circ, \overline{\phi})$ is a left semi-truss with $\overline{M} + \overline{a} \subseteq \overline{a} + \overline{M}$ for all $\overline{a} \in \overline{M}$ and it satisfies the conditions of Proposition 5.4, with $\overline{\phi}(\overline{a}, \overline{b}) = \overline{\lambda}_{\overline{a}}(\overline{b})$, for all $\overline{a}, \overline{b} \in \overline{M}$. In particular, $(\overline{M}, \overline{r})$, where*

$$\overline{r}(\overline{a}, \overline{b}) = (\overline{\lambda}_{\overline{a}}(\overline{b}), \overline{\lambda}^{-1}_{\overline{\lambda}_{\overline{a}}(\overline{b})}(c(\overline{a}, \overline{\lambda}_{\overline{a}}(\overline{b})))),$$

for all $\overline{a}, \overline{b} \in \overline{M}$, is a left non-degenerate set-theoretic solution of the YBE. In particular, $(\overline{X}, \overline{r}_{|\overline{X}})$ is a left non-degenerate solution on the image $\overline{X}$ of X in $\overline{M}$.

We say that a left non-degenerate solution (X, r) of the YBE is injective if the natural map $X \to M/\eta$ is injective. Obvious such examples are irretractable solutions, and in this case $r = \overline{r}|_{\overline{X}^2}$. Note that if r is also bijective and non-degenerate, then this notion corresponds with the one introduced by Lebed and Vendramin in [**24**]. In [**24**] it is also shown that, in this case, several properties of involutive solutions can be generalized to injective ones.

Corollary 5.10. *Any left non-degenerate injective set-theoretic solution (X, r) of the YBE is the restriction of the induced left-non-degenerate solution of the YBE determined by a left cancellative semi-truss $(M, +, \circ, \phi)$ with $M + a \subseteq a + M$ for all $a \in M$.*

However, note that $(\overline{M}, \circ)$ is not necessarily the structure monoid of the solution of $(\overline{X}, \overline{r})$. Indeed, let $X = \mathrm{Sym}_3$ be the symmetric group of degree 3. Let (X, r) be the bijective non-degenerate solution defined by $r(a, b) = (aba^{-1}, a)$ for all $a, b \in X$. Note that the solution (X, r) is non-involutive and irretractable (because the center of Sym_3 is trivial). So, X is naturally embedded in $(\overline{M}, \circ) = (M(X, r)/\eta, \circ)$ and $\overline{r}_{|\overline{X}^2} = r$. Let us denote the multiplication in the structure monoid $M(X, r)$ by $\cdot$. In $(M(X, r), \cdot)$ we have

$$\begin{aligned}(1,2)\cdot(1,2,3)\cdot(1,2,3)\cdot(1,2,3) &= (1,3,2)\cdot(1,3,2)\cdot(1,3,2)\cdot(1,2)\\ &= (1,3,2)\cdot(1,3,2)\cdot(1,3)\cdot(1,3,2)\\ &= (1,3,2)\cdot(2,3)\cdot(1,3,2)\cdot(1,3,2)\\ &= (1,2)\cdot(1,3,2)\cdot(1,3,2)\cdot(1,3,2)\end{aligned}$$

while $(1,2,3)\cdot(1,2,3)\cdot(1,2,3) \neq (1,3,2)\cdot(1,3,2)\cdot(1,3,2)$. Hence, $M(X, r)$ is not left cancellative, while $\overline{M}$ is left cancellative. Thus $\overline{M}$ is not the structure monoid of (X, r).

The following problem remains a challenge.

Question 5.11. *Determine when a left non-degenerate solution (X, r) of the YBE is cancellative injective. If (X, r) is a left non-degenerate solution that is injective, then does there exists a finite left cancellative semi-truss in which X can be embedded naturally? In case r also is finite, bijective, and non-degenerate this has been proven by Lebed and Vendramin in* [**24**].

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References

[1] D. Bachiller, Solutions of the Yang–Baxter equation associated to skew left braces, with applications to racks, *J. Knot Theory Ramifications* **27(8)** (2018), 1850055, 36 pp. DOI: 10.1142/S0218216518500554.

[2] D. Bachiller, F. Cedó, and E. Jespers, Solutions of the Yang–Baxter equation associated with a left brace, *J. Algebra* **463** (2016), 80–102. DOI: 10.1016/j.jalgebra.2016.05.024.

[3] D. Bachiller, F. Cedó, and L. Vendramin, A characterization of finite multipermutation solutions of the Yang–Baxter equation, *Publ. Mat.* **62(2)** (2018), 641–649. DOI: 10.5565/PUBLMAT6221809.

[4] T. Brzeziński, Towards semi-trusses, *Rev. Roumaine Math. Pures Appl.* **63(2)** (2018), 75–89.

[5] F. Cedó, E. Jespers, and J. Okniński, Braces and the Yang–Baxter equation, *Comm. Math. Phys.* **327(1)** (2014), 101–116. DOI: 10.1007/s00220-014-1935-y.

[6] F. Cedó and J. Okniński, Gröbner bases for quadratic algebras of skew type, *Proc. Edinb. Math. Soc. (2)* **55(2)** (2012), 387–401. DOI: 10.1017/S0013091511000447.

[7] F. Chouraqui, Garside groups and Yang–Baxter equation, *Comm. Algebra* **38(12)** (2010), 4441–4460. DOI: 10.1080/00927870903386502.

[8] F. Chouraqui, Left orders in Garside groups, *Internat. J. Algebra Comput.* **26(7)** (2016), 1349–1359. DOI: 10.1142/S0218196716500570.

[9] K. Cvetko-Vah and C. Verwimp, Skew lattices and set-theoretic solutions of the Yang–Baxter equation, *J. Algebra* **542** (2020), 65–92. DOI: 10.1016/j.jalgebra.2019.10.007.

[10] P. Dehornoy, Set-theoretic solutions of the Yang–Baxter equation, RC-calculus, and Garside germs, *Adv. Math.* **282** (2015), 93–127. DOI: 10.1016/j.aim.2015.05.008.

[11] V. G. Drinfeld, On some unsolved problems in quantum group theory, in: *"Quantum Groups"* (Leningrad, 1990), Lecture Notes in Math. **1510**, Springer, Berlin, 1992, pp. 1–8. DOI: 10.1007/BFb0101175.

[12] P. Etingof, T. Schedler, and A. Soloviev, Set-theoretical solutions to the quantum Yang–Baxter equation, *Duke Math. J.* **100(2)** (1999), 169–209. DOI: 10.1215/S0012-7094-99-10007-X.

[13] T. Gateva-Ivanova, Quadratic algebras, Yang–Baxter equation, and Artin–Schelter regularity, *Adv. Math.* **230(4–6)** (2012), 2152–2175. DOI: 10.1016/j.aim.2012.04.016.

[14] T. Gateva-Ivanova and P. Cameron, Multipermutation solutions of the Yang–Baxter equation, *Comm. Math. Phys.* **309(3)** (2012), 583–621. DOI: 10.1007/s00220-011-1394-7.

[15] T. Gateva-Ivanova, E. Jespers, and J. Okniński, Quadratic algebras of skew type and the underlying monoids, *J. Algebra* **270(2)** (2003), 635–659. DOI: 10.1016/j.jalgebra.2003.06.005.

[16] T. Gateva-Ivanova and S. Majid, Matched pairs approach to set theoretic solutions of the Yang–Baxter equation, *J. Algebra* **319(4)** (2008), 1462–1529. DOI: 10.1016/j.jalgebra.2007.10.035.

[17] T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I-type, *J. Algebra* **206(1)** (1998), 97–112. DOI: 10.1006/jabr.1997.7399.

[18] L. Guarnieri and L. Vendramin, Skew braces and the Yang–Baxter equation, *Math. Comp.* **86(307)** (2017), 2519–2534. DOI: 10.1090/mcom/3161.

[19] E. Jespers, Ł. Kubat, and A. Van Antwerpen, The structure monoid and algebra of a non-degenerate set-theoretic solution of the Yang–Baxter equation, *Trans. Amer. Math. Soc.* **372(10)** (2019), 7191–7223. DOI: 10.1090/tran/7837. Corrigendum and addendum to "The structure monoid and algebra of a non-degenerate set-theoretic solution of the Yang–Baxter equation", *Trans. Amer. Math. Soc.* **373(6)** (2020), 4517–4521. DOI: 10.1090/tran/8057.

[20] E. Jespers and J. Okniński, Monoids and groups of I-type, *Algebr. Represent. Theory* **8(5)** (2005), 709–729. DOI: 10.1007/s10468-005-0342-7.

[21] E. Jespers and J. Okniński, *"Noetherian Semigroup Algebras"*, Algebra and Applications **7**, Springer, Dordrecht, 2007. DOI: 10.1007/1-4020-5810-1.

[22] E. Jespers, J. Okniński, and M. Van Campenhout, Finitely generated algebras defined by homogeneous quadratic monomial relations and their underlying monoids, *J. Algebra* **440** (2015), 72–99. DOI: 10.1016/j.jalgebra.2015.05.017.

[23] E. Jespers and M. Van Campenhout, Finitely generated algebras defined by homogeneous quadratic monomial relations and their underlying monoids II, *J. Algebra* **492** (2017), 524–546. DOI: 10.1016/j.jalgebra.2017.09.011.

[24] V. Lebed and L. Vendramin, On structure groups of set-theoretic solutions to the Yang–Baxter equation, *Proc. Edinb. Math. Soc. (2)* **62(3)** (2019), 683–717. DOI: 10.1017/s0013091518000548.

[25] J.-H. Lu, M. Yan, and Y.-C. Zhu, On the set-theoretical Yang–Baxter equation, *Duke Math. J.* **104(1)** (2000), 1–18. DOI: 10.1215/S0012-7094-00-10411-5.

[26] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation, *Adv. Math.* **193(1)** (2005), 40–55. DOI: 10.1016/j.aim.2004.03.019.

[27] W. Rump, Braces, radical rings, and the quantum Yang–Baxter equation, *J. Algebra* **307(1)** (2007), 153–170. DOI: 10.1016/j.jalgebra.2006.03.040.

[28] W. Rump, The brace of a classical group, *Note Mat.* **34(1)** (2014), 115–144. DOI: 10.1285/i15900932v34n1p115.

[29] W. Rump, A covering theory for non-involutive set-theoretic solutions to the Yang–Baxter equation, *J. Algebra* **520** (2019), 136–170. DOI: 10.1016/j.jalgebra.2018.11.007.

[30] A. Smoktunowicz and L. Vendramin, On skew braces (with an appendix by N. Byott and L. Vendramin), *J. Comb. Algebra* **2(1)** (2018), 47–86. DOI: 10.4171/JCA/2-1-3.

[31] A. Soloviev, Non-unitary set-theoretical solutions to the quantum Yang–Baxter equation, *Math. Res. Lett.* **7(5)** (2000), 577–596. DOI: 10.4310/MRL.2000.v7.n5.a4.

[32] M. Takeuchi, Survey on matched pairs of groups—an elementary approach to the ESS-LYZ theory, in: *"Noncommutative Geometry and Quantum Groups"* (Warsaw, 2001), Banach Center Publ. **61**, Polish Acad. Sci. Inst. Math., Warsaw, 2003, pp. 305–331. DOI: 10.4064/bc61-0-19.

[33] A. Weinstein and P. Xu, Classical solutions of the quantum Yang–Baxter equation, *Comm. Math. Phys.* **148(2)** (1992), 309–343. DOI: 10.1007/BF02100863.

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