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# **GREQAM**

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**Real Options under Choquet-Brownian Ambiguity**

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**DT-GREQAM**

## 1. INTRODUCTION

Dealing better with *uncertainty* may contribute to gaining competitive edge and foster value creation. Such ambition is at the very core of the *real options*<sup>1</sup> approach to capital budgeting (Myers, 1977). But what sort of uncertainty entrepreneurs truly face? Is it *objective* or rather a matter of *beliefs* and/or *tastes*? Motivated answers to such questions are key prerequisite before determining if real option models may be significantly impacted by the introduction of uncertainty parameters. Over the last two decades, breakthroughs in decision theory certainly improved the understanding of uncertainty and various definitions of its multiple forms have been proposed. Still, giving axiomatic foundations to what largely derives from various psychological biases remains challenging. Transferring theoretical advances into practical recommendations is even more delicate, especially in dynamic settings. Furthermore, uncertainty remains a somewhat vague notion in the literature, as many different interpretations coexist.

But real options models were precisely developed to allow for a more efficient decision-making in presence of flexibility and irreversibility. As such, they have to be confronted with uncertainty. This issue is not trivial, as illustrated notably by Miao and Wang (2010): indeed a wide range of economic decisions may be reduced to option exercise choices or *optimal stopping problems under uncertainty*. Notice here that “uncertainty” should be used as a generic term, while specific definitions may be given to its different forms, such as the familiar notion of *risk*<sup>2</sup> or that, more elusive, of *ambiguity*<sup>3</sup> (Knight, 1921; Keynes, 1921). Experimental studies in ambiguous settings showed repeatedly (at least since Ellsberg, 1961) that decision makers usually

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<sup>1</sup> A project combining three key characteristics is a standard real option: *irreversibility* (presence of sunk costs), *uncertainty* (at least as regards to future payoffs) and *managerial flexibility* (regarding the timing of option exercise).

<sup>2</sup> A *risky* situation is defined through the existence of a unique probabilistic model, known from the decision maker: It is well aware of the random nature of some elements at least, but remains perfectly confident that no model misspecification needs to be considered. This is equivalent to adopting a rational expectations framework.

<sup>3</sup> *Ambiguity* appears when the decision maker is not fully confident that his beliefs on the possible states of the world are perfect, when uncertainty cannot be reduced to a single Kolmogorov type of probability measure. It may be typically necessary to rely on a *range* of probability measures.

prefer to deal with known probabilities rather than imprecise ones, thereby revealing a form of aversion towards ambiguity. Applying the subjective expected utility framework to a real option model when confronted with ambiguity may consequently be misleading.

Furthermore, real options may appear quite frequently in ambiguous settings (or at least perceived as such by managers). Indeed, getting a quantitative estimate of the cost of opportunity of acting *now* rather than later is particularly relevant in many *irreversible investment* situations, especially in front of large capital budget decisions bearing high uncertainty, such as R&D projects, M&A or intangible asset valuations. Not always easy to apply in practice, real options models may contribute to improving risk analyses by giving management the incentive and ability to actively manage sources of risk and ambiguity<sup>4</sup>, rather than passively following standard DCF threshold methods<sup>5</sup>.

The real option literature so far almost only discusses the impact of an increase in *risk* on valuation of real options and exercise timing<sup>6</sup>. Indeed, very often, especially in the finance literature, uncertainty is reduced to risk only. But as Montesano (2008) points out, ambiguity aversion seems important in financial markets, where agents are deeply concerned over the level of transparency (i.e. the reliability of the probability distribution of outcomes they refer to). Other financial models under uncertainty led to strong conclusions, such as an incompleteness of financial markets (Mukerji and Tallon, 2001) or the unstable nature of portfolio preferences with no equilibrium identifiable (Dow and Verlang, 1992; Epstein and Wang, 1994). In presence of real options, decision makers should be particularly concerned over risk and ambiguity factors.

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<sup>4</sup> Some limits of the standard approach of investment decision (Fisher, 1930; Williams, 1938) are addressed. It has indeed been shown on many occasions (Dean, 1951; Hayes and Garvin, 1982 and others) that discounted cash flows can lead to non-optimal decisions, such as investing too early in projects while waiting would allow to create more value, or conversely to wrongly reject projects, for instance by ignorance of “growth options” (Myers, 1977).

<sup>5</sup> Still, the success of real options theory should not be overestimated, as real options are often seen as too complex to apply and difficult to put into practice. Real option theory has also suffered from being misused to justify unrealistic valuation levels, especially during the internet bubble (for instance with a fraudulent use by Enron).

<sup>6</sup> Contrary to the standard conclusion of neoclassical theory on investment (Markowitz, 1952; Tobin, 1958), an increase in risk (volatility) is almost always shown to have a positive impact on the valuation of real options (see McDonald and Siegel, 1986). Renewed controversy may be found in Sarkar (2000) and Lund (2005).

Obviously, accounting for preferences towards uncertainty in real options models generates additional complexity. Minimizing this likely drawback is fundamental as such models without uncertainty are already often criticized as overly complex! The trade-off between empirical realism vs. tractability of the model needs to be recognized and assumed. Moreover, defining ambiguity and/or aversion to ambiguity remains controversial: ambiguity representations generally translate into non-linearity of probabilities (Feynman, 1963), while many non additive models coexist (Cohen and Tallon, 2000). To expand real options models under ambiguity, a clear choice is required over the terms of the ambiguity representation, whose consequences have to be fully clarified and accounted for, especially when interpreting results. Finally, several technical and theoretical issues have to be solved, especially the difficulty in insuring dynamic consistency in presence of ambiguity<sup>7</sup>.

Recently, a few real option models characterized by the presence of Knightian uncertainty( or ambiguity)<sup>8</sup> have been proposed (Nishimura and Ozaki, 2007; Asano, 2005; Trojanowska and Kort, 2010; Miao and Wang, 2010). Solving uncertainty by application of a maxmin criterion over the potential outcomes of decision, these recursive multiple-priors models<sup>9</sup> identify a significant impact of ambiguity on real option valuations and timing of exercise, but only in the case of extreme aversion towards uncertainty. In the case of a real option to invest, Nishimura and Ozaki (2007) show that ambiguity impacts the value of irreversible investments in a « drastically » different manner from that of “traditional uncertainty” (risk).

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<sup>7</sup> Dynamic consistency implies that decision makers, once committed to a contingent plan, are not changing plans later on during the process. This apparently limits freedom of choices at successive stages, but is a condition for rational inter-temporal behavior, avoiding irrational erratic behaviors (such as *money pumps* or *Dutch books*).

<sup>8</sup> Notice here that these expressions may be used indifferently. Referring to Knightian uncertainty is frequent in the literature, but may seem a little farfetched if Knight’s original proposal is replaced in its context.

<sup>9</sup> The *multiple-priors* preferences (Gilboa and Schmeidler, 1989) approach is based on the *maxmin* criterion (optimization under a worst case scenario): it was adapted to a dynamic setting in continuous time by Chen and Epstein (2002). It uses properties from recursive utility functions and multiple-priors (Wald, 1950; Dreze, 1961) to allow for dynamic consistency, even in the presence of uncertainty (Epstein and Zin, 1989; Duffie and Epstein, 1992; Epstein and Wang, 1994; Epstein and Schneider, 2003). Most recently, Riedel (2009) develops a martingale theory for multiple-priors, generalizing existing optimal stopping theory under multiple-priors uncertainty.

In their model, increasing uncertainty affects negatively the investment value, while an increase in risk raises it. But as regards the timing of exercise, in both cases the value of waiting increases, thereby delaying option exercise. This notable conclusion illustrates the power of introducing ambiguity in real options models. Nevertheless, other articles have been less conclusive as regards the impact of ambiguity (early exercise or no; increased option value or not). Nishimura and Ozaki (2004) themselves showed that in a job search real option model, more ambiguity may lead to earlier option exercise. Other models at least converge in demonstrating that introducing ambiguity is not trivial. In Trojanowska and Kort (2010), ambiguity aversion has an equivocal impact on the value of waiting, accelerating investment only in certain situations. Asano (2005) shows that an increase in uncertainty delays the adoption of environmental policies. Miao and Wang (2010) suggest reconciling some contradictions in these results by considering the moment of resolution (or not) of ambiguity: the prospect of a persisting ambiguity after option exercise may possibly delay option exercise rather than accelerate it. Overall, these models confirm that ambiguity impacts real options valuation and timing of exercise. But by construction only the worst-case scenario is considered in a multiple-priors approach, which reduces the behavioral bias to extreme pessimism<sup>10</sup>. Furthermore, ambiguity and attitude towards ambiguity are mixed and impossible to distinguish<sup>11</sup>. Another stream of literature has been recently associated with the representation of “Knightian ambiguity”, that of *robust control*<sup>12</sup>.

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<sup>10</sup> To avoid ignoring the existence of a whole range of probability measures, several other criteria have been proposed. Ghirardato et al. (2004) following Arrow and Hurwicz (1972) have for instance proposed to combine worst case scenario with best case in a convex combination. See also Chateauneuf et al. (2007) on *neo-additive* capacities, or Schroder’s non dynamically consistent proposal (2008).

<sup>11</sup> It is arguable that through the size of the set of priors a belief towards the level of ambiguity may be expressed but this is much weaker than the dichotomy established for risk: *risk* is determined by the shape of the probability distribution of outcomes, while *risk aversion* results from the curvature of the utility function.

<sup>12</sup> Hansen et al. (2001) shed light on the relation between robust control and aversion to ambiguity, and relate it to the solving of a *maxmin* problem with multiplier preferences. Chen and Epstein (2002) underline some key differences. Klibanoff et al. (2005, 2009) propose a *smooth* version of ambiguity, by adjusting subjective probabilities through a smooth functional, but Skiadas (2009) shows that in continuous-time the smooth ambiguity adjustment will vanish, failing to preserve ambiguity preferences.

This paper aims at expanding real options models under uncertainty to account for the variety of preferences towards ambiguity. Indeed, many experiments and studies confirmed that if aversion may be a prevalent reaction to uncertainty, excluding ambiguity seeking *a priori* may often be unjustified. Consequently, we adopt a *Choquet expected utility* framework<sup>13</sup> and follow Schmeidler's definition of aversion to ambiguity in relation to the convexity of capacities<sup>14</sup>, which allows considering a variety of preferences, including ambiguity loving ones. A few articles used Choquet expected utility to describe the potentially striking impact of ambiguity in the financial markets articles (see for instance the impact of Choquet preferences on portfolio allocation in Basset et al., 2004). In the context of financial options, Montesano (2008) shows that uncertainty aversion with Choquet expected utility leads to decreasing trading volume on the call options market as ambiguity increases.

We leave the Bayesian expected utility and its updating issues aside to look at the impact of ambiguity. We exclude situations where fundamental uncertainty prevails. Moreover, to isolate the effect of uncertainty, the decision maker is assumed to be risk-neutral. Let's underline that we are not to discuss here the normative status of such attitudes towards ambiguity; rather, formalization is given to commonly observed revealed preferences in front of uncertain outcomes, when they challenge the established expected utility framework. Our approach is axiomatic and subjective (the measure derives from the decision maker's preferences), without reference to an objective probability distribution that would be subjectively distorted (although it could be an interpretation).

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<sup>13</sup> The *Choquet expected utility* (CEU) models (Gilboa, 1987; Schmeidler, 1986, 1989) may appear less intuitive than the *maxmin* optimization, but using the Choquet integral to denote the expected utility of beliefs functions allows addressing Allais and Ellsberg paradoxes. Ghirardato and Le Breton (1997, 2000) describes how the usual definition of rationality is expanded to enclose a larger set of beliefs, including non additive beliefs (or *capacities*)

<sup>14</sup> Other definitions coexist: Ghirardato and Marinacci (2002) refer to «the neutrality towards uncertainty *a priori* » and Epstein (1999) to *sophisticated* probabilities. Several competing notions coexists such as « the aversion towards uncertainty » (Châteauneuf, 1994), « pessimism » (Arrow et Hurwicz, 1972) or the preference for « *randomization* » (Eichberger et Kelsey, 1996).

To account for a wider range of preferences towards ambiguity, we rely on dynamically consistent *Choquet-Brownian* processes<sup>15</sup> to model uncertainty. This is the key originality of our model. In our framework, the impact of perceived ambiguity on the expected cash flows from a project is summarized by the value of a parameter  $c$ . It expresses the nature and intensity of the psychological bias revealed by decision makers under ambiguity, that we call *c-ignorance*. The probabilistic case is a special case in our generalized real option model to invest, as well as the multiple-priors.

The remainder of the paper is organized as follows. In section 2, our alternative proposal to the recursive multiple-priors models is described, as dynamically consistent *Choquet-Brownian* motions (CBM) are used to model uncertainty over cash flows of a project. Section 3 applies this approach to the case of a *real option to invest*, solving the optimization problem and identifying the threshold project values. Section 4 provides with a sensitivity analysis illustrating the characteristics of our new optimal investment rule. Some analytical results are established and then complemented by simulations, as standard in the literature on real options models with multiple-priors, in order to get clear comparison of results. Section 5 discusses main results in relation to previous works and presents our concluding remarks.

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<sup>15</sup> A Choquet-Brownian motion (Kast and Lapied, 2008) is a distorted Wiener process, where the distortion derives from the nature and intensity of preferences towards ambiguity. It was shown to be the continuous time limit of a specific kind of random walk, the Choquet Random Walk (CRW). A Choquet Random Walk may be described as a binomial lattice (Bernoulli model) with equal capacities (instead of additive probabilities) on the two states at each node. (See more details on construction in section 2.2)



## 2. FRAMEWORK

### 2.1. Uncertainty over cash flows and recursive multiple-priors:

In the context of irreversible investment decisions or real options, it is typical to consider that a project's profit flow follows a geometric Brownian motion (GBM)  $(\pi_t)_{0 \leq t \leq T}$ , where  $T$  is the expiration date,  $\pi_0 > 0$ ,  $B_t$  is the standard Brownian motion with respect to the original probability measure (towards which the decision maker is perfectly confident) and  $\mu$  and  $\sigma$  are some real numbers, with  $\sigma > 0$  and  $\mu < \rho$ , where  $\rho > 0$ , is the firm discount rate. So far, this set up may be assimilated to that of a classical financial call option, allowing the use of financial options pricing techniques. In the absence of uncertainty, the profit flow is traditionally represented by the following expression: 
$$d\pi_t = \mu\pi_t dt + \sigma\pi_t dB_t \quad (1)$$

Now suppose the decision maker is not perfectly confident about the extent to which the GBM actually models properly the expected profit flow dynamics. Ambiguity is consequently introduced and takes the form of a distortion from the original GBM.

In the multiple-priors approach, Chen and Epstein (2002) proposed the use of *density generators* to build a range of probability measures representing small deviations from the original probability measure. Small deviations only are allowed as the subjective beliefs are constrained by adopting an additional boundary condition. A constant  $\kappa$  is used to limit the scope of the accepted deviations in a range  $[-\kappa, \kappa]$ . Chen and Epstein refer to the level of  $\kappa$ -ignorance, where constant  $\kappa$  derives from a fundamental hypothesis on the domain of acceptable preferences, that of *rectangularity*<sup>16</sup>, in order to guarantee dynamic consistency<sup>17</sup>.

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<sup>16</sup> Beliefs are constrained to a set of "one-step-ahead" conditional probabilities in Chen and Epstein (2002), Epstein and Schneider (2003, 2008). This *rectangularity* property allows for recursivity, which in turn insures dynamic consistency (Sarin and Wakker, 1992). This property is also referred to as *time-consistency* or *stability under pasting*.

<sup>17</sup> Such construction is possible in application of Girsanov's theorem on equivalent probability measures (applying a density generator to a Brownian motion results indeed in another Brownian motion). Notice that Girsanov's theorem applies to finite time intervals only (Karatzas and Shreve, 1991). But it is a common approximation in the literature on financial options when referring to infinite horizon. We adopt it here as well.

Under Knightian uncertainty and recursive multiple-priors model, a set of stochastic differential equations is now to be used. By construction,  $dB_t^\theta = dB_t + \theta_t dt$ , so now we obtain the following modified expression for the profit flow:  $d\pi_t = (\mu - \sigma\theta)\pi_t dt + \sigma\pi_t dB_t^\theta$  (2)

By adopting ambiguity,  $\mu$  is simply replaced by  $(\mu - \sigma\theta)$ . The absence of ambiguity is included as a special case, when  $\theta = 0$ . To sum it up, ambiguity is introduced in a limited way inside an optimal stopping model, through a set of geometric Brownian motions which differ only by their *drift*. This is sufficient to demonstrate that uncertainty has an impact different from risk alone<sup>18</sup>. But as the decision maker considers only the worst case, ambiguity aversion leads to a unique case: only the lowest possible value of the project cash flow growth rate is considered.

## 2.2 Uncertainty through Choquet-Brownian processes

In this paper, we adopt another approach to model uncertainty, in order to avoid some limits inherent to the *maxmin* criterion. As usual, the decision maker expresses preferences relative to the uncertain payoffs generated by a real option project at various dates. But this time they are taken into account in a different way: we refer to *capacities* (instead of additive probabilities) to weight likelihood of events and rely on *discounted Choquet integrals* to compute payoffs value<sup>19</sup>. Let's first clarify these key notions before showing how the dynamics of the real option cash flows will consequently be represented by a distorted kind of Brownian motions (that we call Choquet-Brownian motions) rather than by a standard geometric Brownian.

A capacity on a finite set of states of nature  $S$  is a real-valued function  $\nu$  on the subsets of  $S$  such that:  $\nu(\emptyset) = 0, \nu(S) = 1; A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$ . So one of the key characteristic of a capacity is to be non-additive, which can be used to explain preferences in the absence of objective

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<sup>18</sup> Multiple-priors models helped re-interpret several apparent paradoxes in finance, such as the “equity premium” puzzle (as identified by Merha and Prescott, 1985) or the “home bias” puzzle (Epstein and Miao, 2003). Other articles have presented remarkable contributions, expanding applications to portfolio choices (Epstein and Wang, 1994), contract theory (Mukerji, 1998) or to explain the own-equity effect (Boyle et al., 2003).

<sup>19</sup> The « expected value » of an outcome on a given capacity may be computed through the use of Choquet integrals. Applying Choquet integrals and capacities was suggested in modern decision theory by Schmeidler (1986).

probabilities and to represent a wide range of attitudes towards ambiguity. Why capacities rather than probabilities? Schmeidler (1989) linked the convexity of capacities with a representation of ambiguity aversion<sup>20</sup>. This behavioral interpretation of capacities is at the basis of our construction. Let's note that a capacity is convex if:  $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$ .

When beliefs are represented by capacities, the resulting expected utility cannot be computed through Lebesgue integrals for several reasons. A specific notion of integration is required, which in particular will take into account the rank of outcomes (see rank dependant expected utility models in risky settings). To compute the decision maker preferences, which take the form of cash valuations regarding future uncertain payoffs, we need to use a criterion allowing computation of a certainty equivalent when integrals are non-linear. Using Choquet integrals allows just that in our setting. We refer to Chateauneuf et al. (2001) and Kast and Lapied (2007), for axiomatization of dynamic consistency<sup>21</sup> and discussion of conditioning in this framework. In this setting, preferences of the decision maker for a process of payoffs  $X = (X_0, \dots, X_T)$  are represented by the discounted Choquet expectations, at rate  $r$ , with respect to a capacity  $\nu$ . The certainty equivalent of the process is then:

$$DE(X) = \sum_{t=0}^T r(t) E_{\nu}(X_t), \text{ where: } E_{\nu}(X_t) = \sum_{s_t \in S_t} X_t(s_t) \Delta \nu(s_t), \quad (3)$$

with the usual notation for a Choquet integral for which, if, for instance,  $X_t(s_1) \leq \dots \leq X_t(s_N)$ ,  $\Delta \nu(s_n) = \nu(\{s_n, \dots, s_N\}) - \nu(\{s_{n+1}, \dots, s_N\})$ , with  $\{s_{N+1}\} = \emptyset$ , for notational convenience. It is then possible to compute the Choquet expected value over the Choquet-Brownian motions (CBM) that are to be used to describe dynamics of uncertain future cash flows in our real option to invest model. Let's discuss now the CBM construction itself.

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<sup>20</sup> Epstein (1999) or Ghirardato and Marinacci (2002) discuss the equivalence convexity/ ambiguity aversion.

<sup>21</sup> One of the key axioms used being the property of additivity of Choquet integrals for co-monotone functions.

CBM may be better understood as dynamically consistent continuous-time limits of Choquet Random Walks<sup>22</sup> (CRW). Indeed, CRW are defined in discrete time by referring to a binomial lattice representing uncertainty with equal *capacities* (rather than probabilities) on the two states at each node. In order to characterize a CRW, suppose that for any node  $s_t$  at date  $t$  ( $0 \leq t < T$ ), if  $s_{t+1}^u$  and  $s_{t+1}^d$  are the two possible successors of  $s_t$  at date  $t+1$  (for, respectively, an “up” or a “down” movement in the binomial tree), the conditional capacity is a constant  $c$ . Suppose we consider a *symmetrical* random walk (when “up” and “down” movements have the same likelihood<sup>23</sup>), such that:  $\nu(s_{t+1}^u/s_t) = \nu(s_{t+1}^d/s_t) = c$ , with  $0 < c < 1$ .

The constant conditional capacity  $c$  plays a key role in such setting: it summarizes the decision makers’ attitude towards ambiguity. Indeed dynamics is now described by a discrete time motion in which probability  $1/2$  is replaced by this constant  $c$  : it represents the ambiguous weight that the decision maker is putting both on the event « up » and the event « down » instead of the unambiguous  $1/2$ . Just like  $\kappa$  determines the level of  $\kappa$ -ignorance<sup>24</sup> in the multiple-priors approach, we may use the expression *c-ignorance* in relation to the value of  $c$ .

When the decision maker is ambiguity averse, the capacity is sub-linear: this is the case if and only if parameter  $c < 1/2$ . This relates to Yaari’s definition of aversion to risk as a result of sub-linearity (1969, 1987)<sup>25</sup>. Obviously if  $c=1/2$  then we get back to the probabilistic framework, as a special case. Let’s underline that an *increase* in perceived ambiguity in our setting means that the value of parameter  $c$  is going further away from the central key anchor  $1/2$ : the capacity becomes more convex (increasing ambiguity for an ambiguity averse) or more concave (increasing ambiguity for an ambiguity seeker).

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<sup>22</sup> Just like a standard binomial tree converges to a Brownian motion in continuous time.

<sup>23</sup> Expansion to non symmetrical random walks would be possible in this setting, at least in discrete time.

<sup>24</sup> As a continuous time counterpart, in a different context, to  $\mathcal{E}$ -dissemination, where  $\mathcal{E}$  represents the degree of “contamination” of confidence in the probability measure (Chen and Epstein, 2002). See also the relation with *i.i.d* uncertainty (“independently and indistinguishably distributed”).

<sup>25</sup> See Montesano (1990) for discussion of competing definitions of aversion to risk (*mean preserving spreads* versus *risk premium*) and impact of adopting non-expected utility models to represent it.

Such a symmetrical CRW was shown to converge in continuous time to a general Wiener processes with distorted mean  $m = 2c - 1$  and variance  $s^2 = 4c(1 - c)$ . This allows solving basic optimal investment problems, such as real option models under uncertainty. Overall, not only is taken into account the impact of the intrinsic randomness of trajectories due to the stochastic nature of profit flows and project value (which is already typically achieved by using geometric Brownian motions), but also simultaneously the level of c-ignorance, hence the attitude towards ambiguity. What are the consequences of adopting this framework?

$$\text{The profit flow is modified as follows: } d\pi_t = \mu \pi_t dt + \sigma \pi_t dW_t, \quad (4)$$

with  $dW_t = m dt + s dB_t$ , where  $W_t$  is a general Wiener process with mean  $m = 2c - 1$  and variance  $s^2 = 4c(1 - c)$ . So that we derive the following modified profit flow equation:

$$d\pi_t = (\mu + m \sigma) \pi_t dt + s \sigma \pi_t dB_t \quad (5)$$

This relation is naturally of the same type as the one obtained in the no ambiguity case (1) or with the *maxmin* ambiguity (2). Parameters  $m$  and  $s$ , directly deriving from  $c$ , are introduced to represent the decision maker's attitude towards ambiguity. Some implications appear clearly: if for instance the decision maker is ambiguity averse, then parameter  $c < 1/2$ . Consequently,  $0 < c < 1/2$  implies  $-1 < m < 0$  and  $0 < s < 1$ , and then  $\mu + m\sigma < \mu$  and  $0 < s\sigma < \sigma$ . We already get an insight into the potential impact of *Choquet-Brownian* uncertainty, at least on the profit flow: it introduces a reduction of the instantaneous mean, but also of the volatility in the case of aversion to ambiguity. The last result was not necessarily expected.

Overall, (5) should lead to different results from the case for risk only, as well as from the *maxmin* recursive model, for which the profit flow is also modified but only its drift<sup>26</sup> (Epstein and Schneider, 2003). With *Choquet-Brownian* uncertainty, to the contrary, the effect of the Choquet distortion on the standard profit flow is equivocal, reducing both the instantaneous mean

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<sup>26</sup> Ambiguity into a multiple-priors model is reduced to volatility, which may be questionable.

and the volatility for an ambiguity averse decision maker. Introducing ambiguity with CBM is not neutral, but consequences remain unclear at first. In the next section, we apply this CBM representation of ambiguous cash flows over time to the case of a real option to invest. The subsequent optimization is solved by dynamic programming and use of Ito's Lemma in order to identify the continuation region (values  $V$  of the project for which it is not yet optimal to invest) and the critical value  $V^*$ , such that it is optimal to invest only once  $V \geq V^*$ .

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### 3. A REAL OPTION TO INVEST UNDER CHOQUET-BROWNIAN AMBIGUITY

Suppose a decision maker enjoying an option to invest into a new project (for instance a patent). This project presents the essential characteristics of a real option: it is irreversible (once decided, investment is instantaneous while its cost, noted  $I$  is sunk); it is only affected by time decay; its exercise can be delayed and the choice of timing belongs exclusively to the decision maker. Decision will be taken based exclusively on observed information about stochastic cash flows. In the absence of uncertainty, McDonald and Siegel (1986) presented the seminal version of such irreversible investment decision. Notice that we adopt a continuous time horizon<sup>27</sup>, time being indexed by  $t \geq 0$ . If the project has finite life,  $T$  is the expiration date of the project. All information available at each  $t$  is represented by an increasing filtered probability space  $(\Omega, F_t, P)$ . The decision maker is risk neutral and cash flows are discounted at rate  $\rho > 0$ . Cash flows for the project have to be estimated. They move over time at least partially in a random way, so we rely on some sort of stochastic processes, combining dynamics with uncertainty. Over a given sequence of possible stochastic payoffs, an optimal stopping time has to be identified, maximising the expected overall result<sup>28</sup>.

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<sup>27</sup> Continuous time models leads to more explicit computations, but sometimes by using numerical methods.

<sup>28</sup> Optimal stopping problem grew in the 1960s (see Chow and Robbins (1961, 1963, 1971) following original generalization of sequential analysis by Snell (1952)). In general, stopping rule problems do not have closed form solutions and methods of finding approximate solutions must be used.

**Proposition 1.** *Suppose a decision maker considering a real option to invest in a project at sunk cost  $I$  and facing Knightian uncertainty. Suppose that this uncertainty affects the profit flow  $(\pi_s)_{s \geq t}$ , expected from exercising the option, and that this state variable follows a Choquet-Brownian motion, as characterized earlier in section 2. Then, the project value  $W_t$  at time  $t$ , with expiration time  $T$ , is equal, once exercised, to the expected value  $E^P$  of the discounted<sup>29</sup> cash flows with respect to the probability measure  $P$  conditional on the filtration  $F_t$  defined previously, such that:*

$$W(\pi_t, t) = E^P \left[ \int_t^T e^{-\rho(s-t)} \pi_s ds / F_t \right] \quad (6)$$

*Proof:* Derived and adapted from standard demonstration in the literature since McDonald and Siegel (1986). For more progressive treatment, we refer to Dixit and Pindyck (1994, chapter 4 to 6). Trojanowska and Kort (2010) offer clear and detailed proofs in the context of real options under ambiguity with multiple-priors. The same holds true for proposition 2, 3 and 4.

The decision maker has to determine the optimal moment  $t'$ ,  $t' \in [t, T]$  to exercise the option to invest. This  $F_t$ -optimal stopping time is the one which maximises the value in  $t=0$  of the project, over the whole period considered (principle of optimality), taking into account the discounted cost of investing, at discount rate  $\rho$ . The stopping time is a random variable that described the exercise date of the option. We rely on dynamic programming<sup>30</sup> to identify optimal sequential decision under uncertainty<sup>31</sup>.

**Proposition 2.** *Option value  $V_t$  at time  $t$ , while still not exercised, is the following:*

$$V_t = \max_{t' \in (t, T)} E^P \left[ \int_{t'}^T e^{-\rho(s-t)} \pi_s ds - e^{-\rho(t'-t)} I / F_t \right] \quad (7)$$

*Proof:* As justified earlier.

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<sup>29</sup> At exogenous discount rate  $\rho$ , such that  $\rho > \mu$  in order to avoid triviality.

<sup>30</sup> See Stokey, Lucas and Prescott (1989) for deep treatment of dynamic programming in economic settings.

<sup>31</sup> See Markov stopping rule problems in Chow, Robbins and Siegmund (1972) and Shirayev (1973).

As proved many times, (see for instance Asano, 2005), we obtain from (6):

$$W(\pi_t, t) = \int_t^T \pi_t \exp(-(\rho - \mu)(s - t)) ds = \frac{\pi_t}{\rho - \mu} (1 - e^{-(\rho - \mu)(T - t)}) \quad (8)$$

If the project is *perpetual*, then computation is much eased: it is indeed common assumption to adopt an infinite planning horizon and a never expiring project (cf. Dixit and Pindyck, 1994, or Trigeorgis, 1996). We will then proceed by adopting a stationary model.

**Proposition 3.** *Under stationary hypothesis, the value for the project is the standard expected value of a perpetual profit flow, which can be simplified as such:*  $W(\pi_t) = \frac{\pi_t}{\rho - \mu}$  (9)

*Proof:* See for instance Dixit and Pindyck (1994, p72).

It is not possible to apply ordinary rules of derivation to Ito processes. But the use of Ito's lemma allows differentiation and integration of functions of stochastic processes. In the absence of ambiguity, we obtain the following expression by applying Ito's lemma to (9):

$$dW_t = \mu W_t dt + \sigma W_t dB_t \quad (10)$$

Under Choquet-Brownian ambiguity, this relation is naturally of the same type as for the cases of no ambiguity or *maxmin* ambiguity, only this time  $\mu + m\sigma$  and  $s\sigma$  in place of  $\mu$  and  $\sigma$  (see section 2). Hence, we rewrite the formula (10), as described earlier in previous section:

$$dW_t = (\mu + m\sigma) W_t dt + s\sigma W_t dB_t \quad (11)$$

Project value (9) is consequently rewritten to take into account the presence of ambiguity:

$$W(\pi_t) = \frac{\pi_t}{\rho - (\mu + m\sigma)} \quad (12)$$

**Proposition 4.** *If the project value  $W$  is now technically supposed to be independent from physical time  $t$ , then the option value  $V_t$  only depends on  $W_t$ . Consequently, it is a solution of the following "Hamilton-Jacobi-Bellman" type of function, which will be solved by dynamic programming:*

$$V(W_t) = \max \left\{ W_t - I, E^P [dV_t | F_t] + V(W_t) - \rho V(W_t) dt \right\} \quad (13)$$

*Proof:* As justified earlier. See for instance Nishimura and Ozaki (2007), p681.



We now clearly identify characteristics of the optimal investment strategy: existence of a (unique) critical value  $W^*$  such that option is exercised if and only if  $W_t \geq W^*$ ; if not, the option is kept moving forward, defining a continuation region where  $W_t < W^*$ . In the right side part of (13), the first term  $W_t - I$  represents the value of *investing now* by exercising the option, while the second term corresponds to the *value of waiting*. Notice that both terms on the right hand side of (13) must be equal in the continuation region. Hence, in this continuation region:

$$E^P [dV_t | F_t] = \rho V(W_t) dt \quad (14)$$

Applying Ito's lemma to expand  $dV(W_t)$ , supposing that  $V$  is twice differentiable in the continuation region and  $V' > 0$ , if we now combine (14) and (11) we obtain:

$$dV_t = V'(W_t)((\mu + m\sigma)W_t dt + s\sigma W_t dB_t) + \frac{1}{2}(s\sigma)^2 W_t^2 V''(W_t) dt \quad (15)$$

From (15), the relation obtained is satisfied for every  $dt$ , then we derive the following second-order ordinary differential equation for  $V$ , as:

$$\frac{1}{2}(s\sigma)^2 W_t^2 V''(W_t) + (\mu + m\sigma)W_t V'(W_t) - \rho V(W_t) = 0 \quad (16)$$

Further assumptions are necessary in order to solve equation (16), holding in the continuation region. We adopt the following standard boundary conditions: value matching, smooth pasting and absorbing barrier. If we note  $W^*$  the critical reservation value triggering the option exercise:

$$\left\{ \begin{array}{ll} V(W^*) = W^* - I & \text{“value matching condition”}^{32} \end{array} \right. \quad (17)$$

$$\left\{ \begin{array}{ll} V'(W^*) = 1 & \text{“smooth pasting condition”} \end{array} \right. \quad (18)$$

$$\left\{ \begin{array}{ll} F(0) = 0 & \text{“absorbing barrier condition”} \end{array} \right. \quad (19)$$

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<sup>32</sup> (17) implies that investing in the project gains a net payoff equal to  $V^* - I$ . (18) is derived from the first-order condition when maximizing project value. (19) : if the investment value has no value, then the option is worthless.

We explicitly solve (16) under conditions (17) to (19) in order to get the option value in the continuation region,  $V(W_t)$  as well as the critical value (or free boundary)  $W^*$ , so we obtain<sup>33</sup>:

$$\begin{cases} V(W_t) = \left(\frac{I}{\alpha-1}\right)^{1-\alpha} \alpha^{-\alpha} W_t^\alpha & W_t < W^* \\ \text{and } W^* = \frac{\alpha}{\alpha-1} I \end{cases} \quad (20)$$

$$\quad (21)$$

$$\text{In the exercise region, } W \geq W^*, V(W_t) = W_t - I \quad (22)$$

and  $\alpha$  is a constant<sup>34</sup>, whose value depends on parameters  $\mu, \rho, \sigma, c$ , so defined:

$$\alpha = \frac{-((\mu + m\sigma) - \frac{1}{2}(s\sigma)^2) + \sqrt{\left\{(\mu + m\sigma) - \frac{1}{2}(s\sigma)^2\right\}^2 + 2\rho(s\sigma)^2}}{(s\sigma)^2} \quad (23)$$

with  $m = 2c-1$  and  $s^2 = 4c(1-c)$  (see section 2).

Let's summarize our optimal stopping problem in the context of CBM ambiguity:

**Proposition 5.** *Assuming a real option to invest under Choquet-Brownian ambiguity as defined in propositions 1-4, optimal strategy and value of investment are summarized in (20), (21), (22) and (23).*

This is of course close from what is obtained in the case of absence of uncertainty, which becomes a special case. We observe the introduction in key formulas of parameters  $m$  and  $s$ , directly deriving from  $c$ , which summarizes the attitude of the decision maker towards ambiguity. We may now proceed to a sensitivity analysis to explore impact of changing ambiguity preferences, as well as compare with results from multiple-priors real option models.

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<sup>33</sup> See for instance Dixit and Pindyck for simple treatment (1994, p142-143), getting solution through dynamic programming (using linear combination), as well as description of fundamental quadratic's intuition.

<sup>34</sup>  $\alpha > 1$  so that  $W^*$  and  $V$  are well defined, which in the multiple-priors also holds as  $\rho > \mu$  and  $\kappa > 0$  (see Nishimura and Ozaki, 2007, Annex A.5.).

#### 4. SENSITIVITY ANALYSIS

After identifying the optimal investment rule, what happens if parameters change? We may compare a change in risk (increase in volatility) with a change in Knightian uncertainty (either represented by  $\kappa$  in the multiple-priors or by  $c$  with Choquet-Brownian ambiguity). We obtain several analytical results, more specifically regarding the impacts of risk and ambiguity on project values in the stopping region. Regarding the critical reservation value and the timing of option exercise in the presence of ambiguity, we use some simulations<sup>35</sup> results, as standard in the real option under ambiguity literature, to show the critical impact of parameters and examine the characteristics of the optimal investment rule when analytical results are not easily computable.

##### *4.1. An increase in risk*

In the absence of ambiguity, the well know result of an increase in risk consists in the increase of the value of the project in the continuation region and in the reservation value, while the value of the project once the option has been exercised does not change. Exercise of the option is delayed. If now we introduce ambiguity, we show that a change in risk also impacts the exercised project value. Differences in original attitude towards ambiguity may explain why the same variation in risk may be looked over differently by decision makers revealing different attitudes towards perceived uncertainty.

**Proposition 6.** *In the presence of ambiguity, a change in risk levels impacts project value in the stopping region. An increase in risk leads to an increase in the value of the project once the option has been exercised if and only if the decision maker is ambiguity lover ( $c > 1/2$ ). The opposite holds true if the decision maker is averse to ambiguity ( $c < 1/2$ ).*

*Proof:* See appendix A

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<sup>35</sup> Simulation results have often to be taken with a pinch of salt. Different parameters may likely influence each other and interpreting can be hazardous at time. Nevertheless, simulations are an important tool in practice when dealing with real options. From collecting adequate data on past demand for instance, are generated prospective future demand trajectories. Stochastic dynamic programming in the context of real options relies on the quality of information used, sound data collections, analyses and industry expertise.

#### 4.2. An increase in ambiguity (c-ignorance):

In this subsection we now wish to analyze the effect of a change in c-ignorance on project value, continuation value, reservation value and timing of exercise<sup>36</sup>.

##### 4.2.1. Project value in the stopping region $W_T \geq W^*$ :

In the presence of Choquet-Brownian uncertainty:  $W(\pi_t) = \frac{\pi_t}{\rho - (\mu + m\sigma)}$  (12)

The impact of an increase in c-ignorance in the stopping region is the following: if the decision maker is ambiguity averse,  $\mu + m\sigma \searrow$ , thus project value  $W$  in the stopping region decreases. The opposite holds true for an ambiguity lover. This result generalizes the multiple-priors model, in which an increase in  $\kappa$ -ignorance always translates into a decreased value for the project (indeed the decision maker remaining averse to ambiguity and only considers the lower born).

##### 4.2.2. Project value in the continuation region $W_T < W^*$ :

From now on, computation is not trivial and we cannot derive analytical solutions through simplifying derivatives. Using simulation, here we find that if we fix  $\mu, \sigma, \rho$ : if the decision maker is ambiguity averse, an increase in ambiguity leads to a decrease in project value in the continuation region. The opposite holds true for an ambiguity lover.

##### 4.2.3. Reservation value: $W^* = \frac{\alpha}{\alpha - 1} I$

Let's note that when  $\alpha \nearrow, \alpha > 1$ , then the hysteresis factor  $\frac{\alpha}{\alpha - 1} \searrow$ <sup>37</sup>; this means the reservation value decreases. As shown in the previous section,  $\alpha$  depends on 4 parameters, the degree of c-

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<sup>36</sup> Let's recall first that an *increase* in perceived ambiguity in our setting means that the value of parameter  $c$  is going further away from its central key anchor  $\frac{1}{2}$  (corresponding to the limit probabilistic case, that of an absence of ambiguity). Possible deviations are confined in a range and  $c$  represents the index of the intensity and nature of perceived ambiguity (or c-ignorance).

ignorance  $c$  (which in turn determines the values for  $s$  and  $m$ ), the growth rate  $\mu$ , the discount rate  $\rho$ , as well as the volatility  $\sigma$ .

We may summarize briefly some side results concerning growth rates  $\mu$  and discount rates  $\rho$ , identifying how they also impact reservation values and timing of exercise of option:

- If  $c, \sigma, \rho$  are fixed, then according to our simulation:  $\nearrow \mu \Rightarrow \searrow \alpha$ , that is an increase in the growth rate decreases  $\alpha$ , which in turn means the reservation value  $W^*$  increases (see fig.1 in appendix B). The attitude towards ambiguity (lover, averse, neutral) does not change the direction of the trend, but an ambiguity lover's reservation value is always higher than that of a neutral or averse one.

- Now if  $c, \sigma, \mu$  are fixed, then according to our simulation:  $\nearrow \rho \Rightarrow \nearrow \alpha$ , that is an increase in the discount rate increases  $\alpha$ , meaning the reservation value  $W^*$  decreases (see fig.2 in appendix B). The attitude towards ambiguity does not affect the trend, but an ambiguity lover reservation value is once more always higher than that a neutral or averse one.

Let's now turn to our main point of discussion, the impact of a change in  $c$  when all other parameters are fixed: if  $\mu, \sigma, \rho$  are fixed, then if the decision maker is ambiguity lover, an increase in  $c$ -ignorance will lead to a decrease in  $\alpha$ . Reservation value increases. The opposite holds true for an ambiguity averse (see fig.3 in appendix B). In the case of aversion towards ambiguity, the observed decrease in reservation value is similar to that in Nishimura and Ozaki (2007). But we also establish the opposite result for an ambiguity seeker.

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<sup>37</sup> Let's note that as long as  $\alpha > 1, \frac{\alpha}{\alpha - 1} > 1$ . Hence  $W^* > 1$ , which is sufficient to rule out as incorrect the traditional static NPV criterion.

#### 4.2.4. Value of waiting

Next, we explore the connection between a change in reservation value and the subsequent impact on timing of option exercise. If we reinterpret the reservation value  $W^*$  in terms of a reservation profit flow  $\pi^*$ , then from adapting (12), we get:  $W^* = \frac{\pi^*}{\rho - (\mu + m\sigma)}$  that can be rewritten:  $\pi^* = \{\rho - (\mu + m\sigma)\} W^*$ . In our model<sup>38</sup>, simulations on reservation profit flow  $\pi^*$  show two distinct areas depending on the nature of c-ignorance (seeker or averse):  $\pi^*$  is increasing with the degree of ambiguity for an ambiguity averse (that is the value of waiting increases), while decreasing for an ambiguity seeker. This leads to the adoption of opposite behaviors, with an accelerated (ambiguity seeker) or a delayed (ambiguity averse) option exercise (See fig.4 in appendix B). It does not come as a huge surprise at this stage that preferences towards perceived ambiguity play such a defining role when deciding over the optimal moment of exercise of our real option. Just like project and option valuations are affected by individual preferences, the timing of exercise is modified according to the nature of the attitude of the decision maker towards ambiguity. For an ambiguity averse, the present value effect (decrease in the net present value of the project) dominates the option value effect (the cost of opportunity of acting is reduced), and exercise is delayed. The opposite holds true for an ambiguity seeker. Let's summarize our findings:

**Proposition 7.** *A change in the level of perceived ambiguity has an impact on project value as well as on reservation value, consequently impacting the timing of exercise of real options with Choquet-Brownian motions. While an ambiguity averse decision maker will delay option exercise, an ambiguity seeker will exercise it earlier than if he was neutral towards ambiguity.*

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<sup>38</sup> See Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) for discussion of this relation in the context of multiple-priors. In the latter especially, the impact of Knightian uncertainty on triggers  $W^*$  and  $\pi^*$  appears equivocal for *finite life* projects: an increase in reservation value may not lead to delayed investment (for instance, larger life-times finite projects are negatively associated with investment enhancing). For *perpetual* projects, to the contrary, monotonicity is demonstrated, with  $\pi^*$  increasing with ambiguity.

## 5. CONCLUSION

Few articles within the real options literature have so far explored the impact of ambiguity. Moreover, the few pioneer real options models under ambiguity are all based on the multiple-priors model. They have given great insight on the importance of addressing the existence of preferences towards perceived ambiguity. Unfortunately such models are also very restrictive by definition, as they rely exclusively on a *maxmin* criterion.

In contrast, by introducing a wider spectrum of attitudes towards ambiguity represented through Choquet-Brownian motions, we show that individual preferences matter and lead to significant and contrasted impacts on option valuations and subsequent timing of exercise. Indeed, aversion towards ambiguity will increase the value of waiting and delay exercise, while ambiguity loving preferences will encourage an earlier exercise of a real option to invest. So far in the literature only the result for aversion to ambiguity had been established in the case of a perpetual option (Nishimura and Ozaki, 2007). Real options models under ambiguity so far concur in underlining that ambiguity should not be purely and simply ignored. Considering the limited number of papers on this subject, that some points of debate remain open, such as the moment of resolution of ambiguity (Miao and Wang, 2010), is hardly surprising and actually bodes well for further research. Let's notice that the very effect of risk itself on options may still be subject of debate (see Sarkar's controversial stance, 2000; discussed in Lund, 2005).

Furthermore, in our model a complex inter-relation between risk and ambiguity, often completely ignored in the literature, appears and raises new questions. In the combined presence of ambiguity and risk, individual revealed preferences towards such forms of uncertainty may deeply impact real option valuations and subsequent actions: as we pointed out, even a risk neutral decision maker will react to changes in risk if he is not neutral towards ambiguity. This relation may appear more strikingly here than in the multiple priors' approach where ambiguity is largely "assimilated" to risk.

To conclude, we wish to underline that more research will be necessary to deepen the understanding of ambiguity revealed preferences, to compare their various representations and ponder their respective interest and limits. Even the interpretation of ambiguity itself remains somehow controversial and a better distinction between beliefs and tastes may be desirable. Recently axiomatized Choquet-Brownian motions are tractable enough to be adapted to more complex real options settings, including compound options or multiple sources of ambiguity such as typical stochastic costs (see Kast, Lapied and Roubaud, 2010). Besides, expanding our model to finite life projects would allow comparisons with research on finite life projects in the context of multiple-prior (Trojanowska and Kort, 2010; Gryglewicz, Huisman and Kort, 2008).

As decision makers' preferences towards ambiguity matter, they should often be taken into consideration when examining the timing and valuation of real options projects. Obviously it may be helpful when assessing decisions *ex-post*, to better understand why in practice real options may be exercised much later (or sooner) than predicted in the expected utility framework. But it may also contribute to a better framing *a priori* of the impact of ambiguous key drivers on a real option project, if only through additional sensitivity analyses including ambiguity parameters.

The multiple-priors approach corresponds to a cautious attitude in front of potential model misspecification, a “robust decision rule” for an investment. From a managerial point of view, ambiguity should not necessarily be feared though: embracing it when strategically justified may prove wise and source of competitive advantage, when caution would prevent from undertaking potentially profitable investments.

Obviously the intuition of managers will not be replaced by quantitative estimates, but those who adopted the real options approach often underline that it contributes to better thinking, planning and conducting of projects under uncertainty. Adopting some sort of ambiguity parameter should just help them in doing that in a more explicit way!



Finally, at this early stage of real options models under ambiguity, it may be argued that they already contribute to the idea that dominant models in finance should maybe more often than not take ambiguity preferences into consideration... if only by making the hypothesis of “ambiguity neutrality” at least as explicit than its omnipresent “risk neutral” counterpart!

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## Appendix A. Comparative Statics for risk

### A. An increase in risk in the absence of ambiguity: the standard case

A.1. Project value after exercise,  $W_T \geq W^*$  :

Parameter  $\sigma^2$  representing risk has no impact on the value of the project once launched. Indeed, if there is no ambiguity, then  $m = 2c - 1 = 0$  and  $W(\pi_t) = \frac{\pi_t}{\rho - \mu}$ . In the absence of ambiguity, a change in risk does not modify the project value in the stopping region (the agent is *risk neutral* by hypothesis).

A.2. Project value in the continuation region,  $W_T < W^*$  :

Regarding the option value in the continuation region, let's recall that  $V(W_t)$  is given by (20):

$V(W_t) = \left(\frac{I}{\alpha - 1}\right)^{1-\alpha} \alpha^{-\alpha} W_t^\alpha$ . This time, as parameter  $\sigma^2$  plays a key role in computation of  $\alpha$  in (23), the value of the project will change in the continuation region. We need to look at the sign of a few derivatives to identify the impact of an increase in risk, which implies some calculations (Nishimura and Ozaki, 2007): sign of  $\frac{\partial \alpha}{\partial \sigma^2} < 0$ , sign of  $\frac{\partial V(W_t)}{\partial \alpha} < 0$ ; hence, by combining,  $\frac{\partial V(W_t)}{\partial \sigma^2} > 0$ . An increase in risk increases the value of the project in the continuation region.

A.3. Reservation value  $W^* = \frac{\alpha}{\alpha - 1} I$  :

Again, parameter  $\sigma^2$  plays a key role in computation of  $\alpha$ , so that we again need to establish the signs of:  $\frac{\partial W^*}{\partial \alpha} < 0$ , and  $\frac{\partial \alpha}{\partial \sigma^2} < 0$ , (Nishimura and Ozaki, 2007); hence,  $\frac{\partial W^*}{\partial \sigma^2} > 0$

### B. An increase in risk in the presence of ambiguity: a striking impact on project value!

If the decision maker is not neutral towards ambiguity, a change in risk in the presence of ambiguity will impact the project value in the stopping region (*ceteris paribus*). Indeed, if  $\sigma^2$  increases, then  $(\mu + m\sigma)$  now increases if and only if  $m > 0$ , that is if  $c > 1/2$ , which in turn implies that

$W(\pi_t) = \frac{\pi_t}{\rho - (\mu + m\sigma)}$  increases. Consequently, project value in the stopping region increases for an

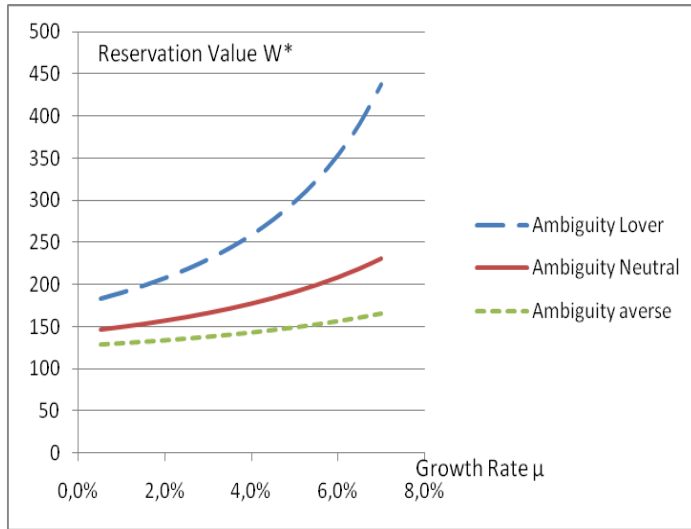
ambiguity seeker when risk increases. The opposite holds true if the decision maker is ambiguity averse<sup>39</sup>.

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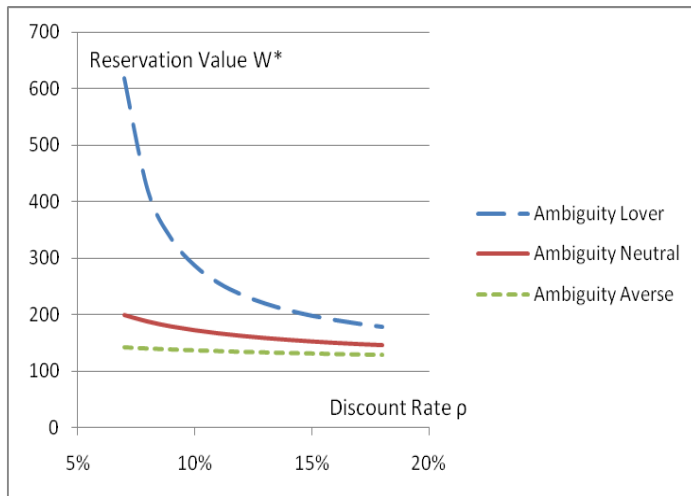
<sup>39</sup> Let's note that in the multiple-priors, as  $\kappa > 0$ , an increase in risk also leads to a decrease in project value, as

$W_t = \frac{\pi_t}{\rho - (\mu - \kappa\sigma)}$ . This is just a special case in our model, that of ambiguity aversion under *maximin*

## Appendix B. Comparative Statics for Ambiguity (1/2)

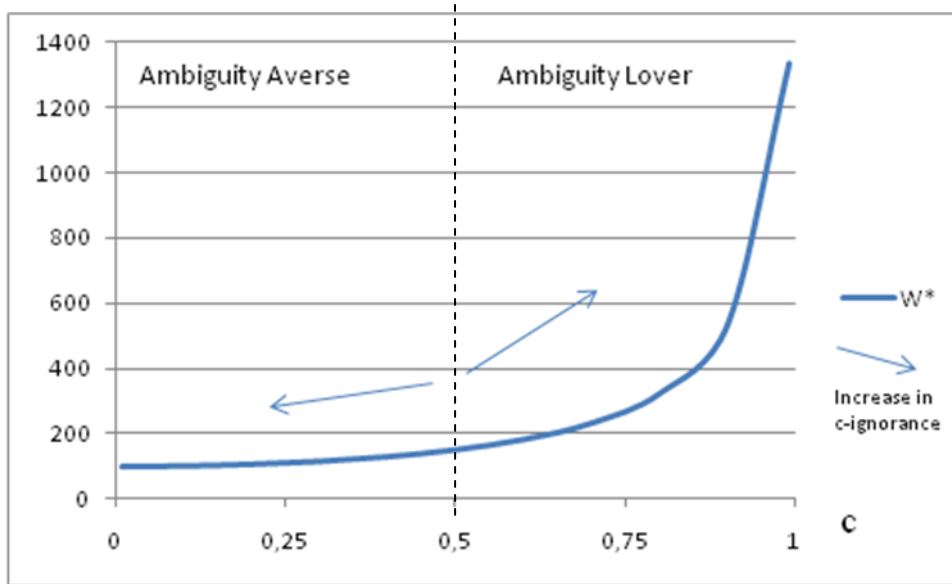


**Fig. 1.** Reservation Value  $W^*$  as a function of  $\mu$  for decision makers expressing various attitudes towards ambiguity (with  $\sigma = 20\%$  ;  $\rho = 15\%$  ;  $\psi = \{-0.1; 0; 0.1\}$  and  $I = 100$ ).

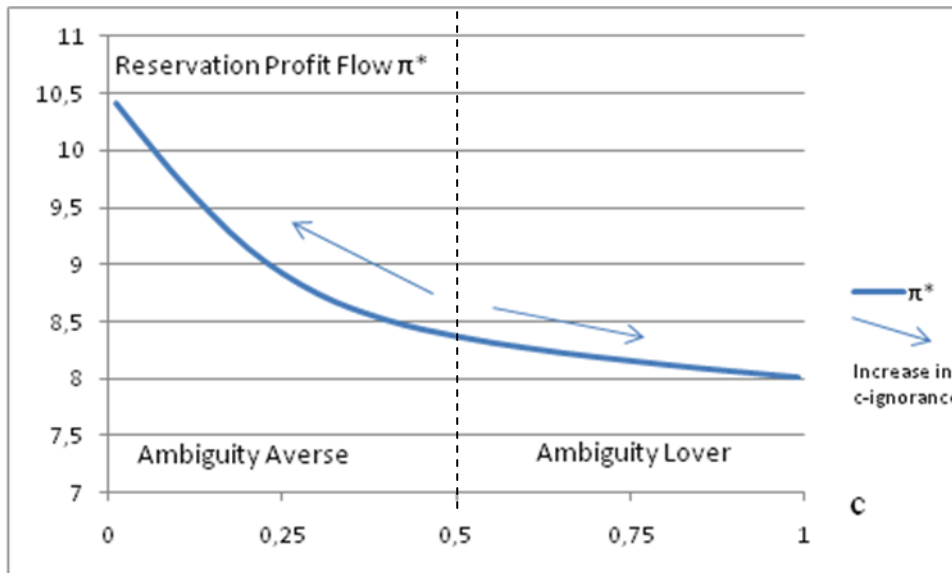


**Fig.2.** Reservation Value  $W^*$  as a function of  $\rho$  for decision makers expressing various attitudes towards ambiguity (with  $\sigma = 20\%$  ;  $\mu = 1.5\%$  ;  $\psi = \{-0.1; 0; 0.1\}$  and  $I = 100$ ).

### Appendix B. Comparative Statics for Ambiguity (2/2)



**Fig.3.** Reservation Value  $W^*$  as a function of the degree of c-ignorance (with  $\sigma = 5\%$ ,  $\mu = 2.5\%$ ,  $\rho = 8\%$  and  $I = 100$ ).



**Fig.4.** Reservation Profit Flow  $\pi^*$  as a function of the degree of c-ignorance (with  $\sigma = 5\%$ ,  $\mu = 2.5\%$ ,  $\rho = 8\%$  and  $I = 100$ ).

## REFERENCES

- Arrow, K. and L. Hurwicz, (1972), "An optimality criterion for decision-making under ignorance". In: Ford, J.L. *Uncertainty and expectations in economics*.
- Asano, T., (2005), "Irreversibilities and the optimal timing of environmental policy under Knightian uncertainty", Discussion paper, Osaka University.
- Basset G., Koenker R. and G. Kordas, (2004), "Pessimistic Portfolio Allocation and Choquet Expected Utility", *Journal of Financial Econometrics*, Vol.2, Oxford University Press.
- Boyle, P., R. Uppal, and T. Wang, (2003), "Ambiguity aversion and the puzzle of own-company stock in pension plans", University of Waterloo, Working paper.
- Chateauneuf, A., (1994), "Modeling Attitudes towards Uncertainty and Risk through the Use of Choquet Intégral", *Annals of Operations Research*, 52 : 3-20.
- Chateauneuf, A., J. Eichberger and S. Grant (2007), "Choice under uncertainty with the best and worst in mind: Neo-additive capacities", *Journal of Economic Theory*, Vol 137, p538-567.
- Chateauneuf A., Kast R. and Lapiéd A., (2001), "Conditioning Choquet integrals: the role of comonotony", *Theory and decision* 51, 367-386.
- Chen, Z. and L. Epstein (2002), "Ambiguity, Risk and Asset Returns in Continuous-Time", *Econometrica*, 70, 1403–1443.
- Cohen, M. and J.M. Tallon, (2000), "Décision dans le risque et l'incertain: l'apport des modèles non-additifs", *Revue d'Economie Politique*, N°110(5), pp.631-681.
- Dean, J. (1951), *Capital Budgeting*. New York: Columbia University Press.
- Dixit, A.K. and R.S. Pindyck (1994). *Investment under Uncertainty*. Princeton University Press, Princeton.
- Dow, J. and Werlang, S.R.C. (1992), "Uncertainty aversion, risk aversion and the optimal choice of portfolio", *Econometrica* 60, 197–204.
- Dreze, J.H. (1961), "*Les fondements logiques de l'utilité cardinale et de la probabilité subjective*" in La Décision, Colloques Internationaux du CNRS.
- Duffie, D. and L. Epstein, (1992), "Stochastic differential utility", *Econometrica*.
- Eichberger, J. and D. Kelsey, (1996) , "Uncertainty Aversion and Preference for Randomisation", *Journal of Economic Theory* 71, pp. 31–43.
- Ellsberg, D. (1961), "Risk, ambiguity and the savage axioms", *Quarterly Journal of Economics* 75(4), 643–669.
- Epstein, L., (1999), "A definition of uncertainty", *Review of Economic Studies* 66, 579–608.
- Epstein, L. and J. Miao, (2003). "A two-person dynamic equilibrium under ambiguity," *Journal of Economic Dynamics and Control*, Elsevier, Volume 27, Number 7, May 2003 , pp. 1253-1288.
- Epstein, L. and M. Schneider, (2003), "Recursive Multiple Priors", *Journal of Economic Theory*, 113, 1–31.

- Epstein, L. and M. Schneider (2008), "Ambiguity, information quality and asset pricing", *Journal of Finance*, 63, 197-228.
- Epstein, L. and T. Wang (1994), "Intertemporal asset pricing under knightian uncertainty", *Econometrica* 62(3), 283–322.
- Epstein, L. and S. Zin, (1989), "Substitution, risk aversion and the temporal behavior of consumption and asset returns: a theoretical framework", *Econometrica* 1989.
- Fisher, I (1930): *The Theory of Interest*. New York: Macmillan.
- Garvin, David A. and Robert H. Hayes, "Managing as if Tomorrow mattered", *Harvard Business Review* 60, no. 3 (May - June 1982).
- Ghiradato, P. and M. Le Breton (1997), "Choquet Rationalizability", Working paper, 1000, California Institute of Technology, Division of the Humanities and Social Sciences.
- Ghiradato, P. and M. Le Breton (2000), "Choquet Rationality", *Journal of Economic Theory*, Elsevier, vol. 90(2), pages 277-285, February.
- Ghiradato, P. and M. Marinacci, (2002), "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory*, 102:251-289.
- Ghirardato, P., Maccheroni, F. and M. Marinacci, (2004), "Differentiating ambiguity and ambiguity attitude", *Journal of Economic Theory* 118, 133–173.
- Gilboa, I., (1987), "Expected Utility with Purely Subjective Non-Additive Probabilities", *Journal of Mathematical Economics*, 16, 65-88.
- Gilboa, I. and D. Schmeidler, (1989), "Maxmin Expected Utility with a Non-Unique Prior", *Journal of Mathematical Economics*, 18, 141-153.
- Gryglewicz, S., Huisman, K.J.M., & Kort, P.M. (2008): "Finite project life and uncertainty effects on investment.", *Journal of Economic Dynamics and Control*, 32(7), 2191-2213
- Hansen, G and T.J. Sargent, G.A. Turmuhambetova and N. Williams, (2001), "Robustness and uncertainty aversion", University of Chicago, Mimeo.
- Karatzas I. and S.E. Shreve, (1991), *Brownian Motion and Stochastic Calculus* (second ed.), Springer, Berlin.
- Kast, R. and A. Lapiéd, (2007), "Dynamically consistent Conditional Choquet Capacities", Working paper, Greqam.
- Kast, R. and A. Lapiéd, (2008), "Dynamically consistent Choquet random walk and real investments", Working paper, Greqam.
- Kast, R., A. Lapiéd and D. Roubaud (2010), "Correlated Choquet-Brownian sources of ambiguity in real options models", Working paper, Greqam.
- Kelsey, D. and J. Eichberger (1996), "Uncertainty Aversion and Preference for Randomisation" *Journal of Economic Theory*, 71, 31-43.
- Keynes, J.M., (1921), *A Treatise on Probability*, Macmillan, London.

- Klibanoff, P., Marinacci, M. and Mukerji, S. (2005), "A smooth model of decision-making under ambiguity", *Econometrica* 73(6), 1849–1892.
- Klibanoff, P., Marinacci, M. and Mukerji, S. (2009), "Recursive smooth ambiguity preferences", *Journal of the Economic Theory*, 144, 930-976.
- Knight, F. (1921). *Risk, Uncertainty and Profit*. Houghton Mifflin, Boston, Mass.
- Lund, D., (2005), "How to analyze the investment-uncertainty relationship in real options models?", *Review of Financial Economics*, 14, 311-322.
- Markowitz, H. (1952) "Portfolio Selection", *Journal of Finance*, Vol. 7, p.77-91.
- McDonald, R.L., Siegel, D., (1986), "The value of waiting to invest". *Quarterly Journal of Economics* 101, 707–727.
- Mehra, R., and Prescott, (1985), "The Equity Premium: A Puzzle", *Journal of Monetary Economics*, 15, 145-161.
- Miao, J. and N. Wang, (2006), "Risk, uncertainty and option exercise", Working paper (first version 2003; most recent : Sept 2010), Boston University and Columbia University.
- Montesano, A. (1990). "On the Definition of Risk Aversion", *Theory and Decision* 29, 53–68.
- Montesano, A., (2008), "Effects of Uncertainty Aversion on the Call Option Market" *Theory and Decision* 65.2 : 97-123.
- Mukerji, S. and Tallon, J.-M. (2001), "Ambiguity aversion and incompleteness of financial markets", *Review of Economic Studies* 68, 883–904.
- Mukerji, S. (1998), "Ambiguity aversion and the incompleteness of contractual form", *American Economic Review*, 88(5):1207–1232.
- Myers, S. (1977), "Determinants of corporate borrowing", *Journal of Financial Economics* 5 (1977), pp. 147–175.
- Nishimura, K. and H. Ozaki, (2004), "Search and Knightian uncertainty", *Journal of Economic Theory* 119, pp. 299–333.
- Nishimura K. and H. Ozaki (2007), "Irreversible investment and Knightian uncertainty", *Journal of Economic Theory*, Volume 136, 668-694.
- Riedel, F. (2009), "Optimal Stopping with Multiple Priors", *Econometrica*, Econometric Society, vol. 77(3), pages 857-908, 05.
- Sarin, R. and Wakker, P. (1992), "A Simple Axiomatization of Nonadditive Expected Utility," *Econometrica*, 60, 1255-1272.
- Sarkar, S. (2000), "On the Investment–Uncertainty Relationship in a Real Options Model". *Journal of Economic Dynamics and Control*, 24, 219–225.
- Schmeidler, D. (1989), "Subjective probability and expected utility without additivity", *Econometrica* 57(3), 571–587.

- Schmeidler D., (1986), *Integral representation without additivity*. Proceedings of the American Mathematical Society 97, 255-261.
- Schroder, D. (2008), "Investment under ambiguity with the best and worst in mind", working paper.
- Shreve, S.E. (2004). *Stochastic Calculus for Finance II: Continuous–Time Models*. Springer, New York.
- Skiadas (2009),
- Stokey, N. and R. E. Lucas with E. C. Prescott, 1989, *Recursive Methods in Economic Dynamics*, Cambridge, MA: Harvard University Press.
- Trigeorgis, L. (1996). *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. Cambridge, NA: The MIT Press.
- Trojanowska, M. and P.M. Kort (2010), "The Worst Case for Real Options", *Journal of Optimization Theory and Applications*, forthcoming.
- Wald A. (1950), *Statistical Decision Functions*, John Willey and Sons, NY, 1950.
- Williams, J.B. (1938), *The Theory of Investment Value*. Harvard University Press 1938.