

FOURIER MULTIPLIERS FOR TRIEBEL-LIZORKIN SPACES ON COMPACT LIE GROUPS

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ABSTRACT. We investigate the boundedness of Fourier multipliers on a compact Lie group when acting on Triebel-Lizorkin spaces. Criteria are given in terms of the Hörmander-Mihlin-Marcinkiewicz condition. In our analysis, we use the difference structure of the unitary dual of a compact Lie group. Our results cover the sharp Hörmander-Mihlin theorem on Lebesgue spaces and also other historical results on the subject.

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1. INTRODUCTION

Let G be a compact Lie group. In this work we study sufficient conditions for the boundedness of Fourier multipliers on the Triebel Lizorkin spaces $F_{p,q}^r(G)$ in terms of the Hörmander-Mihlin condition on their symbols. The Littlewood-Paley theorem states that $L^p(G) \equiv F_{p,2}^0(G)$, so that in view of the classical results of Hörmander-Mihlin type (see Hörmander [34] and Mihlin [36] for instance), Triebel-Lizorkin spaces are a good substitute of L^p -spaces, when considering smoothness of distributions in different scales (see Triebel [50, 51] and [37, 38] for details).

The problem of finding conditions for the boundedness of Fourier multipliers on compact Lie groups started a long of time ago with the study of the theory of Fourier

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series of periodic functions on $[0, 1]$, (functions on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$). Indeed, when classifying the boundedness of multipliers of the Fourier series,

$$Af(x) := \sum_{\xi \in \mathbb{Z}} e^{2\pi i x \cdot \xi} \sigma(\xi) \widehat{f}(\xi), \quad f \in L^1(\mathbb{T}), \quad \widehat{f}(\xi) := \int_0^1 e^{-2\pi i x \cdot \xi} f(x) dx, \quad (1.1)$$

it was observed by Marcinkiewicz in his classical 1939's work [35] that the condition

$$\sup_{\xi \in \mathbb{Z}} |\sigma(\xi)| + \sup_{j \in \mathbb{N}_0} \sum_{2^{j-1} \leq |\xi| < 2^j} |\sigma(\xi + 1) - \sigma(\xi)| < \infty, \quad (1.2)$$

assures the existence of a bounded extension of A on $L^p(\mathbb{T})$, $1 < p < \infty$. Denoting the difference operator on the lattice \mathbb{Z} , by $\Delta\sigma := \sigma(\cdot + 1) - \sigma$, and by Δ^k , $k \in \mathbb{N}_0$, its successive iterations, the Marcinkiewicz condition (1.2) is satisfied by any sequence $(\sigma(\xi))_{\xi \in \mathbb{Z}^n}$ such that

$$|\Delta^k \sigma(\xi)| \lesssim_k |\xi|^{-k}, \quad \xi \neq 0, \quad k = 0, 1, \quad (1.3)$$

which should be, in principle, more easier to verify that (1.2). Another generalisation of Marcinkiewicz's criterion for multipliers of the Fourier transform on \mathbb{R}^n , was done by Mihlin [36], who stated that a function $\sigma \in C^\infty(\mathbb{R}^n \setminus \{0\})$, satisfying estimates of the kind

$$|\partial_\xi^\alpha \sigma(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|}, \quad |\alpha| \leq [n/2] + 1, \quad (1.4)$$

has a multiplier A (of the Fourier transform¹ on \mathbb{R}^n) defined by

$$Af(x) \equiv T_\sigma f(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \sigma(\xi) \widehat{f}(\xi) d\xi, \quad f \in C_0^\infty(\mathbb{R}^n), \quad (1.5)$$

admitting a bounded extension on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$. Subsequent generalisations to Mihlin's theorem were done by Hörmander [34], Calderón and Torchinsky in [4], Taibleson and Weiss [49], Baernstein and Sawyer [3], Seeger [45, 46, 47] and many others. We refer the reader to Grafakos' paper [30] (and reference therein) for a complete historical revision and for recent developments about Mihlin-Hörmander and Marcinkiewicz multiplier theorems on \mathbb{R}^n .

Extensions of Marcinkiewicz, and Hörmander-Mihlin criteria have been proved in the context of Lie groups and several spaces of homogeneous type in the context of spectral multipliers of self-adjoint operators, e.g., sub-Laplacians, or of other operators with heat kernels satisfying Gaussian estimates, with general contexts that go beyond of the objective of this paper. In view of the extensive literature on the field, we will not review it here, but we refer the reader to [1, 2, 7, 13, 14, 19, 48, 53] and to the extensive list of references therein.

In the framework of Fourier multipliers on compact Lie groups, by using the Calderón-Zygmund type theory in Coifman and De Guzmán [16], the L^p -Fourier multipliers for $SU(2)$ were investigated by Coifman and Weiss in their classical works [17, 18]. Subelliptic Spectral multipliers for $L^p(SU(2))$ were also considered in Cowling and Sikora [14]. Later on, criteria for the L^p -boundedness of Fourier multipliers

¹ Defined for $f \in C_0^\infty(\mathbb{R}^n)$, by $\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx$.

for arbitrary compact Lie groups G were given in [44], with a generalisation in [10, Section 5] to L^p -subelliptic Fourier multipliers.

One of the notable questions when studying the qualitative properties for multipliers on Lie groups is to endow (the spaces of discrete functions on) the unitary dual with a difference structure. So, fixing the unitary dual \widehat{G} of an arbitrary compact Lie group G , when generalising the Marcinkiewicz condition on a Fourier multiplier A on G , associated to a sequence (called the symbol of A) $\sigma := \{\sigma(\xi)\}_{[\xi] \in \widehat{G}}$,²

$$Af(x) \equiv T_\sigma f(x) := \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}[\xi(x)\sigma(\xi)\widehat{f}(\xi)], \quad f \in C^\infty(G), \quad (1.6)$$

the following questions arise:

- (Q1): how to define the difference operators Δ^α on \widehat{G} , in such a way that they generalise the usual notion of difference operators on $\mathbb{Z} \cong \widehat{\mathbb{T}}$?
- (Q2): which is the required order for the differences operators applied to σ in order that A admits a bounded extension on $L^p(G)$?

We note that (Q1) was satisfactorily solved in [44] by introducing a family of difference operators $\Delta^\alpha := \mathbb{D}_\xi^\alpha$, (defined in terms of the Fourier transform on G , as it was done in [40]) in terms of the unitary representations $\xi : G \rightarrow \text{End}(\mathbb{C}^\ell)$ ³ of G . About (Q2), the following Marcinkiewicz type theorem was proved in [44].

Theorem 1.1. *Let us assume that G is a compact Lie group of dimension n . Let $\sigma \in \Sigma(\widehat{G})$ ⁴ be a symbol satisfying*

$$\|\mathbb{D}^\alpha \sigma(\xi)\|_{\text{op}} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}, \quad |\alpha| \leq \varkappa := \left\lfloor \frac{n}{2} \right\rfloor + 1. \quad (1.7)$$

Then $A \equiv T_\sigma$ is of weak type $(1,1)$ and bounded on $L^p(G)$ for all $1 < p < \infty$. Moreover,

$$\|A\|_{\mathcal{B}(L^p(G))}, \|A\|_{\mathcal{B}(L^1(G), L^{1,\infty}(G))} \lesssim \max\{C_\alpha : |\alpha| \leq \varkappa\}. \quad (1.8)$$

Remark 1.2. In the symbol condition (1.7),

$$\text{Spect}((1 + \mathcal{L}_G)^{\frac{1}{2}}) := \{\langle \xi \rangle : [\xi] \in \widehat{G}\},$$

is the system of eigenvalues of the Bessel potential operator $(1 + \mathcal{L}_G)^{\frac{1}{2}}$ associated to the Laplacian on G , which can be defined as follows. Taking an arbitrary orthonormal basis $X_{\mathfrak{g}} := \{X_1, \dots, X_n\}$ of the Lie algebra \mathfrak{g} of G , with respect to the Killing form on \mathfrak{g} , $\mathcal{L}_G := -\sum_{i=1}^n X_i^2$. We refer the reader to Remark 2.1 for details about the definition of the difference operators $\mathbb{D}^\alpha = \mathbb{D}_1^{\alpha_1} \dots \mathbb{D}_n^{\alpha_n}$. They are compositions of differences operators \mathbb{D}_j of first order associated to the entries of the matrix-function $\xi_0(\cdot) - I_{d_{\xi_0}}$ for any choice of a unitary representation in every equivalence class $[\xi_0] \in \widehat{G}$.

²Here, $[\xi]$ denotes the equivalence class of a unitary, irreducible and continuous representation $\xi : G \rightarrow \text{Hom}(\mathbb{C}^{d_\xi})$ on G , d_ξ is the dimension of its representation space, $\sigma(\xi) \in \text{Hom}(\mathbb{C}^{d_\xi})$, and $\widehat{f}(\xi) := \int_G f(x)\xi(x)^* dx$, is the Fourier transform on the group G , of $f \in C^\infty(G)$ at $[\xi]$. For instance, in the case of the torus $G = \mathbb{T}^n$, $\xi(x) := e^{i2\pi x \cdot \xi}$, $x \in \mathbb{T}^n$, $d_\xi \equiv 1$, and so $\widehat{G} \cong \mathbb{Z}^n$.

³We will always write $d_\xi := \ell$ for the dimension of the representation space \mathbb{C}^ℓ . Also, I_ℓ is the identity matrix of size $\ell \times \ell$.

⁴ $\Sigma(\widehat{G})$ is the space of functions $\sigma : \widehat{G} \rightarrow \cup_{\ell \in \mathbb{N}_0} (\text{End}(\mathbb{C}^\ell))$.

Remark 1.3. Theorem 1.1 was proved by using the Hörmander-Mihlin Theorem in [44, Page 630], see also Theorem 2.6.

Let us note that for graded Lie groups (e.g. the Heisenberg group, any stratified group and a wide class of nilpotent Lie groups where Rockland operators exist, see [26] for details) the Hörmander-Mihlin and the Marcinkiewicz conditions for L^p , Triebel-Lizorkin and Hardy spaces have been investigated in [11, 12, 8, 25].

In this work we investigate the Marcinkiewicz condition for multipliers on Triebel-Lizorkin spaces $F_{p,q}^r(G)$ on G , extending in Theorem 1.4 to the case of compact Lie groups, the estimate of Seeger [47] for multipliers in Triebel-Lizorkin spaces $F_{p,q}^r(\mathbb{R}^n)$ on \mathbb{R}^n . In order to present our main result, let us define the Triebel-Lizorkin spaces $F_{p,q}^r(G)$, as they were introduced by the second author, Nursultanov and Tikhonov in [38]. So, let us fix $\eta \in C_0^\infty(\mathbb{R}^+, [0, 1])$, $\eta \neq 0$, so that $\text{supp}(\eta) \subset [1/2, 2]$, and such that

$$\sum_{j \in \mathbb{Z}} \eta(2^{-j}\lambda) = 1, \quad \lambda > 0. \quad (1.9)$$

Fixing $\psi_0(\lambda) := \sum_{j=-\infty}^0 \eta_j(\lambda)$, and for $j \geq 1$, $\psi_j(\lambda) := \eta(2^{-j}\lambda)$, we have

$$\sum_{\ell=0}^{\infty} \psi_\ell(\lambda) = 1, \quad \lambda > 0, \quad (1.10)$$

and one can define the family of operators $\psi_j(\mathcal{B})$ using the functional calculus of the elliptic Bessel potential $\mathcal{B} := (1 + \mathcal{L}_G)^{\frac{1}{2}}$. Then, for $0 < q < \infty$, and $1 < p < \infty$, the Triebel-Lizorkin space $F_{p,q}^r(G)$ consists of the distributions $f \in \mathcal{D}'(G)$ such that

$$\|f\|_{F_{p,q}^r(G)} := \left\| \left(\sum_{\ell=0}^{\infty} 2^{\ell r q} |\psi_\ell(\mathcal{B})f|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} < \infty.$$

The weak- $F_{1,q}^r(G)$ space is defined by the distributions $f \in \mathcal{D}'(G)$ such that

$$\|f\|_{\text{weak-}F_{1,q}^r(G)} := \sup_{t>0} t \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} 2^{\ell r q} |\psi_\ell(\mathcal{B})f(x)|^q \right)^{\frac{1}{q}} > t \right\} \right| < \infty. \quad (1.11)$$

Above, for a measurable subset $A \subset G$, $|A|$ denotes its Haar measure. The main results of this work are Theorem 1.4 below and the Hörmander-Mihlin Theorem 3.1.

Theorem 1.4. *Let us assume that G is a compact Lie group of dimension n . Let $\sigma \in \Sigma(\widehat{G})$ be a symbol satisfying*

$$\|\mathbb{D}^\alpha \sigma(\xi)\|_{\text{op}} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}, \quad |\alpha| \leq \varkappa := \left[\frac{n}{2} \right] + 1. \quad (1.12)$$

Then $A \equiv T_\sigma$ extends to a bounded operator from $F_{p,q}^r(G)$ into $F_{p,q}^r(G)$ for all $1 < p, q < \infty$, and all $r \in \mathbb{R}$. For $p = 1$, A admits a bounded extension from $F_{1,q}^r(G)$ into weak- $F_{1,q}^r(G)$. Moreover

$$\|A\|_{\mathcal{B}(F_{p,q}^r(G))}, \|A\|_{\mathcal{B}(F_{1,q}^r(G), \text{weak-}F_{1,q}^r(G))} \lesssim \max\{C_\alpha : |\alpha| \leq \varkappa\}. \quad (1.13)$$

Remark 1.5. The Marcinkiewicz Theorem 1.4 will be deduced, from the Hörmander-Mihlin Theorem 3.1 for Triebel-Lizorkin spaces. In particular, in view of the Littlewood-Paley theorem (see Furioli, Melzi and Veneruso [29]), we have $L^p(G) = F_{p,2}^0(G)$ for $1 < p < \infty$, so that we recover the L^p -bound in Theorem 1.1. Because we are not assuming that the Fourier multipliers are defined by the spectral calculus, Theorem 3.1 extends the main theorem in Weiss [53] and also the historical 1939's result due to Marcinkiewicz [35] (see Remark 4.7 and Corollary 4.8). As before, we refer the reader to Remark 2.1 for details about the definition of the difference operators $\mathbb{D}^\alpha = \mathbb{D}_1^{\alpha_1} \dots \mathbb{D}_n^{\alpha_n}$.

2. PRELIMINARIES

2.1. The unitary dual and the Fourier transform. First, let us record the notion of the unitary dual \widehat{G} of a compact Lie group G . So, let us assume that ξ is a continuous, unitary and irreducible representation of G , this means that,

- $\xi \in \text{Hom}(G, \text{U}(H_\xi))$, for some finite-dimensional vector space $H_\xi \cong \mathbb{C}^{d_\xi}$, i.e. $\xi(xy) = \xi(x)\xi(y)$ and for the adjoint of $\xi(x)$, $\xi(x)^* = \xi(x^{-1})$, for every $x, y \in G$.
- The map $(x, v) \mapsto \xi(x)v$, from $G \times H_\xi$ into H_ξ is continuous.
- For every $x \in G$, and $W_\xi \subset H_\xi$, if $\xi(x)W_\xi \subset W_\xi$, then $W_\xi = H_\xi$ or $W_\xi = \emptyset$.

Let $\text{Rep}(G)$ be the set of unitary, continuous and irreducible representations of G . The relation,

$$\xi_1 \sim \xi_2 \text{ if and only if, there exists } A \in \text{End}(H_{\xi_1}, H_{\xi_2}), \text{ such that } A\xi_1(x)A^{-1} = \xi_2(x),$$

for every $x \in G$, is an equivalence relation and the unitary dual of G , denoted by \widehat{G} is defined via

$$\widehat{G} := \text{Rep}(G)/\sim.$$

By a suitable changes of basis, we always can assume that every ξ is matrix-valued and that $H_\xi = \mathbb{C}^{d_\xi}$. If a representation ξ is unitary, then

$$\xi(G) := \{\xi(x) : x \in G\}$$

is a subgroup (of the group of matrices $\mathbb{C}^{d_\xi \times d_\xi}$) which is isomorphic to the original group G . Thus the homomorphism ξ allows us to represent the compact Lie group G as a group of matrices. This is the motivation for the term ‘representation’. Here, as usually,

$$\widehat{f}(\xi) \equiv (\mathcal{F}f)(\xi) := \int_G f(x)\xi(x)^* dx \in \mathbb{C}^{d_\xi \times d_\xi}, \quad [\xi] \in \widehat{G},$$

is the matrix-valued Fourier transform of $f \in C_0^\infty(G)$ at $\xi = (\xi_{ij})_{i,j=1}^{d_\xi}$.

2.2. Difference operators. Difference operators on compact Lie groups were introduced in [40] to endow the unitary dual of a compact Lie group with a difference structure. In terms of them, Hörmander classes of pseudo-differential operators on a compact Lie group can be characterised, see [41, 42, 43]. Same as in [44], where differences operators were used to study L^p -multipliers we will extend that analysis to the case of Triebel-Lizorkin spaces (see [37, 38] for instance).

We will denote by $\Sigma(\widehat{G})$ the space of matrix-valued functions,

$$\Sigma(\widehat{G}) := \{\sigma \in \mathcal{F}(\mathcal{D}'(G)) =: \mathcal{D}'(\widehat{G}) \mid \sigma : \widehat{G} \rightarrow \cup_{[\xi] \in \widehat{G}} \mathbb{C}^{d_\xi \times d_\xi}\}.$$

By following [44], a difference operator Q_ξ of order k , can be applied to a symbol $\sigma = \widehat{f} \in \mathcal{D}'(\widehat{G})$, via

$$Q_\xi \sigma(\xi) = \widehat{q}f(\xi), \quad [\xi] \in \widehat{G}, \quad (2.1)$$

where Q_ξ is associated with a smooth function q vanishing of order k at the identity $e = e_G$. We will denote by $\text{diff}^k(\widehat{G})$ the set of all difference operators of order k . For a fixed smooth function q , the associated difference operator will be denoted by $\Delta_q := Q_\xi$. We will choose an admissible collection of difference operators (see e.g. [44]),

$$\Delta_\xi^\alpha := \Delta_{q(1)}^{\alpha_1} \cdots \Delta_{q(i)}^{\alpha_i}, \quad \alpha = (\alpha_j)_{1 \leq j \leq i},$$

where

$$\text{rank}\{\nabla q_{(j)}(e) : 1 \leq j \leq i\} = \dim(G), \quad \text{and } \Delta_{q(j)} \in \text{diff}^1(\widehat{G}).$$

We say that this admissible collection is strongly admissible if

$$\bigcap_{j=1}^i \{x \in G : q_{(j)}(x) = 0\} = \{e_G\}.$$

Remark 2.1. A special type of difference operators can be defined by using the unitary representations of G . Indeed, if ξ_0 is a fixed irreducible and unitary representation of G , consider the matrix

$$\xi_0(g) - I_{d_{\xi_0}} = [\xi_0(g)_{ij} - \delta_{ij}]_{i,j=1}^{d_{\xi_0}}, \quad g \in G. \quad (2.2)$$

Then, we associated to the function $q_{ij}(g) := \xi_0(g)_{ij} - \delta_{ij}$, $g \in G$, a difference operator via

$$\mathbb{D}_{\xi_0, i, j} := \mathcal{F}(\xi_0(g)_{ij} - \delta_{ij}) \mathcal{F}^{-1}. \quad (2.3)$$

If the representation is fixed we omit the index ξ_0 so that, from a sequence $\mathbb{D}_1 = \mathbb{D}_{\xi_0, j_1, i_1}, \dots, \mathbb{D}_n = \mathbb{D}_{\xi_0, j_n, i_n}$ of operators of this type we define $\mathbb{D}^\alpha = \mathbb{D}_1^{\alpha_1} \cdots \mathbb{D}_n^{\alpha_n}$, where $\alpha \in \mathbb{N}^n$.

Remark 2.2 (Leibniz rule for difference operators). The difference structure on the unitary dual \widehat{G} , induced by the difference operators acting on the momentum variable $[\xi] \in \widehat{G}$, implies the following Leibniz rule

$$\mathbb{D}^\alpha(a_1 a_2)(x_0, \xi) = \sum_{|\gamma|, |\varepsilon| \leq |\alpha| \leq |\gamma| + |\varepsilon|} C_{\varepsilon, \gamma}(\mathbb{D}^\gamma a_1)(x_0, \xi) (\mathbb{D}^\varepsilon a_2)(x_0, \xi), \quad (x_0, [\xi]) \in G \times \widehat{G},$$

for $a_1, a_2 \in C^\infty(G, \mathcal{S}'(\widehat{G}))$. For details we refer the reader to [44, Page 631].

Remark 2.3 (Difference operators of fractional order and Sobolev spaces on the unitary dual). In the spirit of the Sobolev spaces on the unitary dual of a graded Lie group [24], Sobolev spaces also can be defined for the unitary dual of a compact Lie group. They can be defined as follows: Let Δ_{q_1} , be a difference operator of first order associated to a smooth and non-negative function $q_1 \geq 0$. For $s \in \mathbb{R}$, the Sobolev space $\dot{L}_s^2(\widehat{G})$ consists of all distributions $\sigma = \widehat{f} \in \mathcal{D}'(G)$ such that

$$\|\sigma\|_{\dot{L}_s^2(\widehat{G})} := \|f\|_{L^2(G, q_1^{2s})} = \|q_1^s f\|_{L^2(G)} < \infty. \quad (2.4)$$

For $s \in \mathbb{N}$, observe that,

$$\|\sigma\|_{\dot{L}_s^2(\widehat{G})} \asymp \max_{|\alpha|=s} \|\Delta_\xi^\alpha \sigma\|_{L^2(\widehat{G})} \asymp \max_{|\alpha|=s} \|\mathbb{D}_\xi^\alpha \sigma\|_{L^2(\widehat{G})}. \quad (2.5)$$

So, for every $s \in \mathbb{R}$, the difference operator $\Delta_{q_1}^s := \Delta_{q_1^s}$ of fractional order s , can be defined in terms of the Fourier transform, via:

$$\Delta_{q_1}^s \widehat{f} = q_1^s \widehat{f}, \quad f \in \mathcal{D}'(G). \quad (2.6)$$

So, we have $\|\sigma\|_{\dot{L}_s^2(\widehat{G})} := \|\Delta_{q_1}^s \widehat{f}\|_{L^2(\widehat{G})}$. We will denote

$$\text{diff}^s(\widehat{G}) := \{\Delta_{q_1^s} : \Delta_{q_1} \in \text{diff}^1(\widehat{G})\}. \quad (2.7)$$

2.3. Calderón-Zygmund type estimates for multipliers. In order to provide L^p -estimates for multipliers in the subelliptic context, we will use the techniques developed by the second author and J. Wirth in [44], where a special case (compatible with the notion of difference operators and the difference structure that they provide for the unitary dual) of a statement of Coifman and de Guzmán ([16], Theorem 2) was established. We record it as follows (see [44, p. 630]).

Theorem 2.4 (Hörmander-Mihlin Theorem for $L^p(G)$). *Assume that $A : L^2(G) \rightarrow L^2(G)$ is a left-invariant operator on G satisfying*

$$\|A\psi_{r^{-1}}\|_{L^2(G, \rho(x)^{n(1+\varepsilon)} dx)} := \left(\int_G |A\psi_{r^{-1}}(x)|^2 \rho(x)^{n(1+\varepsilon)} dx \right)^{\frac{1}{2}} \lesssim Cr^{-\frac{\varepsilon}{2}}, \quad (2.8)$$

for some $\varepsilon > 0$, uniformly in $r > 0$. Then A is of weak type $(1, 1)$ and bounded on $L^p(G)$, for all $1 < p < \infty$.

The family $\{\psi_r\}_{r>0}$ that appears in Theorem 2.4 is defined by

$$\psi_r := \phi_r - \phi_{r/2}, \quad (2.9)$$

where the functions in the net $\{\phi_r\}_{r>0}$, satisfy, among other things, the following properties (see [44, Lemma 3.3]):

- $\int_G \phi_r(x) dx = 1$,
- $\int_G \phi_r^2(x) dx = O(\frac{1}{r})$,
- $\phi_r * \phi_s = \phi_s * \phi_r$, $r, s > 0$.

The function $\rho : x \mapsto \rho(x)$, appearing in (2.8), is a suitable pseudo-distance defined on G . If G is semi-simple (this means that the centre of G , $Z(G)$ is trivial), it is defined by

$$\rho(x)^2 := \dim(G) - \mathbf{Tr}(\text{Ad}(x)) = \sum_{\xi \in \Delta_0} (d_\xi - \mathbf{Tr}(\xi(x))), \quad x \in G, \quad (2.10)$$

where $\text{Ad} : G \rightarrow \text{U}(\mathfrak{g})$, and Δ_0 is the system of positive roots. It can be decomposed into irreducible representations as,

$$\text{Ad} = [\text{rank}(G)e_{\widehat{G}}] \oplus \left(\bigoplus_{\xi \in \Delta_0} \xi \right),$$

where $e_{\widehat{G}}$ is the trivial representation. With the consideration on the centre $Z(G) = \{e_G\}$, it can be shown (see Lemma 3.1 of [44]) that

- $\rho^2(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = e_G$.
- $\Delta_{\rho^2} \in \text{diff}^2(\widehat{G})$.

If G is not semi-simple, we refer the reader to [44, Remark 3.2] for the modifications in the definition of ρ , in this particular case. The construction of the functions ϕ_r , is as follows. By choosing $\tilde{\phi} \in C_0^\infty(\mathbb{R})$ such that $\tilde{\phi} \geq 0$, $\tilde{\phi}(0) \equiv 1$, and $\partial_t^\ell \tilde{\phi}(0) = 0$, for $\ell \geq 1$, we define

$$\psi_r(g) := c_r \tilde{\psi}(r^{-\frac{1}{n}} \rho(g)), \quad \int_G \psi_r(g) dg = 1, \quad (2.11)$$

with the normalisation condition used to define c_r . An equivalent statement to the Hörmander-Mihlin Theorem 2.4 will be given in terms of the family of functions $\varphi_{r^{-1}} \equiv \psi_{r^{-n}}$, $r > 0$, (see Theorem 2.6) that provide a continuous dyadic decomposition.

2.4. $L^p(G)$ -boundedness of Fourier multipliers. In this subsection we recall the Hörmander-Mihlin theorem and the Marcinkiewicz-Mihlin theorem for L^p -multipliers on compact Lie groups [44].

Remark 2.5. Let A be a Fourier multiplier with symbol σ . Let us observe that that Theorem 2.4 can be re-written in terms of the Sobolev spaces $\dot{L}_s^2(\widehat{G})$ on the unitary dual, showing that, (2.8) is indeed, an analogue on compact Lie groups of the Hörmander-Mihlin theorem in [34]. So, with the notation in Theorem 2.4, and making use of the Plancherel theorem, we have

$$\begin{aligned} \|A\psi_r\|_{L^2(G, \rho(x)^{n(1+\varepsilon)} dx)}^2 &:= \int_G |A\psi_r(x)|^2 \rho(x)^{n(1+\varepsilon)} dx \\ &= \|\Delta_{\rho(x)}^{n(1+\varepsilon)/2} \sigma(\xi) \widehat{\psi}_r(\xi)\|_{L^2(\widehat{G})}^2 \\ &:= \|\sigma(\xi) \widehat{\psi}_r(\xi)\|_{L_s^2(\widehat{G})}^2, \end{aligned}$$

where $s := n(1 + \varepsilon)/2 > n/2$. Observing that $\frac{\varepsilon}{2} = \frac{1}{n}(s - \frac{n}{2})$, and that $A : L^2(G) \rightarrow L^2(G)$ is bounded if and only if, $\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{[\xi] \in \widehat{G}} \|\sigma(\xi)\|_{\text{op}} < \infty$, Theorem 2.4 is equivalent to the following theorem. We set

$$\varphi_{r^{-1}} \equiv \psi_{r^{-n}} := \phi_{r^{-n}} - \phi_{r^{-n}/2}, \quad r > 0,$$

for the continuous dyadic approximation of the identity in (2.9).

Theorem 2.6 (Hörmander-Mihlin Theorem for $L^p(G)$). *Let G be a compact Lie group and let $s > \frac{n}{2}$. Let $\sigma \in \Sigma(\widehat{G})$ be a symbol satisfying*

$$\|\sigma\|_{l.u., L_s^2(\widehat{G})} := \|\sigma\|_{L^\infty(\widehat{G})} + \sup_{r>0} r^{(s-\frac{n}{2})} \|\sigma \cdot \widehat{\varphi}_{r^{-1}}\|_{L_s^2(\widehat{G})} < \infty. \quad (2.12)$$

Then $A \equiv T_\sigma$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for all $1 < p < \infty$.

We present Theorem 1.1 in the following form.

Theorem 2.7 (Marcinkiewicz-Mihlin Theorem). *Let G be a compact Lie group and let $\varkappa \in 2\mathbb{N}$ be such that $\varkappa > \frac{n}{2}$. Let $\sigma \in \Sigma(\widehat{G})$ be a symbol satisfying*

$$\|\mathbb{D}^\alpha \sigma(\xi)\|_{\text{op}} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}, \quad |\alpha| \leq \varkappa.$$

Then $A \equiv T_\sigma$ is of weak type $(1,1)$ and bounded on $L^p(G)$ for all $1 < p < \infty$.

Remark 2.8. Several properties for the Sobolev spaces $L^2_s(\widehat{G})$ on the unitary dual were established in [23]. In particular, the Hörmander-Mihlin Theorem 2.6 was re-obtained, by observing that the $L^p(G)$ -boundedness and the weak $(1,1)$ type of a Fourier multiplier, also can be obtained if one verifies the following condition:

$$\|\sigma\|'_{l.u., L^2_s(\widehat{G})} := \|\sigma\|_{L^\infty(\widehat{G})} + \sup_{r>0} r^{(s-\frac{n}{2})} \|\sigma \cdot \eta(r^{-1}\langle \xi \rangle)\|_{L^2_s(\widehat{G})} < \infty, \quad (2.13)$$

for any $\eta \neq 0$, $\eta \in C_0^\infty(\mathbb{R}_0^+)$, and $s > n/2$. Both Hörmander-Mihlin conditions ((2.12) or (2.13)) allow us to write Theorem 2.7 in the form of Theorem 1.1.

Remark 2.9. The boundedness of pseudo-differential operators on compact Lie groups, including oscillating Fourier multipliers, on L^p , subelliptic Sobolev and Besov spaces can be found in [8, 15] and [5, 6, 12]. The boundedness of Fourier multipliers on L^p -spaces, Triebel-Lizorkin spaces and Hardy spaces on graded Lie groups can be found in [25, 8] and [33], respectively.

2.5. Triebel-Lizorkin spaces on compact Lie groups. In this section, Triebel-Lizorkin spaces on compact Lie groups are introduced. This family of spaces was introduced by the second author, Nursultanov and Tikhonov in [38] by using a dyadic partition of the spectral resolution of the Bessel potential $\mathcal{B} = (1 + \mathcal{L}_G)^{\frac{1}{2}}$.

As in the introduction, let us fix $\eta \in C_0^\infty(\mathbb{R}^+, [0, 1])$, $\eta \neq 0$, so that $\text{supp}(\eta) \subset [1/2, 2]$, and such that

$$\sum_{j \in \mathbb{Z}} \eta(2^{-j}\lambda) = 1, \quad \lambda > 0. \quad (2.14)$$

Fixing $\psi_0(\lambda) := \sum_{j=-\infty}^0 \eta_j(\lambda)$, and for $j \geq 1$, $\psi_j(\lambda) := \eta(2^{-j}\lambda)$, we have

$$\sum_{\ell=0}^{\infty} \psi_\ell(\lambda) = 1, \quad \lambda > 0. \quad (2.15)$$

Let us define the family of operators $\psi_j(\mathcal{B})$ using the functional calculus. Then, for $0 < q < \infty$, and $1 < p < \infty$, the Triebel-Lizorkin space $F_{p,q}^r(G)$ consists of the distributions $f \in \mathcal{D}'(G)$ such that

$$\|f\|_{F_{p,q}^r(G)} := \left\| \left(\sum_{\ell=0}^{\infty} 2^{\ell r q} |\psi_\ell(\mathcal{B})f|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} < \infty,$$

and for $p = 1$, the weak- $F_{1,q}^r(G)$ space is defined by the distributions $f \in \mathcal{D}'(G)$ such that

$$\|f\|_{\text{weak-}F_{1,q}^r(G)} := \sup_{t>0} t \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} 2^{\ell r q} |\psi_\ell(\mathcal{B})f(x)|^q \right)^{\frac{1}{q}} > t \right\} \right| < \infty. \quad (2.16)$$

In the following theorem we present some embedding properties for Triebel-Lizorkin spaces. For the proof of Theorem 2.10 and for a consistent investigation of Triebel-Lizorkin spaces on compact Lie groups, we refer the reader to [38].

Theorem 2.10. *Let G be a compact Lie group. Then we have the following properties:*

- (1) $F_{p,q_1}^{r+\varepsilon,\mathcal{L}}(G) \hookrightarrow F_{p,q_1}^{r,\mathcal{L}}(G) \hookrightarrow F_{p,q_2}^{r,\mathcal{L}}(G) \hookrightarrow F_{p,\infty}^{r,\mathcal{L}}(G)$, $\varepsilon > 0$, $0 \leq p \leq \infty$, $0 \leq q_1 \leq q_2 \leq \infty$.
- (2) $F_{p,q_1}^{r+\varepsilon,\mathcal{L}}(G) \hookrightarrow F_{p,q_2}^{r,\mathcal{L}}(G)$, $\varepsilon > 0$, $0 \leq p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.
- (3) $F_{p,2}^r(G) = L_r^p(G)$ for all $r \in \mathbb{R}$, and all $1 < p < \infty$, where $L_r^p(G)$ are the standard Sobolev spaces on G .

Remark 2.11. Let us observe that (3) in Theorem 2.10 is a consequence of the Littlewood-Paley theorem. For details we refer the reader to Furioli, Melzi and Veneruso [29].

3. PROOF OF THE MARCINKIEWICZ THEOREM 1.4

Let us fix $\eta \in C_0^\infty(\mathbb{R}^+, [0, 1])$, $\eta \neq 0$, so that $\text{supp}(\eta) \subset [1/2, 2]$, and such that

$$\sum_{j \in \mathbb{Z}} \eta(2^{-j}\lambda) = 1, \quad \lambda > 0. \quad (3.1)$$

By defining $\psi_0(\lambda) := \sum_{j=-\infty}^0 \eta_j(\lambda)$, and for $j \geq 1$, $\psi_j(\lambda) := \eta(2^{-j}\lambda)$, we obviously have

$$\sum_{\ell=0}^{\infty} \psi_\ell(\lambda) = 1, \quad \lambda > 0. \quad (3.2)$$

First, let us deduce the proof of Theorem 1.4 from the following result, which will be proved in Section 4. The distribution $\vartheta_t := \eta(t\mathcal{B})\delta$ denotes the right-convolution kernel of the operator $\eta(t\mathcal{B})$.

Theorem 3.1 (Hörmander-Mihlin Theorem for $F_{p,q}^r(G)$). *Let us consider a Fourier multiplier $A : C^\infty(G) \rightarrow L^2(G)$ satisfying the symbol condition*

$$\|\sigma\|'_{l.u., L_s^2(\widehat{G})} := \|\sigma\|_{L^\infty(\widehat{G})} + \sup_{r>0} r^{(s-\frac{n}{2})} \|\sigma \cdot \eta(r^{-1}\langle \xi \rangle)\|_{L_s^2(\widehat{G})} < \infty, \quad (3.3)$$

with $s > \frac{n}{2}$. Then $A \equiv T_\sigma$ extends to a bounded operator from $F_{p,q}^r(G)$ into $F_{p,q}^r(G)$ for all $1 < p, q < \infty$, and all $r \in \mathbb{R}$. For $p = 1$, A admits a bounded extension from $F_{1,q}^r(G)$ into weak- $F_{1,q}^r(G)$. Moreover

$$\|A\|_{\mathcal{B}(F_{p,q}^r(G))}, \|A\|_{\mathcal{B}(F_{1,q}^r(G), \text{weak-}F_{1,q}^r(G))} \lesssim \|\sigma\|'_{l.u., L_s^2(\widehat{G})}. \quad (3.4)$$

Remark 3.2. Note that the Fourier transform of the distribution $\vartheta_t := \eta(t\mathcal{B})\delta$ is given by

$$\widehat{\vartheta}_t(\xi) := \eta(t\langle \xi \rangle) I_{d_\xi}, \quad [\xi] \in \widehat{G}, \quad (3.5)$$

and that $A : L^2(G) \rightarrow L^2(G)$ is bounded if and only if $\|\sigma\|_{L^\infty(\widehat{G})} < \infty$. Let $\Delta_{q_1^s} \in \text{diff}^s(\widehat{G})$ be a fractional difference operator of order $s := \frac{n+\varepsilon}{2}$, with $\varepsilon > 0$. If $A :$

$L^2(G) \rightarrow L^2(G)$ is bounded and it satisfies the (Coifman-Weiss type) condition

$$\|A\vartheta_{r^{-1}}\|_{L^2(G, q_1(x)^{\varepsilon+n} dx)} := \left(\int_G |A\vartheta_{r^{-1}}(x)|^2 q_1(x)^{\varepsilon+n} dx \right)^{\frac{1}{2}} \lesssim_{\varepsilon} Cr^{-\frac{1}{2}\varepsilon}, \quad (3.6)$$

uniformly in $r > 0$, the Plancherel Theorem makes the hypothesis in Theorem 3.1 equivalent to (3.6). Moreover, (3.4) is equivalent to

$$\|A\|_{\mathcal{B}(F_{p,q}^r(G))}, \|A\|_{\mathcal{B}(F_{1,q}^r(G), \text{weak-}F_{1,q}^r(G))} \lesssim \|\sigma\|_{L^\infty(\widehat{G})} + \sup_{r>0} r^{s-\frac{n}{2}} \|A\vartheta_{r^{-1}}\|_{L^2(G, q_1(x)^{2s} dx)}, \quad (3.7)$$

in view of (3.5).

For our further analysis we will use the following auxiliary estimate.

Lemma 3.3. *Let A be a Fourier multiplier satisfying the hypothesis in Theorem 1.4. Then, the following estimate holds,*

$$\|\sigma(\xi)\psi_\ell(\langle\xi\rangle)\|_{\dot{L}_s^2(\widehat{G})} \lesssim_{s,\psi} 2^{-\ell(s-\frac{n}{2})} \sup_{|\alpha|\leq s} \sup_{\xi\in\widehat{G}} \|(\Delta_\xi^\alpha \sigma(\xi))\langle\xi\rangle^{|\alpha|}\|_{\text{op}}, \quad \ell \in \mathbb{Z}, \quad (3.8)$$

provided that $s > n/2$, $s \in \mathbb{N}$. Moreover, the right-convolution kernel κ_ℓ of $A\psi_\ell$, satisfies the uniform estimate in $\ell \in \mathbb{Z}$,

$$\int_{|x|>4c|z|} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| dx \lesssim 2^{-\ell(s-\frac{n}{2})} \|\sigma\|'_{l.u., \dot{L}_s^2(\widehat{G})}. \quad (3.9)$$

Proof. That (3.9) is a consequence of (3.8) was proved in [23, Page 38]. So, let us continue with the first part of the lemma. Let us observe that, (in view of Proposition 4.13 in [23]), we have the following norm estimates

$$\|\sigma\tau\|_{\dot{L}_s^2(\widehat{G})} \lesssim_s \sum_{s_1+s_2=s} \|\sigma\|_{\dot{L}_{s_1}^\infty(\widehat{G})} \|\tau\|_{\dot{L}_{s_2}^2(\widehat{G})}, \quad \|\sigma\|_{\dot{L}_{s_1}^\infty(\widehat{G})} := \max_{|\alpha|=s_1} \|\Delta_\xi^\alpha \sigma\|_{L^\infty(\widehat{G})}, \quad (3.10)$$

in the sense that if the right-hand side is finite then the left-hand side is finite and the inequality holds. So, applying (3.10) to $\tau = \psi_\ell(\langle\xi\rangle)I_{d_\xi}$, we have

$$\|\sigma\psi_\ell(\langle\xi\rangle)\|_{\dot{L}_s^2(\widehat{G})} \lesssim_s \sum_{s_1+s_2=s} \max_{|\alpha|=s_1} \|\Delta_\xi^\alpha \sigma\|_{L^\infty(\widehat{G})} \|\psi_\ell(\langle\xi\rangle)I_{d_\xi}\|_{\dot{L}_{s_2}^2(\widehat{G})}.$$

Observe that,

$$\|\Delta_\xi^\alpha \sigma\|_{L^\infty(\widehat{G})} \leq \|\Delta_\xi^\alpha \sigma\langle\xi\rangle^{|\alpha|}\|_{L^\infty(\widehat{G})} \|\widehat{\mathcal{B}}(\xi)^{-|\alpha|}\|_{L^\infty(\widehat{G})} \leq \|\Delta_\xi^\alpha \sigma\langle\xi\rangle^{|\alpha|}\|_{L^\infty(\widehat{G})}.$$

Also, in view of Lemma 5.4 of [23], for any positive and smooth function $f \in C_0^\infty(\mathbb{R})$, supported in $[0, a]$ with $a > 0$, one has

$$\|f(t\langle\xi\rangle)I_{d_\xi}\|_{\dot{L}_s^2(\widehat{G})} \lesssim_s t^{s-\frac{n}{2}} \|f\|_{H^{s'}(\mathbb{R})}, \quad s' > s + 1/2, \quad 0 < t \leq 1, \quad (3.11)$$

which applied to $f = \psi$ and $t = 2^{-\ell}$, gives

$$\|\psi_\ell(\langle\xi\rangle)I_{d_\xi}\|_{\dot{L}_{s_2}^2(\widehat{G})} \lesssim \|\psi_\ell(\langle\xi\rangle)I_{d_\xi}\|_{\dot{L}_s^2(\widehat{G})} \lesssim_s 2^{-\ell(s-\frac{n}{2})} \|\psi\|_{H^{s'}(\mathbb{R})}.$$

The analysis above shows that

$$\|\sigma(\xi)\psi_\ell(\langle\xi\rangle)\|_{\dot{L}_s^2(\widehat{G})} \lesssim_s \sup_{|\alpha|\leq s} \|\Delta_\xi^\alpha \sigma\langle\xi\rangle^{|\alpha|}\|_{L^\infty(\widehat{G})} 2^{-\ell(s-\frac{n}{2})}.$$

Thus, the proof is complete. \square

Proof of Theorem 1.4. In view of Lemma 3.3, we have that

$$\begin{aligned} \|\sigma\|'_{l.u., L^2_s(\widehat{G})} &:= \|\sigma\|_{L^\infty(\widehat{G})} + \sup_{r>0} r^{(s-\frac{n}{2})} \|\sigma \cdot \eta(r^{-1}\langle \xi \rangle)\|_{L^2_s(\widehat{G})} \\ &\lesssim_{s,\psi} \sup_{|\alpha| \leq s} \sup_{[\xi] \in \widehat{G}} \|(\Delta_\xi^\alpha \sigma(\xi)) \langle \xi \rangle^{|\alpha|}\|_{\text{op}} \lesssim \sup_{|\alpha| \leq s} C_\alpha. \end{aligned}$$

In view of the Hörmander-Mihlin Theorem 3.1 applied to $s := [\frac{n}{2}] + 1$, $A \equiv T_\sigma$ extends to a bounded operator from $F_{p,q}^r(G)$ into $F_{p,q}^r(G)$ for all $1 < p, q < \infty$, and all $r \in \mathbb{R}$. For $p = 1$, A admits a bounded extension from $F_{1,q}^r(G)$ into weak- $F_{1,q}^r(G)$. \square

4. PROOF OF THE HÖRMANDER-MIHLIN THEOREM 3.1

Let us define the vector-valued operator $W : L^2(G, \ell^2(\mathbb{N}_0)) \rightarrow L^2(G, \ell^2(\mathbb{N}_0))$ by

$$W(\{g_\ell\}_{\ell=0}^\infty) := (\{W_\ell g_\ell\}_{\ell=0}^\infty), \quad W_\ell := A\psi_\ell(\mathcal{B}). \quad (4.1)$$

Observe that W is well-defined (bounded from $L^2(G, \ell^2(\mathbb{N}_0))$ into $L^2(G, \ell^2(\mathbb{N}_0))$), because A admits a bounded extension on $L^2(G)$ and also, in view of the following estimate

$$\begin{aligned} \|W(\{g_\ell\}_{\ell=0}^\infty)\|_{L^2(G, \ell^2(\mathbb{N}_0))}^2 &:= \int_G \sum_{\ell=0}^\infty |A\psi_\ell(\mathcal{B})g_\ell(x)|^2 dx = \sum_{\ell=0}^\infty \int_G |A\psi_\ell(\mathcal{B})g_\ell(x)|^2 dx \\ &\leq \|A\|_{\mathcal{B}(L^2(G))}^2 \sup_\ell \|\psi_\ell(\mathcal{B})\|_{\mathcal{B}(L^2(G))}^2 \sum_{\ell=0}^\infty \int_G |g_\ell(x)|^2 dx \\ &\leq \|A\|_{\mathcal{B}(L^2(G))}^2 \sup_\ell \|\psi_\ell\|_{L^\infty(\mathbb{R})}^2 \sum_{\ell=0}^\infty \int_G |g_\ell(x)|^2 dx \\ &= \|A\|_{\mathcal{B}(L^2(G))}^2 \|\psi\|_{L^\infty(\mathbb{R})}^2 \sum_{\ell=0}^\infty \int_G |g_\ell(x)|^2 dx \\ &\lesssim \|\{g_\ell\}_{\ell=0}^\infty\|_{L^2(G, \ell^2(\mathbb{N}_0))}^2. \end{aligned}$$

So, observe that in order to prove Theorem 3.1 it is enough to prove the following two lemmas and the estimate (3.4).

Lemma 4.1. *Let G be a compact Lie group and let $1 < q < \infty$. Then, $W : L^q(G, \ell^q(\mathbb{N}_0)) \rightarrow L^q(G, \ell^q(\mathbb{N}_0))$ admits a bounded extension.*

Lemma 4.2. *Let G be a compact Lie group and let $1 < q < \infty$. Then, $W : L^1(G, \ell^q(\mathbb{N}_0)) \rightarrow L^{1,\infty}(G, \ell^q(\mathbb{N}_0))$ admits a bounded extension.*

Corollary 4.3. *Let G be a compact Lie group and let $1 < p, q < \infty$. Then, $W : L^p(G, \ell^q(\mathbb{N}_0)) \rightarrow L^p(G, \ell^q(\mathbb{N}_0))$ admits a bounded extension.*

Proof. Indeed, by the Marcinkiewicz interpolation, these two lemmas are enough to show that $W : L^p(G, \ell^q(\mathbb{N}_0)) \rightarrow L^p(G, \ell^q(\mathbb{N}_0))$ admits a bounded extension for all $1 < p \leq q < \infty$. The case $1 < q \leq p < \infty$ follows from the fact that $L^{p'}(G, \ell^{q'}(\mathbb{N}_0))$ is the dual of $L^p(G, \ell^q(\mathbb{N}_0))$ and also that Lemma 4.1 and Lemma 4.2 hold if we change A by its standard L^2 -adjoint. \square

Indeed, we clarify this claim in the following remark.

Remark 4.4 (Lemma 4.2 and Corollary 4.3 imply Theorem 3.1). By observing that $(1 + \mathcal{B})^{\frac{r}{2}} : F_{p,q}^r(G) \rightarrow F_{p,q}^0(G)$ and $(1 + \mathcal{B})^{-\frac{r}{2}} : F_{p,q}^0(G) \rightarrow F_{p,q}^r(G)$ are both isomorphisms, it is enough to prove that A admits a bounded extension from $F_{p,q}^0(G)$ into $F_{p,q}^0(G)$. By defining $\psi_{-1} = \psi_0$, from Corollary 4.3 we have that

$$\begin{aligned} \|Af\|_{F_{p,q}^0(G)} &= \left\| \left(\sum_{l=0}^{\infty} |A\psi_l(\mathcal{B})f|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} \\ &\leq \left\| \left(\sum_{l=0}^{\infty} |A\psi_l(\mathcal{B})[\psi_{l-1}(\mathcal{B}) + \psi_l(\mathcal{B}) + \psi_{l+1}(\mathcal{B})]f|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} \\ &= \|W(\{[\psi_{l-1}(\mathcal{B}) + \psi_l(\mathcal{B}) + \psi_{l+1}(\mathcal{B})]f\}_{l=0}^{\infty})\|_{L^p(\ell^q)} \\ &\lesssim \|\{[\psi_{l-1}(\mathcal{B}) + \psi_l(\mathcal{B}) + \psi_{l+1}(\mathcal{B})]f\}_{l=0}^{\infty}\|_{L^p(\ell^q)} \\ &\lesssim \|f\|_{F_{p,q}^0(G)}. \end{aligned}$$

Also note that from Lemma 4.2, we have

$$\begin{aligned} \|Af\|_{\text{weak-}F_{1,q}^0(G)} &= \left\| \left(\sum_{l=0}^{\infty} |A\psi_l(\mathcal{B})f|^q \right)^{\frac{1}{q}} \right\|_{L^{1,\infty}(G)} \\ &\leq \left\| \left(\sum_{l=0}^{\infty} |A\psi_l(\mathcal{B})[\psi_{l-1}(\mathcal{B}) + \psi_l(\mathcal{B}) + \psi_{l+1}(\mathcal{B})]f|^q \right)^{\frac{1}{q}} \right\|_{L^{1,\infty}(G)} \\ &= \|W(\{[\psi_{l-1}(\mathcal{B}) + \psi_l(\mathcal{B}) + \psi_{l+1}(\mathcal{B})]f\}_{l=0}^{\infty})\|_{L^{1,\infty}(\ell^q)} \\ &\lesssim \|\{[\psi_{l-1}(\mathcal{B}) + \psi_l(\mathcal{B}) + \psi_{l+1}(\mathcal{B})]f\}_{l=0}^{\infty}\|_{L^1(\ell^q)} \lesssim \|f\|_{F_{1,q}^0(G)}. \end{aligned}$$

By having observed that Lemmas 4.1 and 4.2 imply Corollary 4.3, and that these lemmas are enough for proving Theorem 3.1, we will proceed with their proofs. For this, let us recall that on any amenable topological group, and also in several spaces of homogeneous type one has the Calderón-Decomposition lemma.

Remark 4.5. For any non-negative function $f \in L^1(G)$, one has its Calderón-Zygmund decomposition. Indeed, by following Hebisch [31], (whose construction remains valid for any amenable group, in particular, compact Lie groups) one can obtain a suitable family of disjoint open sets $\{I_j\}_{j=0}^{\infty}$ such that

- $f(x) \leq t$, for a.e. $x \in G \setminus \cup_{j \geq 0} I_j$,
- $\sum_{j \geq 0} |I_j| \leq \frac{C}{t} \|f\|_{L^1(G)}$, and
- $t|I_j| \leq \int_{I_j} f(x) dx \leq 2|I_j|t$, for all j .

Now, for every $j \in \mathbb{N}_0$, let us define R_j by

$$R_j := \sup\{R > 0 : B(z_j, R) \subset I_j, \text{ for some } z_j \in I_j\}, \quad (4.2)$$

where $B(z_j, R) = \{x \in I_j : |z_j^{-1}x| < R\}$. Every I_j is bounded, and one can assume that $I_j \subset B(z_j, 2R_j)$, where $z_j \in I_j$. Let us note that by assuming $f(e_G) > t$, (this can be done, just by re-defining $f \in L^1(G)$ at the identity element e_G of G) we should have that

$$e_G \in \bigcup_j I_j, \quad (4.3)$$

because $f(x) \leq t$, for a.e. $x \in G \setminus \bigcup_{j \geq 0} I_j$.

Let us define, for every $x \in I_j$,

$$g(x) := \frac{1}{|I_j|} \int_{I_j} f(y) dy, \quad b(x) = f(x) - g(x), \quad (4.4)$$

and for $x \in G \setminus \bigcup_{j \geq 0} I_j$, $g(x) = f(x)$, $b(x) = 0$. Then, one has the decomposition $f = g + b$, with b having null average on I_j .

Proof of Lemma 4.1. Let us fix $f \in L^1(G)$ to be a non-negative function, and let us consider its Calderón-Zygmund decomposition $f = g + b$ as in Remark 4.5. It is enough to demonstrate that the linear operators W_ℓ , $\ell \in \mathbb{N}_0$, are uniformly bounded on $L^q(G)$, $1 < q < \infty$. This fact is straightforward if $q = 2$, so it suffices (by the duality argument) that the operators W_ℓ are uniformly bounded from $L^1(G)$ into $L^{1,\infty}(G)$. So, we will prove the existence of $C > 0$, independent of $f \in L^1(G)$, and $\ell \in \mathbb{N}_0$, such that

$$|\{x \in G : |W_\ell f(x)| > t\}| \leq \frac{C}{t} \|f\|_{L^1(G)}. \quad (4.5)$$

Let us remark that for every $x \in I_j$,

$$|g(x)| = \left| \frac{1}{|I_j|} \int_{I_j} f(y) dy \right| \leq 2t.$$

By the Minkowski inequality, we have

$$\begin{aligned} |\{x \in G : |W_\ell f(x)| > t\}| &\leq \left| \left\{ x \in G : |W_\ell g(x)| > \frac{t}{2} \right\} \right| \\ &\quad + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right|. \end{aligned}$$

In view of the Chebyshev inequality, we get

$$\begin{aligned} &|\{x \in G : |W_\ell f(x)| > t\}| \\ &\leq \left| \left\{ x \in G : |W_\ell g(x)| > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \\ &= \left| \left\{ x \in G : |W_\ell g(x)|^2 > \frac{t^2}{2^2} \right\} \right| + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \\ &\leq \frac{2^2}{t^2} \int_G |W_\ell g(x)|^2 dx + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{2^2}{t^2} \sup_{\ell} \|W_{\ell}\|_{\mathcal{B}(L^2(G))} \int_G |g(x)|^2 dx + \left| \left\{ x \in G : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right| \\ &\lesssim \frac{2^2}{t^2} \int_G |g(x)|^2 dx + \left| \left\{ x \in G : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right|, \end{aligned}$$

in view of the $L^2(G)$ -boundedness of A and the fact that the operators $\psi_{\ell}(\mathcal{B})$ are $L^2(G)$ -bounded uniformly in $\ell \in \mathbb{N}_0$. Additionally, note that the estimate

$$\begin{aligned} \|g\|_{L^2(G)}^2 &= \int_G |g(x)|^2 dx = \sum_j \int_{I_j} |g(x)|^2 dx + \int_{G \setminus \cup_j I_j} |g(x)|^2 dx \\ &= \sum_j \int_{I_j} |g(x)|^2 dx + \int_{G \setminus \cup_j I_j} |f(x)|^2 dx \\ &\leq \sum_j \int_{I_j} (2t)^2 dx + \int_{G \setminus \cup_j I_j} f(x)^2 dx \lesssim t^2 \sum_j |I_j| + \int_{G \setminus \cup_j I_j} f(x)f(x) dx \\ &\leq t^2 \times \frac{C}{t} \|f\|_{L^1(G)} + t \int_{G \setminus \cup_j I_j} f(x) dx \lesssim t \|f\|_{L^1(G)}, \end{aligned}$$

implies that

$$\left| \left\{ x \in G : |W_{\ell}f(x)| > t \right\} \right| \leq \frac{4}{t} \|f\|_{L^1(G)} + \left| \left\{ x \in G : |W_{\ell}b(x)|^2 > \frac{t}{2} \right\} \right|.$$

Taking into account that $b \equiv 0$ on $G \setminus \cup_j I_j$, we have that

$$b = \sum_k b_k, \quad b_k(x) = b(x) \cdot \mathbf{1}_{I_k}(x). \quad (4.6)$$

Let us assume that I_j^* is an open set, such that $I_j \subset I_j^*$, and $|I_j^*| = K|I_j|$ for some $K > 0$, and $\text{dist}(\partial I_j^*, \partial I_j) \geq 4c \text{dist}(\partial I_j, e_G)$, where $c > 0$ and e_G is the identity element of G . So, by the Minkowski inequality we have

$$\begin{aligned} &\left| \left\{ x \in G : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right| \\ &= \left| \left\{ x \in \cup_j I_j^* : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right| \\ &\leq |\{x \in G : x \in \cup_j I_j^*\}| + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right|. \end{aligned}$$

Consequently, we deduce the estimates

$$\begin{aligned} &\left| \left\{ x \in G : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right| \leq \sum_j |I_j^*| + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right| \\ &= K \sum_j |I_j| + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_{\ell}b(x)| > \frac{t}{2} \right\} \right| \end{aligned}$$

$$\leq \frac{CK}{t} \|f\|_{L^1(G)} + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right|.$$

Now, using the Chebyshev inequality to estimate the right hand side above we obtain

$$\begin{aligned} & \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \leq \frac{2}{t} \int_{G \setminus \cup_j I_j^*} |W_\ell b(x)| dx \\ & \leq \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} |W_\ell b_k(x)| dx. \end{aligned}$$

From now, let us denote by κ_ℓ the right convolution kernel of $W_\ell := A\psi_\ell(\mathcal{B})$. Observe that

$$\begin{aligned} & \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \leq \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} |W_\ell b_k(x)| dx \\ & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} |b_k * \kappa_\ell(x)| dx \\ & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) dz \right| dx. \end{aligned}$$

By using that the average of b_k on I_k is zero, $\int_{I_k} b_k(z) dz = 0$, we have

$$\begin{aligned} & \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) dz \right| dx \\ & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) dz - \kappa_\ell(x) \int_{I_k} b_k(z) dz \right| dx \\ & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \left| \int_{I_k} (\kappa_\ell(z^{-1}x) - \kappa_\ell(x)) b_k(z) dz \right| dx. \end{aligned}$$

Assuming the following uniform estimate

$$M = \sup_k \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| dx < \infty, \quad (4.7)$$

we have

$$\begin{aligned} & \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \leq \frac{2M}{t} \sum_k \int_{I_k} |b_k(z)| dz \\ & = \frac{2M}{t} \|b\|_{L^1(G)} \leq \frac{6M}{t} \|f\|_{L^1(G)}. \end{aligned}$$

So, if we prove the uniform estimate (4.7), we obtain the weak (1,1) inequality for $f \in L^1(G)$, $f \geq 0$. For the proof of (4.7) let us use the estimates of the Calderón-Zygmund kernel of every operator W_ℓ . Before continuing with the proof let us use the following geometrical property of the open sets I_j , $j \geq 0$:

Lemma 4.6. *Let us consider (4.3). Then, with the notation above, for any $x \in G \setminus \cup_j I_j^*$, and for all $z \in I_k$, we have $|z| \lesssim |x|$, i.e. for some $c > 0$, $4c|z| \leq |x|$.*

We will postpone the proof of Lemma 4.6 for a moment to continue with the proof of Lemma 4.1, in order to use the estimate $4c|z| \leq |x|$, for $x \in G \setminus \cup_j I_j^*$, $z \in I_k$, together with (3.9) in Lemma 3.3 for $\ell \geq 0$, as follows,

$$\begin{aligned} M_k &:= \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| dx \lesssim \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{|x| \geq 4c|z|} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| dx \\ &\lesssim \sum_{\ell=0}^{\infty} 2^{-\ell(s-\frac{n}{2})} \|\sigma\|'_{l.u., L^2_s(\widehat{G})}. \end{aligned}$$

The estimate above implies that

$$M_k := \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| dx \lesssim \|\sigma\|'_{l.u., L^2_s(\widehat{G})}. \quad (4.8)$$

So, we have (4.5) for $f \in L^1(G)$ with $f \geq 0$. Note that if $f \in L^1(G)$ is real-valued, one can decompose $f = f^+ - f^-$, as the difference of two non-negative functions, where $f^+, f^- \in L^1(G)$, and $|f| = f^+ + f^-$. Because $f^+, f^- \leq |f|$, we have

$$\begin{aligned} |\{x \in G : |W_\ell f(x)| > t\}| &\leq \left| \left\{ x \in G : |W_\ell f_+(x)| > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G : |W_\ell f_-(x)| > \frac{t}{2} \right\} \right| \\ &\leq \frac{C}{t} \|f_+\|_{L^1(G)} + \frac{C}{t} \|f_-\|_{L^1(G)} \\ &\leq \frac{2C}{t} \|f\|_{L^1(G)}. \end{aligned}$$

A similar analysis, by splitting a complex function $f \in L^1(G)$ into its real and imaginary parts allows to conclude the weak (1,1) inequality (4.5) to complex functions. Thus, the proof of Lemma 4.1 would be complete if we prove Lemma 4.6. That lemma was proved in [8, Page 17] for graded Lie groups and the same proof applies for any amenable group (see Hebisch [31]), we will present the proof here for the case of a compact Lie group G for completeness. \square

Proof of Lemma 4.6. In view of the estimate $\text{dist}(\partial I_j^*, \partial I_j) \geq 4c \text{dist}(\partial I_j, e_G)$, we will prove that for $x \in G \setminus \cup_j I_j^*$, and all $z \in I_k$, $4c|z| = 4c \times \text{dist}(z, e_G) \lesssim \text{dist}(\partial I_k^*, \partial I_k) \leq |x|$. Indeed, fix $\varepsilon > 0$, and let us take $w \in \partial I_k$, and $w' \in \partial I_k^*$ such that $d(w, w') \leq$

$\text{dist}(\partial I_k, \partial I_k^*) + \varepsilon$. Then, from the triangle inequality, we have

$$\begin{aligned}
& d(z, e_G) \\
& \leq d(z, w) + d(w, w') + d(w', e_G) \leq \text{diam}(I_k) + \text{dist}(\partial I_k, \partial I_k^*) + \text{dist}(\partial I_k^*, e_G) + \varepsilon \\
& \lesssim \text{diam}(I_k) + \text{dist}(\partial I_k, \partial I_k^*) + \text{dist}(\partial I_k, e_G) + \varepsilon \\
& \lesssim \text{diam}(I_k) + \text{dist}(\partial I_k, \partial I_k^*) + \frac{1}{4c} \text{dist}(\partial I_k, \partial I_k^*) + \varepsilon \\
& \asymp \text{dist}(\partial I_k, \partial I_k^*) + \varepsilon,
\end{aligned} \tag{4.9}$$

where in the last line we have assumed that $\text{diam}(I_k) \asymp \text{dist}(\partial I_k, \partial I_k^*)$, (with constants of proportionality independent in k) and that $\text{dist}(\partial I_k, \partial I_k^*)$ is proportional to R_k in view of the relation $|I_k^*| = K|I_k|$. Assuming (4.9), one has that for all $\varepsilon > 0$, $d(z, e_G) \lesssim \text{dist}(\partial I_k, \partial I_k^*) + \varepsilon$, which implies that

$$d(z, e_G) \lesssim \text{dist}(\partial I_k, \partial I_k^*). \tag{4.10}$$

To show that the proportionality constant in (4.10) is uniform in k , let us recall the definition of the radii R'_k 's in (4.2), that $B(z_k, R_k) \subset I_k \subset B(z_k, 2R_k)$, and that $B(z_k, R_k/C) \subset I_k^* \subset B(z_k, CR_k)$ for some $C > 2$ independent of k , where for any k , $z_k \in I_k$. From this remark observe that:

- The condition $B(z_k, R_k) \subset I_k \subset B(z_k, 2R_k)$, implies that $2R_k \leq \text{diam}(I_k) \leq 4R_k$.
- That $B(z_k, R_k) \subset I_k \subset I_k^* \subset B(z_k, CR_k)$, implies that

$$\text{dist}(\partial I_k, \partial I_k^*) \leq \text{dist}(\partial B(z_k, R_k), \partial B(z_k, CR_k)) = (C - 1)R_k.$$

On the other hand, by observing that in every step above we can replace $I_k^* := B(z_k, CR_k)$, in view of the inclusion

$$I_k \subset B(z_k, 2R_k) \subset I_k^* := B(z_k, CR_k),$$

we have

$$(C - 2)R_k = \text{dist}(\partial I_k^*, \partial B(z_k, 2R_k)) \leq \text{dist}(\partial I_k, \partial I_k^*).$$

Consequently,

$$\begin{aligned}
& \text{diam}(I_k) \asymp R_k \asymp \text{dist}(\partial B(z_k, 2R_k), \partial B(z_k, CR_k)) \\
& \asymp \text{dist}(\partial I_k, \partial I_k^*).
\end{aligned}$$

To show that $\text{dist}(\partial I_k^*, \partial I_k) \leq |x|$, observe that from Remark 4.3, $e_G \in \cup_j I_j$, and because of $x \in G \setminus \cup_j I_j$,

$$\text{dist}(\partial I_k^*, \partial I_k) \lesssim \text{diam}(\cup_j I_j) \lesssim d(x, e_G) = |x|.$$

So, we have guaranteed the existence of a positive constant, which we again denote by $c > 0$, such that, $\{x \in G : x \in G \setminus \cup_j I_j^*\} \subset \{x \in G : \text{for all } z \in I_k, 4c|z| \leq |x|\}$. \square

Proof of Lemma 4.2. Now, we claim that

$$W : L^1(G, \ell^r(\mathbb{N}_0)) \rightarrow L^{1,\infty}(G, \ell^r(\mathbb{N}_0)), \quad 1 < r < \infty. \tag{4.11}$$

extends to a bounded operator. For the proof of (4.11), we need to show that there exists a constant $C > 0$ independent of $\{f_\ell\} \in L^1(G, \ell^r(\mathbb{N}_0))$ and $t > 0$, such that

$$\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_\ell f_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \leq \frac{C}{t} \|\{f_\ell\}\|_{L^1(G, \ell^r(\mathbb{N}_0))}. \quad (4.12)$$

So, fix $\{f_\ell\} \in L^1(G, \ell^r(\mathbb{N}_0))$ and $t > 0$, and let $h(x) := \left(\sum_{\ell=0}^{\infty} |f_\ell(x)|^r \right)^{\frac{1}{r}}$. Apply the Calderón-Zygmund decomposition Lemma to $h \in L^1(G)$ in Remark 4.5 in order to obtain a disjoint collection $\{I_j\}_{j=0}^{\infty}$ of disjoint open sets such that

- $h(x) \leq t$, for a.e. $x \in G \setminus \cup_{j \geq 0} I_j$,
- $\sum_{j \geq 0} |I_j| \leq \frac{C}{t} \|h\|_{L^1(G)}$, and
- $t \leq \frac{1}{|I_j|} \int_{I_j} h(x) dx \leq 2t$, for all j .

Now, we will define a suitable decomposition of f_ℓ , for every $\ell \geq 0$. Recall that every I_j is bounded, and that $I_j \subset B(z_j, 2R_j)$, where $z_j \in I_j$ (see Remark 4.5). Let us define, for every ℓ , and $x \in I_j$,

$$g_\ell(x) := \frac{1}{|I_j|} \int_{I_j} f_\ell(y) dy, \quad b_\ell(x) = f_\ell(x) - g_\ell(x). \quad (4.13)$$

and for $x \in G \setminus \cup_{j \geq 0} I_j$,

$$g_\ell(x) = f_\ell(x), \quad b_\ell(x) = 0. \quad (4.14)$$

So, for a.e. $x \in G$, $f_\ell(x) = g_\ell(x) + b_\ell(x)$. Note that for every $1 < r < \infty$, $\|\{g_\ell\}\|_{L^r(\ell^r)}^r \leq t^{r-1} \|\{f_\ell\}\|_{L^1(\ell^r)}$. Indeed, for $x \in I_j$, Minkowski integral inequality gives,

$$\begin{aligned} \left(\sum_{\ell=0}^{\infty} |g_\ell(x)|^r \right)^{\frac{1}{r}} &\leq \left(\sum_{\ell=0}^{\infty} \left| \frac{1}{|I_j|} \int_{I_j} f_\ell(y) dy \right|^r \right)^{\frac{1}{r}} \leq \frac{1}{|I_j|} \int_{I_j} \left(\sum_{\ell=0}^{\infty} |f_\ell(y)|^r \right)^{\frac{1}{r}} dy \\ &= \frac{1}{|I_j|} \int_{I_j} h(y) dy \\ &\leq 2t. \end{aligned}$$

Consequently,

$$\sum_{\ell=0}^{\infty} |g_\ell(x)|^r \leq (2t)^r,$$

and from the fact that $h(x) \leq t$, for a.e. $x \in G \setminus \cup_{j \geq 0} I_j$, we have

$$\begin{aligned} \|\{g_\ell\}\|_{L^r(\ell^r)}^r &= \int_G \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx = \sum_j \int_{I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx + \int_{G \setminus \cup_j I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx \\ &= \sum_j \int_{I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx + \int_{G \setminus \cup_j I_j} \sum_{\ell=0}^{\infty} |f_\ell(x)|^r dx \end{aligned}$$

$$\begin{aligned}
&\leq \sum_j \int_{I_j} (2t)^r dx + \int_{G \setminus \cup_j I_j} h(x)^r dx \\
&\lesssim t^r \sum_j |I_j| + \int_{G \setminus \cup_j I_j} h(x)^{r-1} h(x) dx \\
&\leq t^r \times \frac{C}{t} \|h\|_{L^1(G)} + t^{r-1} \int_{G \setminus \cup_j I_j} h(x) dx \lesssim t^{r-1} \|h\|_{L^1(G)} \\
&= t^{r-1} \|\{f_\ell\}\|_{L^1(\ell^r)}.
\end{aligned}$$

Now, by using the Minkowski and the Chebyshev inequality, we obtain

$$\begin{aligned}
&\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_\ell f_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \\
&\leq \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_\ell g_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\
&\leq \frac{2^r}{t^r} \int_G \sum_{\ell=0}^{\infty} |W_\ell g_\ell(x)|^r dx + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|.
\end{aligned}$$

In view of Lemma 4.1, $W : L^r(G, \ell^r(\mathbb{N}_0)) \rightarrow L^r(G, \ell^r(\mathbb{N}_0))$, extends to a bounded operator and

$$\int_G \sum_{\ell=0}^{\infty} |W_\ell g_\ell(x)|^r dx = \|W\{g_\ell\}\|_{L^r(\ell^r)}^r \lesssim \|\{g_\ell\}\|_{L^r(\ell^r)}^r \leq t^{r-1} \|\{f_\ell\}\|_{L^1(\ell^r)}. \quad (4.15)$$

Consequently,

$$\begin{aligned}
&\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_\ell f_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \\
&\lesssim \frac{1}{t} \|\{f_\ell\}\|_{L^1(\ell^r)} + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|.
\end{aligned}$$

Now, we only need to prove that

$$\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \lesssim \frac{1}{t} \|\{f_\ell\}\|_{L^1(\ell^r)}. \quad (4.16)$$

Taking into account that $b_\ell \equiv 0$ on $G \setminus \cup_j I_j$, we have that

$$b_\ell = \sum_k b_{\ell,k}, \quad b_{\ell,k}(x) = b_\ell(x) \cdot 1_{I_k}(x). \quad (4.17)$$

Let us assume that I_j^* is an open set, such that $|I_j^*| = K|I_j|$ for some $K > 0$, and $\text{dist}(\partial I_j^*, \partial I_j) \geq c \text{dist}(\partial I_j, e_G)$, where e_G is the identity element of G . So, by the Minkowski inequality we have,

$$\begin{aligned} & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\ &= \left| \left\{ x \in \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\ &\leq |\{x \in G : x \in \cup_j I_j^*\}| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|. \end{aligned}$$

Taking into account that

$$|\{x \in G : x \in \cup_j I_j^*\}| \leq \sum_j |I_j^*|,$$

we get

$$\begin{aligned} & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ &\leq \sum_j |I_j^*| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ &= K \sum_j |I_j| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ &\leq \frac{CK}{t} \|f\|_{L^1(G, \ell^r)} + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|. \end{aligned}$$

Observe that the Chebyshev inequality implies

$$\begin{aligned} & \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\ &\leq \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^r \right)^{\frac{1}{r}} dx \\ &= \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} \left| \left(W_{\ell} \left(\sum_k b_{\ell,k} \right) \right) (x) \right|^r \right)^{\frac{1}{r}} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \|\{(W_\ell(\sum_k b_{\ell,k})(x))\}_{\ell=0}^\infty\|_{\ell^r(\mathbb{N}_0)} dx \\
&= \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \|\{\sum_k (W_\ell b_{\ell,k})(x)\}_{\ell=0}^\infty\|_{\ell^r(\mathbb{N}_0)} dx \\
&\leq \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^\infty |(W_\ell b_{\ell,k})(x)|^r \right)^{\frac{1}{r}} dx.
\end{aligned}$$

Now, if κ_ℓ is the right convolution Calderón-Zygmund kernel of W_ℓ , and by using that $\int_{I_k} b_{k,\ell}(y) dy = 0$, we have that

$$\begin{aligned}
\left(\sum_{\ell=0}^\infty |(W_\ell b_{\ell,k})(x)|^r \right)^{\frac{1}{r}} &= \left(\sum_{\ell=0}^\infty |b_{\ell,k} * \kappa_\ell(x)|^r \right)^{\frac{1}{r}} \\
&= \left(\sum_{\ell=0}^\infty \left| \int_{I_k} \kappa_\ell(y^{-1}x) b_{\ell,k}(y) dy - \kappa_\ell(x) \int_{I_k} b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}} \\
&= \left(\sum_{\ell=0}^\infty \left| \int_{I_k} (\kappa_\ell(y^{-1}x) - \kappa_\ell(x)) b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}}.
\end{aligned}$$

Now, we will proceed as follows. By using that $|b_{\ell,k}(y)|^r \leq \sum_{\ell'=0}^\infty |b_{\ell',k}(y)|^r$, by an application of the Minkowski integral inequality, we have

$$\begin{aligned}
\left(\sum_{\ell=0}^\infty |(W_\ell b_{\ell,k})(x)|^r \right)^{\frac{1}{r}} &= \left(\sum_{\ell=0}^\infty \left| \int_{I_k} (\kappa_\ell(y^{-1}x) - \kappa_\ell(x)) b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}} \\
&\leq \int_{I_k} \left(\sum_{\ell=0}^\infty |\kappa_\ell(y^{-1}x) - \kappa_\ell(x)|^r |b_{\ell,k}(y)|^r \right)^{\frac{1}{r}} dy \\
&\leq \int_{I_k} \left(\sum_{\ell'=0}^\infty |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} \left(\sum_{\ell=0}^\infty |\kappa_\ell(xy^{-1}) - \kappa_\ell(x)|^r \right)^{\frac{1}{r}} dy.
\end{aligned}$$

Consequently, we deduce,

$$\begin{aligned}
&\frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^\infty |(W_\ell b_{\ell,k})(x)|^r \right)^{\frac{1}{r}} dx \\
&\leq \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \int_{I_k} \left(\sum_{\ell'=0}^\infty |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} \left(\sum_{\ell=0}^\infty |\kappa_\ell(y^{-1}x) - \kappa_\ell(x)|^r \right)^{\frac{1}{r}} dy dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{t} \sum_k \int_{I_k} \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}} dx dy \\
&= \frac{2}{t} \sum_k \int_{I_k} \left(\sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}} dx dy.
\end{aligned}$$

Let us recall that for $x \in G \setminus \cup_j I_j^*$, and $y \in I_k$, in view of Lemma 4.6, we have that $4c|y| \leq |x|$. So,

$$\{x \in G : x \in G \setminus \cup_j I_j^*\} \subset \{x \in G : \text{for all } z \in I_k, 4c|z| \leq |x|\}.$$

Now, in view of (3.9) in Lemma 3.3 with $\ell \geq 0$, we deduce (4.8) and as a consequence we get

$$\begin{aligned}
&\int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}} dx \leq \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)| dx \\
&\leq \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)| dx \\
&\leq \sum_{\ell=0}^{\infty} \int_{|x| > 4c|y|} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)| dx \lesssim \|\sigma\|'_{l.u., L^2_s(\widehat{G})}.
\end{aligned}$$

Thus, we have proved the estimate

$$\begin{aligned}
&\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |W_{\ell} b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \lesssim \frac{2}{t} \sum_k \int_{I_k} \left(\sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} dy \\
&= \frac{2}{t} \int_{\cup_k I_k} \left(\sum_{\ell'=0}^{\infty} |b_{\ell'}(y)|^r \right)^{\frac{1}{r}} dy \\
&\lesssim \frac{1}{t} \|\{f_{\ell}\}\|_{L^1(\ell^r)}.
\end{aligned}$$

Thus, the proof of the weak (1,1) inequality is complete and we have that

$$W : L^1(G, \ell^r(\mathbb{N}_0)) \rightarrow L^{1,\infty}(G, \ell^r(\mathbb{N}_0)), \quad 1 < r < \infty, \quad (4.18)$$

admits a bounded extension. The proof of Lemma 4.2 is complete. \square

Now, having proved Lemma 4.1 and Lemma 4.2, the duality argument in Remark 4.4 proves the Hörmander-Mihlin Theorem 3.1.

Remark 4.7. Let us return to the historical Marcinkiewicz condition on the torus \mathbb{T} ,

$$\sup_{\xi \in \mathbb{Z}} |\sigma(\xi)| + \sup_{j \in \mathbb{N}_0} \sum_{2^{j-1} \leq |\xi| < 2^j} |\sigma(\xi+1) - \sigma(\xi)| < \infty. \quad (4.19)$$

Let $s_0 \in \mathbb{N}$. For a general compact Lie group G of dimension n , one says that a Fourier multiplier $A \equiv T_\sigma$ satisfies the (weak) Marcinkiewicz condition of order s_0 , if its symbol σ satisfies:

$$\|\sigma\|_{L^\infty(\widehat{G})} + \left\| \sigma \cdot 1_{\{[\xi] \in \widehat{G}: 2^{j-1} \leq \langle \xi \rangle < 2^j\}} \right\|_{\dot{L}_{s_0}^1(\widehat{G})} \lesssim_{s_0} 2^{j(n-s_0)}, \quad j \geq 0, \quad (4.20)$$

uniformly in j , with the $\dot{L}_{s_0}^1(\widehat{G})$ -norm defined via:

$$\|\tau\|_{\dot{L}_{s_0}^1(\widehat{G})} := \sum_{|\alpha|=s_0} \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr} |\Delta_\xi^\alpha \tau(\xi)|, \quad |\Delta_\xi^\alpha \tau| := \sqrt{\operatorname{Tr}(\Delta_\xi^\alpha \tau (\Delta_\xi^\alpha \tau)^*)}.$$

It was proved in [23], that for $0 \leq s_0 \leq n$, a symbol satisfying (4.20) also satisfies the Hörmander-Mihlin condition (3.3) for any $s > n/2$. In view of Theorem 3.1, a Fourier multiplier $A \equiv T_\sigma$ satisfying (4.20) extends to a bounded operator from $F_{p,q}^r(G)$ into $F_{p,q}^r(G)$ for all $1 < p, q < \infty$, and all $r \in \mathbb{R}$, and for $p = 1$, A admits a bounded extension from $F_{1,q}^r(G)$ into weak- $F_{1,q}^r(G)$. In the case of the torus $G = \mathbb{T}$, $1 < p, q < \infty$, and $n = 1 = s_0$, (4.20) is just (4.19), and $A \equiv T_\sigma$ extends to a bounded operator from $F_{p,q}^r(\mathbb{T})$ into $F_{p,q}^r(\mathbb{T})$ for all $1 < p, q < \infty$, and all $r \in \mathbb{R}$, and for $p = 1$, A admits a bounded extension from $F_{1,q}^r(\mathbb{T})$ into weak- $F_{1,q}^r(\mathbb{T})$. In view of the Littlewood-Paley theorem, for $1 < p < \infty$, $q = 2$ and $r = 0$, the previous estimate recovers the classical Marcinkiewicz estimate [35]. We summarise this discussion in the following corollary.

Corollary 4.8. *Let us assume that G is a compact Lie group of dimension n . Let $\sigma \in \Sigma(\widehat{G})$ be a symbol satisfying*

$$\|\sigma\|_{L^\infty(\widehat{G})} + \left\| \sigma \cdot 1_{\{[\xi] \in \widehat{G}: 2^{j-1} \leq \langle \xi \rangle < 2^j\}} \right\|_{\dot{L}_{s_0}^1(\widehat{G})} \lesssim 2^{j(n-s_0)}, \quad j \geq 0, \quad (4.21)$$

uniformly in j , for some $0 \leq s_0 \leq n$, $s_0 \in \mathbb{N}$. Then $A \equiv T_\sigma$ extends to a bounded operator from $F_{p,q}^r(G)$ into $F_{p,q}^r(G)$ for all $1 < p, q < \infty$, and all $r \in \mathbb{R}$. For $p = 1$, A admits a bounded extension from $F_{1,q}^r(G)$ into weak- $F_{1,q}^r(G)$.

Proof. That a symbol satisfying (4.21) for some integer s_0 with $0 \leq s_0 \leq n$, satisfies also the Hörmander condition (3.3) for all $s > \frac{n}{2}$ was proved in Theorem 6.16 of [23]. In view of Theorem 3.1 we conclude the proof. \square

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