CONTRIBUTIONS TO THE THEORY OF APOLARITY

being a Thesis presented by

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DECLARATION.

I hereby declare that the following Thesis is based on my own work and that it is my own composition and that it has not been previously presented for a Higher Degree.

The Research was carried out in St. Andrews.

R. Vaidyanathaswamy

CERTIFICATE.

I certify that Mr. R. Vaidyanathaswamy has spent six terms at Research Work under my supervision in the University of St Andrews, and that he has fulfilled the conditions of Ordinance No. 16 (St.Andrews) and that he is qualified to submit the accompanying Thesis in application for the Degree of Ph.D.

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30.5.24

CAREER.

I matriculated in the University of Madras in 1910 and followed a course leading to graduation in Arts; subsequent to it I spent four years as a Research student in the same University.

On Nov 1923. I commenced the research in Algebraic Geometry which is now being submitted as a Ph.D. Thesis.

I am the holder of a Scholarship from the University of Madras and one from the Madras Government; Also a grant from the British Department of Scientific Research.

R. Vaidyanathaswamy

Contributions to the Theory of Apolarity.

The theory of Apolarity which is a principal source of inspiration in Algebraic Geometry, was in great vogue about the middle of last century with a school of German Mathematicians (Clebsch, Meyer, Waelsch, Reye &c.) as a natural consequence of the great advance of Invariant-theory at that time. Some of the main lines of the subject were then laid down, and its more obvious features worked out; but it appears to have fallen since into comparative neglect and disuse, cropping up only on occasions.

Binary Apolarity - which is the special part of Apolarity with which we are concerned here - is a subject which is much more limited in scope, though it makes up for its lack of extent by its corresponding elegance and prettiness. Its great geometrical application is to Rational Curves in general, and in particular to the Norm Curve, which being the prototype of all rational curves may be taken as the rational curve par excellence. Interpreted by means of the Norm Curve, the relations arising out of Apolarity translate themselves into a Binary Geometry of the space of n dimensions. The linear subspaces correspond to linear systems of Binary forms, and speaking generally there is a parallel march between Binary Algebra on one side and the Projective Geometry of the Norm curve on the other ~~curve~~.

It will not be irrelevant in this connexion to draw attention to the fact that various important problems in respect to this correspondence, which would have come up for consideration and solution, had a sustained attention been given to the subject, still remain unsolved, as an evidence of the neglect spoken of above. Thus, though the study of the general correlation, in space of n dimensions in respect of a Norm curve therein, through double-binary forms in cogredient variables, had enabled Waelch [Über Binaren Formen und Correlationen mehrer-dimen-sionaler Räume. Monatsheft für Mathematik und Physik 6. 1895.] to analyse completely the systems of quadrics and linear line-complexes determined by a rational norm curve, still the closely related problem of interpreting geometrically the transvectants of one and two binary forms has never been considered. The particular case of this problem which concerns the specification of binary forms with assigned vanishing transvectants, is closely related to the theory of Rational Automorphic functions and may be expected to throw light from a new quarter on the Polyhedral Forms (Compare Klein. Lectures on the Icosahedron. Page 62).

It remains now to indicate the precise place of the first paper (The Nul Pencil of Binary Quartics) in the above order of ideas. The Rational Norm curve determines in the space of n dimensions a Polarity (or Involutoric Correlation) which is a

Quadric Polarity if n is even, and a Null System if n is odd. Now it is well known that there are infinitely many linear spaces of $I\left(\frac{n-1}{2}\right)$ dimensions lying wholly on a non-singular quadric in n dimensions, and that these linear spaces fall into two algebraically distinct systems if n is odd. In a similar manner, there exist infinitely many linear spaces of $I\left(\frac{n}{2}\right)$ dimensions belonging wholly to a general Null System in n dimensions (that is to say, every line of the space would belong to the linear line-complex determined by the Null System), and these spaces never fall into two systems, in other words, form always an irreducible algebraic manifold. [The existence of two systems of generating planes of a quadric in five dimensions was first noticed by CAYLEY 'on the superlines of a quadric in 5 dimensions' (and also independently by the present writer 'Quadric in five dimensions' Journal of the Indian Mathematical Society (1920); this was later generalised to quadrics in odd-dimensional spaces by Veronese Math. Ann. Bd 19. The corresponding theory for the 'Vollständige' spaces of a line-complex is due to KANTOR Crelle's Journal Bd 118. For full references see Ency. Math. Wiss. III_2 7].

It may be expected that the special systems of binary forms (which I call 'nul systems of binary n-ics' whether n is even or odd), which correspond to the linear spaces of $I\left(\frac{n-1}{2}\right)$ or $I\left(\frac{n}{2}\right)$ dimensions belonging wholly (in the sense defined above) to the polarity of the Norm Curve should have very interesting special properties. These nul systems of binary n-ics are obviously the systems of maximum dimensions consistent with the property that any two of their members are apolar. The simplest case of such systems is the nul pencil of binary cubics which corresponds to a line of the tangent linear complex of a fundamental twisted cubic in space. Two simple properties of this nul pencil are well known :

(A) The Jacobian of the nul pencil is apolar to itself.
(B) The nul pencil coincides with its co-Jacobian pencil.

The next simplest case is the nul pencil of quartics (corresponding to a line lying wholly in the incidence-quadric of the polarity of a norm curve in four dimensions) which is the topic of this paper. It is shown that the characteristic property of this nul pencil is

(C) Its Jacobian is a perfect square.

The paper succeeds in showing that the property (C) also characterises nul systems of binary 2n-ics. What the parallel property of nul systems of binary (2n+1)-ics (i.e. what the corresponding generalisation of (A)) is, is a subject for future research. Thus the paper might be said to have demonstrated one half of a general theorem of unknown form, the simplest cases of which are (A) and (C).

It is necessary to state in this connexion that the property (B) has been extended by Stephanos to pencils of binary n-ics in the memoir cited in the paper. The next paper is formally on a topic of Pure Geometry, but has very intimate relations to the study of systems of binary forms both in its application noticed in Para III and in the manner in which the problem arose for the present writer. It is a fact well known to previous authors (see for instance Meyer's Apolaritát) that the problem of finding the number of systems of binary n-ics with a given Jacobian is exactly equivalent to the enumerative problem of finding the number of regions of r dimensions which meet (n-r)(r+1) regions of n-r-1 dimensions in space of n dimensions. More accurately (and this remark appears to be new) the problem of finding the number (supposed finite) of regions of r dimensions meeting a suitable number of regions of suitable dimensions is the exact counterpart of the problem of finding the number of non-singular systems of binary n-ics determined by a given Jacobian with repeated roots. In particular the general transversal line-problem is the counterpart of the problem of finding the number of non-singular co-Jacobian pencils determined by a given Jacobian with repeated roots. It was from this point of view, and with this purpose, that the problem was attacked by the present writer, resulting in an independent ~~re~~ rediscovery of some of the enumerative methods and results of Schubert.

The last paper (The Rank of the Double-binary form) contains a theory of the double-binary form reminiscent of methods and conceptions used for binary forms. The principal contributions in this paper towards the study of the double-binary form are (1) the concept of Rank and introduction of Rank Covariants; (2) extensions in various directions of known properties of the (2,2) form; (3) a Canonical shape valid in certain cases; (4) a geometrical representation of the general form as a correspondence of a special type between two rational norm curves. This last representation depending as it does on the Geometrical Duality belonging to the norm curve, brings vividly back to one the Binary Geometry of the curve based on the apolar concept. In regard to the canonical form, it is worthy of mention that the canonical shape for the (2,2) form was known to and used by the present writer five years ago. The paper also opens up various questions for further research. In particular, the writer has reason to believe that the determination of the manner in which the rank of the product of two forms depends on the ranks of the individual forms, will bring a very interesting and little-studied class of forms - the 'cyclic' or 'closed' forms, corresponding to the cyclic or closed correspondences - within the purview of the ideas and methods of this paper. This will lead to a very desirable result, namely the greater understanding and insight into various important geometrical configurations depending on such forms.

The Algebraic Geometry of Double-binary forms has not been the subject of much study, except to a certain extent through curves on a quadric surface (or what comes to the same thing, circular curves in a plane). It may be therefore expected that this paper will prove to be of the greatest importance for future study in this direction, as it lays down certain fundamental concepts and lines of investigation which it would be worth while to follow up and extend the scope of. The writer believes (though so far he has not been able to see how) that these methods <u>could</u> be applied with advantage to the study of the (3,3) form and of the interesting unsolved problem of the possibility of its reduction to the sum of four cubes. [<u>Added later</u>. The <u>possibility</u> of this reduction is proved by Prof. Turnbull in Proc. Cam. Phil. Soc. 1924, though the actual reduction has not been performed.].

The Null Pencil of Binary Quartics.

The particular variety of pencils of quartics in which any two members are apolar (which I call the 'Null Pencil of Quartics') does not seem to have received any attention. I set down here some of its fundamental properties, and point out the specialities which it introduces into the usual geometric representations of quartic involutions. I also extend two of its properties to forms of higher order, the more striking of these extensions being:

> 'The Jacobian ('Functional Determinant') of a Null System of binary 2n-ics is a perfect square'

where by a Null System is meant a system of the form $\lambda_1\phi_1+\lambda_2\phi_2+\lambda_3\phi_3+\ldots+\lambda_n\phi_n$ (the ϕ's being linearly independent 2n-ics), possessing the property that any two of its members are apolar.

1. By a 'Null Pencil of Quartics' we shall understand a pencil $\lambda a_x^4+\mu b_x^4$ in which any two members are apolar. The conditions for this are:

$$(a_x^4, a_x^4)^4 = (a_x^4, b_x^4)^4 = (b_x^4, b_x^4)^4 = 0 \quad \ldots\ldots \quad (1)$$

In particular it follows from this definition that every member of a Null Pencil is self-apolar; conversely a pencil containing three self-apolar quartics is composed wholly of such quartics, and is a Null Pencil. For, if

$$F(\lambda,\mu) \equiv \left((\lambda a_x^4+\mu b_x^4), (\lambda a_x^4+\mu b_x^4)\right)^4$$

vanishes for three values of $\lambda:\mu$, it must vanish identically and equations (1) result.

The invariant j of a quartic being of the third degree in the coefficients, it follows that any pencil of quartics must contain three members with vanishing j. Since however, i vanishes for all members of a Null Pencil, it follows that a Null Pencil must contain three members for which $i=j=o$, that is to say, three members possessing a cubed linear factor. Hence

Theorem (A). The Jacobian of Null Pencil is the square of a cubic f.

Now it is known that ~~three~~ there are five pencils of quartics with a given Jacobian (Stephanos Mem de l'Institut, 27, 1883; Meyer Apolaritat §29). We can directly specify these five pencils in the particular case in which the given Jacobian is the square of a cubic f. Supposing the linear factors of f to be x, y, z, these five pencils are (1) the Null Pencil with which we started (2) the Singular pencil $XYZ\alpha_x$, α_x being an arbitrary linear form, (3) the three singular pencils of the type $\lambda xy^3+\mu xz^3$. It would appear therefore that the Null Pencil is the only non-singular pencil which has f^2 for its Jacobian (this is a particular case of the theorem that only one non-singular pencil of binary n-ics is determined by a Jacobian (2n-2)-ic, with only three distinct linear factors, the multiplicity of none of these being greater than n-1). This suggests the converse of Theorem (A), namely:

Theorem (a) <u>A non-singular pencil of quartics is necessarily a Null Pencil, if its Jacobian is a perfect square.</u>

To prove this we have only to remark, that the pencil being non-singular will necessarily have to contain three members, which possess cubed linear factors and are therefore self-apolar. From this the truth of the theorem follows immediately.

The Null Pencil the Jacobian of which is f^2 is thus <u>uniquely</u> determined by the cubic f $(=xyz)$. If x^3x', y^3y', z^3z' (where $x'y'z'$ are linear forms) belong to the Null Pencil, it would <u>therefore</u> follow that $x'y'z'$ is a covariant of f, and hence its cubi-covariant. To verify this, let the linear relation between x,y,z be $x+y+z=0$, so that the linear factors of the cubi-covariant are (except for numerical multipliers) $y-z$, $z-x$, $x-y$. Since

$$\sum x^3(y-z) \equiv -(y-z)(z-x)(x-y)(x+y+z) = 0,$$

it follows that the three quartics $x^3(y-z)$, $y^3(z-x)$, $z^3(x-y)$ belong to a pencil which is precisely the Null Pencil with the Jacobian $f^2=x^2y^2z^2$.

§2. Theorem (B). <u>The Null Pencil defined by the cubic $f=xyz$ consists of all quartics which are apolar to each of the three quartics y^2z^2, z^2x^2, x^2y^2.</u>

For, since there is no linear relation between y^2z^2, z^2x^2, x^2y^2, the quartics apolar to them generate a pencil Γ. Now, that member of Γ which contains the factor X, must (since it is to be apolar to x^2y^2 and x^2z^2) contain the factor X^3. Thus Γ contains three members with the cubed linear factors x^3, y^3, z^3 and is therefore the Null Pencil defined by the cubic $f=xyz$.

Theorem (C). <u>The Null Pencil defined by the cubic f, consists of the Jacobians of f with all cubics ϕ apolar to f.</u>

For, any cubic ϕ apolar to f ($=xyz$, $x+y+z=o$) is a linear combination of x^3, y^3, z^3. Now the Jacobian of xyz and x^3 differs only by a numerical factor from $x^3(y-z)$. Hence, the Jacobian of f and ϕ is of the form $a\,x^3(y-z)+b\,y^3(z-x)+c\,z^3(x-y)$, and is there a member of the Null Pencil defined by f. As a Corollary of this theorem we have:

<u>Two Null Pencils defined respectively by the cubics f and ϕ do, or do not, have a Common member, according as f and ϕ are, or are not, apolar.</u>

§ 3. Geometrical Representation in four dimensions.

The known geometrical representation of binary quartics by points in a four-fold space with a fundamental Norm curve furnishes simple proofs of the above theorems. Thus, let the point $(x_0x_1x_2x_3x_4)$ of S_4 be made to correspond with the binary quartic $(x_0x_1x_2x_3x_4)(x_1, x_2)^4$. The points which correspond to quartics which are perfect fourth powers will then trace the

Normcurve:

$$N_4 :- \rho X_r = t^{n-r} \quad (r=o,\ 1..4),$$

where t is a parameter. It can then be shewn (either analytically or purely from binary considerations of apolarity) that there are four points $t_1 t_2 t_3 t_4$ on N_4, the osculating S_3^1s at which pass through a given point X, and that the $S_3(t_1 t_2 t_3 t_4)$ is the locus of points which correspond to quartics apolar to the quartic 'X'. The correspondence between the point X and the $S_3(t_1 t_2 t_3 t_4)$ is the 'polarity' of the Normcurve.

Further, in this correspondence, self-apolar quartics correspond to points lying on the quadric:

$$N_2 : X_0 X_4 - 4X_1 X_3 + 3X_2^2 = 0.$$

It is well known (and may be immediately verified) that the polarity of this quadric is identical with the polarity of the Normcurve, that is to say, conjugate points with respect to N_2, also correspond to apolar quartics. On this account the quadric N_2 may be called 'the equivalent quadric of the Normcurve N_4', and vice versa, the curve N_4 'an equivalent norm curve of the quadric N_2'.

Now, a pencil of quartics Γ corresponds to a line γ in S_4; the Jacobian of Γ corresponds to the six points on N_4 the osculating planes at which meet γ. If Γ is a Null Pencil then γ is a generating line of the equivalent quadric N_2. The theorems (A) (a) now take the geometrical forms:

Theorem (A_1). <u>Every generating line of the equivalent quadric meets three tangent lines of the Normcurve;</u> or to put it differently, <u>Any generating line of a quadric in S_4 meets three tangent lines of any equivalent normcurve</u>.

Theorem (a_1). <u>Any line which meets three tangents of a Norm Curve N_4 is a generating line of its equivalent quadric</u>.

Since, (as may be easily seen) the tangents of N_4 are generating lines of its equivalent quadric, the truth of (a_1) is geometrically obvious. To prove (A_1), suppose that the generating line γ of the quadric meets the osculating plane P_α at the point α of the norm curve. Now P_α is the polar region (w.r.t the Norm curve and therefore also w.r.t the equivalent quadric) of the tangent T_α at the point α of the curve. Hence P_α touches the equivalent quadric along T_α. Since meeting T_α is equivalent to meeting two consecutive P_α's, it follows that γ meets <u>three</u> tangents of the Norm Curve.

The geometrical proof of Theorem (B) turns upon the evident fact, that the unique line-transversal of three tangents to a Norm curve is the intersection of the three S_3's determined by every two of them.

§ 4. We shall now examine, with reference to the Null Pencil, the three usual modes of representing pencils of quartics geometrically by means of the twisted cubic.

It is known (Meyer Apolarität §19) that the pairs of lines intersecting the tangent-quadruples of an involution of the fourth order on the space-cubic, generate one regulus of a quadric Q (which may be called the Sturm-quadric of the Involution), the other regulus of which belongs to the complex of the curve. If now, the involution be a Null involution, all the tangent-quadruples will be equianharmonic, and therefore their transversal-pairs will coalesce into lines belonging to the complex of the curve. Hence for this case, both reguli of the quadric Q will have to belong to the complex of the curve. This is however not possible unless Q is two coincident planes. Hence :

> The Sturm-quadric of a Null Involution on the twisted cubic is a pair of coincident planes; Conversely the tangent-quadruples of the curve meeting the complex-lines lying in any given plane P constitute a Null Involution the Jacobian of which is the square of the cubic cut out of the curve by P.

It will be noticed that the latter part of the theorem is the direct geometrical expression of Theorem (C).

It will be sufficient merely to state the specialisations which occur in the two other usual modes of representing quartic pencils on the twisted cubic. A pencil Γ of quartics may be represented (a) as the set of tetrahedra inscribed in a twisted cubic C and self-polar with respect to a quadric Σ 'in-polar' to the curve (i.e. apolar as a class quadric to all quadrics containing the curve), (b) as the set of ∞^1 tetrahedra inscribed in C and circumscribed to a second cubic curve C_1. If Γ is a null pencil, the Jacobian of which corresponds to the points X,Y,Z of the curve (the points $X'Y'Z'$ on the curve being the cubi-covariant points of XYZ), then:

(1) the quadric Σ touches the curve at x,y,z; the poles w.r.t Σ of the osculating planes at x,y,z are the points x', y', z'. Further Σ cuts the plane xyz in the polar conic of its nul point w.r.t the triangle XYZ.

(2) the four other quadrics corresponding to the singular Co-Jacobian pencils of Γ are, as loci, three cones touching the curve at two of the points X Y Z and having their vertex at the third, and the squared plane XYZ, and as envelopes, the three squared points X,Y,Z and a conic which has XYZ for a self-polar triangle.

(3) The twisted cubic C_1 touches the lines $X'X$, $Y'Y$, $Z'Z$, at the points X', Y', Z' respectively, and osculates the osculating planes at X,Y,Z to C. The osculating planes at X', Y', Z' to C_1 contain the tangents at X,Y,Z to C.

This relation between the curves C and C_1 may be compared with the corresponding relation which obtains in the case (discussed in detail by Meyer Apolarität No 177) in which Γ contains two perfect fourth powers.

5. It is well known that the conics of a plane which are apolar to a fundamental class conic S, and which cut out the quadruples of an involution on S belong to a pencil. The quadrangle which determines this pencil is self-polar w.r.t S

(i.e. each of the four points is the centre of perspective of the triangle formed by the other three and its polar triangle). This quadrangle together with its polar quadrilateral w.r.t S constitutes a Desergues configuration [D] of ten points and ten lines, out of which there can be formed five self-polar quadrangles (and quadrilaterals) of S. These five quadrangles determine (according to Stephanos l.c-) five Co-Jacobian pencils of quartics on S.

Now one of the Co-Jacobian pencils of the Null Pencil defined by the cubic f=XYZ is the singular pencil $XYZ\alpha_x$. The quadrangle corresponding to this pencil is obviously the quadrangle XYZO, where X,Y,Z are points on S corresponding to the linear forms XYZ, and O is the centre of perspective of XYZ and xyz(x,y,z being the respective tangents at X,Y,Z). The corresponding Desargues configuration consists therefore, of the ten points:X,Y,Z,O,(yz), (zx), (xy), (x, YZ), (y, ZX), (z, XY). The five self-polar quadrangles formes out of these are:

(1) [XYZO]
(2) [X,(yz), (y,ZX), (z,XY)]
(3) [Y,(zx), (z,XY), (x,YZ)]
(4) [Z,(xy), (x,YZ), (y,ZX)]
(5) [O,(yz), (zx), (xy)]

The quadrangles (2), (3), (4) determine the three singular pencils of the type $\lambda XY^3 + \mu XZ^3$; the quadrangle (5) determines the Null

Pencil. The conclusion may be stated thus:

> A Desargues configuration [D] self-reciprocal with respect to a conic S, determines a Null Pencil (and its Co-Jacobian pencils) on S, if and only if three of its points X,Y,Z (and therefore also three of its lines) belong to S; when this happens, the Null Pencil is determined by that particular self-polar quadrangle of [D] which has XYZ for its harmonic triangle.

§ 6. It was already mentioned, that only one non-singular pencil of binary n-ics is determined by a Jacobian of the form $X^p Y^q Z^r$ (where X,Y,Z are linear forms; $p+q+r=2n-2$; $p,q,r<n$). We can specify this unique non-singular pencil in a manner completely analogous to Theorem (B):

Theorem (B′): The unique non-singular pencil of binary n-ics defined by the Jacobian $X^p Y^q Z^r$, is composed of all the n-ics which are apolar to each of the three forms, $Y^{n-q}Z^{n-r}$, $Z^{n-r}X^{n-p}$, $X^{n-p}Y^{n-q}$.

To prove this we notice that the n-ics apolar to the three forms constitute a linear ∞^k-system, where

$$k = 6n-3-2\,(p+q+r)-2n = 1;$$

that is, the system of apolar n-ics is a pencil Γ. Now from a known theorem in Apolarity (Grace and Young Algebra of Invariants, pp. 225, 227), the n-ic $X^{p+1}\alpha_x^{n-p-1}$ is necessarily apolar to X^{n-p} and therefore also to each of the forms $X^{n-p}Y^{n-q}$, $X^{n-p}Z^{n-r}$ (whatever form of order n-p-1 is represented by α_x^{n-p-1}). Further, the condition that $X^{p+1}\alpha_x^{n-p-1}$ is apolar to $Y^{n-q}Z^{n-r}$ gives n-p-1 equations which exactly suffice for determining α_x^{n-p-1} apart from a numerical multiplier. The form $X^{p+1}\alpha_x^{n-p-1}$ thus determined belongs to Γ, so that it follows that Γ contains three members with the respective factors X^{p+1}, Y^{q+1}, Z^{r+1}. Thus Γ is identical with the (non-singular) pencil determined by the Jacobian $X^pY^qZ^r$ – which proves the theorem.

This theorem may be interpreted geometrically in space of n dimensions in the same manner as was already noticed for theorem B.

§7. The geometrical theorem A_1 can be extended to a space S_{2n} of 2n dimensions, and thereby be made to yield an <u>algebraic</u> extension of Theorem A. A Normcurve N_{2n} in S_{2n} determines – as is well known – a polarity of the quadric variety, which corresponds to the apolar relation between binary 2n-ics. The quadric which produces the same polarity – that is to say, the 'equivalent' quadric – contains the Normcurve and is uniquely determined by the further condition that it contains all the (n-1)- dimensional osculating regions of the curve. Now any region P_{n-1} of n-1 dimensions in S_{2n} is met by n(n+1) osculating P_n's of the curve. Suppose now, that P_{n-1} is a generating region of the equivalent quadric, and that it meets O_n^{α}, the osculating region of n dimensions at the point α of the norm curve. Now O_n^{α} is the polar region (w.r.t the Norm curve and therefore also w.r.t the equivalent quadric) of O_{n-1}^{α};

hence since O^{α}_{n-1}, is a generating region, it follows that O^{α}_{n} <u>touches</u> the equivalent quadric along O^{α}_{n-1}. Since P_{n-1} lies wholly in the quadric and meets O^{α}_{n}, it must meet that part of O^{α}_{n} which lies in the quadric; that is, P_{n-1} meets O^{α}_{n-1}. Since meeting O^{α}_{n-1} is equivalent to meeting two consecutive O^{α}_{n}'s, we have finally the theorem:

> Theorem (A'_1): Any generating region (of the maximum number of dimensions n-1) of a quadric in S_{2n} meets $\frac{n(n+1)}{2}$ osculating regions of n-1 dimensions of any equivalent Normcurve.

Since the number $\frac{n(n+1)}{2}$ is just one greater than the number of points necessary to determine a quadric in a P_{n-1} which meets the $\frac{n(n+1)}{2}$ osculating regions, we may conclude that the converse of the theorem is also true; namely:

> Theorem (a'_1): Any P_{n-1} which meets $\frac{n(n+1)}{2}$ osculating regions of n-1 dimensions of a norm curve N_{2n} is a generating region of its equivalent quadric.

In the algebraical interpretation a region P_{n-1} of S_{2n} corresponds to a linear ∞^{n-1}- system of binary 2n-ics; the linear factors of the Jacobian of the system correspond to the points d of the Norm curve such that O^{d}_{n} meets P_{n-1}. A generating region P_{n-1} of the equivalent quadric corresponds to a Null System of binary 2n-ics - that is, a system any two members of which are apolar, and the dimensions of which are the greatest possible consistent with this condition. Hence the algebraic extension of Theorem A derived from A'_1, is:

> Theorem (A'): The Jacobian of a Null System of binary 2n-ics is the square of a form f of order $\frac{n(n+1)}{2}$.

The algebraic form of a'_1, would be:

> Theorem (a'): A linear ∞^{n-1}-system of binary 2n-ics, the Jacobian of which is a perfect square, is necessarily a Null System (in accordance with our definition), provided it is not singular.

[If d be a root of the Jacobian of a linear ∞^{k}-system of binary 2n-ics, then there exists a form p_x^{m-k-1} such that $(dx)^{k+1}p_x^{m-k-1}$ belongs to the system. If there exists more than one such form, d may be called a <u>singular</u> point of the system. A system is <u>singular</u> if it possesses one or more singular points.

A singular point is necessarily a repeated root of the Jacobian. The converse is not true, that is, a repeated root of the Jacobian is not necessarily a singular point of the system. It is however true that a repeated root of the Jacobian J is a singular point of <u>one at least</u> of the Co-Jacobian systems determined by J.

The actual number of Null Systems determined by a given Jacobian f^2 is the number of P_{n-1}'s which meet $\frac{n(n+1)}{2}$ arbitrary P_{n-1}'s in S_{2n}. This number has not been evaluated for the general case. For 2n=6 its value is 16 (<u>Segre</u>, <u>Enc. Math. Wiss.III$_2$ 7.</u> p <u>817 note 157</u>].

[Extracted from the *Proceedings of the Cambridge Philosophical Society*, Vol. XXII. Pt. I.]

On the Number of Lines which meet Four Regions in Hyper-Space.* By R. VAIDYANATHASWAMY. (Communicated by Mr H. W. TURNBULL, Trinity College.)

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The problem of finding the number of r-dimensional regions which are situated in a space S_n of n dimensions, and which satisfy a suitable number of conditions of certain assigned types (called "ground-conditions") has been investigated by Schubert†. A special class of such problems arises when the r-dimensional region is merely required to intersect k regions P_λ of r_λ dimensions $(\lambda = 1, 2 \ldots k)$ situated in general position in S_n, where for the finiteness of the sought number we must have

$$r_1 + r_2 + \ldots + r_k = (k - r - 1)(n - r).$$

These latter may be called "Transversal-Problems" and the required number of r-dimensional regions may be denoted by

$$[n, (r_1, r_2 \ldots r_k), r].$$

The only Transversal Line-problems (that is, transversal-problems in which $r = 1$) which have been solved for general values of n are those in which

$$(1)\quad r_1 = r_2 = \ldots = r_k = n - 2; \quad k = 2n - 2,$$

$$(2)\quad r_1 = r_2 = \ldots = r_{k-1} = n - 2; \quad r_k = k - n,$$

$$(3)\quad r_1 = r_2 = \ldots = r_k = n - 3; \quad k = n - 1.$$

It is the object of this communication to show that a simple and elegant result can be stated for the case $k = 4$.

The symbolism which I have used is in some ways more convenient for transversal-problems than the symbolism of Schubert. The proofs are based upon the Principle of Conservation of Number‡ ("Prinzip der Erhaltung der Anzahl") which is equally the foundation of Schubert's methods. The validity of this principle for the problem of finding the number of lines satisfying assigned ground-conditions has been established in a purely algebraic manner by Max Caspar§.

* *Region* is a convenient term for "*flat subspace*" of a flat space of n dimensions. It is used by Whitehead, *Universal Algebra.*

† *Math. Annalen*, Bd. 26, 38.

‡ For an account of this principle see Schubert, *Kalkul der Abzahlenden Geometrie*, or preferably Zeuthen, "Abzahlenden Methoden" (*Ency. Math. Wiss.*).

§ *Math. Annalen*, Bd. 59.

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I. There are $\infty^{n-1+n-1}$ lines in S_n and of these, ∞^{n-1+r} meet an arbitrary r-dimensional region. Thus a line in S_n which meets a region of r dimensions satisfies thereby $n-1-r$ conditions. Hence for the finiteness of $[n, (r_1 r_2 r_3 r_4), 1]$—which represents the number of lines meeting four base regions P_λ of r_λ dimensions $(\lambda = 1, 2, 3, 4)$—we must have

$$r_1 + r_2 + r_3 + r_4 = 2n - 2 \qquad \ldots\ldots(1).$$

To evaluate $[n, (r_1 r_2 r_3 r_4), 1]$ when this equality is satisfied, consider first the special case in which $r_1 + r_2 = r_3 + r_4 = n - 1$. Let us examine what happens to the transversals in this case when we make P_1 and P_2 intersect in a point p and P_3 and P_4 in a point q. From the Principle of Conservation of Number the number of transversals will either remain unaltered after such intersection or will become infinite. Now since $r_1 + r_2 = n - 1$ and P_1, P_2 intersect in a point, it follows that they are contained in an S_{n-1}; similarly P_3, P_4 are contained in an S'_{n-1}. Hence all transversals of the four regions other than the line pq must be contained in both S_{n-1} and S'_{n-1} and therefore in the S_{n-2} which is their intersection; in fact they will be identical with the transversals of the four regions P_λ' of $r_\lambda - 1$ dimensions in which this S_{n-2} cuts the regions P_λ $(\lambda = 1, 2, 3, 4)$. Since the regions P_λ' are in general position equally with the regions P_λ and since

$$\Sigma\,(r_\lambda - 1) = 2\,(n-2) - 2,$$

it follows from (1) that the number of transversals do not become infinite. Hence the particular mode of intersection of the base regions which we have chosen has no effect on the number of transversals. Consequently

$$[n, (r_1 r_2 r_3 r_4), 1] = 1 + [n-2, (r_1 - 1, r_2 - 1, r_3 - 1, r_4 - 1), 1].$$

Since the relation (1) holds also for the term on the right, this process of reduction could be repeated, giving finally

$$[n, (r_1 r_2 r_3 r_4), 1] = r_1 + [n - 2r_1, (0, r_2 - r_1, r_3 - r_1, r_4 - r_1), 1],$$

on the supposition that r_1 is the least of the numbers r. To evaluate the term on the right, we notice that $r_2 - r_1$ being equal to $n - 2r_1 - 1$ might be cancelled out as it imposes no effective restriction on the transversal line. Thus the term represents the number of lines in S_{n-2r_1} which pass through a point and meet two regions of dimensions $r_3 - r_1$, $r_4 - r_1$ respectively. Since

$$r_3 - r_1 + r_4 - r_1 = n - 2r_1 - 1,$$

this number is unity and we have finally the result that if r_1 be the least of the numbers r,

$$[n, (r_1, r_2, r_3, r_4), 1] = r_1 + 1,$$

when
$$r_1 + r_2 = r_3 + r_4 = n - 1.$$

II. Consider now the general case in which r_1, r_2, r_3, r_4 are any four numbers subject to the relation (1). Supposing that they are in ascending order of magnitude, we may write

$$r_1 + r_2 = n - 1 - \lambda,\ r_3 + r_4 = n - 1 + \lambda;\ \ \lambda = \tfrac{1}{2}(r_3 + r_4 - r_1 - r_2).$$

Since $r_1 + r_2 = n - 1 - \lambda$, P_1 and P_2 are contained in an $S_{n-\lambda}$ which must therefore contain all the possible transversals. Since this $S_{n-\lambda}$ cuts the other two base regions in regions of dimensions $r_3 - \lambda$, $r_4 - \lambda$, we have immediately

$$[n, (r_1 r_2 r_3 r_4), 1] = [n - \lambda, (r_1, r_2, r_3 - \lambda, r_4 - \lambda), 1].$$

But since $r_1 + r_2 = n - \lambda - 1$ the right-hand term is of the type already evaluated and is equal to the smaller of the numbers $r_1 + 1$, $r_3 - \lambda + 1 = n - r_4$. For the general case we have therefore finally the result:

$$[n, (r_1 r_2 r_3 r_4), 1] = \text{smaller of } r_1 + 1,\ n - r_4 \quad (r_1 \leqslant r_2 \leqslant r_3 \leqslant r_4).$$

If we define the *point-wise* and *plane-wise* orders of a region in S_n to be respectively the number of independent points and the number of independent S_{n-1}'s necessary to determine the region, then the point-wise and plane-wise orders of a region of r dimensions would be $r + 1$ and $n - r$ respectively. The theorem obtained may then be stated as follows:

The number of lines meeting four regions in S_n—if finite—is equal to the least of the orders point-wise and plane-wise of the four regions.

III. If r_4 is put equal to $n - 1$ (so that the plane-wise order of P_4 is unity) then P_4 may be cancelled from the list of base regions as it imposes no effective restriction on the transversal line. The theorem then reduces to

$$[n, (r_1, r_2, r_3), 1] = 1;\ r_1 + r_2 + r_3 = n - 1 \quad \ldots\ldots(2),$$

a result which may also be obtained directly. If we suppose that P_1, P_2, P_3 are osculating regions of a rational normal curve in S_n, then, by using the binary coordinate-system, (2) could be transformed into the following theorem relating to Co-Jacobian pencils of binary forms:

One and only one non-singular pencil of binary n-ics is determined by a Jacobian $(2n - 2)$-ic with three distinct linear factors—the multiplicity of no one of these factors being greater than $n - 1$.

IV. Putting $n = 5$, $r_1 = r_2 = r_3 = r_4 = 2$ in the theorem, we have the result:

There are precisely three lines which meet four planes in space of five dimensions.

This result is of particular interest as it can be verified as follows by Line-geometry.

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Represent linear complexes in S_3 by points in S_5. The singular complexes, or—what is the same thing—the lines of S_3 will then correspond to points on a quadric Γ in S_5. A plane in S_5 intersects Γ in a conic which corresponds to one regulus of a quadric in S_3; a pair of mutually polar planes P_1, P_2 of Γ intersect it in a pair of conics which correspond to the two complementary reguli of a quadric Q in S_3. If q_1q_2 be a chord of Γ which intersects P_1 and P_2, then q_1, q_2 correspond to mutually polar lines of the quadric Q in S_3. Hence if P_3, P_4 be another pair of polar planes of Γ corresponding to the quadric Q' in S_3, there will be as many common transversals of P_1, P_2, P_3, P_4 as there are pairs of common polar lines of Q and Q' in S_3. But the only common polar line-pairs of two quadrics in S_3 are the pairs of opposite edges of their common self-polar tetrahedron. Thus P_1, P_2, P_3, P_4 have exactly three transversals. The same is true of *any* four regions, for since a quadric in S_5 involves 20 constants, any two pairs of planes are always polar pairs with respect to a quadric.

V. Supposing now that $r_4 = 1$ and none of r_1, r_2, r_3 is equal to 0 or $n-1$, we have from the main theorem

$$[n, (r_1, r_2, r_3, r_4), 1] = 2; \quad r_1 + r_2 + r_3 = 2n - 3, \; r_4 = 1.$$

Now it is easily seen that the regions P_1, P_2, P_3 have ∞^{n-2} transversals. The locus Q of these transversals contains therefore ∞^{n-1} points and must be a quadric since it cuts an arbitrary line P_4 in two points. Now since

$$r_1 + r_2 + r_3 = 2n - 3; \quad r_1, r_2, r_3 < n - 1,$$

it follows that the sum of any two of the numbers r is equal to or greater than $n-1$ and therefore a transversal can always be drawn from any point in any one of the regions P_1, P_2, P_3 to the other two. Thus the quadric Q contains P_1, P_2, P_3, and further, any quadric containing P_1, P_2, P_3 must contain their transversals and is therefore identical with Q. We have thus the theorem:

Any three regions in S_n of dimensions r_1, r_2, r_3, such that $r_1 + r_2 + r_3 = 2n - 3$, are contained in a unique quadric Q.

This is an extension of the theorem that three lines in three dimensions are contained in a unique quadric. It may be observed that the theorem may be supposed to hold also for the boundary case $r_3 = n - 1$; for here Q may be considered to have degenerated into P_3 and the S_{n-1} determined by P_1 and P_2.

The quadric Q has a singular region of $n-2$ dimensions in the boundary case $r_3 = n-1$ and a singular region of $n-4$ dimensions in every other case.

The first part is obvious as Q splits up into two $(n-1)$-dimensional regions in the boundary case. To prove the second part, let $r_1 r_2 r_3$ be in ascending order of magnitude. Projecting from the $(r_2 + r_3 - n)$-

dimensional region common to P_2 and P_3, Q projects into a quadric Q' in a space of $r_1 + 2$ dimensions determined by three regions P_1', P_2', P_3' of the respective dimensions r_1, $n - 1 - r_3$, $n - 1 - r_2$. If P_1' and P_3' do not intersect, then $r_2 = n - 2$ and therefore $r_3 = n - 2$, $r_1 = 1$ and the quadric Q' becomes a non-singular quadric in three dimensions. Thus the singular region of Q is the common region of P_2 and P_3 and is of $r_2 + r_3 - n = n - 4$ dimensions. If P_1' and P_3' intersect, then projecting again from their common region of $n - 3 - r_2$ dimensions, Q' projects into a quadric Q'' in a space of $n - r_3 + 1$ dimensions determined by three regions P_1'', P_2'', P_3'' of the respective dimensions

$$n - 1 - r_3,\ n - 1 - r_3,\ 1.$$

If P_1'' and P_2'' do not intersect, then $r_3 = n - 2$, Q'' is a non-singular quadric in three dimensions, and the singular region of Q is of $r_2 + r_3 - n + n - 3 - r_2 + 1 = n - 4$ dimensions. If P_1'' and P_2'' intersect, then projecting from their common region of $n - 3 - r_3$ dimensions, Q'' projects into a non-singular quadric determined by three lines in three dimensions. Hence the singular region of Q is of $r_2 + r_3 - n + n - 3 - r_2 + n - 3 - r_3 + 2 = n - 4$ dimensions.

It is clear that in every case the singular region of Q is the join of the regions of intersection of P_1, P_2, P_3 two and two.

On the Rank of the Double-binary Form.

The point of view which is adopted in this paper towards the double-binary form, is one which attempts to study it by means of binary forms.* This is the method naturally suggested by Geometry. In this point of view the rank-concept plays a fundamental rôle.

The paper is mainly occupied with the properties, and the geometrical interpretation of the Rank-Covariants, and their reciprocity for the equiform. Section (5) studies an important method of representing a double-binary form, as a correspondence of a special type between two rational norm curves. In Section (6) a useful canonical shape is obtained for double-binary forms which are of even order in each of the variables. It is seen from the last section, that this canonical shape is more advantageous for the (2,2) form, in point of symmetry, than the numerical canonical types which are in ordinary use.

[The two variables of the double-binary form are considered throughout as independent, i.e not cogredient].

* The recent study of the (2,2) form and its concomitants by Prof Turnbull (Proc. Roy. Soc. Edin. Vol XLIV 1924, esp. pp 36-40) is interesting as a special case of this point of view, as it obtains simple interpretations of the unsymmetrical (2,4) and (4,2) covariants.

ON THE RANK OF THE DOUBLE-BINARY FORM.

R. Vaidyanathaswamy.

1. Definition of Rank.

Let a double-binary form $F = a_x^m b_y^n$ be written out in the shape :

$$\begin{aligned} F &= \sum a_{rs} x_1^{m-r} x_2^r y_1^{n-s} y_2^s \\ &= y_1^n F_0(x) + y_1^{n-1} y_2 F_1(x) + \cdots + y_2^n F_n(x) \\ &= x_1^m \Phi_0(y) + x_1^{m-1} x_2 \Phi_1(y) + \cdots + x_2^m \Phi_m(y) \quad \cdots\cdots (1) \end{aligned}$$

where

$$F_k(x) = \sum_r a_{rk} x_1^{m-r} x_2^r \; ; \; \Phi_k(y) = \sum_s a_{ks} y_1^{n-s} y_2^s .$$

Let Δ stand for the matrix $\left| \begin{matrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & \cdot & \cdot & a_{1n} \\ \vdots & & & \\ a_{m0} & \cdot & \cdot & a_{mn} \end{matrix} \right)$.

<u>We define the rank of the double-binary form F to be the rank of the matrix Δ.</u>

If Δ has the rank r, then only r of its rows or columns are linearly independent, and therefore only r of the binary forms $F_k(x)$ or $\Phi_k(y)$ are linearly independent. On expressing the n+1 forms $F_k(x)$ linearly in terms of r independent forms $f_1(x)$, $f_2(x) \ldots f_r(x)$, it follows from (1), that F is expressible in the shape :

$$F(x,y) = f_1(x)\phi_1(y) + f_2(x)\phi_2(y) + \cdots + f_r(x)\phi_r(y) \quad \cdots\cdots (2)$$

It should be noticed that not only the forms f but also the forms ϕ in (2), are linearly independent; for since the forms

$\Phi_k(y)$ are obviously linear combinations of $\phi_1, \phi_2, \cdots \phi_r$, any linear relation between these latter would make the rank of Δ less than r, contrary to our hypothesis.

§ 2. The Rank Covariants.

The form $F(x,y) = a_{0x}^m\, b_{0y}^n = a_{1x}^m\, b_{1y}^n = \ldots\ldots$ is ordinarily of rank equal to the lesser of the numbers m+1, n+1. In order to find the condition that it may be of any given rank r (< m+1, n+1), we observe that when F is of rank r and is expressed in the form (1), any r+1 linear combinations of the forms $\Phi_k(y)$ must be connected by a linear relation. Hence the (r+1) r^{th} polars :

$$\frac{\partial^r F(x,y)}{\partial x_1^k\, \partial x_2^{r-k}} \quad (k=0,1,\cdots r),$$

regarded as forms in y must be subject to a linear relation. Now the symbolic condition (Compare Grace and Young Algebra of Invariants pp 220, 221) that the forms

$$c_{0y}^n,\ c_{1y}^n \cdots \quad c_{ry}^n,$$

be linearly related is the identical vanishing of their Jacobian, namely the covariant,

$$(c_0c_1)\cdots(c_0c_r)(c_1c_2)\cdots(c_{r-1}c_r)\, c_{0y}^{n-r}\, c_{1y}^{n-r} \cdots c_{ry}^{n-r}.$$

Hence the necessary and sufficient condition for a linear relation between the (r+1) r^{th} polars,

$$a_{01}^r\, a_{0x}^{m-r}\, b_{0y}^n,\ a_{11}^{r-1} a_{12}\, a_{1x}^{m-r}\, b_{1y}^n, \cdots\ a_{r2}^r\, a_{rx}^{m-r}\, b_{ry}^n,$$

regarded as forms in y, is the identical vanishing of

$$(b_0b_1)\cdots\cdots(b_{r-1}b_r)\, b_{0y}^{n-r}\cdots b_{ry}^{n-r}\, a_{0x}^{m-r}\cdots a_{rx}^{m-r}\cdot a_{01}^r\cdot a_{11}^{r-1}a_{12}\cdots a_{r2}^r.$$

By deranging the equivalent symbols in all possible ways, this expression is seen to be equal to

$$\frac{1}{(r+1)!}(b_0 b_1)\cdots(b_{r-1} b_r)\, b_{0_y}^{n-r}\cdots b_{r_y}^{n-r}\, a_{0_x}^{m-r}\cdots a_{r_x}^{m-r} \begin{vmatrix} a_{01}^{r} & a_{01}^{r-1}a_{02} & \cdot & \cdot & \cdot & a_{02}^{r} \\ a_{11}^{r} & \cdot & \cdot & \cdot & \cdot & a_{12}^{r} \\ \cdot & & & & & \\ a_{r1}^{r} & a_{r_1}^{r-1}a_{r2} & \cdot & \cdot & \cdot & a_{r2}^{r} \end{vmatrix}$$

$$= \frac{1}{(r+1)!}(a_0 a_1)\cdots(a_{r-1} a_r)(b_0 b_1)\cdots(b_{r-1} b_r)\, a_{0_x}^{m-r}\cdots a_{r_x}^{m-r}\, b_{0_y}^{n-r}\cdots b_{r_y}^{n-r}$$

$$= \frac{1}{(r+1)!} G_r \text{ (Say)}.$$

Thus if F is of rank r, the covariant G_r (which we call the r^{th} rank covariant of F) vanishes identically. Conversely if G_r vanishes identically, but not G_{r-1}, then there is a linear relation between any r+1 linear combinations of the forms Φ_k, but not between any r. Hence F is of rank r. We have thus the theorem :

> The necessary and sufficient condition that F may be of rank r, is that G_r be the first of the rank Covariants $G_1, G_2, \ldots$ which vanishes identically...(3).

In particular, a form of rank one is of the shape f $(x)\phi(y)$; that is, it reduces to a product of two binary forms. Since the covariant G_1 is the (1,1) transvectant $(F,F)^{1,1}$, we have the result :

> The necessary and sufficient condition that a double-binary form may reduce to the product of two binary forms is the identical vanishing of its (1,1) transvectant. - - - - - - - - - - (4).

§ 3. If the binary form $a_{0_x}^{m+n} = a_{1_x}^{m+n} = \ldots$ possesses an apolar r-ic f_x^r, then all its $(m+n-r)^{th}$ polars are apolar to f_x^r, and therefore it is directly evident that the rank of the double-binary form $a_{o_x}^{m+n-r}\, a_{o_y}^r$ is less than r+1. From what has been proved, the condition for this is the vanishing of the r^{th} rank covariant G_r, which for this form reduces to

$$(a_0 a_1)^2 (a_0 a_2)^2 \cdots (a_{r-1} a_r)^2 a_{o_x}^{m+n-2r} \cdots a_{r_x}^{m+n-2r}.$$

This is the known covariant (Grace and Young p. 232) the vanishing of which is the condition for the existence of an apolar r-ic of the binary form $a_{o_x}^{m+n} = a_{1_x}^{m+n} = \ldots.$ Thus the rank-theory of the double-binary form may be regarded from the symbolic view-point, as an extension of the theory of binary forms apolar to a given binary form. On this view, theorem (4) would be the extension of the binary theorem, that the identical vanishing of the Hessian is the necessary and sufficient condition for a form to be a power of a linear form.

§ 4. Reciprocity of Rank Covariants of (n,n) forms.

The symbolic expression obtained for the rank covariant G_r, of $F = a_{o_x}^m b_{o_y}^n = a_{1_x}^m b_{1_y}^n = \ldots$ was $G_r = (a_0 a_1)\cdots(a_{r-1} a_r)(b_0 b_1)\cdots(b_{r-1} b_r)\, a_{o_x}^{m-r} \cdots b_{o_y}^{n-r} \cdots$. If in this expression, the equivalent symbols are considered to refer to r+1 distinct forms $F_0 F_1 \ldots F_r$, where

$$F_k = f_k(x)\,\phi_k(y) \quad (k=0,1 \ldots r),$$

it is evident that the symbolic expression of G_r will reduce to the product of the Jacobian of the r+1 forms $f_k(x)$ and the

Jacobian of the r+1 forms $\phi_k(y)$. <u>Hence</u> it follows, that if the general (m,n) form F is written in the shape :

$$F = f_1(x)\phi_1(y) + \cdots + f_t(x)\phi_t(y),$$

its r^{th} rank covariant G_r is the sum of the products of the Jacobians of every r+1 of the forms $f_1(x), \ldots f_t(x)$ and the corresponding r+1 forms $\phi_1(y), \ldots \phi_t(y)$.

Applying this result to the (n,n) form F, of rank n+1, where

$$F = f_0(x)\phi_0(y) + \cdots + f_n(x)\phi_n(y),$$

we see that the r^{th} rank covariant is given by

$$G_r = \Sigma\ J_{01\ldots r}(x)\ J'_{01\ldots r}(y),$$

where $J_{01..r}$ and $J^1_{01..r}$ are the respective Jacobians of $f_0 f_1 \ldots f_r$, and $\phi_0\phi_1\phi_r$.

In particular, if we define the forms $F_k(x)$, $\Phi_k(y)$ by :

$$F_k(x) = (-1)^k\ J_{01\ldots(k-1)(k+1)\ldots n}(x)$$
$$\Phi_k(y) = (-1)^k\ J'_{01\cdot(k-1)(k+1)\cdot\cdot n}(y),$$

then the $(n-1)^{th}$ rank covariant of F is given by

$$G_{n-1} = F_0(x)\,\Phi_0(y) + \cdots + F_n(x)\,\Phi_n(y) \,.$$

<u>Apolar sets of binary forms.</u>

The set of n+1 binary n-ics $(F_0, F_1, \ldots F_n)$ derived as above from the set $(f_0, f_1, \ldots f_n)$, may be called <u>the apolar set</u> of the latter. These sets stand in very important mutual relationships. Any form F_k is from its definition, apolar to every form f except f_k.

It is also obvious that the apolar invariant

$$(f_k, F_k)^n = \text{the invariant } J_{01..n} = j \text{ (say)},$$

for any K.

It is easy to shew that if the point-equations of a rational norm curve R_n, in n dimensions, are :

$$X_r = f_r(x) \quad (r = 0, 1 \dots n)$$

then the dual (or 'plane') equations of the same curve are :

$$U_r = F_r(x) \quad (r = 0, 1 \dots n).$$

From this relation of apolar sets to Geometrical Duality, we shall deduce two important consequences. From the two sets of equations to the norm curve, it follows that the expression $\sum_r f_r(x) F_r(y)$ can vanish only if the point x on the norm curve lies on the osculating region of n-1 dimensions at the point y. Since the curve is of order n, this can happen when and only when the points x and y coincide. Thus $\sum f_r(x) F_r(y)$ must be a constant multiple of $(xy)^n$. The constant multiplier is easily identified as the invariant $j = J_{01..n}$, and we arrive at the fundamental identity of apolar sets :

$$f_0(x) F_0(y) + f_1(x) F_1(y) + \dots + f_n(x) F_n(y) \equiv j\,(xy)^n \quad \dots \quad (A)$$

As a second consequence, consider the k-dimensional osculating region at the point x of the norm curve. The point-coordinates of this region are the Jacobians of the forms $f_o(x)$, ... $f_n(x)$, k+1 at a time; the dual coordinates of the same region are the Jacobians of the forms $F_o(x)$,..$F_n(x)$, n-k at a time. From the known relation between point and dual region-coordinates, we have the result :

> The Jacobians of the type $J_{01..k}(x)$, of the forms $f_0, f_1, ..f_k$, k+1 at a time, differ only by a constant multiplier from the Jacobians of the type $J'_{(k+1)(k+2)..n}(x)$ of the forms $F_0, F_1, ..F_n$, n-k at a time..........(B)

From consideration of dimensions, it is seen that the constant multiplier is a numerical multiple of $\frac{1}{J^{n-k-1}}$; the actual form of the numerical factor is complicated, and is quite irrelevant for our purpose.

We are now in a position to establish the reciprocity of the rank covariants of the (n,n) form F, of rank n+1, where

$$F = f_0(x)\phi_0(y) + \cdots + f_n(x)\phi_n(y).$$

If $(F_o, F_1, ..F_n)$, $(\Phi_o, ..\Phi_n)$ be the apolar sets of $(f_o,..f_n)$, $(\phi_o,..\phi_n)$ respectively, the $(n-1)^{th}$ rank covariant of F is given by :

$$G_{n-1} = F_0(x)\,\Phi_0(y) + \cdots + F_n(x)\,\Phi_n(y).$$

It follows from (B) that the K^{th} rank covariant of F, which is the product-sum of the Jacobians of the forms (f) and (ϕ), k + 1 at a time, can only differ by a constant factor λ_k from the $(n-k-1)^{th}$ rank covariant of G_{n-1}, which is the product-sum of the Jacobians of the forms of the apolar sets, n-k at a time. Here λ_k is a numerical multiple of $\frac{1}{J^{n-k-1}J'^{n-k-1}} = \frac{1}{G_n^{n-k-1}}$

j and j^1 being the Jacobians of all the forms f and ϕ, respectively. We have therefore the result :

<u>The $(n-k-1)^{th}$ rank covariant of G_{n-1} is a numerical multiple of $G_n^{n-k-1} G_k$.</u> - - - - - - - - - - (5)

In particular, the $(n-1)^{th}$ rank covariant of G_{n-1} is a multiple of F itself. The simplest case of this reciprocity occurs for the (2,2) form F, for which we have

$$G_1 = (F, F)^{1,1}\,; \quad (G_1, G_1)^{1,1} = \tfrac{1}{3} G_2 F.$$

Another important relation between F and G_{n-1} may be deduced from the fundamental identity (A). Considering F and G_{n-1} to be binary forms in x, their apolar invariant is

$$\sum_k (f_k, F_k)^n \phi_k(y) \Phi_k(y) = j \sum \phi_k(y) \Phi_k(y).$$

This vanishes in virtue of (A), giving the result :

<u>The forms F and G_{n-1} are identically apolar, when considered as binary forms in either x or y.</u> (The apolarity of this theorem is <u>binary</u> apolarity, and is not to be confused with <u>double-binary</u> apolarity. Thus F and G_{n-1} are <u>not</u> apolar, as double-binary forms, their double-binary apolar invariant being in fact equal to G_n.) - - - - - - - - - - - - - - - - - - - (6)

This theorem is the extension of the known identical binary apolarity of a (2,2) form and its (1,1) transvectant (See <u>COOLIDGE</u> <u>Treatise on the Circle and Sphere Page 210</u>; also <u>KASNER</u> cited hereafter).

5. Geometrical Representation of Rank Covariants.

Let C_m^1, C_n be two rational norm curves, the former of which is situated in a space S_m of m dimensions, and the latter in a subspace S_n of S_m. Further, consider C_m^1, C_n to be carriers of the respective binary variables x,y. Let Γ be a general correlation in S_m, involving $(m+1)^2$ constants. The S_{m-1} which corresponds to any point x of C_m^1 in Γ, cuts C_n in n points. Also, the locus of points whose corresponding S_{m-1}'s pass through an assigned point, is itself a S_{m-1} which cuts C_m^1 in m points. Thus Γ institutes a (m,n) correspondence F between the curves C_m^1, C_n. To shew that F is a general (m,n) correspondence, let Γ' be the general correlation (involving (m-n) (m+1) constants), in which the S_{m-1} corresponding to <u>every</u> point contains S_n. Then the correlations Γ and $\Gamma+\Gamma'$ are such, that the two S_{m-1}'s which they make correspond to any point, cut S_n in the same S_{n-1}. It is therefore clear that $\Gamma+\Gamma'$ is the most general correlation which produces the same (m,n) correspondence F, as Γ. Hence the number of arbitrary constants in F is

$$(m+1)^2-(m-n)(m+1)=(m+1)(n+1).$$

Hence F is a general (m,n) correspondence.

Now as the point x traces C_m^1, the S_{m-1} corresponding to it in Γ osculates another norm curve C_m at a point whose parameter on C_m may also be supposed to be x. Since C_m^1 and C_n have no special relation towards one another, the same must be true of C_m and C_n. It follows therefore that the general form $F = a_x^m b_y^n$ $(m \geqq n)$, of rank n+1, can be represented as a correspondence of a special type between two suitably chosen norm curves C_m, C_n; namely that, in which any point x on C_m corresponds to those points y, in which the osculating S_{m-1} at x cuts C_n.

This representation of $F = a_x^m b_y^n$ has the advantage that the rank covariants of F also receive a simple interpretation therein. From the symbolic derivation of the covariant G_r of F, it is seen that G_r is the binary Jacobian of the following forms in y:

$$a_x^m b_y^n, \quad a_{x_1}^m b_y^n, \quad \ldots \quad a_{x_r}^m b_y^n,$$

where $x_1, x_2 \ldots x_r$ differ infinitesimally from x. Hence the points y which correspond in G_r to a given point x on C_m, are the roots of the Jacobian of the r+1 y-groups, which correspond in F to r+1 points infinitely near x. From the correspondence-representation adopted for F, it follows that these y-groups are cut out of C_n by S_{m-1}'s passing through the osculating S_{m-1-r} at x; therefore the Jacobian of these y-groups corresponds to those points y on C_n the osculating S_r's at which intersect the osculating S_{m-1-r} at x. Thus the correspondence defined by G_r is that, in which corresponding points x,y on C_m, C_n are such that the osculating S_{m-1-r} at x and the osculating S_r at y meet each other.

This representation of the rank covariants also puts in evidence their relation to the rank of the form F. For instance, if G_n=o, the <u>correspondence</u> G_n must become indeterminate, and therefore the osculating S_{m-1-n} at any point x of C_m must meet the osculating S_n at any point y of C_n. But this osculating S_n is simply the S_n containing C_n, and it cannot meet an infinite number of osculating S_{m-1-n}'s of C_m, unless C_n is contained in a space of n-1 dimensions in S_n. When this happens, the correspondence F obviously suffers a reduction in rank from n+1 to n. A similar argument holds for the case G_r = o.

<u>Reciprocity of Rank Covariants for the equiform.</u>

When m = n, the form $F = a_x^n\, b_y^n$ would be represented in the above manner, as the correspondence between a point x on C_n and the points y in which the osculating S_{n-1} at x meets a second fixed norm curve C_n^1. The rank covariant G_{n-1} is then the correspondence between a point x on C_n and the points of contact y of the osculating S_{n-1}'s from x to C_n^1. This dual symmetry of F and G_{n-1} shews, that if we attempt to form the correspondence representing the successive rank covariants of

G_{n-1}, we would only retrace in reverse order the rank covariants of F. The reciprocity of the rank covariants is thus geometrically established.

The identical binary apolarity of F and G_{n-1} is also put in evidence by the geometrical representation. For, the y-group corresponding to x in F, is cut out of C_n^1 by a S_{n-1} through x, and the y-group corresponding to the same x in G_{n-1} is cut out of C_n^1 by the polar S_{n-1} of x with respect to C_n^1. The apolarity of these y-groups follows from the fundamental property of the norm curve.

A special case of interest is the (2,2) form F. This can be represented as a (2,2) correspondence between the conics C_2, C_2' in the same plane, in which any point x on C_2 corresponds to the two points y in which the tangent at x cuts C_2^1. The (1,1) transvectant G_1 (which is the first rank covariant of F) is then represented by the correspondence between a point x on C_2, and the two points of contact y of tangents from x to C_2^1.

6. Canonical Form.

The shape

$$F = f_1(x)\phi_1(y) + \cdots + f_r(x)\phi_r(y) = a_x^m b_y^n,$$

obtained for the form F of rank r, is not unique. If in fact, we write

$$f_p(x) = \sum_q \lambda_{pq} f_q'(x)$$
$$\phi_p(y) = \sum_q \mu_{pq} \phi_q'(y) \quad ; \quad p, q = 1, 2 \cdots r,$$

then we would have identically,

$$F \equiv \Sigma f_p(x)\phi_p(y) \equiv \Sigma f'_p(x)\phi'_p(y),$$

provided the linear substitutions $|\lambda|, |\mu|$ are contragredient.

Let us now write

$$(f_p, f_q)^m = \alpha_{pq}\ ;\ (f'_p, f'_q)^m = \alpha'_{pq}$$
$$(\phi_p, \phi_q)^n = \delta_{pq}\ ;\ (\phi'_p, \phi'_q)^n = \delta'_{pq}.$$

If $|\lambda|', |\mu|'$ denote the transposed matrices of $|\lambda|, |\mu|$ we have the matrix-equations :

$$|\alpha_{pq}| = |\lambda|\,|\alpha'_{pq}|\,|\lambda'| \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot \qquad (1)$$
$$|\delta_{pq}| = |\mu|\,|\delta'_{pq}|\,|\mu'|.$$

Since $|\lambda|, |\mu|$ are contragredient, $|\lambda| = (|\mu|')^{-1}$ and the second of these equations may be written

$$|\delta_{pq}|^{-1} = |\lambda|(|\delta_{pq}|')^{-1}|\lambda|' \quad - \ - \ - \qquad (2)$$

If we now suppose that both the partial orders m, n are even, then the matrices $|\alpha|, |\delta|, |\alpha'|, |\delta'|$ as well as their inverses, are all symmetric. By comparing equations (1), (2), with the known theory of reduction of two quadratic forms, we conclude, in this case, that it is in general possible to choose $|\lambda|$ uniquely, in such wise that $|\alpha|'$ as well as $(|\delta|')^{-1}$ (and therefore also $|\delta|'$) are in normal shape (that is, have vanishing non-diagonal elements). If $|\alpha|'$ and $|\delta|'$ are both in normal shape, then in the transformed shape of F, the forms f^1 as well as the forms ϕ' will be mutually apolar. Hence we have the theorem :

A double-binary form F of rank r, whose partial orders m,n are both even, can in general be reduced to the shape :

$$F = f_1(x)\phi_1(y) + \cdots + f_r(x)\phi_r(y),$$

in which the forms f are mutually apolar m-ics, and the forms ϕ, mutually apolar n-ics; and this reduction is in general unique in the sense, that the only other sets f^1, ϕ', with this property are given by :

$$f_k' = \lambda_k f_k \; ; \; \phi_k' = \frac{1}{\lambda_k}\phi_k \qquad \text{------------(7).}$$

The actual possibility of this reduction in any particular case, as well as its uniqueness when possible, will ofcourse depend on the invariant-factors of $|d| + \lambda\,|\delta|^{-1}$.

If one or both the partial orders m,n, are odd, then the reduction of F will depend upon the simultaneous reduction of two bilinear forms, which are either symmetric and skew-symmetric, or both skew-symmetric. There is however no simple type of general canonical reduction for these cases.

As a special case, suppose m=n=3. Then $|d|$ and $|\delta|^{-1}$ are skew-symmetric matrices of the fourth order, and therefore correspond to two linear complexes in space. These complexes have in general a unique pair of common polar lines; by choosing these as edges of the tetrahedron of reference, $|d|$ and $|\delta|$ can be reduced simultaneously to the shape in which

$$d_{13} = d_{14} = d_{23} = d_{24} = 0$$

$$\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0 .$$

Hence the (3,3) form can be reduced (but not uniquely) to the shape

$$f_1(x)\phi_1(y) + f_2(x)\phi_2(y) + f_3(x)\phi_3(y) + f_4(x)\phi_4(y),$$

where each of f_1, f_2 is apolar to f_3, f_4, and each of ϕ_1, ϕ_2 is apolar to each of ϕ_3; ϕ_4.

§ 7. Canonical shape for (2,2) form.

From the theorem of the last section, the (2,2) form of rank three can be put into the shape

$$F = a_x^2 b_y^2 = f_1'(x)\phi_1'(y) + f_2'(x)\phi_2'(y) + f_3'(x)\phi_3'(y),$$

where (f_1', f_2', f_3') and $(\phi_1', \phi_2', \phi_3')$ are mutually apolar triads of quadratics. Now none of the quadratics f^1(or ϕ') can be a perfect square; for, $f_1' f_2' f_3'$ which form a mutually apolar triad, would thereby obtain a common factor, which would reduce the rank of the form. Hence we can replace the quadratics f', ϕ' by numerical multiples of themselves having unit discriminants, and write

$$F = \lambda_1 f_1(x)\phi_1(y) + \lambda_2 f_2(x)\phi_2(y) + \lambda_3 f_3(x)\phi_3(y) \quad \ldots \text{(C)}$$

where
$$(f_r, f_s)^2 = (\phi_r, \phi_s)^2 = 0$$
$$(f_r, f_r)^2 = (\phi_r, \phi_r)^2 = 1 \ : \quad r, s = 1, 2, 3.$$

From the fact that f, ϕ are mutually apolar triads of quadratics with <u>unit</u> discriminants, we have the further relations :

$$f_1^2 + f_2^2 + f_3^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 = 0$$
$$(f_2, f_3)' = \delta f_1 \ , (\phi_2, \phi_3)' = \delta' \phi_1 \text{ etc,}$$

where δ, δ' are the determinants of the triads f, ϕ and are given by :

$$2\delta^2 = 2\delta'^2 = 1.$$

We may also suppose that δ and δ' have the same sign, so that $\delta = \delta'$.

The form (C) is very convenient for the study of the (2,2) form, and has the advantage of not being too specialised. The two discriminants of the form are (The nomenclature used for the concomitants is the same as in KASNER, <u>The Invariant Theory of the Inversion Group</u>, <u>Trans. Am. Math. Soc.</u> 1900) :

$$D_1(x) = (F, F)^{0,2} = \lambda_1^2 f_1^2 + \lambda_2^2 f_2^2 + \lambda_3^2 f_3^2$$
$$D_2(y) = (F, F)^{2,0} = \lambda_1^2 \phi_1^2 + \lambda_2^2 \phi_2^2 + \lambda_3^2 \phi_3^2,$$

and their linear equivalenece is apparent. The principal equi-binary concomitants of F are :

$$G = (F, F)^{1,1} = 2\delta\delta' \Sigma \lambda_2 \lambda_3 f_1 \phi_1 = \Sigma \lambda_2 \lambda_3 f_1 \phi_1$$
$$-2I = (F, F)^{2,2} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$3J = (F, G)^{2,2} = 3\lambda_1 \lambda_2 \lambda_3$$
$$H = -\frac{I}{2} F - (F, G)^{1,1} = \frac{1}{4} \Sigma \lambda_1 (\lambda_1^2 - \lambda_2^2 - \lambda_3^2) f_1 \phi_1.$$
$$4K = (F, H)^{2,2} = \frac{1}{4}(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2\lambda_2^2\lambda_3^2 - 2\lambda_3^2\lambda_1^2 - 2\lambda_1^2\lambda_2^2).$$

3J is the second rank covariant, the vanishing of which is the condition that F may be of rank two. The (1,1) transvectant of G may be immediately verified to be JF. The equi-transvectants of F, G, H may be easily found from the above values. The two discriminants (or <u>branch quartics</u>) of H are seen to be K $D_1(x)$ and K $D_2(y)$.

If F is interpreted geometrically as a cyclic, the condition that it is a pair of circles, is that both its branch quartics are perfect squares (KASNER loc. cit.), that is,

$$(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)=0$$

The condition that it is a repeated circle, is the identical vanishing of both the branch quartics $D_1(x)$, $D_2(y)$; that is,

$$\lambda_1^2=\lambda_2^2=\lambda_3^2.$$

The covariant system of cyclics a F+b G+c H is thus seen to contain <u>four</u> repeated circles, namely the <u>director circles</u> of F (and also of every cyclic of the system). To study this system, and also the <u>confocal system</u> of cyclics, we may represent $F=\lambda_1 f_1\phi_1+\lambda_2 f_2\phi_2+\lambda_3 f_3\phi_3$ by the point F, whose trilinear coordinates with respect to a fundamental triangle ABC are $\lambda_1 \lambda_2 \lambda_3$. In this representation, the points ABC will correspond to the forms of rank <u>one</u> (namely $f_1\phi_1, f_2\phi_2, f_3\phi_3$) in the co-director system, and points on the sides of ABC, to forms of rank two. The in- and ex-centres of ABC, P $P_1P_2P_3$ will correspond to the four repeated circles - namely the common director circles - and the points on the internal and external bisectors of the angles A, B, C will correspond to pairs of circles. The points F, G are seen to form a pair of the Steinerian involution determined by the quadrangle P $P_1P_2P_3$; that is, they are <u>isogonal conjugates</u> with respect to the triangle ABC, Since any line contains a unique pair of isogonal conjugates, we have :

> Any pencil of cyclics of a co-director system contains two cyclics F, G, such that $G = p(F,F)^{\prime,\prime}$; it also contains six pairs of circles (That the pencil contains six pairs of circles appears to have been first noticed by CAYLEY).

A confocal system of cyclics (that is, a system every member of which has the same branch of quartics) corresponds to a conic through $P\,P_1P_2P_3$; the confocal system of F corresponds to the conic $P\,P_1P_2P_3F$. Since F, G are isogonal conjugates, F G is the tangent at F to this conic; the point H may be constructed either as the point of contact of the other tangent from G to this conic, or as the point which corresponds to the third polar of F with respect to the quadruple $P\,P_1P_2P_3$ on the conic.