# DISTRIBUTED FORMATION CONTROL OF NETWORKED DYNAMIC MULTI-AGENT SYSTEMS

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE FACULTY OF ENGINEERING AND PHYSICAL SCIENCES

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## The University of Manchester

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**Doctor of Philosophy** 

Distributed Formation Control of Networked Dynamic Multi-Agent Systems

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This study considers the distributed affine formation control problem of networked multi-agent systems. In affine formation manoeuvre control, the agents are to be capable of producing specified geometric patterns and simultaneously accomplish required manoeuvres, such as scales, translations and rotations. Here, the formation control problem is studied using the stress matrix approach which has similar properties as the Laplacian matrix of a graph. The major difference is that the edge weights can have positive or negative values and can be considered as the generalized Laplacian matrix of a graph.

In this study, we commence by considering the scenario where the dynamics of the agents are defined using triple-integrator dynamics. This is inspired by the consideration that a broad range of systems can be modelled by triple-integrator dynamics. For instance, the DC motor which serves as actuator in most mechanical control systems. The longitudinal dynamics of individual vehicles in an *n*-vehicle system travelling on a single lane in a drive-train model is approximated by triple-integrator dynamics in some existing literature. It is therefore important to widen the application area by considering triple-integrator agent dynamics. Here, the cases where the inter-agent communications are in continuous-time and sampled-data are considered. Under the proposed control laws, the group of agents are able to track time-varying targets that are affine transformations of a given nominal formation, and the desired formation maneuvers are only known by the leaders.

Furthermore, the affine formation control problem of general linear systems with uncertainty is considered. A variety of control laws are presented to address different cases. The proposed laws consider the general linear case, the case with uncertainty and the fully distributed case using robust and adaptive strategies. Under the proposed laws, the collection of agents can track any targets that are affine transforms of a defined reference configuration. Experimental results are presented to demonstrate the effectiveness of the proposed control laws.

# Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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# **First Author Publications**

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- 2). O. Onuoha, H. Tnunay, Z Li, and Z. Ding, "Affine Formation Algorithms and Implementation Based on Triple-Integrator Dynamics," Unmanned Systems Journal, vol.07, no. 01, pp. 33-45, 2019.
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To My loving mum and family

# Chapter 1

# Introduction

### 1.1 Background

Control of multi-agent systems has attracted researchers in the control system community in the last two decades because of its potential to resolve some complicated control problems, both theoretical and application problems. For instance, a complex control problem involving multiple tasks could be broken down into a collection of smaller sub-tasks and each set of sub-task can be modelled as an agent of the overall multi-agent system. A distributed control protocol can then be developed to govern the entire multi-agent system causing the overall goal to be achieved.

Foundational work on multi-agent system coordination has been carried out by several researchers. For instance, consensus control of multi-agent system described by continuous-time first order dynamics is studied in [1–4] and [4–6] addressed the discrete-time version. Following these results, there was a need to study the consensus control problem of agents with double-integrator dynamics since a broad range of practical problems are modelled by double-integrator dynamics. The control gain under which the interaction topology is stable for double-integrator dynamics is derived in [7]. Formation keeping problem of a multi-agent system is studied in [8] while flocking is studied in [9–11]. These double-integrator studies are limited to continuous-time dynamics setting. Consensus based formation control for discrete-time double-integrator dynamics is studied in [12].

Some practical applications of multi-agent systems (MASs) control includes unmanned aerial vehicles (UAV) control, smart grids control, sensor networks, medicine, military observation, etc [13–15].

### 1.2 Related Works

#### 1.2.1 Consensus Control Protocol

The consensus protocol is a fundamental technique in the coordination of multi-agent systems. It has applications in a broad area of multi-agent coordination schemes such as distributed optimization [16–23], estimation [24, 25] and formation control. Some reviews on consensus-based formation control are presented in [26–33]. Consensus control protocols aim to synchronize the states of the agents of a multi-agent system to some common state. Some studies on consensus coordination are carried out in [34–37].

The problem of constrained optimization has been studied by researchers in recent times. For example, consensus subject to communication constraint (e.g. delay) is studied in [38–50]. The leader-following consensus is studied in [51–59]. Group consensus is studied in [60–72]. Consensus based on event-trigger mechanism is considered in [73–81]. Finite-time consensus is studied in [82,83].

#### **1.2.2** Average Consensus (For Balanced Graphs)

The consensus value of a set of agents is a function of the left eigenvector of the Laplacian matrix derived from the agent's communication graph. Unfortunately, allowing the consensus value to depend on the respective graph topology is not always desirable. From [84–86], the consensus value for balanced graphs is given by

$$c = \frac{1}{N} \sum_{i=1}^{N} x_{i(0)} \tag{1.1}$$

where c is independent of the graph structure. It is worth noting that a graph is said to be (weight) balanced if the "in-degree" and "out-degree" are equal for each respective node of a given graph. The term in-degree has been used to denote the number of agents a given agent receives information from and the out-degree denotes the number of agents it sends information to.

#### **1.2.3** Formation Control

The multi-agent system (MAS) formation control problem has attracted increased attention from researchers in recent times because of its application to a wide range of areas. This includes the control of unmanned aerial vehicles, satellite clusters, coordination of teams of mobile robots, and so on. The ultimate goal of formation control is to design distributed control laws that ensure that the agents of a given MAS both form desired geometric patterns and collectively achieve any required maneuver.

Different strategies have been proposed to address the formation control problem of MASs. For example, the consensus-based strategy is used to address the formation control problem in [87], [88]. This strategy is grouped into three approaches by convention. They are the bearing-based [89], displacement-based [90] and distance-based [91] approaches. These approaches achieve the defined formation by defining constant constraints on the inter-agent bearing, displacement and distance. These predefined (preset) constant offsets in turn negatively impose limitations on the maneuvers the formation, as a whole, can carry out. For instance, consensus-based formation control laws based on the displacement approach can track target functions having time-varying translations [92], [93] but carrying out scales on the formation would require a redesign of the displacements. Similarly, formation control laws based on distance can track formation targets with time-varying translations and orientations [94], [95], but have difficulties in tracking target formations having time-varying scales. Also, the bearing-based formation control laws can track time-varying formation translations

and scales [91], [89], but have difficulties in tracking time-varying formation orientations. Thus, simultaneously achieving translation, rotation and scaling maneuvers by any of these methods are involving. The Complex Laplacian-based strategy [96], [97], has been proposed to extend the maneuverability of formation of MASs. This strategy is able to carry out maneuvers such as rotation, translation and scaling. Unfortunately, the strategy is only able to address two-dimensional systems. This inadequacy motivated further research which led to the development of the strategy based on the stress matrix.

The prospect of the stress matrix based strategy is good with the promise of being able to achieve general formation maneuvers in all dimensions. Both Laplacian and stress matrices have similar properties. However, the respective edge entries producing the stress matrix need to be jointly determined by the formation configuration. Furthermore, the respective values of the weight (edge) entries of the stress matrix can be either positive or negative.

#### **1.2.4** Affine Formation Control

Recently, the formation control strategy based on the stress matrix has been applied to the affine formations control of MASs. An affine transformation is viewed as a general linear transformation corresponding to translation, rotation, shearing, scaling, or any combination of them. The affine formation control problem based on the stress matrix with stationary leaders is studied for agent dynamics described by single-integrator in [98]. Formation scaling is considered in [99]. The affine formation control problem where the leaders are dynamic or modelled with double-integrator agent dynamics is addressed in [100]. These studies only considered the case where inter-agent communication (or sensing) occur continuously in time and the agents' dynamics are limited to double-integrators. Furthermore, they all assumed the dynamics of the agents are governed by mere network of integrators.

### 1.3 Adopted Approach

This study adopts a leader-follower strategy to accomplish the required formation maneuver control in a distributed manner based on stress matrices. Here, only the leaders need to know the exact maneuvers to be accomplished in advance. This is achieved using matching knowledge of the  $\Upsilon(t)$  and b(t) matrices for the leaders. The time-varying affine target position of agent *i* is defined by

$$p_i^*(t) = \Upsilon(t)r_i + b(t),$$

where both  $\Upsilon(t) \in \mathbb{R}^{d \times d}$  and  $b(t) \in \mathbb{R}^d$  vary with time and  $r_i \in \mathbb{R}^d$  denotes a constant reference (or nominal) configuration of the *i*th agent. Note that the required affine transformation is accomplished for the leaders by varying the  $\Upsilon(t)$  and b(t) matrices for the leaders. The followers do not need to know these matrices, but will track their own corresponding position using the variety of distributed control protocol proposed in this study. The control protocol facilitates the corresponding affine transformations for the followers without using the  $\Upsilon(t) \in \mathbb{R}^{d \times d}$  and  $b(t) \in \mathbb{R}^d$  terms.

In practical situations, the exact values of the  $\Upsilon(t)$  and b(t) matrices can be defined based on the required maneuvers and the leaders can be virtual nodes. The followers only require a knowledge of their own states and those of their respective neighbours.

### **1.4** Main Contributions

This research investigates the formation control problem of networked multi-agent systems. The approach of this study is to use stress matrices to accomplish formation (unlike the Laplacian matrix based approach). The approach allows the use of a single protocol to accomplish several formation maneuvers of the agents as a whole in any dimension. The possible maneuvers that can be accomplished by a single protocol using the approach includes scaling, shearing, translation, rotation, and all other maneuvers that are affine transforms of a defined reference configuration. The main novelty of the approach is the ability to use a single control protocol for general affine formation control in any dimension unlike other approaches. The main contributions of this study are summarised as follows.

- The affine formation control problem of networked multi-agent systems with periodic inter-agent communication is studied.
  - 1. The formation control case where the networked multi-agent systems are governed by single integrator dynamics are studied. Two control laws are proposed to address the cases where the leaders are stationary and dynamic are presented. Sufficient conditions to guarantee the stability of the proposed laws are derived.
  - 2. The case where agent dynamics are governed by double-integrator dynamics is studied. A control law is proposed to accomplish the required formation control. Sufficient conditions to guarantee the stability of the proposed law are derived.
- The affine formation control problem of networked multi-agent systems with models described with triple-integrator dynamics is studied.
  - The formation control case where the networked multi-agent systems communicate in continuous-time is studied. A control law is proposed to accomplish the required formation control. Sufficient conditions to guarantee the stability of the proposed law are derived.
  - The case where the inter-agent communication is periodic is similarly investigated. A control law is proposed to accomplish the required formation control. Sufficient conditions to guarantee the stability of the proposed law are derived.
  - Steps on the implementation of the proposed control laws are provided. Simulation studies are derived.
- The affine formation control problem of networked general linear multi-agent systems is investigated.

- A fundamental case of the problem is studied. Two control laws are proposed to address the problem. Sufficient conditions to guarantee the stability of the proposed law are derived.
- 2. The problem is studied using an adaptive scheme to address the problem of the coupling gain requiring global information. This is to facilitate the design of a fully distributed control law. The stability analysis is carried out using Lyapunov theory.
- 3. The case where the agent dynamics contain uncertainties are investigated. A control law is proposed to accomplish the required formation control. Sufficient conditions to guarantee the stability of the proposed law are derived.
- 4. Experiments are carried out to demonstrate the procedure for the implementation of the proposed control laws.

### 1.5 Thesis Organization

This section presents the structural organization of the entire thesis. In this research, the affine formation control problem of networked multi-agent systems with tripleintegrator dynamics is first studied. The study is then extended to the general affine formation control problem of general linear multi-agent systems. Further studies are then carried out to consider the case with uncertainty.

To facilitate an easy study of this thesis, a general overview of the thesis structure is presented as follows.

**Chapter 1** presents a brief background on networked multi-agent systems coordination. A brief review of consensus and formation control schemes are presented. The contributions of the research are also discussed.

Chapter 2 presents notations and some preliminaries used throughout this thesis.

Chapter 3 studies the problem of affine formation control of networked multi-agent

systems for stationary and dynamic leaders with periodic inter-agent communication. Three control laws are proposed to address cases of single- and double-integrator dynamics. The stability analysis of the proposed laws are given.

**Chapter 4** studies the problem of affine formation control of networked multi-agent systems with triple-integrator dynamics. The cases of continuous-time and sampled-data inter-agent communications are considered. Two control laws are presented to address the two cases. Sufficient conditions to guarantee the system stability are derived. Procedures for the implementation of the proposed control laws are provided.

Chapter 5 studies the affine formation control problem of general linear multi-agent systems.

**Chapter 6** further investigates the problem of affine formation control of general linear multi-agent systems. The problem of the coupling gains requiring global information is addressed by using an adaptive scheme. The cases of the model containing uncertainties are also studied. A variety of control laws are proposed to deal with the different cases are presented. The stability of the proposed laws are studied using Lyapunov theory. Finally, experimental study results are presented.

Chapter 7 presents suggestions for future work.

# Chapter 2

# Preliminaries

#### 2.1 Notations

This Section presents some notations used in the rest of this thesis. The notations are standard and have been used in the existing literature, e.g. in [101, 102]. Let  $\mathbb{R}^{n \times m}$ ,  $\mathbb{R}^n$ , and  $\mathbb{R}$  denote the sets of real matrices (of dimensions  $n \times m$ ), real vectors (of dimensions n) and real numbers, respectively. The vector of dimension n with elements of all ones and the  $n \times n$  identity matrix are denoted with  $\mathbf{1}_n$  and  $I_n$ , respectively. The Kronecker product is denoted with  $\otimes$  and has the following properties  $(A \otimes B)(G \otimes$  $H) = (AG) \otimes (BH)$ , and  $(A \otimes B)^T = A^T \otimes B^T$ , where A, B, G and H are matrices of proper dimensions.

### 2.2 Graph Theory

The dynamic behaviour of a multi-agent system can be effectively modelled using graph theory. This makes the study of graph theory important for the study of multiagent systems. This section discusses some important areas of graph theory necessary for a study on multi-agent systems.

#### 2.2.1 Graph Basics

Some very fundamental definitions in graph theory are presented in this subsection. Graph, Node and Edge

A graph G is a set of Nodes (also called Vertices, V) and Edges, E. In the study of multi-agent system, the agents are represented as nodes (vertices) and the communication between nodes are represented as the edges of the graph.

#### Edge

With a graph having  $v_1$  and  $v_2$  as vertices,  $v_1v_2$  refers to a connection from  $v_1$  to  $v_2$ and not from  $v_2$  to  $v_1$ . If this connection from  $v_1$  to  $v_2$  exists, then  $v_1v_2$  is said to be an edge and the edge is said to be "in-coming to  $v_2$  and out-going from  $v_1$ ." Also, if  $v_1v_2$  and  $v_2v_1$  are both edges, "the edge" is said to be undirected but if only one pair is an edge, it is said to be directed. The neighbours of a vertice  $(N_i)$  is a collection of vertices the given vertice can receive information from directly.



Figure 2.1: A simple graph

As shown in figure 2.1,  $v_1, v_2, ..., v_5$  are all nodes (vertices) of the graph and  $v_1v_5$  is an edge but  $v_5v_1$  is not.  $v_2$  has  $v_1$  and  $v_3$  as its neighbours.

#### **Connection Notations**

In a graph, a *path* is an ordered sequence of nodes, with edges between each pair of consecutive nodes in the sequence. A given graph is said to be *connected* if there is a path that has every pair of nodes as its end nodes in the graph. If otherwise, the graph is said to be *disconnected*. A cycle is a path that contains two or more nodes and it starts and ends at the same node. A connected graph is considered to be an undirected graph if the graph is connected and all the edges are undirected. The term connected mixed graph has been used by some researchers to refer to connected graphs with both directed and undirected edges, and then the term directed graph to refer to connected graphs with directed edges. In this study, mixed graphs are considered to be directed graphs. A graph that is connected and has no cycle is called a *tree*. It is also worth noting that a connected graph is a tree if and only if the number of nodes equals the number edges plus one, that is, |N| = |E| + 1. A directed graph is said to be strongly connected if a path exists between every node in the graph. That is, from any chosen node, every other node can be reached. A graph is *complete* if there is an edge between every two nodes of the graph. That is, there is an undirected connection between every two nodes of the graph. A rooted directed graph is a graph in which every node has exactly one parent node except a node, called the root node, that has no parent and has a path to every other node in the graph. A spanning tree is a connected, rooted directed graph.

#### 2.2.2 Graph Matrices

Every graph can be represented by a matrix, and the structural properties of the given graph are also contained in the matrix. The study of graphs through the study of their corresponding matrix is called *algebraic graph theory*.

A given graph can be represented by its *adjacency* (or *connectivity*) matrix, A, given by  $A = [a_{ij}]$ , where  $a_{ij}$  is the weight of the edge  $(v_j, v_i)$ .  $a_{ij} > 0$  if  $a_{ij} \in E$  and 0 otherwise.

#### Laplacian Matrix

A given graph can be conveniently represented as a Laplacian Matrix, which is a

matrix representation of the given graph. Here, it is defined by

$$L = D - A, \tag{2.1}$$

where L is the Laplacian matrix, A is the Adjacency matrix and D the Diagonal matrix defined by:

$$D = \operatorname{diag}\left\{\sum_{j=1}^{N} a_{ij}\right\}$$
(2.2)

For example, given the graph shown in Figure 2.2, assuming the weight of each edge is 1, then the corresponding Adjacency matrix, A, Diagonal matrix, D and Laplacian matrix, L, are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$L = D - A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

### 2.3 Graph Theory for Affine Formation Control

Consider a multi-agent system composed of n agents. The number of leaders and followers are denoted by  $n_l$  and  $n_f$ , respectively. Denote the first  $n_l$  agents as leaders. This study assumes that each agent knows its own states and those of its neighbours. A *configuration* refers to a collection of nodes described by their positions in the Euclidean coordinate space  $p = [p_1^T, ..., p_n^T]^T \in \mathbb{R}^{nd}$ , where  $p_i \in \mathbb{R}^d$ . A *framework*  $\mathcal{F} = (\mathcal{G}, p)$  in  $\mathbb{R}^d$  denotes a graph defined with its configuration. Frameworks,  $(\mathcal{G}, q)$ and  $(\mathcal{G}, p)$  are considered equivalent, i.e.,  $(\mathcal{G}, q) \equiv (\mathcal{G}, p)$ , if and only if



Figure 2.2: Graph with four nodes

 $\parallel q_i - q_j \parallel = \parallel p_i - p_j \parallel, \quad \forall (i, j) \in \mathcal{E}.$ 

Frameworks,  $(\mathcal{G}, q)$  and  $(\mathcal{G}, p)$ , are considered congruent, i.e.,  $(\mathcal{G}, p) \cong (\mathcal{G}, q)$ , iff

$$|| q_i - q_j || = || p_i - p_j ||, \quad \forall i, j \in \mathcal{V}.$$

A framework in  $\mathbb{R}^d$  is said to be globally rigid if  $(\mathcal{G}, p) \equiv (\mathcal{G}, q)$  implies  $(\mathcal{G}, p) \cong (\mathcal{G}, q)$ . This implies, any framework in  $\mathbb{R}^d$  equivalent to  $(\mathcal{G}, p)$  is congruent to it also. A configuration p is considered universally rigid if for all  $\mathbb{R}^{d_1}$ , where  $d_1$  denotes any positive integer,  $(\mathcal{G}, p) \equiv (\mathcal{G}, q)$  implies  $(\mathcal{G}, p) \cong (\mathcal{G}, q)$ . That is, universal rigidity implies global rigidity. However, global rigidity does not confer universal rigidity. More details can be found in [98, 101, 103].

### 2.4 Affine Span

A collection of points,  $\{p_i\}_{i=1}^n \in \mathbb{R}^d$ , have an affine span  $\mathcal{S}$  given by

$$\mathcal{S} = \left\{ \sum_{i=1}^{n} a_i p_i : a_i \in \mathbb{R} \, \forall i \text{ and } \sum_{i=1}^{n} a_i = 1 \right\}.$$

To affinely span any d-dimensional space, a set of d+1 affinely independent points are required. Note that, the affine span of any two distinct points is a line connecting the points. Also, three unique points that are not collinear have an affine span that is a 2-dimensional plane passing through the points. Higher dimensions follow the analogy.

A given affine span can be translated to contain the origin, and hence, a linear space having the same dimension as the affine space. Thus, given a *d*-dimensional affine span, one can say they span  $\mathbb{R}^d$  affinely. In this study, d + 1 leaders that span  $\mathbb{R}^d$  are to be chosen for the affine formation manoeuvre control of a *d*-dimensional space [101].

### 2.5 Stress Matrix

Given a framework  $\mathcal{F} = (\mathcal{G}, p)$  for affine formation control, we denote the stress as the collection of scalars associated with the weights  $w_{ij}$  of the edges (i, j) of the graph. A stress satisfying

$$-\sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = 0, \quad i\in\mathcal{V},$$
(2.3)

is considered an equilibrium stress [104], [103]. The concept can be visualized when considered in the light of mechanically stress where  $w_{ij} > 0$  denotes an attractive force and  $w_{ij} < 0$  denotes a repulsive force, and the equilibrium force is given by (2.3). Equation (2.3) can be written in matrix form as

$$-(\Omega \otimes I_d)p = 0,$$

where the stress matrix,  $\Omega \in \mathbb{R}^{n \times n}$ , is given by

$$\Omega_{ij} = \begin{cases} \sum_{j \in \mathcal{N}_i} w_{ij(k)}, & \text{for } i = j, \\ -w_{ij}, & \text{for } i \neq j, \quad (j,i) \in \mathcal{E}, \\ 0, & \text{for } i \neq j, \quad (j,i) \notin \mathcal{E}. \end{cases}$$

$$(2.4)$$

The stress matrix  $\Omega$  is commonly partitioned, for convenience, as

$$\Omega = \begin{bmatrix} \Omega_{ll} & \Omega_{lf} \\ \Omega_{fl} & \Omega_{ff} \end{bmatrix}, \qquad (2.5)$$

where  $\Omega_{ff}$  and  $\Omega_{ll}$  are respectively  $n_f \times n_f$  and  $n_l \times n_l$  sub-matrices. Note that the value of every  $w_{ij}$  is to be computed.

#### 2.6 Stress Matrix Design

Stress-matrix plays a pivotal role in affine formation control. For an affine formation to be stabilizable, the underlining framework  $(\mathcal{G}, p)$  needs to be universally rigid. The universal rigidity of a framework guarantees its uniqueness in all dimensions. Therefore, to guarantee the accomplishment of affine formation control, the framework needs to be unique. That is, the framework needs to be rigid (universally) to facilitate proper stabilization. The required universal rigidity is achieved via the design of a suitable stress matrix. Note that, a framework,  $\mathcal{F} = (\mathcal{G}, p)$ , denotes a communication graph defined along with the positions of the nodes. Rigidity generally plays a crucial role in stress-matrix based formation control. Universal rigidity is stricter than global rigidity. Global rigidity guarantees the uniqueness of a framework in the entire space of a defined dimension. Unfortunately, the necessary and sufficient conditions to guarantee universal (or global) rigidity is still lacking in existing literature.

Studies mostly concerned with the coordination of multi-agent systems commonly focus on the special universal and global rigidity cases where the framework has a generic configuration. A configuration is considered generic if it has coordinates that are algebraically independent over the integer [105,106]. For frameworks having generic configurations, studies, e.g., in [104–106] present sufficient conditions to guarantee universal (or global) rigidity.

A key feature of a generic framework (or configuration) is that the necessary and sufficient conditions for universal rigidity is that the associated stress matrix is positive semi-definite with a rank of n-d-1 [107]. Here, n and d have been used to respectively denote the number of agents (or nodes) and the dimension of the space considered. Note that in the traditional consensus control (or consensus-based formation control), the Laplacian matrix of a connected graph is positive semi-definite and has rank n-1. The entries of both the Laplacian and stress matrices are populated in similar manner, i.e., the off-diagonal entries are populated with  $-w_{ij}$  and the diagonal entries with  $\sum_{j \in \mathcal{N}_i} w_{ij}$ . However, since the  $w_{ij}$  entries of the stress matrix can have either a positive or negative entry, there is no guarantee that the stress matrix would be positive semidefinite, unless the matrix is carefully designed subject to some constraints. An obvious requirement is that d + 1 nodes (denoted as leaders) need to be selected so that the other nodes can be uniquely defined [107]. In the rest of this study, we assume that our configurations are generic.

Consider a framework with an undirected communication graph, universally rigid and with a generic configuration. Let  $\varpi_1, ..., \varpi_d$  denote the 1st, ..., dth components of the configuration having *d*-dimensional nodes. For instance, consider a 3-dimensional configuration (x, y and z) of three unique nodes,  $p_1(4, 0, 3)$ ,  $p_2(9, 1, 3)$  and  $p_3(7, 0, 0)$ , i.e.,  $p_i(x, y, z)$ . This follows that  $\varpi_1 = [4, 9, 7]^T$ ,  $\varpi_2 = [0, 1, 0]^T$  and  $\varpi_3 = [3, 3, 0]^T$ . Note that  $\mathbf{1}_n, \varpi_1, ..., \varpi_d$  are linearly independent considering that the configuration is generic.

**Lemma 2.6.1.** [104–106]: A framework  $(\mathcal{G}, p)$  whose graph is undirected and has a generic configuration in  $\mathbb{R}^d$  with  $n \ge d+2$  nodes is universally rigid if and only if its communication graph is (d+1)-connected with a stress matrix,  $\Omega$ , that is positive semi-definite and has a rank of n - d - 1.

Assumption 1. The framework  $(\mathcal{G}, p)$  is assumed to be generically universally rigid.

**Remark 2.6.1.** Assumption 1 guarantees that the rank( $\Omega$ ) of the stress matrix is n - d - 1.

Assumption 2. [108] Let Assumption 1 hold, then all the eigenvalues of  $\Omega_{ff}$  have positive real parts.

A method for the computation of the stress matrix as given in [100] is presented. Let w denote the stress vector of the reference formation whose communication is modelled with an undirected graph having m undirected edges. Choose any orientation for the graph and let the incidence matrix be denoted by  $H \in \mathbb{R}^{m \times n}$ . Let  $h_i \in \mathbb{R}^m$  denote the *i*th column of  $H^T$ , so that  $H^T = [h_1, ..., h_n]$ . Choose

$$Z = \begin{bmatrix} \bar{P}^T(r) H^T \operatorname{diag}(h_1) \\ \vdots \\ \bar{P}^T(r) H^T \operatorname{diag}(h_n) \end{bmatrix}, \in \mathbb{R}^{n(d+1) \times m},$$

where  $\bar{P}(r)$  denotes the matrix  $[\varpi_1, ..., \varpi_d, \mathbf{1}_n]^T$ . Denote the basis of the null space of Z, null(Z) by  $z_1, ..., z_n \in \mathbb{R}^m$ . Let the singular value decomposition (SVD) of  $\bar{P}(r) = U \sum V$ . Let  $U = [U_1 \ U_2]$ , where the first d + 1 columns of U are used to compose  $U_1$ . By defining

$$\Psi_i = U_2^T H^T \operatorname{diag}(z_i) H U_2, \quad \forall i = 1, .., \varpi$$

and choosing  $c_1, ..., c_{\varpi}$  such that,

$$c_i \Psi_i > 0,$$

the stress vector is given by

$$w = \sum_{i=1}^{\varpi} c_i z_i. \tag{2.6}$$

See [100] for more details. Furthermore, studies in [109] present some useful guide on constructing universally rigid frameworks.

### 2.7 Affine Realizability and Leaders Selection

This subsection presents a guideline on choosing the leaders for affine formation control. Note that, for control in a d-dimensional space, d + 1 number of leaders that span the entire d space affinely need to be selected [104–106].

**Lemma 2.7.1.** [100]: Let the reference formation of the framework  $(\mathcal{G}, p)$  comprise of  $n_l$  leaders and  $n_f$  followers. Then, the target positions of the followers  $p_f^*$  can be uniquely computed from the relation

$$p_f^*(t) = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d)p_l^*(t),$$

for any  $p = [p_l^T, p_f^T]^T$  belonging to the set of affine transform of the reference and, if  $\Omega_{ff}$  is nonsingular.

Note that we have used  $p_f$  and  $p_l$  to respectively denote the positions of the followers and leaders,  $p_f^*$  and  $p_l^*$  to respectively denote the target positions of the followers and leaders. Let the tracking errors of the followers be defined by

$$\delta_{p_f}(t) = p_f(t) - p_f^*(t) = p_f(t) + (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d) p_l^*(t).$$
(2.7)

Next, an assumption on the leaders is presented.

Assumption 3. Assume that the d + 1 (number of) leaders are selected such that they span the  $\mathbb{R}^d$  space affinely.
# Chapter 3

# Distributed Affine Formation Control of Multi-Agent Systems with Periodic Communication

## 3.1 INTRODUCTION

In this study, a leader-follower strategy is used to study the affine formation control of multi-agent systems with periodic information exchange among the agents. In existing literature, affine formation control of multi-agent system has been studied in continuous-time settings for single- and double-integrator dynamics. However, in practical situations, agents may only be able to communicate in a periodic time interval. In this study, we proposed a variety of distributed laws for the coordination of multi-agent systems with single- and double-integrator dynamics. We show conditions on the control gain and sampling period for the overall stability of the system. The proposed control protocols are globally stable and able to track time-varying formations targets that are affine images of the nominal formation. The results contained in this chapter form part of the results being considered for submission in a paper provisionally titled "Distributed Affine Formation Control of Multi-Agent Systems with Periodic Communication."

### 3.2 Problem Formulation

Consider a MAS with n agents. Let the position of the *i*th agent be denoted by  $x_1, ..., x_n \in \mathbb{R}^d$ , so that the *i*th agent's target position in the time-varying formation is given by

$$x_i^*(t) = A(t)r_i + b(t)$$
(3.1)

where both  $b(t) \in \mathbb{R}^d$  and  $A(t) \in \mathbb{R}^{d \times d}$  are time-varying and the nominal (constant reference) configuration is denoted by  $r_i \in \mathbb{R}^d$ . Equation (3.1) is written in global form as

$$r^*(t) = [I_n \otimes A(t)]r + \mathbf{1}_n \otimes b(t)$$
(3.2)

where  $r = [r_1^T, ..., r_n^T] \in \mathbb{R}^{nd}$  and  $x^*(t) \in \mathbb{R}^{nd}$  respectively denote the reference configuration and the targets (time-varying) to be tracked. The affine image is the set of all affine transform of the nominal configuration. Note that the tracked time-varying targets are affine images of the nominal configuration r.

We define the affine image as a collection of all the affine transformation of the reference configuration r. Note that the time-varying targets are affine images of the reference configuration. The affine image is given in global form by [98]

$$\mathcal{A}(r) = \{ x \in \mathbb{R}^{dn} : x = (I_n \otimes A)r + \mathbf{1}_n \otimes b, A \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d \}.$$
(3.3)

The overall goal is to find conditions that guarantees that

$$\lim_{x \to \infty} x(t) = x^*(t), \ \forall x^*(t) \in \mathcal{A}(r).$$

### 3.2.1 Preamble

Consider a multi-agent system comprising of n agents. Assume that the agents have continuous-time dynamics, but sense their neighbours at discrete sampling time intervals and have control inputs that are based on zero-order hold. Such that,

$$u_{i(t)} = u_{i[k]}, \quad kT \le t < (k+1)T,$$

where  $u_{i(t)}$ , k, T and  $u_{i[k]}$  respectively denote control input at time t (continuous-time), discrete-time index, sampling period, and control input at t = kT. The following Lemmas would be required in this study.

Lemma 3.2.1. The polynomial

$$s + a = 0, \tag{3.4}$$

where  $a \in \mathbb{C}$ , has its root within a unit circle if and only if

$$(a+1)t - (a-1) = 0 \tag{3.5}$$

has its roots in the open left half plane.

**Proof 3.2.1.** : By using bilinear transformation  $s = \frac{t+1}{t-1}$  [110], (3.4) can be rewritten as

$$(a+1)t - (a-1) = 0 (3.6)$$

Note that there is a one-to-one mapping of the open left half plane of a polynomial onto the interior of unit circle using bilinear transformation.

Lemma 3.2.2. [111]: The polynomial

$$s^2 + as + b = 0, (3.7)$$

where  $b, c \in \mathbb{C}$ , has all of its root with within a unit circle if and only if

$$(1+a+b)t^{2} + 2(1-b)t + b - a + 1 = 0$$
(3.8)

has all of its roots in the open left half plane.

We assume that the multi-agent system is composed of  $n_l$  leaders and  $n_f$  followers and begin our study with the case were the agents are modelled using single-integrator dynamics.

### 3.2.2 Single-integrator System

Consider a multi-agent with each agent modelled with single-integrator dynamics described respectively in continuous- and discrete-time by

$$\dot{x}_{i(t)} = u_{i(t)}, \quad i = 1, ..., n$$
(3.9)

and

$$x_{i(k+1)} = x_{i(k)} + Tu_{i(k)}, \quad i = 1, ..., n,$$
(3.10)

where  $x_i$  denotes the state of agent *i*. Note that (3.10) can be written in global form for all agents as

$$x_{(k+1)} = x_{(k)} + Tu_{(k)},$$

and for all follower agents as

$$x_{f(k+1)} = x_{f(k)} + Tu_{f(k)}, \quad \forall i \in f$$
 (3.11)

Note that  $x = [x_l^T, x_f^T]^T$ , where the first d + 1 agents denote the set of leaders and their states are denoted by  $x_l^T$ . The remaining agents are the followers and their states are denoted by  $x_f^T$ .

We now consider the case where the leaders are respectively stationary and dynamic.

### **Stationary Leaders**

In this case, the leaders are stationary, i.e.  $x_{i(k+1)} = x_{i(k)}$ ,  $\forall i \in \mathcal{V}_l$ . In the continuoustime setting, this is given by  $\dot{x}_i = 0$ ,  $\forall i \in \mathcal{V}_l$ . Note that  $\mathcal{V}_l$  and  $\mathcal{V}_f$  respectively denote the sets of leaders and followers. We study the sampled-data affine formation control problem using the protocol

$$u_{i(k)} = -\sum w_{ij}(x_{i(k)} - x_{j(k)}), \quad i \in \mathcal{V}_f.$$
(3.12)

Equation (3.12) can be written in global form for all agents as

$$u_{(k)} = -(\Omega \otimes I_d) x_{(k)}. \tag{3.13}$$

By noting how the stress matrix is partitioned in (2.5), repeated for convenience here as

$$\Omega = \begin{bmatrix} \Omega_{ll} & \Omega_{lf} \\ \Omega_{fl} & \Omega_{ff} \end{bmatrix}, \qquad (3.14)$$

we can write (3.13) for only the followers as

$$u_{f(k)} = -(\Omega_{fl} \otimes I_d) x_{l(k)}^* - (\Omega_{ff} \otimes I_d) x_{f(k)}.$$

$$(3.15)$$

Using (3.15), (3.11) can be written in matrix form for all followers as

$$x_{f(k+1)} = [I_{dn_f} - T(\Omega_{ff} \otimes I_d)] x_{f(k)} - T(\Omega_{fl} \otimes I_d) x_{l(k)}^*$$
(3.16)

**Theorem 3.2.1.** Assume that the leaders are stationary, i.e.  $x_{i[k]} = x_{i[k+1]} \quad \forall i \in \mathbb{V}_l$ ; the communication graph of the agents is universally rigid, such that the rank of the stress matrix  $rank(\Omega) = n - d - 1$ ; and the d + 1 leaders have been chosen such that they span the  $\mathbb{R}^d$  space. Let  $\mu_i$  denote the *i*th eigenvalue of  $-\Omega_{ff}$ . Then, by choosing  $T\mu_{\min} > -2$ , control law (3.16) stabilizes each respective follower to the desired target.

**Proof 3.2.2.** Define the global disagreement of all followers by

$$\delta_{x_f[k+1]} = x_{f[k+1]} - x_{f[k+1]}^* = x_{f[k+1]} - \left[ -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d) \right] x_{l[k+1]}^*$$
$$= x_{f[k+1]} + (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d) x_{l[k+1]}^*$$
(3.17)

Similarly,

$$\delta_{x_f[k]} = x_{f[k]} + (\Omega_{ff}^{-1} \Omega_{fl} \otimes I_d) x_{l[k]}^*.$$
(3.18)

Substitute for  $x_{f[k+1]}$  in (3.17) using (3.16) and note that  $x_{l[k+1]} = x_{l[k]} = x_{l^*[k]}$  (since the leaders are already at their target positions), to obtain the expression

$$\delta_{x_{f[k+1]}} = x_{f[k]} - T(\Omega_{ff} \otimes I_d) x_{f[k]} - T(\Omega_{fl} \otimes I_d) x_{l[k]}^* + (\Omega_{ff}^{-1} \Omega_{fl} \otimes I_d) x_{l[k]}^*$$
  
=  $-T(\Omega_{ff} \otimes I_d) [x_{f[k]} + (\Omega_{ff}^{-1} \Omega_{fl} \otimes I_d) x_{l[k]}^*] + [x_{f[k]} + (\Omega_{ff}^{-1} \Omega_{fl} \otimes I_d) x_{l[k]}^*]$   
=  $-T(\Omega_{ff} \otimes I_d) \delta_{f[k]} + \delta_{f[k]} = [(-T\Omega_{ff} + I_{n_f}) \otimes I_d] \delta_{f[k]}.$  (3.19)

Note that (3.18) has been used to obtain (3.19).

To ensure that the followers track their targets, (3.19) needs to be stabilized to the origin. This is achieved if the term  $(-T\Omega_{ff} + I_{n_f})$  have negative eigenvalues. Its characteristic polynomial is given by  $det(sI_{nf} - I_{nf} + T\Omega_{ff}))$ . Let  $\mu_i$  denote the *i*th eigenvalue of  $-\Omega_{ff}$ , so that

$$det(sI_{nf} - I_{nf} + T\Omega_{ff})) = \prod_{i=1}^{n} (s - 1 - T\mu_i)$$

This requires that

$$s - 1 - T\mu_i = 0. (3.20)$$

By substituting a for  $(-1 - T\mu_i)$  in (3.20) and considering Lemma (3.2.1), we obtain

$$-T\mu_i t + T\mu_i + 2 = 0. (3.21)$$

Since,  $-T\mu_i > 0$ , then  $T\mu_i + 2$  needs to be greater than zero to have the roots of (3.21) in the left half plane and (3.20) within a unit circle. This implies  $T\mu_{\min} > -2$ , where  $\mu_{\min}$  is chosen to denote the lower bound.

Next, we present another control law to deal with the case where the leaders positions are dynamic.

#### **Dynamic Leaders**

The control law proposed in (3.16) is unable to guarantee that the tracking errors reduce to zero for systems with dynamic leaders. To deal with this, we propose another control law. Using the algorithm,

$$u_{i[k]} = -\frac{1}{\gamma} \sum_{j \in \mathcal{N}_i} w_{ij} (x_{i[k]} - x_{j[k]} - \frac{1}{T} (x_{j[k+1]} - x_{j[k]})), \quad i \in \mathcal{V}_f,$$
(3.22)

where  $\gamma = \sum_{j \in \mathcal{N}_i} w_{ij}$ . By substituting for  $u_{i[k]}$  in (3.11) using (3.22) and performing some algebraic simplification, the closed loop expression

$$\sum_{j \in \mathcal{N}_i} w_{ij}(x_{i[k+1]} - x_{j[k+1]}) = (1 - T) \sum_{j \in \mathcal{N}_i} w_{ij}(x_{i[k]} - x_{j[k]}), \quad i \in \mathcal{V}_f,$$
(3.23)

is obtained.

Next, the stability of control law (3.23) is analysed.

**Theorem 3.2.2.** Assume that the communication graph of the agents is universally rigid, such that the rank of the stress matrix  $rank(\Omega) = n-d-1$ ; and the d+1 leaders have been chosen such that they span the  $\mathbb{R}^d$  space. Then, by choosing T < 2, the tracking error of all follower nodes is stabilized to the origin by the action of control law (3.23).

**Proof 3.2.3.** Equation (3.23) can be written in global form as

$$(\Omega \otimes I_d) x_{(k+1)} = (1-T)(\Omega \otimes I_d) x_{(k)}.$$
(3.24)

By noting (3.14), (3.24) can be written for the followers as

$$(\Omega_{fl} \otimes I_d) x_{l(k+1)} + (\Omega_{ff} \otimes I_d) x_{f(k+1)} = (1-T) [(\Omega_{fl} \otimes I_d) x_{l(k+1)} + (\Omega_{ff} \otimes I_d) x_{f(k+1)}].$$

Multiplying through by  $(\Omega_{ff}^{-1} \otimes I_d)$  from the left hand side, the expression

$$(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d) x_{l(k+1)} + x_{f(k+1)} = (1-T) [(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d) x_{l(k+1)} + x_{f(k+1)}]$$
(3.25)

is obtained. Define the error as  $\delta_{x_{f[k]}} = x_{f[k]} - x_{f[k]}^* = x_{f[k]} + \Omega_{ff}^{-1} \Omega_{fl} x_{l[k]}^*$ . This allows us to write control law (3.25) in terms of the disagreements as  $\delta_{x_f[k+1]} = (1-T) I_{dn_f} \delta x_{f[k]}$ . The characteristic equation satisfies s+T-1 = 0. Applying the bilinear transformation  $s = \frac{t+1}{t-1}$ , to obtain

$$Tt - T + 2 = 0.$$

Since Tt > 0, then -T+2 needs to be greater then zero to satisfy the systems stability requirement. This is satisfied by choosing the sampling period T, such that T < 2.

### 3.2.3 Double-Integrator Dynamics

Consider the case where each agent is modelled with double-integrator dynamics described in continuous-time by:

$$\dot{x}_{i(t)} = v_{i(t)}, \quad \dot{v}_{i(t)} = u_{i(t)}.$$
(3.26)

Discretizing (3.26) (based on Taylor series second order integral approximation) results in:

$$\begin{cases} x_{i[k+1]} = x_{i[k]} + Tv_{i[k]} + \frac{T^2}{2}u_{i[k]} \\ v_{i[k+1]} = v_{i(k)} + Tu_{i[k]}, \qquad i \in \mathcal{V}_f, \end{cases}$$

$$(3.27)$$

where  $x_i$  and  $v_i$  respectively denote the position and velocity of agent *i*. We now consider the cases where the leaders' acceleration are constantly zero and dynamically changing.

### Dynamic Leaders with Zero Acceleration

Here, the case where the acceleration of the leaders are constantly zero is considered. Consider the control protocol

$$u_{i[k]} = -\sum_{j \in \mathcal{N}_i} w_{ij} [(x_{i[k]} - x_{j[k]}) + \beta (v_{i[k]} - v_{j[k]})], \quad i \in \mathcal{V}_f,$$
(3.28)

whose global form is given by

$$u_{(k)} = -(\Omega \otimes I_d) x_{(k)} - \beta(\Omega \otimes I_d) v_{(k)}.$$
(3.29)

By considering how the stress matrix  $\Omega$  is partitioned in (3.14), (3.29) can be rewritten for the followers as

$$u_f = -[(\Omega_{ff} \otimes I_d)x_{f[k]} + (\Omega_{fl} \otimes I_d)x_{l[k]}] - \beta[(\Omega_{ff} \otimes I_d)v_{f[k]} + (\Omega_{fl} \otimes I_d)v_{l[k]}] \quad (3.30)$$

With (3.30), (3.27) can be written in matrix-vector form as

$$\begin{cases} x_{f[k+1]} = [((I_{nf} - \frac{T^2}{2}\Omega_{ff}) \otimes I_d)x_{f[k]} - \frac{T^2}{2}(\Omega_{fl} \otimes I_d)x_{l[k]}^*] \\ +[((TI_{nf} - \frac{T^2\beta}{2}\Omega_{ff}) \otimes I_d)v_{f[k]} - \frac{T^2\beta}{2}(\Omega_{fl} \otimes I_d)v_{l[k]}^*] \\ v_{f[k+1]} = -T[(\Omega_{ff} \otimes I_d)x_{f[k]} + (\Omega_{fl} \otimes I_d)x_{l[k]}^*] + [(I_{nf} - T\beta\Omega_{ff}) \otimes I_d]v_{f[k]} \\ -T\beta(\Omega_{fl} \otimes I_d)v_{l[k]}^* \end{cases}$$
(3.31)

i.e.,

$$\begin{bmatrix} x_{f[k+1]} \\ v_{f[k+1]} \end{bmatrix} = \left( \begin{bmatrix} (I_{nf} - \frac{T^2}{2}\Omega_{ff}) & (TI_{nf} - \frac{\beta T^2}{2}\Omega_{ff}) \\ -T\Omega_{ff} & (I_{nf} - \beta T\Omega_{ff}) \end{bmatrix} \otimes I_d \right) \begin{bmatrix} x_{f[k]} \\ v_{f[k]} \end{bmatrix} + \left( \begin{bmatrix} -\frac{T^2}{2}\Omega_{fl} & -\frac{\beta T^2}{2}\Omega_{fl} \\ -T\Omega_{fl} & -\beta T\Omega_{fl} \end{bmatrix} \otimes I_d \right) \begin{bmatrix} x_{l[k]} \\ v_{l[k]} \end{bmatrix}$$
(3.32)

where  $v_f \in \mathbb{R}^{dn_f}$  and  $v_l^* \in \mathbb{R}^{dn_l}$  respectively denote the velocities of the followers and leaders. Next, the stability of control law (3.32) is analysed.

**Theorem 3.2.3.** Assume that the communication graph of the agents is universally rigid, such that the rank of the stress matrix  $rank(\Omega) = n - d - 1$ ; and the d + 1 leaders have been chosen such that they span the  $\mathbb{R}^d$  space, and the leaders have zero acceleration, i.e.,  $v_{i[k]} = v_{i[k+1]} \forall i \in \mathbb{V}_l$ , then by choosing T and  $\beta$  such that

$$\begin{cases} -(T^{2}\mu_{i} + 2\beta\mu_{i}T) < 6, \text{ and} \\ -(T\mu_{i} + T^{2}\mu_{i} - T^{2}\beta\mu_{i}^{2} - T^{3}\beta\mu_{i}^{2}) < 4. \end{cases}$$

control law (3.32) stabilize the tracking error of all followers to zero.

Proof 3.2.4. By defining the velocity and position disagreements respectively as

$$\begin{cases} \delta_{v_f[k+1]} = v_{f[k+1]} - v_{f[k+1]}^*, \\ \delta_{x_f[k+1]} = x_{f[k+1]} - x_{f[k+1]}^*, \end{cases}$$
(3.33)

control law (3.32) can then be rewritten in terms of the follower agents disagreements as

$$\begin{bmatrix} \delta_{x_f[k+1]} \\ \delta_{v_f[k+1]} \end{bmatrix} = \left( \underbrace{\begin{bmatrix} (I_{n_f} - \frac{T^2}{2}\Omega_{ff}) & (TI_{n_f} - \frac{\beta T^2}{2}\Omega_{ff}) \\ -T\Omega_{ff} & (I_{n_f} - \beta T\Omega_{ff}) \end{bmatrix}}_{H} \otimes I_d \right) \begin{bmatrix} \delta_{x_f[k]} \\ \delta_{v_f[k]} \end{bmatrix}$$
(3.34)

The characteristic polynomial of H in (3.34) is  $det(sI_{2n_f} - H)$ , i.e.,

$$det \left( \begin{bmatrix} sI_{n_f} - (I_{n_f} - \frac{T^2}{2}\Omega_{ff}) & (TI_{n_f} - \frac{\beta T^2}{2}\Omega_{ff}) \\ T\Omega_{ff} & sI_{n_f} - (I_{n_f} - \beta T\Omega_{ff}) \end{bmatrix} \right)$$

$$= [sI_{n_f} - (I_{n_f} - \frac{T^2}{2}\Omega_{ff})][sI_{n_f} - (I_{n_f} - \beta T\Omega_{ff})] - [T\Omega_{ff}(TI_{n_f} - \frac{\beta T^2}{2}\Omega_{ff})]$$
  
$$= s^2 I_{n_f} - (I_{n_f} - \frac{T^2}{2}\Omega_{ff} + I_{n_f} - \beta T\Omega_{ff})s + (I_{n_f} - \frac{T^2}{2}\Omega_{ff})(I_{n_f} - \beta T\Omega_{ff})$$
  
$$- T\Omega_{ff}(TI_{n_f} - \frac{\beta T^2}{2}\Omega_{ff}).$$
(3.35)

Denoting the *i*th eigenvalue of  $-\Omega_{ff}$  by  $\mu_i$ , so that  $det(sI_n + \Omega_{ff}) = \prod_{i=1}^n (s - \mu_i)$ , it follows that (3.35) must satisfy

$$s^{2} + \left(\frac{T^{2}\beta\mu_{i}^{2}}{2} - T^{2}\mu_{i} - \frac{T\mu_{i}}{2} + \frac{T^{3}\beta\mu_{i}^{2}}{2} - T\beta\mu_{i} + 1\right)s + \left(-\frac{T^{2}\mu_{i}^{2}}{2} - T\beta\mu_{i} - 2\right) = 0.$$
(3.36)

By setting  $a = \frac{T^2 \beta \mu_i^2}{2} - T^2 \mu_i - \frac{T \mu_i}{2} + \frac{T^3 \beta \mu_i^2}{2} - T \beta \mu_i + 1$ ,  $b = -\frac{T^2 \mu_i^2}{2} - T \beta \mu_i - 2$ and considering lemma 3.2.2, using the bi-linear transformation  $s = \frac{t+1}{t-1}$  for (3.36), the expression

$$\frac{1}{2} \left( T^2 \beta \mu_i^2 - 3T^2 \mu_i - T \mu_i + T^3 \beta \mu_i^2 - 4T \beta \mu_i \right) t^2 + \left( T^2 \mu_i + 2\beta \mu_i T + 6 \right) + \frac{1}{2} \left( T \mu_i + T^2 \mu_i - T^2 \beta \mu_i^2 - T^3 \beta \mu_i^2 + 4 \right) = 0$$
(3.37)

is derived. Note that  $T, \beta > 0$  and  $\mu_i < 0$ . Also note that all the roots of (3.35) are within a unit circle if all the roots of (3.37) are in the open left half plane. Since  $(T^2\beta\mu_i^2 - 3T^2\mu_i - T\mu_i + T^3\beta\mu_i^2 - 4T\beta\mu_i) > 0$ , considering that  $\mu_i$  is always negative, then ensuring that

$$\begin{cases} -(T^{2}\mu_{i} + 2\beta\mu_{i}T) < 6, \text{ and} \\ -(T\mu_{i} + T^{2}\mu_{i} - T^{2}\beta\mu_{i}^{2} - T^{3}\beta\mu_{i}^{2}) < 4. \end{cases}$$

ensures that the roots of (3.35) are within the unit circle.

### 3.2.4 Summary

In this Chapter, we have studied the affine formation control of multi-agent systems with single- and double-integrator dynamics in the sampled-data setting. We showed condition on control gains and sampling time to guarantee the stability of the overall system and proposed control laws that are distributed and able to track time-varying targets that are affine formation of the nominal formations.

# Chapter 4

# Affine Formation Algorithms and Implementation Based on Triple-Integrator Dynamics

## 4.1 INTRODUCTION

In this chapter, we study the affine formation maneuver control problem of multi-agent systems (MASs) described by triple-integrator agent dynamics. Previous studies on affine formation control of MASs only considered the case where inter-agent communication (or sensing) occur continuously in time and the agents' dynamics are limited to double-integrators. In real-life situations, however, agents may only communicate in periodic time intervals and may have agent dynamics described by triple-integrators. Triple-integrator agent dynamics have found applications in robot motion control, aircraft control, lifts and a wide range of mechanical control systems. For example, the DC motor which features as an actuator in a vast majority of mechanical control systems is normally modelled with triple-integrator agent dynamics are used to approximate the individual agent dynamics in an *n*-vehicle system of travelling along a single lane for a drive-train model in [112]. A broad range of systems could be modelled using triple-integrator

agent dynamics, see [113], [114]. Therefore, it is important to broaden the application area by considering the stability conditions for MASs with triple-integrator agent dynamics.

Here, we propose two control laws based on periodic and continuous communications. Sufficient conditions are presented to guarantee the global stability of the proposed control laws. The proposed laws are implemented for four cases consisting of two scenarios each considered for both sampled-data and continuous-time agent communication cases. Please not that the results contained in this chapter have been published in the paper titled "Affine Formation Maneuver Control of Multi-Agent Systems with Triple-Integrator Dynamics, and "Affine Formation Algorithms and Implementation Based on Triple-Integrator Dynamics."

### 4.2 **Problem Formulation**

Consider a MAS with n agents. Let the position of the *i*th agent be denoted by  $x_1, ..., x_n \in \mathbb{R}^d$ , so that the *i*th agent's target position in the time-varying formation is given by

$$x_i^*(t) = A(t)r_i + b(t)$$
(4.1)

where both  $b(t) \in \mathbb{R}^d$  and  $A(t) \in \mathbb{R}^{d \times d}$  are time-varying and the nominal (constant reference) configuration is denoted by  $r_i \in \mathbb{R}^d$ . Equation (4.1) is written in global form as

$$x^*(t) = [I_n \otimes A(t)]r + \mathbf{1}_n \otimes b(t)$$
(4.2)

where  $r = [r_1^T, ..., r_n^T] \in \mathbb{R}^{nd}$  and  $x^*(t) \in \mathbb{R}^{nd}$  respectively denote the reference configuration and the targets (time-varying) to be tracked. The affine image is the set of all affine transform of the nominal configuration. Note that the tracked time-varying targets are affine images of the nominal configuration r.

We define the affine image as a collection of all the affine transformation of the reference configuration r. Note that the time-varying targets are affine images of the reference

configuration. The affine image is given in global form by [98]

$$\mathcal{A}(r) = \{ x \in \mathbb{R}^{dn} : x = (I_n \otimes A)r + \mathbf{1}_n \otimes b, A \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d \}.$$
(4.3)

The overall goal is to find conditions that guarantees that

$$\lim_{x \to \infty} x(t) = x^*(t), \ \forall x^*(t) \in \mathcal{A}(r).$$

# 4.3 Affine Formation Control Law for Continuoustime Coordination of Multi-agent Systems Described by Triple-Integrator Agent Dynamics

Consider the MAS where the agents communicate continuously in time, their interagent communication is modelled using an undirected graph, and each agent has dynamics described using triple-integrator, such that

$$\begin{cases} \dot{x}_{i(t)} = v_{i(t)}, \\ \dot{v}_{i(t)} = a_{i(t)}, \\ \dot{a}_{i(t)} = -\sum_{j \in \mathcal{N}_i} w_{ij} [k_x(x_{i(t)} - x_{j(t)}) + k_v(v_{i(t)} - v_{j(t)}) \\ + k_a(a_{i(t)} - a_{j(t)})], \quad i \in \mathcal{V}_f, \end{cases}$$

$$(4.4)$$

where  $k_a$ ,  $k_v$  and  $k_x$  are positive constant control gains. Note that, for brevity, we have dropped the subscript (t) in (4.4) for the remainder of this section. System (4.4) can be given in matrix-vector form by

$$\begin{cases} \dot{x}_{f} = v_{f}, \\ \dot{v}_{f} = a_{f}, \\ \dot{a}_{f} = -k_{x}[(\Omega_{ff} \otimes I_{d})x_{f} + (\Omega_{fl} \otimes I_{d})x_{l}^{*}] \\ -k_{v}[(\Omega_{ff} \otimes I_{d})v_{f} + (\Omega_{fl} \otimes I_{d})v_{l}^{*}] \\ -k_{a}[(\Omega_{ff} \otimes I_{d})a_{f} + (\Omega_{fl} \otimes I_{d})a_{l}^{*}], \end{cases}$$

$$(4.5)$$

where the states  $x_f, v_f, a_f \in \mathbb{R}^{dn_f}$  respectively denote the position, velocity and acceleration of the followers while  $x_l^*, v_l^* = \dot{x}_l$  and  $a_l^* = \dot{v}_l$  have been used to denote those of the leaders respectively. Define the position tracking error as

$$\delta_{x_f} = x_f - x_f^* = x_f + (\Omega_{ff}^{-1} \Omega_{fl} \otimes I_d) x_l^*,$$

where  $x_f^*$  denotes the target positions of the followers. Next, the stability of control law (4.5) is investigated. Let  $\mu_i$  denote the *i*th eigenvalue of  $-\Omega_{ff}$ .

**Theorem 4.3.1.** Assume that Assumptions 1, 2 and 3 (the Assumptions are in Sections 2.6 and 2.7) hold, and the leaders' jerk  $\dot{a}_l$  are constantly zero, then by choosing the control gains such that  $-k_v k_a \mu_i > k_p$ ,  $\forall i$ , control law (4.5) guarantees that the errors in the followers positions  $\delta_{x_f}$  converge to zero.

*Proof.* Define the disagreements of the position, velocity and acceleration respectively by  $\delta_{x_f} = x_f + (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d) x_l^*$ ,  $\delta_{v_f} = v_f + (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d) v_l^*$  and  $\delta_{a_f} = a_f + (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d) a_l^*$ so that (4.5) can be re-written as functions of their disagreements as

$$\begin{bmatrix} \dot{\delta}_{x_f} \\ \dot{\delta}_{v_f} \\ \dot{\delta}_{a_f} \end{bmatrix} = \left( \underbrace{ \begin{pmatrix} \mathbf{0}_{n_f \times n_f} & I_{n_f} & \mathbf{0}_{n_f \times n_f} \\ \mathbf{0}_{n_f \times n_f} & \mathbf{0}_{n_f \times n_f} & I_{n_f} \\ -k_p \Omega_{ff} & -k_v \Omega_{ff} & -k_a \Omega_{ff} \end{bmatrix}}_{F_1} \otimes I_d \right) \begin{bmatrix} \delta_{x_f} \\ \delta_{v_f} \\ \delta_{a_f} \end{bmatrix} + \left( \begin{pmatrix} \mathbf{0}_{n_f \times n_f} \\ \mathbf{0}_{n_f \times n_f} \\ \Omega_{ff} \Omega_{fl} \end{bmatrix} \otimes I_d \right) \dot{a}_l^*.$$
(4.6)

Note that  $\dot{a}_l^* = 0$  for this study. The system matrix of (4.6) has the characteristic polynomial given by

$$det([sI_{3n_{f}} - F_{1}] \otimes I_{d}) = \\ det \begin{pmatrix} sI_{n_{f}} & -I_{n_{f}} & \mathbf{0}_{n_{f} \times n_{f}} \\ \mathbf{0}_{n_{f} \times n_{f}} & sI_{n_{f}} & -I_{n_{f}} \\ k_{p}\Omega_{ff} & k_{v}\Omega_{ff} & (sI_{n_{f}} + k_{a}\Omega_{ff}) \end{pmatrix} \otimes I_{d} \end{pmatrix}.$$

Note that  $\mu_i$  denotes the *i*th eigenvalue of  $-\Omega_{ff}$ . Thus,

$$det(sI_n + \Omega_{ff}) = \prod_{i=1}^n (s - \mu_i).$$
(4.7)

This implies that

$$det([sI_{3n_f} - F_1] \otimes I_d) = det(sI_{3n_f} - F_1)$$

By noting (4.6) and (4.7), the expression

$$det([sI_{3n_f} - F_1] \otimes I_d) = \prod_{i=1}^n \left[ s^3 - k_a \mu_i s^2 - k_v \mu_i s - k_p \mu_i \right].$$

is obtained. Therefore, the eigenvalues satisfy

$$s^{3} - k_{a}\mu_{i}s^{2} - k_{v}\mu_{i}s - k_{p}\mu_{i} = 0.$$
(4.8)

Since  $k_a, k_v, k_x > 0$  and  $-\mu_i > 0, \forall_i$ , all errors converge to zero if the control gains  $k_a, k_v$  and  $k_x$  are chosen such that  $-k_v k_a \mu_i > k_p, \forall_i \in \mathcal{V}_f$ .

**Remark 4.3.1.** The choice of  $-k_v k_a \mu_{max} > k_p$  satisfies the convergence requirement  $-k_v k_a \mu_i > k_p$  for all followers. Note that  $\mu_{max}$  has been used to denote the greatest eigenvalue of  $-\Omega_{ff}$ , i.e., the greatest  $\mu_i$ .

**Remark 4.3.2.** Note that the case of time-varying jeck can be studied using the protocol

$$\begin{cases} \dot{x}_{i(t)} = v_{i(t)}, \\ \dot{v}_{i(t)} = a_{i(t)}, \\ \dot{a}_{i(t)} = -\frac{1}{\gamma} \sum_{j \in \mathcal{N}_i} w_{ij} [k_x(x_{i(t)} - x_{j(t)}) + k_v(v_{i(t)} - v_{j(t)}) \\ + k_a(a_{i(t)} - a_{j(t)}) - \dot{a}_j], \quad i \in \mathcal{V}_f, \end{cases}$$

where  $\gamma = \sum_{j \in \mathcal{N}_i}$ . Further stability analysis is not included here for brevity.

# 4.4 Affine Formation Control Law for Sampled-data Coordination of Multi-agent Systems Described by Triple-Integrator Agent Dynamics

Consider the MAS coordination case where the agents are described with tripleintegrator dynamics modelled in continuous-time by

$$\dot{x}_{i(t)} = v_{i(t)}, \quad \dot{v}_{i(t)} = a_{i(t)}, \quad \dot{a}_{i(t)} = u_{i(t)}.$$
(4.9)

Assume that each agent has continuous-time dynamics, but communicate with (or sense) their neighbours at periodic time intervals and their control inputs are zero-order hold based. Such that, following [115],

$$u_i(t) = u_i(kT), \quad kT \le t < (k+1)T,$$

where T, k,  $u_{i(t)}$ , and  $u_{i[k]}$  respectively denote sampling period, discrete-time index, control input at time t (continuous-time), and control input at t = kT.

Discretization of (4.9) yields

$$\begin{cases} x_{i[k+1]} = x_{i[k]} + Tv_{i[k]} + \frac{T^2}{2}a_{i[k]} + \frac{T^3}{6}u_{i[k]}, \\ v_{i[k+1]} = v_{i[k]} + Ta_{i[k]} + \frac{T^2}{2}u_{i[k]}, \\ a_{i[k+1]} = a_{i[k]} + Tu_{i[k]}, \qquad i \in \mathcal{V}_f, \end{cases}$$

$$(4.10)$$

where  $a_i$ ,  $v_i$  and  $x_i$  have respectively been used to denote the acceleration, velocity and position of agent *i*. Next, we study the sampled-data case where the jerk of leaders are constantly zero. Consider the protocol,

$$u_{i[k]} = -\sum_{j \in \mathcal{N}_i} w_{ij} [k_x(x_{i[k]} - x_{j[k]}) + k_v(v_{i[k]} - v_{j[k]}) + k_a(a_{i[k]} - a_{j[k]})], \quad i \in \mathcal{V}_f,$$

which is given in closed-loop form by

$$u_{f} = -k_{x} [(\Omega_{ff} \otimes I_{d})x_{f} + (\Omega_{fl} \otimes I_{d})x_{l}^{*}] -k_{v} [(\Omega_{ff} \otimes I_{d})v_{f} + (\Omega_{fl} \otimes I_{d})v_{l}^{*}] -k_{a} [(\Omega_{ff} \otimes I_{d})a_{f} + (\Omega_{fl} \otimes I_{d})a_{l}^{*}].$$

$$(4.11)$$

Using (4.11), (4.10) can be re-written in matrix form as

$$\begin{bmatrix} x_{f[k+1]} \\ v_{f[k+1]} \\ a_{f[k+1]} \end{bmatrix} = \\ \begin{pmatrix} \left[ (I_{nf} - \frac{T^{3}}{6} k_{x} \Omega_{ff}) & (TI_{nf} - \frac{T^{3}}{6} k_{v} \Omega_{ff}) & (\frac{T^{2}}{2} I_{nf} - \frac{T^{3}}{6} k_{a} \Omega_{ff}) \\ -\frac{T^{2}}{2} k_{x} \Omega_{ff} & (I_{nf} - \frac{T^{2}}{2} k_{v} \Omega_{ff}) & (TI_{nf} - \frac{T^{2}}{2} k_{a} \Omega_{ff}) \\ -Tk_{x} \Omega_{ff} & -Tk_{v} \Omega_{ff} & (I_{nf} - Tk_{a} \Omega_{ff}) \end{bmatrix} \\ \times \begin{bmatrix} x_{f[k]} \\ v_{f[k]} \\ a_{f[k]} \end{bmatrix} \\ + \begin{pmatrix} \left[ -\frac{T^{3}}{6} k_{x} \Omega_{fl} & -\frac{T^{3}}{6} k_{v} \Omega_{fl} & -\frac{T^{3}}{6} k_{a} \Omega_{fl} \\ -\frac{T^{2}}{2} k_{x} \Omega_{fl} & -\frac{T^{2}}{2} k_{v} \Omega_{fl} & -\frac{T^{2}}{2} k_{a} \Omega_{fl} \\ -Tk_{x} \Omega_{fl} & -Tk_{v} \Omega_{fl} & -Tk_{a} \Omega_{fl} \end{bmatrix} \otimes I_{d} \begin{bmatrix} x_{l[k]}^{*} \\ v_{l[k]} \\ a_{l[k]}^{*} \end{bmatrix} \end{cases}$$
(4.12)

where  $v_l^* \in \mathbb{R}^{dn_l}$  and  $v_f \in \mathbb{R}^{dn_f}$  is used to respectively denote the velocities of the leaders and followers. Define the position tracking error as

$$\delta_{f[k]} = x_{f[k]} - x_{f[k]}^* = x_{f[k]} + (\Omega_{ff}^{-1} \Omega_{fl} \otimes I_d) x_{l[k]}^*,$$

where  $x_{f[k]}^*$  denotes the target positions of the followers. Next, we analyse the stability of control law (4.12).

**Theorem 4.4.1.** Assume that Assumptions 1, 2 and 3 (the Assumptions are in Sections 2.6 and 2.7) hold, and the leaders jerk is constantly zero, by choosing  $k_x$ ,  $k_a$ ,  $k_v$ 

and T such that  $(2k_v - Tk_x) > 0$  and  $k_v(3Tk_v - T^2k_x - 6k_a)\mu_i - 6k_x > 0$ , the tracking error  $\delta_{x_f[k]}$  of all followers is guaranteed to stabilize to the origin by control law (4.12).

*Proof.* Let the state errors (or disagreements) of the followers be defined by

$$\left[\delta_{x_{f}[k]}^{T}, \delta_{v_{f}[k]}^{T}, \delta_{a_{f}[k]}^{T}\right]^{T} = \left[x_{f[k]}^{T}, v_{f[k]}^{T}, a_{f[k]}^{T}\right]^{T} - \left[x_{f[k]}^{*T}, v_{f[k]}^{*T}, a_{f[k]}^{*T}\right]^{T},$$
(4.13)

where  $a_f^*, v_f^*, x_f^* \in \mathbb{R}^{dn_f}$  respectively denote the target accelerations, velocities and positions of the followers. By taking (4.13) and control law (4.12) into consideration, the system matrix of the error dynamics is defined by

$$\underbrace{\begin{bmatrix} (I_{nf} - \frac{T^{3}}{6}k_{x}\Omega_{ff}) & (TI_{nf} - \frac{T^{3}}{6}k_{v}\Omega_{ff}) & (\frac{T^{2}}{2}I_{nf} - \frac{T^{3}}{6}k_{a}\Omega_{ff}) \\ -\frac{T^{2}}{2}k_{x}\Omega_{ff} & (I_{nf} - \frac{T^{2}}{2}k_{v}\Omega_{ff}) & (TI_{nf} - \frac{T^{2}}{2}k_{a}\Omega_{ff}) \\ -Tk_{x}\Omega_{ff} & -Tk_{v}\Omega_{ff} & (I_{nf} - Tk_{a}\Omega_{ff}) \end{bmatrix}}_{F_{2}} \otimes I_{d}.$$
(4.14)

The characteristic polynomial of  $F_2 \otimes I_d$  in (4.14) is given by  $det([sI_{3nf} - F_2] \otimes I_d)$ , which equals  $det(sI_{3nf} - F_2)$ . Note that  $\mu_i$  is used to denote the *i*th eigenvalue of  $-\Omega_{ff}$ . Thus,  $det(sI_n + \Omega_{ff}) = \prod_{i=1}^n (s - \mu_i)$ .

Therefore, the characteristic equation of (4.14) satisfies

$$s^{3} - \left(\frac{T^{3}}{6}k_{x}\mu_{i} + \frac{T^{2}}{2}k_{v}\mu_{i} + Tk_{a}\mu_{i} + 3\right)s^{2} + \left(-\frac{2T^{3}}{3}k_{x}\mu_{i} + 2Tk_{a}\mu_{i} + 3\right)s - \left(\frac{T^{3}}{6}k_{x}\mu_{i} - \frac{T^{2}}{2}k_{v}\mu_{i} + Tk_{a}\mu_{i} + 1\right) = 0.$$

$$(4.15)$$

Using the bi-linear transform  $s = \frac{t+1}{t-1}$ , for (4.15), the expression

$$-3T^{3}k_{x}\mu_{i}t^{3} + (3T^{3}k_{x}\mu_{i} - 6T^{2}k_{v}\mu_{i})t^{2} + (T^{3}k_{x}\mu_{i} + 6T^{2}k_{v}\mu_{i} - 12Tk_{a}\mu_{i})t + (-T^{3}k_{x}\mu_{i} + 12Tk_{a}\mu_{i} + 24) = 0$$
(4.16)

is obtained. Thus, a sufficient condition for the stability of (4.12) is for all the roots of (4.16) to be on the left half plane. Note that the entire roots of (4.15) fall within a unit circle if the entire roots of (4.16) are situated in the open left half plane. Note that  $-3T^3k_x\mu_i > 0$ , because  $\mu_i$  is always negative. Thus, based on the Routh-Hurwitz stability criterion, ensuring that

$$3T^{3}k_{x}\mu_{i} - 6T^{2}k_{v}\mu_{i} > 0, \quad \text{and}$$

$$(3T^{3}k_{x}\mu_{i} - 6T^{2}k_{v}\mu_{i})(T^{3}k_{x}\mu_{i} + 6T^{2}k_{v}\mu_{i} - 12Tk_{a}\mu_{i})$$

$$+ 3T^{3}k_{x}\mu_{i}(-T^{3}k_{x}\mu_{i} + 12Tk_{a}\mu_{i} + 24) > 0,$$

guarantees that the roots of (4.15) are within the unit circle. By further algebraic simplification, these requirements are respectively reduced to  $(2k_v - Tk_x) > 0$  and  $k_v(3Tk_v - T^2k_x - 6k_a)\mu_i - 6k_x > 0.$ 

### 4.5 Implementation

The first step in the implementation of the proposed algorithms is the design of a reference formation that satisfies Assumptions 1 and 2. That is, the reference formation should be both generically universally rigid and have at least d + 1 nodes, that span the  $\mathbb{R}^d$  space affinely, to be selected as leaders. The next step is to compute the stress matrix  $\Omega$ . Equation (2.6) is used to compute the stress matrix in this study.

The simulations were done in obstacle avoidance scenarios. The acceleration varied slowly at some points, however, the control algorithms are still effective in these cases. We now present the results of our four implemention cases grouped into two scenarios. Note that, the agents' connections in the framework (e.g. in Fig.4.6) are denoted with straight lines and (2,0) is used to denote that the agent's position on the x- and y-axis are respectively 2 and 0.

### 4.5.1 Scenario One

Consider a five-agent MAS where each agent is modelled as triple-integrators. Assume that the agents are denoted by i = 1, 2, ..., 5. Let Fig. 4.1 denote the communication graph of the agents.



Figure 4.1: Framework showing agent communication along with their 2-dimensional reference positions.

Agents 1 to 3 are selected as leaders while the rest are followers. The stress matrix of the graph is given by

$$\Omega = \begin{bmatrix} \Omega_{ll} & \Omega_{lf} \\ \Omega_{fl} & \Omega_{ff} \end{bmatrix}$$

where

$$\begin{cases} \Omega_{ll} = \begin{bmatrix} 0.2919 & -0.2919 & -0.2919 \\ -0.2919 & 0.3544 & 0.3544 \\ -0.2919 & 0.3544 & 0.3544 \end{bmatrix}, \\ \Omega_{ff} = \begin{bmatrix} 1.2919 & -0.5000 \\ -0.5000 & 0.2500 \end{bmatrix}, \\ \Omega_{fl} = \begin{bmatrix} 0.2919 & -0.5419 & -0.5419 \\ 0 & 0.1250 & 0.1250 \end{bmatrix} \text{ and } \\ \Omega_{lf} = \begin{bmatrix} 0.2919 & 0 \\ -0.5419 & 0.1250 \\ -0.5419 & 0.1250 \end{bmatrix}. \end{cases}$$

The largest eigenvalue of  $-\Omega_{ff}$ ,  $\mu_{max} = -0.049$ . The leaders' initial positions are their respective nominal positions while the followers are initialized to the following positions:  $P_4(-3, 1)$  and  $P_5(-1, 2)$ , where  $P_5(-1, 2)$  implies that the position of agent 5 is (-1, 2). We now present the results of two simulation studies where the leaders' paths are generated in advance.

#### Continuous-time Inter-Agent Communication Case

This study considers the case where inter-agent communication occurs continuously in time. We use control law (4.5) in this study. The control parameters are chosen to satisfy the requirement of Theorem 6.3, i.e.,  $-k_v k_a \mu_i > k_p$ ,  $\forall i$ . Here, we consider  $-k_v k_a \mu_{max} > k_p$ , which is the lower bound. From (4.17),  $\mu_{max} = -0.049$  for  $-\Omega_{ff}$ . It is easy to verify that the choice of the control gains  $k_x = 0.8$ ,  $k_v = 8$  and  $k_a = 10$  with  $\mu_{max} = -0.049$  satisfy the requirement of Theorem 6.3 ( $-k_v k_a \mu_i > k_p$ ,  $\forall i$ ). These parameters are used in the simulation.

In the simulation study, the velocities and accelerations of the leaders are estimated using discrete differentiators (zero-order hold based). Note that the trajectories of the leaders is assumed to be piecewise continuous and differentiable. The simulation result showing the agents' positions is presented in Fig. 4.2. It shows the formation maneuver around obstacles on its path. The trajectories of the agents' velocities and accelerations are presented in Fig. 4.12 and the tracking errors of the followers' positions are presented in Fig. 4.3.



Figure 4.2: Simulation illustrating the agents' positions in the continuous-time case of scenario one, based on control law (4.5). The simulation is carried out in a collision scenario. Here, the leaders are able to sense obstacles but the followers can not. The leaders are to steer the followers through the journey and simultaneously avoid any obstacles on their path. The leaders formation is changed to maneuver the overall formation. The followers are able to track their new formation following any change based on control law (4.5).

#### Sampled-data Inter-Agent Communication Case

This study considers the case where inter-agent communication occurs in periodic time intervals. We use control law (4.12) in this study. The control parameters are chosen to satisfy the requirements of Theorem 4.4.1. Denote the least eigenvalue of  $-\Omega_{ff}$ by  $\mu_{min}$ . It can be verified from (4.17), that  $\mu_{max} = -0.049$  and  $\mu_{min} = -1.493$ in this case. It is easy to verify that the choices of T = 0.1,  $k_x = 0.8$ ,  $k_v = 8$ and  $k_a = 8$  satisfy the requirements of Theorem 4.4.1, i.e.,  $(2k_v - Tk_x) > 0$  and  $k_v(3Tk_v - T^2k_x - 6k_a)\mu_i - 6k_x > 0$ ,  $\forall i$ . Note that both  $\mu_{max}$  and  $\mu_{min}$  are used because they form the boundary.



Figure 4.3: Simulation illustrating the positions error dynamics for the continuoustime case of scenario one

The simulation result showing the agents' positions is presented in Fig. 4.4. It shows the formation maneuver around obstacles on its path. The trajectories of the agents' velocities and accelerations are presented in Fig. 4.13 and the tracking errors of the followers' positions are shown in Fig. 4.5.

### 4.5.2 Scenario Two

Consider a seven-agent MAS where each agent is modelled as triple-integrators. Assume that the agents are denoted by i = 1, 2, ..., 7. Let Fig. 4.6 denote the communication graph of the agents.

Agents 1 to 3 are selected as leaders while the rest are followers. The stress matrix of the graph is given by [100].

$$\Omega = \begin{bmatrix} \Omega_{ll} & \Omega_{lf} \\ \Omega_{fl} & \Omega_{ff} \end{bmatrix}$$

where



Figure 4.4: Simulation illustrating the agents' positions in the sampled-data case of scenario one, based on control law (4.12). The simulation is carried out in a collision scenario. Here, the leaders are able to sense obstacles but the followers can not. The leaders are to steer the followers through the journey and simultaneously avoid any obstacles on their path. The leaders formation is changed to maneuver the overall formation. The followers are able to track their new formation following any change based on control law (4.12).

$$\begin{cases} \Omega_{ll} = \begin{bmatrix} 0.2742 & -0.2741 & -0.2741 \\ -0.2741 & 0.6853 & 0 \\ -0.2741 & 0 & 0.6853 \end{bmatrix}, \\ \Omega_{ff} = \begin{bmatrix} 0.7538 & -0.0685 & -0.2741 & 0 \\ -0.0685 & 0.7538 & 0 & -0.2741 \\ -0.2741 & 0 & 0.2741 & -0.1370 \\ 0 & -0.2741 & -0.1370 & 0.2741 \end{bmatrix}, \\ \Omega_{fl} = \begin{bmatrix} 0.1370 & -0.5482 & 0 \\ 0.1370 & 0 & -0.5482 \\ 0 & 0 & 0.1370 \\ 0 & 0.1370 & 0 \end{bmatrix} \text{ and } \\ \Omega_{lf} = \begin{bmatrix} 0.1370 & 0 & 0 \\ 0 & 0 & 0.1370 \\ 0 & 0 & 0.1370 \\ -0.5482 & 0.1370 & 0 \end{bmatrix}.$$



Figure 4.5: Simulation illustrating the positions error dynamics for the sampled-data case of scenario one.



Figure 4.6: Framework showing agent communication along with their 2-dimensional reference positions.

The largest eigenvalue of  $-\Omega_{ff}$ ,  $\mu_{max} = -0.024$ . The leaders' initial positions are their respective nominal positions while the followers are initialized to the following

positions:  $P_4(0,2)$ ,  $P_5(0,-2)$ ,  $P_6(-1,3)$  and  $P_7(-1,3)$ , where  $P_5(0,-2)$  implies that the position of agent 5 is (0,-2). We now present the results of two simulation studies where the leaders' paths are generated in advance.

#### Continuous-time Inter-Agent Communication Case

This study considers the case where inter-agent communication occurs continuously in time. We use control law (4.5) in this study. Note that in this case  $\mu_{max} = -0.024$ . The control gains  $k_x$ ,  $k_v$  and  $k_a$  are respectively chosen to be 0.8, 8 and 9. It is easy to verify that this satisfies the requirement of Theorem 6.3, i.e.,  $-k_v k_a \mu_i > k_p$ ,  $\forall i$ . These parameters are used in the simulation.

In the simulation study, the velocities and accelerations of the leaders are estimated using discrete differentiators (zero-order hold based). Note that the trajectories of the leaders is assumed to be piecewise continuous and differentiable.

The simulation result showing the agents' positions is presented in Fig. 4.7. It shows the formation maneuver around obstacles on its path. The trajectories of the agents' velocities and accelerations are presented in Fig. 4.14 and the tracking errors of the followers' positions are presented in Fig. 4.8.

#### Sampled-data Inter-Agent Communication Case

This study considers the case where inter-agent communication occurs in periodic time intervals. We use control law (4.12) in this study. Denote the least eigenvalue of  $-\Omega_{ff}$  by  $\mu_{min}$ . It can be verified from (4.18) that  $\mu_{max} = -0.024$  and  $\mu_{min} = -0.96$ in this case. Thus, the choices of T = 0.1,  $k_x = 0.8$ ,  $k_v = 8$  and  $k_a = 8$  can be verified to satisfy the requirements of Theorem 4.4.1, i.e.,  $(2k_v - Tk_x) > 0$  and  $k_v(3Tk_v - T^2k_x - 6k_a)\mu_i - 6k_x > 0$ ,  $\forall i$ . Note that both  $\mu_{max}$  and  $\mu_{min}$  are used because they form the boundary.

The simulation result showing the agents' positions is presented in Fig. 4.9. It shows the formation maneuver around obstacles on its path. The trajectories of the agents'



Figure 4.7: Simulation illustrating the agents' positions in the continuous-time case of scenario two, based on control law (4.5). The simulation is carried out in a collision scenario. Here, the leaders are able to sense obstacles but the followers can not. The leaders are to steer the followers through the journey and simultaneously avoid any obstacles on their path. The leaders formation is changed to maneuver the overall formation. The followers are able to track their new formation following any change based on control law (4.5).



Figure 4.8: Simulation illustrating the positions error dynamics for the continuoustime case of scenario two.

velocities and accelerations are presented in Fig. 4.11 and the tracking errors of the followers' positions are presented in Fig. 4.10.



Figure 4.9: Simulation illustrating the agents' positions in the sampled-data case of scenario two, based on control law (4.12). The simulation is carried out in a collision scenario. Here, the leaders are able to sense obstacles but the followers can not. The leaders are to steer the followers through the journey and simultaneously avoid any obstacles on their path. The leaders formation is changed to maneuver the overall formation. The followers are able to track their new formation following any change based on control law (4.12).

### 4.6 Summary

In this chapter, we have studied the affine formation control of MASs with tripleintegrator agent dynamics in both sampled-data and continuous-time settings. Algorithms, based on stress matrix, are proposed to accomplish formation control in each setting. Sufficient conditions on the sampling intervals and control gains for the overall stability of the formation are presented. Implementations were carried out for four cases. The results obtained in all four cases are in agreement with the proposed algorithms. The proposed control algorithms are capable of tracking time-varying transformations that are the affine transform of the nominal formation if the jerk of



Figure 4.10: Simulation illustrating the positions error dynamics for the sampled-data case of scenario two.

the agents is zero. An ongoing study is on extending the scheme to the general linear multi-agent system case.



Figure 4.11: Simulation depicting the agents' velocity and acceleration trajectories for the sampled-data case of Scenario two.



Figure 4.12: Simulation depicting the agents' velocity and acceleration trajectories for the continuous-time case of Scenario one.



Figure 4.13: Simulation depicting the agents' velocity and acceleration trajectories for the sampled-data case of Scenario one.



Figure 4.14: Simulation depicting the agents' velocity and acceleration trajectories for the continuous-time case of Scenario two.

## Chapter 5

# Optimal Affine Formation Control of Linear Multi-agent Systems

This study considers the affine formation control problem of linear multi-agent systems. Previous studies are limited to the cases where the dynamics of the agents are described by mere network of integrators. Note that the results contained in this chapter have been published in the paper titled "Optimal Affine Formation Control of Linear Multiagent Systems."

## 5.1 Problem Formulation

Consider a group of multi-agent system composed of n agents ( $n_l$  leaders and  $n_f$  followers) with every agent described by a similar linear time invariant dynamic equation

$$\dot{x}_i = Ax_i + Bu_i,\tag{5.1}$$

where  $u_i$  and  $x_i \in \mathbb{R}^d$  respectively denote the control input and the state of agent i while A and B are constant matrices of appropriate dimensions.

**Assumption 4.** Assume that the pair (A, B) is controllable.

Equation (5.1) is given in global form by

$$\dot{x} = (I_n \otimes A)x + (I_n \otimes B)u. \tag{5.2}$$

Let  $r = [r_1^T, \ldots, r_n^T]^T = [r_l^T, r_f^T]^T \in \mathbb{R}^{nd}$  denote a constant configuration, where  $r_l$  and  $r_f$  respectively denote the configuration of the leaders and followers. We define the target configuration,  $r^*$  as

$$r^* = (I_n \otimes \Lambda)r + \mathbf{1_n} \otimes \beta,$$

knowing that by choosing appropriate values for  $\Lambda$  and  $\beta$  we can achieve our desired target configuration. By singular value decomposition, a real matrix  $\Lambda$  can be decomposed into  $U_{\Sigma}V$ , where V and U are unitary matrices and  $\Sigma$  is a diagonal  $n \times n$ matrix. Thus, the desired target configuration is attained by an appropriate translation  $\beta$ , followed by rotation V, scaling  $\Sigma$  and then rotation U. All these together achieve the affine transformation.

We propose a following affine formation control protocol for multi-agent systems modelled with undirected communication graph under the distributed control protocol

$$\xi_i = -cK_2 \sum w_{ij}(x_i - x_j), \quad i \in f,$$
(5.3)

where c > 0 and  $K_2 \in \mathbb{R}^{m \times n}$  respectively denote the coupling gain and local feedback gain.

The agents are considered to have achieved affine formation if

$$\lim_{t \to \infty} \|x_i - x_i^*\| = 0,$$

where  $x_i^*$  is the target position of agent *i*.

The goal of this study is to design a control in the form  $u = -c(\Omega \otimes K_2)x$  that ensures each follower agent in the affine formation tracks its target state using a distributed framework that is globally stabilizing and optimal with respect to an LQR performance index.

### 5.2 Main Results

The globally optimal affine formation control problem is to design a distributed protocol  $u_i$  for all follower nodes, such that each follower node tracks its target state in the affine formation and simultaneously optimize some global performance indexes.

Consider a multi-agent system with each agent having similar dynamic model described by (5.1). Consider the protocol

$$\xi_i = -\sum w_{ij}(x_i - x_j), \ \forall i \in f,$$

whose closed loop form is given by

$$\xi_f = -(\Omega_{ff} \otimes I_d) x_f - (\Omega_{fl} \otimes I_d) x_l \tag{5.4}$$

Note that  $\xi = [\xi_l^T, \xi_f^T]^T$  where  $\xi_f(\xi_l)$  denotes the set of  $\xi_i$  for all *i* that are follower (leader) nodes.

Define the error of the followers as

$$e = x_f - x_f^*$$
  
=  $x_f + (\Omega_{ff}^{-1} \Omega_{fl} \otimes I_d) x_l^*,$ 

so that (5.4) can be re-written in terms of the error as  $\xi_f = -(\Omega_{ff} \otimes I_d)e$ . Thus, a suitable control that guarantees stabilization at each node is given in global form by

$$u = -c\Omega_{ff} \otimes K_2 e, \tag{5.5}$$

where c and  $K_2$  are respectively coupling and local feedback gains. To accomplish global stabilization of the follower agents to their targets, (5.2) is redefined in terms of the global error dynamics of the followers as

$$\dot{e} = (I_{n_f} \otimes A)e - (I_{n_f} \otimes B)u, \tag{5.6}$$

and by considering (5.5), the expression

$$\dot{e} = (I_{n_f} \otimes A - c\Omega_{ff} \otimes BK_2)e \tag{5.7}$$
is obtained. For the desired affine formation to be achieved, (5.7) needs to be stabilised to the origin, this is equivalent to the follower agents attaining their respective targets in the formation. Next, sufficient conditions to guarantee the required stabilizations are given.

**Theorem 5.2.1.** Assume that the error dynamics of the system is given by (5.6) and Assumptions 1, 2 and 3 hold (the Assumptions are in Sections 2.6 and 2.7). Let  $P_1$ ,  $P_2$ ,  $R_1$ ,  $R_2$  and  $Q_2$  be symmetric positive definite matrices such that

$$P_1 = cR_1\Omega_{ff},\tag{5.8}$$

$$0 = P_2 A + A^T P_2 + Q_2 - P_2 B R_2^{-1} B^T P_2.$$
(5.9)

Note that under Assumption 1 - 3,  $\Omega_{ff}$  is positive definite and symmetric since the communication graph is undirected.

By choosing

$$c > \frac{\sigma_{max}(R_1\Omega_{ff} \otimes (Q_2 - P_2 B R_2^{-1} B^T P_2))}{\sigma_{min}(\Omega_{ff}^T R_1 \Omega_{ff} \otimes P_2 B R_2^{-1} B^T P_2)} \quad \text{and} \quad (5.10)$$

$$K_2 = R_2^{-1} B^T P_2, (5.11)$$

the control  $u = -c(\Omega_{ff} \otimes K_2)e$  is stabilizing to the target state of each follower (i.e. the origin of (5.6)) and globally optimal with respect to the LQR performance index

$$J = \int_0^\infty (e^T Q e + u^T R u) dt$$

where  $\sigma_{max}(.)$  and  $\sigma_{min}(.)$  respectively denote the largest and smallest eigenvalues of (.),  $P = P_1 \otimes P_2$ ,  $R = R_1 \otimes R_2$  and  $Q = -(P_1 \otimes (P_2A + A^TP_2) - P_1R_1^{-1}P_1 \otimes P_2BR_2^{-1}B^TP_2)$ . Note that Q is guaranteed to be positive definite, assuming that (5.10) holds.

**Proof 5.2.1.** Let Q be a symmetric positive definite matrix. Also, let  $P = P_1 \otimes P_2$ , where  $P_2$  is nonsingular. Since both  $P_1$  and  $P_2$  are nonsingular, the Lyapunov function  $V_e = e^T P e = e^T (P_1 \otimes P_2) e > 0$ , since  $P = (P_1 \otimes P_2)$  is positive definite. The regular form of Algebraic Riccati Equation (ARE),  $Q + PA + A^T P - PBR^{-1}B^T P = 0$  is written in the global form for the follower agents as

$$Q + (P_1 \otimes P_2)(I_{nf} \otimes A) + (I_{nf} \otimes A^T)(P_1 \otimes P_2) - (P_1 \otimes P_2)(I_{nf} \otimes B)(R_1^{-1} \otimes R_2^{-1})(I_{nf} \otimes B^T)(P_1 \otimes P_2) = 0 \quad (5.12)$$

or

$$Q = -(P_1 \otimes (P_2 A + A^T P_2) - P_1 R_1^{-1} P_1 \otimes P_2 B R_2^{-1} B^T P_2)$$
  
= -(cR\_1 \Omega\_{ff} \otimes (P\_2 A + A^T P\_2) - c^2 \Omega\_{ff}^T R\_1 \Omega\_{ff} \otimes P\_2 B R\_2^{-1} B^T P\_2). (5.13)

Rearranging (5.10), we have

$$c(\sigma_{min}(\Omega_{ff}^T R_1 \Omega_{ff} \otimes P_2 B R_2^{-1} B^T P_2)) > \sigma_{max}(R_1 \Omega_{ff} \otimes (Q_2 - P_2 B R_2^{-1} B^T P_2)),$$

which is equivalent to

$$c^{2}(\sigma_{min}(\Omega_{ff}^{T}R_{1}\Omega_{ff} \otimes P_{2}BR_{2}^{-1}B^{T}P_{2})) > c\sigma_{max}(R_{1}\Omega_{ff} \otimes (Q_{2} - P_{2}BR_{2}^{-1}B^{T}P_{2})).$$
(5.14)

Comparing this to (5.13), it follows that  $Q \succ \mathbf{0}$ . Hence,  $e^T Q e > 0$ .

Equation (5.12) which denotes the system's ARE is satisfied by  $P = P_1 \otimes P_2$ . The resulting global optimal control

$$u = -R^{-1}B^T P e = -(R_1^{-1} \otimes R_2^{-1})(I_{nf} \otimes B^T)(P_1 \otimes P_2)e,$$
  
=  $-R_1^{-1}P_1 \otimes R_2^{-1}B^T P_2 e = -c\Omega_{ff} \otimes K_2 e.$ 

Furthermore,

$$\dot{V}_{(e)} = \dot{e}^T P e + e^T P \dot{e},$$

$$= e^T (I_{nf} \otimes A - c\Omega_{ff} \otimes BK_2)^T P e,$$

$$+ e^T P (I_{nf} \otimes A - c\Omega_{ff} \otimes BK_2) e,$$

$$= e^T (P_1 \otimes (A^T P_2 + P_2 A) - c\Omega_{ff}^T P_1 \otimes K_2^T B^T P_2,$$

$$- cP_1 \Omega_{ff} \otimes P_2 BK_2) e.$$

Simplification using (5.8) and (5.13) yields

$$\dot{V}_{(e)} = -e^T (Q + c^2 \Omega_{ff}^T R_1 \Omega_{ff} \otimes K_2^T R_2 K_2) e < 0,$$

which shows asymptotic stability to the desired target formation. It is easy to verify that the LQR optimality requirements are satisfied.

# 5.3 Simulation Study

Consider a multi-agent system composed of 5 agents with every agent having similar linear time invariant dynamics. Each agent's dynamics is discribed by

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, 2, \dots, 5,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 4 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$
(5.15)

The agents communication graph along with their configuration for the nominal formation is given in Fig. 4.1.



Figure 5.1: Formation of agents

Agents 1 to 3 are selected as leaders while the rest are followers. The stress matrix of the graph is given by [116].



Figure 5.2: Error of follower nodes

$$\Omega = \begin{bmatrix} 0.292 & -0.292 & -0.292 & 0.292 & 0 \\ -0.292 & 0.354 & 0.354 & -0.542 & 0.125 \\ -0.292 & 0.354 & 0.354 & -0.542 & 0.125 \\ 0.292 & -0.542 & -0.542 & 1.292 & -0.500 \\ 0 & 0.125 & 0.125 & -0.500 & 0.250 \end{bmatrix}$$

Compute the feedback gain,

$$K_2 = \begin{bmatrix} 1.7755 & 0.3204 \\ 0.6407 & 0.2907 \end{bmatrix}$$

from (5.9) and (5.11) by choosing  $R_2 = I_2$ , and  $Q_2 = I_2$ . Let  $R_1 = I_2$  then the choice c = 900 satisfies (5.10). The initial states of the leaders are their nominal configurations. Fig. 5.1 shows the follower nodes attain the desired targets in the formation while Fig. 5.2 shows the followers error converge to zero.

# 5.4 Summary

In this chapter, the problem of optimal formation control of the linear time-invariant multi-agent system with single-integrator dynamics and undirected graph have been studied. A control law design that guarantees both global optimality for some LQR performance index and stability for the entire follower nodes of a multi-agent system has been presented. The result extends earlier studies on affine formation control to some cases where the input and system matrices may not respectively be the identity and zero matrices. Ongoing research is on extending the scheme to systems with sampled-data communication.

# Chapter 6

# Fully Distributed Affine Formation Control of General Linear Systems with Uncertainty

# 6.1 Introduction

This chapter studies the distributed adaptive affine formation manoeuvre control problem of MASs with general linear dynamics and parametric uncertainty. The main contributions of this study are in three folds. Firstly, we consider the problem for systems with static, and dynamic coupling gains. Two control laws are proposed to address the different cases. The study extends our previous work in chapter 4, which only considers the static coupling gain case requiring global information in its design. Secondly, the case of linear systems with uncertainties are considered. Two control laws with robustness to uncertainties are presented to address systems with static and dynamic coupling gains. Finally, an experimental study is used to verify the effectiveness of the design. Note that the results contained in this chapter have been accepted for publication in the paper titled "Fully Distributed Affine Formation Control of General Linear Systems with Uncertainty."

# 6.2 Problem Description

Consider a multi-agent system comprising n subsystems referred to as agents. Let the n-agents comprise of  $n_l$  leaders and  $n_f$  followers (note that  $n_f = n - n_l$ ) where each follower agent (for the case without uncertainties) has similar linear time-invariant dynamics described by

$$\dot{p}_i = Ap_i + Bu_i. \tag{6.1}$$

In (6.1), A and B denote constant matrices of appropriate dimensions,  $u_i$  and  $p_i \in \mathbb{R}^d$ respectively denote the control input and position (state) of agent *i*. Equation (6.1) can be written in global form for all agents as

$$\dot{p} = (I_n \otimes A)p + (I_n \otimes B)u, \tag{6.2}$$

and for all follower agents as

$$\dot{p}_f = (I_{n_f} \otimes A)p_f + (I_{n_f} \otimes B)u_f, \tag{6.3}$$

where 
$$p = [p_1^T, ..., p_n^T]^T$$
,  $p_f = [p_{n_l+1}^T, ..., p_n^T]^T$ ,  $u = [u_1, ..., u_n]^T$  and  $u_f = [u_{n_l+1}, ..., u_n]^T$ .

In this study, the time-varying affine target position of agent i is defined by

$$p_i^*(t) = \Upsilon(t)r_i + b(t), \tag{6.4}$$

where both  $\Upsilon(t) \in \mathbb{R}^{d \times d}$  and  $b(t) \in \mathbb{R}^d$  vary with time and  $r_i \in \mathbb{R}^d$  denotes a constant reference (or nominal) configuration of *i*th agent. Equation (6.4) can be written in global form for all agents as

$$p^*(t) = [I_n \otimes \Upsilon(t)]r + \mathbf{1}_n \otimes b(t), \tag{6.5}$$

and for all follower agents as

$$p_f^*(t) = [I_{n_f} \otimes \Upsilon(t)]r_{n_f} + \mathbf{1}_{n_f} \otimes b(t), \tag{6.6}$$

where  $\mathbf{1}_n$ ,  $p^*(t) \in \mathbb{R}^{nd}$  and  $r = [r_1^T, ..., r_n^T] \in \mathbb{R}^{nd}$  denote an *n*-length vector whose elements (entries) are all ones, the time-varying affine targets to be tracked and the nominal (reference) configuration of the agents respectively. Note that a trivial strategy of accomplishing any required affine formation control is to define the  $\Upsilon(t)$  and b(t) (for the entire tractory) for all agents in (6.4). A challenge with this strategy is that all the agents will require knowledge of  $\Upsilon(t)$  and b(t) values in advance. This, in turn, limits the ability of the formation of agents to respond to any unexpected situation, e.g., an unanticipated obstacle requiring a change in the formation to avoid. This would require the values of  $\Upsilon(t)$  and b(t) to be simultaneously changed for all agents.

Let the affine image of the agents denote the set of all the affine transform of the defined reference configuration r. Note that the time-varying targets are always affine images of the defined reference configuration. The affine image is given in the global form as [98]

$$\mathcal{A}(r) = \{ p \in \mathbb{R}^{dn} : p = (I_n \otimes A)r + \mathbf{1}_n \otimes b, A \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d \}.$$
(6.7)

Therefore, the overall aim is to steer the entire followers such that as  $t \to \infty$ ,  $p(t) = p^*(t)$ . That is,

$$\lim_{t \to \infty} p_f(t) = p_f^*(t), \ \forall p_f^*(t) \in \mathcal{A}(r).$$

This study make the following assumption.

**Assumption 5.** For every agent, the pair (A, B) is controllable.

**Remark 6.2.1.** Assumption 5 is to facilitate full manipulation of the states using the control input.

**Remark 6.2.2.** The agent dynamics given in (6.1) can be used to study the case of heterogeneous multi-agent systems, i.e., the multi-agent system where the A and B matrices may not be identical. However, when they are different, the kronecker product  $\otimes$  used in e.g. (6.2) is not suitable for use. To simplify our presentation in this study, the case where A and B matrices are the same is used.

For brevity,  $p_i(t)$  (including,  $p_f(t)$ ,  $p_l^*(t)$ , ...) is written as  $p_i$  (including,  $p_f$ ,  $p_l^*$ , ...) in the rest of this chapter.

# 6.3 General Linear Multi-agent System Case

Consider a multi-agent system comprising n subsystems referred to as agents. The nagents comprise of  $n_l$  leaders and  $n_f$  followers each having similar linear-time invariant
dynamics described by (6.1) repeated here as

$$\dot{p}_i = Ap_i + Bu_i.$$

Let the control input be defined by

$$u_i = \varphi_i K \sum_{j \in \mathcal{N}_i} w_{ij} (p_i - p_j), \ \forall i \in f,$$
(6.8)

where f denotes the set of followers. Assume that the leaders are at their targets, i.e.,  $p_l = p_l^*$ , where  $p_l^*$  denotes the target of the leaders. Then, the closed loop form of (6.8) is given by

$$u_f = (\Phi\Omega_{ff} \otimes K)p_f + (\Phi\Omega_{fl} \otimes K)p_l^*, \tag{6.9}$$

where  $\varphi_i$ ,  $\Phi$  and K denote the coupling gain of agent i, diag $[\varphi_{n_{l+1}}, ..., \varphi_n]$  and local feedback gain, respectively.

The followers are considered to have accomplished the required affine formation if

$$\lim_{t \to \infty} \|p_i - p_i^*\| = 0, \quad \forall i \in f,$$
(6.10)

where  $p_i^*$  is the target position of agent *i*. The objective of this Section is to design distributed control  $u_f$ , such that (6.10) is achieved.

Using (6.9), network (6.3) can be written as

$$\dot{p}_f = (I_n \otimes A + \Phi \Omega_{ff} \otimes BK)p_f + (\Phi \Omega_{fl} \otimes BK)p_l^*$$
(6.11)

Next, we present sufficient conditions to guarantee (6.10) is accomplished.

**Theorem 6.3.1.** Under Assumptions 1 - 3 (the Assumptions are in Sections 2.6 and 2.7), by choosing  $\varphi_i > \frac{1}{2\min \Re(\lambda_i)}$  and  $K = -B^T Q$ , where  $\min \Re(\lambda_i)$  denotes the minimum real eigenvalue of  $\Omega_{ff}$  and  $Q = Q^T > 0$  is the solution of

$$QA + A^T Q - QBB^T Q < 0, (6.12)$$

control law (6.9) solves the affine formation control problem by ensuring that the tracking error of the followers  $\delta_{p_f}$  are stabilised to the origin.

*Proof.* Define the disagreements of the followers as

$$\delta_{p_f} = p_f - p_f^*$$
  
=  $p_f + (\Omega_{ff}^{-1} \Omega_{fl} \otimes I_d) p_l^*,$  (6.13)

(disagreement as defined in (2.7)) where d denotes the dimension of the space. Using (6.9) and (6.13), (6.3) can be written as

$$\dot{\delta}_{p_f} = (I_{nf} \otimes A + \Phi \Omega_{ff} \otimes BK) \delta_{p_f}.$$
(6.14)

Therefore, the task of achieving (6.10) reduces to ensuring that the disagreement in (6.14) stabilises to the origin. Define the unitary matrix U, such that

$$U^T \Omega_{ff} U = \Lambda, \tag{6.15}$$

where  $\Lambda$  is a diagonal matrix with diagonal entries  $\lambda_i$ , such that  $i = 1, ..., n_f$ . Let

$$\tilde{\delta}_{p_f} = (U^T \otimes I)\delta_{p_f}.$$
(6.16)

So that by using (6.16), (6.14) can be written as

$$\dot{\tilde{\delta}}_{p_f} = (I_{nf} \otimes A + \Phi \Lambda \otimes BK) \tilde{\delta}_{p_f}$$

or for each agent as

$$\dot{\tilde{\delta}}_i = (A + \varphi_i \lambda_i BK) \tilde{\delta}_i, \quad \forall i \in f.$$

Let the eigenvalues of  $\Omega_{ff}$  be  $\lambda_i = \eta_i + k\sigma_i$ , where  $\eta_i = \Re(\lambda_i)$ ,  $k^2 = -1$  and  $\sigma_i$  denotes the imaginary part of  $\lambda_i$ . By Lemma 2,  $\eta_i > 0$ . There exists a Q, such that

$$Q(A + \varphi_i \lambda_i BK) + (A + \varphi_i \lambda_i BK)^* Q < 0, \tag{6.17}$$

by noting that Assumption 5 implies that (6.1) is stabilisable. Straightforward algebraic manipulation of (6.17) using  $\lambda_i = \eta_i + k\sigma_i$ ,  $K = -B^T Q$  and (6.12) gives

$$-(2\varphi_i\eta_i - 1)QBB^TQ < 0. \tag{6.18}$$

It is easy to verify that (6.18) holds by considering that  $\varphi_i > \frac{1}{2\eta_i}$ .

Thus, by choosing K and  $\varphi_i$  as presented in Theorem 6.3.1, control law (6.11) guarantees that the respective target functions of the followers are tracked.

**Remark 6.3.1.** Equation (6.12) can also be written in the form  $QA + A^TQ - QB^TBQ + Q_2 = 0$ , where  $Q_2 = Q_2^T > 0$ . Therefore, an algebraic manipulation of (6.17) using  $\lambda_i = \eta_i + k\sigma_i$ ,  $K = -B^TQ$  and (6.12) gives

$$-Q_2 - (2\varphi_i\eta_i - 1)QBB^TQ < 0.$$

### 6.4 Distributed General Linear System

The control law designed in the previous section requires the knowledge of the smallest eigenvalue of  $\Omega_{ff}$ , which is considered to be a global parameter, in the choice of suitable coupling gains. This section considers an adaptive technique which obviates the need for this global information in designing the coupling gain  $\varphi_i$ . To simplify the presentation,  $\varphi_i$  is rewritten for the agents state as  $\varphi_{p_i}$  and the diagreement as  $\varphi_{\delta_i}$ .

Consider a multi-agent system with each agent governed by similar dynamics described by

$$\dot{p}_i = Ap_i + Bu_i,\tag{6.19}$$

and protocol

$$\begin{cases} u_i = \varphi_{p_i} K \sum_{j \in \mathcal{N}_i} w_{ij} (p_i - p_j), \\ \dot{\varphi}_{p_i} = \left[ \sum_{j \in \mathcal{N}_i} w_{ij} (p_i - p_j)^T \right] \Gamma \left[ \sum_{j \in \mathcal{N}_i} w_{ij} (p_i - p_j) \right], \ \forall i \in f, \end{cases}$$

$$(6.20)$$

where  $\varphi_{p_i}$  is a coupling gain and  $\Gamma$  is a local feedback gain and the other terms are as defined in (6.1) and (6.3). Networks (6.19) and (6.20) can be rewritten as

$$\begin{cases} \dot{p}_{f} = (I_{n_{f}} \otimes A)p_{f} + (I_{n_{f}} \otimes B)u_{f}, \\ u_{f} = (\Phi\Omega_{ff} \otimes K)p_{f} + (\Phi\Omega_{fl} \otimes K)p_{l}^{*}, \\ \dot{\varphi}_{p_{i}} = \left(\sum_{j=1}^{n} \Omega_{ij}p_{j}^{T}\right)\Gamma\left(\sum_{j=1}^{n} \Omega_{ij}p_{j}\right), \ \forall i \in f, \end{cases}$$

$$(6.21)$$

where  $\Omega_{ij} = -w_{ij}$ . With some algebraic manipulations, (6.21) can be written as

$$\begin{cases} \dot{p}_f = (I_{n_f} \otimes A + \Phi \Omega_{ff} \otimes BK) p_f + (\Phi \Omega_{fl} \otimes BK) p_l^*, \\ \dot{\varphi}_{p_i} = \left(\sum_{j=1}^n \Omega_{ij} p_j^T\right) \Gamma\left(\sum_{j=1}^n \Omega_{ij} p_j\right), \ \forall i \in f, \end{cases}$$
(6.22)

Next, sufficient conditions to guarantee the global stability of the network (6.22) are presented.

**Theorem 6.4.1.** Under Assumptions 1 - 3 (the Assumptions are in Sections 2.6 and 2.7), by choosing  $K = -B^T Q$  and  $\Gamma = QBB^T Q$ , where  $Q = Q^T > 0$  is the solution of

$$QA + A^TQ + I - QBB^TQ = 0,$$

control law (6.20) solves the affine formation control problem.

*Proof.* Define the disagreements of the followers as  $\delta_{p_f} = p_f - p_f^* = p_f + (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d)p_l^*$  (defined in (6.13)). Therefore, network (6.22) can be written in terms of the disagreement dynamics as

$$\begin{cases} \dot{\delta}_{p_f} = (I_{n_f} \otimes A + \Phi \Omega_{ff} \otimes BK) \delta_{p_f}, \\ \dot{\varphi}_{\delta_i} = \left(\sum_{j=1}^n \Omega_{ij} \delta_j^T\right) \Gamma \left(\sum_{j=1}^n \Omega_{ij} \delta_j\right), \ \forall i \in f \end{cases}$$
(6.23)

Therefore, the task of achieving (6.10) reduces to ensuring that the disagreement in (6.23) stabilises to the origin. Consider the Lyapunov function candidate

$$V_{2} = \frac{1}{2} \delta_{p_{f}}^{T} (\Omega_{ff} \otimes Q) \delta_{p_{f}} + \frac{1}{2} \sum_{i=n_{l}+1}^{n} (\varphi_{\delta_{i}} - \beta_{2})^{2}, \qquad (6.24)$$

where  $\beta_2$  is a parameter to be defined later. It is easy to verify that (6.24) is positive definite by noting Lemma 2 and that Q > 0. The derivative of  $V_2$  along the trajectory of (6.23) implies that

$$\dot{V}_{2} = \delta_{p_{f}}^{T} [(\Omega_{ff} \otimes Q)(I_{n_{f}} \otimes A + \Phi \Omega_{ff} \otimes BK)] \delta_{p_{f}} + \sum_{i=n_{l}+1}^{n} [\varphi_{\delta_{i}} - \beta_{2}] \left[ (\sum_{j=1}^{n} \Omega_{ij} \delta_{j}^{T}) \Gamma(\sum_{j=1}^{n} \Omega_{ij} \delta_{j}) \right].$$
(6.25)

By noting that  $K = -B^T Q$  and  $\Gamma = QBB^T Q$ , it can be verified that

$$\delta_{p_f}^T [(\Omega_{ff} \Phi \Omega_{ff} \otimes QBK)] \delta_{p_f} = \sum_{i=n_l+1}^n \varphi_{\delta_i} (\sum_{j=1}^n \Omega_{ij} \delta_j^T) QBB^T Q(\sum_{j=1}^n \Omega_{ij} \delta_j), \quad (6.26)$$

where  $\Phi = \text{diag}(\varphi_{\delta_{n_l+1}}, ..., \varphi_{\delta_n}).$ 

Considering (6.26), (6.25) can be rewritten as

$$\dot{V}_2 = \delta_{p_f}^T (\Omega_{ff} \otimes QA - \beta_2 \Omega_{ff}^2 \otimes QBB^T Q) \delta_{p_f}.$$
(6.27)

Let  $\tilde{\delta}_{p_f} = (U^T \otimes I) \delta_{p_f}$ , where U is as defined in (6.15). It follows that, (6.27) can be written as

$$\begin{split} \dot{V}_2 &= \tilde{\delta}_{p_f}^T (\Lambda \otimes QA - \beta_2 \Lambda^2 \otimes QBB^T Q) \tilde{\delta}_{p_f} \\ &= \frac{1}{2} \tilde{\delta}_{p_f}^T [\Lambda \otimes (QA + A^T Q) - 2\beta_2 \Lambda^2 \otimes QBB^T Q] \tilde{\delta}_{p_f}, \end{split}$$

that is,

$$\dot{V}_2 = \frac{1}{2} \sum_{\forall i \in f} \tilde{\delta}_i^T [\lambda_i \otimes (QA + A^T Q - 2\beta_2 \lambda_i QBB^T Q)] \tilde{\delta}_i.$$
(6.28)

By choosing  $\beta_2 \geq \frac{1}{2\lambda_i}$  and noting that  $QA + A^TQ + I - QBB^TQ = 0$ , it can be verified that  $\dot{V}_2 < 0$ . Therefore, by choosing K and  $\Gamma$  as presented in Theorem 6.5.1,  $V_2 > 0$ and  $\dot{V}_2 < 0$  ensuring that the disagreements(errors) of all follower nodes asymptotically stabilise to the origin by the action of (6.21) and guarantees that the affine formation control is accomplished.

### 6.5 Systems with Uncertainty

This section considers general linear systems with uncertainty.

#### 6.5.1 General Linear System with Uncertainty

Consider the case where each agent has a model given by

$$\dot{p}_i = (A + \Delta A_i)p_i + Bu_i, \quad \forall i \in f, \tag{6.29}$$

where  $\Delta A_i$  denotes a time-varying uncertainty associated with agent *i* and considered to be in the form

$$\Delta A_i = D\varepsilon_i M,\tag{6.30}$$

where  $\varepsilon_i \in \mathbb{R}^{y \times z}$  is an uncertainty satisfying

$$\varepsilon_i^T \varepsilon_i \le \alpha^2 I, \quad \forall i \in f,$$
(6.31)

 $\alpha$  is a constant while D and M are known matrices characterizing the structure of  $\Delta A_i$ . Assume that each agent communicate with its neighbours using protocol (6.8), which is given in closed loop form for all followers as (6.9). Using (6.9) and (6.30), (6.29) can be written in global form for all followers as

$$\dot{p}_{f} = [I_{n_{f}} \otimes A + \Phi \Omega_{ff} \otimes BK + (I_{n_{f}} \otimes D)\Delta(I_{n_{f}} \otimes M)]p_{f} + [\Phi \Omega_{ff} \otimes BK]p_{l}^{*}, \qquad (6.32)$$

where  $\Delta = \text{diag}(\varepsilon_i), \forall i \in f; D \text{ and } M$  are defined in (6.30); and then  $\Phi$  and K are defined in (6.9). Before moving on, we introduce an important concept and some Lemmas on quadratic stability.

**Definition 1.** [117]: System (6.29) with  $u_i = 0$  is quadratically stable if a common Lyapunov matrix with P > 0 exist for all admissible uncertainty  $\Delta A_i$ , such that,

$$(A + \Delta A_i)^T P + P(A + \Delta A_i) > 0.$$

**Lemma 6.5.1.** [118]: System (6.29) with  $u_i = 0$  is quadratically stable for the entire admissible uncertainties  $\varepsilon_i$  satisfying (6.30) if and only if:

- 1. the matrices  $A + \varphi_i \lambda_i B K$  are Hurwitz, and
- 2.  $||M(sI A)^{-1}D||_{\infty} < \frac{1}{\alpha}$

where  $\alpha > 0$ .

Lemma 6.5.2 (Bounded Real Lemma [119]). The following statements are equivalent:

- 1. if the matrix  $A + \varphi_i \lambda_i BK$  are Hurwitz and  $\|M(sI A)^{-1}D\|_{\infty} < \frac{1}{\alpha}$ .
- 2. Then, there exists a Q > 0 such that

$$A^T Q + QA + \alpha^2 DD^T + QM^T MQ < 0.$$

**Theorem 6.5.1.** Under Assumptions 1-3, by choosing  $\varphi_i \geq \frac{1}{2\Re(\lambda_i)}$  and  $K = -B^T Q$ , network (6.32) solves the affine formation control problem for all admissible uncertainties  $\varepsilon_i$ ,  $\forall i \in f$ , satisfying (6.31) if there exists a  $Q = Q^T > 0$  such that

$$A^T Q + QA - QBB^T Q + \alpha^2 DD^T + QM^T MQ < 0.$$

$$(6.33)$$

*Proof.* Let the disagreement be as given in (6.13) so that (6.32) can be re-written in terms of disagreements as

$$\dot{\delta}_{p_f} = [I_{n_f} \otimes A + \Phi \Omega_{ff} \otimes BK + (I_{n_f} \otimes D) \Delta (I_{n_f} \otimes M)] \delta_{p_f}, \tag{6.34}$$

so that the task of accomplishing the required affine formation reduces to quadratically stabilising (6.34). Consider a unitary matrix U, as defined in (6.15), and let  $\tilde{\delta}$  be as given in (6.16) so that (6.34) can be written as

$$\dot{\tilde{\delta}}_{p_f} = [I_{n_f} \otimes A + \Phi \Lambda \otimes BK + (I_{n_f} \otimes D\varepsilon_i M)]\delta_{p_f},$$
(6.35)

that is,

$$\dot{\tilde{\delta}}_{i} = \sum_{i=n_{l}+1}^{n} (A + \varphi_{i}\lambda_{i}BK + D\varepsilon_{i}M)\delta_{i}.$$
(6.36)

Note that (6.35) is block diagonal and is quadratically stable if every *i* in (6.36) is quadratically stable. Thus, a necessary requirement for quadratic stability of (6.34) for all admissible uncertainties satisfying (6.31) is that for every following *i*th agent,  $A + \varphi_i \lambda_i BK$  is Hurwitz and  $||M(sI - A - \varphi_i \lambda_i BK)^{-1}D||_{\infty} < \frac{1}{\alpha}$  by considering Lemma 6.5.1. We are to design *K* to satisfy these requirements. By considering Lemma 6.5.2, these requirements can be satisfied if there exits a *Q* such that

$$(A + \varphi_i \lambda_i BK)^* Q + Q(A + \varphi_i \lambda_i BK) + \alpha^2 DD + QM^T MQ < 0$$

That is,

$$A^{T}Q + QA + 2\varphi_{i}\Re(\lambda_{i})K^{T}B^{T}Q + \alpha^{2}DD + QM^{T}MQ < 0.$$
(6.37)

(Sufficiency) Choosing  $\varphi_i \geq \frac{1}{2\Re(\lambda_i)}$  and  $K = -B^T Q$  simplifies (6.37) to

$$A^T Q + QA - QBB^T Q + \alpha^2 DD + QM^T MQ < 0,$$

which by Lemma 6.5.2 implies that  $A + \varphi_i \lambda_i BK$  is Hurwitz and  $\|M(sI - A - \varphi_i \lambda_i BK)^{-1}D\|_{\infty} < \frac{1}{\alpha}$ , or in global form for all followers implies  $I_{n_f} \otimes A + \Phi \Omega_{ff} \otimes BK$  is Hurwitz and

 $\|(I_{n_f} \otimes M)(sI_{n_f} - I_{n_f} \otimes A - \Phi \Omega_{ff} \otimes BK)^{-1}(I_{n_f} \otimes D)\|_{\infty} < \frac{1}{\alpha}$ . Thus, quadratically stable by Lemma 6.5.1. Therefore, the disagreement of each follower nodes are stabilised to the origin and hence the tracking of the respective follower targets are accomplished by the action of (6.32) with K and  $\varphi_i$  chosen as presented in Theorem 6.5.1.

#### 6.5.2 Distributed Linear System with Uncertainties

Consider a multi-agent system with each agent governed by similar dynamics described by

$$\dot{p}_i = (A + \Delta A_i)p_i + Bu_i, \tag{6.38}$$

and the protocol

$$\begin{cases} u_i = \varphi_{p_i} K \sum w_{ij} (p_i - p_j), \\ \dot{\varphi}_{p_i} = \left[ \sum w_{ij} (p_i - p_j)^T \right] \Gamma \left[ \sum w_{ij} (p_i - p_j) \right], \quad \forall i \in f, \end{cases}$$

$$(6.39)$$

where the terms are as defined in (6.1), (6.8) and (6.29). Control laws (6.38) and (6.39) can be rewritten in closed loop form for all followers as

$$\begin{cases} \dot{p}_{f} = [I_{n_{f}} \otimes A + (I_{n_{f}} \otimes D)\Delta(I_{n_{f}} \otimes M)]p_{f} + (I_{n_{f}} \otimes B)u_{f}, \\ u_{f} = (\Phi\Omega_{ff} \otimes K)p_{f} + (\Phi\Omega_{fl} \otimes K)p_{l}^{*}, \\ \dot{\varphi}_{p_{f}} = [(\Omega_{ff} \otimes I_{d})p_{f}]^{T}\Gamma[(\Omega_{ff} \otimes I_{d})p_{f}] + [(\Omega_{fl} \otimes I_{d})p_{l}^{*}]^{T}\Gamma[(\Omega_{fl} \otimes I_{d})p_{l}^{*}], \end{cases}$$

$$(6.40)$$

and with some algebraic manipulations, (6.40) can be written as

$$\begin{cases} \dot{p}_{f} = [I_{n_{f}} \otimes A + \Phi \Omega_{ff} \otimes BK + (I_{n_{f}} \otimes D)\Delta(I_{n_{f}} \otimes M)]p_{f} + [\Phi \Omega_{ff} \otimes BK]p_{l}^{*}, \\ \dot{\varphi}_{p_{f}} = p_{f}^{T}(\Omega_{ff}^{2} \otimes \Gamma)p_{f} + p_{l}^{*T}(\Omega_{fl}^{2} \otimes \Gamma)p_{l}^{*}, \end{cases}$$

$$(6.41)$$

**Theorem 6.5.2.** Under Assumptions 1 - 3, by choosing  $K = -B^T Q$ , control law (6.39) solves the affine formation control problem for all admissible uncertainties  $\varepsilon_i, \forall i \in f$ , satisfying (6.31) if there exists a  $Q = Q^T > 0$  such that

$$A^T Q + QA - QBB^T Q + \alpha^2 DD^T + QM^T MQ < 0.$$

$$(6.42)$$

*Proof.* The proof of this Theorem is similar to the combination of the proofs of Theorems 6.4.1 and 6.5.1. Thus, it is omitted for brevity.  $\Box$ 

# 6.6 Experimental Study with Mona Mobile Robots

#### 6.6.1 Experimental Platform

An experimental testbed comprising Mona robots [120] as agents, a network of six cameras and a central computer for robots localization and mapping are used. Each robot has a unique arrangement of markers to facilitate its identification (see Fig. 6.6). The camera network videos real-time motion of the robots and transmits the image to a central computer for storage. A computer monitor (see Fig. 6.3) is connected to the computer to facilitate remote monitoring of the activities of the robots.

#### 6.6.2 Kinematic Model of Mona Robot

This experiment utilises a Mona robot whose model is feedback linearized. Some details are provided as follows. Let the position of the robot in the global inertial frame be as described in Fig. 6.1, where  $(\varkappa_i, y_i), v_i, w_i$  and  $\theta_i$  respectively denote the cartesian coordinates, linear velocity, angular velocity and orientation of robot *i*. Note that  $x = (\varkappa_i, y_i)$  and  $\hat{x} = (\varkappa_{h_i}, y_{h_i})$  respectively denote the position and head position of the robot in the cartesian coordinate. Then, we can describe the Mona robot kinematic model as

$$\begin{bmatrix} \dot{\varkappa}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{bmatrix} = \begin{bmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix}.$$
(6.43)

By considering the nonholonomic constraint of our robot, (6.43) can be rewritten in terms of the head position as

$$\begin{bmatrix} \dot{\varkappa}_{h_i} \\ \dot{y}_{h_i} \\ \dot{\theta}_i \end{bmatrix} = \begin{bmatrix} \cos\theta_i & -r\sin\theta_i \\ \sin\theta_i & r\cos\theta_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix}, \qquad (6.44)$$

where r is the distance between the inertial position and the head position (see Fig. 6.1), and

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\frac{1}{r}\sin\theta_i & \frac{1}{r}\cos\theta_i \end{bmatrix} \begin{bmatrix} u_{\varkappa_i} \\ u_{y_i} \end{bmatrix}.$$
 (6.45)

Note that  $u_i = [u_{\varkappa_i}, u_{y_i}]^T$  and  $\dot{x}_i = u_i$ . Thus, compared to (6.1), it follows that

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
(6.46)

For more details on Mona mobile robots, see [120].



Figure 6.1: Mona robot

#### 6.6.3 Formation Control with Seven Robots

Normally, the first step in the implementation of the proposed control laws is the design of suitable stress matrices. For this, a system comprising seven agents is chosen. Figure 6.2 is chosen as the reference configuration with agents 1-3 designated leaders. Following the procedure presented in Subsection 2.6, the stress matrix is computed as

$$\Omega = \begin{bmatrix} \Omega_{ll} & \Omega_{lf} \\ \Omega_{fl} & \Omega_{ff} \end{bmatrix}$$



Figure 6.2: Agents' configuration and communication graph. The inter-agent communications are denoted by straight lines

where

$$\begin{split} \Omega_{ll} &= \begin{bmatrix} 0.1368 & -0.2052 & -0.2052 \\ -0.2052 & 0.7182 & 0 \\ -0.2052 & 0 & 0.7182 \end{bmatrix}, \\ \Omega_{ff} &= \begin{bmatrix} 0.7524 & -0.0684 & -0.2052 & 0 \\ -0.0684 & 0.7524 & 0 & -0.2052 \\ -0.2052 & 0 & 0.2052 & -0.1026 \\ 0 & -0.2052 & -0.1026 & 0.2052 \end{bmatrix}, \\ \Omega_{fl} &= \begin{bmatrix} 0.1368 & -0.6156 & 0 \\ 0.1368 & 0 & -0.6156 \\ 0 & 0.1026 & 0 \end{bmatrix} \text{ and } \\ \Omega_{lf} &= \begin{bmatrix} 0.1368 & 0.1368 & 0 & 0 \\ -0.6156 & 0 & 0 & 0.1026 \\ 0 & -0.6156 & 0.1026 & 0 \end{bmatrix}. \end{split}$$

The experiment is carried out using control law (6.11). To satisfy Theorem 6.3 for the A and B given in (6.46), compute Q using (6.12), modified here as  $QA + A^TQ$  –  $QB^TBQ + 0.22I_2 = 0$ . Then, compute the feedback gain

$$K = \begin{bmatrix} -0.47 & 0\\ 0 & -0.47 \end{bmatrix},$$

from the relation  $K = -B^T Q$ . Also, compute the minimum eigenvalue of  $\Omega_{ff}$  as 0.0375, thus choose  $\varphi_i = 13.90, \forall i \in f$ .

### 6.6.4 Result

A scenario to provide dynamic landing platforms for multiple unmanned aerial vehicles (UAVs) is considered. The platform is to be able to handle UAVs having landing platforms of different widths.



Figure 6.3: Image display of monitoring system



Figure 6.4: Formations of robot. The robots first produced the reference formation (depicted with squares) and then execute scales to produce another formation (depicted with circles)

•



Figure 6.5: Follower agents error dynamics over two formations. At the beginning of the experiment, the leaders were first placed at their target positions in the reference formation and follower tracked their own respective targets. Then, after about 75 seconds, the leaders changed there positions and the followers tracked their corresponding positions in the new formation.



Figure 6.6: Mona robot

Pictures of experiment illustrating team of robots executing affine formation control under the action of Theorem 6.3. A short video of the formation control is available at https://www.youtube.com/watch?v=54xi6l4KBTs



Figure 6.7: Initial positions of robots



Figure 6.8: Robots form reference formation



Figure 6.9: The leader execute scaling formation

The plot of the robots as they first form their reference formation, from their initial positions, followed by a scaling formation manoeuvre is presented in Fig. 6.4. The dynamics of the error is presented in Fig. 6.5 and the photographs of these are presented in Figs. 6.7, 6.8 and 6.9. The errors of the followers stabilise to the origin with each manoeuvre and the follower tracked their respective targets. The follower agents (robots) in the experiment are able to scale to any matching width of the UAVs landing platform. From the experimental results, our proposed method demonstrates that the desired formation of robots can be achieved from any arbitrary initial positions of the followers. Furthermore, instead of specifying the coordinates of all robots, one could easily transform the desired formation pattern only by changing the positions of the leaders.

# 6.7 Summary

In this chapter, we have investigated the affine formation control of general linear multi-agent system based on the stress matrix approach which allows a team of agents to simultaneously accomplish general formation manoeuvres. Four control laws are presented to deal with basic general linear systems case, the case with uncertainties, a fully distributed case, and the fully distributed case with uncertainties. Sufficient conditions are presented to guarantee the global stability of the proposed control laws. Results of experimental studies are presented to demonstrating the effectiveness of the study. Future work focuses on achieving absolute controllability for multi-agent systems.

# Chapter 7

# **Conclusions and Future Works**

This chapter presents a general conclusions as well as the perspective of future works considered interesting.

# 7.1 Conclusions

This thesis addresses the affine formation control problem of multi-agent systems (MASs).

Firstly, the affine formation control of multi-agent systems with periodic inter-agent communication is studied. The cases where the agent dynamics are modelled as singleand double-integrators are studied. A variety of control laws are provided. Sufficient conditions to guarantee the system stability are established.

Secondly, the cases where the agents of a multi-agent system are described by tripleintegrator dynamics are considered. Two control laws are given to address the situations where the inter-agent communications are in continuous-time and discrete-time settings. Sufficient conditions are derived to guarantee the overall stability of the system. Procedures of the implementation of the system are also given. Simulations are used to demonstrate the efficacy of the formulated control laws.

Thirdly, the case of agents with general linear system dynamics are considered. The

scenarios where the agent dynamics have uncertainties are also considered. A variety of control laws are given to deal with the different scenarios. Sufficient conditions are given to guarantee the stability of the overall system. The stability analysis are carried out using Lyapunov theorem. Experiments are carried out to demonstrate the efficacy of the proposed control laws.

### 7.2 Future Research

The following are some interesting open problems in affine formation control of multiagent systems.

- 1. How to reduce the number of leaders that need to have advance knowledge of the maneuvres to be executed. As shown in the studies, to carry out maneuvres in *d*-dimensional space, d+1 number of leaders need to be selected for the affine formation control. A strategy that incorporates learning algorithms for some leaders to estimate their trajectories will help reduce the number of leaders that need advance knowledge of the maneuvres. For example, devising a method for *d* leaders to estimate their trajectories will reduce the number agents (or leaders) to just one and this can improve the ease of maneuvring the entire formation.
- 2. How to construct stress matrices for formation control in real-time. As presented in this study, the current affine formation control is limited to the affine transformation of reference configuration used in the design of the stress matrix. This implies that the formation control that can be accomplished with the scheme is limited to the affine images of the chosen reference configuration. Therefore, a method of constructing stress matrices in real-time will facilitate the accomplishment of a broader class of formations. This is possible since to accomplish a desired formation we would just construct a stress matrix whose affine transform contains the desired target formation.

# 7.3 Reflections

The proposed approach to formation control studied in this work allows a single control protocol to be used to simultaneously accomplish general affine formation control of multi-agent formations in any dimension. This is very significant in formation control. However, the work has some limitation that can be improved on. Some of the limitations are as follows.

- 1. The construction of the stress matrix is not fully distributed.
- 2. The communication requirement, through the number of required inter-agent connections due to the Assumptions, is high.

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