ALGEBRA STRUCTURES ON THE COMPARISON OF THE REDUCED BAR CONSTRUCTION AND THE REDUCED *W*-CONSTRUCTION

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For a simplicial augmented algebra K, Eilenberg–Mac Lane constructed a chain map $g : \overline{B}_N(K_N) \to \overline{W}_N(K)$. They proved that g is a reduction (homology isomorphism) and conjectured that it is also the injection of a contraction (special homotopy equivalence). The contraction $C_{\overline{W}-\overline{B}} : \{\overline{W}_N(K), \overline{B}_N(K_N), f, g, \phi\}$ is followed at once by using homological perturbation techniques. If K is commutative, Eilenberg–Mac Lane proved that g is a morphism of DGA-algebras. The present article is devoted to proving that f and ϕ satisfy certain multiplicative properties (weaker than g) and showing how they can be used for computing in an economical way the homology of twisted cartesian products of two Eilenberg–Mac Lane spaces.

1. INTRODUCTION

Eilenberg–Mac Lane introduced the bar and *W*-constructions in [5] as important algebraic tools for computing the groups $H_*(\pi, n)$. They showed that for any simplicial augmented algebra, K, there is a "reduction" of the reduced normalized *W*-construction $\overline{W}_N(K)$ of K onto the reduced normalized bar construction $\overline{B}_N(K_N)$ of the normalized chain algebra K_N of K. They then raised the question whether this reduction, $g : \overline{B}_N(K_N) \to \overline{W}_N(K)$, is in fact part of a contraction. The homological perturbation theory answers this question by establishing such a contraction as $C_{\overline{W}-\overline{B}} : \{\overline{W}_N(K), \overline{B}_N(K_N), f, g, \phi\}$ (see [15, 17, 19]). The basic tool is the "perturbation lemma" exploited in [8, 9].

If the algebra K is commutative, it is well known that $\overline{W}_N(K)$ and $\overline{B}_N(K_N)$ are commutative DGA-algebras as well. Although Eilenberg–Mac Lane showed that g is a DGA-algebra morphism, nothing is known so far about the multiplicative behavior of the other morphisms that compose the contraction between $\overline{W}_N(K)$ and

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$\overline{B}_N(K_N)$ (the projection f and the homotopy operator ϕ). Taking as a starting point Real's work in [15], we prove here that f and ϕ are a "quasi algebra" projection and a "quasi algebra" homotopy, respectively. The result, apart from being interesting in its own right, provides a step towards designing "economical" algorithms for computing the homology of twisted cartesian products of two Eilenberg–Mac Lane spaces.

Our approach for proving the main statement of this article depends deeply upon a suitable development of the homological perturbation theory (HPT) [8, 9], particular to the category of DGA-algebras [16]. The cornerstone of HPT is the Basic Perturbation Lemma (BPL), which gives an explicit formula for transferring a perturbation of the differential of a chain complex to an equivalent chain subcomplex. [9, 11] prove that the BPL also holds true in the category of algebra, assuming that the data has the "maximum" degree of compatibility with regards to the algebra structures. [16] assumed weaker hypotheses for the data involved in the BPL, the analogous result for the algebra category was obtained wherein the resulting contraction had weaker properties as well. The result, which is essential in this article, is stated in Theorem 4.1.

The article is organized as follows. Section 2 describes the main algebraic and simplicial objects we intend to deal with, it also contains the objects studied in the homological perturbation theory. Section 3 is devoted to the main ideas of Real's work [15] on the construction of the contraction $C_{\overline{W}-\overline{B}}$ from $\overline{W}_N(K)$ to $\overline{B}_N(K_N)$ hoped for by Eilenberg–Mac Lane. In Section 4 we analyze the multiplicative behavior of the morphisms involved in the contraction $C_{\overline{W}-\overline{B}}$, under the hypothesis that K is a commutative augmented simplicial algebra. This study relies heavily on the outline of the construction of $C_{\overline{W}-\overline{B}}$ given in Section 3 and uses Theorem 4.1 as a principal tool. Section 5 is devoted to describing how to use the multiplicative properties of the morphisms involved in the contraction $C_{\overline{W}-\overline{B}}$ for reducing the cost of computing the homological models of twisted cartesian products of two Eilenberg–Mac Lane spaces.

2. PRELIMINARIES

We will quickly review some basic notions on homological algebra specific to HPT, and introduce the notation and terminology used throughout the remainder of this article. More details can be found in [13]. Let Λ be a commutative ring with nonzero unit, taken henceforth as ground ring and fixed throughout, and A be an augmented differential graded algebra over Λ, briefly DGA-algebra. The differential, product, augmentation, and coaugmentation of A will be denoted, respectively, by d_A, μ_A, ϵ_A, and η_A. Nevertheless, we will sometimes write them simply as d, μ, ϵ, and η, when no confusion can arise. In what follows, the Koszul sign conventions will be used. A morphism $\rho : A_* \to A_{*-1}$ is called *derivation* if it is compatible with the algebra structures on A. The degree of an element $a \in A$ is denoted by $|a|$. If B is a DGA-algebra, then $A \otimes B$ has an algebra structure by means of the canonical morphism $\mu_{A \otimes B} = (\mu_A \otimes \mu_B)(1_A \otimes T \otimes 1_B)$, where $T(b \otimes a) = (-1)^{|b||a|} a \otimes b$. If the DG-algebra A is connected, that is $A_0 = \Lambda$ and $d_1 : A_1 \to A_0$ is zero, then there is a canonical augmentation $\epsilon_A = 1_\Lambda : A_0 \to \Lambda$.

Here we use the *normalized reduced bar construction* $\overline{B}_N(A)$ of a DGA-algebra A. Recall that it is defined as the connected DGA-coalgebra,

$\overline{B}_N(A) = T^c(S(\overline{A})))$, where $T^c(\ )$ is the tensor coalgebra, $S(\ )$ is the suspension functor and $\overline{A} = \mathrm{Ker}\ \epsilon_A$ is the augmentation ideal of A. The element on zero degree of $\overline{B}_N(A)$ corresponding to the identity element of Λ is denoted by $[\ ]$, and the element $S\bar{a}_1 \otimes \cdots \otimes S\bar{a}_n$ of $\overline{B}_N(A)$ is denoted by $[a_1|\cdots|a_n]$. The tensor and simplicial degrees of the element $[a_1|\cdots|a_n]$ are $|[a_1|\cdots|a_n]|_t = \sum |a_i|$ and $|[a_1|\cdots|a_n]|_s = n$, respectively; its total degree is the sum of its tensor and simplicial degrees. The tensor and simplicial differentials are defined by

$$d_t([a_1 \mid \cdots \mid a_n]) = -\sum_i (-1)^{e_{i-1}}[a_1 \mid \cdots \mid d_A(a_i) \mid \cdots \mid a_n],$$

and

$$d_s([a_1 \mid \cdots \mid a_n]) = \sum_i (-1)^{e_i}[a_1 \mid \cdots \mid \mu_A(a_i \otimes a_{i+1}) \mid \cdots \mid a_n],$$

where

$$e_i = i + |a_1| + \cdots + |a_i|.$$

The augmentation $\varepsilon_{B(A)} : \overline{B}_N(A) \to \Lambda$ is determined by

$$\varepsilon_{B(A)}([\ ]) = 1, \qquad \varepsilon_{B(A)}(u) = 0; \qquad \text{for } u \in \overline{B}_{q,N}(A), q > 0.$$

If the product of A is commutative, a product $*$ (called *shuffle product*) can be defined on $\overline{B}_N(A)$. In this way, the reduced bar construction has a commutative Hopf algebra structure.

Now we recall some basic concepts of simplicial topology.

A simplicial Λ-algebra (or a simplicial algebra) is a simplicial object in the category of Λ-algebras ([13, p. 233]); in other words, a simplicial algebra K is a simplicial set $K = (K_n, \partial_i, s_i)$ where every K_n is an algebra and every face or degeneracy operator is compatible with the algebra structures (in a similar way, a simplicial module is defined). If every K_n is a commutative algebra, then K is a *commutative* simplicial algebra. The *inner* product of two elements a and b of a simplicial algebra is defined only if a and b have the same dimension n, and the product is the product of a and b in K_n. An *augmentation* in a simplicial algebra K is a morphism of algebras $\varepsilon : K_0 \to \Lambda$ satisfying the relation $\varepsilon\partial_0 = \varepsilon\partial_1$.

A simplicial algebra defines a differential graded algebra (DG-algebra) where the underlying graded module is the direct sum $\bigoplus_n K_n$; the product, which is called the *external* product, is defined by the formula

$$a_p \diamond b_q = \sum_{(\alpha,\beta)} (-1)^{sg(\alpha,\beta)} \ (s_{\beta_q} \ldots s_{\beta_1} a_p) \cdot (s_{\alpha_p} \ldots s_{\alpha_1} b_q), \qquad a_p \in K_p, \quad b_q \in K_q.$$

In this formula, the pair (α, β) runs through the set of all (p, q)-shuffles and sg $(\alpha, \beta) = \sum_{i=1}^p (\alpha_i - (i - 1))$. Finally, the differential d is the boundary operator

$$d(a_p) = \sum_{i=0}^p (-1)^i \partial_i(a_p), \qquad a_p \in K_p.$$

The corresponding normalized chain complex K_N is also a DG-algebra. If the initial simplicial algebra is commutative, the induced DG-algebra is commutative as well. An augmentation of simplicial algebras induces an augmentation of DG-algebras.

Each simplicial algebra K has a canonical *base point* K^u, a simplicial subalgebra, where $K_n^u = \Lambda$, the subalgebra generated by the unit $1_{n,K}$ of K_n, the face and degeneracy operators being the identity. An augmentation $\varepsilon : K_0 \to \Lambda$ induces a simplicial augmentation $\varepsilon^s : K \to K^u$ defined by $\varepsilon_n^s(a_n) = \varepsilon(\partial_0^n a_n)1_{n,K}$ if a_n is an n-simplex of K; each ε_n^s is the augmentation of the corresponding K_n. The elements of K^u are called the *scalar* elements of K_n.

The kernel $\overline{K}$ of this simplicial augmentation is a simplicial submodule of K; each $\overline{K}_n$ is the augmentation ideal of K_n. This simplicial submodule is also an ideal of K for the external product, that is, for the DG-algebra structure. Furthermore, $K = K^u \oplus \overline{K}$.

The *simplicial augmented module* $\overline{W}(K)$ is defined as

$$\overline{W}_n(K) = K_{n-1} \otimes \cdots \otimes K_0 \otimes \Lambda$$

(in particular, $\overline{W}_0(K) = \Lambda$). An n-dimensional element $a_{n-1} \otimes \cdots \otimes a_0 \otimes 1_\Lambda$ will be denoted by $(a_{n-1}, \ldots, a_0)$ if $n > 0$ and 1_Λ is the unique generator of $\overline{W}_0(K)$.

The *face* and *degeneracy* operators of $\overline{W}(K)$ are defined as follows:

$$\partial_0(a_{n-1}, \ldots, a_0) = \varepsilon^s(a_{n-1})(a_{n-2}, \ldots, a_0),$$

$$\partial_i(a_{n-1}, \ldots, a_0) = (\partial_{i-1}a_{n-1}, \partial_{i-2}a_{n-2}, \ldots, (\partial_0 a_{n-i})a_{n-i-1}, a_{n-i-2}, \ldots, a_0) \qquad 0 < i < n,$$

$$\partial_n(a_{n-1}, \ldots, a_0) = (\partial_{n-1}a_{n-1}, \partial_{n-2}a_{n-2}, \ldots, \partial_1 a_1)\varepsilon(a_0), \qquad \text{and}$$

$$s_i(a_{n-1}, \ldots, a_0) = (s_{i-1}a_{n-1}, \ldots, s_0 a_{n-i}, 1_{(n-i,K)}, a_{n-i-1}, \ldots, a_0).$$

The *augmentation* of $\overline{W}(K)$ is the identity $\varepsilon_{\overline{w}} : \overline{W}_0(K) = \Lambda \to \Lambda$.

The simplicial module $\overline{W}(K)$, or more precisely the underlying DGA-module, carries a natural coalgebra structure with the same augmentation (or counit) and a coproduct which is defined as follows:

$$\Delta : \overline{W}(K) \to \overline{W}(K) \otimes \overline{W}(K)$$

$$\Delta(a_{n-1}, \ldots, a_0) = \sum_{i=0}^{n} (\hat{\partial}^{n-i}a_{n-1}, \ldots, \hat{\partial}^{n-i}a_{n-i}) \otimes (a_{n-i-1}, \ldots, a_0),$$

where the operator $\hat{\partial}$ is the Whitney operator $\hat{\partial}(x) = \partial_n x$ if x is n-dimensional and $\hat{\partial}^{n-i} = \hat{\partial} \overset{n-i}{\cdots} \hat{\partial}$.

The normalization $\overline{W}_N(K)$ of $\overline{W}(K)$ is defined in the usual way and inherits a DGA-coalgebra structure. $\overline{W}_{n,N}(K)$ denotes the nth-level of $\overline{W}_N(K)$ constituted by the elements of degree n. Let us note that if $(a_{n-1}, \ldots, a_0) \in \overline{W}_{n,N}(K)$, then $a_{n-1} \in \overline{K}_{n-1}$. Otherwise, this element of $\overline{W}_n(K)$ is in fact s_0-degenerated.

If the simplicial algebra K is commutative, the (trivial) inner product:

$$(a_{n-1}, \ldots, a_0) \cdot (b_{n-1}, \ldots, b_0) = (a_{n-1}b_{n-1}, \ldots, a_0 b_0)$$

provides $\overline{W}(K)$ with a structure of commutative simplicial algebra; indeed the commutativity property of K allows this inner product to be compatible with the simplicial structure. In such a case, as explained previously, an algebra structure is also defined on $\overline{W}(K)$; this structure is compatible with the coalgebra structure, so that $\overline{W}(K)$ (and also $\overline{W}_N(K)$) becomes a Hopf algebra.

If G is a simplicial monoid, and if $\Lambda[G]$ is the usual Λ-algebra of G, then $\overline{W}(\Lambda[G])$ is nothing but the chain complex associated to the classifying space of G.

Now, we define precisely the objects studied in homological perturbation theory and sketch a familiar example.

A *contraction* (see [5, 11]) is a data set $c : \{N, M, f, g, \phi\}$ where $f : N \to M$ and $g : M \to N$ are morphisms of DG-modules (called, respectively, *the projection* and *the inclusion*), and $\phi : N \to N$ is a morphism of graded modules of degree $+1$ (called *the homotopy operator*). These data are required to satisfy the rules: (c1) $fg = 1_M$, (c2) $\phi d_N + d_N \phi + gf = 1_N$, (c3) $\phi\phi = 0$, (c4) $\phi g = 0$, and (c5) $f\phi = 0$. These last three are called the *side conditions* [12]. In fact, these may always be assumed to hold, since the homotopy ϕ can be altered to satisfy these conditions [8]. These formulas imply that both chain complexes N and M have the same homology. We will also denote a contraction c by $(f, g, \phi) : N \Rightarrow M$.

The Eilenberg–Zilber theorem [7] provides the most classic example of a contraction of chain complexes. An Eilenberg–Zilber contraction is defined in [6] by the data set

$$EZ_{F,B} : \{C_N(F \times B), C_N(F) \otimes C_N(B), AW_{F,B}, EML_{F,B}, SHI_{F,B}\}, \tag{1}$$

where F and B are simplicial sets. The results of the present article depend heavily on these morphisms. We follow May's book [14] for the terminology and notation on Simplicial Topology.

We recall several results about constructions of contractions that will be used later. If we have two contractions (f_i, g_i, ϕ_i) from N_i to M_i, for $i = 1, 2$ then, the following contractions can be constructed (see [7]):

i) The *suspension* contraction $(f_i, g_i, -\phi_i)$ from $S(N_i)$ to $S(M_i)$;
ii) The *direct sum* contraction $(f_1 \oplus f_2, g_1 \oplus g_2, \phi_1 \oplus \phi_2)$ from $N_1 \oplus N_2$ to $M_1 \oplus M_2$;
iii) The *tensor product* contraction $(f_1 \otimes f_2, g_1 \otimes g_2, \phi_1 \otimes g_2 f_2 + 1_{N_1} \otimes \phi_2)$ from $N_1 \otimes N_2$ to $M_1 \otimes M_2$;
iv) If $N_2 = M_1$, the *composition* contraction $(f_2 f_1, g_1 g_2, \phi_1 + g_1 \phi_2 f_1)$ from N_1 to M_2.

One of the cornerstones of the homological perturbation theory is the Basic Perturbation Lemma. It provides a beautiful way of unifying many disparate results in Algebraic Topology concerning chain homotopy equivalences, and it can be used to find new results as well.

Now, we recall the concept of a perturbation datum. Let N be a graded module, and let $f : N \to N$ be a morphism of graded modules. The morphism f is *pointwise nilpotent* if for all $x \in N (x \neq 0)$, a positive integer n exists (in general, the number n depends on the element x) such that $f^n(x) = 0$. A *perturbation of a DG-module N* is a morphism of graded modules $\delta : N \to N$ of degree -1, such that $(d_N + \delta)^2 = 0$ and $\delta_1 = 0$. A *perturbation datum of the contraction* $c : \{N, M, f, g, \phi\}$

is a perturbation δ of the DGA-module N verifying that the composition $\phi\delta$ is pointwise nilpotent.

A *Transference Problem* consists of a contraction $c : \{M, N, f, g, \phi\}$ together with a perturbation δ of the DG-module N. The problem is to determine new morphisms $d_\delta, f_\delta, g_\delta$, and ϕ_δ such that $c_\delta : \{(N, d_N + \delta), (M, d_M + d_\delta), f_\delta, g_\delta, \phi_\delta\}$ is a contraction.

The Basic Perturbation Lemma ([3, 8, 9, 16]) gives an explicit solution to the Transference Problem, assuming that δ is a perturbation datum of c.

Theorem 2.1 (BPL). *Let* $c : \{N, M, f, g, \phi\}$ *be a contraction and* $\delta : N \to N$ *a perturbation datum of c. Then, a new contraction*

$$c_\delta : \{(N, d_N + \delta), (M, d_M + d_\delta), f_\delta, g_\delta, \phi_\delta\}$$

is defined by the formulas: $d_\delta = f\delta\Sigma_c^\delta g$, $f_\delta = f(1 - \delta\Sigma_c^\delta\phi)$, $g_\delta = \Sigma_c^\delta g$, $\phi_\delta = \Sigma_c^\delta\phi$, *where*

$$\Sigma_c^\delta = \sum_{i\geq 0}(-1)^i(\phi\delta)^i = 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i(\phi\delta)^i + \cdots.$$

Let us note that $\Sigma_c^\delta(x)$ is a finite sum for each $x \in N$, because of the pointwise nilpotency of the composition $\phi\delta$. Moreover, it is obvious that the morphism d_δ is a perturbation of the DG-module (M, d_M).

The twisted Eilenberg–Zilber theorem can be seen as an important example of the usefulness of this lemma (see [18]). It solves the Transference Problem for twisted cartesian products. In the sequel, we give the ingredients for a concrete statement of this result.

Consider two simplicial sets F, B and a simplicial group G which operates on F from the left. A *twisted cartesian product* E with fibre F, base B, and structural group G consists of a simplicial set $E_n = F_n \times B_n$ and

$$\partial_0(f, b) = (\tau b \star \partial_0 f, \partial_0 b)$$

$$\partial_i(f, b) = (\partial_i f, \partial_i b), \qquad \text{for } i > 0$$

$$s_i(f, b) = (s_i f, s_i b), \qquad \text{for } i \geq 0,$$

as face and degeneracy operators. Here $\star : G \times F \to F$ is the action of G on F and τ is a *twisting function*, i.e., $\tau_n : B_n \to G_{n-1}, n \geq 1$ satisfies

$$\partial_0\tau(b) = [\tau(\partial_0 b)]^{-1} \cdot \tau(\partial_1 b)$$

$$\partial_i\tau(b) = \tau(\partial_{i+1}b), \qquad \text{for } i > 0$$

$$s_i\tau(b) = \tau(s_{i+1}b), \qquad \text{for } i \geq 0$$

$$\tau(s_0 b) = e_n,$$

where e_n denotes the identity element of the corresponding group G_n. We write $E = F \times_\tau B$.

Theorem 2.2 (Twisted Eilenberg–Zilber Theorem)[18]. *The morphism*

$$\delta(f, b) = (\tau b \star \partial_0 f, \partial_0 b) - (\partial_0 f, \partial_0 b), \qquad (f, b) \in C_N(F \times B)$$

is a perturbation datum of the contraction,

$$EZ_{F,B} : \{C_N(F \times B), C_N(F) \otimes C_N(B), AW_{F,B}, EML_{F,B}, SHI_{F,B}\}.$$

From these data, a new contraction (*called the twisted Eilenberg–Zilber contraction*) *is obtained by applying BPL*

$$TEZ_{F,B} : \{C_N(F \times_\tau B), (C_N(F) \otimes C_N(B), d^\otimes + d_\delta), (AW_{F,B})_\delta, (EML_{F,B})_\delta, (SHI_{F,B})_\delta\},$$

where the bigger chain complex is associated to the Twisted Cartesian Product $F \times_\tau B$ with fibre F, base B, and structural group G. And the smaller one consists of a twisted tensor product along the twisting cochain t, $C_N(F) \otimes_t C_N(B)$, for $t = p \circ d_\delta \circ \rho$

$$C_N(B) \xrightarrow{\rho} C_N(G) \otimes C_N(B) \xrightarrow{d_\delta} C_N(G) \otimes C_N(B) \xrightarrow{p} C_N(G)$$

where

$$\rho(x) = e_0 \otimes x, \qquad e_0 \text{ being the identity element of } G_0 \text{ and } p(y \otimes x) = y \cdot \epsilon_{C_N(B)} x.$$

Hence,

$$d_\delta = t \cap = (\mu_{C_N(F)} \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta_{C_N(B)}),$$

where $\mu_{C_N(F)}$ is the module action induced by the the action $\star : G \times F \to F$.

3. A CONTRACTION BETWEEN $\overline{W}_N(K)$ AND $\overline{B}_N(K_N)$

We outline the construction of a contraction between $\overline{W}_N(K)$ and $\overline{B}_N(K_N)$ given in [15] based on Eilenberg–Mac Lane's work in [5]. To this end, we recall some properties on the reduced bar and reduced W-constructions extracted from [5]. In this section, K is a given simplicial augmented algebra.

The decomposition $K = K^u \oplus \overline{K}$ shows that $\overline{W}(K)$ is spanned by elements $(a_{n-1}, \ldots, a_0)$ wherein each entry a_i is either an identity or an element of $\overline{K}$. The simplicial submodule $\overline{W}^{(k)}(K)$ is spanned by all such elements with at most k "nontrivial" entries. We thus have the family of simplicial submodules

$$\overline{W}^{(0)}(K) \subset \cdots \subset \overline{W}^{(k)}(K) \subset \cdots \subset \overline{W}(K),$$

and the quotients

$$\widetilde{W}^{(k)}(K) = \frac{\overline{W}^{(k)}(K)}{\overline{W}^{(k-1)}(K)}$$

which are simplicial submodules. Normalizing the above family, we have the following filtration of the normalized complex $\overline{W}_N(K)$

$$\overline{W}_N^{(0)}(K) \subset \cdots \subset \overline{W}_N^{(k)}(K) \subset \cdots \subset \overline{W}_N(K).$$

We also have the quotients,

$$\widetilde{W}_N^{(k)}(K) = \frac{\overline{W}_N^{(k)}(K)}{\overline{W}_N^{(k-1)}(K)}$$

which coincide with the DG-modules obtained by normalizing the simplicial modules $\widetilde{W}^{(k)}(K)$.

The DGA-module $\overline{B}_N(K_N)$ carries a filtration defined by the simplicial degree. The underlying module of the kth element, $\overline{B}_N^{(k)}(K_N)$, of this filtration is the direct sum $\bigoplus_{i=0}^{k} \overline{K}_N^{\otimes i}$

$$\overline{B}_N^{(0)}(K_N) \subset \cdots \subset \overline{B}_N^{(k)}(K_N) \subset \cdots \subset \overline{B}_N(K_N).$$

Denote the successive quotients

$$\widetilde{B}_N^{(k)}(K_N) = \frac{\overline{B}_N^{(k)}(K_N)}{\overline{B}_N^{(k-1)}(K_N)}$$

which are DG-modules with underlying modules $\overline{K}_N^{\otimes k}$. The differential operator is that which is induced by the residual tensor differential of the bar construction.

The following lemma gives a recursive description of the complexes $\widetilde{W}_N^{k+1}(K)$ and $\widetilde{B}_N^{k+1}(K_N)$, which will be used extensively in the sequel.

Lemma 3.1 [5]. *There are two DG-modules isomorphisms*:

$$\varphi : \widetilde{W}_N^{k+1}(K) \to S((\overline{K} \times \widetilde{W}^{(k)}(K))_N) \ \textit{defined}$$
$$\textit{by } \varphi(a_q \otimes u) = p(a_q) \times u, \quad \textit{where } a_q \otimes u \in \widetilde{W}_N^{k+1}(K), \qquad (2)$$

and

$$\psi : \widetilde{B}_N^{k+1}(K_N) \to S(\overline{K}_N \otimes \widetilde{B}_N^{(K)}) \ \textit{defined}$$
$$\textit{by } \psi([a|u]) = p(a) \otimes u, \quad \textit{where } a \in K_N, \ \textit{and } u \in \overline{B}_N(K_N). \qquad (3)$$

The morphism $p : K \to \overline{K}$ is defined as

$$p(a_l) = a_l - \varepsilon^s(a_l); \qquad a_l \in K_l.$$

From Eilenberg–Mac Lane's work [5] on the reduction $g : \overline{B}_N(K_N) \to \overline{W}_N(K)$, it follows at once that, for each $k \geq 0$ a contraction, $r^{(k)} : (\tilde{f}_k, \tilde{g}_k, \tilde{\phi}_k)$ from $\widetilde{W}_N^{(k)}(K)$ into $\widetilde{B}_N^{(k)}(K_N)$ can be constructed. In the sequel, we only show a sketch of this construction.

Both $\widetilde{B}_N^{(0)}(K_N)$ and $\widetilde{W}_N^{(0)}(K)$ are the complex consisting of the ground ring in dimension 0. Hence we have the stated result for $k = 0$. For $k = 1$, again we have a DG-module isomorphism where

$$\tilde{f}_1(a_q \otimes 1_{q,\overline{w}}) = [a_q] \quad \text{if } a_q \otimes 1_{q,\overline{w}} \in \widetilde{W}_{q+1,N}^{(1)}(K),$$

$$\tilde{g}_1[a_q] = a_q \otimes 1_{q,\tilde{w}} \quad \text{if } [a_q] \in \widetilde{B}_{q+1,N}^{(1)}(K_N) \quad \text{and} \quad \tilde{\phi}_1 = 0.$$

For $k > 1$, we have the diagram

Here $(AW_{k-1}, EML_{k-1}, SHI_{k-1})$ are the morphisms integrating the Eilenberg–Zilber contraction between $(\overline{K} \times \widetilde{W}^{(k-1)}(K))_N$ and $\overline{K}_N \otimes \widetilde{W}_N^{k-1}(K)$. It can be proved inductively that the following data set is a contraction:

$$r^{(k)} : \{\widetilde{W}_N^{(k)}(K), \widetilde{B}_N^{(k)}(K_N), \tilde{f}_k, \tilde{g}_k, \tilde{\phi}_k\}$$

where

$$\tilde{f}_k = \psi^{-1} (1 \otimes \tilde{f}_{k-1}) AW_{k-1} \varphi,$$

$$\tilde{g}_k = \varphi^{-1} EML_{k-1} (1 \otimes \tilde{g}_{k-1}) \psi,$$

$$\tilde{\phi}_k = -\varphi^{-1} SHI_{k-1} \varphi - \varphi^{-1} EML_{k-1} (1 \otimes \tilde{\phi}_{k-1}) AW_{k-1} \varphi.$$

Thus, we can consider the direct sum of contractions

$$\bigoplus_{k \geq 0} r^{(k)} : \left\{ \bigoplus_{k \geq 0} \widetilde{W}_N^{(k)}(K), \bigoplus_{k \geq 0} \widetilde{B}_N^{(k)}(K_N), \bigoplus_{k \geq 0} \tilde{f}_k, \bigoplus_{k \geq 0} \tilde{g}_k, \bigoplus_{k \geq 0} \tilde{\phi}_k \right\}. \tag{4}$$

It follows at once that $\bigoplus_{k \geq 0} \tilde{g}_k = g$ (g is the morphism given by Eilenberg–Mac Lane).

It is clear that, from the viewpoint of the underlying graded module structure, $\bigoplus_{k \geq 0} \widetilde{W}_N^{(k)}(K)$ and $\bigoplus_{k \geq 0} \widetilde{B}_N^{(k)}(K_N)$ coincide with $\overline{W}_N(K)$ and $\overline{B}_N(K_N)$, respectively.

Hence, given $w \in \overline{W}_N(K)$, there exists $k \geq 0$ such that $w \in \widetilde{W}_N^{(k)}(K)$. The differential operator of $\overline{W}_N(K)$ can be decomposed into two parts

$$d_{\overline{W}}(w) = \rho^{k,k-1}(w) \oplus \rho^k(w)$$

by the definition of the face operators. Where $\rho^{k,k-1}(w) \in \widetilde{W}_N^{(k-1)}(K)$ and $\rho^k(w) \in \widetilde{W}_N^{(k)}(K)$. Moreover, $\rho^k(w) = d_{\widetilde{W}_N^{(k)}}(w)$ with $d_{\widetilde{W}_N^{(k)}}$ as the differential in $\widetilde{W}_N^{(k)}$. In general,

$$d_{\overline{W}} = \bigoplus_{k \geq 0}(\rho^{k,k-1} + \rho^k),$$

and we denote,

$$\rho = \bigoplus_{k \geq 1} \rho^{k,k-1} \quad \text{and} \quad d = \bigoplus_{k \geq 0} \rho^k.$$

Obviously, d is the differential operator over $\bigoplus_{k \geq 0} \widetilde{W}_N^{(k)}(K)$ and ρ a perturbation of this DG-module.

Similarly, the differential operator of $\overline{B}_N(K_N)$ is also constituted by two summands: d_t the tensorial differential, and d_s the simplicial differential. Clearly, d_t is the differential operator of $\bigoplus_{k \geq 0} \widetilde{B}_N^{(k)}(K_N)$.

By a simple inspection, we have

$$\tilde{\phi}_{k-1} \circ \rho^{k,k-1}(w) \in \widetilde{W}_N^{(k-1)}(K), \qquad \forall w \in \widetilde{W}_N^{(k)}(K).$$

Therefore, ρ is a perturbation datum of $\bigoplus_{k \geq 0} r^{(k)}$.

Using BPL with input data the contraction $\bigoplus_{k \geq 0} r^{(k)}$ and the perturbation ρ, it follows that

$$\left(\bigoplus_{k \geq 0} r^{(k)}\right)_\rho : \left\{\overline{W}_N(K), \left(\bigoplus_{k \geq 0} \widetilde{B}_N^{(k)}(K_N), d_t + d_\rho\right), \left(\bigoplus_{k \geq 0} \tilde{f}_k\right)_\rho, \left(\bigoplus_{k \geq 0} \tilde{g}_k\right)_\rho, \left(\bigoplus_{k \geq 0} \tilde{\phi}_k\right)_\rho\right\}.$$

It may be readily checked that

$$\rho^{k,k-1}\tilde{g}_k = g_{k-1}d_s, \qquad \forall k \geq 1,$$

thus it follows at once that $d_\rho = d_s$ and $\left(\bigoplus_{k \geq 0} \tilde{g}_k\right)_\rho = \bigoplus_{k \geq 0} \tilde{g}_k$. Therefore, the desired contraction $C_{\overline{W}-\overline{B}}$, from $\overline{W}_N(K)$ to $\overline{B}_N(K_N)$ has been described.

4. THE MAIN THEOREM

In this section we assume that K denotes a commutative simplicial augmented algebra. Therefore, the algebra structures are defined on $\overline{W}_N(K)$ and on $\overline{B}_N(K_N)$. ∇ denotes the external product on $\overline{W}_N(K)$ and $*$ the shuffle product on $\overline{B}_N(K_N)$. These products induce an algebra structure on the complexes $\bigoplus_{k \geq 0} \widetilde{W}_N^{(k)}(K)$ and $\bigoplus_{k \geq 0} \widetilde{B}_N^{(k)}(K_N)$, respectively.

It may be readily verified that the following identities hold:

1. If $x^{(i)}$ denotes an element of $\widetilde{W}_N^{(i)}(K)$, then $x^{(p)} \nabla x^{(q)} \in \widetilde{W}_N^{(p+q)}(K)$;

2. $\rho^k \nabla = \nabla(1 \otimes \rho^{k-i} + \rho^i \otimes 1), \forall k - i \geq 0, \forall i \geq 0;$
3. $\rho^{k,k-1} \nabla = \nabla(1 \otimes \rho^{k-i,k-i-1} + \rho^{i,i-1} \otimes 1), \forall k - i \geq 1, \forall i \geq 1.$

Hence the DGA-module $\bigoplus_{k \geq 0} \widetilde{W}_N^{(k)}(K)$ (we recall that $d = \bigoplus_{k \geq 0} \rho^k$ is its differential) becomes a commutative DGA-algebra with regard to the product ∇. Moreover, the perturbation datum $\rho = \bigoplus_{k \geq 0} \rho^{k,k-1}$ is also a derivation.

Similar properties hold for the DGA-module $\bigoplus_{k \geq 0} \widetilde{B}_N^{(k)}(K_N)$:

1. If $y^{(i)}$ denotes an element of $\widetilde{B}_N^{(i)}(K_N)$, then $y^{(p)} * y^{(q)} \in \widetilde{B}_N^{(p+q)}(K_N)$;
2. $d_t * = *(1 \otimes d_t + d_t \otimes 1).$

Hence the DGA-module $\bigoplus_{k \geq 0} \widetilde{B}_N^{(k)}(K_N)$ (we recall that d_t is its differential) becomes a commutative DGA-algebra with regards to the product $*$.

The goal of this section is to prove that the projection f and the homotopy operator ϕ of the contraction $C_{\overline{W}-\overline{B}} : \{\overline{W}_N(K), \overline{B}_N(K_N), f, g, \phi\}$ satisfy certain multiplicative properties. For this purpose, we use some results of the homological perturbation theory extended to the category of the DGA-algebra. We begin by extending the notion of contraction to the proper category of DGA-algebra.

Let A and A' be two DGA-algebras. Given a contraction $c : \{A, A', f, g, \phi\}$ we can distinguish different types of algebra contractions in function of the degree of compatibility of the morphisms $f, g,$ and ϕ with regards to the products μ_A and $\mu_{A'}$ (for details see [16]). We say that c is:

i) *A full algebra contraction* if f and g are morphisms of DGA-algebras and ϕ is an algebra homotopy;
ii) *An almost-full algebra contraction* if f and g are morphisms of DGA-algebras and ϕ is a quasi algebra homotopy. An example of this type is the Eilenberg–Zilber contraction (1), when F and B are simplicial groups;
iii) *A semi-full algebra contraction* if f is a quasi algebra, projection, g is a morphism of DGA-algebras, and ϕ is a quasi algebra homotopy.

Here the projection f is said to be a *quasi algebra projection* if the following conditions hold:

$$f\mu_A(\phi \otimes \phi) = 0, \qquad f\mu_A(\phi \otimes g) = 0, \qquad f\mu_A(g \otimes \phi) = 0.$$

The homotopy operator ϕ is said to be a *quasi algebra homotopy* if the following conditions hold:

$$\phi\mu_A(\phi \otimes \phi) = 0, \qquad \phi\mu_A(\phi \otimes g) = 0, \qquad \phi\mu_A(g \otimes \phi) = 0.$$

We also recall that ϕ is called a *homotopy algebra* if

$$\phi\mu_A = \mu_A \phi^{[\otimes 2]}$$

where $\phi^{[\otimes 2]} = 1 \otimes \phi + \phi \otimes gf.$

One can see immediately that, in the context of algebra contractions, the designation of full implies almost-full, in the same way that almost-full implies semi-full. Let us observe that each of these classes is closed under composition and

tensor product of contractions. Examples of all these types of contractions appear in [16]. The following result tells us that the set of semi-full algebra contractions is closed under homological perturbation. This theorem is used in the proof of one of the main theorems of this article.

Theorem 4.1 (SF-APL) ([16]). *Taking as input data a semi-full algebra contraction* $c : \{A, A', f, g, \phi\}$ *and an algebra perturbation datum* δ *of* c, *the perturbed contraction* c_δ *is a semi-full algebra contraction, where the product on* A'_δ *is the original product* $\mu_{A'}$.

The previous result is the key which will lead us to the desired goal of this section. To this end, we need to prove the following facts:

  i) The contraction $\bigoplus_{k \geq 0} r^{(k)}$ is a semi-full algebra contraction;
 ii) The perturbation datum ρ is a derivation.

We focus on the first one, as the second has already been proved above.

Theorem 4.2. $\bigoplus_{k \geq 0} r^{(k)} : \{\bigoplus_{k \geq 0} \widetilde{W}_N^{(k)}(K), \bigoplus_{k \geq 0} \widetilde{B}_N^{(k)}(K_N), \bigoplus_{k \geq 0} \tilde{f}_k, \bigoplus_{k \geq 0} \tilde{g}_k, \bigoplus_{k \geq 0} \tilde{\phi}_k\}$ *is an almost-full algebra contraction.*

Proof. From now on, we will identify both $\widetilde{W}_N^{(k+1)}(K)$ and $S((\overline{K} \times \widetilde{W}^{(k)}(K))_N)$ via the isomorphism φ (see (2)). To simplify the notation, we use the same letter for referring to an element in either $\widetilde{W}_N^{(k+1)}(K)$ or $S((\overline{K} \times \widetilde{W}^{(k)}(K))_N)$. The context will clarify the meaning of the term when it will be necessary. In the same way, we identify in $\widetilde{B}_N^{(k+1)}(K_N)$ and $S(\overline{K}_N \otimes \widetilde{B}_N^{(k)}(K_N))$ via the isomorphism ψ (see (3)).

Firstly, we prove that $\tilde{f}_k$ is a DGA-algebra morphism for each $k \geq 0$. That is,

$$\tilde{f}_k \nabla = *(\tilde{f}_{k-i} \otimes \tilde{f}_i), \qquad 0 \leq i \leq k. \tag{5}$$

The proof is by induction on k, and is immediate for $k = 0$ since $\tilde{f}_0$ is the identity map of Λ. Assume that the identity (5) holds for $k < n$; we now prove that it holds for $k = n$. Let n be equal to $h + h'$, where h and h' denote non-negative integers, getting $a_p \otimes u_p \in \widetilde{W}_{p+1,N}^{(h)}(K)$ and $b_q \otimes v_q \in \widetilde{W}_{q+1,N}^{(h')}(K)$ with $a_p \in \overline{K}_p$ and $b_q \in \overline{K}_q$; it is clear that $(a_p \otimes u_p) \nabla (b_q \otimes v_q) \in \widetilde{W}_{p+q+2,N}^{(h+h')}(K)$.

Expanding the right term of (5). We get

$$\tilde{f}_h(a_p \otimes u_p) * \tilde{f}_{h'}(b_q \otimes v_q) = \big((1 \otimes \tilde{f}_{h-1})AW(a_p \otimes u_p)\big) * \big((1 \otimes \tilde{f}_{h'-1})AW(b_q \otimes v_q)\big)$$

(the Alexander–Whitney operator $AW : C_N(F \times B) \to C_N(F) \otimes C_N(B)$ is defined by $AW(f_n \times b_n) = \sum_{i=0}^{n} \hat{\partial}^{n-i} f_n \otimes \partial_0^i b_n$, where F and B are simplicial sets and $\hat{\partial}$ is the last face operator and ∂_i^j means $\partial_i \overset{j \text{ times}}{\cdots} \partial_i$)

$$= \left(\sum_{i=0}^{p} \hat{\partial}^{p-i} a_p \otimes \bar{f}_{h-1}(\partial_0^i u_p) \right) * \left(\sum_{j=0}^{q} \hat{\partial}^{q-j} b_q \otimes \bar{f}_{h'-1}(\partial_0^j v_q) \right)$$

(taking into account that each term of the above shuffle product can be considered as an element of $\widetilde{B}_N^{(n)}(K_N)$ and applying the next identity (see [5, p. 74])):

$$[a \,|\, u] * [b \,|\, v] = [a] \,|\, [u * (b \,|\, v)] + (-1)^{|b|_{\overline{B}}|[a \,|\, u]|_{\overline{B}}}[b] \,|\, [(a \,|\, u) * v],$$

where $[a_1|\ldots|a_n]\,|\,[b_1|\ldots|b_m] = [a_1|\ldots|a_n|b_1|\ldots|b_m])$

$$= \sum_{i=0}^{p} [\hat{\partial}^{p-i} a_p]\,|\,[\bar{f}_{h-1}(\partial_0^i u_p) * \bar{f}_{h'}(b_q \otimes v_q)]$$

$$+ \sum_{j=0}^{q} (-1)^{(p+1)(j+1)} [\hat{\partial}^{q-j} b_q]\,|\,[\bar{f}_h(a_p \otimes u_p) * \bar{f}_{h'-1}(\partial_0^j v_q)]$$

(by the induction assumption)

$$= \sum_{i=0}^{p} [\hat{\partial}^{p-i} a_p]\,|\,[\bar{f}_{h+h'-1}(\partial_0^i u_p \nabla(b_q \otimes v_q))] \tag{6}$$

$$+ \sum_{j=0}^{q} (-1)^{(p+1)(j+1)} [\hat{\partial}^{q-j} b_q]\,|\,[\bar{f}_{h+h'-1}((a_p \otimes u_p)\nabla\partial_0^j v_q)]. \tag{7}$$

As a preliminary to the next step, it is necessary to note the following. Let (α, β) be a (p, q)-shuffle. If $\alpha_1 = 0$, i.e., $(\alpha, \beta) = (0, \alpha_2, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$, then $\bar{\alpha}$ denotes the set obtained from α by omitting $\alpha_1 = 0$, i.e., $\bar{\alpha} = (\alpha_2, \ldots, \alpha_p)$; in an analogous way, $\bar{\beta}$ may be defined. The notation $\alpha - 1$ defines the set obtained from α by subtracting 1, i.e., $\alpha - 1 = (\alpha_1 - 1, \ldots, \alpha_p - 1)$; the term $\beta - 1$ is similarly defined.

Assuming the above notations, the succeeding assertions are satisfied [14]:

1. Let (α, β) be a (p, q)-shuffle, with $\alpha_1 = 0$. Then $(\bar{\alpha} - 1, \beta - 1)$ becomes a $(p - 1, q)$-shuffle and $\mathrm{sg}(\bar{\alpha} - 1, \beta - 1) = \mathrm{sg}(\alpha, \beta)$. Furthermore, every $(p - 1, q)$-shuffle may be constructed in this way.
2. Let (α, β) be a (p, q)-shuffle, with $\beta_1 = 0$. Then, $(\alpha - 1, \bar{\beta} - 1)$ is a $(p, q - 1)$-shuffle and $\mathrm{sg}(\alpha - 1, \bar{\beta} - 1) = \mathrm{sg}(\alpha, \beta) + p$. Furthermore, every $(p, q - 1)$-shuffle may be constructed in this way.

It is convenient to write

$$s_\alpha = s_{\alpha_p} \ldots s_{\alpha_1}, \qquad s_\beta = s_{\beta_q} \ldots s_{\beta_1}.$$

With these notations at hand, we expand the left term of (5),

$$\bar{f}_{h+h'}((a_p \otimes u_p)\nabla(b_q \otimes v_q)) = \bar{f}_{h+h'}\left(\sum_{(\mu,v)} (-1)^{\mathrm{sg}(\mu,v)} s_v(a_p \otimes u_p) \cdot s_\mu(b_q \otimes v_q) \right), \tag{8}$$

where the (μ, v) index runs over all the $(p + 1, q + 1)$-shuffles. Taking into account the following identities on the degeneracy operators of $\overline{W}(K)$ (see [5, p. 90]):

$$s_0(c_r \otimes w_r) = 1_{r+1} \otimes (c_r \otimes w_r), \qquad s_{i+1}(c_r \otimes w_r) = s_i c_r \otimes s_i w_r, \quad i = 0, \ldots, r \tag{9}$$

with $c_r \in K_r, w_r \in \overline{W}_r(K)$.

By (9), it is possible to rewrite (8) as

$$\bar{f}_{h+h'}\left(\sum_{\substack{(\mu,v)\\ \mu_1=0}}(-1)^{\mathrm{sg}(\mu,v)}(s_{v-1}a_p \otimes s_{v-1}u_p)\cdot(s_{\bar{\mu}}(1_{q+1}\otimes b_q \otimes v_q))\right.$$

$$\left.+\sum_{\substack{(\mu,v)\\ v_1=0}}(-1)^{\mathrm{sg}(\mu,v)}(s_{\bar{v}}(1_{p+1}\otimes a_p \otimes u_p))\cdot(s_{\mu-1}b_q \otimes s_{\mu-1}v_q))\right)$$

(the first sum is taken over all $(p+1, q+1)$-shuffles (μ, v) with $\mu_1 = 0$, and the second is taken over all $(p+1, q+1)$-shuffles (μ, v) with $v_1 = 0$)

$$=\bar{f}_{h+h'}\left(\sum_{(\bar{\mu}-1,v-1)}(-1)^{\mathrm{sg}(\bar{\mu}-1,v-1)}s_{v-1}a_p \otimes (s_{v-1}u_p \cdot s_{\bar{\mu}-1}(b_q \otimes v_q))\right)$$

$$+\bar{f}_{h+h'}\left(\sum_{(\mu-1,\bar{v}-1)}(-1)^{\mathrm{sg}(\mu-1,\bar{v}-1)+p+1}s_{\mu-1}b_q \otimes (s_{\bar{v}-1}(a_p \otimes u_p)\cdot s_{\mu-1}v_q)\right)$$

$$=\sum_{i=0}^{p+q+1}\sum_{(\bar{\mu}-1,v-1)}(-1)^{\mathrm{sg}(\bar{\mu}-1,v-1)}(\hat{\partial}^{p+q+1-i}s_{v-1}a_p \tag{10}$$

$$\otimes \bar{f}_{h+h'-1}(\partial_0^i s_{v-1}u_p \cdot \partial_0^i s_{\bar{\mu}-1}(b_q \otimes v_q)))$$

$$+\sum_{i=0}^{p+q+1}(-1)^{p+1}\sum_{(\mu-1,\bar{v}-1)}(-1)^{\mathrm{sg}(\mu-1,\bar{v}-1)}(\hat{\partial}^{p+q+1-i}s_{\mu-1}b_q$$

$$\otimes \bar{f}_{h+h'-1}(\partial_0^i s_{\bar{v}-1}(a_p \otimes u_p)\cdot \partial_0^i s_{\mu-1}v_q)). \tag{11}$$

At this point, we need to recall the same results from [6, pp. 57–58] which will play an essential role in the sequel. Let (α, β) be any (p, q)-shuffle and i any integer with $0 \le i \le p+q$. The conditions

$$\alpha_r < i, \qquad \alpha_{r+1} \ge i; \qquad \beta_t < i, \qquad \beta_{t+1} \ge i$$

(or, $r = 0$, if $\alpha_1 \ge i$; $r = p$, if $\alpha_p < i$, and similarly for s) uniquely determine a pair of integers (r, t) satisfying:

$$0 \le r \le p, \qquad 0 \le t \le q, \qquad r+t = i, \tag{12}$$

and yield an (r, t)-shuffle

$$(\alpha', \beta') = (\alpha_1, \dots \alpha_r; \beta_1, \dots, \beta_t),$$

and a $(p-r, q-t)$-shuffle

$$(\alpha'', \beta'') = (\alpha_{r+1} - i, \dots \alpha_p - i; \beta_{t+1} - i, \dots, \beta_q - i).$$

We then say that (α, β) has been *split* at i into the two shuffles (α', β') and (α'', β''). Conversely, given integers r, t as in (12), an (r, t)-shuffle (α', β'), and a $(p-r, q-t)$-

shuffle (α'', β''), there exists a unique (p, q)-shuffle (α, β) which splits at $i = r + t$ into (α', β') and (α'', β''). The signatures of these shuffles are related by the equation

$$\mathrm{sg}(\alpha', \beta') + \mathrm{sg}(\alpha'', \beta'') = \mathrm{sg}(\alpha, \beta) - t(p - r). \tag{13}$$

If (α, β) is split at i, the face and degeneracy operators satisfy the following properties

$$\partial_0^i s_\beta = s_{\beta''} \partial_0^r \qquad \partial_0^i s_\alpha = s_{\alpha''} \partial_0^t$$

$$\hat{\partial}^{p+q-i} s_\beta = s_{\beta'} \hat{\partial}^{p-r} \qquad \hat{\partial}^{p+q-i} s_\alpha = s_{\alpha'} \hat{\partial}^{q-t}.$$

Upon splitting each shuffle at i, (10) and (11) can be rewritten as

$$\sum_{i=0}^{p+q+1} \sum_{(\bar\mu-1, v-1)} (-1)^{\mathrm{sg}(\bar\mu-1, v-1)} s_{(v-1)'} \hat{\partial}^{p-r} a_p \otimes \bar{f}_{h+h'-1}$$

$$(s_{(v-1)''} \partial_0^r u_p \cdot s_{(\bar\mu-1)''} \partial_0^t (b_q \otimes v_q)) \tag{14}$$

$$+ \sum_{i=0}^{p+q+1} (-1)^{p+1} \sum_{(\mu-1, \bar v-1)} (-1)^{\mathrm{sg}(\mu-1, \bar v-1)} \big(s_{(\mu-1)'} \hat{\partial}^{q-t} b_q$$

$$\otimes \bar{f}_{h+h'-1} (s_{(\bar v-1)''} \partial_0^r (a_p \otimes u_p) \cdot s_{(\mu-1)''} \partial_0^t v_q) \big). \tag{15}$$

Let us observe that $s_{(v-1)'} \hat{\partial}^{p-r} a_p$ appears in (14). The only case where it is a nondegenerate element in $\overline{K}$ is when $(v-1)'$ is the empty set, i.e., $t = 0$. In other words, it is the only summand which is not null modulo degeneracies. If $t = 0$, then $((\bar\mu - 1)', (v - 1)')$ is an $(r, 0)$-shuffle, and $((\bar\mu - 1)'', (v - 1)'')$ is a $(p - r, q + 1)$-shuffle. Moreover, by (13) we have $\mathrm{sg}((\bar\mu - 1)'', (v - 1)'') = \mathrm{sg}(\bar\mu - 1, v - 1)$.

Thus, with the above results at hand, (14) can be rewritten as

$$\sum_{r=0}^{p} \sum_{((\bar\mu-1)'', (v-1)'')} (-1)^{\mathrm{sg}((\bar\mu-1)'', (v-1)'')} \hat{\partial}^{p-r} a_p \otimes \bar{f}_{h+h'-1}(s_{(v-1)''} \partial_0^r u_p \cdot s_{(\bar\mu-1)''}(b_q \otimes v_q))$$

$$= \sum_{r=0}^{p} [\hat{\partial}^{p-r} a_p] \, \Big| \, \Big[\bar{f}_{h+h'-1} \Big(\sum_{((\bar\mu-1)'', (v-1)'')} (-1)^{\mathrm{sg}((\bar\mu-1)'', (v-1)'')}$$

$$\times s_{(v-1)''} \partial_0^r u_p \cdot s_{(\bar\mu-1)''}(b_q \otimes v_q) \Big) \Big]$$

$$= \sum_{r=0}^{p} [\hat{\partial}^{p-r} a_p] \, | \, [\bar{f}_{h+h'-1}(\partial_0^r u_p \nabla(b_q \otimes v_q))]. \tag{16}$$

Let us observe that $s_{(\mu-1)'} \hat{\partial}^{q-t} b_q$ appears in (15). Once again, the only case that we have to study is for $r = 0$; in this case, $((\mu - 1)', (\bar v - 1)')$ is a $(0, t)$-shuffle, and $((\mu - 1)'', (\bar v - 1)'')$ is a $(p + 1, q - t)$-shuffle. Furthermore, $\mathrm{sg}(\mu - 1, \bar v - 1) =$

$\text{sg}((\mu - 1)'', (\bar{v} - 1)'') + t(p + 1)$. Hence (15) becomes

$$\sum_{t=0}^{q} (-1)^{p+1} \sum_{((\mu-1)'',(\bar{v}-1)'')} (-1)^{\text{sg}((\mu-1)'',(\bar{v}-1)''))+t(p+1)}$$

$$\times \left(\hat{\partial}^{q-t} b_q \otimes \bar{f}_{h+h'-1}(s_{(\bar{v}-1)''}(a_p \otimes u_p) \cdot s_{(\mu-1)''} \partial_0^t v_q)\right)$$

$$= \sum_{t=0}^{q} (-1)^{(p+1)(t+1)} [\hat{\partial}^{q-t} b_q] \left| \left[\bar{f}_{h+h'-1}\left(\sum_{((\mu-1)'',(\bar{v}-1)'')} (-1)^{\text{sg}((\mu-1)'',(\bar{v}-1)''))} \right.\right.\right.$$

$$\left.\left.\left. \times s_{\bar{v}-1)''}(a_p \otimes u_p) \cdot s_{(\mu-1)''} \partial_0^t v_q \right) \right] \right]$$

$$= \sum_{t=0}^{q} (-1)^{(p+1)(t+1)} [\hat{\partial}^{q-t} b_q] \mid [\bar{f}_{h+h'-1}((a_p \otimes u_p) \nabla \partial_0^t v_q)]. \tag{17}$$

The terms (16) and (6) are equal to each other; analogously, for (17) and (7). Hence, we can state that $\bar{f}_k$ is multiplicative for each non-negative integer k, and $\bigoplus_{k\geq 0} \bar{f}_k$ is a morphism of DGA-algebras.

Secondly, the inclusion $\bigoplus_{k\geq 0} \bar{g}_k$ is a DGA-algebra morphism, since Eilenberg–Mac Lane proved in [5] that $g_{\overline{W}-\overline{B}}$ is a DGA-algebra morphism and as was seen in the previous section $g_{\overline{W}-\overline{B}} = \bigoplus_{k\geq 0} \tilde{g}_k$.

Thirdly, we are going to show that $\tilde{\phi}_k$ is a quasi algebra homotopy, for each non-negative integer k. That is,

$$\bar{\phi}_k \nabla(\bar{\phi}_{k-i} \otimes \bar{g}_i) = 0, \qquad \bar{\phi}_k \nabla(\bar{g}_{k-i} \otimes \bar{\phi}_i) = 0, \qquad \bar{\phi}_k \nabla(\bar{\phi}_{k-i} \otimes \bar{\phi}_i) = 0.$$

Now, we make some useful observations to prove the desired result:

1. We will use the fact that the Eilenberg–Zilber contraction, $EZ_{A,A'} : \{(A \times A')_N, A_N \otimes A'_N, AW_{A,A'}, EML_{A,A'}, SHI_{A,A'}\}$ is a semi-full algebra contraction where A and A' are commutative simplicial algebras (see [1]);

2. From the definition of EML, it follows that

$$1_p \times a_p = EML(1_0 \otimes a_p); \tag{18}$$

3. A property involving the different products that appear in our setting:

$$(a_p \otimes u_p)\nabla(b_q \otimes v_q) = (a_p \otimes u_p)\nabla'(1_{q+1} \otimes b_q \otimes v_q)$$

$$+ (-1)^{p+1}(1_{p+1} \otimes a_p \otimes u_p)\nabla'(b_q \otimes v_q)$$

$$= (a_p \times u_p)\heartsuit(1_{q+1} \times (b_q \otimes v_q))$$

$$+ (-1)^{p+1}(1_{p+1} \times (a_p \otimes u_p))\heartsuit(b_q \times v_q), \tag{19}$$

where $\heartsuit$ denotes the external product in $K \times \overline{W}(K)$. Here we use

$$c_p \diamond d_q = c_p \diamond' s_0 d_q + (-1)^p s_0 c_p \diamond' d_q,$$

where $c_p \in A_p$, $d_q \in A_q$, A is a simplicial algebra, $\diamond$ is the external product over A and $\diamond'$ is its derived product (see [5, (6.10) p. 71]).

Remark 4.3. Let us observe that $\overline{K} \times (\bigoplus_{k \geq 0} \widetilde{W}^{(k)}(K))$ is an ideal of $K \times \overline{W}(K)$ with regards to the inner product and the external product.

4. Let us recall that the product on $\overline{K} \otimes (\bigoplus_{k \geq 0} \widetilde{W}^{(k)}(K))$ is $(\diamond \otimes \nabla)(1 \otimes T \otimes 1)$, where $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$.

We begin by proving that

$$\bar{\phi}_k \nabla(\bar{\phi}_{k-i} \otimes \bar{g}_i) = 0 \qquad \text{for all } k \geq 0. \tag{20}$$

We proceed by induction on k.

If $k = 0$ or $k = 1$, then the identity (20) is immediate, since $\tilde{\phi}_0$ and $\tilde{\phi}_1$ are both null morphisms. Assume the formula (20) holds for $k \leq n$; we will prove it for $k = n$. Suppose that $n = h + h'$, where h y h' are non-negative integers, $a_p \otimes u_p \in \widetilde{W}^{(h)}_{p+1,N}(K)$ and $[b_q] \, | \, v \in \widetilde{B}^{(h')}_{l,N}(K_N)$ with $a_p \in \overline{K}_p$ and $b_q \in \overline{K}_q$. Computing the left term of the identity (20) on $(a_p \otimes u_p) \otimes ([b_q] \, | \, v)$. We have:

$$\bar{\phi}_n(\bar{\phi}_h(a_p \otimes u_p) \nabla \bar{g}_{h'}([b_q] \, | \, v))$$

(using (19) and the definitions of the morphisms, the above expression can be rewritten)

$$\begin{aligned}
= \, & (-SHI_{n-1} - EML_{n-1}(1 \otimes \bar{\phi}_{n-1})AW_{n-1})(-1)^{p+2} \\
& \times \left((1_{p+2} \times \bar{\phi}_h(a_p \otimes u_p)) \heartsuit EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v)\right) \\
& + (-SHI_{n-1} - EML_{n-1}(1 \otimes \bar{\phi}_{n-1})AW_{n-1}) \\
& \times \left(\bar{\phi}_h(a_p \otimes u_p) \heartsuit (1_l \times EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))\right)
\end{aligned} \tag{21}$$

(expanding both summands)

$$= (-1)^{p+1} SHI_{n-1}\left((1_{p+2} \times \bar{\phi}_h(a_p \otimes u_p)) \heartsuit EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v)\right) \tag{22}$$

$$+ (-1)^{p+1}(EML_{n-1}(1 \otimes \bar{\phi}_{n-1})AW_{n-1})$$
$$\times \left((1_{p+2} \times \bar{\phi}_h(a_p \otimes u_p)) \heartsuit EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v)\right) \tag{23}$$

$$- SHI_{n-1}\left(\bar{\phi}_h(a_p \otimes u_p) \heartsuit (1_l \times EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))\right) \tag{24}$$

$$- EML_{n-1}(1 \otimes \bar{\phi}_{n-1})AW_{n-1}\left(\bar{\phi}_h(a_p \otimes u_p)\right.$$
$$\left. \heartsuit (1_l \times EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))\right). \tag{25}$$

In the sequel, we shall prove that each of the summands are zero.
Using (18), we have

$$(1_{p+2} \times \bar{\phi}_h(a_p \otimes u_p)) = EML_h(1_0 \otimes \bar{\phi}_h(a_p \otimes u_p)). \tag{26}$$

Taking into account the above identity, the summand (22) can be written as

$$(-1)^{p+1} SHI_{n-1}\big(EML_h(1_0 \otimes \bar{\phi}_h(a_p \otimes u_p))\heartsuit EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v)\big)$$

(using the fact that EML is a morphism of DGA-algebras, then, up to sign)

$$= SHI_{n-1} EML_{n-1}\big((1_0 \diamond b_q) \otimes (\bar{\phi}_h(a_p \otimes u_p)\nabla \bar{g}_{h'-1})(v))\big) = 0$$

since $SHI_{n-1}EML_{n-1} = 0$.

Let us prove that (23) vanishes. By (26), (23) can be written as

$$(-1)^{p+1} EML_{n-1}(1 \otimes \bar{\phi}_{n-1}) AW_{n-1}$$
$$\times \big(EML_h(1_0 \otimes \bar{\phi}_h(a_p \otimes u_p))\heartsuit EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v)\big)$$

(taking into account that EML is a morphism of DGA-algebras and $AW_{n-1}EML_{n-1}$ is the identity morphism, up to sign)

$$= EML_{n-1}(1_0 \diamond b_q \otimes \bar{\phi}_{n-1}(\bar{\phi}_h(a_p \otimes u_p)\nabla \bar{g}_{h'-1}(v))) = 0,$$

(since by the inductive assumption $\bar{\phi}_{n-1}(\bar{\phi}_h(a_p \otimes u_p)\nabla \bar{g}_{h'-1}(v)) = 0$).

On the other hand, using (18)

$$1_l \times EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v) = EML_{h'}(1_0 \otimes EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v)). \quad (27)$$

Hence (24) may be rewritten as

$$-SHI_{n-1}\big(\bar{\phi}_h(a_p \otimes u_p)\heartsuit EML_{h'}(1_0 \otimes EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))\big)$$

(by definition of $\bar{\phi}_h$)

$$= -SHI_{n-1}\big(SHI_{h-1}(a_p \otimes u_p)\heartsuit EML_{h'}(1_0 \otimes EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))\big)$$
$$- SHI_{n-1}(EML_{h-1}(1 \otimes \bar{\phi}_{h-1}) AW_{h-1}(a_p \otimes u_p)$$
$$\heartsuit EML_{h'}(1_0 \otimes EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))).$$

Using the fact that SHI_{n-1} is a quasi algebra homotopy, we conclude that the first summand is zero. The second one vanishes as well, since EML is a morphism of DGA-algebras and $SHI_{n-1}EML_{n-1} = 0$. Thereby (24) vanishes.

Finally, let us show that (25) vanishes as well. In stating (27), we can write (25) as

$$-EML_{n-1}(1 \otimes \bar{\phi}_{n-1}) AW_{n-1}\big(\bar{\phi}_h(a_p \otimes u_p)\heartsuit EML_{h'}(1_0 \otimes EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))\big)$$

(by the definition of $\bar{\phi}_h$)

$$= -EML_{n-1}(1 \otimes \bar{\phi}_{n-1}) AW_{n-1}$$
$$\times \big(SHI_{h-1}(a_p \otimes u_p)\heartsuit EML_{h'}(1_0 \otimes EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))\big)$$

$$- EML_{n-1}(1 \otimes \bar{\phi}_{n-1})AW_{n-1}$$
$$\times \left(EML_{h-1}(1 \otimes \bar{\phi}_{h-1})AW_{h-1}(a_p \otimes u_p) \right.$$
$$\heartsuit EML_{h'}(1_0 \otimes EML_{h'-1}(1 \otimes \bar{g}_{h'-1})(b_q \otimes v))).$$

The first summand is zero since

$$AW_{n-1}(SHI_{h-1} \heartsuit EML_{h'}) = 0.$$

This identity is deduced from the following facts, AW is a DGA-algebra morphism and $AW\,SHI = 0$.

Taking into account that EML is a morphism of DGA-algebras and $AW_{n-1}EML_{n-1}$ is the identity morphism, the second summand can be rewritten, up to sign, as

$$\sum_{i=0}^{p} EML_{n-1}\left((\hat{\partial}^{p-i} a_p \diamond 1_0) \otimes (\bar{\phi}_{n-1}(\bar{\phi}_{h-1}(\partial_0^i u_p) \nabla \bar{g}_{h'}(b_q \otimes v)) \right).$$

But this summand is zero since $\bar{\phi}_{n-1} \nabla(\bar{\phi}_{h-1} \otimes \bar{g}_{h'}) = 0$, by the inductive hypothesis. This concludes the proof of (20).

The second identity,

$$\bar{\phi}_k \nabla(\bar{g}_{k-i} \otimes \bar{\phi}_i) = 0, \tag{28}$$

is evident since ∇ is commutative; therefore, both identities (28) and (20) are equal to each other. Finally, the third identity,

$$\bar{\phi}_k \nabla(\bar{\phi}_{k-i} \otimes \bar{\phi}_i) = 0,$$

may be proven in an analogous way to (20). Hence, $\bigoplus_{k \geq 0} \tilde{\phi}_k$ is a quasi algebra homotopy. $\qquad \square$

Theorem 4.4. *The contraction,* $C_{\overline{W}-\overline{B}} : \{\overline{W}_N(K), \overline{B}_N(K_N), f_{\overline{W}-\overline{B}}, g_{\overline{W}-\overline{B}}, \phi_{\overline{W}-\overline{B}}\}$, *is a semi-full algebra contraction.*

Proof. a) Theorem 4.2 states that $\sum_{k \geq 0} r^{(k)}$ is an almost-full algebra contraction, and at the beginning of this section we proved that the morphism ρ is a derivation.

b) This $\sum_{k \geq 0} r^{(k)}$ contraction, can be perturbed into $C_{\overline{W}-\overline{B}}$ taking ρ as a perturbation datum.

Hence, using Theorem 4.1, we obtain the desired conclusion. $\qquad \square$

5. HOMOLOGY OF TWISTED CARTESIAN PRODUCTS OF EILENBERG–MAC LANE SPACES

We find that a remarkable consequence of Theorem 4.4 in the computation of the homology of Eilenberg–Mac Lane spaces, $K(\pi, n)$, where π is an abelian

group and n is a non-negative integer. In the sequel, we follow the iterative simplicial description of $K(\pi, n)$ in terms of the classifying complex (W) given in [14, Theorem 23.2]. Taking as a starting point Eilenberg–Mac Lane's work in [5, 6], Real states in [15] a contraction $C_{\pi,n}$ from the normalized chain complex $C_N(K(\pi, n))$ canonically associated to $K(\pi, n)$ to a tensor product of Cartan's elementary complexes (see [4]), denoted by $G(n, \pi)$. For this purpose, the following chain of contractions is established:

$$C_N(K(\pi, 1)) \Rightarrow \overline{W}_N(C(K(\pi, 0))) \Rightarrow \overline{B}_N(\Lambda(\pi)) \Rightarrow G(\pi, 1).$$

Using the iterative simplicial description of $K(\pi, n)$, we have

$$C_N(K(\pi, n+1)) \Rightarrow \overline{W}_N(\overline{W}^n(C(K(\pi, 0)))) \Rightarrow \overline{B}_N(\overline{W}_N^n(C(K(\pi, 0))))$$
$$\Rightarrow \overline{B}_N(\overline{B}_N^n(\Lambda(\pi))) \Rightarrow \overline{B}_N(G(\pi, n)) \Rightarrow G(\pi, n+1).$$

In order to establish the last contraction of the above chain, Real needed to fix $\mathbf{Z}$ localised over a prime p as ground ring. In [16] it is proven that the last three contractions are, at least, semi-full algebra contractions. Now, taking into account that the first one is an algebra isomorphism (see [10, p. 81]), and using Theorem 4.4 we showed that all the contractions involved in the precedent chain are semi-full algebra contractions. Hence we can state the following result.

Theorem 5.1. *Taking $\mathbf{Z}$ localised over a prime p as ground ring, given an abelian group π and a non-negative integer n. There exists a semi-full algebra contraction, $C_{\pi,n}$, from $C_N(K(\pi, n))$ to $G(\pi, n)$.*

Now, we try to take advantage of this result for achieving economical homological computations of some fiber bundles. To this end, we need the following theorem.

Theorem 5.2 [1]. *Let G'' be a simplicial group operating on the left of G and $\tau : G' \to G''$ be a twisting function such that the twisted cartesian product (TCP) $G \times_\tau G'$ is a simplicial group with regards to the trivial group multiplication. Then, the contraction associated to $G \times_\tau G'$ by applying Theorem 2.2 is a semi-full algebra contraction.*

It is well known (see [14, Lemma 25.1]) that if $f : \pi \to \pi'$ is a homomorphism of groups, then f induces a homomorphism of groups between $K(\pi, n)$ and $K(\pi', n)$, which we denote by f as well. It is apparent that if $f : \pi \to \pi'$ is a homomorphism of groups, then the twisted cartesian product $E_f = K(\pi, n) \times_{\tau_u \circ f} K(\pi', n+1)$ induced by f from the universal (principal) TCP $K(\pi, n) \times_{\tau_u} K(\pi, n+1)$ is a simplicial group. Then, E_f provides examples of TCP satisfying the hypothesis of Theorem 5.2.

Combining both Theorems 5.1 and 5.2 we conclude the following proposition.

Proposition 5.3. *Let $E = K(\pi, n) \times_\tau K(\pi', n')$ be a TCP, where the structural group is reduced and working in $\mathbf{Z}$ localised over a prime p as ground ring. If E is a group*

with regards to the group multiplication, then there exists a semi-full algebra contraction from

$$C_N(K(\pi, n) \times_\tau K(\pi', n'))$$

to a twisting tensor product $G(\pi, n)\tilde{\otimes}G(\pi', n')$.

Proof. We only give the main ideas of the proof. By Theorem 5.2, we have the following semi-full algebra contraction:

$$C_N(E) \Rightarrow C_N(K(\pi, n)) \otimes_t C_N(K(\pi, n')). \tag{29}$$

Furthermore, $C_N(K(\pi, n)) \otimes_t C_N(K(\pi, n'))$ is DGA-algebra with regards to the natural product of $C_N(K(\pi, n)) \otimes C_N(K(\pi', n'))$. Thus $t\cap$ is a derivation.

Using Theorem 5.1 and the tensor product of contractions, we can establish the following semi-full algebra contraction:

$$C_N(K(\pi, n)) \otimes C_N(K(\pi', n')) \Rightarrow G(\pi, n) \otimes G(\pi', n').$$

Taking $t\cap$ as the perturbation datum, this contraction can be perturbed into

$$C_N(K(\pi, n)) \otimes_t C_N(K(\pi', n')) \Rightarrow \big(G(\pi, n) \otimes G(\pi', n'), d^\otimes + d_{t\cap}\big). \tag{30}$$

The pointwise nilpotency condition is guaranteed (see [12, Lemma 3.4.]) using the standard filtration on $C_N(K(\pi, n)) \otimes C_N(K(\pi', n'))$. Once the structural group is reduced, t vanishes on one-simplices (see [14]).

Finally, composing contractions (29) and (30), the desired result is concluded.

□

In the following example (extracted from [1]) we show how it is possible to take advantage of the multiplicative properties of the chain complexes in order to compute in an economical way homological models for some TCP of two Eilenberg–Mac Lane spaces.

Example 5.4. Let $E = K(\mathbf{Z}_2, 1) \times_{\tau_u f} K(\mathbf{Z}, 2)$ be the principal TCP induced from $K(\mathbf{Z}_2, 1) \times_{\tau_u} K(\mathbf{Z}_2, 2)$ by $f : \mathbf{Z} \to \mathbf{Z}_2$. It is clair that E satisfies the hypotheses of Proposition 5.3. Fix $\mathbf{Z}$ as ground ring. Combining the results of [6] and [16] it is possible to establish the following semi-full algebra contraction

$$C_N(K(\mathbf{Z}_2, 1)) \Rightarrow E(u_1, 1) \otimes \Gamma(v_1, 2),$$

where $d(\gamma_j(v_1)) = 2u_1 \gamma_{j-1}(v_1)$. (It is, actually, almost-full.)

And

$$C_N(K(\mathbf{Z}, 2)) \Rightarrow \Gamma(v_2, 2).$$

Here $E(u, 1)$ denotes the exterior algebra with one generator u in degree 1, and $\Gamma(v, 2)$ denotes the divided power algebra with on generator v in degree 2. In this case, the contraction (30) can be written as

$$C_N(K(\mathbf{Z_2}, 1)) \otimes_t C_N(K(\mathbf{Z}, 2)) \Rightarrow \big((E(u_1, 1) \otimes \Gamma(v_1, 2)) \otimes \Gamma(v_2, 2), d^\otimes + d_{t\cap}\big)$$

Let us observe that the smaller complex of this contraction is a DGA-algebra with regards to the same product of $((E(u_1, 1) \otimes \Gamma(v_1, 2)) \otimes \Gamma(v_2, 2), d^\otimes)$. Thus, $d_{t\cap}$ is a derivation. This fact has important consequences in order to compute $d_{t\cap}$ on the whole set of additive generators of the homological model (as a module). By Basic Perturbation Lemma, we have

$$d_{t\cap} = f(t\cap)\Sigma_c^{t\cap}g. \tag{31}$$

By direct computation, we obtain

$$d_{t\cap}(\gamma_1(v_2)) = u_1.$$

Taking into account that

$$\gamma_i(v_2)\gamma_j(v_2) = \binom{i+j}{i}\gamma_{i+j}(v_2),$$

and that $d_{t\cap}$ is a derivation, we have

$$d_{t\cap}(\gamma_j(v_2)) = u_1\gamma_{j-1}(v_2).$$

By a similar argument, we can calculate $d_{t\cap}$ on any additive generator of the module; in this way, it is easy to see that $d_{t\cap}$ is null on all additive generators except on $\gamma_j(v_2)$.

Summing up, in order to compute $d_{t\cap}$ over all additive generators of the module, we only use the formula (31) over the generators of the algebra: $u_1, v_1,$ and v_2. For the remainder of the additive generators we use the multiplicative property of $d_{t\cap}$. Let us observe that this represents an economical way for computing $d_{t\cap}$, alternative to the formula (31).

6. CONCLUSIONS AND FURTHER WORK

In this article, we presented a technique that improves the computation of the homology of certain twisted cartesian products of two Eilenberg–Mac Lane spaces.

One of our next goals is to find a manageable formula for the projection of the contraction between $\overline{W}_N(K)$ and $\overline{B}_N(K_N)$. This formula will allow us to compute a cohomology class $\zeta \in H^m(K(\pi, n), G)$ (we have exploited this idea in [2] to obtain a generating set for the second cohomology group in terms of representative 2-cocycles over a group π). These cohomology classes can define the twisted cartesian products of two Eilenberg–Mac Lane spaces, $K(G, m-1) \times_{\tau(\zeta)} K(\pi, n)$ (see [14, Lemmas 21.9 and 24.3]). Another of our goals is to study the algebra structure on this kind of twisted cartesian products of Eilenberg–Mac Lane spaces since they might be an interesting source of examples where we can apply our technique.

REFERENCES

[1] Álvarez, V., Armario, J. A., Frau, M. D., Real, P. (2007). Algebra structures on the twisted Eilenberg–Zilber theorem. *Communications in Algebra* 35:3273–3291.

[2] Álvarez, V., Armario, J. A., Frau, M. D., Real, P. (2009). The homological reduction method for computing cocyclic Hadamard matrices. *Journal of Symbolic Computation* 44.

[3] Brown, R. (1967). The twisted Eilenberg–Zilber theorem. *Celebrazioni Archimedee del secolo XX. Simposio di topologia* 33–37.

[4] Cartan, H. (1956). *Algèbres d'Eilenberg–Mac Lane*. Séminaire H. Cartan 1954/55, (expos é 2 à 11). Ecole Normale Superieure, Paris.

[5] Eilenberg, S., Mac Lane, S. (1953). On the groups $H(\pi, n)$, I. *Annals of Math.* 58:55–106.

[6] Eilenberg, S., Mac Lane, S. (1954). On the groups $H(\pi, n)$, II. *Annals of Math.* 66:49–149.

[7] Eilenberg, S., Zilber, J. A. (1953). On products of complexes. *Am. J. Math.* 75:200–204.

[8] Gugenheim, V. K. A. M., Lambe, L. A. (1989). Perturbation theory in differential homological algebra I. *Illinois J. Math.* 33(4):566–582.

[9] Gugenheim, V. K. A. M., Lambe, L. A., Stasheff, J. D. (1991). Perturbation theory in differential homological algebra II. *Illinois J. Math.* 35(3):357–373.

[10] Gugenheim, V. K. A. M., May, J. P. (1974). On the theory and application of differential torsion products. *Memo. Amer. Math. Soc.* 142.

[11] Huebschmann, J., Kadeishvili, T. (1991). Small models for chain algebras. *Math. Z.* 209:245–280.

[12] Lambe, L. A., Stasheff, J. D. (1987). Applications of perturbation theory to iterated fibrations. *Manuscripta Math.* 58:367–376.

[13] Mac Lane, S. (1995). *Homology. Classics in Mathematics.* Berlin. Springer-Verlag: Reprint of the 1975 edition.

[14] May, J. P. (1967). *Simplicial Objects in Algebraic Topology*. Princeton, NJ: Van Nostrad.

[15] Real, P. (1993). Algorithms for Computing the Effective Homology of Classifying Spaces. Ph.D. thesis. University of Seville. Seville, Spain.

[16] Real, P. (2000). Homological perturbation theory and associativity. *Homology, Homotopy and Applications* 2:51–88.

[17] Real, P., Sergeraert, F. (1994). A conjecture of Eilenberg–Mac Lane. *Institut Fourier* (Preprint).

[18] Shih, W. (1962). Homologie des espaces fibrés. *Inst. Hautes Etudes Sci.* 13:93–176.

[19] Wong, S. C. (1985). *Comparison Between the Reduced Bar Construction and the Reduced W-Construction*. Diplomarbeit. Math. Institut der Universität Heidelberg, Heidelberg, Germany.