Adaptive Neural Control of a Class of Uncertain State and Input Delayed Systems with Input Magnitude and Rate Constraints

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***Abstract*—This paper aims at proposing an adaptive neural control strategy for a class of nonlinear time-delay systems with input delays and unknown control directions. Different from previous researches that investigated delays and constraints separately, the novelty of this paper lies in that it simultaneously considers delays (state and input delays) and input constraints (magnitude and rate constraints) for a class of uncertain nonlinear systems. In this paper, the uncertain states and input delays are handled by integrating a constructed auxiliary system that functions as an observer with neural networks (NNs), with which the adverse effects caused by the uncertain states and input delays can be approximated and compensated. By involving smooth hyperbolic tangent functions in the designed auxiliary system, the problem of magnitude and rate constraints of the control input is fully addressed. Then the backstepping technique runs through the entire control designing process, which allows the designed adaptive neural control strategy to handle the input constraints and delays at the same time. Furthermore, Nussbaum functions are employed to resolve the problem of unknown control directions. Due to introducing an input-driven filter, only the output of the system is required to be measured as the control feedback, which promotes the applicability of the designed controller. Under the proposed control scheme, semi-global, uniform and ultimate boundedness of all signals of the closed-loop sytem is realized with uncertain control directions, input and state delays, and guaranteed magnitude and rate constraints of control inputs. Finally, simulation results are illustrated to verify the effectiveness of the presented control method.**

***Index Terms*—State and input delays; input constraints; unknown control direction; neural networks.**

I. Introduction

Due to inevitable environmental perturbations, signal transmission, limited devices, and non-standard operation, time delay arises in almost all engineering control systems, especially the systems in the biological field, such as combustion systems and physical networks, which may deteriorate the control performance and result in system instability. Therefore, developing proper control algorithms to deal with the time-delay problem is increasingly explored over the past few decades [1]–[3].

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In control process, it generally takes time to send control signals to actuators and feedback signals to sensors, which causes the input and state delays of the system. Consequently, time delay is mainly embodied in the transmission of input signals and state signals of the control systems. A variety of control methodologies have been contemplated for stabilizing heterogeneous systems with state delay, such as TS fuzzy time-delay systems, markovian jump systems, switched delay systems, stochastic time-delay systems, etc., to avoid quenching phenomenon mentioned by [4]. Besides, various researches have been focused on systems that suffer from constant delay of control signals [5]–[7]. Based on the assumption of delay invariance, control laws designed in [5]–[7] are greatly simplified. However, in practice, input delay of physical systems is more likely to vary over time. Motivated by such practical challenges, several control schemes taking into account time-dependent variation of input delay are presented. In [8]–[10], Pade approximation method is adopted to deal with the time delay of the control input by converting the input delay into a compensation system. However, the disadvantage of this method lies in that it is only effective for short-time input delay and the approximation is not satisfied when the delay time is longer than 0.1s. To overcome this shortcoming, a compensation system is proposed in [11] and a control scheme is addressed using backstepping technique to settle the case of long input delay. Although some control strategies have been formed for compensating the negative effect of input and state delays of nonlinear systems [12]–[14], they are all constructed based on the presupposition of known control directions. In such condition, the application of researches in [12]–[14] is greatly restricted. In practical industrial engineering, systems with unknown control directions are frequently encountered and have extremely wide applicability. Consequently, studies on control design of nonlinear systems with uncertain control directions and time delays on control inputs and states are still inadequate and need to be thrashed out.

Due to hardware limitations of actuators and realistic restrictions of inputs, input constraint is ubiquitous in real systems. Without being handled properly, input constraint may give rise to degraded control precision or even system damage. For this reason, the influence of input magnitude constraint is concerned in the control design in [15]–[17], in which saturation or hyperbolic tangent functions are generally utilized to describe the saturated nonlinearity of the control input. Besides magnitude, varying rate of control signal is also a popular concern in control design for safety use of actuators.

Once the actual varying rate of the control input exceeds the limited value, fatigue damage is prone to be caused and undesired oscillation may be induced, which will deteriorate the control performance and even lead to a catastrophic accident especially for systems with high-performance requirements, such as aircrafts and spacecrafts. For these reasons, many researches have devoted a great deal of attention to dealing with the input rate constraints for control systems. On the basis of [15]–[18], in addition to magnitude, varying rate of input is also limited by priori-fixed constants using control strategies designed in [19] and [20].

Although a considerable amount of effort has been made above, the control problem for nonlinear input and state delayed systems considering input constraint is rarely reported. Several remarkable researches on saturated control design for nonlinear systems subject to input delay are discussed in [21] and [22]. However, control methods in [21] and [22] only take into consideration the magnitude constraint of the input signal for the system with input delay, regardless of the input rate constraint and the state delay. Consequently, there is a tremendous need to design a control method that can stabilize nonlinear input and state delayed systems with both input magnitude and rate limits.

Inspired by the deficiency of the above-mentioned studies, an adaptive neural output feedback control scheme is concerned for a class of nonlinear time-delay systems with unknown control directions, input delays, and constraints of both input magnitude and rate. The undesired nonlinearity and uncertainties are approximated and compensated by NNs so that the ultimately uniform boundedness of system states and system stability are ensured under the developed control. Major contributions of this paper are addressed as:

i) Different from previous researches, which focused on dealing with input and state delays but neglected input constraints or only considered the magnitude limit of the control input, an auxiliary system functioning as an observer is introduced and combined with NNs to handle the state and input delays of a class of nonlinear systems. Hyperbolic tangent function is employed for strictly constraining the magnitude and rate of input signals as anticipated. To make the designed control has a desired capability of simultaneously handling input constraints and delays, backstepping technology that is widely used in the control field is carried out during the entire control design process;

ii) Uncertain control direction of the nonlinear system studied in this paper and disturbance are allowed due to the utilization of Nussbaum functions and NNs;

iii) The control scheme designed in this paper is independent of the state information of the nonlinear system under study, which is realized by the developed input-driven filter with state estimation. In such case, only the output signals of the system are required in the control design, which renders easier control implementation.

Since the designed adaptive neural controller has versatile functions and is able to deal with delays of inputs and states, magnitude and rate constraints of inputs, unknown control directions and disturbances for a class of generalized nonlinear systems, it has a broad application prospect in the control area.

This paper is organized as follows. The problem is expounded in Section II. Section III proposes an adaptive neural control for a class of uncertain state and input delayed systems with input magnitude and rate constraints, followed by simulation results which are included in Section IV. Concluding remarks are presented in Section V.

Notations 1: In this paper, $\underline{\lambda}(\cdot)$ and $\bar{\lambda}(\cdot)$ denote minimum and maximum eigenvalues of matrix $(\cdot)$, respectively. Besides, $\|*\|$ denotes the Euclidean norm of vector $*$.

II. Problem Formulation and Preliminaries

A. Control Problem for Nonlinear State and Input Delayed Systems with Unknown Control Directions

Consider the following uncertain finite n-dimensional nonlinear systems with time-varying delays in both state and input

$$\begin{cases} \dot{x}_i(t) = \mu_i x_{i+1}(t) + \omega_i(\bar{\boldsymbol{x}}_i(t)) + d_i(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\bar{\tau}_i(t))), \\ \qquad i = 1, \cdots n-1 \\ \dot{x}_n(t) = \mu_n u(t-\tau(t)) + \omega_n(\bar{\boldsymbol{x}}_n(t)) + d_n(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\bar{\tau}_n(t))) \\ \qquad y(t) = x_1(t) \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) = [x_1(t), \cdots, x_n(t)]^{\mathrm{T}} \in \mathbb{R}^{\mathrm{n}}$ is the state vector of the above nonlinear system; $\bar{\boldsymbol{x}}_i(t) = [x_1(t), \cdots, x_i(t)]^{\mathrm{T}} \in \mathbb{R}^{\mathrm{i}}$, $i = 1, \cdots, n$, and $\bar{\boldsymbol{x}}_n(t) = \boldsymbol{x}(t)$; $\omega_i(\bar{\boldsymbol{x}}_i(t))$ and $d_i(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\bar{\tau}_i(t)))$ are unknown smooth nonlinear functions with $\omega_i(0) = 0$, which respectively denote the state-related modeling uncertainty and environmental disturbance; $u(t-\tau(t))$ and $y(t)$ are delayed input and output of the system; $\tau(t)$ is an unknown non-negative time-varying delay of the control input and $\bar{\tau}_i(t)$ is an unknown non-negative delay function of the ith subsystem of $\boldsymbol{x}(t)$; $\mu_i(t)$ is a bounded uncertain non-zero time-varying parameter with an unknown sign and $\mu_n(t)$ represents the control direction.

In the following formulation, (t) in the variable is omitted for the readability, i.e., $(*)(t)$ is abbreviated as $(*)$, and $(*)(t-\tau(t))$ as well as $(*)(t-\bar{\tau}_i(t))$ is abbreviated as $(*)(t-\tau)$ and $(*)(t-\bar{\tau}_i)$, respectively.

The control goal of this paper lies in proposing a NN control algorithm such that the system output y can be stabilized and all signals of the closed-loop control system are uniformly and ultimately bounded.

Remark 1: The nonlinear system in Eq. (1) with unknown control coefficients has significant applications in actual engineering, such as robot manipulators [23] and vehicle motion systems [24]. Besides, it has a more general form than systems in [25], [26] and can represent nonlinear systems with unknown control directions (uncertain sign of μ_n) and degradation of actuator (uncertain value of μ_n) [27] which commonly exist in industry.

Assumption 1: [28] The nonlinear function $d_i(\cdot)$ in Eq. (1) meets

$$|d_i(t, \boldsymbol{x}, \boldsymbol{x}(t-\bar{\tau}_i))| \leq \widehat{\ell}_{i1}(x_1) + \widehat{\ell}_{i2}(x_1(t-\bar{\tau}_i)) \tag{2}$$

for $i=1,\cdots,n$, in which $\widehat{\ell}_{i1}(\cdot)$ and $\widehat{\ell}_{i2}(\cdot)$ are positive functions satisfying $\widehat{\ell}_{i1}(x_1)=x_1\breve{\ell}_{i1}(x_1)$ and $\widehat{\ell}_{i2}(x_1)=x_1\breve{\ell}_{i2}(x_1)$ where $\breve{\ell}_{i1}(\cdot)$ and $\breve{\ell}_{i2}(\cdot)$ are unknown functions.

Assumption 1 is common in many researches on control design of time-delay systems [29], [30]. To make Assumption 1 more general, a positive constant term D_i is added to the upper bound of $|d_i(t,\boldsymbol{x},\boldsymbol{x}(t-\bar{\tau}_i))|$, in which $\boldsymbol{x}$ is used to replace $\boldsymbol{r}$. Thus inequality (2) can be rewritten as

$$|d_i(t,\boldsymbol{x},\boldsymbol{x}(t-\bar{\tau}_i))|\leq\widehat{\ell}_{i1}(x_1)+\widehat{\ell}_{i2}(x_1(t-\bar{\tau}_i))+D_i \tag{3}$$

such that

$$\begin{aligned}&|d_i(t,\boldsymbol{x},\boldsymbol{x}(t-\bar{\tau}_i))|^2\\&\leq 3\widehat{\ell}^2_{i1}(y)+3\widehat{\ell}^2_{i2}(y(t-\bar{\tau}_i))+3D_i^2\\&=3y^2\breve{\ell}^2_{i1}(y)+3y^2(t-\bar{\tau}_i)\breve{\ell}^2_{i2}(y(t-\bar{\tau}_i))+3D_i^2\end{aligned} \tag{4}$$

which is used in the following proof of system stability. In reality, when $d_i(\cdot)$ is time-invariant and only related to y and $y(t-\bar{\tau}_i)$ [31], Assumption 1 always holds. In this case, D_i can be regarded as zero and inequality (3) is equivalent to inequality (2).

Assumption 2: The plant in Eq. (1) under input constraints is controllable.

Remark 2: In actual engineering, the information of upper bound functions $\widehat{\ell}_{i1}(y)$, $\widehat{\ell}_{i2}(y(t-\bar{\tau}_i))$, and D_i in inequality (4) may be unclear and hard to obtain. Thus NN is applied in the following control design to approximate these functions and the control scheme is constructed without concerning detailed information of $\widehat{\ell}_{i1}(y)$, $\widehat{\ell}_{i2}(y(t-\bar{\tau}_i))$, $\breve{\ell}_{i1}(y)$, $\breve{\ell}_{i2}(y(t-\bar{\tau}_i))$, and D_i.

Assumption 3: [13] The state delay function $\bar{\tau}_i$ is bounded by $|\bar{\tau}_i|\leq\tau^*$ for $i=1,\cdots n$, in which τ^* is a positive constant. Besides, its time derivative is constrained by $\left|\dot{\bar{\tau}}_i\right|\leq\widehat{\tau}$ in which $\widehat{\tau}>0$.

B. System Transformation for Control Design

To facilitate control design, define

$$\mu=\mu_1\mu_2\cdots\mu_n \tag{5}$$

and

$$\mu_{in}=\mu_i\mu_{i+1}\cdots\mu_n, \tag{6}$$

then the nonlinear system in Eq. (1) can be transformed into

$$\begin{cases}\dot{\zeta}_i=\zeta_{i+1}+g_i(\bar{\zeta}_i)+h_i(t,\zeta,\zeta(t-\bar{\tau}_i)),\ i=1,\cdots n-1\\ \dot{\zeta}_n=u(t-\tau)+g_n(\bar{\zeta}_n)+h_n(t,\zeta,\zeta(t-\bar{\tau}_n))\\ y=\mu\zeta_1\end{cases} \tag{7}$$

where $\zeta_i=\mu_{in}^{-1}x_i$, $\bar{\zeta}_i=[\zeta_1,\cdots,\zeta_i]^{\mathrm{T}}\in\mathbb{R}^{\mathrm{i}}$, $\zeta=\bar{\zeta}_n=[\zeta_1,\cdots,\zeta_n]^{\mathrm{T}}\in\mathbb{R}^{\mathrm{n}}$, $g_i(\bar{\zeta}_i)=\mu_{in}^{-1}\omega_i(f(\bar{\zeta}_i))$, $f(\bar{\zeta}_i)=[\mu_{1n}\zeta_1,\cdots,\mu_{in}\zeta_i]^{\mathrm{T}}\in\mathbb{R}^{\mathrm{i}}$, $h_i(t,\zeta,\zeta(t-\bar{\tau}_i))=\mu_{in}^{-1}d_i(t,f(\zeta),f(\zeta(t-\bar{\tau}_i)))$, $f(\zeta)=[\mu_{1n}\zeta_1,\cdots,\mu_{nn}\zeta_n]^{\mathrm{T}}\in\mathbb{R}^{\mathrm{n}}$, and $f(\zeta(t-\bar{\tau}_i))=[\mu_{1n}\zeta_1(t-\bar{\tau}_i),\cdots,\mu_{nn}\zeta_n(t-\bar{\tau}_i)]^{\mathrm{T}}\in\mathbb{R}^{\mathrm{n}}$, $i=1,\cdots,n$.

According to inequality (4), the following inequality holds

$$\begin{aligned}&|h_i(t,\zeta,\zeta(t-\bar{\tau}_i))|^2\\&\leq 3\breve{\mu}_i\widehat{\ell}^2_{i1}(y)+3\breve{\mu}_i\widehat{\ell}^2_{i2}(y(t-\bar{\tau}_i))+3\breve{\mu}_iD_i^2\\&=3\breve{\mu}_iy^2\breve{\ell}^2_{i1}(y)+3\breve{\mu}_iy^2(t-\bar{\tau}_i)\breve{\ell}^2_{i2}(y(t-\bar{\tau}_i))+3\breve{\mu}_iD_i^2\end{aligned} \tag{8}$$

where $\breve{\mu}_i$ is a constant and larger than the upper bound of μ_{in}^{-2} which exists due to the boundedness of μ_i.

Since the transformed system in Eq. (7) is equivalent to the previous system in Eq. (1), Eq. (7) is applied to design the controller in following sections.

C. NN Approximation

Given the universal approximation property of radial basis function neural network (RBFNN), it is employed in this paper.

For any uncertain continuous function $\vartheta(\boldsymbol{X}):\mathbb{R}^{\mathrm{p}}\to\mathbb{R}$ defined over a compact set $\Omega_x\subseteq\mathbb{R}^{\mathrm{p}}$ and specified accuracy $\bar{\varepsilon}>0$, an RBFNN ϑ_{nn} given as

$$\vartheta_{nn}=\boldsymbol{W}^{\mathrm{T}}\boldsymbol{S}(\boldsymbol{X}) \tag{9}$$

where $\boldsymbol{W}\in\mathbb{R}^{\mathrm{m}}$ and $m\geq 1$ is the node number, such that

$$\vartheta(\boldsymbol{X})=\boldsymbol{W}^{*\mathrm{T}}\boldsymbol{S}(\boldsymbol{X})+\sigma(\boldsymbol{X}),\boldsymbol{X}\in\Omega_x\subseteq\mathbb{R}^{\mathrm{p}} \tag{10}$$

where $\sigma(\boldsymbol{X})$ is the approximation error bounded by $|\sigma(\boldsymbol{X})|\leq\bar{\varepsilon}$; $\boldsymbol{W}^*\in\mathbb{R}^{\mathrm{m}}$ is an idealized weight vector and $\boldsymbol{S}(\boldsymbol{X})=[s_1(\boldsymbol{X}),\cdots,s_m(\boldsymbol{X})]^{\mathrm{T}}\in\mathbb{R}^{\mathrm{m}}$ is a Gaussian basis function vector; $\boldsymbol{W}^*$ and s_i are respectively defined as

$$\boldsymbol{W}^*:=\arg\min_{\boldsymbol{W}\in\mathbb{R}^{\mathrm{m}}}\left\{\sup_{\boldsymbol{X}\in\Omega_x}\left|\vartheta(\boldsymbol{X})-\boldsymbol{W}^{\mathrm{T}}\boldsymbol{S}(\boldsymbol{X})\right|\right\} \tag{11}$$

$$s_i(\boldsymbol{X}):=\exp\left[-\frac{(\boldsymbol{X}-\boldsymbol{q}_i)^{\mathrm{T}}(\boldsymbol{X}-\boldsymbol{q}_i)}{\varsigma_i^2}\right] \tag{12}$$

in which $\boldsymbol{q}_i=[q_{i1},\cdots,q_{ip}]^{\mathrm{T}}\in\mathbb{R}^{\mathrm{p}}$ is the center of the receptive field and ς_i, $i=1,\cdots,m$, is the width of the Gaussian function.

D. Nussbaum Function

If a function $N(\nu)$ has following properties

$$\lim_{s\to\pm\infty}\sup\frac{1}{s}\int_0^s N(\nu)\mathrm{d}\nu=\infty \tag{13}$$

$$\lim_{s\to\pm\infty}\inf\frac{1}{s}\int_0^s N(\nu)\mathrm{d}\nu=-\infty, \tag{14}$$

it can be regarded as a Nussbaum function.

Many functions meet the conditions mentioned by Eqs. (13) and (14), e.g., $e^{\nu^2}\cos\nu$, $\nu^2\cos\nu$, $\nu^2\sin\nu$, $\cosh\nu\sin\nu$, etc. Thus they all belong to Nussbaum functions which are generally used to handle unknown control dynamics, such as unknown control directions, uncertainties, and unpredictable disturbances. In this paper, the function of the Nussbaum function lies in handling the uncertain control direction of the investigated nonlinear system.

Lemma 1: [32] Define ν and $\widehat{V}$ as two smooth functions over $[0,t_f)$ with $\widehat{V} \geq 0$ and $\forall t \in [0,t_f)$. If following conditional inequality holds

$$\widehat{V} \leq e^{-bt}\int_0^t \kappa(s)N(\nu)\dot{\nu}e^{bs}\mathrm{d}s + e^{-bt}\int_0^t \dot{\nu}e^{bs}\mathrm{d}s + \mathrm{c} \quad (15)$$

where $b>0$, c is an appropriate constant, κ is time-varying and takes values in uncertain closed intervals $\Delta := \begin{bmatrix} T^- & T^+ \end{bmatrix}$ with $0 \notin \Delta$, then V, ν and $\int_0^t \kappa(s)N(\nu)\dot{\nu}\mathrm{d}s$ are bounded over $[0,t_f)$.

III. Adaptive Neural Control Using Backstepping Approach

An adaptive neural control strategy is constructed for the uncertain system governed by Eq. (7) with time delays and input constraints. The control flow chart is given in Fig. 1.

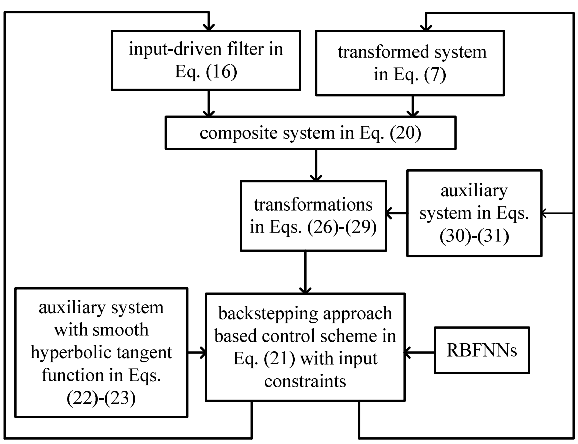


Fig. 1. Control flow chart.

A. Input-Driven Filter Design

An input-driven filter is first presented for the following control design

$$\begin{cases} \dot{\hat{\zeta}}_i = \hat{\zeta}_{i+1} - l_i\hat{\zeta}_1, \ i = 1,\cdots,n-1 \\ \dot{\hat{\zeta}}_n = u(t-\tau) - l_n\hat{\zeta}_1 \end{cases} \quad (16)$$

in which $\hat{\zeta}_i$, $i=1,\cdots,n$, is the estimated value of the system state and l_i is a positive constant.

Define the estimated error as $e_i = \zeta_i - \hat{\zeta}_i$ and then the error system is attained using Eqs. (7) and (16) as

$$\begin{cases} \dot{e}_i = -l_ie_1 + e_{i+1} + l_i\zeta_1 + g_i(\bar{\zeta}_i) + h_i(t,\zeta,\zeta(t-\bar{\tau}_i)), \\ \qquad i = 1,\cdots,n-1 \\ \dot{e}_n = -l_ne_1 + g_n(\bar{\zeta}_n) + h_n(t,\zeta,\zeta(t-\bar{\tau}_n)) + l_n\zeta_1 \end{cases} \quad (17)$$

which can be reexpressed as

$$\dot{\boldsymbol{e}} = \boldsymbol{A}_c\boldsymbol{e} + \boldsymbol{G} + \boldsymbol{H} \quad (18)$$

where $\boldsymbol{e} = [e_1,\cdots,e_n]^{\mathrm{T}} \in \mathbb{R}^{\mathrm{n}}$, $\boldsymbol{G} = [g_1(\bar{\zeta}_1) + l_1\zeta_1,\cdots,g_n(\bar{\zeta}_n) + l_n\zeta_1]^{\mathrm{T}} \in \mathbb{R}^{\mathrm{n}}$, $\boldsymbol{H} = [h_1(t,\zeta,\zeta(t-\bar{\tau}_1)),\cdots,h_n(t,\zeta,\zeta(t-\bar{\tau}_n))]^{\mathrm{T}} \in \mathbb{R}^{\mathrm{n}}$, and

$$\boldsymbol{A}_c = \begin{bmatrix} -l_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_{n-1} & 0 & \cdots & 1 \\ -l_n & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}.$$

Choose l_i appropriately to make $\boldsymbol{A}_c$ be a strict Hurwitz matrix such that there exists a symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ for any given symmetric positive definite matrix $\boldsymbol{Q} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ satisfying

$$\boldsymbol{A}_c^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_c = -\boldsymbol{Q}. \quad (19)$$

Then the composite system can be written as

$$\begin{cases} \dot{\boldsymbol{e}} = \boldsymbol{A}_c\boldsymbol{e} + \boldsymbol{G} + \boldsymbol{H} \\ \dot{\hat{\zeta}}_i = \hat{\zeta}_{i+1} - l_i\hat{\zeta}_1, \ i = 1,\cdots,n-1 \\ \dot{\hat{\zeta}}_n = u(t-\tau) - l_n\hat{\zeta}_1 \\ \dot{y} = \mu\zeta_2 + \mu g_1(\zeta_1) + \mu h_1(t,\zeta,\zeta(t-\bar{\tau}_1)). \end{cases} \quad (20)$$

Remark 3: In this paper, an input-driven is used for estimating the states of the system, and then only the filter state $\hat{\zeta}_i$ and the output y are employed in the following adaptive neural control scheme. Thus system states do not need to be measured in the following proposed control algorithm, which solves the immeasurable state problem and greatly improves the feasibility of the proposed control scheme.

B. NNs-Based Adaptive Control Design

In this subsection, the NNs-based adaptive control is p-resented for the system in Eq. (7) using the backstepping technique.

As the system under investigation is subject to input magnitude and rate constraints, the control law is proposed utilizing a smooth hyperbolic tangent function for satisfying both magnitude and rate limits as follows

$$u = \delta(\upsilon) = u_M \tanh\left(\frac{\upsilon}{u_M}\right) \quad (21)$$

with an auxiliary system

$$\dot{\upsilon} = v_M \tanh\left(\frac{\varpi}{v_M}\right)\left(\frac{\partial\delta}{\partial\upsilon}\right)^{-1} = \left(\frac{\partial\delta}{\partial\upsilon}\right)^{-1} f(\varpi) \quad (22)$$

$$\dot{\varpi} = \left(\frac{\partial f(\varpi)}{\partial\varpi}\right)^{-1}\bar{u} \quad (23)$$

where $f(\varpi) = v_M \tanh\left(\frac{\varpi}{v_M}\right)$, u_M and v_M are known positive constraints of input magnitude and rate respectively, and

$$0 < \overline{\delta} \leq \frac{\partial\delta}{\partial\upsilon} = \frac{4}{\left(e^{\upsilon/u_M} + e^{-\upsilon/u_M}\right)^2} \leq 1 \quad (24)$$

$$0 < \overline{\varpi} \leq \frac{\partial f(\varpi)}{\partial\varpi} = \frac{4}{\left(e^{\varpi/v_M} + e^{-\varpi/v_M}\right)^2} \leq 1, \quad (25)$$

where δ and ϖ are two positive constants.

Then following transformations are selected

$$z_1 = y \quad (26)$$

$$z_i = \hat{\zeta}_i - \eta_i - \alpha_{i-1} \tag{27}$$

$$z_{n+1} = \delta(\upsilon) - \alpha_n \tag{28}$$

$$z_{n+2} = f(\varpi) - \alpha_{n+1} \tag{29}$$

where η_i is the state of the following auxiliary system that functions as an observer

$$\dot{\eta}_i = \eta_{i+1} - \gamma_i \eta_i,\ i = 1, \cdots, n-1 \tag{30}$$

$$\dot{\eta}_n = -\gamma_n \eta_n + u(t-\tau) - u \tag{31}$$

in which γ_i and γ_n are positive constants, and α_1, α_i, α_{n+1}, $\bar{u}$ are virtual controls designed as

$$\begin{aligned}\alpha_1 =& N(\chi)\left[\left(\xi_1 + \frac{1}{2} + \widehat{p} + \frac{\hat{\omega}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1)}{2k_1}\right)z_1\right.\\ &\left.+\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)\right]\end{aligned} \tag{32}$$

$$\alpha_i = -z_i\left[\left(\xi_i + \frac{1}{2}\right) + \frac{\hat{\omega}\boldsymbol{S}_i^{\mathrm{T}}(\boldsymbol{X}_i)\boldsymbol{S}_i(\boldsymbol{X}_i)}{2k_i}\right],\ i = 2, \cdots, n \tag{33}$$

$$\begin{aligned}\alpha_{n+1} =& -\left(\xi_{n+1} + \frac{1}{2}\right)z_{n+1} - z_n\\ &- \frac{\hat{\omega}\boldsymbol{S}_{n+1}^{\mathrm{T}}(\boldsymbol{X}_{n+1})\boldsymbol{S}_{n+1}(\boldsymbol{X}_{n+1})z_{n+1}}{2k_{n+1}}\end{aligned} \tag{34}$$

$$\begin{aligned}\bar{u} =& -\left(\xi_{n+2} + \frac{1}{2}\right)z_{n+2} - z_{n+1}\\ &- \frac{\hat{\omega}\boldsymbol{S}_{n+2}^{\mathrm{T}}(\boldsymbol{X}_{n+2})\boldsymbol{S}_{n+2}(\boldsymbol{X}_{n+2})z_{n+2}}{2k_{n+2}}\end{aligned} \tag{35}$$

with

$$\begin{aligned}\dot{\chi} =& \left(\xi_1 + \frac{1}{2} + \widehat{p}\right)z_1^2 + z_1\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)\\ &+ \frac{\hat{\omega}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1)}{2k_1}z_1^2\end{aligned} \tag{36}$$

$$\dot{\boldsymbol{W}}_z = \hbar\boldsymbol{S}_z(z_1)z_1 - \iota\hbar\boldsymbol{W}_z \tag{37}$$

$$\dot{\hat{\omega}} = \sum_{j=1}^{n+2}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j) - c_\omega\hat{\omega} \tag{38}$$

where $N(\chi) = \chi^2\cos\chi$, $\boldsymbol{W}_z \in \mathbb{R}^{\mathrm{m_z}}$, $\boldsymbol{S}_z(z_1) \in \mathbb{R}^{\mathrm{m_z}}$, $\boldsymbol{X}_1 = z_1$, $\boldsymbol{X}_i = \left[y, \hat{\zeta}_1, \cdots, \hat{\zeta}_i, \eta_1, \cdots, \eta_i, \chi, \hat{\omega}, \boldsymbol{W}_z^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{X}_{n+1} = \left[y, \hat{\zeta}_1, \cdots, \hat{\zeta}_n, \eta_1, \cdots, \eta_n, \chi, \hat{\omega}, \upsilon, \boldsymbol{W}_z^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{X}_{n+2} = \left[y, \hat{\zeta}_1, \cdots, \hat{\zeta}_n, \eta_1, \cdots, \eta_n, \chi, \hat{\omega}, \upsilon, \varpi, \boldsymbol{W}_z^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{S}_1(z_1) \in \mathbb{R}^{\mathrm{m_1}}$, $\boldsymbol{S}_i(\boldsymbol{X}_i) \in \mathbb{R}^{\mathrm{m_i}}$, $\boldsymbol{S}_{n+1}(\boldsymbol{X}_{n+1}) \in \mathbb{R}^{\mathrm{m_{n+1}}}$, $\boldsymbol{S}_{n+2}(\boldsymbol{X}_{n+2}) \in \mathbb{R}^{\mathrm{m_{n+2}}}$, $m_z, m_1, m_i, m_{n+1}, m_{n+2} > 1$, and k_1, k_i, k_{n+1}, k_{n+2}, $\hbar$, $\widehat{p}$, ι, ξ_1, ξ_i, ξ_{n+1}, ξ_{n+2}, λ, c_ω are positive constants.

According to Eqs. (21) and (22), the input signal of the system in Eq. (7) always satisfies the constraints of both magnitude and rate, namely, $|u| \le u_M$ and $|\dot{u}| = \left|\left(\frac{\partial\delta}{\partial\upsilon}\right)\dot{\upsilon}\right| = |f(\varpi)| \le v_M$.

Remark 4: Smooth hyperbolic tangent function is used in this paper to restrict control input magnitude and rate constraints instead of the discontinuous sign function [33] to avoid jump phenomena while using backstepping technique. Although the hyperbolic tangent function is widely used in the control field, the auxiliary design system using hyperbolic tangent functions in Eqs. (21)-(38) is firstly proposed in this paper and endows the designed controller with the capability of simultaneously handling delays (state and input delays) and input constraints (magnitude and rate constraints), this is also one of the major contributions of this paper.

Remark 5: Backstepping technique and NNs are adopted for the control design. For the system in Eq. (7), the backstepping procedure has $n+2$ steps. At Step i, $1 \le i \le n+1$, the virtual control α_i is first proposed to stabilize the ith subsystem using RBFNNs for approximating and compensating uncertainties. At Step $n+2$, the auxiliary control $\bar{u}$ is attained. Using the virtual control $\bar{u}$ given in Eq. (35) and the augmented auxiliary system in Eqs. (22) and (23), the variable υ is obtained such that u in Eq. (21) can be finally achieved satisfying magnitude and rate constraints.

Theorem 1: Considering the nonlinear input-delay system in Eq. (1), the proposed NN adaptive output feedback control scheme u in Eq. (21) with the input-driven filter in Eq. (16), intermediate virtual controllers α_i and $\bar{u}$, $i = 1, \cdots, n+1$, in Eqs. (32)-(36), NN adaptive laws $\boldsymbol{W}_z$ and $\hat{\omega}$ in Eqs. (37) and (38), and auxiliary systems (22)-(23) and (30)-(31) in which the former one is used for the guarantee of input saturation, can stabilize the uncertain system. Denote a compact set by $\Omega_y = \left\{y \in \mathbb{R} \middle| \lim_{t\to\infty}|y| \le \bar{y}\right\}$. Then all states of the closed-loop system are semi-globally, uniformly, and ultimately bounded with $y \in \Omega_y$ whenever input magnitude and rate never violate the limits, i.e., $|u| \le u_M$ and $|\dot{u}| \le v_M$ always hold for $\forall t > 0$.

Proof. Before beginning the backstepping procedure, an uncertain constant is defined as

$$\omega^* = \max_{i=1,\cdots,n+2}\left\{m_0\|\boldsymbol{W}_0^*\|^2, \|\boldsymbol{W}_i^*\|^2\right\} \tag{39}$$

in which $\left\|\boldsymbol{W}_0^*\right\|$ and $\|\boldsymbol{W}_i^*\|$ are unknown, and $\boldsymbol{W}_0^* \in \mathbb{R}^{m_0}$ and $\boldsymbol{W}_i^* \in \mathbb{R}^{\mathrm{m_i}}$ in which $m_0 > 1$ will be defined in following steps.

Then the direct Lyapunov function approach is applied for stability analysis [34]–[37].

Step 1. Choose the following Lyapunov function

$$V_0 = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e} + \frac{1}{2\lambda}\tilde{\omega}^2 + \frac{1}{2}z_1^2 \tag{40}$$

where $\tilde{\omega} = \omega^* - \hat{\omega}$ and $\hat{\omega}$ is the estimation of ω^*.

Differentiating V_0 leads to

$$\begin{aligned}\dot{V}_0 =& -\boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{e} + 2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{G}+\boldsymbol{H}) - \frac{1}{\lambda}\tilde{\omega}\dot{\hat{\omega}}\\ &+ \mu z_1(\alpha_1 + e_2 + z_2 + \eta_2 + g_1 + h_1)\end{aligned} \tag{41}$$

where h_1 and g_1 are abbreviations of $h_1(t, \zeta, \zeta(t-\bar{\tau}_1))$ and $g_1(\zeta_1)$ respectively.

For any $\varepsilon_0 > 0$, there exists a NN such that

$$\|\boldsymbol{G}\| = \boldsymbol{W}_0^{*\mathrm{T}}\boldsymbol{S}_0(\zeta) + \sigma_0(\zeta) \tag{42}$$

where $\boldsymbol{S}_0(\zeta) \in \mathbb{R}^{\mathrm{m}_0}$ and $|\sigma_0(\zeta)| \le \varepsilon_0$.

Now, using Eq. (42) and

$$\|\boldsymbol{H}\|^2 \le 3\sum_{i=1}^{n} \breve{\mu}_i \left(\widehat{\ell}_{i1}^{2}(z_1) + \widehat{\ell}_{i2}^{2}(z_1(t-\bar{\tau}_i)) + D_i^2\right) \tag{43}$$

which can be obtained by inequality (8), we have

$$\begin{aligned} &2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{G}+\boldsymbol{H}) \\ &\le 2\|\boldsymbol{e}\|^2 + \|\boldsymbol{P}\|^2\left(m_0\|\boldsymbol{W}_0^*\|^2 + \varepsilon_0^2\right) + \varphi^{-1}e^{\varepsilon\tau^*}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{P}\boldsymbol{e} \\ &\quad + 3\varphi e^{-\varepsilon\tau^*}\sum_{i=1}^{n}\breve{\mu}_i\left(\widehat{\ell}_{i1}^{2}(z_1) + \widehat{\ell}_{i2}^{2}(z_1(t-\bar{\tau}_i)) + D_i^2\right) \end{aligned} \tag{44}$$

where φ and ε are positive constants.

With

$$\mu z_1 h_1 \le \frac{3}{4}\mu^2(n+2)e^{\varepsilon\tau^*}z_1^2 + \frac{1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2, \tag{45}$$

we have

$$\begin{aligned} \dot{V}_0 \le &-\underline{\mathrm{q}}_1\|\boldsymbol{e}\|^2 + z_1(\mu\alpha_1 + G_1(z_1)) + \frac{1}{2}z_2^2 \\ &+ \frac{1}{2}\eta_2^2 - \frac{1}{\lambda}\tilde{\omega}\dot{\hat{\omega}} + \underline{\theta}_1 + \frac{1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2 \\ &+ 3\varphi e^{-\varepsilon\tau^*}\sum_{i=1}^{n}\breve{\mu}_i\left(\widehat{\ell}_{i1}^{2}(z_1) + \widehat{\ell}_{i2}^{2}(z_1(t-\bar{\tau}_i))\right) \end{aligned} \tag{46}$$

where $\underline{\mathrm{q}}_1 = \underline{\lambda}(\boldsymbol{Q}) - \frac{5}{2} - \varphi^{-1}e^{\varepsilon\tau^*}\bar{\lambda}(\boldsymbol{P}\boldsymbol{P})$, $G_1(z_1) = \mu g_1 + z_1\left[\frac{3}{2}\mu^2 + \frac{3}{4}\mu^2(n+2)e^{\varepsilon\tau^*}\right]$, and $\underline{\theta}_1 = \|\boldsymbol{P}\|^2\left(\omega^* + \varepsilon_0^2\right) + 3\varphi e^{-\varepsilon\tau^*}\sum_{i=1}^{n}\breve{\mu}_i D_i^2$.

The unknown function $G_1(z_1)$ needs to be approximated by RBFNN and is expressed by

$$G_1(z_1) = \boldsymbol{W}_1^{*\mathrm{T}}\boldsymbol{S}_1(z_1) + \sigma_1(z_1) \tag{47}$$

where $|\sigma_1(z_1)| \le \varepsilon_1$, and $\varepsilon_1 > 0$, such that the following inequality satisfies

$$z_1 G_1(z_1) \le \frac{z_1^2\omega^*}{2k_1}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1) + \frac{k_1}{2} + \frac{z_1^2}{2} + \frac{\varepsilon_1^2}{2}. \tag{48}$$

Applying inequalities (46) and (48) yields the following inequality

$$\begin{aligned} \dot{V}_0 \le &-\underline{\mathrm{q}}_1\|\boldsymbol{e}\|^2 + z_1\Bigg[\mu\alpha_1 + \frac{z_1\omega^*}{2k_1}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1) + \frac{z_1}{2} \\ &+ 3\varphi z_1\sum_{i=1}^{n}\breve{\mu}_i\left(e^{-\varepsilon\bar{\tau}_i}\breve{\ell}_{i1}^{2}(z_1) + \frac{1}{1-\widehat{\tau}}\breve{\ell}_{i2}^{2}(z_1)\right) \\ &+ \breve{\mu}_1 z_1\left(e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^{2}(z_1) + \frac{1}{1-\widehat{\tau}}\breve{\ell}_{12}^{2}(z_1)\right)\Bigg] \\ &+ \frac{1}{2}z_2^2 + \frac{1}{2}\eta_2^2 - \frac{1}{\lambda}\tilde{\omega}\dot{\hat{\omega}} + \theta_1 + \frac{1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2 \\ &- 3\varphi z_1^2\sum_{i=1}^{n}\breve{\mu}_i\left(e^{-\varepsilon\bar{\tau}_i}\breve{\ell}_{i1}^{2}(z_1) + \frac{1}{1-\widehat{\tau}}\breve{\ell}_{i2}^{2}(z_1)\right) \\ &- \breve{\mu}_1 z_1^2\left(e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^{2}(z_1) + \frac{1}{1-\widehat{\tau}}\breve{\ell}_{12}^{2}(z_1)\right) \\ &+ 3\varphi e^{-\varepsilon\tau^*}\sum_{i=1}^{n}\breve{\mu}_i\left(\widehat{\ell}_{i1}^{2}(z_1) + \widehat{\ell}_{i2}^{2}(z_1(t-\bar{\tau}_i))\right) \end{aligned} \tag{49}$$

where $\theta_1 = \underline{\theta}_1 + \frac{k_1}{2} + \frac{\varepsilon_1^2}{2}$.

Consider a Lyapunov-Krasovskii functional Δ as

$$\begin{aligned} \Delta = \frac{e^{-\varepsilon t}}{1-\widehat{\tau}}\Bigg(&3\varphi\sum_{i=1}^{n}\int_{t-\bar{\tau}_i}^{t}\breve{\mu}_i e^{\varepsilon\Theta}\widehat{\ell}_{i2}^{2}(z_1(\Theta))\,\mathrm{d}\Theta \\ &+ \int_{t-\bar{\tau}_1}^{t}\breve{\mu}_1 e^{\varepsilon\Theta}\widehat{\ell}_{12}^{2}(z_1(\Theta))\,\mathrm{d}\Theta\Bigg) \end{aligned} \tag{50}$$

whose derivative is given as

$$\begin{aligned} \dot{\Delta} \le &3\varphi\sum_{i=1}^{n}\breve{\mu}_i\left(\frac{1}{1-\widehat{\tau}}\widehat{\ell}_{i2}^{2}(z_1) - e^{-\varepsilon\bar{\tau}_i}\widehat{\ell}_{i2}^{2}(z_1(t-\bar{\tau}_i))\right) \\ &+ \breve{\mu}_1\left(\frac{1}{1-\widehat{\tau}}\widehat{\ell}_{12}^{2}(z_1) - e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^{2}(z_1(t-\bar{\tau}_1))\right) - \varepsilon\Delta. \end{aligned} \tag{51}$$

Define

$$V_1 = V_0 + \Delta \tag{52}$$

and then using inequalities (49) and (51) yields

$$\begin{aligned} \dot{V}_1 \le &-\underline{\mathrm{q}}_1\|\boldsymbol{e}\|^2 + z_1\Bigg[\mu\alpha_1 + \frac{z_1\omega^*}{2k_1}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1) \\ &+ \frac{z_1}{2} + G_z(z_1)\Bigg] + \frac{1}{2}z_2^2 + \frac{1}{2}\eta_2^2 - \frac{1}{\lambda}\tilde{\omega}\dot{\hat{\omega}} + \theta_1 \\ &+ \frac{1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2 - \breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^{2}(z_1) \\ &- \breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^{2}(z_1(t-\bar{\tau}_1)) - \varepsilon\Delta \end{aligned} \tag{53}$$

where $G_z(z_1) = 3\varphi z_1\sum_{i=1}^{n}\breve{\mu}_i\left(e^{-\varepsilon\bar{\tau}_i}\breve{\ell}_{i1}^{2}(z_1) + \frac{1}{1-\widehat{\tau}}\breve{\ell}_{i2}^{2}(z_1)\right) + \breve{\mu}_1 z_1\left(e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^{2}(z_1) + \frac{1}{1-\widehat{\tau}}\breve{\ell}_{12}^{2}(z_1)\right)$.

Let

$$G_z(z_1) = \boldsymbol{W}_z^{*\mathrm{T}}\boldsymbol{S}_z(z_1) + \sigma_z(z_1) \tag{54}$$

where $\boldsymbol{W}_z^* \in \mathbb{R}^{\mathrm{m}_z}$, $|\sigma_z(z_1)| \le \varepsilon_z$ and $\varepsilon_z > 0$.

Substituting Eqs. (32) and (36) into inequality (53) gives

$$\begin{aligned} \dot{V}_1 \le &-\underline{\mathrm{q}}_1\|\boldsymbol{e}\|^2 - \left(\xi_1 + \widehat{p}\right)z_1^2 + (\mu N(\chi) + 1)\dot{\chi} \\ &+ \frac{1}{\lambda}\tilde{\omega}\left(\frac{\lambda z_1^2}{2k_1}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1) - \dot{\hat{\omega}}\right) \\ &+ \frac{1}{2}z_2^2 + \frac{1}{2}\eta_2^2 + \theta_1 + \frac{1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2 \\ &- \breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^{2}(z_1) - \breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^{2}(z_1(t-\bar{\tau}_1)) \\ &+ z_1 G_z(z_1) - z_1\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1) - \varepsilon\Delta. \end{aligned} \tag{55}$$

Step $i = 2, \cdots, n-1$. Based on Step 1, the following Lyapunov function is selected

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 \tag{56}$$

whose derivative with respect to time can be given as

$$\begin{aligned} \dot{V}_i \le &-\underline{\mathrm{q}}_{i-1}\|\boldsymbol{e}\|^2 - \sum_{j=1}^{i-1}\xi_j z_j^2 + (\mu N(\chi) + 1)\dot{\chi} \\ &+ \sum_{j=2}^{i-1} z_j\left(\Phi_j(\boldsymbol{X}_j) - \frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right) + z_i\left[z_{i+1} + \alpha_i\right. \end{aligned}$$

$$-l_i\hat{\zeta}_1+\gamma_i\eta_i-\mu\frac{\partial\alpha_{i-1}}{\partial y}(e_2+h_1)-\frac{\partial\alpha_{i-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}-\psi_{i-1}\Bigg]$$
$$+\frac{1}{2}z_i^2+\frac{1}{2}\eta_2^2+\theta_{i-1}-\widehat{p}z_1^2-\breve{\mu}_1z_1^2e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)$$
$$-\breve{\mu}_1e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))+\frac{i-1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2$$
$$+z_1G_z(z_1)-z_1\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)-\varepsilon\Delta$$
$$+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{i-1}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right) \tag{57}$$

where $\psi_1=\mu\frac{\partial\alpha_1}{\partial y}\left(\hat{\zeta}_2+g_1\right)+\left(\frac{\partial\alpha_1}{\partial\boldsymbol{W}_z^{\mathrm{T}}}\right)\left(\hbar\boldsymbol{S}_z(z_1)z_1-\hbar\iota\boldsymbol{W}_z\right)+\frac{\partial\alpha_1}{\partial\chi}\left[\left(\xi_1+\frac{1}{2}+\widehat{p}\right)z_1+\frac{\hat{\omega}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1)}{2k_1}z_1+\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)\right]z_1$, $\psi_{i-1}=\mu\frac{\partial\alpha_{i-1}}{\partial y}\left(\hat{\zeta}_2+g_1\right)+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\eta_j}\left(\eta_{j+1}-\gamma_j\eta_j\right)+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\zeta}_j}\left(\hat{\zeta}_{j+1}-l_j\hat{\zeta}_1\right)+\left(\frac{\partial\alpha_{i-1}}{\partial\boldsymbol{W}_z^{\mathrm{T}}}\right)\left(\hbar\boldsymbol{S}_z(z_1)z_1-\hbar\iota\boldsymbol{W}_z\right)+\frac{\partial\alpha_{i-1}}{\partial\chi}\left[\left(\xi_1+\frac{1}{2}+\widehat{p}\right)z_1+\frac{\hat{\omega}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1)}{2k_1}z_1+\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)\right]z_1$ for $i=3,\cdots,n-1$.

Remark 6: Since $\hat{\omega}$ in Eq. (38) is designed in the last step in reality, the term $\frac{\partial\alpha_{i-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}$ in inequality (57) cannot be expressed in the ith step. Consequently, a function $\Phi_j(\boldsymbol{X}_j)$ is introduced from Step 3 in inequality (57), which means that terms $\sum_{j=2}^{1}z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)$ in inequality (57) and $\Phi_1(\boldsymbol{X}_1)$ equal zero in Step 2, and $\Phi_j(\boldsymbol{X}_j)$ will be designed later for compensating the term $\frac{\partial\alpha_{i-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}$.

In the light of the following inequality

$$z_i\left(z_{i+1}-\mu\frac{\partial\alpha_{i-1}}{\partial y}(e_2+h_1)\right)$$
$$\leq\left[\frac{1}{2}+\frac{1}{2}\mu^2\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2+\frac{3}{4}\mu^2(n+2)e^{\varepsilon\tau^*}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\right]z_i^2$$
$$+\frac{1}{2}\left(z_{i+1}^2+\|\boldsymbol{e}\|^2+\frac{2}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2\right), \tag{58}$$

inequality (57) is further rewritten in the following form

$$\dot{V}_i\leq-\underline{q}_i\|\boldsymbol{e}\|^2-\sum_{j=1}^{i-1}\xi_jz_j^2+(\mu N(\chi)+1)\dot{\chi}$$
$$+\sum_{j=2}^{i}z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)+z_i(\alpha_i+G_i(\boldsymbol{X}_i))$$
$$+\frac{1}{2}z_{i+1}^2-\widehat{p}z_1^2+\frac{1}{2}\eta_2^2+\theta_{i-1}-\breve{\mu}_1z_1^2e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)$$
$$-\breve{\mu}_1e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))+\frac{i}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2$$
$$+z_1G_z(z_1)-z_1\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)-\varepsilon\Delta$$
$$+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{i-1}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right) \tag{59}$$

where $\underline{q}_i=\underline{q}_{i-1}-\frac{1}{2}$, $G_i(\boldsymbol{X}_i)=-l_i\hat{\zeta}_1+\gamma_i\eta_i-\psi_{i-1}-\Phi_i(\boldsymbol{X}_i)+z_i\left[1+\frac{1}{2}\mu^2\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2+\frac{3}{4}\mu^2(n+2)e^{\varepsilon\tau^*}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\right]$ for $i=2,\cdots,n-1$.

Let $G_i(\boldsymbol{X}_i)$ be

$$G_i(\boldsymbol{X}_i)=\boldsymbol{W}_i^{*\mathrm{T}}\boldsymbol{S}_i(\boldsymbol{X}_i)+\sigma_i(\boldsymbol{X}_i) \tag{60}$$

with $|\sigma_i(\boldsymbol{X}_i)|\leq\varepsilon_i$ and $\varepsilon_i>0$.

Substituting Eq. (33) into inequality (59), it is inferred that

$$\dot{V}_i\leq-\underline{q}_i\|\boldsymbol{e}\|^2-\sum_{j=1}^{i}\xi_jz_j^2+(\mu N(\chi)+1)\dot{\chi}$$
$$+\sum_{j=2}^{i}z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)+\frac{1}{2}z_{i+1}^2+\frac{1}{2}\eta_2^2+\theta_i$$
$$-\widehat{p}z_1^2-\breve{\mu}_1z_1^2e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)-\breve{\mu}_1e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))$$
$$+\frac{i}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2+z_1G_z(z_1)-z_1\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)-\varepsilon\Delta$$
$$+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{i}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right) \tag{61}$$

where $\theta_i=\theta_{i-1}+\frac{k_i}{2}+\frac{\varepsilon_i^2}{2}$.

Step $i=n$. Construct a Lyapunov function as

$$V_n=V_{n-1}+\frac{1}{2}z_n^2+\frac{1}{2}\sum_{i=1}^{n}\eta_i^2+\frac{1}{2\hbar}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z \tag{62}$$

where $\tilde{\boldsymbol{W}}_z=\boldsymbol{W}_z-\boldsymbol{W}_z^*$.

By taking the derivative of V_n, it yields

$$\dot{V}_n\leq-\underline{q}_{n-1}\|\boldsymbol{e}\|^2-\sum_{j=1}^{n-1}\xi_jz_j^2+(\mu N(\chi)+1)\dot{\chi}$$
$$+\sum_{j=2}^{n-1}z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)+z_n\Bigg[u-l_n\hat{\zeta}_1+\gamma_n\eta_n$$
$$-\mu\frac{\partial\alpha_{n-1}}{\partial y}(e_2+h_1)-\frac{\partial\alpha_{n-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}-\psi_{n-1}\Bigg]+\frac{1}{2}z_n^2+\frac{1}{2}\eta_2^2$$
$$+\theta_{n-1}+\sum_{j=1}^{n-1}\eta_i(\eta_{i+1}-\gamma_i\eta_i)+\eta_n(-\gamma_n\eta_n+u(t-\tau)-u)$$
$$-\widehat{p}z_1^2-\breve{\mu}_1z_1^2e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)-\breve{\mu}_1e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))$$
$$+\frac{n-1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2+\varepsilon_z|z_1|-\varepsilon\Delta-\iota\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\boldsymbol{W}_z$$
$$+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n-1}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right) \tag{63}$$

where $\psi_{n-1}=\mu\frac{\partial\alpha_{n-1}}{\partial y}\left(\hat{\zeta}_2+g_1\right)+\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\eta_j}\left(\eta_{j+1}-\gamma_j\eta_j\right)+\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\zeta}_j}\left(\hat{\zeta}_{j+1}-l_j\hat{\zeta}_1\right)+\left(\frac{\partial\alpha_{n-1}}{\partial\boldsymbol{W}_z^{\mathrm{T}}}\right)\left(\hbar\boldsymbol{S}_z(z_1)z_1-\hbar\iota\boldsymbol{W}_z\right)+\frac{\partial\alpha_{n-1}}{\partial\chi}\left[\left(\xi_1+\frac{1}{2}+\widehat{p}\right)z_1+\frac{\hat{\omega}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1)}{2k_1}z_1+\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)\right]z_1$.

Using Eq. (28) and combining

$$-\mu z_n\frac{\partial\alpha_{n-1}}{\partial y}(e_2+h_1)$$
$$\leq\frac{1}{2}\mu^2\left[1+\frac{3}{2}(n+2)e^{\varepsilon\tau^*}\right]\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^2z_n^2$$
$$+\frac{1}{2}\left(\|\boldsymbol{e}\|^2+\frac{2}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2\right), \tag{64}$$

one can get

$$\begin{aligned}\dot{V}_n \le & -\underline{q}_n\|e\|^2-\sum_{j=1}^{n-1}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}\\ &+\sum_{j=2}^{n} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2+\underline{\theta}_{n-1}\\ &+z_n(z_{n+1}+\alpha_n+G_n(\boldsymbol{X}_n))-\breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)\\ &-\breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))+\frac{n}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2+\varepsilon_z|z_1|\\ &-\widehat{p}z_1^2-\varepsilon\Delta-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z+\frac{\iota}{2}\left\|\boldsymbol{W}_z^*\right\|^2\\ &+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n-1}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right)\end{aligned} \tag{65}$$

where $\underline{q}_n=\underline{q}_{n-1}-\frac{1}{2}$, $\underline{\theta}_{n-1}=\theta_{n\text{-}1}+u_M^2$, $\bar{\gamma}_1=\gamma_1-\frac{1}{2}$, $\bar{\gamma}_2=\gamma_2-\frac{3}{2}$, $\bar{\gamma}_i=\gamma_i-1$, $i=3,\cdots,n-1$, $\bar{\gamma}_n=\gamma_n-\frac{3}{2}$, and $G_n(\boldsymbol{X}_n)=-l_n\hat{\zeta}_1+\gamma_n\eta_n-\psi_{n-1}-\Phi_n(\boldsymbol{X}_n)+\frac{1}{2}\left[1+\mu^2\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^2+\frac{3}{2}(n+2)\mu^2e^{\varepsilon\tau^*}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^2\right]z_n$.

For any $\varepsilon_n>0$, we have

$$G_n(\boldsymbol{X}_n)=\boldsymbol{W}_n^{*\mathrm{T}}\boldsymbol{S}_n(\boldsymbol{X}_n)+\sigma_n(\boldsymbol{X}_n) \tag{66}$$

where $|\sigma_n(\boldsymbol{X}_n)|\le\varepsilon_n$.

Then using Eq. (33) and inequality (65), we have

$$\begin{aligned}\dot{V}_n \le & -\underline{q}_n\|e\|^2-\sum_{j=1}^{n}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}\\ &+\sum_{j=2}^{n} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)+z_nz_{n+1}-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2\\ &+\theta_n-\varepsilon\Delta-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z-\breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)\\ &-\breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))+\frac{n}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2\\ &+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right)\end{aligned} \tag{67}$$

in which $\theta_n=\underline{\theta}_{n-1}+\frac{k_n}{2}+\frac{\varepsilon_n^2}{2}+\frac{\varepsilon_z^2}{4\widehat{p}}+\frac{\iota}{2}\left\|\boldsymbol{W}_z^*\right\|^2$.

Step $i=n+1$. Consider the following Lyapunov function

$$V_{n+1}=V_n+\frac{1}{2}z_{n+1}^2. \tag{68}$$

It follows from Eqs. (22), (28), and (29) that

$$\dot{z}_{n+1}=\left(\frac{\partial\delta}{\partial\upsilon}\right)\dot{\upsilon}-\dot{\alpha}_n=f(\varpi)-\dot{\alpha}_n=z_{n+2}+\alpha_{n+1}-\dot{\alpha}_n. \tag{69}$$

Substituting virtual control signal α_{n+1} in Eq. (34) to inequality (67) gives $\dot{V}_{n+1}$ as

$$\begin{aligned}\dot{V}_{n+1} \le & -\underline{q}_n\|e\|^2-\sum_{j=1}^{n+1}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}\\ &+\sum_{j=2}^{n} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2+\theta_n\\ &+z_{n+1}\left[-\frac{1}{2}z_{n+1}+z_{n+2}-\frac{\partial\alpha_n}{\partial\hat{\omega}}\dot{\hat{\omega}}-\mu\frac{\partial\alpha_n}{\partial y}(e_2+h_1)\right.\\ &\left.-\frac{z_{n+1}}{2k_{n+1}}\hat{\omega}\boldsymbol{S}_{n+1}^{\mathrm{T}}(\boldsymbol{X}_{n+1})\boldsymbol{S}_{n+1}(\boldsymbol{X}_{n+1})\right]+|z_{n+1}||\psi_n|\\ &-\varepsilon\Delta-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z-\breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)\\ &-\breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))+\frac{n}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2\\ &+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right)\end{aligned} \tag{70}$$

where $\psi_n=\mu\frac{\partial\alpha_n}{\partial y}\left(\hat{\zeta}_2+g_1\right)+\sum_{j=1}^{n-1}\frac{\partial\alpha_n}{\partial\eta_j}\left(\eta_{j+1}-\gamma_j\eta_j\right)+\sum_{j=1}^{n-1}\frac{\partial\alpha_n}{\partial\hat{\zeta}_j}\left(\hat{\zeta}_{j+1}-l_j\hat{\zeta}_1\right)+\left|\frac{\partial\alpha_n}{\partial\hat{\zeta}_n}\right|\left(u_M+l_n\left|\hat{\zeta}_1\right|\right)+\left|\frac{\partial\alpha_n}{\partial\eta_n}\right|(\gamma_n|\eta_n|+2u_M)+\left(\frac{\partial\alpha_n}{\partial\boldsymbol{W}_z^{\mathrm{T}}}\right)\left(\hbar\boldsymbol{S}_z(z_1)z_1-\hbar\iota\boldsymbol{W}_z\right)+\frac{\partial\alpha_n}{\partial\chi}\left[\left(\xi_1+\frac{1}{2}+\widehat{p}\right)z_1+\frac{\hat{\omega}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1)}{2k_1}z_1+\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)\right]z_1$.

Then it can be obtain from inequality (70) that

$$\begin{aligned}\dot{V}_{n+1} \le & -\underline{q}_{n+1}\|e\|^2-\sum_{j=1}^{n+1}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}-\varepsilon\Delta\\ &+\sum_{j=2}^{n+1} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2+\theta_n\\ &+z_{n+1}\left(-\frac{1}{2}z_{n+1}-\frac{z_{n+1}}{2k_{n+1}}\hat{\omega}\boldsymbol{S}_{n+1}^{\mathrm{T}}(\boldsymbol{X}_{n+1})\boldsymbol{S}_{n+1}(\boldsymbol{X}_{n+1})\right.\\ &\left.+z_{n+2}+G_{n+1}(\boldsymbol{X}_{n+1})\right)+\frac{n+1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2\\ &+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right)-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z\\ &-\breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)-\breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))\end{aligned} \tag{71}$$

where $\underline{q}_{n+1}=\underline{q}_n-\frac{1}{2}$ and $G_{n+1}(\boldsymbol{X}_{n+1})=\operatorname{sgn}(z_{n+1})|\psi_n|-\Phi_{n+1}(\boldsymbol{X}_{n+1})+\frac{1}{2}\mu^2\left[1+\frac{3}{2}(n+2)e^{\varepsilon\tau^*}\right]\left(\frac{\partial\alpha_n}{\partial y}\right)^2 z_{n+1}$.

Let

$$G_{n+1}(\boldsymbol{X}_{n+1})=\boldsymbol{W}_{n+1}^{*\mathrm{T}}\boldsymbol{S}_{n+1}(\boldsymbol{X}_{n+1})+\sigma_{n+1}(\boldsymbol{X}_{n+1}) \tag{72}$$

where $|\sigma_{n+1}(\boldsymbol{X}_{n+1})|\le\varepsilon_{n+1}$ and $\varepsilon_{n+1}>0$ such that

$$\begin{aligned}\dot{V}_{n+1} \le & -\underline{q}_{n+1}\|e\|^2-\sum_{j=1}^{n+1}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}\\ &+\sum_{j=2}^{n+1} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2+\theta_{n+1}\\ &+z_{n+1}z_{n+2}-\varepsilon\Delta-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z+\frac{n+1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2\\ &-\breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^2(z_1)-\breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\widehat{\ell}_{12}^2(z_1(t-\bar{\tau}_1))\\ &+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n+1}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right)\end{aligned} \tag{73}$$

where $\theta_{n+1}=\theta_n+\frac{k_{n+1}}{2}+\frac{\varepsilon_{n+1}^2}{2}$.

Step $i=n+2$. Design a Lyapunov function as

$$V_{n+2}=V_{n+1}+\frac{1}{2}z_{n+2}^2. \tag{74}$$

Then considering Eqs. (35), (23), and (29), it yields

$$\begin{aligned}\dot{V}_{n+2} \leq & -\underline{q}_{n+1}\|\boldsymbol{e}\|^2-\sum_{j=1}^{n+2}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}\\ &+\sum_{j=2}^{n+1} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2\\ &+z_{n+2}\left[-\frac{1}{2}z_{n+2}-\frac{\partial\alpha_{n+1}}{\partial\hat{\omega}}\dot{\hat{\omega}}-\mu\frac{\partial\alpha_{n+1}}{\partial y}(e_2+h_1)\right.\\ &\left.-\frac{z_{n+2}}{2k_{n+2}}\hat{\omega}\boldsymbol{S}_{n+2}^{\mathrm{T}}(\boldsymbol{X}_{n+2})\boldsymbol{S}_{n+2}(\boldsymbol{X}_{n+2})\right]+|z_{n+2}||\psi_{n+1}|\\ &+\theta_{n+1}-\varepsilon\Delta-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z-\breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^{2}(z_1)\\ &-\breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\overset{\frown}{\ell}{}_{12}^{2}(z_1(t-\bar{\tau}_1))+\frac{n+1}{3(n+2)}e^{-\varepsilon\tau^*}h_1^2\\ &+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n+1}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right)\end{aligned}\tag{75}$$

where $\psi_{n+1} = \mu\frac{\partial\alpha_{n+1}}{\partial y}\left(\hat{\zeta}_2+g_1\right) + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n+1}}{\partial\eta_j}\left(\eta_{j+1}-\gamma_j\eta_j\right) + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n+1}}{\partial\hat{\zeta}_j}\left(\hat{\zeta}_{j+1}-l_j\hat{\zeta}_1\right) + \left|\frac{\partial\alpha_{n+1}}{\partial\eta_n}\right|(\gamma_n|\eta_n|+2u_M) + \left|\frac{\partial\alpha_{n+1}}{\partial\hat{\zeta}_n}\right|\left(u_M+l_n\left|\hat{\zeta}_1\right|\right) + \frac{\partial\alpha_{n+1}}{\partial\upsilon}\left(\frac{\partial\delta}{\partial\upsilon}\right)^{-1}f(\varpi) + \left(\frac{\partial\alpha_{n+1}}{\partial\boldsymbol{W}_z^{\mathrm{T}}}\right)\left(\hbar\boldsymbol{S}_z(z_1)z_1-\hbar\iota\boldsymbol{W}_z\right) + \frac{\partial\alpha_{n+1}}{\partial\chi}\left[\left(\xi_1+\frac{1}{2}+\hat{p}\right)z_1+\frac{\hat{\omega}\boldsymbol{S}_1^{\mathrm{T}}(z_1)\boldsymbol{S}_1(z_1)}{2k_1}z_1+\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)\right]z_1$.

Inequality (75) implies that following inequality holds

$$\begin{aligned}\dot{V}_{n+2} \leq & -\underline{q}_{n+2}\|\boldsymbol{e}\|^2-\sum_{j=1}^{n+2}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}\\ &+\sum_{j=2}^{n+2} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2\\ &+z_{n+2}\left(-\frac{1}{2}z_{n+2}-\frac{z_{n+2}}{2k_{n+2}}\hat{\omega}\boldsymbol{S}_{n+2}^{\mathrm{T}}(\boldsymbol{X}_{n+2})\boldsymbol{S}_{n+2}(\boldsymbol{X}_{n+2})\right.\\ &\left.+G_{n+2}(\boldsymbol{X}_{n+2})\right)+\theta_{n+1}-\varepsilon\Delta-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z+\frac{1}{3}e^{-\varepsilon\tau^*}h_1^2\\ &-\breve{\mu}_1 z_1^2 e^{-\varepsilon\bar{\tau}_1}\breve{\ell}_{11}^{2}(z_1)-\breve{\mu}_1 e^{-\varepsilon\bar{\tau}_1}\overset{\frown}{\ell}{}_{12}^{2}(z_1(t-\bar{\tau}_1))\\ &+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n+1}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right)\end{aligned}\tag{76}$$

where $\underline{q}_{n+2}=\underline{q}_{n+1}-\frac{1}{2}$ and $G_{n+2}(\boldsymbol{X}_{n+2})=\operatorname{sgn}(z_{n+2})|\psi_{n+1}|-\Phi_{n+2}(\boldsymbol{X}_{n+2})+\frac{1}{2}\mu^2\left[1+\frac{3}{2}(n+2)e^{\varepsilon\tau^*}\right]\left(\frac{\partial\alpha_{n+1}}{\partial y}\right)^2 z_{n+2}$.

Making

$$G_{n+2}(\boldsymbol{X}_{n+2})=\boldsymbol{W}_{n+2}^{*\mathrm{T}}\boldsymbol{S}_{n+2}(\boldsymbol{X}_{n+2})+\sigma_{n+2}(\boldsymbol{X}_{n+2})\tag{77}$$

where $|\sigma_{n+2}(\boldsymbol{X}_{n+2})|\leq\varepsilon_{n+2}$ and $\varepsilon_{n+2}>0$, thus the time derivative of V_{n+2} is rewritten as

$$\begin{aligned}\dot{V}_{n+2} \leq & -\underline{q}_{n+2}\|\boldsymbol{e}\|^2-\sum_{j=1}^{n+2}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}\\ &+\frac{1}{\lambda}\tilde{\omega}\left(\sum_{j=1}^{n+2}\frac{\lambda z_j^2}{2k_j}\boldsymbol{S}_j^{\mathrm{T}}(\boldsymbol{X}_j)\boldsymbol{S}_j(\boldsymbol{X}_j)-\dot{\hat{\omega}}\right)\\ &+\sum_{j=2}^{n+2} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)\\ &+\theta_{n+2}-\varepsilon\Delta-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2\end{aligned}\tag{78}$$

in which $\theta_{n+2}=\theta_{n+1}+\frac{k_{n+2}}{2}+\frac{\varepsilon_{n+2}^2}{2}+\breve{\mu}_1 D_1^2 e^{-\varepsilon\tau^*}$.

Then substituting adaptive law in Eq. (38) to inequality (78) achieves

$$\begin{aligned}\dot{V}_{n+2} \leq & -\underline{q}_{n+2}\|\boldsymbol{e}\|^2-\sum_{j=1}^{n+2}\xi_j z_j^2+(\mu N(\chi)+1)\dot{\chi}\\ &+\frac{c_\omega}{\lambda}\tilde{\omega}\hat{\omega}+\sum_{j=2}^{n+2} z_j\left(\Phi_j(\boldsymbol{X}_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\right)\\ &-\sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2+\theta_{n+2}-\varepsilon\Delta-\frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z.\end{aligned}\tag{79}$$

The unknown function $\Phi_j(\boldsymbol{X}_j)$ occurring in Eq. (57) is now proposed as

$$\begin{aligned}\Phi_j(\boldsymbol{X}_j)= & -c_\omega\hat{\omega}\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}-\frac{\lambda m_j z_j}{2k_j}\sum_{i=2}^{j}\left|z_i\frac{\partial\alpha_{i-1}}{\partial\hat{\omega}}\right|\\ &+\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\sum_{i=1}^{j-1}\frac{\lambda}{2k_i}z_i^2\boldsymbol{S}_i^{\mathrm{T}}(\boldsymbol{X}_i)\boldsymbol{S}_i(\boldsymbol{X}_i)\end{aligned}\tag{80}$$

where $j=2,\cdots,n+2$.

In the light of

$$\begin{aligned}&-\sum_{j=2}^{n+2}z_j\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\\ =&\sum_{j=2}^{n+2}c_\omega z_j\hat{\omega}\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\\ &-\sum_{j=2}^{n+2}\left[z_j\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\left(\sum_{i=1}^{j-1}\frac{\lambda z_i^2}{2k_i}\boldsymbol{S}_i^{\mathrm{T}}(\boldsymbol{X}_i)\boldsymbol{S}_i(\boldsymbol{X}_i)\right)\right]\\ &-\sum_{j=2}^{n+2}\left[z_j\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\left(\sum_{i=j}^{n+2}\frac{\lambda z_i^2}{2k_i}\boldsymbol{S}_i^{\mathrm{T}}(\boldsymbol{X}_i)\boldsymbol{S}_i(\boldsymbol{X}_i)\right)\right]\end{aligned}\tag{81}$$

and

$$\begin{aligned}&-\sum_{j=2}^{n+2}\left[z_j\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\left(\sum_{i=j}^{n+2}\frac{\lambda z_i^2}{2k_i}\boldsymbol{S}_i^{\mathrm{T}}(\boldsymbol{X}_i)\boldsymbol{S}_i(\boldsymbol{X}_i)\right)\right]\\ \leq&\sum_{j=2}^{n+2}\left[\left|z_j\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\right|\left(\sum_{i=j}^{n+2}\frac{\lambda m_i z_i^2}{2k_i}\right)\right]\\ =&\sum_{j=2}^{n+2}\left[\frac{\lambda m_j z_j^2}{2k_j}\left(\sum_{i=2}^{j}\left|z_i\frac{\partial\alpha_{i-1}}{\partial\hat{\omega}}\right|\right)\right],\end{aligned}\tag{82}$$

it produces

$$\begin{aligned}&-\sum_{j=2}^{n+2}z_j\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\dot{\hat{\omega}}\\ \leq&\sum_{j=2}^{n+2}z_j\left[c_\omega\hat{\omega}\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}-\frac{\partial\alpha_{j-1}}{\partial\hat{\omega}}\left(\sum_{i=1}^{j-1}\frac{\lambda z_i^2}{2k_i}\boldsymbol{S}_i^{\mathrm{T}}(\boldsymbol{X}_i)\boldsymbol{S}_i(\boldsymbol{X}_i)\right)\right.\\ &\left.+\frac{\lambda m_j z_j}{2k_j}\left(\sum_{i=2}^{j}\left|z_i\frac{\partial\alpha_{i-1}}{\partial\hat{\omega}}\right|\right)\right]\end{aligned}$$

$$= -\sum_{i=2}^{n+2} z_j \Phi_j(\boldsymbol{X}_j). \tag{83}$$

Due to the inequality $\frac{c_\omega}{\lambda}\tilde{\omega}\hat{\omega} \le \frac{c_\omega}{2\lambda}\left(\omega^{*2} - \tilde{\omega}^2\right)$, it can be obtained

$$\dot{V}_{n+2} \le -\underline{q}_{n+2}\|\boldsymbol{e}\|^2 - \sum_{j=1}^{n+2}\xi_j z_j^2 + (\mu N(\chi)+1)\dot{\chi} - \frac{c_\omega}{2\lambda}\tilde{\omega}^2 - \sum_{j=1}^{n}\bar{\gamma}_j\eta_j^2 + \underline{\theta}_{n+2} - \varepsilon\Delta - \frac{\iota}{2}\tilde{\boldsymbol{W}}_z^{\mathrm{T}}\tilde{\boldsymbol{W}}_z \tag{84}$$

where $\underline{\theta}_{n+2} = \theta_{n+2} + \frac{c_\omega}{2\lambda}\omega^{*2}$, and thus

$$\dot{V}_{n+2} \le -\rho V_{n+2} + (\mu N(\chi)+1)\dot{\chi} + \underline{\theta}_{n+2} \tag{85}$$

where
$\rho = \min\left\{\frac{\underline{q}_{n+2}}{\bar{\lambda}(\boldsymbol{P})}, 2\xi_1, \cdots, 2\xi_{n+2}, c_\omega, 2\bar{\gamma}_1, \cdots, 2\bar{\gamma}_n, \varepsilon, \iota\hbar\right\} > 0.$

Remark 7: The design parameters l_j and γ_j need to be selected appropriately such that $\underline{q}_{n+2}$ and $\bar{\gamma}_j$, $j = 1,\cdots,n$, are positive constants resulting a positive ρ in inequality (85).

Multiplying inequality (85) by $e^{\rho t}$, one gets

$$\frac{\mathrm{d}\left(e^{\rho t}V_{n+2}\right)}{\mathrm{d}t} \le e^{\rho t}(\mu N(\chi)+1)\dot{\chi} + e^{\rho t}\underline{\theta}_{n+2}. \tag{86}$$

Then integrating and rearranging inequality (86) produce

$$\begin{aligned} V_{n+2} \le\, & e^{-\rho t}\int_0^t (\mu(s)N(\chi)+1)\dot{\chi}e^{\rho s}\mathrm{d}s \\ & + e^{-\rho t}\int_0^t \underline{\theta}_{n+2}e^{\rho s}\mathrm{d}s + e^{-\rho t}V_{n+2}(0) \\ =\, & e^{-\rho t}\int_0^t (\mu(s)N(\chi)+1)\dot{\chi}e^{\rho s}\mathrm{d}\tau + \frac{\underline{\theta}_{n+2}}{\rho} \\ & - \frac{\underline{\theta}_{n+2}}{\rho}e^{-\rho t} + e^{-\rho t}V_{n+2}(0) \\ \le\, & e^{-\rho t}\int_0^t \mu(s)N(\chi)\dot{\chi}e^{\rho s}\mathrm{d}s + e^{-\rho t}\int_0^t \dot{\chi}e^{s\tau}\mathrm{d}s \\ & + \frac{\underline{\theta}_{n+2}}{\rho} + V_{n+2}(0). \end{aligned} \tag{87}$$

Since $\rho > 0$, $\underline{\theta}_{n+2}$, and $V_{n+2}(0)$ are constants, inequality (87) has the same form as inequality (15) given in Lemma 1. Consequently, Lemma 1 can be fully applied to inequality (87), and functions V_{n+2}, χ, and $\int_0^t (\mu(s)N(\chi)+1)\dot{\chi}\mathrm{d}s$ are bounded over a finite interval $\left[0, t_f\right)$ with $t_f \in [0,\infty)$, which means that z_i, η_i, $\hat{\omega}$, α_i, $\hat{\zeta}_i$, x_i, ζ_i, $i = 1,\cdots,n$, and z_{n+1}, z_{n+2}, α_{n+1}, $\bar{u}$ are all bounded with both input magnitude and rate constraints.

Using inequality (87), we have

$$|y| \le 2\sqrt{e^{-\rho t}\int_0^t (\mu(s)N(\chi)+1)\dot{\chi}e^{\rho s}\mathrm{d}s + \frac{\underline{\theta}_{n+2}}{\rho} + V_{n+2}(0)}, \tag{88}$$

that is, if $V_{n+2}(0)$ is bounded, it produces

$$|y| \le \bar{y} \tag{89}$$

where $\bar{y} = 2\sqrt{\bar{N} + \frac{\underline{\theta}_{n+2}}{\rho} + \bar{V}}$, $\bar{V}$, $\bar{N}$ are positive constants satisfying $|V_{n+2}(0)| \le \bar{V}$ and $\int_0^t |(\mu(s)N(\chi)+1)\dot{\chi}|\mathrm{d}s \le \bar{N}$.

Thus Theorem 1 is completely given.

Remark 8: Due to the finiteness of interval $\left[0, t_f\right)$ defined in Lemma 1, the boundedness of all signals can only be guaranteed in a finite interval instead of infinite interval with control. However, that does not hamper the application of Lemma 1 in the control field and the finite $t_f \in [0,\infty)$ can be regarded to be large enough to guarantee the boundedness of all the closed-loop signals during control process as expected.

Remark 9: According to the definition of $\underline{\theta}_{n+2}$, which is proportional to k_i and $\frac{\varepsilon_z^2}{4\widehat{p}}$, $i = 1,\cdots,n+2$, it is known that decreasing the designed control parameter k_i and increasing $\widehat{p}$ can improve the control effect with a reduced control error. Besides, by observing inequality (89), it is obvious that increasing γ_j, which is equivalent to increasing $\bar{\gamma}_j$ for $j = 1,\cdots,n$, or increasing ξ_i to increase ρ may also lead to a smaller control error. Nevertheless, when parameters k_i becomes smaller, and ξ_i and $\widehat{p}$ are selected to be larger, the overall control energy may also grow simultaneously. Consequently, it needs to choose proper control parameters for practical consideration and requirement.

IV. Simulation Results

Effectiveness and performance of the proposed control algorithm are illustrated by two simulation cases in this section. The selection of parameters of the controller should be in the light of Remark 9 according to various needs of users to achieve the ideal control performance.

Example 1. In this example, a second-order Lagrangian system, which is commonly used to depict various industrial systems, such as autonomous ground vehicles (AGVs) and robot manipulator, is considered as an inertial system in the following form [38]

$$m_q\ddot{q}_0 + c_q\dot{q}_0 + k_q q_0 = u(t-\tau) + f_q \tag{90}$$

where q_0 and $\dot{q}_0$ are position/angle and velocity/angular velocity, respectively; m_q, c_q, k_q are system parameters; f_q is the term of dynamic uncertainties of the system.

Then Eq. (90) can be converted to

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{m_q}u(t-\tau) - \frac{c_q}{m_q}x_2 - \frac{k_q}{m_q}x_1 + \frac{1}{m_q}f_q \\ y = \mu x_1 \end{cases} \tag{91}$$

where $x_1 = q_0$, $x_2 = \dot{q}_0$, and μ is an uncertain coefficient as defined in Eq. (7) which is used to simulate the unknown control direction of the system.

System parameters are taken as $m_q = 2$, $c_q = 3$, $k_q = 6$; τ is the time delay of the input signal; the term of dynamic uncertainties is set as $f_q = 0.1x_1(t-\bar{\tau})x_1 + 0.1\sin t$ which is related to states of the system with time delay $\bar{\tau}$. Five NNs $\boldsymbol{W}_z^{\mathrm{T}}\boldsymbol{S}_z(z_1)$ and $\boldsymbol{W}_i^{\mathrm{T}}\boldsymbol{S}_i(\boldsymbol{X}_i)$, $i = 1,2,3,4$, are used and each of them contains 5 hidden nodes, i.e., $n = 5$, with width 0.05. The centers of NNs evenly space on domains $[-2,2]$ and $[-2,2]$, $\prod_1^{12}[-2,2]$, $\prod_1^{13}[-2,2]$, $\prod_1^{14}[-2,2]$, respectively. System initial states are set as $x_1(0) = 0.5$, $x_2(0) = 0.1$, $\hat{\zeta}_1(0) = 0$, $\hat{\zeta}_2(0) = 0$, $\eta_1(0) = 0$, $\eta_2(0) = 0$, $\upsilon(0) = 0$, $\varpi(0) = 0$,

$\chi(0)=0$, $\hat{\omega}(0)=0$, and $\boldsymbol{W}_z=[0,0,0,0,0]^{\mathrm{T}}$. Besides, the uncertain coefficient μ is set as $\mu=-0.7$. The controller is with parameters $l_1=50$, $l_2=50$, $\gamma_1=0.2$, $\gamma_2=0.1$, $\xi_1=2.5$, $\xi_2=4$, $\xi_3=30$, $\xi_4=30$, $k_i=2$, $c_\omega=1$, $\overset{\frown}{p}=1.5$, $\hbar=1$, $\iota=1$, $\lambda=5$, $u_M=1$, and $v_M=20$, $i=1,2,3,4$.

Since the sign of the control coefficient denotes the control direction [11], if the sign of the control coefficient varies between positive and negative values, namely the control direction randomly varies without a priori knowledge and without taking any proper measurement, the control performance will be adversely affected and the control system may eventually be out of control and be unstable with time [39]. To show the capability of our proposed method in dealing with the change of the control direction, the coefficient of the designed control input μ is initially set as $\mu=1$ and dynamically jumps to $\mu=-0.7$ at $t=1$s.

For physical control systems, especially for control systems with uncertainties that are sensitive to environmental changes, even small delays of control and states can directly degrade the control performance and control accuracy [3]. Compared with [8]–[10], delays are set to be larger and various here to cater to the practical delays caused by the signal transmission. To be more diverse, delays in cases 1-3 of Examples 1 and 2 are respectively designed to be constant and time-varying, and the delay time is gradually increased to simulate varying degrees of delays in practice. Following three cases of delays are deliberated in Example 1:

case a: $\tau=0.1$s and $\bar{\tau}=0.5$s;

case b: $\tau=1$s and $\bar{\tau}=2$s;

case c: $\tau=2$s and $\bar{\tau}=4$s.

To show the advantage of our proposed control more clearly, the performance of the traditional PD control that is given as $u=-k_p x_1-k_d x_2$ with $k_p=3$ and $k_d=0.2$ is also portrayed in the presence of delays.

Considering the plant in Eq. (91), Figs. 2-4 clearly show the simulation results of a second-order Lagrangian system in cases a-c of Example 1. Time responses of x_1 and x_2 under PD and proposed controls are respectively given in Figs. 2(a) and 2(b). Figs. 3(a) and 3(b) illustrate the spatial trajectory of the closed-loop system with these two different types of controls in the plane. As revealed by Figs. 2 and 3, the proposed control scheme has more prominent performance than the PD controller when the control system is subject to longer input and state delays in case c, which demonstrates that the delays occurring to the system indeed adversely deteriorate the control effect and urgently require to be effectively compensated by designing proper control algorithms. According to Figs. 2 and 3, although there exist input and state delays, states x_1 and x_2 can still be stabilized with the proposed control scheme within 12s with satisfying performance, which indicates its superiority in dealing with different sorts of delays.

Typical PD and proposed control inputs and their varying rates are respectively given by Figs. 4(a) and 4(b). Observing Fig. 4, it indicates that the varying rate of the traditional PD control far outstrips the specified constraint value. Conversely, the magnitude and varying rate of the proposed control input are satisfactorily constrained within the given bounds with a pulse at t = 1s in u, which is caused by the change of the control direction. The pulse in Fig. 4(b) also indicates that the proposed control is able to effectively deal with the control direction change with timely regulation.

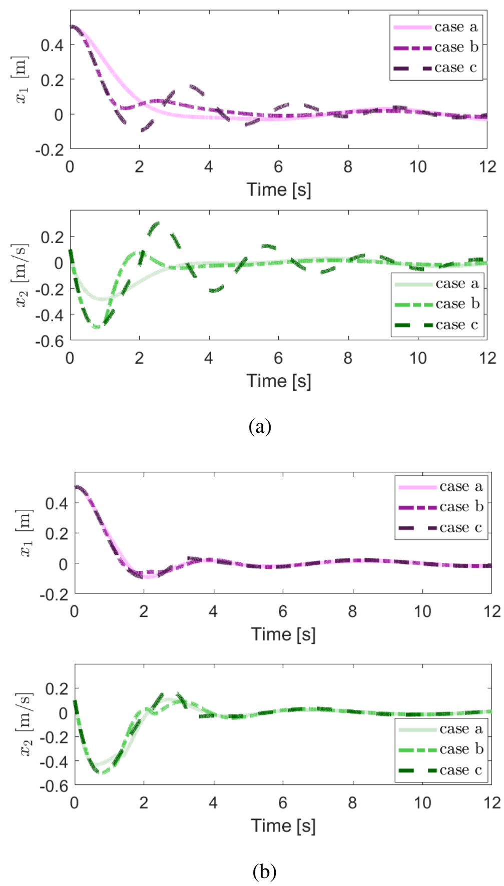


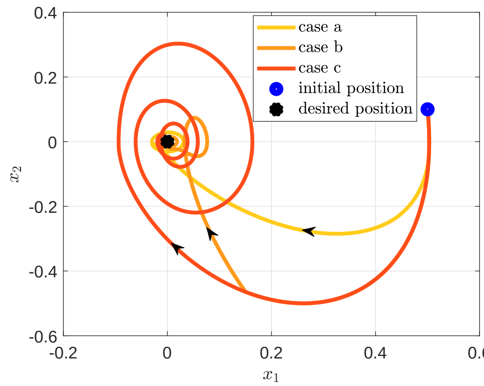
Fig. 2. States x_1 and x_2 in cases a-c of Example 1 with the traditional PD control in (a) and the proposed control in (b).

Example 2. [40] Dynamics of a single-link manipulator considering motor dynamics is given in this illustration and its state-space form can be expressed by

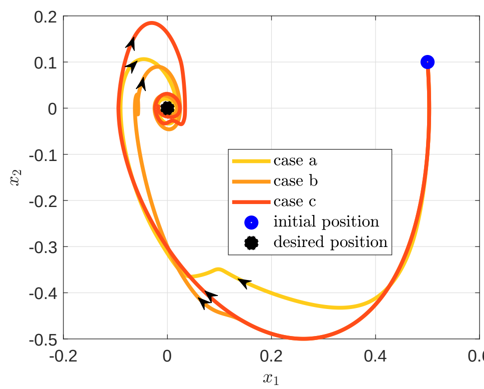
$$\begin{cases} \dot{x}_1=x_2+f_1 \\ J\dot{x}_2=x_3-\theta_1 x_2-\theta_2\sin x_1+f_2 \\ L\dot{x}_3=u(t-\tau)-\theta_3 x_2-\theta_4 x_3+f_3 \\ y=\mu x_1 \end{cases} \tag{92}$$

in which x_1 and x_2 respectively denote position and velocity of the link; x_3 represents the torque; μ is an unknown non-zero constant set as $\mu=0.8$; τ is the input delay; f_1, f_2, f_3 are additional disturbances which are given as $f_1=0.1\sin(x_2 x_1(t-\bar{\tau}_1))$, $f_2=0.1x_1(t-\bar{\tau}_2)+\frac{0.5\sin(x_2)x_1^2(t-\bar{\tau}_2)}{1+x_1^2(t-\bar{\tau}_2)}+0.05\sin(5t)$, $f_3=0.1x_2x_3+0.2x_1^2(t-\bar{\tau}_3)+\frac{0.1x_1^3(t-\bar{\tau}_3)}{1+x_1^2(t-\bar{\tau}_3)}$, where $\bar{\tau}_1$, $\bar{\tau}_2$, $\bar{\tau}_3$ are state delays of the system; J is the moment of inertia; L is the armature inductance; θ_1 is the frictional coefficient; θ_2 is a positive constant determined by the gravity coefficient and payload mass; θ_3 is the back EMF coefficient; θ_4 is the armature resistance of motor.

In the simulation, we use $J=1\mathrm{kgm}^2$, $L=0.5$mH, $\theta_1=0.3$Nms/rad, $\theta_2=1$, $\theta_3=2$Nm/A, and $\theta_4=6\Omega$, with initial

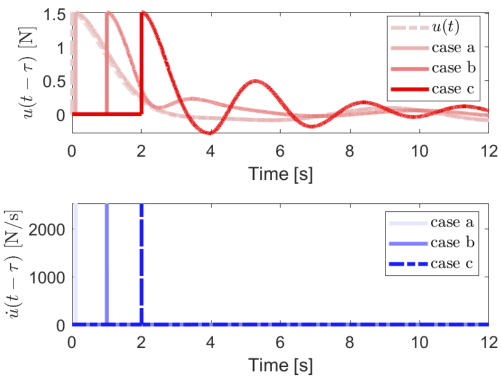


(a)

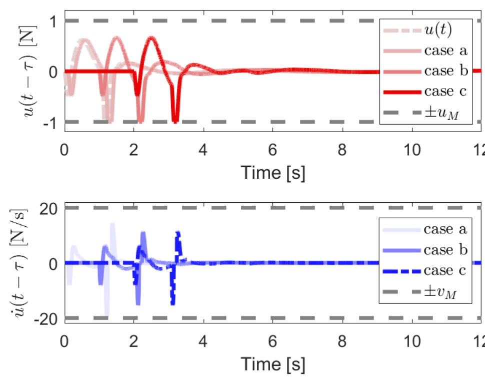


(b)

Fig. 3. Trajectory of the system in cases a-c of Example 1 with the PD control in (a) and the proposed control in (b) in 2D space.



(a)



(b)

Fig. 4. PD control u and its changing rate $\dot{u}$ in (a) and proposed control u and its changing rate $\dot{u}$ in (b) in cases a-c of Example 1.

conditions $x_1(0) = 0.5$m, $x_2(0) = 0$m/s, and $x_3(0) = 0$Nm. Widths and centers of Gaussian basis functions of RBFNNs are selected as same as the ones in Example 1 for $i = 1, \cdots, 5$ and centers of NNs $\boldsymbol{W}_z^{\mathrm{T}} \boldsymbol{S}_z(z_1)$ and $\boldsymbol{W}_i^{\mathrm{T}} \boldsymbol{S}_i(\boldsymbol{X}_i)$ randomly distribute on domains $[-5,5]$, $[-5,5]$, $\prod_1^{12}[-5,5]$, $\prod_1^{14}[-5,5]$, $\prod_1^{15}[-5,5]$, $\prod_1^{16}[-5,5]$, respectively. Initial conditions of auxiliary system, i.e., $\hat{\zeta}_i(0)$, $\eta_i(0)$, $\upsilon(0)$, $\varpi(0)$, $\chi(0)$, $\hat{\omega}(0)$, and $\boldsymbol{W}_z$, $i = 1,2,3$, are all set as zero.

In this case, control parameters are set as $l_1 = l_2 = l_3 = 5$, $\gamma_1 = \gamma_2 = \gamma_3 = 1$, $\xi_1 = 2$, $\xi_2 = 1$, $\xi_3 = 1$, $\xi_4 = 500$, $\xi_5 = 500$, $k_i = 2$, $\frown{p} = 0.1$, $\hbar = 0.1$, $\imath = 0.1$, $c_\omega = 1$, $\lambda = 1$, $u_M = 0.5$, and $v_M = 8$, $i = 1, \cdots, 5$.

To enhance the diversity of delay patterns, delays of control and system states are selected as time-varying functions in this example and three cases are considered as below:

case a: $\tau = (0.24 + 0.06\sin t)$ s and $\bar{\tau}_i = (0.4 + 0.1\sin t)$ s;

case b: $\tau = (0.47 + 0.13\sin 1.5t)$ s and $\bar{\tau}_i = (1.2 + 0.3\cos 0.5t)$ s;

case c: $\tau = (1.8 + 0.7\cos 1.5t)$ s and $\bar{\tau}_i = (3 + \cos 0.5t)$ s, $i = 1,2,3$.

Similar with Example 1, the proposed control coefficient μ is dynamically changed from $\mu = 1$ to $\mu = 0.5$ at $t = 2$s. In such case, only the magnitude of μ varies without sign change to simulate the actuator degradation fault [41].

Additionally, the PD control used for comparison is designed as $u = -k_P x_1 - k_D x_2$ with $k_p = 2$ and $k_d = 0.5$.

Simulation results of the controlled single-link manipulator in cases a-c of Example 2 are elaborated by Figs. 5-7, where the system states x_1, x_2 and x_3 with PD and proposed controls are displayed in Fig. 5 and the spatial 3D trajectory curves are illustrated by Fig. 6. Control signals u and $\dot{u}$ are explicitly depicted by Fig. 7. By observing Figs. 5 and 6, it is clearly seen that the traditional PD control lacks the capability of handling the undesired effects caused by the longer delays in cases b and c with relative larger tracking errors. On the contrary, the designed adaptive neural control can realize desirable stabilization with various delays of states and control inputs in cases a-c, and both magnitude and rate constraints can be strictly ensured during the control process. With the proposed control laws, system states converge to small neighborhoods of zero within 14s after compensating the undesired effect caused by the change of the control direction. The manipulator can finally achieve the target position so that the closed-loop system in Figs. 5(b) and 6(b) has satisfactory performance even subject to delays and uncertain control directions.

Simulation results of Examples 1 and 2 reveal that the proposed adaptive neural control is superior than the typical PD control in dealing with the input and state delays. With the proposed control scheme, all signals of the closed-loop system are bounded and stabilization is realized in spite of various

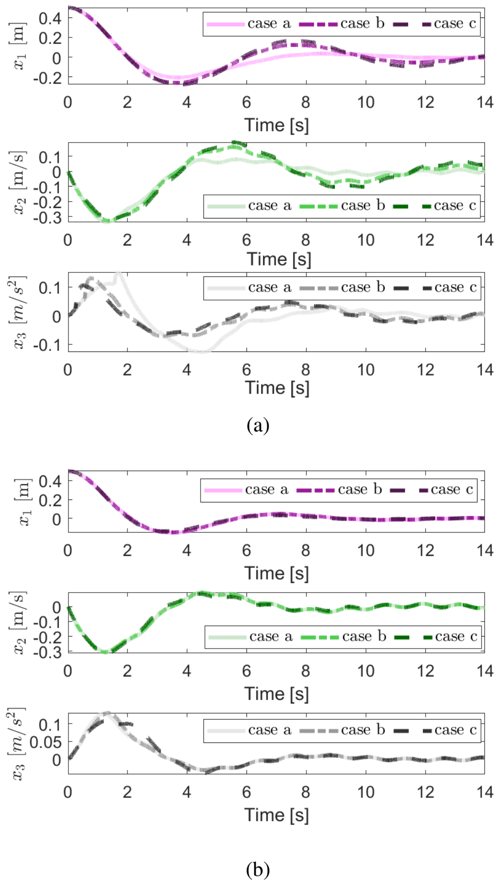


Fig. 5. States x_1, x_2 and x_3 in cases a-c of Example 2 with the traditional PD control in (a) and the proposed control in (b).

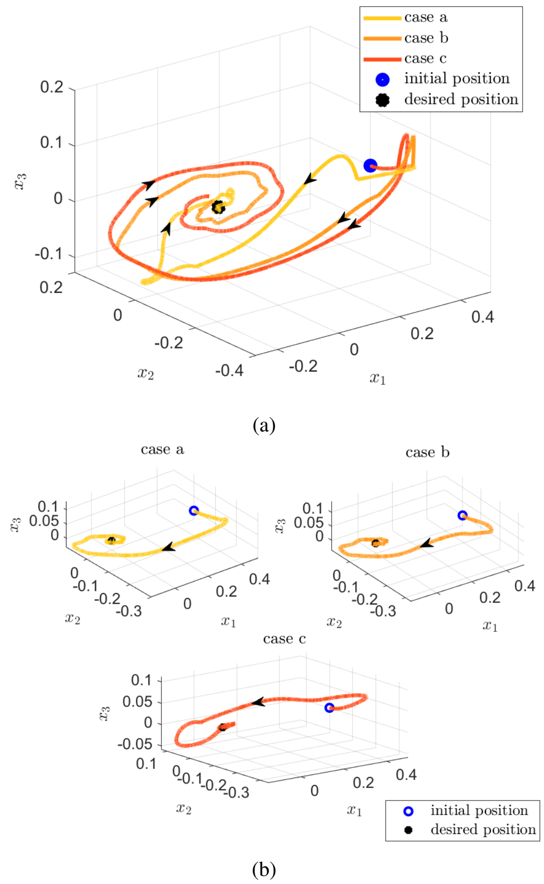


Fig. 6. Trajectory of the system in cases a-c of Example 2 with the PD control in (a) and the proposed control in (b) in 3D space.

input and state time delays with guarantee of input magnitude and rate constraints, which verifies the effectiveness and merits of the designed strategy.

V. Conclusion

In this paper, a NNs-based control algorithm is addressed for a class of nonlinear systems for compensating delays of inputs and states, and constraining control inputs. Both input magnitude and rate limits of the control signals are strictly satisfied during the control course by modeling the magnitude and rate saturations of control inputs with continuous hyperbolic tangent functions. Nussbaum-type function is employed such that the priori knowledge of the control direction is nonessential and the control direction allows to be uncertain. In view of the possible difficulty in system state measurement, an input-driven filter is introduced for the state estimation with which only the output signal is required for control. Using the backstepping technique, the derivation of the control algorithm is implemented. With the help of the developed control, states of the closed-loop system with unknown control directions and disturbances can remain semi-globally, uniformly and ultimately bounded, and eventually converge to residual sets even in the presence of input magnitude and rate saturations and delays of both states and control inputs. Simulation examples are provided and agree with theoretical results. However, the biggest deficiency of our proposed control method is that it can only achieve the bounded stability and the final control performance depends on the selection of control parameters. In the future work, we plan to improve the control method to completely compensate the undesired nonlinearity caused by the uncertainties, and the closed-loop system can thus be asymptotically stable without any residual errors. Besides, the switched nonlinear system [42], [43] and flexible wing systems [44]–[46] with input constraints and time delay will also be our future research subject.

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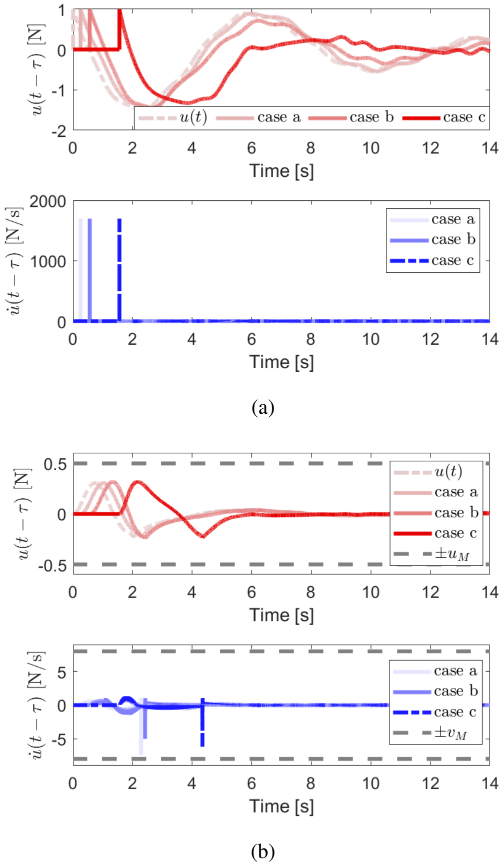

Fig. 7. PD control u and its changing rate $\dot{u}$ in (a) and proposed control u and its changing rate $\dot{u}$ in (b) in cases a-c of Example 2.

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