New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

PAPER • OPEN ACCESS

Two-particle quantum correlations in stochastically-coupled networks

To cite this article: Roberto de J León-Montiel *et al* 2019 *New J. Phys.* **21** 053041

IOP Publishing New J. Phys. 21 (2019) 053041 https://doi.org/10.1088/1367-2630/ab1c79

New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

PAPER

Two-particle quantum correlations in stochastically-coupled networks

Roberto de J León-Montiel[1,6], **Vicenç Méndez**[2,6], **Mario A Quiroz-Juárez**[1], **Adrian Ortega**[3], **Luis Benet**[3], **Armando Perez-Leija**[4,5] **and Kurt Busch**[4,5]

[1] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, 04510 Cd. Mx., México
[2] Grup de Física Estadística, Departament de Física, Universitat Autònoma de Barcelona, E-08193 Barcelona, Spain
[3] Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, 62251 Cuernavaca, Morelos, México
[4] Max-Born-Institut, Max-Born-Straße 2A, D-12489 Berlin, Germany
[5] Humboldt-Universität zu Berlin, Institut für Physik, AG Theoretische Optik Photonik, Newtonstraße 15, D-12489 Berlin, Germany
[6] These authors contributed equally to this work.

E-mail: roberto.leon@nucleares.unam.mx

CrossMark

OPEN ACCESS

RECEIVED
9 January 2019

REVISED
25 March 2019

ACCEPTED FOR PUBLICATION
25 April 2019

PUBLISHED
29 May 2019

Keywords: many-particle quantum correlations, quantum networks, off-diagonal dynamical disorder

Abstract

Quantum walks in dynamically-disordered networks have become an invaluable tool for understanding the physics of open quantum systems. Although much work has been carried out considering networks affected by diagonal disorder, it is of fundamental importance to study the effects of fluctuating couplings. This is particularly relevant in materials science models, where the interaction forces may change depending on the species of the atoms being linked. In this work, we make use of stochastic calculus to derive a master equation for the dynamics of one and two non-interacting correlated particles in tight-binding networks affected by off-diagonal dynamical disorder. We show that the presence of noise in the couplings of a quantum network creates a pure-dephasing-like process that destroys all coherences in the single-particle Hilbert subspace. Moreover, we show that when two or more correlated particles propagate in the network, coherences accounting for particle indistinguishability are robust against the impact of off-diagonal noise, thus showing that it is possible, in principle, to find specific conditions for which many indistinguishable particles can traverse stochastically-coupled networks without losing their ability to interfere.

1. Introduction

The study of quantum random walks in noisy environments have played a fundamental role in understanding non-trivial quantum phenomena observed in an interdisciplinary framework of studies ranging from biology [1, 2], chemistry [3], materials science [4] and electronics [5], to photonics [6–9] and ultracold matter [10, 11]. For many years, most of the research efforts had been focused on the propagation of single particles [12]; however, a great interest in describing the dynamics of correlated particles in noisy systems has recently arisen [13–16], mainly because it has been recognized that many-particle quantum correlations can be preserved in noisy networks by properly controlling the initial state of the particles, their statistics, indistinguishability or their type of interaction [17, 18].

In general, the interesting features in the dynamics of quantum correlated particles traversing noisy networks are due to the tunneling amplitudes in the associated Hamiltonians. Therefore, including noise into the off-diagonal elements of the Hamiltonian allows one to assess the effects of decoherence and noise. On many occasions, when describing the evolution of correlated particles in network systems affected by non-dissipative noise, a physically accurate result can be obtained after averaging over many realizations of the noisy walks. In other words, in most cases, one does not have a master equation to analytically describe the phenomenon under study. Indeed, this represents a serious problem, specially in cases where the number of particles or network sites is extremely large. In such scenarios, computing the evolution of the system quickly becomes a computationally demanding task, which can only be tackled by developing sophisticated computer algorithms [19]. Consequently, most of the work is generally focused on optimizing numerical approaches, and the physical interpretation of the noise effects are sometimes overlooked.

In the present work we introduce an exact analytical approach to study quantum walks in noisy systems. We use stochastic calculus to derive a master equation for the propagation of two correlated particles in a quantum network affected by off-diagonal dynamical disorder. By using our results, we show that off-diagonal noise produces an effective pure-dephasing-like process that destroys all coherences in a single-particle quantum walk. Remarkably, we find that when two or more indistinguishable particles propagate in a noisy system, coherences accounting for particle indistinguishability are robust against the dephasing-like process. These results elucidate the role of particle indistinguishability in the preservation of quantum coherence in systems that interact with a noisy environment.

2. Single-particle dynamics

We start by describing the dynamics of a single particle in a quantum network affected by random fluctuations in the coupling between sites. In this situation, the time evolution of the single-particle wavefunction at the nth site, ψ_n, is given by the stochastic Schrödinger equation (with $\hbar = 1$)

$$\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = -\mathrm{i}\omega_n\psi_n - \mathrm{i}\sum_{m\neq n}\kappa_{nm}(t)\psi_m, \tag{1}$$

where ω_n stands for the energy of the nth site, and the coupling between them is given by $\kappa_{nm}(t) = \kappa_{nm} + \phi_{nm}(t)$, with $\phi_{nm}(t) = \phi_{mn}(t)$ describing a white-noise process with zero average, that is, $\langle\phi_{nm}(t)\rangle = 0$, and $\langle\phi_{nm}(t)\phi_{jl}(t')\rangle = \gamma_{nm}\delta_{nm,jl}\delta(t-t')$. Here $\delta_{nm,jl} = \delta_{nj}\delta_{ml} + \delta_{nl}\delta_{mj}$, with δ_{nm} being the Kronecker delta. γ_{nm} denotes the noise intensity, that is, how strong the stochastic fluctuations are, and $\langle\cdots\rangle$ denotes averaging over the noise realizations.

Following a treatment equivalent to the one used in [20, 21], where fluctuations are introduced in the site-energies rather than the couplings, we can obtain a master equation for a stochastically-coupled network by taking the time derivative of $\rho_{nm}(t) = \langle\psi_n\psi_m^*\rangle$. Thus, by using equation (1), we can write

$$\begin{aligned}\frac{\mathrm{d}\rho_{nm}}{\mathrm{d}t} &= \left\langle \psi_n\frac{\mathrm{d}\psi_m^*}{\mathrm{d}t} + \psi_m^*\frac{\mathrm{d}\psi_n}{\mathrm{d}t}\right\rangle,\\ &= -\mathrm{i}(\omega_n-\omega_m)\rho_{nm} + \mathrm{i}\sum_j\kappa_{mj}\rho_{nj} - \mathrm{i}\sum_j\kappa_{nj}\rho_{jm}\\ &\quad - \mathrm{i}\sum_j\sqrt{\gamma_{mj}}\langle\psi_n\psi_j^*\eta_{mj}(t)\rangle + \mathrm{i}\sum_j\sqrt{\gamma_{nj}}\langle\psi_j\psi_m^*\eta_{nj}(t)\rangle,\end{aligned} \tag{2}$$

where we have defined a new stochastic variable $\eta_{nm}(t) = -\phi_{nm}(t)/\sqrt{\gamma_{nm}}$, which satisfies the conditions $\langle\eta_{nm}(t)\rangle = 0$, and $\langle\eta_{nm}(t)\eta_{jl}(t')\rangle = \delta_{nm,jl}\delta(t-t')$. Notice that equation (2) is not yet complete, as it remains to compute the correlation functions of the last two terms. To do so, we employ the Novikov's theorem [22], which for the fourth term on the right hand side of equation (2) takes the form

$$\begin{aligned}\langle\psi_n\psi_j^*\eta_{mj}(t)\rangle &= \sum_{pq}\int \mathrm{d}t'\langle\eta_{mj}(t)\eta_{pq}(t')\rangle\left\langle\frac{\delta[\psi_n(t)\psi_j^*(t)]}{\delta\eta_{pq}(t')}\right\rangle,\\ &= \frac{1}{2}\sum_{pq}\delta_{mj,pq}\left\langle\frac{\delta[\psi_n(t)\psi_j^*(t)]}{\delta\eta_{pq}(t)}\right\rangle,\end{aligned} \tag{3}$$

where the operator $\delta/\delta\eta_{pq}(t)$ stands for the functional derivative with respect to the stochastic process. At this point, it is worth remarking that the Novikov's theorem can be used for solving different types of Gaussian stochastic processes, such as non-Markovian colored noise [23]. Now, to solve the functional derivative in equation (3), we make use of the expression

$$\begin{aligned}\psi_n(t)\psi_m^*(t) = \int_0^t \mathrm{d}t'\Big[& f(\psi_n\psi_m^*,\ldots) - \mathrm{i}\sum_r\sqrt{\gamma_{mr}}\psi_n\psi_r^*\eta_{mr}(t)\\ & + \mathrm{i}\sum_r\sqrt{\gamma_{nr}}\psi_r\psi_m^*\eta_{nr}(t)\Big],\end{aligned} \tag{4}$$

which corresponds to the formal integration of equation (2) without the noise average. Note that, in equation (4), the function $f(\psi_n\psi_m^*,\ldots)$ contains all terms that do not depend on stochastic variables.

Thus, by using equation (4) we obtain

$$\frac{\delta[\psi_n(t)\psi_j^*(t)]}{\delta\eta_{pq}(t)} = -\mathrm{i}\sum_r \sqrt{\gamma_{jr}}\,\psi_n\psi_r^*\delta_{jr,pq} + \mathrm{i}\sum_r \sqrt{\gamma_{nr}}\,\psi_r\psi_j^*\delta_{nr,pq}, \tag{5}$$

where we have used of the relation $\delta\eta_{jr}/\delta\eta_{pq} = \delta_{jr,pq}$. We can now substitute equation (5) into (3) to find

$$\langle\psi_n\psi_j^*\eta_{mj}(t)\rangle = -\frac{\mathrm{i}}{2}\sum_r \sqrt{\gamma_{jr}}\,\rho_{nr}\delta_{jr,mj} + \frac{\mathrm{i}}{2}\sum_r \sqrt{\gamma_{nr}}\,\rho_{rj}\delta_{nr,mj}. \tag{6}$$

Similarly, the fifth term on the right hand side of equation (2) is found to be

$$\langle\psi_j\psi_m^*\eta_{nj}(t)\rangle = -\frac{\mathrm{i}}{2}\sum_r \sqrt{\gamma_{mr}}\,\rho_{jr}\delta_{mr,nj} + \frac{\mathrm{i}}{2}\sum_r \sqrt{\gamma_{jr}}\,\rho_{rm}\delta_{jr,nj}. \tag{7}$$

Finally, by substituting equations (6) and (7) into (2), we obtain

$$\begin{aligned}\frac{\mathrm{d}\rho_{nm}}{\mathrm{d}t} = &-\left[\mathrm{i}(\omega_n - \omega_m) + \frac{1}{2}\sum_j(\gamma_{nj} + \gamma_{mj})\right]\rho_{nm}\\ &+ \mathrm{i}\sum_j(\kappa_{mj}\rho_{nj} - \kappa_{nj}\rho_{jm}) + \gamma_{nm}\rho_{nm} + \delta_{nm}\sum_j\sqrt{\gamma_{nj}\gamma_{mj}}\,\rho_{jj},\end{aligned} \tag{8}$$

which corresponds to a master equation for the time evolution of a single particle in a stochastically-coupled quantum network. Note that equation (8) can be cast into a Lindblad equation [12] by noticing that the deterministic part of equation (1) corresponds to the dynamics of a quantum network described by a tight-binding Hamiltonian of the form

$$H_d = \sum_n \omega_n|n\rangle\langle n| + \sum_{m\neq n}\kappa_{nm}|n\rangle\langle m|, \tag{9}$$

with $|n\rangle$ denoting the particle (or excitation) being at the nth site of the network. Then, by making use of equation (9), we can write equation (8) as

$$\frac{\mathrm{d}\rho_{nm}}{\mathrm{d}t} = -\mathrm{i}[H_{\mathrm{d}}, \rho]_{nm} + \mathcal{L}_{\mathrm{deph}}[\rho]_{nm}, \tag{10}$$

where [,] stands for the commutator. Notice that the effect of the fluctuating coupling is captured by the pure-dephasing-like operator

$$\mathcal{L}_{\mathrm{deph}}[\rho]_{nm} = -\left[\frac{1}{2}\sum_j(\gamma_{nj} + \gamma_{mj}) - \gamma_{nm}\right]\rho_{nm} + \delta_{nm}\sum_j\sqrt{\gamma_{nj}\gamma_{mj}}\,\rho_{jj}, \tag{11}$$

with γ_{nm} describing the dephasing rate introduced in the coupling between the nth and the mth site.

To elucidate the effect of the stochastic coupling between sites, we now compute the dynamics of a single excitation in a fully connected network composed by three sites. We have chosen this configuration, because it constitutes the simplest quantum network that one can investigate both theoretically and experimentally [5, 24]. The energies of the sites are arbitrarily chosen to be $\omega_1 = \omega_2 = \omega_3 = 5\ \mathrm{ps}^{-1}$, whereas the coupling between them are set to $\kappa_{12} = 1\ \mathrm{ps}^{-1}$, and $\kappa_{13} = \kappa_{23} = 0.5\ \mathrm{ps}^{-1}$. Figure 1 shows some examples of platforms where single-excitation stochastic networks have been successfully implemented, namely optical tweezers [25], waveguide arrays [24], superconducting circuits [26–28], and electrical-circuit arrays [29]. Quite recently, it has been shown that stochastic quantum networks can also be implemented in fascinating platforms, such as nuclear magnetic resonance systems [30], and ion traps [31, 32].

The time evolution of the diagonal (populations) and off-diagonal (coherences) elements of the system's density matrix, solved by means of equation (8), is shown in figure 2. In all figures, the dephasing rate is set to $\gamma_{12} = \gamma_{13} = \gamma_{23} = 0.38\ \mathrm{ps}^{-1}$. For the sake of comparison, we have included the numerical solution (dashed lines) of equation (1), which corresponds to the average of 10 000 random realizations, where the dephasing coefficient is defined by means of the relation [33, 34]: $\gamma_{nm} = \sigma_{nm}^2\Delta t$, with σ_{nm}^2 being the variance of the Gaussian distribution containing the values of the stochastic variable $\phi_{nm}(t)$, and Δt the correlation time. Notice that the effect of the fluctuating coupling is a pure-dephasing-like process, described by the operator $\mathcal{L}_{\mathrm{deph}}[\rho]$, that destroys the coherence between the network's sites. This phenomenon is equivalent to the quantum Zeno effect [35], a process in which the system is driven by its environment into a steady state where the regular hopping of the wavefunction is no longer sustained, i.e. a state in which the initial excitation becomes incoherently delocalized [36, 37].

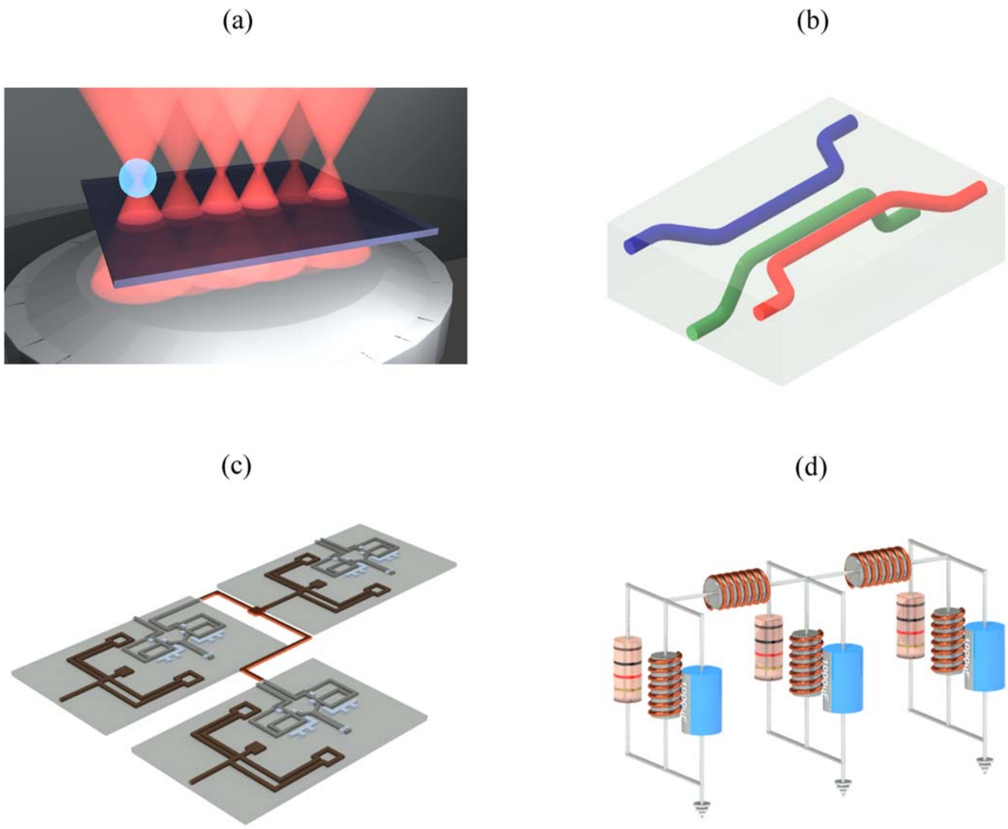

Figure 1. Schematic representation of photonic and electronic platforms where single-excitation stochastic networks have been investigated: (a) optical tweezers, (b) waveguides, (c) superconducting circuits, and (d) electrical-circuit arrays.

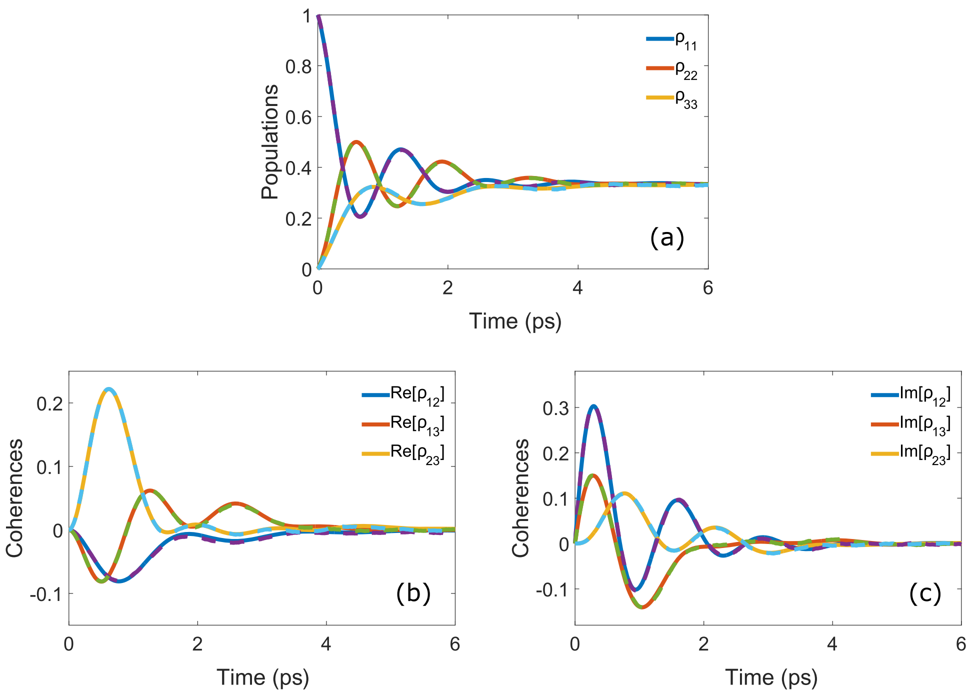

Figure 2. Dynamics of a single excitation injected into site 1 of a stochastically-coupled three-site quantum network. (a) Time evolution of the population in each of the sites; (b) and (c) show the real and imaginary parts of the coherence (off-diagonal) terms, respectively. Note that the effect of the fluctuating coupling is a non-dissipative pure-dephasing-like process that destroys coherence between the network's sites. The solid line corresponds to the solution using our derived master equation (equation (8)); whereas the dashed line shows the numerical solution of equation (1) obtained by averaging 10 000 realizations. In both cases, we have set the dephasing rates to $\gamma_{12} = \gamma_{13} = \gamma_{23} = 0.38$ ps^{-1}.

3. Two-particle wavefunction dynamics

We now turn our attention to the description of two-particle correlation dynamics. To this end, we use the concept of two-particle probability amplitude [24, 38], and derive the corresponding equations of motion for finite tight-binding networks comprising N sites.

We start by noting that the probability amplitudes for a quantum particle, initialized at a site n, are governed by the equations [24, 38]: $\frac{\mathrm{d}U_{p,n}}{\mathrm{d}t} = -\mathrm{i}\omega_n U_{p,n} - \mathrm{i}\sum_{r=1}^{N}\kappa_{pr}(t)U_{r,n}$, where $U_{p,n}$ stands for the impulse response of the system, that is, the unitary probability amplitude for a single particle traveling from site n to site p. As in the previous section, the coupling $\kappa_{pr}(t)$ represents a Gaussian Markov process with zero average. We can then write, in terms of single-particle probability amplitudes, the two-particle probability amplitudes at sites p and q as: $\psi_{p,q}(t) = \sum_{m=1,n=1}\xi_{m,n}[U_{p,n}(t)U_{q,m}(t) \pm U_{p,m}(t)U_{q,n}(t)]$, where $\xi_{m,n}$ is the initial probability amplitude profile that fulfills the conditions $\sum_{m=1,n=1}|\xi_{m,n}|^2 = 1$. Notice that the sign $\pm$ determines whether the particles are bosons $(+)$ or fermions $(-)$, respectively. Then, by taking the time derivative of the two-particle wavefunction, we obtain the equation

$$\frac{\mathrm{d}\psi_{p,q}}{\mathrm{d}t} = -\mathrm{i}(\omega_p + \omega_q)\psi_{p,q} - \mathrm{i}\sum_r[\kappa_{pr}(t)\psi_{r,q} + \kappa_{qr}(t)\psi_{p,r}], \tag{12}$$

which describes the dynamics of two-particle quantum correlations. Notice that two-particle quantum states evolve in a Hilbert space composed by a discrete set of N^2-mode states occupied by the two particles. One important fact to highlight regarding equation (12) is the presence of the term $(\omega_p + \omega_q)\psi_{p,q}$, which implies that during evolution the wavefunction $\psi_{p,q}$ acquires a phase that a single particle acquires when it traverses the same network twice [39]. Indeed, such effects can be expected since we are dealing with two correlated particles [40]. Finally, we remark that the modulus squared of the two-particle wavefunction gives the probability of finding one particle at site p and the other at q [41–46].

We can now follow the same procedure as in the previous section to obtain a master equation for the two-particle wavefunction dynamics by taking the time derivative of $\rho_{pq,p'q'} = \langle\psi_{pq}\psi^*_{p'q'}\rangle$. Thus, by using equation (12), we obtain (see appendix A for details)

$$\begin{aligned}\frac{\mathrm{d}\rho_{pq,p'q'}}{\mathrm{d}t} = \Bigg[&-\mathrm{i}(\omega_p + \omega_q - \omega_{p'} - \omega_{q'}) - \gamma_{pq} - \gamma_{p'q'}\\
&- \frac{1}{2}\sum_l(\gamma_{lp} + \gamma_{lq} + \gamma_{lp'} + \gamma_{lq'})\Bigg]\rho_{pq,p'q'}\\
&- \mathrm{i}\sum_l(\kappa_{lq}\rho_{pl,p'q'} + \kappa_{lp}\rho_{lq,p'q'})\\
&+ \mathrm{i}\sum_l(\kappa_{lq'}\rho_{pq,p'l} + \kappa_{lp'}\rho_{pq,lq'})\\
&- \sum_l(\delta_{pq}\sqrt{\gamma_{lq}\gamma_{lp}}\rho_{ll,p'q'} + \delta_{p'q'}\sqrt{\gamma_{lp'}\gamma_{lq'}}\rho_{pq,ll})\\
&+ \sum_l(\delta_{qq'}\sqrt{\gamma_{lq}\gamma_{lq'}}\rho_{pl,p'l} + \delta_{qp'}\sqrt{\gamma_{lq}\gamma_{lp'}}\rho_{pl,lq'})\\
&+ \sum_l(\delta_{pq'}\sqrt{\gamma_{lp}\gamma_{lq'}}\rho_{lq,p'l} + \delta_{pp'}\sqrt{\gamma_{lp}\gamma_{lp'}}\rho_{lq,lq'})\\
&+ \gamma_{qq'}\rho_{pq',p'q} + \gamma_{qp'}\rho_{pp',qq'}\\
&+ \gamma_{pp'}\rho_{p'q,pq'} + \gamma_{pq'}\rho_{q'q,p'p},\end{aligned} \tag{13}$$

which is the master equation that describes the time evolution of two correlated particles in a stochastically-coupled quantum network. Note that we can write equation (13) in its equivalent Lindblad form by following the procedure described above and noticing that the Hamiltonian for the two-particle system is given by $\mathcal{H} = H_d\otimes\mathbb{1} + \mathbb{1}\otimes H_d$, where $\otimes$ stands for the tensor product and $\mathbb{1}$ is the identity matrix, whose dimension is the same as the single-particle Hamiltonian.

Before considering particular examples, it is worth noting that in the following we will use the compact notation $|1_n, 1_m\rangle$ to represent the states where one particle is populating the site n and another the site m, i.e. $|1_n\rangle \otimes |1_m\rangle$, whereas states $\propto(|1_n, 1_m\rangle + |1_m, 1_n\rangle)$ are symmetrized wavefunctions.

We now examine the evolution of two-particle correlations in a three-site fully-connected network. As initial states we consider three different bosonic cases: (i) two indistinguishable particles in the separable state $|\psi(0)\rangle = (|1_1, 1_2\rangle + |1_2, 1_1\rangle)/\sqrt{2}$, (ii) an incoherent two-distinguishable-particle state represented by $\rho(0) = (|1_1, 1_2\rangle\langle 1_1, 1_2| + |1_2, 1_1\rangle\langle 1_2, 1_1|)/2$, and (iii) two particles in an entangled state

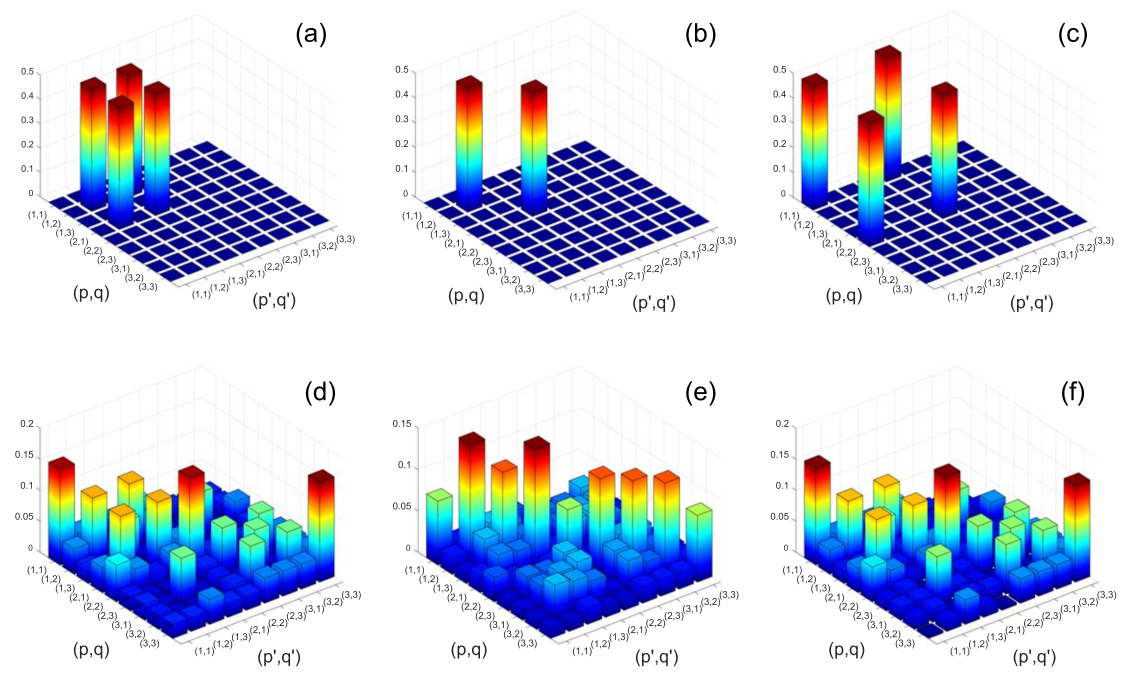

Figure 3. Density matrices (absolute value) for (a), (d) separable, (b), (e) incoherent and (c), (f) entangled states at $t = 0$ ps and $t = 1$ ps, respectively. Note that the stochastic coupling affects the system in such a way that when indistinguishable particles are injected into the system (figures 3(a) and (c)) the probability of finding both particles in the same site is the largest (bunching effect); whereas for distinguishable photons (figure 3(b)), the probability of finding them in different sites becomes larger (anti-bunching effect). The parameters used for the quantum network—namely site-energies, couplings and dephasing rates—are set to be $\omega_1 = \omega_2 = \omega_3 = 5$ ps^{-1}; $\kappa_{12} = 2$ ps^{-1}, $\kappa_{13} = \kappa_{23} = 1$ ps^{-1}, and $\gamma_{12} = \gamma_{13} = \gamma_{23} = 0.38$ ps^{-1}.

$|\psi(0)\rangle = (|1_1, 1_1\rangle + |1_2, 1_2\rangle)/\sqrt{2}$. Figure 3 shows the evolution of the initial states at $t = 1$ ps. Notice that the stochastic fluctuations affect the system in such a way that, when indistinguishable particles (figures 3(a), (d) and (c), (f)) are injected in the system, the probability of finding both particles in the same site is the largest, that is, the photons bunch in all sites with the same probability. This effect could be thought of as a generalized Hong-Ou-Mandel effect produced by the pure-dephasing-like process. In striking contrast, when distinguishable photons are injected in the system (figures 3(b), (e)), the probability of finding them in different sites becomes larger, thus leading to an anti-bunching effect. An important aspect to point out regarding the results shown in figure 3 is the computation time required for obtaining them. Remarkably, solving the master equation in equation (13) takes ~0.521 s; whereas the pure numerical solution of equation (12) takes ~2.4 h. This implies that our derived master equation improves the computation time by at least four orders of magnitude, while providing the maximum accuracy possible (see appendix B for details).

Recently, it has been shown that coherences arising from particle indistinguishability are robust against noise [24, 38]. By making use of our model, we have verified that in the steady-state, coherences accounting for particle indistinguishability do survive the impact of stochastic fluctuations in the coupling between sites (see appendix C for details). These results imply that it is possible, in principle, to find specific conditions for which many indistinguishable particles can traverse stochastically-coupled networks without losing their ability to interfere.

Finally, notice that the generalization of our results to N correlated particles is straightforward following similar steps as above by introducing the N-particle probability amplitude

$$\Psi_{p,q,r,\ldots}(t) = \sum_{a,b,c,\ldots}^{N} \varphi_{a,b,c,\ldots}[\chi_{a,b,c,\ldots}^{p,q,r,\ldots} + \chi_{a,b,c,\ldots}^{\mathrm{per}} + \ldots], \tag{14}$$

with $\chi_{a,b,c,\ldots}^{p,q,r,\ldots} = U_{p,a}(t)U_{q,b}(t)U_{r,c}(t)\ldots$, where $U_{m,n}$ represents the probability amplitude for each particle at site n when it is injected into channel m. The superscript 'per' stands for the cyclic permutations of the subscripts p, q, r, … in the corresponding transition amplitudes.

4. Conclusions

In this work, we have derived a master equation for the propagation of correlated particles in quantum networks affected by off-diagonal dynamical disorder. Unlike commonly-used computational methods, where many stochastic trajectories are needed, our equation allows one to find the average trajectory of correlated particles in a single calculation. By using our results, we showed that the effect of introducing noise in the couplings of a quantum network leads to a dephasing-like process that destroy all coherences in the single-particle Hilbert subspace. Interestingly, we found that when two or more correlated particles propagate in a stochastically-coupled network, coherences accounting for the indistinguishability of the particles endure the impact of noise. These results may help elucidating the role of particle indistinguishability to preserve quantum coherence and entanglement propagating through complex dynamically-disordered systems.

Acknowledgments

This work was supported by DGAPA-UNAM under the project UNAM-PAPIIT IA100718, and by CONACYT under the project CB-2016-01/284372. Armando Perez-Leija and Kurt Busch acknowledge financial support by the Deutsche Forschungsgemeinschaft within the priority program SPP 1839 'Tailored Disorder' (PE2602/2-2 and BU 1107/12-2). VM acknowledges financial support by Ministerio de Economia y Competitividad through Grant No. CGL2016-78156-C2-2-R.

Appendix A. Derivation of the two-particle master equation

We start by writing the expression for the probability amplitude dynamics of a quantum particle initiated at site n

$$\frac{\mathrm{d}U_{q,n}}{\mathrm{d}t} = -\mathrm{i}\omega_q U_{q,n} - \mathrm{i}\sum_r \kappa_{rq}(t) U_{r,n}, \tag{A.1}$$

where ω_n stands for the energy of the nth site, and the coupling between the rth and qth sites is given by $\kappa_{rq}(t) = \kappa_{rq} + \phi_{rq}(t)$, with $\phi_{rq}(t) = \phi_{qr}(t)$ describing a Gaussian Markov process with zero average, that is

$$\langle \phi_{rq}(t) \rangle = 0, \tag{A.2}$$

$$\langle \phi_{rq}(t)\phi_{jl}(t') \rangle = \gamma_{rq}\delta_{rq,jl}\delta(t - t'). \tag{A.3}$$

Here $\delta_{rq,jl} = \delta_{rj}\delta_{ql} + \delta_{rl}\delta_{qj}$, with δ_{rq} being the Kronecker delta. γ_{rq} denotes the noise intensity, that is, how strong the stochastic fluctuations are, and $\langle \cdots \rangle$ denotes stochastic averaging. By defining the stochastic variable $\phi_{rq}(t) = -\sqrt{\gamma_{rq}}\,\xi_{rq}(t)$, we can write

$$\frac{\mathrm{d}U_{q,n}}{\mathrm{d}t} = -\mathrm{i}\omega_q U_{q,n} - \mathrm{i}\sum_r \kappa_{rq} U_{r,n} + \mathrm{i}\sum_r \sqrt{\gamma_{rq}}\,\xi_{rq}(t) U_{r,n}, \tag{A.4}$$

with the properties of the stochastic variable ξ_{rq} given by

$$\langle \xi_{rq}(t) \rangle = 0, \tag{A.5}$$

$$\langle \xi_{rq}(t)\xi_{jl}(t') \rangle = \delta_{rq,jl}\delta(t - t'). \tag{A.6}$$

Notice that because noise (dynamic disorder) is introduced in the couplings, we must keep in mind that $r \neq q$ and, consequently, $j \neq l$.

Now, to compute the evolution of the two-particle density matrix $\rho_{pq,p'q'} = \langle \psi_{pq}\psi^*_{p'q'} \rangle$, with $\psi_{p,q}(t) = \sum_{m=1,n=1} \xi_{m,n}[U_{p,n}(t)U_{q,m}(t) \pm U_{p,m}(t)U_{q,n}(t)]$, we first write

$$\begin{aligned}\frac{\mathrm{d}(\psi_{pq}\psi^*_{p'q'})}{\mathrm{d}t} &= -\mathrm{i}[\omega_p + \omega_q - \omega_{p'} - \omega_{q'}]\psi_{pq}\psi^*_{p'q'} \\ &- \mathrm{i}\sum_l \kappa_{lq}\psi_{pl}\psi^*_{p'q'} - \mathrm{i}\sum_l \kappa_{lp}\psi_{lq}\psi^*_{p'q'} \\ &+ \mathrm{i}\sum_l \kappa_{lq'}\psi_{pq}\psi^*_{p'l} + \mathrm{i}\sum_l \kappa_{lp'}\psi_{pq}\psi^*_{lq'} \\ &- \mathrm{i}\sum_l \sqrt{\gamma_{lq}}\,\psi_{pl}\psi^*_{p'q'}\xi_{lq}(t) - \mathrm{i}\sum_l \sqrt{\gamma_{lp}}\,\psi_{lq}\psi^*_{p'q'}\xi_{lp}(t) \\ &+ \mathrm{i}\sum_l \sqrt{\gamma_{lq'}}\,\psi_{pq}\psi^*_{p'l}\xi_{lq'}(t) + \mathrm{i}\sum_l \sqrt{\gamma_{lp'}}\,\psi_{pq}\psi^*_{lq'}\xi_{lp'}(t).\end{aligned} \tag{A.7}$$

We can formally integrate equation (A.7), and obtain

$$\begin{aligned}\psi_{pq}\psi^{*}_{p'q'} = \int_0^t \mathrm{d}t' \Big\{ & f(\psi_{pq}\psi^{*}_{p'q'},\ldots)\\ & - \mathrm{i}\sum_l \sqrt{\gamma_{lq}}\,\psi_{pl}(t')\psi^{*}_{p'q'}(t')\xi_{lq}(t')\\ & - \mathrm{i}\sum_l \sqrt{\gamma_{lp}}\,\psi_{lq}(t')\psi^{*}_{p'q'}(t')\xi_{lp}(t')\\ & + \mathrm{i}\sum_l \sqrt{\gamma_{lq'}}\,\psi_{pq}(t')\psi^{*}_{p'l}(t)\xi_{lq'}(t')\\ & + \mathrm{i}\sum_l \sqrt{\gamma_{lp'}}\,\psi_{pq}(t')\psi^{*}_{lq'}(t')\xi_{lp'}(t') \Big\},\end{aligned} \tag{A.8}$$

where $f(\cdots)$ is a function that contains all terms that do not depend on the stochastic variables. Concurrently, we can write the average of equation (A.7) as

$$\begin{aligned}\frac{\mathrm{d}\langle\psi_{pq}\psi^{*}_{p'q'}\rangle}{\mathrm{d}t} = & -\mathrm{i}[\omega_p + \omega_q - \omega_{p'} - \omega_{q'}]\langle\psi_{pq}\psi^{*}_{p'q'}\rangle\\ & - \mathrm{i}\sum_l \kappa_{lq}\langle\psi_{pl}\psi^{*}_{p'q'}\rangle - \mathrm{i}\sum_l \kappa_{lp}\langle\psi_{lq}\psi^{*}_{p'q'}\rangle\\ & + \mathrm{i}\sum_l \kappa_{lq'}\langle\psi_{pq}\psi^{*}_{p'l}\rangle + \mathrm{i}\sum_l \kappa_{lp'}\langle\psi_{pq}\psi^{*}_{lq'}\rangle\\ & - \mathrm{i}\sum_l \sqrt{\gamma_{lq}}\langle\psi_{pl}\psi^{*}_{p'q'}\xi_{lq}(t)\rangle\\ & - \mathrm{i}\sum_l \sqrt{\gamma_{lp}}\langle\psi_{lq}\psi^{*}_{p'q'}\xi_{lp}(t)\rangle\\ & + \mathrm{i}\sum_l \sqrt{\gamma_{lq'}}\langle\psi_{pq}\psi^{*}_{p'l}\xi_{lq'}(t)\rangle\\ & + \mathrm{i}\sum_l \sqrt{\gamma_{lp'}}\langle\psi_{pq}\psi^{*}_{lq'}\xi_{lp'}(t)\rangle.\end{aligned} \tag{A.9}$$

It is clear that in order to obtain the master equation for $\rho_{pq,p'q'}(t)$, we must evaluate the correlation functions in the last four terms of equation (A.9). To do so, we invoke the Novikov's theorem [22, 47], which for the first correlation function in equation (A.9) takes the form

$$\begin{aligned}\langle\psi_{pl}\psi^{*}_{p'q'}\xi_{lq}(t)\rangle &= \sum_{rs}\int \mathrm{d}t'\langle\xi_{lq}(t)\xi_{rs}(t')\rangle\left\langle\frac{\delta[\psi_{pl}(t)\psi^{*}_{p'q'}(t)]}{\delta\xi_{rs}(t')}\right\rangle,\\ &= \sum_{rs}\int \mathrm{d}t'\delta_{lq,rs}\delta(t-t')\left\langle\frac{\delta[\psi_{pl}(t)\psi^{*}_{p'q'}(t)]}{\delta\xi_{rs}(t')}\right\rangle,\\ &= \frac{1}{2}\sum_{rs}\delta_{lq,rs}\left\langle\frac{\delta[\psi_{pl}(t)\psi^{*}_{p'q'}(t)]}{\delta\xi_{rs}(t)}\right\rangle.\end{aligned} \tag{A.10}$$

Here, we have taken into account the fact that, in the Stratonovich interpretation [48], $\int\delta(t) = 1/2$. We can then use equation (A.8) to write the functional derivative as

$$\begin{aligned}\frac{\delta[\psi_{pl}(t)\psi^{*}_{p'q'}(t)]}{\delta\xi_{rs}(t)} = & -\mathrm{i}\sum_\sigma \sqrt{\gamma_{\sigma l}}\,\psi_{p\sigma}(t)\psi^{*}_{p'q'}(t)\delta_{\sigma l,rs}\\ & - \mathrm{i}\sum_\sigma \sqrt{\gamma_{\sigma p}}\,\psi_{\sigma l}(t)\psi^{*}_{p'q'}(t)\delta_{\sigma p,rs}\\ & + \mathrm{i}\sum_\sigma \sqrt{\gamma_{\sigma q'}}\,\psi_{pl}(t)\psi^{*}_{p'\sigma}(t)\delta_{\sigma q',rs}\\ & + \mathrm{i}\sum_\sigma \sqrt{\gamma_{\sigma p'}}\,\psi_{pl}(t)\psi^{*}_{\sigma q'}(t)\delta_{\sigma p',rs},\end{aligned} \tag{A.11}$$

where we used the relation $\delta\xi_{\sigma l}/\delta\xi_{rs} = \delta_{\sigma l,rs}$. By substituting this result into equation (A.10), we can write

$$\begin{aligned}\langle\psi_{pl}\psi^*_{p'q'}\xi_{lq}(t)\rangle = &-\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma l,lq}\sqrt{\gamma_{\sigma l}}\,\rho_{p\sigma,p'q'}\\ &-\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma p,lq}\sqrt{\gamma_{\sigma p}}\,\rho_{\sigma l,p'q'}\\ &+\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma q',lq}\sqrt{\gamma_{\sigma q'}}\,\rho_{pl,p'\sigma}\\ &+\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma p',lq}\sqrt{\gamma_{\sigma p'}}\,\rho_{pl,\sigma q'}.\end{aligned} \tag{A.12}$$

Similarly, the remaining correlation functions are given by

$$\begin{aligned}\langle\psi_{lq}\psi^*_{p'q'}\xi_{lp}(t)\rangle = &-\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma q,lp}\sqrt{\gamma_{\sigma q}}\,\rho_{l\sigma,p'q'}\\ &-\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma l,lp}\sqrt{\gamma_{\sigma l}}\,\rho_{\sigma q,p'q'}\\ &+\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma q',lp}\sqrt{\gamma_{\sigma q'}}\,\rho_{lq,p'\sigma}\\ &+\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma p',lp}\sqrt{\gamma_{\sigma p'}}\,\rho_{lq,\sigma q'},\end{aligned} \tag{A.13}$$

$$\begin{aligned}\langle\psi_{pq}\psi^*_{p'l}\xi_{lq'}(t)\rangle = &-\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma q,lq'}\sqrt{\gamma_{\sigma q}}\,\rho_{p\sigma,p'l}\\ &-\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma p,lq'}\sqrt{\gamma_{\sigma p}}\,\rho_{\sigma q,p'l}\\ &+\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma l,lq'}\sqrt{\gamma_{\sigma l}}\,\rho_{pq,p'\sigma}\\ &+\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma p',lq'}\sqrt{\gamma_{\sigma p'}}\,\rho_{pq,\sigma l},\end{aligned} \tag{A.14}$$

$$\begin{aligned}\langle\psi_{pq}\psi^*_{lq'}\xi_{lp'}(t)\rangle = &-\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma q,lp'}\sqrt{\gamma_{\sigma q}}\,\rho_{p\sigma,lq'}\\ &-\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma p,lp'}\sqrt{\gamma_{\sigma p}}\,\rho_{\sigma q,lq'}\\ &+\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma q',lp'}\sqrt{\gamma_{\sigma q'}}\,\rho_{pq,l\sigma}\\ &+\frac{\mathrm{i}}{2}\sum_\sigma \delta_{\sigma l,lp'}\sqrt{\gamma_{\sigma l}}\,\rho_{pq,\sigma q'}.\end{aligned} \tag{A.15}$$

Finally, by substituting equations (A.12)–(A.15) into equation (A.9) we obtain

$$\begin{aligned}\frac{\mathrm{d}\rho_{pq,p'q'}}{\mathrm{d}t} = &-\mathrm{i}(\omega_p+\omega_q-\omega_{p'}-\omega_{q'})\rho_{pq,p'q'}\\ &-\frac{1}{2}\sum_l[(\gamma_{lp}+\gamma_{lq}+\gamma_{lp'}+\gamma_{lq'})-\gamma_{pq}-\gamma_{p'q'}]\rho_{pq,p'q'}\\ &-\mathrm{i}\sum_l(\kappa_{lq}\rho_{pl,p'q'}+\kappa_{lp}\rho_{lq,p'q'}-\kappa_{lq'}\rho_{pq,p'l}-\kappa_{lp'}\rho_{pq,lq'})\\ &-\sum_l(\delta_{pq}\sqrt{\gamma_{lq}\gamma_{lp}}\,\rho_{ll,p'q'}+\delta_{p'q'}\sqrt{\gamma_{lp'}\gamma_{lq'}}\,\rho_{pq,ll})\\ &+\sum_l(\delta_{qq'}\sqrt{\gamma_{lq}\gamma_{lq'}}\,\rho_{pl,p'l}+\delta_{qp'}\sqrt{\gamma_{lq}\gamma_{lp'}}\,\rho_{pl,lq'})\\ &+\sum_l(\delta_{pq'}\sqrt{\gamma_{lp}\gamma_{lq'}}\,\rho_{lq,p'l}+\delta_{pp'}\sqrt{\gamma_{lp}\gamma_{lp'}}\,\rho_{lq,lq'})\\ &+\gamma_{qq'}\rho_{pq',p'q}+\gamma_{qp'}\rho_{pp',qq'}+\gamma_{pp'}\rho_{p'q,pq'}+\gamma_{pq'}\rho_{q'q,p'p},\end{aligned} \tag{A.16}$$

which is the result shown in equation (13) of the main text.

Appendix B. Comparison between master equation and the direct stochastic numerical simulation

We now provide a quantitative comparison between the time evolution of a two-particle state obtained by means of our derived master equation and by directly implementing the stochastic equations. Figure B1 shows the

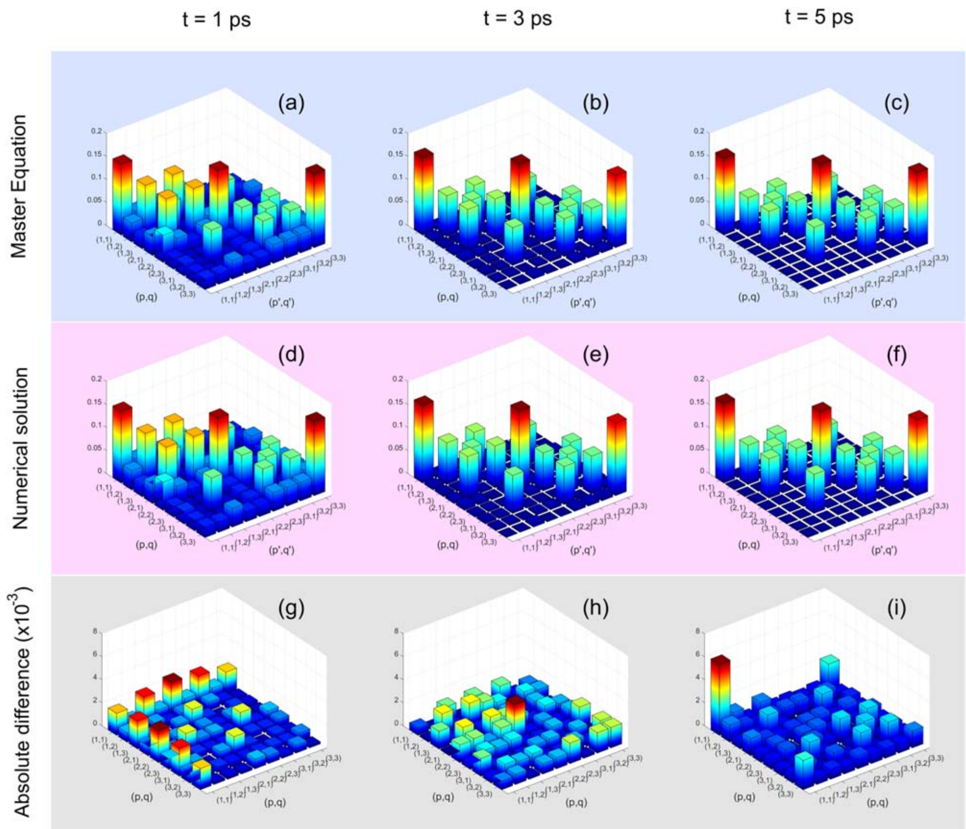

Figure B1. Density matrices (absolute value) for a separable state, $|\psi(0)\rangle = (|1_1,\ 1_2\rangle + |1_2,\ 1_1\rangle)/\sqrt{2}$, at $t = 1$ ps, $t = 3$ ps, and $t = 5$ ps, obtained by means of the derived master equations (a)–(c), and by the direct numerical evaluation of the stochastic equations (d)–(f). Figures 1(g)–(i) show the absolute difference between both solutions, $\Delta\rho = |\rho^{(\mathrm{master})}_{pq,p'q'} - \rho^{(\mathrm{numerical})}_{pq,p'q'}|$, at the corresponding evolution times.

evolution of a separable state, $|\psi(0)\rangle = (|1_1,\ 1_2\rangle + |1_2,\ 1_1\rangle)/\sqrt{2}$, propagating in a dynamically-disordered three-site network. The parameters used for the quantum networks—namely site-energies, couplings and dephasing rates—are the same as those used for obtaining figure 3 of the main text. Figures B1(a)–(c) show the results obtained by using the derived master equation (equation (13) of the main text), whereas figures B1(d)–(f) show the results obtained by numerically solving equation (12) of the main text using the Taylor Integration package [49]. The latter were obtained by averaging over 10 000 different realizations of the two-particle random walk. It is important to highlight that the computation time required for each case was $T_c^{(\mathrm{master})} = 0.521$ s, and $T_c^{(\mathrm{numerical})} = 2.4$ h for the master equation and direct stochastic evaluation, respectively. Clearly, our derived equation improves the computation time by at least four orders of magnitude, while providing the maximum accuracy possible. For the sake of completeness, in figures B1(g)–(i), we have included the absolute difference between the absolute value of the density matrix elements obtained from the master equation and the numerical solution, i.e. $\Delta\rho = ||\rho^{(\mathrm{master})}_{pq,p'q'}|-|\rho^{(\mathrm{numerical})}_{pq,p'q'}||$. Finally, we would like to remark that while the derived master equation provides the exact solution, the accuracy of the stochastic-computation solution strongly depends on the number of realizations being used for the average, which implies that many realizations (and therefore longer computation times) are required in order to obtain reliable numerical results. This is the reason why, when possible, one should use master equations instead of direct stochastic numerical simulations.

Appendix C. Quantitative analysis of two-particle coherence preservation

To quantify the amount of surviving coherence in the steady states, we use two different coherence measures, the physically intuitive norm of coherence [50]: $C_N(\rho) = \sum_{i\neq j}|\rho_{i,j}|$, and the relative entropy of coherence [51] $C_E(\rho) = S(\rho_{\mathrm{diag}}) - S(\rho)$, with S representing the von Neumann entropy and ρ_{diag} the matrix obtained from the density matrix ρ after removing all off-diagonal elements. Note that, in both measures, a totally mixed (or incoherent) state is signaled by a vanishing coherence measure. Figure C1 shows the evolution of the norm of coherence and the entropy of coherence for the initial separable (solid line), path-entangled (dashed line), and

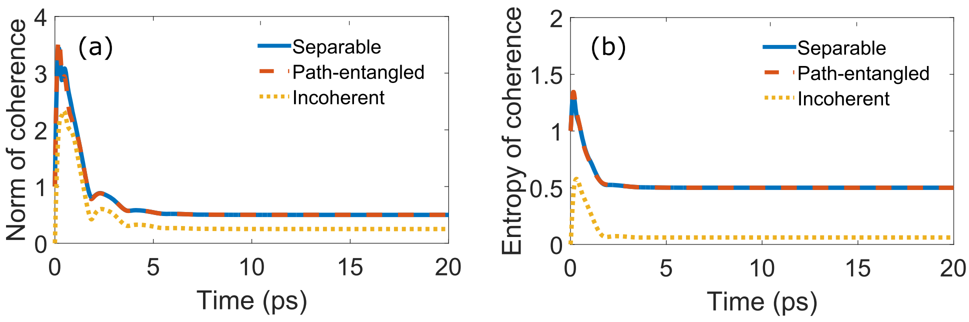

Figure C1. Time evolution of (a) the norm of coherence $C_N(\rho)$, and (b) the relative entropy of coherence $C_E(\rho)$, considering three different initial states, namely separable (solid line), path-entangled (dashed line), and incoherent (dotted line) states.

incoherent (dotted line) states. Interestingly, when distinguishable particles (incoherent states) are injected into the system ($C_N = C_E = 0$), coherence due to propagation-induced indistinguishability rapidly emerges and the system evolves into a steady state where $C_N = 0.250$ and $C_E = 0.061$. Remarkably, the same results as those shown in figure C1 can be observed for any value of the dephasing rates. Therefore, we can convincingly say that under the influence of fluctuating couplings, identical particles always evolve into a steady state in which coherences due to indistinguishability perpetually prevail.

ORCID iDs

Roberto de J León-Montiel https://orcid.org/0000-0002-3045-3604
Vicenç Méndez https://orcid.org/0000-0002-2605-6606
Mario A Quiroz-Juárez https://orcid.org/0000-0002-5995-9510
Luis Benet https://orcid.org/0000-0002-8470-9054
Armando Perez-Leija https://orcid.org/0000-0002-6218-295X

References

[1] Rebentrost P, Mohseni M, Kassal I, Lloyd S and Aspuru-Guzik A 2009 *New J. Phys.* **11** 033003
[2] Chin A, Datta A, Caruso F, Huelga S and Plenio M 2010 *New J. Phys.* **12** 065002
[3] Park H *et al* 2016 *Nat. Mater.* **15** 211
[4] Ziman J M 1979 *Models of Disorder: The Theoretical Physics of Homogeneously Disordered Systems* (Cambridge: Cambridge University Press)
[5] León-Montiel R J, Quiroz-Juárez M A, Quintero-Torres R, Domínguez-Juárez J L, Moya-Cessa H M, Torres J P and Aragón J L 2015 *Sci. Rep.* **5** 17339
[6] Saikin S K, Eisfeld A, Valleau S and Aspuru-Guzik A 2013 *Nanophotonics* **2** 21
[7] Viciani S, Lima M, Bellini M and Caruso F 2015 *Phys. Rev. Lett.* **115** 083601
[8] Biggerstaff D N, Heilmann R, Zecevik A A, Grafe M, Broome M A, Fedrizzi A, Nolte S, Szameit A, White A G and Kassal I 2016 *Nat. Commun.* **7** 11282
[9] Caruso F, Crespi A, Ciriolo A G, Sciarrino F and Osellame R 2016 *Nat. Commun.* **7** 11682
[10] Schonleber D W, Eisfeld A, Genkin M, Whitlock S and Wuster S 2015 *Phys. Rev. Lett.* **114** 123005
[11] Trautmann N and Hauke P 2018 *Phys. Rev.* A **97** 023606
[12] Breuer H P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[13] Rigovacca L and Franco C D 2016 *Sci. Rep.* **6** 22052
[14] Siloi I, Benedetti C, Piccinini E, Piilo J, Maniscalco S, Paris M G A and Bordone P 2017 *Phys. Rev.* A **95** 022106
[15] Rossi M A C, Benedetti C, Borrelli M, Maniscalco S and Paris M G A 2017 *Phys. Rev.* A **96** 040301
[16] de Vega I and Alonso D 2017 *Rev. Mod. Phys.* **89** 015001
[17] Bromley T R, Cianciaruso M and Adesso G 2015 *Phys. Rev. Lett.* **114** 210401
[18] Beggi A, Buscemi F and Bordone P 2016 *Quantum Inf. Process.* **15** 3711
[19] Piccinini E, Benedetti C, Siloi I and Paris M G A 2017 *Comput. Phys. Commun.* **215** 235
[20] Eisfeld A and Briggs J S 2012 *Phys. Rev.* E **85** 046118
[21] León-Montiel R J and Torres J P 2013 *Phys. Rev. Lett.* **110** 218101
[22] Novikov E A 1965 *Sov. Phys.—JETP* **20** 1290
[23] Strunz W T and Yu T 2004 *Phys. Rev.* A **69** 052115
[24] Perez-Leija A, Guzman-Silva D, León-Montiel R J, Graefe M, Heinrich M, Moya-Cessa H, Busch K and Szameit A 2018 *njp Quantum Inf.* **4** 024002
[25] León-Montiel R J and Quinto-Su P A 2017 *Sci. Rep.* **7** 44287
[26] Mostame S, Rebentrost P, Eisfeld A, Kerman A J, Tsomokos D I and Aspuru-Guzik A 2012 *New J. Phys.* **14** 105013
[27] Mostame S, Huh J, Kreisbeck C, Kerman A J, Fujita T, Eisfeld A and Aspuru-Guzik A 2017 *Quantum Inf. Process.* **16** 44
[28] Potocnik A *et al* 2018 *Nat. Commun.* **9** 904

[29] Quiroz-Juárez M A, Aragón J L, León-Montiel R J, Vázquez-Medina R, Domínguez-Juárez J L and Quintero-Torres R 2016 *Europhys. Lett.* **116** 50004
[30] Wang B X, Tao M J, Ai Q, Xin T, Lambert N, Ruan D, Cheng Y C, Nori F, Deng F G and Long G L 2018 *njp Quantum Inf.* **4** 52
[31] Gorman D J, Hemmerling B, Megidish E, Moeller S A, Schindler P, Sarovar M and Haffner H 2018 *Phys. Rev.* X **8** 011038
[32] Maier C, Brydges T, Jurcevic P, Trautmann N, Hempel C, Lanyon B P, Hauke P, Blatt R and Roos C F 2019 *Phys. Rev. Lett.* **122** 050501
[33] Laing C and Lord G J 2008 *Stochastic Methods in Neuroscience* (Oxford: Clarendon)
[34] León-Montiel R J, Svozilík J and Torres J P 2014 *Phys. Rev.* E **90** 012108
[35] Misra B and Sudarshan E C G 1977 *J. Math. Phys.* **18** 756
[36] Levi F and Mintert F 2014 *New J. Phys.* **16** 033007
[37] León-Montiel R J, Vallés A, Moya-Cessa H M and Torres J P 2015 *Laser Phys. Lett.* **12** 085204
[38] Perez-Leija A, León-Montiel R J, Sperling J, Moya-Cessa H, Szameit A and Busch K 2017 *J. Phys. B: At. Mol. Opt. Phys.* **51** 024002
[39] Giuseppe G D, Martin L, Perez-Leija A, Keil R, Dreisow F, Nolte S, Szameit A, Abouraddy A F, Christodoulides D N and Saleh B E A 2013 *Phys. Rev. Lett.* **110** 150503
[40] Klyshko D N 1982 *Sov. Phys.—JETP* **56** 753
[41] Abouraddy A F, Saleh B E A, Sergienko A V and Teich M C 2001 *Phys. Rev. Lett.* **87** 123602
[42] Saleh B E A, Teich M C and Sergienko A V 2005 *Phys. Rev. Lett.* **94** 223601
[43] Abouraddy A F, Saleh B E A, Sergienko A V and Teich M C 2002 *J. Opt. Soc. Am.* B **19** 1174
[44] Bromberg Y, Lahini Y, Morandotti R and Silberberg Y 2009 *Phys. Rev. Lett.* **102** 253904
[45] Lebugle M, Gräfe M, Heilmann R, Perez-Leija A, Nolte S and Szameit A 2015 *Nat. Commun.* **6** 8273
[46] Weimann S, Perez-Leija A, Lebugle M, Keil R, Tichy M, Gräfe M, Heilmann R and Nolte S 2016 *Nat. Commun.* **7** 11027
[47] Méndez V, Horsthemke W, Mestres P and Campos D 2011 *Phys. Rev.* E **84** 041137
[48] van Kampen N G 1981 *J. Stat. Phys.* **24** 175
[49] Perez J A and Benet L 2017 Taylorintegration.jl https://github.com/PerezHz/TaylorIntegration.jl
[50] Streltsov A, Adesso G and Plenio M B 2017 *Rev. Mod. Phys.* **89** 041003
[51] Baumgratz T, Cramer M and Plenio M B 2014 *Phys. Rev. Lett.* **113** 140401