

TROPICAL IDEALS DO NOT REALISE ALL BERGMAN FANS

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ABSTRACT. Every tropical ideal in the sense of Maclagan-Rincón has an associated tropical variety, a finite polyhedral complex equipped with positive integral weights on its maximal cells. This leads to the realisability question, ubiquitous in tropical geometry, of which weighted polyhedral complexes arise in this manner. Using work of Las Vergnas on the non-existence of tensor products of matroids, we prove that there is no tropical ideal whose variety is the Bergman fan of the direct sum of the Vámos matroid and the uniform matroid of rank two on three elements, and in which all maximal cones have weight one.

1. INTRODUCTION

An ideal in a polynomial ring over a field with a non-Archimedean valuation gives rise to a tropical variety, either by taking all weight vectors whose initial ideals do not contain a monomial or, equivalently if the field and the value group are large enough [Dra08, Theorem 4.2], by applying the coordinate-wise valuation to all points in the zero set of the ideal. In the middle of this construction sits a *tropical ideal*, obtained by applying the valuation to all polynomials in the ideal. This ideal is a purely tropical object, in that it does not know about the field or the valuation, and it contains more information than the tropical variety itself. For these reasons, tropical ideals, axiomatised in [MR18], were proposed as the correct algebraic structures on which to build a theory of tropical schemes. We review the relevant definitions below.

It was proved in [MR18] that tropical ideals, while not finitely generated as ideals—nor in any sense that we know of!—have a rational Hilbert series, satisfy the ascending chain condition, and define a tropical variety: a finite weighted polyhedral complex. Later in [MR], it was shown that the top-dimensional parts of these varieties are always balanced polyhedral complexes. This leads to the following realisability question.

Question 1.1. Which pure-dimensional balanced polyhedral complexes are the variety of some tropical ideal?

If the tropical ideal is the tropicalisation of a prime classical ideal, then the tropical variety is pure-dimensional and balanced [MS15, Theorem 3.3.5]. The question of which balanced polyhedral complexes are realised by classical ideals has received much attention, especially in the case of curves (see, e.g., [Spe14, BS15, BGS17]). But for general tropical ideals, very little is known about Question 1.1: for instance, no natural algebraic criterion that ensures that the variety is pure-dimensional is

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known. In fact, until recently we had no intuition as to whether tropical ideals are flexible enough that they can realise basically any balanced polyhedral complex, or rather more rigid, like algebraic varieties. In view of the following theorem, we now lean towards the latter intuition.

Theorem 5.2. Let M and N be loopless matroids of ranks a and b that do not have a quasi-product of rank $a \cdot b$. Then there exists no tropical ideal whose tropical variety is the Bergman fan of the direct sum of M and N , with all maximal cones having weight 1.

In particular, there exists no tropical ideal whose tropical variety is the Bergman fan of the direct sum of the Vámos matroid V_8 and the uniform matroid $U_{2,3}$ of rank two on three elements, with all maximal cones having weight 1.

In this theorem, a *quasi-product* of two loopless matroids is a matroid analogue of tensor products; see Section 4. The fact that the Vámos matroid V_8 and the uniform matroid $U_{2,3}$ have no quasi-product of rank 8 was proved by Las Vergnas in [LV81].

We believe that this theorem marks the beginning of an interesting research programme, which, in addition to the pureness and balancing questions mentioned above, asks which tropical ideals define matroids on the set of variables, and which matroids are, in this sense, tropically algebraic—See Problem 3.5 and Question 3.6.

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2. DEFINITIONS AND BASIC RESULTS ON TROPICAL IDEALS

Consider the tropical semifield $(\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with $\oplus := \min$ and $\odot := +$. Let R be a sub-semifield of $\overline{\mathbb{R}}$. The example most relevant to us is the Boolean semifield $\mathbb{B} := \{0, \infty\}$, which is not only a sub-semifield but also a quotient of $\overline{\mathbb{R}}$.

Definition 2.1. Let N be a finite set. A set $L \subseteq R^N$ is a **tropical linear space** if it is an R -submodule (i.e., $(\infty, \dots, \infty) \in L$ and $f, g \in L, c \in R \Rightarrow (c \odot f) \oplus g \in L$) and if, moreover, L satisfies the following elimination axiom: for $i \in N$ and $f, g \in L$ with $f_i = g_i \neq \infty$, there exists an $h \in L$ with $h_i = \infty$ and $h_j \geq f_j \oplus g_j$ for all $j \in N$, with equality whenever $f_j \neq g_j$. The $\overline{\mathbb{R}}$ -submodule $L_{\overline{\mathbb{R}}}$ of $\overline{\mathbb{R}}^N$ generated by L is a tropical linear space in $\overline{\mathbb{R}}^N$, and has the structure of a finite polyhedral complex; we denote its dimension as such by $\dim L$.

If K is a field equipped with a non-Archimedean valuation onto R and if $V \subseteq K^N$ is a linear subspace, then the image of V under the coordinate-wise valuation is a tropical linear space in R^N , but not all tropical linear spaces arise in this manner. Tropical linear spaces are well-studied objects in tropical geometry and matroid theory: the definition above is equivalent to that of [Spe08], except that we allow some coordinates to be ∞ . A tropical linear space L gives rise to a matroid $M(L)$ in which the independent sets are those subsets $A \subseteq N$ for which $L \cap (R^A \times \{\infty\}^{N \setminus A}) = \{\infty\}^N$, and L is the set of **vectors** (R -linear combinations of valuated circuits) of a **valuated matroid** on $M(L)$ [MT01]. With this setup, $\dim L = |N| - \text{rk}(M(L))$.

We will freely alternate between these different characterisations of tropical linear spaces.

Set $\mathbb{N} := \{0, 1, 2, \dots\}$, and let $n \in \mathbb{N}$. Denote by $R[x_1, \dots, x_n]$ the semiring of polynomials in the variables x_1, \dots, x_n with coefficients in R . We write Mon_d and $\text{Mon}_{\leq d}$ for the set of monomials in x_1, \dots, x_n of degree equal to d and at most d , respectively, and we identify a polynomial in $R[x_1, \dots, x_n]$ of degree at most d with its coefficient vector in $R^{\text{Mon}_{\leq d}}$.

Definition 2.2. A subset $I \subseteq R[x_1, \dots, x_n]$ is a **tropical ideal** if $x_i \circ I \subseteq I$ for all $i = 1, \dots, n$ and if for each $d \in \mathbb{N}$ the set $I_{\leq d} := \{f \in I : \deg(f) \leq d\}$ is a tropical linear space in $R^{\text{Mon}_{\leq d}}$.

This definition is equivalent to [MR18, Definition 1.1]. Indeed, there, in addition to the requirement that $I_{\leq d}$ be a tropical linear space, it is required that I is an ideal in the semiring $R[x_1, \dots, x_n]$. This is equivalent to the statement that I is closed under tropical multiplication by each x_i and closed under tropical addition. However, as tropical linear spaces are already closed under tropical addition, this does not need to be included as an explicit axiom.

If I is homogeneous, then the latter condition is equivalent to the condition that for each d the set I_d of homogeneous polynomials in I of degree d is a tropical linear space in R^{Mon_d} . There is a natural notion of tropical ideals in the Laurent polynomial ring $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ that we will also use, and if I is a tropical ideal in $R[x_1, \dots, x_n]$ then the set $I' := \{f/\mathbf{x}^{\mathbf{u}} \mid f \in I, \mathbf{u} \in \mathbb{N}^n\}$ is a tropical ideal in $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Tropical ideals were introduced by Maclagan and Rincón in [MR18] as a framework for developing algebraic foundations for tropical geometry. Tropical ideals are much better behaved than general ideals of the polynomial semiring $R[x_1, \dots, x_n]$, as we explain below.

Definition 2.3. For $\mathbf{w} \in \mathbb{R}^n$ and $f = \bigoplus_{\mathbf{u}} c_{\mathbf{u}} \circ \mathbf{x}^{\mathbf{u}} \in R[x_1, \dots, x_n]$, define the **initial part** of f relative to \mathbf{w} as

$$\text{in}_{\mathbf{w}}(f) := \bigoplus_{\mathbf{u} : c_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{w} = f(\mathbf{w})} \mathbf{x}^{\mathbf{u}} \in \mathbb{B}[x_1, \dots, x_n].$$

For a tropical ideal I define its **initial ideal** relative to \mathbf{w} as

$$\text{in}_{\mathbf{w}} I := \langle \text{in}_{\mathbf{w}} f \mid f \in I \rangle_{\mathbb{B}}.$$

Note that in this paper we only consider weights \mathbf{w} in \mathbb{R}^n , not in $\overline{\mathbb{R}}^n$ as in [MR18]. In other words, we do geometry only inside the tropical torus.

Definition 2.4. The **Hilbert function** of a tropical ideal $I \subseteq R[x_1, \dots, x_n]$ is the map $H_I : \mathbb{N} \rightarrow \mathbb{N}$ given by $d \mapsto \binom{n+d}{d} - \dim I_{\leq d}$.

Note that, as usual in commutative algebra, the Hilbert function measures the codimension of $I_{\leq d}$ in its ambient space $R^{\text{Mon}_{\leq d}}$. A homogeneous variant of this Hilbert function applies only to homogeneous ideals and measures the codimension of I_d in R^{Mon_d} . The Hilbert function of a not necessarily homogeneous ideal I in $R[x_1, \dots, x_n]$ equals the homogeneous Hilbert function of its homogenisation in $R[x_0, \dots, x_n]$.

The following is a special case of [MR18, Corollary 3.6].

Theorem 2.5. *For a homogeneous tropical ideal $I \subseteq R[x_1, \dots, x_n]$ and any $\mathbf{w} \in \mathbb{R}^n$, $\text{in}_{\mathbf{w}} I \subseteq \mathbb{B}[x_1, \dots, x_n]$ is a homogeneous tropical ideal, and $H_{\text{in}_{\mathbf{w}} I} = H_I$.*

Theorem 2.5 allows one to pass to monomial initial ideals and show that the Hilbert function $H_I(d)$ of a homogeneous tropical ideal I becomes a polynomial in d for sufficiently large d , and also that homogeneous tropical ideals satisfy the ascending chain condition. Via homogenisation, one sees that both statements also hold for non-homogeneous tropical ideals (but, as in the classical setting, the theorem does not apply directly, since for instance, when $n = 1$, $\text{in}_{(1)}(0 \oplus x_1) = 0$ generates an ideal—the entire semiring—with a smaller Hilbert function than any tropical ideal containing $0 \oplus x_1$ but not 0).

Furthermore, Maclagan and Rincón prove that tropical ideals have tropical varieties that are finite polyhedral complexes [MR18, Theorem 5.11].

Theorem 2.6. *If $I \subseteq R[x_1, \dots, x_n]$ is a tropical ideal then its (tropical) variety*

$$V(I) := \{\mathbf{w} \in \mathbb{R}^n : \text{in}_{\mathbf{w}} I \text{ contains no monomial}\}$$

is the support of a finite polyhedral complex.

Indeed, if I is homogeneous, they show that the sets of \mathbf{w} where $\text{in}_{\mathbf{w}} I$ is constant form the relatively open polyhedra of a polyhedral complex with support \mathbb{R}^n called the **Gröbner complex** of I , and that the cells where $\text{in}_{\mathbf{w}} I$ contains no monomial form a subcomplex with support $V(I)$. By homogeneity, all cells then contain in their lineality space the linear span of the all-ones vector $\mathbf{1}$. In the case where $I \subseteq R[x_1, \dots, x_n]$ is not necessarily homogeneous, let I^h be its homogenisation in $R[x_0, x_1, \dots, x_n]$. Then $\mathbf{w} \mapsto (0, \mathbf{w})$ is a bijection between $V(I)$ and the intersection of $V(I^h)$ with the zeroth coordinate hyperplane, and we give $V(I)$ the corresponding polyhedral complex structure.

The variety of a tropical ideal comes equipped with positive integral weights on its maximal polyhedra; this is inspired by [MS15, Lemma 3.4.7] and studied more in depth in [MR].

Definition 2.7. Let $I \subseteq R[x_1, \dots, x_n]$ be a tropical ideal, let σ be a maximal polyhedron of $V(I)$, and let \mathbf{w} be in the relative interior of σ . The **multiplicity** of σ in $V(I)$ is defined as follows. First, let $I' \subseteq R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the (tropical) ideal in the Laurent polynomial ring generated by I . After an automorphism of the Laurent polynomial ring given by $\mathbf{x}^{\mathbf{u}} \mapsto \mathbf{x}^{A\mathbf{u}}$ with $A \in \text{GL}_n(\mathbb{Z})$, we can assume that the affine span of σ is a translate of $\text{span}(\mathbf{e}_1, \dots, \mathbf{e}_d)$ for some d . In this case, by [MR, Lemma 6.2], the tropical ideal $J := \text{in}_{\mathbf{w}}(I') \cap \mathbb{B}[x_{d+1}, \dots, x_n]$ is zero-dimensional, i.e., $H_J(e)$ is a constant for $e \gg 0$. The multiplicity of σ is defined to be equal to this constant, called the degree of J .

Remark 2.8. A more coordinate-free version of Definition 2.7 is the following. Consider the linear span of σ , defined as

$$\text{span}(\sigma) := \mathbb{R}_{\geq 0}\{\mathbf{v} - \mathbf{v}' \mid \mathbf{v}, \mathbf{v}' \in \sigma\}.$$

Let $S \subseteq \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the sub-semiring spanned by monomials $\mathbf{x}^{\mathbf{u}}$ of \mathbf{w} -weight $\mathbf{w} \cdot \mathbf{u}$ equal to zero for all $\mathbf{w} \in \text{span}(\sigma)$. Then S itself is isomorphic to a Laurent polynomial semiring in $n - d$ variables. The multiplicity of σ is the degree of the zero-dimensional tropical ideal $\text{in}_{\mathbf{w}}(I') \cap S$.

We will need the following results.

Lemma 2.9. *Let I be a tropical ideal in $R[x_1, \dots, x_n]$. Denote by I' the ideal generated by I in $R[x_1^{\pm 1}, \dots, x_n^{\pm n}]$, and set $I^{\text{sat}} := I' \cap R[x_1, \dots, x_n]$. Then $I^{\text{sat}} \supseteq I$ is a tropical ideal, and $V(I^{\text{sat}}) = V(I)$ as weighted polyhedral complexes.*

We call I^{sat} the **saturation** of I with respect to $m := x_1 \cdots x_n$, and we call I **saturated** with respect to m if $I^{\text{sat}} = I$.

Proof. That I^{sat} is a tropical ideal containing I is straightforward from the definition. Since $I^{\text{sat}} \supseteq I$ we have $V(I^{\text{sat}}) \subseteq V(I)$. Conversely, let $\mathbf{w} \in V(I)$ and $f \in I^{\text{sat}}$. Then $\mathbf{x}^{\mathbf{u}} \circ f \in I$ for some $\mathbf{u} \in \mathbb{N}^n$, hence $\text{in}_{\mathbf{w}}(\mathbf{x}^{\mathbf{u}} \circ f)$ is not a monomial, and therefore neither is $\text{in}_{\mathbf{w}} f$. This shows that $V(I) = V(I^{\text{sat}})$. That the multiplicities are the same follows from the fact that the multiplicities in $V(I)$ are defined using I' . \square

If Σ is a polyhedral complex in \mathbb{R}^n and σ is a polyhedron in Σ , the **star** $\text{star}_{\sigma} \Sigma$ of Σ at σ is a weighted polyhedral fan, whose cones are indexed by the cones τ of Σ containing σ . The cone indexed by such τ is

$$\bar{\tau} := \mathbb{R}_{\geq 0} \{ \mathbf{v} - \mathbf{w} \mid \mathbf{v} \in \tau \text{ and } \mathbf{w} \in \sigma \},$$

with weight equal to the weight of τ in Σ .

The following can be found in [MR, Corollary 2.11 and Proposition 6.4].

Proposition 2.10. *Let I be a tropical ideal in $R[x_1, \dots, x_n]$, σ be a polyhedron in $V(I)$, and \mathbf{w} be in the relative interior of σ . Then $\text{in}_{\mathbf{w}} I \subseteq \mathbb{B}[x_1, \dots, x_n]$ is homogeneous with respect to every vector $\mathbf{v} \in \text{span}(\sigma)$, and $V(\text{in}_{\mathbf{w}} I) = \text{star}_{\mathbf{w}} V(I)$ as weighted polyhedral complexes.*

3. THE INDEPENDENCE COMPLEX OF A TROPICAL IDEAL

Definition 3.1. Let $I \subseteq R[x_1, \dots, x_n]$ be a tropical ideal. The **independence complex** of I is the simplicial complex

$$(3.1) \quad \mathcal{I}(I) := \{ A \subseteq \{1, \dots, n\} : I \cap R[x_i : i \in A] = \{\infty\} \}.$$

When $\mathcal{I}(I)$ is the collection of independent sets of a matroid M , we will say that I is a **matroidal tropical ideal**, and that M is its associated **algebraic matroid**.

The independence complex of a tropical ideal I can be recovered from its variety $V(I)$, at least if $R = \overline{\mathbb{R}}$.

Proposition 3.2. *If $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal then*

$$(3.2) \quad \mathcal{I}(I) = \{ A \subseteq \{1, \dots, n\} : \pi_A(V(I)) = \mathbb{R}^A \},$$

where $\pi_A: \mathbb{R}^n \rightarrow \mathbb{R}^A$ is the coordinate projection onto the coordinates indexed by A . In particular, the independence complex $\mathcal{I}(I)$ depends only on the variety $V(I)$.

Proof. Let $A \subseteq \{1, \dots, n\}$. If $A \notin \mathcal{I}(I)$ then there exists $f \in I \cap \overline{\mathbb{R}}[x_i : i \in A]$ such that $f \neq \infty$, and $V(I) \subseteq V(f)$. We then have $\pi_A(V(I)) \subseteq \pi_A(V(f)) \subsetneq \mathbb{R}^A$, as claimed. For the reverse inclusion, suppose that $\pi_A(V(I)) \subsetneq \mathbb{R}^A$, and let $\mathbf{w} \in \mathbb{R}^A \setminus \pi_A(V(I))$. For any polynomial $f \in \overline{\mathbb{R}}[x_1, \dots, x_n]$, denote by $f|_{\mathbf{w}}$ the polynomial in $\overline{\mathbb{R}}[x_i : i \notin A]$ obtained by specializing each variable x_i with $i \in A$ to $w_i \in \mathbb{R}$. Consider the ideal $I|_{\mathbf{w}} \subseteq \overline{\mathbb{R}}[x_i : i \notin A]$ defined as $I|_{\mathbf{w}} := \{ f|_{\mathbf{w}} : f \in I \}$. By [MR, Theorem 3.6],

the ideal $I|_{\mathbf{w}}$ is a tropical ideal. Moreover, we must have $V(I|_{\mathbf{w}}) = \emptyset$, as any point $\mathbf{v} \in V(I|_{\mathbf{w}})$ would lift to the point $(\mathbf{v}, \mathbf{w}) \in V(I)$, contradicting that $\mathbf{w} \notin \pi_A(V(I))$. By the weak Nullstellensatz [MR18, Corollary 5.17], the tropical ideal $I|_{\mathbf{w}}$ must contain the constant polynomial 0. But then $0 = f|_{\mathbf{w}}$ for some $f \in I$, which in particular implies that $f \in I \cap \overline{\mathbb{R}}[x_i : i \in A]$ and $f \neq \infty$. \square

Proposition 3.2 also follows from the fact that a coordinate projection of the variety of a tropical ideal is the variety of the corresponding elimination ideal [MR, Theorem 4.7].

Recall that the Hilbert function $H_I(e)$ of a tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ eventually agrees with a polynomial in e , called the **Hilbert polynomial** of I [MR18, Proposition 3.8]. The **dimension** $\dim(I)$ of I is defined as the degree of its Hilbert polynomial.

Corollary 3.3. *For any tropical ideal I we have*

$$\dim \mathcal{I}(I) + 1 = \dim V(I) = \dim I.$$

Proof. From (3.2) it is clear that $\dim V(I) \geq \dim \mathcal{I}(I) + 1$. Now, if $V(I)$ contains a polyhedron σ of dimension d then there is some coordinate projection $\pi_A(\sigma)$ that is d -dimensional, and thus from (3.1) we see that $A \in \mathcal{I}(I)$ and thus $\dim \mathcal{I}(I) + 1 \geq d$. This shows that $\dim \mathcal{I}(I) + 1 = \dim V(I)$. The equality $\dim V(I) = \dim I$ is proved in [MR, Theorem 4.3]. \square

In the classical setting, primality of an ideal implies matroidality. We do not know about a similarly appealing sufficient condition for matroidality of general tropical ideals.

Example 3.4. If $J \subseteq K[x_1, \dots, x_n]$ is a prime ideal, where K is a field with a non-Archimedean valuation, then $\text{trop}(J)$ is a matroidal tropical ideal. Its associated algebraic matroid is the matroid that captures algebraic independence among the coordinate functions x_1, \dots, x_n in the field of fractions of $K[x_1, \dots, x_n]/J$. \diamond

Problem 3.5. Find algebraic conditions on a tropical ideal that imply matroidality.

As shown in Example 3.4, any (classically) algebraic matroid is the algebraic matroid of a tropical ideal. However, in principle, it is possible that the class of matroids that are “tropically algebraic” is strictly larger than the usual class of algebraic matroids.

Question 3.6. Which matroids arise as the algebraic matroid of a tropical ideal?

4. QUASI-PRODUCTS OF MATROIDS

To motivate the definition of quasi-products, let v_1, \dots, v_m be nonzero vectors in a vector space V and let w_1, \dots, w_n be nonzero vectors in a vector space W over the same field. The v_i define a matroid M with ground set $[m]$ in which $S \subseteq [m]$ is dependent if and only if the set $\{v_i : i \in S\}$ is linearly dependent. Similarly, the w_j define a matroid N with ground set $[n]$. Now consider the vectors $v_i \otimes w_j \in V \otimes W$, $i \in [m], j \in [n]$. In the same manner, these define a matroid P with ground set $[m] \times [n]$. One can check that P is in general not determined by M and N , i.e., the linear dependencies among the $v_i \otimes w_j$ cannot be read off from those among the v_i and those among the w_j . However, some features of P are predicted

by M and N : for each fixed $i \in [m]$, the linear dependencies among the vectors $v_i \otimes w_j$, $j \in [n]$ are precisely those recorded by N ; here we use that v_i is nonzero. Similarly, for each $j \in [n]$, the restriction of P to $[m] \times \{j\}$ is isomorphic to M . Furthermore, if B is a basis of M and C is a basis of N , then $B \times C$ is a basis of P . In particular, the rank of P is the product of the ranks of M and N . Following Las Vergnas, we use these observations to define quasi-products of general matroids, as follows.

Definition 4.1 ([LV81]). Let M, N be loopless matroids with ground sets $[m], [n]$, respectively. A *quasi-product* of M and N is a matroid P with ground set $[m] \times [n]$ with the property that for each $i \in [m]$ the map $[n] \rightarrow [m] \times [n], j \mapsto (i, j)$ is an isomorphism from M to the restriction of P to $\{i\} \times [n]$ and for each $j \in [n]$ the map $[m] \rightarrow [m] \times [n], i \mapsto (i, j)$ is an isomorphism from M to the restriction of P to $[m] \times \{j\}$.

The properties of a quasi-product P of M and N imply that if $B \subseteq [m]$ is a basis of M and $C \subseteq [n]$ is a basis of N , then $B \times C$ is a spanning set of P , so the rank of P is at most the product of the ranks of M and N . By the discussion above, two matroids that are representable over the same field always admit a quasi-product whose rank is the product of their ranks. In general, however, a quasi-product with this property need not exist.

Theorem 4.2 ([LV81]). *Any quasi-product of the rank-4 Vámos matroid V_8 and the rank-2 uniform matroid $U_{2,3}$ has rank at most $7 < 4 \cdot 2$.*

5. NOT EVERY BERGMAN FAN IS THE VARIETY OF A TROPICAL IDEAL

We now prove that not every balanced polyhedral complex can be obtained as the variety of a tropical ideal. Our counterexample will be the Bergman fan of a matroid; see [AK06] for details.

Definition 5.1. Let M be a loopless matroid of rank d on the ground set $\{1, \dots, n\}$. The **Bergman fan** $\mathcal{B}(M)$ of M is the pure d -dimensional polyhedral fan in \mathbb{R}^n consisting of the cones of the form

$$\sigma_{\mathcal{F}} := \text{cone}(\mathbf{e}_{F_1}, \mathbf{e}_{F_2}, \dots, \mathbf{e}_{F_k}) + \mathbb{R} \cdot \mathbf{e}_{\{1, \dots, n\}}$$

where $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \subsetneq \{1, \dots, n\}\}$ is a chain of flats in the lattice of flats $\mathcal{L}(M)$ of M , and where \mathbf{e}_S stands for the sum of the standard basis vectors \mathbf{e}_i with i running through S . The Bergman fan of any matroid is given the structure of a balanced polyhedral complex by defining the multiplicity of each maximal cone to be equal to 1.

Bergman fans of matroids are the tropical linear spaces (more specifically, their part inside the torus \mathbb{R}^n) that correspond to valuated matroids where the basis valuations all take values in \mathbb{B} .

The following is our main result.

Theorem 5.2. *Let M be a loopless matroid of rank a with ground set $[m]$ and let N be a loopless matroid of rank b with ground set $[n]$. Suppose that every quasi-product of M and N has rank strictly less than $a \cdot b$. Then there exists no tropical ideal $I \subseteq \mathbb{R}[x_1, \dots, x_m, y_1, \dots, y_n]$ such that $V(I)$ is equal to $\mathcal{B}(M \oplus N)$ as weighted polyhedral complexes, even up to common refinement.*

In particular, there is no tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_3, y_1, \dots, y_8]$ such that $V(I)$ is equal to $\mathcal{B}(U_{2,3} \oplus V_8)$ as weighted polyhedral complexes, even up to common refinement.

Note that we do not require the polyhedral structure on $V(I)$ coming from the Gröbner complex of the homogenisation of I to be equal to the fan structure on the Bergman fan described above.

To prove the theorem, in addition to the fundamental results from Section 2, we will need results relating $V(I)$ to H_I for any tropical ideal I .

Lemma 5.3. *Let $L, L' \subseteq R^N$ be tropical linear spaces. If $\dim L + \dim L' > |N|$, then $L \cap L' \neq \{(\infty, \dots, \infty)\}$.*

Proof. The notion of stable intersection for tropical linear spaces was studied by Speyer in [Spe08] when the underlying matroids of both tropical linear spaces were uniform matroids, and later generalized by Mundinger [Mun] for arbitrary tropical linear spaces in R^N . The stable intersection $L \cap_{\text{st}} L'$ is a tropical linear space contained in both L and L' , and it has dimension at least $\dim L + \dim L' - |N| > 0$, which implies the desired result. \square

Proposition 5.4. *Let $I \subseteq R[x_1, \dots, x_n]$ be a tropical ideal. If the independence complex $\mathcal{I}(I)$ contains a subset A of size r , then $H_I(d) \geq \binom{r+d}{d}$ for all $d \in \mathbb{N}$.*

Proof. The space $R[x_i : i \in A]_{\leq d}$ is a tropical linear space in $R^{\text{Mon}_{\leq d}}$ of dimension $\binom{r+d}{d}$ and, by assumption, it does not intersect $I_{\leq d}$. Hence by Lemma 5.3, $\dim I_{\leq d} \leq \binom{n+d}{d} - \binom{r+d}{d}$, and therefore $H_I(d) \geq \binom{r+d}{d}$. \square

Proposition 5.5. *Let $I \subsetneq R[x_1, \dots, x_n]$ be a tropical ideal, and set $r := H_I(1) - 1$. Then $H_I(d) \leq \binom{n+d}{d}$ for all $d \in \mathbb{N}$.*

Proof. Let $I^h \subseteq R[x_0, \dots, x_n]$ be the homogenisation of I . Then $\dim(I^h)_d = \dim I_{\leq d}$ for all $d \in \mathbb{N}$, and in particular $\dim(I^h)_1 = \dim I_{\leq 1} = n + 1 - H_I(1) = n - r$. Moreover, by applying Theorem 2.5 with a sufficiently general weight vector \mathbf{w} , the Hilbert function of I^h is also that of some monomial ideal J . We find that J contains precisely $n - r$ of the $n + 1$ variables x_0, \dots, x_n , and therefore all their multiples. This implies that $\dim J_d \geq \binom{n+d}{d} - \binom{r+d}{d}$, where the last term counts monomials in the remaining $r + 1$ variables of degree d . We then have

$$H_I(d) = \binom{n+d}{d} - \dim I_{\leq d} = \binom{n+d}{d} - \dim J_d \leq \binom{n+d}{d} - \binom{n+d}{d} + \binom{r+d}{d},$$

as desired. \square

The following proposition shows that the algebraic matroid of a Bergman fan $\mathcal{B}(M)$ (as in Proposition 3.2) is equal to the matroid M .

Proposition 5.6 ([Yu17, Lemma 3]). *The independence complex of the Bergman fan $\mathcal{B}(M)$ of a loopless matroid M is the same as the independence complex of M .*

We now present a key step towards proving our main result.

Proposition 5.7. *Let M be a loopless matroid on the ground set $\{1, \dots, n\}$. Suppose $J \subseteq \mathbb{B}[x_1, \dots, x_n]$ is a homogeneous tropical ideal, saturated with respect to $x_1 \cdots x_n$, whose variety $V(J)$ has a common refinement, as weighted polyhedral complexes,*

with the Bergman fan $\mathcal{B}(M)$ (with weight 1 in all its maximal cones). Then the matroid $M(J_1)$ is equal to M , under the identification $x_i \leftrightarrow i$ of ground sets.

Proof. Let $B = \{b_1, \dots, b_d\}$ be a basis of M . For $0 \leq i \leq d$, consider the flat F_i of M obtained as the closure of the set $\{b_1, \dots, b_i\}$, and let σ be the maximal cone of $\mathcal{B}(M)$ corresponding to the chain of flats $\emptyset = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_{d-1} \subsetneq F_d = \{1, \dots, n\}$. Let $\tau \subseteq \sigma$ be a maximal cone in a common refinement of both $V(J)$ and $\mathcal{B}(M)$. The linear span $\text{span}(\tau) = \text{span}(\sigma)$ consists of all vectors $\mathbf{w} \in \mathbb{R}^n$ for which $w_i = w_j$ whenever $\{i, j\} \subseteq F_k \setminus F_{k-1}$ for some $k = 1, \dots, d$. A monomial $\mathbf{x}^{\mathbf{u}}$ in $\mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has \mathbf{w} -weight equal to zero for all such \mathbf{w} if and only if for every k we have $\sum_{i \in F_k \setminus F_{k-1}} u_i = 0$. As in Remark 2.8, let S be the subsemiring of $\mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ consisting of all polynomials involving only such monomials, and let J' be the (tropical) ideal in $\mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generated by J .

Take \mathbf{v} to be a vector in the relative interior of τ . Since τ has multiplicity 1 in $V(J)$, $\text{in}_{\mathbf{v}}(J') \cap S$ is zero-dimensional of degree 1, and contains no monomials. Hence for any pair of distinct monomials $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{u}'}$ in S , $\text{in}_{\mathbf{v}}(J') \cap S$ contains the binomial $\mathbf{x}^{\mathbf{u}} \oplus \mathbf{x}^{\mathbf{u}'}$. In particular, if $\{i \neq j\} \subseteq F_k \setminus F_{k-1}$ for some k then $0 \oplus x_i^{-1} x_j \in \text{in}_{\mathbf{v}}(J') \cap S$, and thus $x_i \oplus x_j \in \text{in}_{\mathbf{v}}(J')$. As J is homogeneous and saturated with respect to $x_1 \cdots x_n$, this implies that there is a polynomial of the form $x_i \oplus x_j \oplus f$ in J_1 where f is a sum of variables all contained in F_{k-1} . It follows that x_i is in the closure of $F_{k-1} \cup \{x_j\}$ in the matroid $M(J_1)$. We conclude that $\{b_1, \dots, b_d\}$ is a generating set in the matroid $M(J_1)$, and thus $\text{rank}(M(J_1)) \leq \text{rank}(M)$. Now, the tropical prevariety cut out by the linear polynomials in J is equal to $\mathcal{B}(M(J_1))$, so we have $\mathcal{B}(M(J_1)) \supseteq V(J) = \mathcal{B}(M)$. It follows from [Rin12, Lemma 7.4] that $\mathcal{B}(M(J_1)) = \mathcal{B}(M)$, and thus $M(J_1) = M$, completing the proof. \square

We conclude with the proof of the main theorem.

Proof of Theorem 5.2. Suppose that such an I exists, and denote $O := M \oplus N$. We first argue that we may replace I by an ideal J that is homogeneous as well as saturated. To this end, let σ be a polyhedron in $V(I)$ whose affine span is $\mathbb{R} \cdot \mathbf{1}$ (which is contained in the lineality space of $\mathcal{B}(O)$), and let \mathbf{w} be in the relative interior of σ . Set $J' := \text{in}_{\mathbf{w}} I \subseteq \mathbb{B}[x_1, \dots, x_m, y_1, \dots, y_n]$. By Proposition 2.10, the tropical ideal J' is homogeneous (with respect to $\mathbf{1}$) and has variety $V(J') = \text{star}_{\mathbf{w}} V(I)$, which is equal to $\mathcal{B}(O)$ up to common refinement. Consider the homogeneous ideal $J := (J')^{\text{sat}}$. By Lemma 2.9, we have that $V(J)$ is also equal to $\mathcal{B}(O)$ up to common refinement.

Now, by Proposition 5.7, $M(J_1)$ is equal to O . Since $\text{rk } O = a + b$, we find that $H_J(1) = 1 + a + b$ and thus, by Proposition 5.5, $H_J(d) \leq \binom{a+b+d}{d}$ for all d . On the other hand, since $V(J) = \mathcal{B}(O)$, by Propositions 5.6 and 3.2 the tropical ideal J is matroidal, with associated algebraic matroid $O = M \oplus N$. Hence, by Proposition 5.4 we have $H_J(d) \geq \binom{a+b+d}{d}$. We conclude that $H_J(d) = \binom{a+b+d}{d}$.

Denote $Q := M(J_2)$. The matroid Q has rank $H_J(2) - H_J(1) = \binom{a+b+1}{2}$ on the ground set $S_1 \sqcup S_2 \sqcup S_3$, where $S_1 := \{x_i x_j \mid 1 \leq i \leq j \leq m\}$, $S_2 := \{y_i y_j \mid 1 \leq i \leq j \leq n\}$, and $S_3 := \{x_i y_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. The restriction $Q|_{S_1}$ is spanned by all products of two elements in a basis of $M(J_1)|_{\{x_1, x_2, \dots, x_m\}}$, hence has rank at most $\binom{a+1}{2}$. Similarly, the restriction $Q|_{S_2}$ has rank at most $\binom{b+1}{2}$. Hence $Q|_{S_3}$ has rank at least $\binom{a+b+1}{2} - \binom{a+1}{2} - \binom{b+1}{2} = ab$.

Since J is saturated, for each $1 \leq i \leq m$, multiplication by x_i yields an isomorphism between the matroid $M(J_1)|\{y_1, \dots, y_n\} \cong N$ and the restriction of Q to $x_i \cdot \{y_1, \dots, y_n\} \subseteq S_3$. Similarly, for each $1 \leq j \leq n$, the restriction of Q to $y_j \cdot \{x_1, \dots, x_m\}$ is isomorphic to M . Hence $Q|S_3$ is a quasi-product of M and N in the sense of Definition 4.1. But the assumption in the theorem is that such a quasi-product has rank strictly less than $a \cdot b$, a contradiction. Hence no such ideal I exists.

The second part of the main theorem is a direct consequence of the first part and Theorem 4.2 by Las Vergnas. \square

6. CONCLUDING REMARKS

Using the result by Las Vergnas that $U_{2,3}$ and V_8 do not have a quasi-product of rank 8, we have showed that the Bergman fan of their direct sum is not the tropical variety of any tropical ideal, with weight 1 on all the maximal cones.

We do not know whether there exists a tropical ideal whose tropical variety is the Bergman fan of $U_{2,3} \oplus V_8$ as a set, without the condition that all weights be 1.

We also do not know whether $\mathcal{B}(V_8)$ itself is the tropical variety of any tropical ideal with weight one on the maximal cones. To study this question for a matroid M , one needs to develop the theory of *symmetric squares* of matroids, in a fashion similar to Las Vergnas's quasi-products from Section 4. But already for V_8 this seems considerably harder than quasi-products of $U_{2,3}$ with V_8 .

Finally, we'd like to point out that for any $m \geq 3$, the matroids $U_{2,m}$ and V_8 do not admit a quasi-product of rank 8. Indeed, if P were such a quasi-product on $[m] \times [8]$, then for any basis $C \subseteq [8]$ of V_8 the set $[2] \times C$, which spans P , would have to be a basis. But then the restriction of P to $[3] \times [8]$ would be a quasi-product of $U_{2,3}$ and V_8 of rank 8, a contradiction to Las Vergnas's Theorem 4.2. This simple observation yields infinitely many matroids to which our Theorem 5.2 applies. However, it would be interesting to find more intricate families of pairs of matroids that do not admit quasi-products of the correct rank.

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