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Lie symmetry and μ-symmetry methods for nonlinear generalized Camassa–Holm equation

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Abstract

In this paper, a Lie symmetry method is used for the nonlinear generalized Camassa–Holm equation and as a result reduction of the order and computing the conservation laws are presented. Furthermore, μ-symmetry and μ-conservation laws of the generalized Camassa–Holm equation are obtained.

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1 Introduction

Partial differential equations (PDEs) with suitable solutions are among the most important topics in various branches of mathematical physics [1–6]. The most accurate methods for order reduction and computation conservation rules are the classical Lie theory [7–9], the general theorem [7], the direct method [10], the μ-symmetries method [11], and the Noether theorem [7, 12].

Using Lie transformation group theory for order reduction and constructing solutions of nonlinear PDEs with integer order [8, 9] or fractional order partial differential equations [6, 13–20] and ordinary differential equations is one of the most efficient fields of research in the theory of nonlinear PDEs. The λ-symmetry method has been presented by Muriel and Romero, which is a modern method to order reduction of ODEs [21]. Gaeta and Morando have extended the λ-symmetries approach for ODEs to the μ-symmetries method for PDEs [22, 23].

The concepts of variational problem and conservation law in the case of symmetry have been developed by Muriel, Romero and Olver to λ-symmetries of ODEs via proper formulation of Noether's theorem for λ-symmetry of ODEs [12, 24]. These results have been generalized by Cicogna and Gaeta to the framework of μ-symmetries for PDEs. Also, the conservation law called the μ-conservation law in μ-symmetry of the Lagrangian topic has been addressed [12].

The nonlinear generalized Camassa–Holm equation (gC-H) equation with the nonlinearity parameter p is discussed in this article,

$$\Delta_u \equiv u_t - u_{txx} - \frac{1}{2}(p+1)(p+2)u^p u_x + \frac{1}{2}p(p-1)u^{p-2}u_x^3 + 2pu^{p-1}u_x u_{xx} + u^p u_{xxx} = 0. \tag{1}$$

In the case that $p = 1$, the above equation reduces to the Camassa–Holm equation. In the case that $p = 2$, we have the case generalized of the Camassa–Holm equation possesses bi-Hamiltonian structure [25]. This Camassa–Holm equation is a mathematical model for shallow water waves that explains wave breaking for a major class of solutions in which the wave inclination blows up in a limited time while the amplitude of wave is confined [26, 27]. Integrity, possessing a Lax pair, bi-Hamiltonian structure property and infinite hierarchy of symmetries are important features of the Camassa–Holm equation [26, 28].

The outline of this paper is as follows: Lie symmetry analysis, reduction, optimal system of one-dimensional subalgebras and differential invariant of Eq. (1) are presented in Sect. 2. Some conservation laws for Eq. (1) will be found in Sect. 3. In Sect. 4, we calculate the μ-symmetry and order reduction of Eq. (1) and Sect. 5 deals with Lagrangian of Eq. (1) in potential form and we will apply it to obtain μ-conservation laws of Eq. (1).

2 Lie symmetries of gC-H equation

r independent variables $x = (x^1, \dots, x^r)$ and s dependent variables $u = (u^1, \dots, u^s)$ construct the total space $M = X \times U$, and the kth order jet space $M^{(k)}$, including M and the derivatives of dependent variables up to order k. Let $\Delta(x, u^k) = 0$ be a PDE on M and G be a local group transformation that acts on M. Suppose $\nu = \sum_{i=1}^{r} \eta^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{s} \psi^\alpha(x,u)\partial_{u^\alpha}$ is an infinitesimal transformation of G in g that acts on $X \times U \times U^{(1)}$. Now, to calculate the symmetry group via an infinitesimal symmetry condition, the kth prolongation of ν must acts on $\Delta(x, u^k)$, i.e.,

$$\mathrm{Pr}^{(k)}\,\nu\big[\Delta\big(x,u^{(k)}\big)\big] \equiv 0, \quad \bmod \Delta\big(x,u^{(k)}\big),$$

where

$$\mathrm{Pr}^{(k)}\,\nu = \nu + \sum_{\alpha=1}^{s} \sum_{J=(j_1,\dots,j_k)} \left[\mathrm{D}_J \left(\psi_\alpha - \sum_{h=1}^{r} \eta^h u_h^\alpha \right) + \sum_{h=1}^{r} \eta^h u_{J,h}^\alpha \right] \partial_{u_J^\alpha}, \quad 1 \le k \le p.$$

(See for more details [7].) The symmetry group of $\Delta(x, u^k)$ is obtained by solving these system [7, 9]. Let $\mathrm{v} = \eta(x,t,u)\partial_x + \gamma(x,t,u)\partial_t + \psi(x,t,u)\partial_u$ be an infinitesimal generator of the classical Lie point symmetry groups for the gC-H equation, To calculation of the symmetry group for the gC-H equation, first applying the $\mathrm{Pr}^{(3)}\,\mathrm{v}$ on Eq. (1), we have $\mathrm{Pr}^{(3)}\,\mathrm{v}[\Delta_u] = 0$, then substituting

$$u_{txx} + (1/2)(p+1)(p+2)u^p u_x - (1/2)p(p-1)u^{p-2}u_x^3 - 2pu^{p-1}u_x u_{xx} - u^p u_{xxx}$$

for u_t, whatever remains is a polynomial equation involving the various derivatives of $u(x,t)$, whose coefficients are certain derivatives of η, γ and ψ. By solving this multivariate system, we will have

$$\eta = b_3,$$
$$\gamma = b_1 t + b_2,$$
$$\psi = -\frac{b_1}{p}u,$$

where b_1, b_2 and b_3 are arbitrary constants.

Corollary 1 *The Lie algebras of infinitesimal projectable symmetries of the gC-H equation is spanned by the vector fields*

$$\nu_1 = \partial_x,$$
$$\nu_2 = \partial_t,$$
$$\nu_3 = t\partial_t - \frac{u}{p}\partial_u.$$

The one-parameter groups B_i, generated by the vector fields ν_1, ν_2 and ν_3, each of them giving the transformed point $\exp(\kappa\nu_i)(x,t,u)$, are

$$B_1 : (x,t,u) \mapsto (x+\kappa, t, u),$$
$$B_2 : (x,t,u) \mapsto (x, t+\kappa, u),$$
$$B_3 : (x,t,u) \mapsto \bigl(x, te^{\kappa}, ue^{-\kappa/p}\bigr).$$

If $u = f(x,t)$ is a solution of Eq. (1), since each group G_i is a symmetry group, we conclude that the functions $u_1 = f(x-\epsilon, t)$, $u_2 = f(x, t-\epsilon)$ and $u_3 = f(x, t/e^{\varepsilon})e^{-\varepsilon/p}$, are solutions of Eq. (1).

Invariant solutions are a family of solutions that all correspond to the one-parameter subgroup of the symmetry group of a PDE, and for computing an optimal system of subalgebras, an optimal system of subgroups must be calculated. Usually, to solve the classification problem of one-dimensional subalgebras, the same problem of classifying the orbits is considered [7]. Table 1 shows the commutation table of Lie algebra g for the gC-H equation. The solubility property of g is significant.

Table 1 The commutator table of Eq. (1)

$[\nu_i,\nu_j]$	ν_1	ν_2	ν_3
ν_1	0	0	0
ν_2	0	0	ν_2
ν_3	0	$-\nu_2$	0

Table 2 Adjoint representation table of Eq. (1)

$\mathrm{Ad}(\exp(\varepsilon\nu_i)\nu_j)$	ν_1	ν_2	ν_3
ν_1	ν_1	ν_2	ν_3
ν_2	ν_1	ν_2	$\nu_3 - \varepsilon\nu_2$
ν_3	ν_1	$\nu_2 + \varepsilon\nu_2$	ν_3

Table 2 shows adjoint representation of the gC-H equation, such that

$$\mathrm{Ad}\big(\exp(\varepsilon\nu_i)\big)\nu_j = \nu_j - \varepsilon[\nu_i,\nu_j] + \frac{\varepsilon^2}{2}\big[\nu_i,[\nu_i,\nu_j]\big] - \cdots.$$

Theorem 1 *An optimal system of one-dimensional Lie algebras of Eq.* (1) *is provided by* $a_1\nu_1+\nu_3$, $a_1\nu_1+\nu_2$ *and* $a_1\nu_1$.

Proof Suppose the symmetry algebra g of the gC-H equation is spanned by the vector fields ν_1, ν_2 and ν_3. A linear map

$$\begin{cases} F_i^\varepsilon : g \longrightarrow g \quad (i=1,2,3), \\ \nu \mapsto \mathrm{Ad}(\exp(\varepsilon\nu_i))\nu, \end{cases}$$

is assumed. The matrices B_i^ε of F_i^ε, with respect to basis $\{\nu_1,\nu_2,\nu_3\}$ are

$$B_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every one-dimensional subalgebra of g is determined by a nonzero vector $\nu = c_1\nu_1 + c_2\nu_2 + c_3\nu_3$, where c_i are arbitrary constants. We will make coefficients c_i as simple as possible through acting with these matrices on ν.

Case 1. Suppose that $c_3 \neq 0$. So, we can assume that $c_3 = 1$. Acting with B_1^ε and B_2^ε on ν, the coefficient of ν_2 vanishes and the coefficient of ν_1 does not change. Then ν is reduced to $\nu = a_1\nu_1 + \nu_3$.

Case 2. Suppose that $c_3 = 0$ and $c_2 \neq 0$. So, we can assume that $c_2 = 1$. Acting with B_i^ε on ν, the coefficient of ν_1 does not change and ν is reduced to $\nu = c_1\nu_1 + \nu_2$.

Case 3. Suppose that $c_3 = 0$ and $c_2 = 0$. Acting with B_i^ε on ν, the coefficient of ν_1 does not change. Then $\nu = c_1\nu_1$. □

Table 3 Invariant of Eq. (1)

Operator	y	w	u
$\alpha\nu_1$	t	u	$w(y)$
$\alpha\nu_1+\nu_2$	$x-\alpha t$	u	$w(y)$
$\alpha\nu_1+\nu_3$	$te^{-x/\alpha}$	$ue^{x/\alpha p}$	$w(y)e^{-x/\alpha p}$

Table 4 Reduction of Eq. (1)

Operator	Reduced equations
$\alpha\nu_1$	$w_y = 0$
$\alpha\nu_1+\nu_2$	$-\alpha w_y + \alpha w_{yyy} - \frac{1}{2}(p+1)(p+2)w^p w_y + \frac{1}{2}p(p-1)w^{p-2}w_y^3$ $+ 2pw^{p-1}w_y w_{yy} + w^p w_{yy} = 0$
$\alpha\nu_1+\nu_3$	$-\frac{1}{\alpha^3}(y^3 + w^p y^4)w_{yyy} + (\frac{2}{\alpha^3}w^p y^2 - \frac{3p+2}{\alpha p}y^2 - \frac{3p+3}{\alpha^3 p}w^p y^3)w_{yy}$ $- \frac{2p}{\alpha^3}w^{p-1}y^3 w_y w_{yy} - \frac{p(p-1)}{2\alpha^3}w^{p-2}y^4 w_y^3 - \frac{(p+1)}{2\alpha^3}w^{p-1}y^3 w_y^2$ $+ (\frac{\alpha^2p^2-(p+1)^2}{\alpha^2p^2}y + \frac{p+7}{2\alpha^3 p} - \frac{p^2+3p+3}{\alpha^3p^2})w^p y^2 w_y g$ $+ (\frac{p+5}{2\alpha^3p^2} - \frac{1}{\alpha^3p^3} - \frac{(p+1)(p+2)}{2\alpha p})w^{p+1}y = 0$

Invariants associated with the symmetry operators can be computed, which are calculated by integrating the characteristic equations. For example, the characteristic equation of the operator $\alpha \mathrm{v}_1 = \alpha\partial_x$ is $\frac{dx}{\alpha} = \frac{dt}{0} = \frac{du}{0}$, and the corresponding invariants are $y = t$ and $w = u$. The derivatives of u are given in terms of y and $w(y)$ as $u_t = w_y$, $u_{txx} = 0$, $u_x = 0$, $u_{xx} = 0$, $u_{xxx} = 0$. Substituting them into Eq. (1), the ordinary differential equation $w_y = 0$ is obtained. Tables 3 and 4 show all results in this step.

3 Conservation laws of the gC-H equation

A conservation law in physics describes a measured property of an isolated system that does not alter over time. The following divergence expression shows a local conservation law of a system $\Delta(x, u^k) = 0$:

$$\operatorname{Div}\vartheta = D_1\vartheta^1\big(x, u^{(k)}\big) + \cdots + D_r\vartheta^r\big(x, u^{(k)}\big) = L.\Delta, \quad L = (L_1, \ldots, L_n),$$

where $\vartheta = (\vartheta^1, \ldots, \vartheta^r)$ is a r-tuple of smooth functions on $M^{(k)}$, and $\vartheta^i s$ and L are the fluxes and characteristic of conservation law. Also,

$$E_{\alpha^j} = \frac{\partial}{\partial\alpha^j} - D_i\frac{\partial}{\partial\alpha^j} + \cdots + (-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial\alpha^j_{i_1\cdots i_s}} + \cdots$$

is the Euler operator with respect to α^j. The equations

$$E_{\alpha^j}F\big(x, \alpha, \partial_\alpha, \ldots, \partial_\alpha^s\big) \equiv 0, \quad j = 1, \ldots, s,$$

hold for arbitrary $\alpha(x)$ if and only if $F(x, \alpha, \partial_\alpha, \ldots, \partial_\alpha^s) \in \operatorname{Div}$ [10]. Suppose

$$E_{\alpha^j}\big(\Lambda_\nu\big(x, \alpha, \partial_\alpha, \ldots, \partial_\alpha^r\big)\Delta_\nu\big(x, u^{(k)}\big)\big) \equiv 0, \quad j = 1, \ldots q.$$

Finally, this set of equations holds for arbitrary functions $\alpha(x)$ and $\{\Lambda_\nu\}_{\nu=1}^l$ yields a local conservation law for the system [10].

Now, all the rules in the form $\Lambda = \Lambda(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$ of Eq. (1) are obtained, and the solution of the determining system is

$$\Lambda_1 = 1, \qquad \Lambda_2 = u,$$

where Λ determines a pair of nontrivial local conservation law of (ρ, ϱ), where

$$\begin{aligned} &D_t\rho + D_x\varrho \\ &\quad\equiv \Lambda\left(u_t - u_{txx} - \frac{1}{2}(p+1)(p+2)u^p u_x + \frac{1}{2}p(p-1)u^{p-2}u_x^3 + 2pu^{p-1}u_x u_{xx} + u^p u_{xxx}\right). \end{aligned}$$

To calculate (ρ, ϱ), we can use the strong 2-dimensional homotopy operator

$$D_t\rho + D_x\varrho = D_t\mathrm{H}^{(t)}_{u(x,t)}f + D_x\mathrm{H}^{(x)}_{u(x,t)}f = 0.$$

Table 5 Conservation laws for Eq. (1)

Λ	
$\Lambda_1 = 1$	$\Upsilon_u^{(x)} = -\frac{1}{2}(p+1)(p+2)u^{p+1} + \frac{1}{2}(p)(p+1)u^{p-1}u_x^2 + (p+1)u^p u_{xx} - \frac{2}{3}u_{xt}$ $\Upsilon_u^{(t)} = u - \frac{1}{3}u_{xx}$ $\rho = -\frac{1}{2}(p+2)u^{p+1} + \frac{1}{2}(p)u^{p-1}u_x^2 + u^p u_{xx} - \frac{2}{3}u_{xt}$ $\varrho = u - \frac{1}{3}u_{xx}$
$\Lambda_2 = U$	$\Upsilon_u^{(x)} = -\frac{1}{2}(p+1)(p+2)u^{p+2} + \frac{1}{2}(p-1)(p+2)u^p u_x^2 + (p+2)u^{p+1}u_{xx} - \frac{4}{3}uu_{xt} + \frac{2}{3}u_x u_t$ $\Upsilon_u^{(t)} = u^2 + \frac{1}{3}u_x^2 - \frac{2}{3}u_{xx}u$ $\rho = -\frac{1}{2}(p+1)(p+2)u^{p+2} + \frac{1}{2}(p-1)(p)u^{p-1}u_x^3 + u^{p+1}u_{xx} - \frac{2}{3}uu_{xt} + \frac{1}{3}u_x u_t$ $\varrho = \frac{1}{2}u^2 + \frac{1}{6}u_x^2 - \frac{1}{3}u_{xx}u$

Definition 1 The homotopy operator is a pair vector operator of $(\mathrm{H}_{u(x,t)}^{(x)}f, \mathrm{H}_{u(x,t)}^{(t)}f)$, where

$$\mathrm{H}_{u(x,t)}^{(x)}f = \int_0^1 \left(\sum_{j=1}^{q} \Upsilon_{u^j}^{(x)} f \right) [\kappa u] \frac{d\kappa}{\kappa}, \qquad \mathrm{H}_{u(x,t)}^{(t)}f = \int_0^1 \left(\sum_{j=1}^{q} \Upsilon_{u^j}^{(t)} f \right) [\kappa u] \frac{d\kappa}{\kappa}.$$

The x-integrand, $\Upsilon_{u^j(x,t)}^{(x)}f$ and the t-integrand, $\Upsilon_{u^j(x,t)}^{(t)}f$ are

$$\Upsilon_{u^j}^{(x)}f = \sum_{\iota_1=1}^{N_1^j} \sum_{\iota_2=0}^{N_2^j} \left(\sum_{r_1=0}^{\iota_1-1} \sum_{r_2=0}^{\iota_2} \mathbf{J}^{(x)} u_{x^{r_1}t^{r_2}}^j (-D_x)^{\iota_1-r_1-1} (-D_t)^{\iota_2-r_2} \right) \frac{\partial f}{\partial u_{x^{\iota_1}t^{\iota_2}}^j},$$

$$\Upsilon_{u^j}^{(t)}f = \sum_{\iota_1=0}^{N_1^j} \sum_{\iota_2=1}^{N_2^j} \left(\sum_{r_1=0}^{\iota_1} \sum_{r_2=0}^{\iota_2-1} \mathbf{J}^{(x)} u_{x^{r_1}t^{r_2}}^j (-D_x)^{\iota_1-r_1} (-D_t)^{\iota_2-r_2-1} \right) \frac{\partial f}{\partial u_{x^{\iota_1}t^{\iota_2}}^j},$$

where N_1^j, N_2^j are the orders of the derivatives u in x and t and

$$\mathbf{J}^{(x)} = \mathbf{J}(r_1, r_2, \iota_1, \iota_2) = \frac{C(r_1 + r_2, r_1)C(\iota_1 + \iota_2 - r_1 - r_2 - 1, \iota_1 - r_1 - 1)}{C(\iota_1 + \iota_2, \iota_1)}.$$

Also, $\mathbf{J}^{(t)} = \mathbf{J}(r_2, r_1, \iota_2, \iota_1)$. Table 5 shows the local conservation law multipliers for the gC-H equation.

4 μ-Symmetry method for the gC-H equation

Suppose that $\lambda_i : J^{(1)}M \longrightarrow \mathbb{R}$ and $\mu = \lambda_i\, dx^i$ is a horizontal one-form on $(M^{(1)}, \pi, M)$ which is compatible, i.e., $D_i\lambda_j - D_j\lambda_i = 0$ [22]. Let $\Delta(x, u^{(k)}) = 0$ be a scalar PDE of order n, involving r independent variables $x = (x^1, \dots, x^r)$ and one dependent variable $u = u(x^1, \dots, x^r)$. Suppose that $X = \sum_{i=1}^{p} \phi^i(x,u)\partial_{x^i} + \varphi(x,u)\partial_u$ is a vector field. Then the vector field $Y = X + \sum_{J=1}^{n} \chi^J \partial_{u_J}$ is a μ-prolongation of X on $M^{(k)}$, if for $\chi^0 = \varphi$, its coefficient satisfies the μ-prolongation formula, i.e.,

$$\chi^{J,i} = (D_i + \lambda_i)\chi^J - u_{J,m}(D_i + \lambda_i)\phi^m. \tag{2}$$

Let the solution manifold for Δ be denoted by $\mathcal{S}$. If $Y : \mathcal{S} \longrightarrow T\mathcal{S}$, we say that X is a μ-symmetry for Δ. In general, if $\mu = 0$, ordinary prolongation and ordinary symmetry are one and the calculation of both is similar. Now to obtain μ-symmetry of a system, apply Y to Δ, and restrict the obtained results to the solution manifold $\mathcal{S}_\Delta \subset M^{(k)}$ that will be up to ϕ, φ and λ_i. If we consider the λ_i as functions on $M^{(k)}$ and compatibility conditions between the λ_i, a system of all the dependence on u_J form the determining equation [22]. If X is a vector field on M, then $V = e^{(\int \mu)}X$ is an exponential vector field. X is a μ-symmetry for Δ if and only if V is a general symmetry for Δ.

Theorem 2 (Order reduction of PDEs under μ-symmetry method) $\Delta(x, u^k)$ *be a scalar PDE and* $X = \eta^i(x,u)\partial_{x^i} + \psi(x,u)\partial_u$ *be a vector field on* M, *with characteristic* $Q = \psi - u_i\eta^i$, *and let* Y *be the* μ-*prolong of order* n *of* X. *If* X *is a* μ-*symmetry for* Δ, *then* $Y : \mathcal{S}_X \longrightarrow T\mathcal{S}_X$, *where* $\mathcal{S}_X \subset M^{(k)}$ *is the solution manifold for the system* Δ_X *made of* Δ *and* $E_J := D_J Q = 0$ *for all* J *with* $|J| = 0, 1, \dots, n-1$ [22].

To calculate the μ-symmetry of Eq. (1), suppose we have a horizontal one-form $\mu = \lambda_1\,dx + \lambda_2\,dt$ such that $D_t\lambda_1 = D_x\lambda_2$ when $\Delta_u = 0$. Let $X = \eta\partial_x + \tau\partial_t + \psi\partial_u$ be a vector field on total space and Y be a μ-prolongation of order 3 of X. So,

$$Y = X + \Psi^x\partial_{u_x} + \Psi^t\partial_{u_t} + \Psi^{xx}\partial_{u_{xx}} + \cdots + \Psi^{ttt}\partial_{u_{ttt}},$$

where the coefficients Y are

$$\begin{aligned}
&\Psi^x = (D_x + \lambda_1)\psi - u_x(D_x + \lambda_1)\eta - u_t(D_x + \lambda_1)\tau,\\
&\Psi^t = (D_t + \lambda_2)\psi - u_x(D_t + \lambda_2)\eta - u_t(D_t + \lambda_2)\tau,\\
&\quad\dots.
\end{aligned} \tag{3}$$

Applying Y to Eq. (1) and combining terms, we obtain the following system:

$$\begin{aligned}
&-3\xi_u u^p = 0,\\
&2\tau_u = 0,\\
&4\tau_{uu} = 0,\\
&-3u^p\tau_u + 3\xi_u = 0.
\end{aligned} \tag{4}$$

Suppose that λ_1 and λ_2 are any choice of the type

$$\lambda_1 = D_x\bigl[M(x,t)\bigr] + N(x), \qquad \lambda_2 = D_t\bigl[M(x,t)\bigr] + P(t),$$

and $M(x,t)$, $N(x)$ and $P(t)$ are arbitrary functions. For example, taking $N(x) = 0$, $P(t) = 0$ and $M(x,t) = -\ln(G(x,t))$ into λ_1 and λ_2, then substituting $\lambda_1 = -G_x(x,t)/G(x,t)$ and $\lambda_2 = -G_t(x,t)/G(x,t)$ and solving (4), we deduce that $\eta = G(x,t)$, $\tau = 0$, $\psi = 0$, where $G(x,t)$ is an arbitrary positive function. Hence, the vector field $X = G(x,t)\partial_x$ is a μ-symmetry of Eq. (1) and the vector field $V = \exp(\int \lambda_1\,dx + \lambda_2\,dt)X = \exp(\int -\frac{G_x(x,t)}{G(x,t)}\,dx - \frac{G_t(x,t)}{G(x,t)}\,dt)X$, is a general symmetry of exponential type corresponds to X. Now, using Theorem 2, the order reduction of Eq. (1) is $Q = \psi - \eta u_x - \tau u_t = -G(x,t)u_x$. Other modes are presented in Tables 6 and 7.

Table 6 μ-symmetry of Eq. (1)

$N(x)$, $P(t)$	λ_1, λ_2	μ-symmetry
$N(x)=0$	$\lambda_1 = -\frac{G_x(x,t)}{G(x,t)}$	$X = G(x,t)\partial_x$
$P(t)=0$	$\lambda_2 = -\frac{G_t(x,t)}{G(x,t)}$	
$N(x)=0$	$\lambda_1 = -\frac{G_x(x,t)}{G(x,t)}$	$X = G(x,t)(\partial_t + \frac{u}{c_1-pt}\partial_u)$
$h(t)=\frac{p}{pt-c_1}$	$\lambda_2 = -\frac{G_t(x,t)}{G(x,t)} + \frac{p}{pt-c_1}$	
$N(x)=0$	$\lambda_1 = -\frac{G_x(x,t)}{G(x,t)}$	$X = G(x,t)(\frac{-1}{c_1t+c_2}\partial_x$
$P(t)=\frac{c_1}{c_1t+c_2}$	$\lambda_2 = -\frac{G_t(x,t)}{G(x,t)} + \frac{c_1}{c_1t+c_2}$	$+ \partial_t - \frac{c_1u}{(c_1t+c_2)p}\partial_u)$

Table 7 Order reduction of Eq. (1)

Symmetry of exponential type	Order reduction
$V = \exp(\int -\frac{G_x(x,t)}{G(x,t)}\,dx - \frac{G_t(x,t)}{G(x,t)}\,dt)X$	$-G(x,t)u_x = 0$
$V = \exp(\int -\frac{G_x(x,t)}{G(x,t)}\,dx - (\frac{G_t(x,t)}{G(x,t)} - \frac{p}{pt-c_1})\,dt)X$	$\frac{-G(x,t)}{pt-c_1}((pt-c_1)u_t + u) = 0$
$V = \exp(\int -\frac{G_x(x,t)}{G(x,t)}\,dx - (\frac{G_t(x,t)}{G(x,t)} - \frac{c_1}{c_1t+c_2})\,dt)X$	$\frac{-G(x,t)}{(c_1t+c_2)p}((c_1t+c_2)pu_t + c_1u - pu_x) = 0$

5 μ-Conservation laws of the gC-H equation

Let $\mu = \lambda_i\,dx^i$ be a horizontal one-form such that $D_i\lambda_j = D_j\lambda_i$. A μ-conservation law is

$$(D_i + \lambda_i)P^i = 0,$$

where the μ-conserved vector P^i is a matrix-valued M-vector.

Theorem 3 ([12]) *Let $\mathcal{L}(x,u^n)$ represent the nth order Lagrangian. X is a μ-symmetry for $\mathcal{L}$, in other words, $Y[\mathcal{L}] = 0$ if and only if there exists M-vector P^i such that $(D_i + \lambda_i)P^i = 0$.*

To calculate the P^i, let $\mathcal{L}(x,u^2)$ be a second order Lagrangian, and the vector field $X = \psi(\partial/\partial u)$ be a μ-symmetry for $\mathcal{L}$. Then the M-vector P^i is obtained as follows [12]:

$$P^i := \psi\frac{\partial\mathcal{L}}{\partial u_i} + \big((D_j + \lambda_j)\psi\big)\frac{\partial\mathcal{L}}{\partial u_{ij}} - \psi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}}. \tag{5}$$

The Frechet derivative of a system is self-adjoint, i.e., $D^*_\Delta = D_\Delta$ iff the system accepts a variational formulation [7].

Theorem 4 *Let $\Delta = 0$ be a PDE and $\mathfrak{L} = \int L\,dx$ is a variational problem. Then $\Delta = E(L)$, i.e., Δ is the Euler–Lagrange expression for L, iff the Frechet derivative D_Δ is self-adjoint. Also, using the homotopy formula $L[u] = \int_0^1 u.\Delta(\kappa u)\,d\kappa$, the Lagrangian for Δ can be precisely constructed.*

The gC-H equation (Δ_u) is of odd order and its Frechet derivative is

$$\begin{aligned} D_{\Delta_u} &= -\frac{1}{2}(p+1)(p+2)u^{p-1}u_x + \frac{1}{2}p(p-1)u^{p-3}u_x^3 + 2pu^{p-2}u_xu_{xx} \\ &\quad + u^{p-1}u_{xxx} + \left(\frac{-1}{2}(p+1)(p+2)u^p + \frac{1}{2}p(p-1)u^{p-2}u_x^2 + 2pu^{p-1}u_{xx}\right)D_x \\ &\quad + 2pu^{p-1}u_xD_x^2 + u^pD_x^3 + D_t - D_x^2D_t. \end{aligned}$$

Note that $D^*_{\Delta_u} \neq D_{\Delta_u}$. So, Δ_u does not admit a variational problem. The gC-H equation in potential form Δ_v obtained by the well-known differential substitution $u = v_x$ and its Frechet derivative are

$$\begin{aligned}
\Delta_v &= v_{xt} - v_{txxx} - \frac{1}{2}(p+1)(p+2)v_x^p v_{xx} + \frac{1}{2}p(p-1)v_x^{p-2}v_{xx}^3 \\
&\quad + 2pv_x^{p-1}v_{xx}v_{xxx} + v_x^p v_{xxxx} = 0, \\
D_{\Delta_v} &= D_xD_t - D_x^3D_t + \biggl(-\frac{1}{2}(p+1)(p+2)v_x^{p-1}v_{xx} + \frac{1}{2}p(p-1)v_x^{p-3}v_{xx}^3 \\
&\quad + 2pv_x^{p-2}v_{xx}v_{xxx} + v_x^{p-1}v_{xxxx}\biggr)D_x + \biggl(-\frac{1}{2}(p+1)(p+2)v_x^p \\
&\quad + \frac{1}{2}p(p-1)v_x^{p-2}v_{xx}^2 + 2pv_x^{p-1}v_{xxx}\biggr)D_x^2 + 2pv_x^{p-1}v_{xx}D_x^3 + v_x^pD_x^4,
\end{aligned}$$

which is self-adjoint and the Lagrangian of Δ_v is

$$\begin{aligned}
\mathcal{L}[v] &= \int_0^1 v.\Delta_v(\kappa v)\,d\kappa = -\frac{1}{2}\bigl(v_xv_t + v_{xx}v_{xt} - v_x^pv_{xx}^2 - v_x^{p+2}\bigr) + \operatorname{Div} P. \\
&= -\frac{1}{2}\bigl(v_xv_t + v_{xx}v_{xt} - v_x^pv_{xx}^2 - v_x^{p+2}\bigr).
\end{aligned}$$

For calculation of the μ-conservation law of $\Delta_v = E(\mathcal{L}[v])$, let $X = \psi\,\partial_v$ be a vector field for $\mathcal{L}[v]$, and $\mu = \lambda_1\,dx + \lambda_2\,dt$ be a horizontal one-form such that $D_t\lambda_1 = D_x\lambda_2$ when $\Delta_v = 0$. Now, using (2), Y and its coefficients are

$$\begin{aligned}
&Y = \psi\,\partial_v + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} + \Psi^{xx}\partial_{v_{xx}} + \Psi^{xt}\partial_{v_{xt}}/ + \Psi^{tt}\partial_{v_{tt}}, \\
&\Psi^x = (D_x + \lambda_1)\psi, \\
&\Psi^t = (D_t + \lambda_2)\psi, \\
&\Psi^{xx} = (D_x + \lambda_1)\Psi^x, \\
&\Psi^{xt} = (D_t + \lambda_2)\Psi^x, \\
&\Psi^{tt} = (D_t + \lambda_2)\Psi^t.
\end{aligned}$$

Applying the μ-prolongation Y on the $\mathcal{L}[v]$, and substituting $(-v_{xx}v_{xt} + v_x^pv_{xx}^2 + v_x^{p+2})/v_x$ for v_t into it, we obtain the system

$$\begin{aligned}
&(1/2)p\psi_v = 0, \\
&-(1/2)\psi_{vv} = 0, \\
&(1/2)(\lambda_1\psi + \psi_x) = 0, \\
&\ldots.
\end{aligned} \tag{6}$$

Consider $\varphi = G(x,t)$, where $G(x,t)$ is an arbitrary positive function and $\mathcal{L}[v] = 0$. Now, a special solution of system (6) is

$$\lambda_1 = -\frac{G_x(x,t)}{G(x,t)}, \qquad \lambda_2 = -\frac{G_t(x,t)}{G(x,t)}. \tag{7}$$

Therefore, $X = G(x,t)\partial_v$ is a μ-symmetry for $\mathcal{L}[v]$ and there exists an M-vector P^i which is μ-conservation law, i.e., $(D_i + \lambda_i)P^i = 0$. The M-vector P^i for $\mathcal{L}[v]$ is obtained:

$$\begin{aligned} P^1 &= -\frac{1}{2}\big(v_t - 2v_{txx} - (p+2)v_x^{p+1} + pv_x^{p-1}v_{xx}^2 + 2v_x^p v_{xxx}\big)G(x,t), \\ P^2 &= -\frac{1}{2}v_x G(x,t). \end{aligned} \tag{8}$$

So, μ-conservation law for second order Lagrangian $\mathcal{L}[v]$ is the form $(D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 = 0$.

Corollary 2 *The μ-conservation law for the gC-H equation in potential form $\Delta_v = E(\mathcal{L}[v])$ is $D_xP^1 + D_tP^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0$, where P^1 and P^2 are the M-vectors P^i of* (8).

Remark 1 The gC-H equation in potential form Δ_v satisfies Noether's theorem for μ-symmetry and μ-conservation law, i.e.

$$\begin{aligned} (D_i + \lambda_i)P^i &= (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 \\ &= G(x,t)\bigg(v_{xt} - v_{txxx} - \frac{1}{2}(p+1)(p+2)v_x^p v_{xx} \\ &\quad + \frac{1}{2}p(p-1)v_x^{p-2}v_{xx}^3 + 2pv_x^{p-1}v_{xx}v_{xxx} + v_x^p v_{xxxx}\bigg) \\ &= QE\big(\mathcal{L}[v]\big). \end{aligned}$$

To compute the μ-conservation law of the gC-H equation Δ_u, we can use the gC-H equation in potential form Δ_v. The Δ_v corresponds to $D_x(v_t - v_{txx} - (1/2)(p+2)v_x^{p+1} + (1/2)pv_x^{p-1}v_{xx}^2 + v_x^p v_{xxx}) = 0$, or equivalently $v_t - v_{txx} - (1/2)(p+2)v_x^{p+1} + (1/2)pv_x^{p-1}v_{xx}^2 + v_x^p v_{xxx} = h(t)$, where $h(t)$ is an arbitrary function. If we substitute

$$h(t) + v_{txx} + (1/2)(p+2)v_x^{p+1} - (1/2)pv_x^{p-1}v_{xx}^2 - v_x^p v_{xxx}$$

for v_t and substitute u for v_x into (8), then we obtain the M-vectors P^1 and P^2 as follows:

$$\begin{aligned} P^1 &= -\frac{1}{2}\bigg(h(t) - u_{tx} - \frac{1}{2}(p+2)u^{p+1} + \frac{1}{2}pu^{p-1}u_x^2 + u^p u_{xx}\bigg)G(x,t), \\ P^2 &= -\frac{1}{2}uG(x,t). \end{aligned} \tag{9}$$

Corollary 3 *The μ-conservation law for the gC-H equation Δ_u is $D_xP^1 + D_tP^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0$, where P^1 and P^2 are the M-vectors P^i of* (9).

Remark 2 The gC-H equation Δ_u satisfies the characteristic form, i.e.,

$$\begin{aligned} (D_i + \lambda_i)P^i &= (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 \\ &= F(x,t)\bigg(u_t - u_{txx} - \frac{1}{2}(p+1)(p+2)u^p u_x \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}p(p-1)u^{p-2}u_x^3+2pu^{p-1}u_xu_{xx}+u^pu_{xxx}\Big)\\
&=Q\Delta_u.
\end{aligned}
$$

6 Conclusion

We attempted to study the generalized Camassa–Holm equation using the symmetry approach in this paper. The general form of an infinitesimal generator admitted for the generalized Camassa–Holm equation and transformed solutions are calculated. Then the optimal system, one-dimensional subalgebras, Lie invariants, reduced equations, differential invariants and the conservation laws corresponding to the infinitesimal symmetries of the gC-H equation are obtained. In the second part, μ-symmetry and order reduction, Lagrangian of generalized Camassa–Holm equation in potential form and μ-conservation laws of the gC-H equation are calculated. In [29], by defining the new variable, the symmetries of the generalized Camassa–Holm equation are obtained. Also in [30–32], the equations derived from the generalized Camassa–Holm equation are discussed in several ways.

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