CONSERVED ENERGIES FOR THE ONE DIMENSIONAL GROSS-PITAEVSKII EQUATION

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Abstract. We prove the global-in-time well-posedness of the one dimensional Gross-Pitaevskii equation in the energy space, which is a complete metric space equipped with a newly introduced metric and with the energy norm describing the H^s regularities of the solutions. We establish a family of conserved energies for the one dimensional Gross-Pitaevskii equation, such that the energy norms of the solutions are conserved globally in time. This family of energies is also conserved by the complex modified Korteweg-de Vries flow.

Keywords: Gross-Pitaevskii equation, modified Korteweg-de Vries equation, well-posedness, transmission coefficient, conserved energies, metric space

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Contents

1. Introduction

We consider the one dimensional Gross-Pitaevskii equation

$$i\partial_t q + \partial_{xx} q = 2q(|q|^2 - 1), \tag{1.1}$$

where $(t,x) \in \mathbb{R}^2$ denote the time and space variables and $q = q(t,x) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$ denotes the unknown complex-valued wave function.

It was used by E.P. Gross [21] and L.P. Pitaevskii [35] to describe the oscillations of a Bose gas at zero temperature. In nonlinear optics, the equation (1.1) models the propagation of a monochromatic wave in a defocusing medium and in particular the dark/black solitons with $|q| = 1$ at infinity arise as solutions of (1.1). See the review paper [29] for more physical interpretations.

The Gross-Pitaevskii equation (1.1) can be viewed as the defocusing cubic nonlinear Schrödinger equation (NLS), but with a nonstandard boundary condition at infinity: $|q| \to 1$ as $|x| \to \infty$. This nonzero boundary condition brings a substantial difference between (1.1) and (NLS) (for which we assume zero boundary condition at infinity): For example, the former equation has soliton solutions (e.g. the black soliton solution $q(t,x) = \tanh(x)$) while the latter equation possesses scattering phenomenon. One will see below that the solution space for the Gross-Pitaevskii equation (1.1) is much more delicate and we will derive a family of conserved energies which describe all the H^s, $s > \frac{1}{2}$ regularities of the solutions in a nonstandard way.

The equation (1.1) can be viewed as a Hamiltonian evolutionary equation associated to the Ginzburg-Landau energy

$$\mathcal{E}_{GL}(q) = \frac{1}{2} \int_{\mathbb{R}} \left((|q|^2 - 1)^2 + |\partial_x q|^2 \right) \mathrm{d}x \,, \tag{1.2}$$

with respect to the symplectic form $\omega(u,v) = \mathrm{Im} \int_{\mathbb{R}} u\bar{v} \, \mathrm{d}x$. P.E. Zhidkov [38] proved the local-in-time well-posedness of the Gross-Pitaevskii equation (1.1) in the so-called Zhidkov's space Z^k, $k = 1, 2, \cdots$, which is the closure of the space $\{q \in C^k(\mathbb{R}) \cap L^\infty(\mathbb{R}) \,|\, \partial_x q \in H^{k-1}(\mathbb{R})\}$ for the norm

$$\|q\|_{Z^k} = \|q\|_{L^\infty} + \sum_{1 \le l \le k} \|\partial_x^l q\|_{L^2}, \tag{1.3}$$

and in particular when $k = 1$, under the initial finite-energy assumption $\mathcal{E}_{GL}(q_0) < \infty$, the finite-energy solution exists globally in time. See also [3, 6, 15, 16, 38, 39] for more results in the n-dimensional case, with $Z^k = Z^k(\mathbb{R})$ above replaced by $Z^k(\mathbb{R}^n)$, $k > \frac{n}{2}$. In dimension $n = 2$ or 3, P. Gérard [17] showed the global-in-time well-posedness of (1.1) in the energy space $Y^1 = \{q \in H^1_{\mathrm{loc}}(\mathbb{R}^n) : |q|^2 - 1 \in L^2(\mathbb{R}^n),\, \nabla q \in L^2(\mathbb{R}^n)\}$, endowed with the metric distance

$$d_{Y^1}(p,q) = \|p - q\|_{Z^1 + H^1} + \left\| |p|^2 - |q|^2 \right\|_{L^2},$$
$$\text{with } \|u\|_{A+B} = \inf\{\|u_1\|_A + \|u_2\|_B \,|\, u = u_1 + u_2,\, u_1 \in A,\, u_2 \in B\},$$

and more topological properties of this complete metric space Y^1 can be found in [18]. Particular attention has been paid to show the existence (or non-existence) of the travelling wave solutions in [7, 10, 20, 34] and there is also rich literature contributed to their stability or instability issues: See [4, 5, 32] and the references therein. Most authors in the study of the stability issues adopt the following metric distance in the energy space:

$$d_E(p,q) = \|p - q\|_{L^2(\{x \in \mathbb{R}^n : |x| \le 1\})} + \||p|^2 - |q|^2\|_{L^2(\mathbb{R}^n)} + \|\nabla p - \nabla q\|_{L^2(\mathbb{R}^n)}.$$

In higher dimensional case $n \geq 4$, [22] (see [23] for the results when $n = 2, 3$) established the scattering theory for the Gross-Pitaevskii equation with the initial data of form $1 + \varphi$ and $\varphi \in H^s$, $s \geq \frac{n}{2} - 1$ sufficiently small.

In this paper we take the general *energy spaces* as follows

$$X^s = \{q \in H^s_{\mathrm{loc}}(\mathbb{R}) : |q|^2 - 1 \in H^{s-1}(\mathbb{R}), \quad \partial_x q \in H^{s-1}(\mathbb{R})\}/\mathbb{S}^1, \quad s \geq 0, \tag{1.4}$$

where $\mathbb{S}^1$ denotes the unit circle, i.e. we identify functions which differ by a multiplcative constant of modulus 1. Recall that for $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R})$ consists of tempered distributions f with finite $H^s(\mathbb{R})$-norm which is defined as follows:

$$\|f\|_{H^s(\mathbb{R})} = \Big(\int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \Big)^{\frac{1}{2}},$$

where $\widehat{f}(\xi)$ denotes the Fourier transform of $f(x)$. We endow the set of functions X^s with the following metric $d^s(\cdot, \cdot)$ [1]:

$$d^s(p, q) = \Big(\int_{\mathbb{R}} \inf_{|\lambda|=1} \big\| \operatorname{sech}(\cdot - y)(\lambda p - q) \big\|^2_{H^s(\mathbb{R})} \, \mathrm{d}y \Big)^{\frac{1}{2}}, \tag{1.5}$$

where $\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$ [2] and we will prove in Section 6 the following theorem:

Theorem 1.1. *Let X^s, $d^s(\cdot, \cdot)$, $s \geq 0$ be defined in (1.4) and (1.5). Then the space $(X^s, d^s(\cdot, \cdot))$ is a complete metric space, with the following topological properties:*

- *The subset $\{q \,|\, q - 1 \in C_0^\infty(\mathbb{R})\}$ is dense in X^s and hence $(X^s, d^s(\cdot, \cdot))$ is separable.*
- *Any ball $B^s_r(q) = \{p \in X^s \,|\, d^s(p, q) < r\}$, $r \in \mathbb{R}^+$, $q \in X^s$, in X^s is contractible.*
- *Any set $\{q \in H^s_{\mathrm{loc}}(\mathbb{R}) : \|\partial_x q\|_{H^{s-1}} + \||q|^2 - 1\|_{H^{s-1}} < C\}$ is contained in some ball $B^s_r(1)$ with r depending on C.*
- *Any closed ball $\overline{B^s_r(q)}$ in X^s, $s > 0$ is weakly sequentially compact.*
- *There is an analytic structure on X^s (see Theorem 6.2 for details).*

In the following we will define the solution of the Gross-Pitaevskii equation (1.1). The initial data $q_0 \in X^s$ has a representative $\tilde{q}_0$. A solution $q(t, \cdot) \in X^s$ with $t \in I$ the time interval will be the projections of some function in t: $\tilde{q}(t, \cdot) \in X^s$. We define the notion of a solution in terms of the representative.

Definition 1.1 (Solutions)**.** *We call $q \in \mathcal{C}(I; X^s)$, $s \geq 0$ to be a solution of the Gross-Pitaevskii equation (1.1) with the initial data $q|_{t=0} = q_0 \in X^s$ on the open time interval $I \ni 0$, if there is $\tilde{q} : I \to H^s_{loc}$ which satisfies that*

$$I \ni t \to \tilde{q}(t) - \tilde{q}(0) \in L^2, \tag{1.6}$$

is weakly continuous and

$$\|\tilde{q}(\cdot) - \tilde{q}_{0,\varepsilon}\|_{L^4([a,b] \times \mathbb{R})} \leq C, \tag{1.7}$$

for some regularized initial data $\tilde{q}_{0,\varepsilon}$ of $\tilde{q}(0)$ and all $0 \in [a, b] \subset I$, such that the equation (1.1) holds in the distributional sense on $I \times \mathbb{R}$ and $\tilde{q}(t)$ projects to $q(t)$.

We have the following well-posedness results.

[1] If $p, q \in X^s$, then we indeed have $\big\||p|^2 - |q|^2\big\|_{H^{s-1}(\mathbb{R})} \leq c d^s(p, q)$ by (6.10) below.

[2] We can take any other strictly positive smooth function which decays fast at infinity instead of $\operatorname{sech}(x)$.

Theorem 1.2. *Let $s \geq 0$. The Gross-Pitaevskii equation* (1.1) *is loally-in-time well-posed in the metric space (X^s, d^s) in the following sense: For any initial data $q_0 \in X^s$, there exists a positive time $\bar{t} \in (0, \infty)$ and a unique local-in-time solution $q \in \mathcal{C}((-\bar{t}, \bar{t}); X^s)$ of* (1.1) *and for any $t \in (0, \bar{t})$, the Gross-Pitaevskii flow map $X^s \ni q_0 \mapsto q \in \mathcal{C}([-t, t]; X^s)$ is continuous. Let $s > \frac{1}{2}$, then the above holds for all $\bar{t} \in \mathbb{R}^+$ and hence the Gross-Pitaevskii equation* (1.1) *is globally-in-time well-posed in the metric space (X^s, d^s).*

Remark 1.1. *Compared with the distance function d^s introduced for the* nonlinear *energy space X^s here, the Zhidkov's norm $\| \cdot \|_{Z^k}$ or the metric d_{Y^1} is more rigid and the subset $\{v \,|\, v - 1 \in \mathcal{S}(\mathbb{R})\}$ is not dense in Z^k or Y^1. The known global well-posedness result in Z^1 does not cover the above global well-posedness result in X^1.*

The equation (1.1) is completely integrable by means of the inverse scattering method. According to the seminal paper by Zakharov-Shabat [37], the equation (1.1) can be viewed as the compatibility condition for the two systems

$$
\begin{aligned}
u_x &= \begin{pmatrix} -i\lambda & q \\ \bar{q} & i\lambda \end{pmatrix} u, \\
u_t &= i \begin{pmatrix} -2\lambda^2 - (|q|^2 - 1) & -2i\lambda q + \partial_x q \\ -2i\lambda \bar{q} - \partial_x \bar{q} & 2\lambda^2 + (|q|^2 - 1) \end{pmatrix} u,
\end{aligned}
\tag{1.8}
$$

where $u : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}^2$ is the unknown vector and $\lambda \in \mathbb{C}$ can be viewed as parameter. More precisely, if we fix $\lambda \in C$ then the (1.1) is the compatibility condition. On the other hand, if (1.1) holds then the compatibility condition is satisfied for all complex numbers λ. The first system in (1.8) can be written in the form of a spectral problem $Lu = \lambda u$ of the so-called Lax operator

$$
L = \begin{pmatrix} i\partial_x & -iq \\ i\bar{q} & -i\partial_x \end{pmatrix}, \tag{1.9}
$$

and correspondingly the second system of (1.8) reads as a differential operator as follows (by eliminating λ using the relation $\lambda u = Lu$)

$$
P = i \begin{pmatrix} 2\partial_x^2 - (|q|^2 - 1) & -q\partial_x - \partial_x q \\ \bar{q}\partial_x + \partial_x \bar{q} & -2\partial_x^2 + (|q|^2 - 1) \end{pmatrix}.
$$

A formal calculation shows that $q(t, x)$ solves the equation (1.1) if and only if there holds the operator evolution equation $L_t = [P; L] := PL - LP$, i.e. the two operators (L, P) form the so-called Lax-pair, which *formally* implies the invariance of the spectra of L by time evolution. Indeed, let the skewadjoint operator P generate a unitary family of evolution operators $U(t', t)$, then

$$
L(t) = U^*(t', t) L(t') U(t', t)
$$

and $L(t)$ and $L(t')$ are similar. The inverse scattering transform relates the evolution of the Gross-Pitaevskii flow to the study of the spectral and scattering property of the Lax operator L. In the classical framework where $q - 1$ is Schwartz function, the self-adjoint operator L has essential spectrum $(-\infty, -1] \cup [1, \infty)$ and at most countably many *simple real* eigenvalues $\{\lambda_m\}$ on $(-1, 1)$. See [1, 11, 12, 13, 14, 19, 37] for more discussions between the potential q and the spectral information of L.

It is interesting to notice that the *complex* defocusing modified Korteweg-de Vries equation

$$\psi_t + \psi_{xxx} - 6|\psi|^2\psi_x = 0, \quad \psi : \mathbb{R}\times\mathbb{R}\to\mathbb{C}, \tag{1.10}$$

possesses also a Lax-pair structure and shares the same Lax operator (1.9): $L_{\mathrm{mKdV}} = \begin{pmatrix} i\partial_x & -i\psi \\ i\overline{\psi} & -i\partial_x \end{pmatrix}$ as the Gross-Pitaevskii equation, although the corresponding matrix/operator P reads differently as

$$P_{\mathrm{mKdV}} = \begin{pmatrix} -4i\lambda^3 - 2i\lambda|\psi|^2 + (\overline{\psi}\psi_x - \psi\overline{\psi}_x) & 4\lambda^2\psi + 2i\lambda\psi_x - \psi_{xx} + 2|\psi|^2\psi \\ 4\lambda^2\overline{\psi} - 2i\lambda\overline{\psi}_x - \overline{\psi}_{xx} + 2|\psi|^2\overline{\psi} & 4i\lambda^3 + 2i\lambda|\psi|^2 - (\overline{\psi}\psi_x - \psi\overline{\psi}_x) \end{pmatrix}. \tag{1.11}$$

In this paper we focus on the first system in (1.8), i.e. the spectral problem $Lu = \lambda u$ for the Lax operator L. It is not hard to see that L is selfadjoint. We study in particular the *time-independent* transmission coefficient $T^{-1}(\lambda)$ associated to it. For the cubic nonlinear Schrödinger equation case, Koch-Tataru [31] (see also [28]) made use of the corresponding invariant transmission coefficient to establish a family of conserved energies which are equivalent to the H^s, $s > -\frac{1}{2}$-norms of the solutions and hence all the H^s-regularities are preserved a priori for regular initial data. We will adopt the idea in [31] to formulate the conserved energies for the Gross-Pitaevskii equation (1.1) and the defocusing modified Korteweg-de Vries equation (1.10).

The first obstacles on the way are the mass $\mathcal{M}$ and momentum $\mathcal{P}$:

$$\mathcal{M} = \int_{\mathbb{R}} (|q|^2 - 1)\,\mathrm{d}x\,, \quad \mathcal{P} = \mathrm{Im}\int_{\mathbb{R}} q\partial_x\bar{q}\,\mathrm{d}x\,, \tag{1.12}$$

which are only well-defined under more integrability assumptions on $|q|^2-1, \partial_x q$, rather than the mere L^2-type boundedness assumptions for $q \in X^s$. In the classical setting where $(q-1)$ is a Schwartz function, we have the following expansion for the logarithm of the transmission coefficient (see [14]): There exist countably many *real* numbers $\{\mathcal{H}^n\}_{n\geq 0}$ such that for any $k \geq 1$,

$$\begin{aligned} &\ln T^{-1}(\lambda) = i\sum_{l=0}^{k-1} \mathcal{H}^l (2z)^{-l-1} + (\ln T^{-1}(\lambda))^{\geq k+1}, \quad \mathrm{Im}\,\lambda > 0, \\ &\text{with } |(\ln T^{-1}(\lambda))^{\geq k+1}| = O(|\lambda|^{-k-1}) \text{ as } |\lambda|\to\infty, \end{aligned} \tag{1.13}$$

where (λ, z) stays on the upper sheet of a Riemann surface $\{(\lambda,z)\in\mathbb{C}^2\,|\,\lambda^2 - z^2 = 1, \mathrm{Im}\,z > 0\}$ (see Subsection 3.1 for more details). We also have the corresponding expansion for $\ln T^{-1}(\lambda)$ as $|\lambda|\to\infty$ for $\mathrm{Im}\,\lambda < 0$, by use of the symmetry $\ln T^{-1}(\lambda) = \overline{\ln T^{-1}(\overline{\lambda})}$. The first three coefficients $\mathcal{H}^0, \mathcal{H}^1, \mathcal{H}^2$ in (1.13) are the conserved mass, momentum and energy (see (1.2)) for the Gross-Pitaevskii equation (1.1) (and hence also for the mKdV (1.10)) respectively:

$$\mathcal{H}^0 = \mathcal{M}, \quad \mathcal{H}^1 = \mathcal{P}, \quad \mathcal{H}^2 = 2\mathcal{E}_{GL} = \int_{\mathbb{R}} \big((|q|^2-1)^2 + |\partial_x q|^2\big)\,\mathrm{d}x\,,$$

and the fourth conserved Hamiltonian $\mathcal{H}^3$ reads (see also Remark 5.1)

$$\mathcal{H}^3 = \mathrm{Im}\int_{\mathbb{R}} \big(\partial_x q\,\partial_{xx}\bar{q} + 3(|q|^2-1)q\partial_x\bar{q}\big)\,\mathrm{d}x\, - \mathcal{P}.$$

The momentum $\mathcal{P}$ is not defined on X^s for any $s \geq 0$ and hence in our L^2-framework $q \in X^s$ we have to consider the *renormalised* transmission coefficient $T_{\rm c}^{-1}(\lambda)$ which will be $T^{-1}(\lambda)$ modulo the mass and momentum (see Theorem 3.1 below for more details).

In constrast to the Nonlinear Schrödinger equation we cannot scale solutions of the Gross-Pitaevskii equation because of the boundary condition at infinity. Hence there is no scaling invariance property for the Gross-Pitaevskii equation and it does not suffice to consider small data. In order to handle the large energy case we introduce the frequence-rescaled Sobolev norm $H^s_\tau(\mathbb{R})$, $\tau \geq 2$ which is equivalent to $H^s(\mathbb{R})$-norm as follows [3]

$$\|f\|^2_{H^s_\tau(\mathbb{R})} = \int_{\mathbb{R}} (\tau^2 + \xi^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \,. \tag{1.14}$$

For any $q \in X^s$, we introduce the following notation

$$\mathbf{q} := (|q|^2 - 1, \partial_x q), \text{ with } \| \mathbf{q} \|^2_{H^s_\tau(\mathbb{R})} = \| |q|^2 - 1 \|^2_{H^s_\tau(\mathbb{R})} + \|\partial_x q\|^2_{H^s_\tau(\mathbb{R})}, \tag{1.15}$$

and we define the associated *energy* $E^s_\tau(q)$ as

$$E^s_\tau(q) := \| \mathbf{q} \|_{H^{s-1}_\tau(\mathbb{R})}, \tag{1.16}$$

which describes the H^s-regularity of q and in particular when $\tau = 2$ we denote simply

$$E^s(q) := E^s_2(q). \tag{1.17}$$

We also introduce the Banach space $l^2_\tau DU^2 = DU^2 + \tau^{\frac{1}{2}} L^2 \supset H^{s-1}$, $s > \frac{1}{2}$ (which can be viewed as a replacement of $H^{-\frac{1}{2}}$ and see Subsection 4.1 below for more details) and the norm

$$\| \mathbf{q} \|_{l^2_\tau DU^2} = \left(\big\| |q|^2 - 1 \big\|^2_{l^2_\tau DU^2} + \big\| \partial_x q \big\|^2_{l^2_\tau DU^2} \right)^{\frac{1}{2}}.$$

It is straightforward to check that the $H^s_\tau(\mathbb{R})$-norm and the $l^2_\tau DU^2(\mathbb{R})$-norm have the following scaling invariance property:

$$\|f\|_{H^s_\tau(\mathbb{R})} = \tau^{s+\frac{1}{2}} \|f_\tau\|_{H^s(\mathbb{R})}, \quad \|f\|_{l^2_\tau DU^2(\mathbb{R})} = \|f_\tau\|_{l^2_1 DU^2(\mathbb{R})}, \quad f_\tau = \frac{1}{\tau} f(\frac{\cdot}{\tau}). \tag{1.18}$$

We establish a family of conserved energy functionals $(\mathcal{E}^s_\tau)_{\tau \geq 2}$ as follows:

Theorem 1.3. *Let $s > \frac{1}{2}$. There exist a constant $C \geq 2$ (depending only on s) and a family of analytic energy functionals $(\mathcal{E}^s_\tau)_{\tau \geq 2} : X^s \mapsto [0, \infty)$, such that*

- *$\mathcal{E}^s_\tau(q)$ is equivalent to $(E^s_\tau(q))^2$ in the following sense:*

$$\begin{aligned} &|\mathcal{E}^s_\tau(q) - (E^s_\tau(q))^2| \leq \frac{C}{\tau} \| \mathbf{q} \|_{l^2_\tau DU^2} (E^s_\tau(q))^2, \\ &\text{if } q \in X^s \text{ such that } \frac{1}{\tau} \| \mathbf{q} \|_{l^2_\tau DU^2} < \frac{1}{2C}, \end{aligned} \tag{1.19}$$

- *$\mathcal{E}^s_\tau(\cdot)$, $\tau \geq 2$ is conserved by the one-dimensional Gross-Pitaevskii flow* (1.1).

[3] $\tau^{-s}\|f\|_{H^s_\tau}$ is the semiclassical Sobolev norm $\big(\int_{\mathbb{R}} (1 + (\hbar\xi)^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\big)^{1/2}$, $\hbar = \tau^{-1}$. We always take $\tau \geq 2$ in the frequency-rescaled Sobolev norms, in order to avoid the possible zeros on $(-1, 1)$ of the transmission coefficient in the formulation of the conserved energies (see (5.3) below).

Correspondingly, for any initial data $q_0 \in X^s$, there exists $\tau_0 \geq C$ depending only on $E^s(q_0)$ such that the unique solution $q \in \mathcal{C}(\mathbb{R}; X^s)$ (given in Theorem 1.2) of the Gross-Pitaevskii equation (1.1) *satisfies the following energy conservation law:*

$$E^s_{\tau_0}(q(t)) \leq 2E^s_{\tau_0}(q_0), \quad \frac{1}{\tau_0}\|\,\mathbf{q}(t)\|_{l^2_{\tau_0}DU^2} < \frac{1}{2C}, \quad \forall t \in \mathbb{R}\,. \tag{1.20}$$

Remark 1.2. *1) One can find the precise definition and the trace formula of the energies $\mathcal{E}^s_\tau(q)$ in Theorem 5.1.*

For example, we have the following trace formula for the conserved energies $\mathcal{E}^s_\tau(q)$ when $s = n \geq 1$ is an integer (recalling $\mathcal{H}^l$ in (1.13)*)*

$$\mathcal{E}^n_\tau(q) = \sum_{l=0}^{n-1} \tau^{2(n-1-l)} \binom{n-1}{l} \mathcal{H}^{2l+2},$$
$$\mathcal{H}^{2l+2} = \frac{1}{\pi}\int_{\mathbb{R}} \xi^{2l+2}\frac{1}{2}\sum_{\pm} \ln|T_c^{-1}|(\pm\sqrt{\xi^2/4+1})d\xi - \frac{1}{2l+3}\sum_m \mathrm{Im}\,(2z_m)^{2l+3},$$

where T_c^{-1} is the renormalised *transmission coefficient defined for any $q \in X^s$ in Theorem 3.1 and $z_m = i\sqrt{1-\lambda_m^2} \in i(0,1]$ with $\{\lambda_m\}_m \subset (-1,1)$ being the possible countably many zeros of the holomorphic function $T_c^{-1}(\lambda)$ and hence the possible eigenvalues of the Lax operator L.*

In particular if $q - 1 \in \mathcal{S}(\mathbb{R})$, then by changing of variables $\xi \to \lambda$ with $\lambda^2 = \xi^2 + 1$ and noticing the symmetry in Subsection 3.2.2: $\ln|T^{-1}|(\lambda + i0) = \ln|T^{-1}|(\lambda - i0)$ for $\lambda \in \mathcal{I}_{cut} = (-\infty, 1] \cup [1, \infty)$,

$$\mathcal{H}^{2l+2} = \frac{2^{2l+3}}{\pi}\int_{\mathcal{I}_{cut}} |\lambda|\sqrt{\lambda^2-1}^{2l+1}\ln|T^{-1}|(\lambda)d\lambda - \frac{1}{2l+3}\sum_m \mathrm{Im}\,(2z_m)^{2l+3},$$

for $l \geq 0$. This can be compared with $2^{2l+3}c_{2l+3,\varrho}$ on Pages 76 in [14].

2) For any ball $B^s_r(q_0) = \{p_0 \in X^s \mid d^s(q_0,p_0) < r\}$, $r > 0$, in X^s such that (see Lemma 6.1 below) for any $p_0 \in B^s_r(q_0)$,

$$E^s(p_0) \leq E^s(q_0) + c(1+E^s(q_0))^{\frac{1}{2}}d^s(q_0,p_0) + c(d^s(q_0,p_0))^2 \leq C(E^s(q_0), r),$$

there exists τ_0 (depending only on $E^s(q_0), r$) such that all the solutions $p \in \mathcal{C}(\mathbb{R}; X^s)$ of the Gross-Pitaevskii equation (1.1) *with the corresponding initial data $p_0 \in B^s_r(q_0)$ satisfy the energy conservation law* (1.20).

3) The idea of the proof of Theorem 1.3 is similar as in [31]*, however due to the nonzero background, the proof requires substantial new ideas and concepts and the characterised quantities in the energy space are the nonlinear function of q: $|q|^2 - 1$ and its derivative q' rather than the solution q itself.*

4) In the proof showing the asymptotic approximation of the Gross-Pitaevskii equation by the Korteweg-de Vries equations in long-wave regime, [6] *made use of the uniform bounds of $E^k(q), k = 1, 2, 3, 4$ which were derived from a linear (and not obvious at all) combination of the first nine energy conservation laws $\mathcal{H}^0, \cdots, \mathcal{H}^8$. Theorem 1.3 here is a first existence result of infinitely many conserved quantities which control $E^k(q), k = 1, 2, \cdots$ of the solutions of the Gross-Pitaevskii equation (and mKdV with the same boundary condition at ∞).*

We also have the following results for the modified KdV equation (1.10). We recall that we define wellposedness in terms of the existence of a representative.

Theorem 1.4. *The complex modified KdV equation* (1.10) *is globally-in-time well-posed in the metric space* (X^s, d^s), $s > \frac{3}{4}$ *in the following sense (as in Theorem 1.2): For any initial data* $\psi_0 \in X^s$, *there exists a unique solution* $\psi \in \mathcal{C}(\mathbb{R}; X^s)$ *(by which we mean that the flow map on* $1 + \mathcal{S}$ *extends continuously to* X^s*) and the flow map* $X^s \ni \psi_0 \mapsto \psi \in \mathcal{C}(\mathbb{R}; X^s)$ *is continuous. The energy functionals* $(\mathcal{E}^s_\tau(\cdot))_{s>\frac{1}{2},\tau\geq 2}$ *established in Theorem 1.3 are also conserved by the modified KdV flow* (1.10).

For real data the flow map extends to a continuous map from X^s *to* $\mathcal{C}(\mathbb{R}; X^s)$ *for* $s \geq 0$.

The following sections are organised as follows:

- In Section 2 we state and prove Theorem 2.1 (*resp.* Theorem 2.2), which states the local-in-time well-posedness of the Gross-Pitaevskii equation (1.1) (*resp.* the modified KdV equation (1.10)) in the energy space (X^s, d^s), $s \geq 0$ (*resp.* $s > \frac{3}{4}$ in the complex case and $s \geq 0$ in the real case): For any initial data $q_0 \in X^s$, there exists a unique solution $q \in \mathcal{C}([-t_0, t_0]; X^s)$ of (1.1) (*resp.* (1.10)), such that the flow map is continuous and the existence time t_0 depends only on $E^s(q_0)$.
- In Section 3 we state Theorem 3.1, where we introduce the renormalised transmission coefficient $T_{\mathrm{c}}^{-1}(\lambda)$ and show the conservation of $T_{\mathrm{c}}^{-1}(\lambda; q(t))$ by the Gross-Pitaevskii flow on the existence time interval I for any solution $q \in \mathcal{C}(I; X^s)$, $s > \frac{1}{2}$.
- Section 4 is devoted to the proof of Theorem 3.1.
- In Section 5 we state and prove Theorem 5.1, where we establish a family of energy functionals $(\mathcal{E}^s_\tau : X^s \mapsto [0,\infty))_{s>\frac{1}{2},\tau\geq 2}$ in terms of $\ln T_{\mathrm{c}}^{-1}$, which satisfies the equivalence relation (1.19).
- Section 6 is devoted to the proof of Theorem 1.1.
- In the Appendix we calculate the quadratic term in the expansion of $\ln T_{\mathrm{c}}^{-1}(\lambda)$ on the imaginary axis.

At the end of this introduction, we prove our main Theorems 1.2 and 1.3 concerning the Gross-Pitaevskii equation (1.1) by use of the results from Theorems 2.1, 3.1 and 5.1. Since the modified KdV equation (1.10) shares the same Lax operator as the Gross-Pitaevskii equation, Theorem 1.4 follows from Theorems 2.2, 3.1 and 5.1 exactly in the same way.

We first state the relations between $E^s_\tau = \|\,\mathbf{q}\,\|_{H^{s-1}_\tau}$, $E^s = E^s_2$ and $\frac{1}{\tau}\|\,\mathbf{q}\,\|_{l^2_\tau DU^2}$.

Lemma 1.1. *There exists a family of constants* $(C_s)_{s>\frac{1}{2}}$ *with* $C_s \geq 1$ *and* $C_s = C_1$, $s \geq 1$ *such that whenever* $\tau \geq 2$, *for all* $s > \frac{1}{2}$,

$$\begin{aligned} &\frac{1}{\tau}\|\,\mathbf{q}\,\|_{l^2_\tau DU^2} \leq C_s \tau^{-\frac{1}{2}-s} E^s_\tau, \quad E^s_\tau \leq C_s \tau^{\max\{0,s-1\}} E^s, \\ &\textit{and hence } \frac{1}{\tau}\|\,\mathbf{q}\,\|_{l^2_\tau DU^2} \leq C_s \tau^{-\frac{1}{2}-\min\{s,1\}} E^s. \end{aligned} \tag{1.21}$$

Proof. We derive from the scaling property (1.18) and the embedding $H^{s-1}(\mathbb{R}) \hookrightarrow l^2_1 DU^2(\mathbb{R})$, $s > \frac{1}{2}$ that

$$\begin{aligned} \frac{1}{\tau}\|\,\mathbf{q}\,\|_{l^2_\tau DU^2} &= \frac{1}{\tau}\Big(\big\|(|q|^2-1)_\tau\big\|^2_{l^2_1 DU^2} + \big\|(\partial_x q)_\tau\big\|^2_{l^2_1 DU^2}\Big)^{\frac{1}{2}}, \text{ with } f_\tau = \frac{1}{\tau} f(\frac{\cdot}{\tau}) \\ &\leq C_s \frac{1}{\tau}\Big(\big\|(|q|^2-1)_\tau\big\|^2_{H^{s-1}} + \big\|(\partial_x q)_\tau\big\|^2_{H^{s-1}}\Big)^{\frac{1}{2}} = C_s \tau^{-\frac{1}{2}-s} E^s_\tau(q). \end{aligned}$$

By virtue of the fact that $\tau \mapsto E^s_\tau$ is decreasing if $s \in (\frac{1}{2}, 1]$ and $E^s_\tau \le (\tau/2)^{s-1} E^s$ if $s \ge 1$, we have $E^s_\tau \le C_s \tau^{\max\{0,s-1\}} E^s$. □

We are going to prove the global-in-time wellposedness result (Theorem 1.2) and the energy conservation law (1.20) (Theorem 1.3) simultaneously, for the initial data $q_0 \in X^s$, $s > \frac{1}{2}$ by use of the following facts from Theorems 2.1, 3.1 and 5.1:

- There exists a unique solution $q \in \mathcal{C}([-t_0, t_0]; X^s)$ of the Gross-Pitaevskii equation with $t_0 > 0$ depending only on $E^s(q_0)$ (by Theorem 2.1);
- The renormalised transmission coefficient $T_c^{-1}(\lambda; q(t))$ is conserved by the Gross-Pitaevskii flow on $[-t_0, t_0]$ (by Theorem 3.1);
- The energy functional $\mathcal{E}^s_{\tau_0}$, which is constructed in terms of $\ln T_c^{-1}$, is also conserved by the Gross-Pitaevskii flow, and furthermore, the equivalence relation (1.19) holds (by Theorem 5.1).

For the initial data $q_0 \in X^s$, we take τ_0 depending only on $E^s(q_0)$ such that (with the constant C given in (1.19))
(1.22)
$$C_s^2 \tau_0^{-\frac{1}{2}-\min\{s,1\}} \big(2E^s(q_0)\big) < \frac{1}{2C} \text{ and hence by Lemma 1.1, } \frac{1}{\tau_0} \| \mathbf{q}_0 \|_{l^2_{\tau_0} DU^2} < \frac{1}{2C}.$$

The equivalence relation (1.19) implies initially $E^s_{\tau_0}(q_0) \le \sqrt{2\mathcal{E}^s_{\tau_0}(q_0)} \le 2E^s_{\tau_0}(q_0)$.

By the equivalence relation (1.19) and the conservation of the energy $\mathcal{E}^s_{\tau_0}(q(t))$, the solution $q \in \mathcal{C}([-t_0, t_0]; X^s)$ satisfies the conservation law (1.20) on the existence time interval $t \in [-t_0, t_0]$ as follows (noticing also (1.21), (1.22)):

$$\begin{aligned} E^s_{\tau_0}(q(t)) &\le \sqrt{2\mathcal{E}^s_{\tau_0}(q(t))} = \sqrt{2\mathcal{E}^s_{\tau_0}(q_0)} \le 2E^s_{\tau_0}(q_0), \\ \frac{1}{\tau_0} \| \mathbf{q}(t) \|_{l^2_{\tau_0} DU^2} &\le C_s \tau_0^{-\frac{1}{2}-s} E^s_{\tau_0}(q(t)) \le C_s \tau_0^{-\frac{1}{2}-s} (2E^s_{\tau_0}(q_0)) \\ &\le C_s^2 \tau_0^{-\frac{1}{2}-\min\{s,1\}} (2E^s(q_0)) < \frac{1}{2C}. \end{aligned}$$

By a continuity argument, the solution q exists globally in time and satisfies the energy conservation law (1.20): $E^s_{\tau_0}(q(t)) \le 2E^s_{\tau_0}(q_0)$, $\forall t \in \mathbb{R}$. Indeed, if not and suppose $I \ne \mathbb{R}$ is the maximal existence time interval for the solution q, then by the above argument we have $E^s_{\tau_0}(q(t)) \le 2E^s_{\tau_0}(q_0)$ for all $t \in I$. By Theorem 2.1 we can extend the solution to a strictly larger time interval than I, which is a contradiction of the maximality of I.

2. Local well-posedness

We prove the locally-in-time well-posedness for the Gross-Pitaevskii equation (see Theorem 2.1) and for the modified Korteweg-de Vries equation (see Theorem 2.2) respectively in this section.

Theorem 2.1. *The Gross-Pitaevskii equation* (1.1) *is locally-in-time well-posed in the metric space* (X^s, d^s), $s \ge 0$ *in the following sense (as in Theorem 1.2):*

- *For any initial data* $q_0 \in X^s$, *there exists* $t_0 > 0$ *depending only on* $E^s(q_0) = \| \mathbf{q}_0 \|_{H_2^{s-1}}$, $\mathbf{q}_0 = \big(|q_0|^2 - 1, q_0'\big)$, *and a unique solution* $q \in \mathcal{C}((-t_0, t_0); X^s)$ *(defined in Definition 1.1) of the Gross-Pitaevskii equation;*
- *For the neighbourhood* $B^s_r(q_0) = \{p_0 \in X^s \,|\, d^s(q_0, p_0) < r\}$, $r > 0$, *of the initial data* $q_0 \in X^s$, *there exists* $t_1 > 0$ *depending only on* $E^s(q_0), r$ *such that the flow map* $B^s_r(q_0) \ni p_0 \mapsto p \in \mathcal{C}((-t_1, t_1); X^s)$ *is continuous.*

We begin with a lemma. Let $\varepsilon \in (0,1]$, with $\varepsilon = 1$ being a legitimate and most natural choice in what follows. We regularize a function f by taking its convolution with the mollifier $\rho_\varepsilon = \varepsilon^{-1}\rho(\varepsilon^{-1}x)$, $0 \le \rho \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \rho = 1$ as $f_\varepsilon := f * \rho_\varepsilon$.

Lemma 2.1. *Let $q \in X^s$, $s \ge 0$ and let $\tilde q$ be a representative. Then*

(1) $\tilde q_\varepsilon \in X^\sigma$ for all $\sigma \ge 0$ and $\tilde q_\varepsilon \to \tilde q$ in X^s as $\varepsilon \to 0$. The map $X^s \ni \tilde q \to \tilde q_\varepsilon \in X^\sigma$ is Lipschitz continuous.

(2) Let $q \in X^s$, and ϕ a Schwartz function such that for one (and hence for all) representative $\tilde q$ there holds $\int_{\mathbb{R}} \tilde q \phi \,\mathrm{d}x \neq 0$. Then there is a neighborhood of q such that this remains true. We fix the representatives with $\int_{\mathbb{R}} \tilde q\phi\,\mathrm{d}x \in (0,\infty)$, then the map

$$X^s \ni q \to \tilde q - \tilde q_\varepsilon \in H^s$$

is continuous.

(3) The map

$$H^s \ni b \to \tilde q + b \in X^s$$

is Lipschitz continuous.

Proof. We derive from

$$\begin{aligned}(f - f_\varepsilon)(x) &= \int_{\mathbb{R}} (f(x) - f(x-y))\rho_\varepsilon(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \int_0^y f'(x-a)\ \mathrm{d}a\ \rho_\varepsilon(y)\,\mathrm{d}y = \int_{\mathbb{R}} f'(x-a) \int_{A(a)} \rho_\varepsilon(y)\,\mathrm{d}y\ \mathrm{d}a\end{aligned}$$

where

$$A(a) = \begin{cases} (a,\infty) & \text{if } a > 0 \\ (-\infty, a) & \text{if } a < 0 \end{cases}$$

that

$$\|f - f_\varepsilon\|_{L^2} \lesssim \|f'\|_{H^{-1}} \tag{2.1}$$

with an absolute implicit constant.

We choose $\eta \in C_0^\infty$. Then

$$\int \eta |f_\varepsilon|^2 dx = \int \eta(|f|^2 - 1)dx + \int \eta dx - \int \eta |f - f_\varepsilon|^2 dx - 2\mathrm{Re} \int \eta \bar f_\varepsilon (f - f_\varepsilon) dx,$$

and hence

$$\int \eta |f_\varepsilon|^2 dx \le 2\||f|^2 - 1\|_{H^{-1}}\|\eta\|_{H^1} + 2\|\eta\|_{L^1} + 3\|f - f_\varepsilon\|_{L^2}^2 \|\eta\|_{sup}.$$

Choosing η appropriately we see that there exists $C > 0$ so that for all $x \in \mathbb{R}$

$$\|f_\varepsilon\|_{L^2([x,x+1])} \le C(1 + \||f|^2 - 1\|_{H^{-1}}^{\frac12} + \|f'\|_{H^{-1}}). \tag{2.2}$$

We may choose $\tilde\rho = \rho * \rho$ and obtain with a small abuse of notation

$$\|f_\varepsilon\|_{L^\infty} \le c\varepsilon^{-1/2} C(1 + \||f|^2 - 1\|_{H^{-1}}^{\frac12} + \|f'\|_{H^{-1}}). \tag{2.3}$$

Using the embedding $L^1 \hookrightarrow H^{-1}$ we estimate using a partition of unity

$$\begin{aligned}\||f_\varepsilon|^2 - 1\|_{H^{-1}} &\le \||f|^2 - 1\|_{H^{-1}} + c\sum_{k\in\mathbb{Z}} \||f_\varepsilon|^2 - |f|^2\|_{L^1((k-1,k+1))} \\ &\le \||f|^2 - 1\|_{H^{-1}} + c\sup{}_{k\in\mathbb{Z}} \left(\|f_\varepsilon\|_{L^2((k-1,k+1))} + \|f\|_{L^2((k-1,k+1))}\right)\|f_\varepsilon - f\|_{L^2} \\ &\le c\Big(\||f|^2 - 1\|_{H^{-1}} + (1 + \|f'\|_{H^{-1}})\|f'\|_{H^{-1}}\Big)\end{aligned}$$

Notice that for any $\sigma \in \mathbb{R}$,

$$\|f_\varepsilon'\|_{H^\sigma} \le C(\varepsilon,\sigma)\|f'\|_{H^{-1}}, \tag{2.4}$$

and hence

$$\begin{aligned} \||f_\varepsilon|^2-1\|_{H^{\sigma+1}} + \|f_\varepsilon'\|_{H^{\sigma-1}} &\lesssim \||f_\varepsilon|^2-1\|_{H^{-1}} + \|\bar f_\varepsilon f_\varepsilon'\|_{H^\sigma} + \|f'\|_{H^{-1}} \\ &\le C(\varepsilon,\sigma,E^0(f))E^0(f). \end{aligned} \tag{2.5}$$

Therefore $\tilde q_\varepsilon \in X^\sigma$ for all $\sigma \ge 0$ and the convergence $\tilde q_\varepsilon \to \tilde q$ in X^s, i.e. $d^s(\tilde q_\varepsilon,\tilde q) \to 0$, as $\varepsilon \to 0$ follows immediately from these considerations. Indeed,

$$\begin{aligned} (d^s(\tilde q_\varepsilon,\tilde q))^2 = \int_{\mathbb{R}} \inf_{|\lambda|=1} \|\operatorname{sech}(x-y)(\tilde q - \lambda \tilde q_\varepsilon)\|_{H^s}^2 \,\mathrm{d}y \;&\le \int_{\mathbb{R}} \|\operatorname{sech}(x-y)(\tilde q-\tilde q_\varepsilon)\|_{H^s}^2\,\mathrm{d}y \\ &\le c_1 \int_{\mathbb{R}} \|\operatorname{sech}(x-y)\tilde q'\|_{H^{s-1}}^2 dy \\ &\le c_2 \|\tilde q'\|_{H^{s-1}}^2 < \infty. \end{aligned}$$

Given $\delta > 0$ we can restrict the above y integration to a compact interval with an error at most δ. The convergence on the compact y interval is immediate.

We turn to the proof of Lipschitz continuity of the map

$$*\rho_\varepsilon : X^s \ni \tilde q \mapsto \tilde q_\varepsilon \in X^\sigma.$$

Indeed, recalling the metric distance function d^s in (1.5), we first calculate $d^s(\tilde q_\varepsilon, \tilde p_\varepsilon)$. We have the following commutator formulae:

$$[\operatorname{sech}(\cdot - y), *\rho_\varepsilon](\lambda\tilde q - \tilde p) = \int_{\mathbb{R}} \int_0^m \operatorname{sech}'(\cdot - y - a)(\lambda \tilde q - \tilde p)(\cdot - m)\,\mathrm{d}a\; \rho_\varepsilon(m)\,\mathrm{d}m\,,$$

where $[\operatorname{sech}(\cdot-y), *\rho_\varepsilon]f = \operatorname{sech}(\cdot - y)\big(f*\rho_\varepsilon\big) - \big(\operatorname{sech}(\cdot-y)f\big)*\rho_\varepsilon$. Hence we derive the Lipschitz continuity of the map $*\rho_\epsilon : X^s \mapsto X^s$ as follows:

$$\begin{aligned} \big(d^s(\tilde q * \rho_\epsilon, \tilde p*\rho_\epsilon)\big)^2 &= \int_{\mathbb{R}} \inf_{|\lambda|=1} \|\operatorname{sech}(\cdot-y)\big((\lambda\tilde q-\tilde p)*\rho_\epsilon\big)\|_{H^s}^2\,\mathrm{d}y \\ &\lesssim \int_{\mathbb{R}} \inf_{|\lambda|=1}\Big(\|\operatorname{sech}(\cdot-y)(\lambda\tilde q-\tilde p)\|_{H^s}^2 + \|\operatorname{sech}'(\cdot-y)(\lambda\tilde q-\tilde p)\|_{H^s}^2\Big)\,\mathrm{d}y \le C(d^s(\tilde q,\tilde p))^2. \end{aligned}$$

We turn to the proof that

$$X^s \ni q \to \tilde q - \tilde q_\varepsilon \in H^s$$

is continuous. First, it is not hard to see that, if ϕ is a Schwartz function and

$$\int \tilde q \phi dx \ne 0,$$

then there is a neighborhood so that this remains true. Now let $\varepsilon_0 > 0$. There exists a smaller neighborhood and a compact interval I so that

$$\|\tilde q - \tilde q_\varepsilon\|_{H^s(\mathbb{R}\setminus I)} \lesssim \|\tilde q'\|_{H^{s-1}(\mathbb{R}\setminus I)} < \varepsilon_0/2$$

for all representative of functions in this smaller neighborhood. Clearly there exists $\delta > 0$ so that

$$\|p - \tilde q\|_{H^s(I)} < \varepsilon_0/2, \quad \forall p \in B_\delta^s(\tilde q).$$

Now let $\tilde q$ be a representative and consider

$$H^s \ni b \to \widetilde{q+b} \in X^s.$$

It suffices to prove that there exists C so that

$$d^s(q+b,q) \leq C\|b\|_{H^s},$$

which follows immediately from the definition of d^s. □

Proof of Theorem 2.1 . If q solves the Gross-Pitaevskii equation (1.1) with the initial data $q_0 \in X^0$ and $\tilde{q}$, $\tilde{q}_0$ are the corresponding representatives of $q(t), q_0$, then $b = \tilde{q} - \tilde{q}_{0,\varepsilon}$ satisfies the following nonlinear Schrödinger-type equation

$$i\partial_t b + \partial_{xx} b = g(b), \quad b|_{t=0} = b_0 = \tilde{q}_0 - \tilde{q}_{0,\varepsilon} \in H^s, \tag{2.6}$$

where

$$g(b) = 2|b|^2 b + 4\tilde{q}_{0,\varepsilon}|b|^2 + 2\overline{\tilde{q}_{0,\varepsilon}} b^2 + (4|\tilde{q}_{0,\varepsilon}|^2 - 2)b + 2(\tilde{q}_{0,\varepsilon})^2 \bar{b} + 2\tilde{q}_{0,\varepsilon}(|\tilde{q}_{0,\varepsilon}|^2 - 1) - \tilde{q}''_{0,\varepsilon}.$$

Vice versa: If b satisfies this equation then $\tilde{q}$ satisfies (1.1).

We claim that there exist a positive time t^0 and a positive constant C^0 depending only on $E^0(q_0), \varepsilon$ and a unique solution of (2.6): $b \in \mathcal{C}([-t^0, t^0]; L^2)$ such that

$$\|b\|_{t^0} := \|b(t)\|_{L^\infty([-t^0,t^0];L^2_x)} + \|b\|_{L^8([-t^0,t^0];L^4(\mathbb{R}_x))} + \|b\|_{L^6([-t^0,t^0]\times\mathbb{R})} \leq C^0 E^0(q_0).$$

Indeed, recall the Strichartz estimates for the Schrödinger semigroup $S(t) = e^{it\partial_{xx}}$:

$$\|S(t)b_0\|_T \lesssim \|b_0\|_{L^2}, \quad \Big\| \int_0^t S(t-t')g(t')dt' \Big\|_T \lesssim \|g\|_{L^1([-T,T];L^2_x)}.$$

Since we derive from the estimates (2.5) that

$$\begin{aligned} \|g(b)\|_{L^1([-T,T];L^2_x)} \lesssim\, & T^{\frac12}\|b\|_T^3 + T^{\frac34}\|\tilde{q}_{0,\varepsilon}\|_{L^\infty_x}\|b\|_T^2 \\ & + T(\|\tilde{q}_{0,\varepsilon}\|^2_{L^\infty_x} + 1)\big(\|b\|_T + \||\tilde{q}_{0,\varepsilon}|^2 - 1\|_{L^2_x} + \|\tilde{q}''_{0,\varepsilon}\|_{L^2_x}\big) \\ \leq\, & C(\varepsilon, E^0(q_0))\Big(T^{\frac12}\|b\|_T^3 + T^{\frac34}\|b\|_T^2 + T(\|b\|_T + 1)\Big), \end{aligned}$$

there exist a small enough positive time t^0 and a positive constant C^0 (depending only on $\varepsilon, E^0(q_0)$) such that the map

$$b \mapsto S(t)b_0 + \int_0^t S(t-t')g(b(t'))dt',$$

is a contraction map in the complete metric space $\{b \in \mathcal{C}([-t^0,t^0];L^2) \,|\, \|b\|_{t^0} \leq C^0 E^s(q_0)\}$, and hence its fixed point is the unique solution of (2.6). It is easy to see that the flow map $b_0 \mapsto b(t)$ is locally Lipschitz in L^2. Correspondingly there exists a solution (in Definition 1.1) with $\tilde{q} = \tilde{q}_{0,\varepsilon} + b \in \mathcal{C}((-t^0,t^0); X^s + L^2) = \mathcal{C}((-t^0,t^0); X^0)$ of the Gross-Pitaevskii equation (1.1) with the initial data $q_0 \in X^0$, such that $\|\tilde{q}(t) - \tilde{q}_0\|_{L^2} \leq \|\tilde{q}_{0,\varepsilon} - \tilde{q}_0\|_{L^2} + \|b(t)\|_{L^2} \leq (C + C^0)E^0(q_0)$, $\forall t \in [-t^0, t^0]$ and $\tilde{q}(t) - \tilde{q}_{0,\varepsilon} = b(t) \in L^4([-t^0,t^0]\times\mathbb{R})$.

Consider two solutions (in Definition 1.1) $q_1, q_2 \in \mathcal{C}((-t^0,t^0); X^0)$ of the Gross-Pitaevskii equation (1.1) with the initial data $q_0 \in X^0$. We may choose for both solutions the same representative $\tilde{q}_0$. Then on any compact time interval $I \ni 0$ in $(-t^0, t^0)$ their difference $b_{12} = \tilde{q}_1 - \tilde{q}_2 \in L^\infty(I; L^2) \cap L^4(I \times \mathbb{R})$ with zero initial data satisfies in the distribution sense the following equation similar as (2.6):

$$i\partial_t b_{12} + \partial_{xx} b_{12} = 2|b_{12}|^2 b_{12} + 4\tilde{q}_2|b_{12}|^2 + 2\overline{\tilde{q}_2}(b_{12})^2 + (4|\tilde{q}_2|^2 - 2)b_{12} + 2(\tilde{q}_2)^2 \bar{b}_{12},$$

which has a unique solution 0 in $L^\infty([a,b];L^2)$, by virtue of the energy inequality [4].Hence $\tilde q_1=\tilde q_2=\tilde q_{0,\varepsilon}+b$ with b satisfying (2.6).

If $s\in(0,1)$, we decompose $g=g(b)$ into

$$g=g_2(b)+g_1(b)+g_0,\quad g_1(b)=(4|\tilde q_{0,\varepsilon}|^2-2)b+2(\tilde q_{0,\varepsilon})^2\bar b,\quad g_0=2\tilde q_{0,\varepsilon}(|\tilde q_{0,\varepsilon}|^2-1)-\tilde q_{0,\varepsilon}''.$$

Recall the definition of the Besov-norm for $s\in(0,1)$:

$$\|f\|_{\dot B^s_{\alpha,r}}=\Big\|\frac{\|f(x-y)-f(x)\|_{L^\alpha_x}}{|y|^s}\Big\|_{L^r(\mathbb{R};\frac{dy}{|y|})},\quad \|f\|_{B^s_{\alpha,r}}=\|f\|_{L^\alpha}+\|f\|_{\dot B^s_{\alpha,r}},$$

and in particular $\dot B^s_{2,2}=\dot H^s$. We apply the previous construction to the finite differences, and integrate the estimates for fixed y according to the Besov norm above. It follows from these construction that the time of existence is the same for all $s\in[0,1)$.

The case $s\ge 1$ follows similarly.

Therefore the Gross-Pitaevskii flow map $X^s\ni\tilde q_0\mapsto\tilde q_0*\rho_\varepsilon+b\in X^s$ is continuous on the existence time interval $[-t^0,t^0]$. Indeed, by the Lipschitz continuity of the flow (2.6), for any two solutions $\tilde q_1(t)=\tilde q_{1,\varepsilon}+b_1(t)$ and $\tilde q_2(t)=\tilde q_{2,\varepsilon}+b_2(t)$,

$$d^s(\tilde q_1(t),\tilde q_2(t))\le d^s(\tilde q_{1,\varepsilon},\tilde q_{2,\varepsilon})+C\|b_1(0)-b_2(0)\|_{H^s},$$

and the continuity of the GP flow follows from Lemma 2.1.

□

We complete this section by a discussion of the flow defined by modified Korteweg-de Vries equation (1.10): $q_t+q_{xxx}-6|q|^2q_x=0$.

Theorem 2.2. *The complex modified KdV equation* (1.10) *is locally-in-time well-posed in the metric space* (X^s,d^s), $s>\frac34$ *in the following sense (as in Theorem 1.2):*

- *For any initial data* $q_0\in X^s$*, there exists* $t_0>0$ *depending only on* $E^s(q_0)=\|\,\mathbf{q}_0\,\|_{H^{s-1}_2}$, $\mathbf{q}_0=\big(|q_0|^2-1,q_0'\big)$*, and a unique solution* $q\in\mathcal{C}((-t_0,t_0);X^s)$*, by which we mean that the flow map on* $1+\mathcal{S}$ *extends continuously to* X^s*.*
- *For the neighbourhood* $B^s_r(q_0)=\{p_0\in X^s\,|\,d^s(q_0,p_0)<r\}$, $r>0$*, of the initial data* $q_0\in X^s$*, there exists* $t_1>0$ *depending only on* $E^s(q_0),r$ *such that the flow map* $B^s_r(q_0)\ni p_0\mapsto p\in\mathcal{C}((-t_1,t_1);X^s)$ *is continuous.*

For real data the flow map extends to a continuous map from $X^s_{\mathbb{R}}$ *to* $\mathcal{C}((-t_1,t_1);X^s)$ *for* $s\ge 0$*. Here* $X^s_{\mathbb{R}}$ *denotes the subspace of real valued functions.*

Proof. We proceed in the same fashion as for the Gross-Pitaevskii equation. Now $b=\tilde q-\tilde q_{0,\varepsilon}$ satisfies

$$b_t+b_{xxx}=g(b) \tag{2.7}$$

where

$$\begin{aligned}g(b)=&6|b|^2b_x+12\mathrm{Re}\,(b\overline{\tilde q_{0,\varepsilon}})b_x+6b_x+6(|\tilde q_{0,\varepsilon}|^2-1)b_x+6|b|^2\tilde q_{0,\varepsilon}'\\&+12\mathrm{Re}\,(b\overline{\tilde q_{0,\varepsilon}}_x)\tilde q_{0,\varepsilon}+6|\tilde q_{0,\varepsilon}|^2\partial_x\tilde q_{0,\varepsilon}-\tilde q_{0,\varepsilon}^{(3)}.\end{aligned}$$

[4]We can follow the standard regularizing procedure to derive the energy inequality: We regularize the b_{12}-equation by convolution with ρ_δ, take the $L^2(\mathbb{R})$-inner product between the regularized equation and $b_{12,\delta}=b_{12}*\rho_\delta$, take the imaginary part and finally we use Gronwall's inequality and let $\delta\to 0$.

Changing coordinates $(t,x)\to(t,y)$ with $y=x+6t$ we remove the term $6b_x$. Notice that

$$6|\tilde q_{0,\varepsilon}|^2\partial_x\tilde q_{0,\varepsilon}-\tilde q_{0,\varepsilon}^{(3)}\in L^2$$

and $\tilde q_{0,\varepsilon}$ is together with all its derivatives uniformly bounded. The most critical terms are $6|b|^2b_x$, $12\mathrm{Re}\,(b\overline{\tilde q_{0,\varepsilon}})b_x$ and $6(|\tilde q_{0,\varepsilon}|^2-1)b_x$.

We claim that (2.7) is locally wellposed in H^s, $s>\frac{3}{4}$, and that the solution is continuous with values in H^s. Indeed, this follows from a contraction argument as for the Korteweg-de Vries equation

$$u_t+u_{xxx}-6uu_x=0 \tag{2.8}$$

by Kenig, Ponce and Vega [25, 26]. More precisely their arguments allow to deal with $|b|^2b_x$ and $\mathrm{Re}\,(b\overline{\tilde q_{0,\varepsilon}})b_x$. Since $|\tilde q_{0,\varepsilon}|^2-1\in H^N$ for all N the term

$$(|\tilde q_{0,\varepsilon}|^2-1)b_x$$

is covered by the same estimates as the previous terms.

For real initial data we use a different argument. Let $s\ge 0$. Then again $|q_{0,\varepsilon}|^2-1\in H^N$ for all N. Since it is also real we must have one of the following alternatives for fixed N:

(1) $q_{0,\varepsilon}-1\in H^N$
(2) $q_{0,\varepsilon}-\tanh(x)\in H^N$
(3) $q_{0,\varepsilon}+1\in H^N$
(4) $q_{0,\varepsilon}+\tanh(x)\in H^N$.

Replacing q by $-q$ if necessary it remains to consider two situations:

(i) $q+1\in H^s$
(ii) $q+\tanh(x)\in H^s$.

It is easy to see that $q\in X^s$ if one these situations holds. We recall the definition of the Miura map

$$M(q)=q_x+q^2.$$

Then the following lemma holds.

Lemma 2.2. *A) The map*

$$H^s\ni w\to\big(M(w-1)-1\big)\in H^{s-1}$$

is a diffeomorphism of H^s to its range

$$\{u\in H^{s-1}:-\partial_{xx}+u \text{ has no eigenvalue } \le -1\}.$$

B) The map

$$H^s\times(0,\infty)\ni(w,\lambda)\to\big(M(w-\lambda\tanh(\lambda x))-\lambda^2\big)\in H^{s-1}$$

is a diffeomorphism to its range

$$\{u\in H^{s-1}:-\partial_{xx}+u \text{ has a negative eigenvalue}\}.$$

Moreover $-\lambda^2$ is the lowest eigenvalue.

C) In both cases A) and B), let $q=w-1$ resp. $q=w-\tanh(x)$, then $q:\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$ satisfies the real modified KdV (1.10) iff

$$u=M(q)-1=q_x+q^2-1$$

satisfies the KdV equation (2.8):

$$u_t-6u_x+u_{xxx}-6uu_x=0. \tag{2.9}$$

Since the KdV equation (2.8) is wellposed in H^{-1} [27] (in the sense that the flow map extends continuously), if $w = q + 1 \in H^s$ then it follows from A) and C) that $u = M(q) - 1 \in H^{s-1} \subset H^{-1}$ and (1.10) is well-posed in H^s, $s \geq 0$. Similarly Theorem 2.2 follows in the case (ii): $w = q + \tanh(x) \in H^s$.

It remains to prove Lemma 2.2. Part B) of the Lemma has been proven by Buckmaster and the first author [8]. If q satisfies (1.10) then $u = q_x + q^2$ satisfies (2.8). Now suppose that u satisfies KdV (2.9) (the term $6u_x$ is inessential, and can be removed by a Galilean transform). Since the preimage is unique it has to be a solution to mKdV, at least if the initial data is sufficiently smooth. This can be achieved by an approximation argument.

It remains to prove A). It is easy to see (compare [8]) that $w \in H^s$ implies $M(w-1) - 1 \in H^{s-1}$ for $s \geq 0$. Moreover this map is clearly analytic. The derivative at w_0 is

$$\dot{w} \to \dot{w}_x + 2(w_0 - 1)\dot{w}$$

which has the (right) inverse

$$(Tf)(x) = -\int_x^\infty e^{2\int_x^y w_0 d\tau - 2(y-x)} f(y) dy.$$

It is easy to see that T maps H^{s-1} to H^s for all $s \geq 0$ and $w_0 \in H^s$. Moreover the linearization is injective. Indeed, suppose that $\dot{w} \in L^2$ satisfies

$$\dot{w}_x + 2(w_0 - 1)\dot{w} = 0.$$

Then $\dot{w}$ is absolutely continuous and decays to 0 as $x \to \infty$. The variation of constants formula and a limit argument show that $\dot{w}$ vanishes.

To verify injectivity of the nonlinear map we assume that w_0 and w_1 are mapped to the same function. Then, with $\dot{w} = w_1 - w_0$

$$\dot{w}_x + (w_0 + w_1)\dot{w} - 2\dot{w} = 0$$

and hence $\dot{w} = 0$ by the same argument as for the injectivity of the linearization.

The argument for surjectivity is based on Kappeler *et al* [30]. Let $u \in H^{-1}$ be a function so that the spectrum of $-\partial^2 + u$ is contained in $(-1, \infty)$. According to [30] there exists a bounded positive function ϕ which satisfies

$$-\phi'' + u\phi + \phi = 0. \tag{2.10}$$

Let $v = \frac{d}{dx} \ln \phi$. A straightforward calculation shows that

$$v' + v^2 = u + 1, \text{ i.e. } M(v) - 1 = u.$$

Let $\tilde{v} = v * \rho$ where $\rho \in C_0^\infty$ is supported in $[-1, 1]$ with integral 1. It suffices to find $\tilde{v}$ so that

$$\lim_{x \to -\infty} \tilde{v}(x) = -1, \quad \lim_{x \to \infty} \tilde{v}(x) = -1.$$

Now we use [8] to see that $\tilde{v}$ has a limit in $\{\pm 1\}$ as $x \to \pm\infty$, possibly different on both sides. If

$$\lim_{x \to -\infty} \tilde{v}(x) = 1, \lim_{x \to \infty} \tilde{v}(x) = -1,$$

then $\phi \in L^2$ and it were an eigenfunction of the eigenvalue -1, which contradicts our assumption. Thus, if

$$\lim_{x \to \infty} \tilde{v}(x) = -1,$$

then $\lim_{x\to-\infty}\tilde{v}(x) = -1$ and we found the preimage in case A). Hence suppose that

$$\lim_{x\to\infty}\tilde{v}(x) = 1.$$

Then

$$\phi_1 = \phi(x)\int_x^\infty \phi(y)^{-2}dy$$

is a nonnegative solution of (2.10) and is bounded for positive x. Hence $v_1 = \frac{d}{dx}\ln\phi_1$ satisfies (with $\tilde{v}_1 = v_1 * \rho$ as above)

$$\lim_{x\to\infty}\tilde{v}_1(x) = -1$$

and, by the previous considerations of our assumption,

$$\lim_{x\to-\infty}\tilde{v}_1(x) = -1.$$

With this we have found the preimage v_1 in case A). □

3. The transmission coefficient

We introduce the renormalised transmission coefficient $T_c^{-1}(\lambda)$ and state its properties in Theorem 3.1 in this section.

We will first recall the definition of the transmission coefficient T^{-1} associated to the Lax operator (1.9), i.e. the Lax equation:

$$u_x = \begin{pmatrix} -i\lambda & q \\ \bar{q} & i\lambda \end{pmatrix} u, \tag{3.1}$$

on the Riemann surface $\mathcal{R}$ (see Subsection 3.1 below for the definition), in the classical functional setting where $q-1$ is Schwartz function in Subsection 3.2.

With the notations introduced in Subsection 3.3, we will give an *asymptotic expansion* of the transmission coefficient T^{-1} in Subsection 3.4, which will play a key role in the analysis of T^{-1}.

Finally in Subsection 3.5 we discuss the renormalisation of the transmission coefficient and give Theorem 3.1 stating the well-definedness and the asymptotic expansion of the renormalised transmission coefficient T_c^{-1} in our finite energy setting $q \in X^s$, $s > \frac{1}{2}$, whose proof will be postponed in Section 4.

3.1. **A Riemann surface.**

We define a Riemann surface by

$$\{(\lambda, z) \in \mathbb{C}^2 \mid \lambda^2 = 1 + z^2\}.$$

If infinity is added, its genus is 0 and it is indeed the Riemann sphere with respect to the complex variable $\zeta := \lambda + z$.

We typically choose the upper sheet $\mathcal{R}$ of this Riemann surface:

$$\begin{aligned} &\mathcal{R} = \big\{(\lambda, z) \mid \lambda \in \mathcal{V}, \quad z = z(\lambda) = \sqrt{\lambda^2-1} \in \mathcal{U}\big\}, \\ &\text{where } \mathcal{V} := \mathbb{C}\setminus\mathcal{I}_{\text{cut}},\ \mathcal{I}_{\text{cut}} := (-\infty,-1]\cup[1,+\infty),\ \mathcal{U} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}, \end{aligned} \tag{3.2}$$

and we can take simply $\lambda \in \mathcal{V}$ as the coordinate on $\mathcal{R}$. We notice the following symmetry of $\mathcal{R}$

$$(\lambda, z) \in \mathcal{R} \Leftrightarrow (\overline{\lambda}, -\overline{z}) \in \mathcal{R}. \tag{3.3}$$

In particular, the points $(\pm\sqrt{1-\tau^2/4}, i\tau/2)$, $\tau \in (0,2]$ and the purely imaginary points $(\pm i\sigma, i\tau/2)$, $\tau \in [2,\infty)$ stay on $\mathcal{R}$:

$$(\pm i\sigma, i\tau/2) \in \mathcal{R} \text{ whenever } \tau \geq 2 \text{ and } \sigma = \sqrt{\tau^2/4 - 1} \in \mathbb{R}\,. \tag{3.4}$$

We can define a conformal mapping from $(\lambda, z) \in \mathcal{R}$ to $\zeta \in \mathcal{U}$ the upper half-plane by $\zeta = \zeta(\lambda) = \lambda + z$. The mapping takes the cuts $\lambda \in \mathcal{I}_{\text{cut}}$ to the real axis $\zeta \in \mathbb{R}$ and the neighbourhood of ∞ for $\operatorname{Im}\lambda < 0$ to a neighbourhood of $\zeta = 0$. The inverse mapping is given by the so-called Zukowsky mapping $\zeta \mapsto \lambda = \lambda(\zeta) = \frac{1}{2}(\zeta + \frac{1}{\zeta})$ and hence $1 = (\lambda - z)\zeta$, $z = z(\zeta) = \frac{1}{2}(\zeta - \frac{1}{\zeta})$, $\frac{1}{\zeta} = \bar{\lambda} - \bar{z} \in \mathcal{U}$.

3.2. **Jost solutions and the transmission coefficient.** In this subsection we assume the classical functional setting (as in [14, 37])

$$q = 1 + q^0, \quad q^0 \in \mathcal{S}(\mathbb{R}) \text{ Schwartz function,} \tag{3.5}$$

and we are going to introduce the Jost solution of the Lax equation (3.1) as well as the associated transmission coefficient.

3.2.1. *Real line case* $(\lambda, z) = (\hat{\xi}, \xi) \in \mathbb{R}^2$. Let $0 \neq z = \xi \in \mathbb{R}$ and $\lambda = \hat{\xi} \in \mathbb{R}$ such that $\hat{\xi} = (1+\xi^2)^{\frac{1}{2}} > 1$. Then under the assumption (3.5) on the potential q, $\pm i\xi$ are the two eigenvalues of the matrix in (3.1) at infinity: $\begin{pmatrix} -i\hat{\xi} & 1 \\ 1 & i\hat{\xi} \end{pmatrix}$.

Let the Jost solution u_l solve the Lax equation (3.1) (viewing $\lambda = \hat{\xi}$ as parameter) satisfying the following boundary conditions at $-\infty$

$$u_l(x) = e^{-i\xi x} \begin{pmatrix} 1 \\ i(\hat{\xi} - \xi) \end{pmatrix} + o(1) \text{ as } x \to -\infty.$$

Then there exist two complex numbers $T^{-1}, R \in \mathbb{C}$ such that u_l takes the following asymptotic at $+\infty$:

$$u_l(x) = e^{-i\xi x} T^{-1} \begin{pmatrix} 1 \\ i(\hat{\xi} - \xi) \end{pmatrix} + e^{i\xi x} R T^{-1} \begin{pmatrix} 1 \\ i(\hat{\xi} + \xi) \end{pmatrix} + o(1) \text{ as } x \to +\infty.$$

These two complex numbers T^{-1}, R are called the transmission coefficient and the right reflection coefficient respectively [5].

Observe that if $u = (u^1, u^2)^T$ is the solution of (3.1), then the quantity $|u^1|^2 - |u^2|^2$ is constant. We compare the asymptotic behaviours of the Jost solution u_l at $\pm\infty$ respectively to acquire

$$|T|^2 = 1 - (\hat{\xi} + \xi)^2 |R|^2 \leq 1, \quad \text{if } (\lambda, z) = (\hat{\xi}, \xi) \in \mathbb{R}^2\,. \tag{3.6}$$

---

[5] In this paper we call T^{-1} the transmission coefficient while its reciprocal T is the physical relevant transmission coefficient. We can define similarly the left reflection coefficient by considering the asymptotic at $-\infty$ of the Jost solution with the boundary condition $e^{i\xi x} \begin{pmatrix} 1 \\ i(\hat{\xi} + \xi) \end{pmatrix} + o(1)$ as $x \to +\infty$.

3.2.2. *Upper Riemann sheet case* $(\lambda, z) \in \mathcal{R}$. The Jost solution $u_l(x;\lambda)$ defined above on the "real axis" $(\lambda, z) = (\hat{\xi}, \xi) \in \mathbb{R}^2$ can be analytically continued to the upper Riemann sheet $(\lambda, z) \in \mathcal{R}$, taking the following asymptotics

$$
\begin{aligned}
u_l(\lambda, x, t) &= e^{-izx} \begin{pmatrix} 1 \\ i(\lambda - z) \end{pmatrix} + o(1) e^{(\operatorname{Im} z)x} \text{ as } x \to -\infty, \\
u_l(\lambda, x, t) &= e^{-izx} T^{-1}(\lambda) \begin{pmatrix} 1 \\ i(\lambda - z) \end{pmatrix} + o(1) e^{(\operatorname{Im} z)x} \text{ as } x \to +\infty.
\end{aligned} \tag{3.7}
$$

Under the potential assumption (3.5), $T^{-1}(\lambda)$ is a holomorphic function on $\mathcal{R}$ and $\lim_{|\lambda|\to\infty} T^{-1}(\lambda) = 1$.

The possible zeros of $T^{-1}(\lambda)$ for $\lambda \in \mathbb{C} \setminus (\mathbb{R} \setminus (-1,1))$ are located on the interval $(-1,1) \subset \mathbb{R}$. Indeed, if $T^{-1}(\lambda) = 0$, then $\lambda \in \mathcal{V} = \mathbb{C} \setminus \mathcal{I}_{\text{cut}}$ by (3.6). Thus $z = \sqrt{\lambda^2 - 1} \in \mathcal{U}$ has strictly positive imaginary part such that λ, u_l (with the asymptotics (3.7) and with $T^{-1}(\lambda) = 0$) are the eigenvalues and the corresponding eigenfunctions of the self-adjoint Lax operator L given in (1.9), and thus $\lambda \in (-1,1) \subset \mathbb{R}$. Let $\lambda \in (-1,1)$ be an eigenvalue, then $\pm iz = \mp\sqrt{1-\lambda^2}$ are negative and positive real numbers. By checking the characteristic exponents of the ODE (3.1): $Lu = \lambda u$ near infinity, the geometric multiplicity of λ is 1. Since the Lax operator L is self-adjoint, the algebraic multiplicity is also 1, and all eigenvalues in $(-1,1)$ are simple. As a consequence, the root λ of T^{-1} has multiplicity 1.

We denote these at most countably many zeros on $(-1,1)$ by $\{\lambda_m\}_m$ and

$$
z_m = i\sqrt{1 - (\lambda_m)^2} \in i(0,1], \quad m \in \mathbb{N}. \tag{3.8}
$$

We have the following symmetry for T^{-1}:

$$
\overline{T}^{-1}(\lambda) = T^{-1}(\overline{\lambda}), \quad (\lambda, z),\, (\overline{\lambda}, -\overline{z}) \in \mathcal{R}. \tag{3.9}
$$

Indeed, the symmetry of the Lax equation (3.1) implies that $\overline{u_l}^T := (\overline{u_l^2}, \overline{u_l^1})$ with the asymptotics

$$
\begin{aligned}
\overline{u_l}^T &= -i(\overline{\lambda} - \overline{z}) e^{i\overline{z}x} \begin{pmatrix} 1 \\ i(\overline{\lambda} + \overline{z}) \end{pmatrix} + o(1) e^{\operatorname{Im} zx} \text{ as } x \to -\infty, \\
\overline{u_l}^T &= -i(\overline{\lambda} - \overline{z}) e^{i\overline{z}x} \overline{T}^{-1}(\lambda) \begin{pmatrix} 1 \\ i(\overline{\lambda} + \overline{z}) \end{pmatrix} + o(1) e^{\operatorname{Im} zx} \text{ as } x \to +\infty,
\end{aligned}
$$

satisfies the Lax equation (3.1) with $(\lambda, z) \in \mathcal{R}$ replaced by $(\overline{\lambda}, -\overline{z}) \in \mathcal{R}$, which itself possesses a Jost solution with the following asymptotics:

$$
\begin{aligned}
u_l(\overline{\lambda}) &= e^{i\overline{z}x} \begin{pmatrix} 1 \\ i(\overline{\lambda} + \overline{z}) \end{pmatrix} + o(1) e^{\operatorname{Im} zx} \text{ as } x \to -\infty, \\
u_l(\overline{\lambda}) &= e^{i\overline{z}x} T^{-1}(\overline{\lambda}) \begin{pmatrix} 1 \\ i(\overline{\lambda} + \overline{z}) \end{pmatrix} + o(1) e^{\operatorname{Im} zx} \text{ as } x \to +\infty.
\end{aligned}
$$

By uniqueness we deduce (3.9). Therefore

- For $\hat{\xi} \in \mathcal{I}_{\text{cut}}$, the limits $\lim_{\lambda \to \hat{\xi}+i0} |T^{-1}(\lambda)|$ and $\lim_{\lambda \to \hat{\xi}-i0} |T^{-1}(\lambda)|$ are the same. Hence the subharmonic function $\ln |T^{-1}(\lambda)|$ on $\mathcal{V}$ is continuous on $\mathcal{I}_{\text{cut}}$ and generally $\ln |T^{-1}(\lambda)| = \ln |T^{-1}(\overline{\lambda})|$ for $(\lambda, z) \in \mathcal{R}$;
- For $(\lambda_\pm, z) = (\pm i\sigma, i\tau/2) \in \mathcal{R}$ with $\tau \geq 2$ and $\sigma = \sqrt{\tau^2/4 - 1} \in \mathbb{R}$,

$$
\frac{1}{2} \sum_{\pm} \operatorname{Re} \ln T^{-1}(\lambda_\pm) = \frac{1}{2} \operatorname{Re} \left(\ln T^{-1}(\lambda_+) + \ln T^{-1}(\overline{\lambda_+}) \right) = \operatorname{Re} \ln T^{-1}(\lambda_+). \tag{3.10}
$$

Let us take the time variable into account. We multiply u_l by $e^{-iz(2\lambda)t}$ such that the time evolutionary equation in (1.8) ensures

$$\partial_t(T^{-1}) = 0, \quad \partial_t R = 4iz\lambda R.$$

That is, the transmission coefficient $T^{-1}(\lambda)$ is conserved by the Gross-Pitaevskii flow and we will make use of it to define the conserved energies for the Gross-Pitaevskii equation (1.1).

Similarly, $e^{-iz(4\lambda^2+2)t}u_l$ satisfies (1.8) with $q=\psi$ and with the matrix in $(1.8)_2$ replaced by P_{mKdV} in (1.11). The same transmission coefficient $T^{-1}(\lambda)$ as for the Gross-Pitaevskii equation is conserved by the modified KdV flow (1.10).

3.3. **Notations.** Let $q \in X^s$, $s > \frac{1}{2}$. Let $(\lambda, z) \in \mathcal{R}$ and $\zeta = \lambda + z \in \mathcal{U}$ the upper half-plane be as in Subsection 3.1. Then

$$|q|^2 - \zeta^2 \neq 0, \text{ and } \frac{1}{\big||q|^2-\zeta^2\big|} \leq \frac{1}{(\operatorname{Im}\zeta)^2}, \quad \frac{|\zeta|}{\big||q|^2-\zeta^2\big|} \leq \frac{1}{\operatorname{Im}\zeta}. \tag{3.11}$$

Indeed, if $|\operatorname{Re}\zeta| \geq \operatorname{Im}\zeta$, then

$$\frac{1}{\big||q|^2-\zeta^2\big|} \leq \frac{1}{|\operatorname{Im}\zeta^2|} \leq \frac{1}{2(\operatorname{Im}\zeta)^2} \text{ and } \frac{|\zeta|}{\big||q|^2-\zeta^2\big|} \leq \frac{\sqrt{2}|\operatorname{Re}\zeta|}{2|\operatorname{Re}\zeta|\operatorname{Im}\zeta} \leq \frac{1}{\operatorname{Im}\zeta},$$

while if $|\operatorname{Re}\zeta| \leq \operatorname{Im}\zeta$, then

$$\frac{1}{\big||q|^2-\zeta^2\big|} = \Big(\big(|q|^2 + (\operatorname{Im}\zeta)^2 - (\operatorname{Re}\zeta)^2\big)^2 + (2\operatorname{Re}\zeta\operatorname{Im}\zeta)^2\Big)^{-\frac{1}{2}} \leq \frac{1}{|\zeta|^2}.$$

We introduce the following functions which will play an essential role in the analysis of the transmission coefficient:

$$(*)\qquad \begin{aligned} q_1 &= \frac{i\zeta(|q|^2-1) - \bar{q}q'}{|q|^2-\zeta^2}, \quad q_2 = \frac{i\zeta q' + (|q|^2-1)q}{|q|^2-\zeta^2}, \quad q_3 = \frac{-i\zeta\bar{q}' + (|q|^2-1)\bar{q}}{|q|^2-\zeta^2}, \\ q_4 &= \frac{2i\zeta(|q|^2-1) + q\bar{q}' - \bar{q}q'}{|q|^2-\zeta^2}, \quad \varphi(x) = 2izx + \int_0^x q_4(x_1)\,\mathrm{d}x_1. \end{aligned}$$

As in [31], let the symbols $\diagup$, $\diagdown$ correspond to the ordered integrals with respect to the functions $e^{-\varphi(x)}q_3(x)$ and $e^{\varphi(y)}q_2(y)$ respectively in the following way

$$\cap := \int_{x_1<y_1} e^{\varphi(y_1)-\varphi(x_1)} q_3(x_1)q_2(y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1,$$

$$\cap^j := \int_{x_1<y_1<\cdots<x_j<y_j} \prod_{n=1}^{j} e^{\varphi(y_n)-\varphi(x_n)} q_3(x_n)q_2(y_n)\,\mathrm{d}x\,\,\mathrm{d}y\,,$$

$$\diagup\!\cap\!\diagdown := \int_{t_1<t_2<t_3<t_4} e^{\varphi(t_4)+\varphi(t_3)-\varphi(t_2)-\varphi(t_1)} q_3(t_1)q_3(t_2)q_2(t_3)q_2(t_4)dt,$$

$$\diagup\!\diagup\!\cap\!\diagdown\!\diagdown = \int_{t_1<\cdots<t_6} e^{\varphi(t_6)+\varphi(t_5)+\varphi(t_4)-\varphi(t_3)-\varphi(t_2)-\varphi(t_1)} q_3(t_1)q_3(t_2)q_3(t_3)q_2(t_4)q_2(t_5)q_2(t_6)dt,$$

$$\diagup\!\cap\cap\!\diagdown = \int_{t_1<\cdots<t_6} e^{\varphi(t_6)+\varphi(t_5)-\varphi(t_4)+\varphi(t_3)-\varphi(t_2)-\varphi(t_1)} q_3(t_1)q_3(t_2)q_2(t_3)q_3(t_4)q_2(t_5)q_2(t_6)dt,$$

and so on. In particular, a symbol of form $_{2j}$, where under the arc $\cap$ consists of $(j-1)$ non-interacting symbols $\cap$, is said to be *connected* of degree $2j$. We will simply omit the subscript $2j$ in $_{2j}$ when the degree is clear. For example, $\cap$, $\diagup\!\cap\!\diagdown$

are connected symbols of degree $2, 4$ respectively, and ,  are connected symbols of degree 6, while is not a connected symbol.

We introduce the operator S as follows:

$$(Sf)(t) = \int_{x<y<t} e^{\varphi(y)-\varphi(x)} q_2(y)(q_3 f)(x)\,\mathrm{d}x\ \mathrm{d}y\,, \tag{3.12}$$

such that we can express $\cap^j = \lim_{t\to\infty}(S^j 1)(t)$.

3.4. **Asymptotic expansion of the transmission coefficient T^{-1}.** Let $q-1 \in \mathcal{S}(\mathbb{R})$. Recall the definition of the transmission coefficient $T^{-1}(\lambda) = \lim_{x\to\infty} e^{izx}u^1(x)$ in Subsection 3.2, where the Jost solution u satisfies the Lax equation (3.1) and the asymptotics (3.7). We are going to solve the initial value problem (3.1)-$(3.7)_1$ iteratively, to derive the asymptotic expansion of the transmission coefficient T^{-1}.

More precisely, we are going to follow the following procedure: We will first make change of variables $u \mapsto w$ (see (3.16) below) to renormalise the problem (3.1)-$(3.7)_1$ into the following ODE, where q_1, q_2, q_3, q_4 are given in $(*)$:

$$w_x = \begin{pmatrix} 0 & 0 \\ 0 & 2iz \end{pmatrix} w + \begin{pmatrix} 0 & q_2 \\ q_3 & q_4 \end{pmatrix} w, \quad \lim_{x\to-\infty} w(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{ODE}$$

such that

$$e^{\int_{-\infty}^{\infty} q_1\,\mathrm{d}m}T^{-1}(\lambda) = \lim_{x\to\infty} w^1(x) \text{ the asymptotic of the first component of } w.$$

Then we formally solve (ODE) iteratively as follows

$$\begin{aligned} &w = \sum_{n=0}^{\infty} w_n, \quad w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ &w_n(x) = \int_{-\infty}^{x} \begin{pmatrix} 0 & q_2(x_1) \\ e^{2iz(x-x_1)+\int_{x_1}^{x} q_4\,\mathrm{d}m} q_3(x_1) & 0 \end{pmatrix} w_{n-1}(x_1)\,\mathrm{d}x_1, \end{aligned} \tag{3.13}$$

to derive the following formal asymptotic expansion for T^{-1}:

$$e^{\int_{-\infty}^{\infty} q_1\,\mathrm{d}m}T^{-1}(\lambda) = \lim_{x\to\infty} w^1(x) = \sum_{n=0}^{\infty} \lim_{x\to\infty} w_n^1(x) = 1 + \sum_{j=1}^{\infty} \cap^j,$$

where we noticed $\lim_{x\to\infty} w_{2j-1}^1(x) = 0$ and $\lim_{x\to\infty} w_{2j}^1(x) = \cap^j$, $j \geq 1$, with $\cap^j$ given in Subsection 3.3.

Proposition 3.1. *Let $q-1 \in \mathcal{S}$. Let $(\lambda, z) \in \mathcal{R}$ with $\zeta = \lambda + z \in \mathcal{U}$ as in Subsection 3.1. Recall the notations in Subsection 3.3. Then the transmission coefficient T^{-1} defined in Subsection 3.2 expands asymptotically as follows:*

$$e^{\int_{-\infty}^{\infty} q_1\,\mathrm{d}x}T^{-1}(\lambda) = 1 + \sum_{j=1}^{\infty} T_{2j}(\lambda), \quad T_{2j} = \cap^j, \tag{3.14}$$

and its logarithm expands asymptotically as

$$\int_{-\infty}^{\infty} q_1\,\mathrm{d}x + \ln T^{-1}(\lambda) = T_2 + \sum_{j=2}^{\infty} \tilde{T}_{2j}, \tag{3.15}$$

where $\tilde{T}_{2j}$ is linear combination of connected *symbols* $_{2j}$ *of degree $2j$.*

Proof. It is convenient to rewrite the Lax equation (3.1) for the Jost solution u_l (we will omit the subscript l) in several steps.

A straightforward calculation shows that

$$\frac{1}{|q|^2-\zeta^2}\Big[\begin{pmatrix}-i\zeta & q\\ \bar q & i\zeta\end{pmatrix}\begin{pmatrix}-i\lambda & q\\ \bar q & i\lambda\end{pmatrix}-\begin{pmatrix}|q|^2-1 & 0\\ 0 & |q|^2-1\end{pmatrix}\Big]\begin{pmatrix}-i\zeta & q\\ \bar q & i\zeta\end{pmatrix}=\begin{pmatrix}-iz & 0\\ 0 & iz\end{pmatrix}.$$

We define the renormalised Jost solution of u as

$$v=\begin{pmatrix}-i\zeta & q\\ \bar q & i\zeta\end{pmatrix}u,\ \text{such that}\ u=\frac{1}{|q|^2-\zeta^2}\begin{pmatrix}-i\zeta & q\\ \bar q & i\zeta\end{pmatrix}v.$$

Hence v solves (with q_1,q_2,q_3,q_4 defined in $(*)$)

$$\begin{aligned}v_x=&\frac{1}{|q|^2-\zeta^2}\Big[\begin{pmatrix}-i\zeta & q\\ \bar q & i\zeta\end{pmatrix}\begin{pmatrix}-i\lambda & q\\ \bar q & i\lambda\end{pmatrix}\begin{pmatrix}-i\zeta & q\\ \bar q & i\zeta\end{pmatrix}+\begin{pmatrix}0 & q_x\\ \bar q_x & 0\end{pmatrix}\begin{pmatrix}-i\zeta & q\\ \bar q & i\zeta\end{pmatrix}\Big]v\\ =&\begin{pmatrix}-iz & 0\\ 0 & iz\end{pmatrix}v+\begin{pmatrix}-q_1 & q_2\\ q_3 & q_4-q_1\end{pmatrix}v.\end{aligned}$$

We want to remove the upper left entries of the two matrices: Let

$$w=-\frac{1}{2iz}e^{izx+\int_{-\infty}^x q_1\,\mathrm{dm}}v=-\frac{1}{2iz}e^{izx+\int_{-\infty}^x q_1\,\mathrm{dm}}\begin{pmatrix}-i\zeta & q\\ \bar q & i\zeta\end{pmatrix}u, \tag{3.16}$$

then it satisfies the renormalised (ODE) above. In other words, the renormalized Jost solution w satisfies the following integral equation

$$w(x)=\begin{pmatrix}1\\0\end{pmatrix}+\int_{-\infty}^x\begin{pmatrix}0 & q_2(x_1)\\ e^{2iz(x-x_1)+\int_{x_1}^x q_4\,\mathrm{dm}}q_3(x_1) & 0\end{pmatrix}w(x_1)\,dx_1,$$

with the following asymptotics as $x\to\pm\infty$ (recalling u's asymptotics (3.7)):

$$\begin{aligned}w(\lambda,x,t)&=\begin{pmatrix}1\\0\end{pmatrix}+o(1)\ \text{as}\ x\to-\infty,\\ w(\lambda,x,t)&=\begin{pmatrix}e^{\int_{-\infty}^{\infty}q_1\,\mathrm{dm}}T^{-1}(\lambda)\\ 0\end{pmatrix}+o(1)\ \text{as}\ x\to+\infty.\end{aligned}$$

Hence we use the iterative procedure in (3.13) to derive the formal asymptotic expansion (3.14) of $e^{\int_{-\infty}^{\infty}q_1\,\mathrm{dm}}T^{-1}(\lambda)=\lim_{x\to\infty}w^1(x)=\sum_{n=0}^{\infty}\lim_{x\to\infty}w^1_n(x)$.

Finally, it follows from Theorem 3.3 [6] in [31] that whenever we have the *formal* expansion as in (3.14): $g=1+\sum_{j=1}^{\infty}T_{2j},\ T_{2j}=\frown\!\!\smile^j$, we will have the formal expansion of its logarithm in (3.15): $\ln g=T_2+\sum_{j=2}^{\infty}\tilde T_{2j}$.

□

3.5. **The renormalised transmission coefficient $T_{\mathbf{c}}^{-1}$.** Let $q\in X^s$, $s>\frac12$ and let $(\lambda,z)\in\mathcal{R}$, $\zeta=\lambda+z\in\mathcal{U}$.

Recall $q_1=\frac{i\zeta(|q|^2-1)-\bar q q'}{|q|^2-\zeta^2}$ defined in $(*)$, then its integral $\int_{-\infty}^{\infty}q_1\,\mathrm{d}x$ (which appears in the expansion of T^{-1}) may not be well-defined for $q\in X^s$. More precisely, by view of the following fact coming from $\lambda^2-z^2=1$ and $\zeta=\lambda+z$:

$$\frac{1}{1-\zeta^2}=\frac{-1}{2z\zeta},\quad \frac{\zeta}{1-\zeta^2}=\frac{-1}{2z},\quad \frac{1}{|q|^2-\zeta^2}=\frac{1}{1-\zeta^2}-\frac{|q|^2-1}{(1-\zeta^2)(|q|^2-\zeta^2)}, \tag{3.17}$$

[6]Indeed, we can endow the ring of the formal power series of the unknown symbols $\frown$, $\smile$ with a shuffle product and a coproduct, and we typically take a commutative subalgebra H where the symbols $\frown$, $\smile$ appear in non interacting pairs, such that g is a group-like element and $\ln g$ is a primitive element in H. See also [31, 33, 36] for more shuffle algebra theory.

we rewrite the integral of q_1 when $q-1\in\mathcal{S}(\mathbb{R})$ as follows:

$$\int_{-\infty}^{\infty} q_1\,\mathrm{d}x = -\frac{i}{2z}\mathcal{M} - \frac{i}{2z\zeta}\mathcal{P} + \frac{i}{2z}\int_{\mathbb{R}} \frac{(|q|^2-1)^2}{|q|^2-\zeta^2}\,\mathrm{d}x - \frac{1}{2z\zeta}\int_{\mathbb{R}} \frac{\bar{q}q'(|q|^2-1)}{(|q|^2-\zeta^2)}\,\mathrm{d}x\,, \tag{3.18}$$

where $\mathcal{M}=\int_{\mathbb{R}}(|q|^2-1)\,\mathrm{d}x$, $\mathcal{P}=\operatorname{Im}\int_{\mathbb{R}} q\bar{q}'\,\mathrm{d}x$ are the mass and momentum (in (1.12)), which can be well-defined only under further integrability assumptions on $|q|^2-1, q'$.

We hence introduce the *renormalised* transmission coefficient T_c^{-1} which is the transmission coefficient T^{-1} module the mass and momentum, such that it is well-defined for $q\in X^s$. More precisely, we have

Theorem 3.1. *Let $q\in X^s$, $s>\frac{1}{2}$. Then there exists a renormalised transmission coefficient $T_c^{-1}(\lambda)$ which is holomorphic on the Riemann surface $\mathcal{R}$ (defined in* (3.2)*), such that*

- *If $q=1+q^0$, $q^0\in\mathcal{S}(\mathbb{R})$, then*

$$\begin{aligned} &T_c^{-1}(\lambda) = e^{-i\mathcal{M}(2z)^{-1}-i\mathcal{P}(2z\zeta)^{-1}}T^{-1}(\lambda), \\ &\textit{i.e.}\ \ -\ln T_c(\lambda) = -\ln T(\lambda) - i\mathcal{M}(2z)^{-1} - i\mathcal{P}(2z\zeta)^{-1}, \quad \zeta=\lambda+z, \end{aligned} \tag{3.19}$$

  where T^{-1} is the transmission coefficient defined in Subsection 3.2 and $\mathcal{M}=\int_{\mathbb{R}}(|q|^2-1)\,\mathrm{d}x$ and $\mathcal{P}=\operatorname{Im}\int_{\mathbb{R}} q\bar{q}'\,\mathrm{d}x$ are the conserved mass and momentum associated to the Gross-Pitaevskii equation, such that

$$|T_c^{-1}(\lambda)|\ge 1 \text{ if } \lambda\in\mathcal{I}_{cut}=(-\infty,-1]\cup[1,\infty), \quad T_c^{-1}\to 1 \text{ as } |\lambda|\to\infty. \tag{3.20}$$

- *For any fixed $(\lambda,z)\in\mathcal{R}$, the renormalised transmission coefficient $T_c^{-1}(\lambda;q)$ is extended uniquely to an analytic function in $q\in X^s$ (with respect to the analytic structure in Theorem 6.2) and $T_c^{-1}(\lambda;q(t))$ is conserved by the Gross-Pitaevskii flow on the existence time interval of the solution $q(t)$ (defined in* (1.1)*).*
- *$T_c^{-1}(\lambda)$ has the following asymptotic expansion*

$$T_c^{-1}(\lambda) = e^{\Phi(\lambda)}\Big(1+\sum_{j=1}^{\infty} T_{2j}(\lambda)\Big), \quad T_{2j} = \wedge^j, \tag{3.21}$$

  and its logarithm expands asymptotically as

$$\ln T_c^{-1}(\lambda) = \Phi(\lambda) + T_2(\lambda) + \sum_{j=2}^{\infty}\tilde{T}_{2j}(\lambda), \tag{3.22}$$

  where

$$\Phi(\lambda) := -\frac{i}{2z}\int_{\mathbb{R}}\frac{(|q|^2-1)^2}{|q|^2-\zeta^2}\,\mathrm{d}x + \frac{1}{2z\zeta}\int_{\mathbb{R}}\frac{\bar{q}q'(|q|^2-1)}{(|q|^2-\zeta^2)}\,\mathrm{d}x\,, \tag{3.23}$$

  and $\tilde{T}_{2j}$ is linear combination of connected *symbols $_{2j}$ of degree $2j$. Here the symbols are defined in Subsection 3.3.*
- *T_c^{-1} satisfies the following properties:*

$$\begin{aligned} &\operatorname{Re} T_c^{-1}(\lambda) = \operatorname{Re} T_c^{-1}(\bar{\lambda}) \text{ if } (\lambda,z)=(i\sigma,\pm i\tfrac{\tau}{2})\in\mathcal{R},\ \tau\ge 2,\ \sigma=\sqrt{\tfrac{\tau^2}{4}-1}, \\ &T_c^{-1} \text{ has at most countably many simple zeros } \{\lambda_m\}\subset(-1,1). \end{aligned} \tag{3.24}$$

We can define a superharmonic function $G(z)$ on the upper half plane $\mathcal{U}$ as follows

$$G(z) := \frac{1}{2}\sum_{\pm} \mathrm{Re}\left(4z^2 \ln T_c^{-1}\big(\pm\sqrt{z^2+1}\big)\right), \quad \mathrm{Im}\, z > 0, \quad \mathrm{Im}\sqrt{z^2+1} \geq 0, \tag{3.25}$$

such that $G \geq 0$ on the upper half plane $\mathcal{U}$ and $-\Delta G \geq 0$ is a nonnegative measure on the upper half plane $\mathcal{U}$ as follows

$$\nu_G(z) = -\Delta_z G(z) = -\pi \sum_m (2z)^2 \delta_{z=z_m} \geq 0,\ z_m = i\sqrt{1-\lambda_m^2} \in i(0,1], \tag{3.26}$$

where $\{\lambda_m\}$ are the simple zeros of $T_c^{-1}(\lambda)$ in (3.24).

The proof of this theorem will be demonstrated in next section where the functional spaces $l^p_\tau U^2$, $l^p_\tau V^2$, $l^p_\tau DU^2$ will come into play.

4. Proof of Theorem 3.1

In this section we will prove Theorem 3.1 concerning the well-definedness and the property of the renormalised transmission coefficient $T_{\mathrm{c}}^{-1}(\lambda)$ in the energy framework $q \in X^s$, $s > \frac{1}{2}$ in the following steps:

- In Subsection 4.1 we introduce the function spaces $l^p_\tau U^2, l^p_\tau V^2, l^p_\tau DU^2$.
- We derive some preliminary estimates in Subsection 4.2.1. Then we solve the renormalised Lax equation (ODE) rigorously when $q \in X^s$ and define the renormalised transmission coefficient $T_{\mathrm{c}}^{-1}(\lambda)$ in Subsection 4.2.2.
- We study the nonnegative superharmonic function $G(z)$ and conclude the proof of Theorem 3.1 in Subsection 4.3.

4.1. **Function spaces.** In this subsection we will briefly recall the function spaces U^2, V^2, DU^2 and the inhomogeneous norms $\|\cdot\|_{l^p_\tau V}$, $V = U^2, V^2, DU^2$. See [24, 31] for more details of the U^2, V^2 theory.

4.1.1. *Spaces U^2, V^2, DU^2.* We denote the bounded functions on $\mathbb{R}$ by $B(\mathbb{R})$. We use the spaces $U^2, V^2 \subset B(\mathbb{R})$ and DU^2 to substitute the Sobolev spaces $\dot{H}^{\frac{1}{2}} \not\hookrightarrow L^\infty$ and $\dot{H}^{-\frac{1}{2}}$. The space V^2 is defined as follows

$$V^2 = \Big\{ v \,\Big|\, \|v\|_{V^2} = \sup_{-\infty<t_1<\cdots<t_N=\infty} \Big(\sum_{j=1}^{N-1} |v(t_{j+1}) - v(t_j)|^2\Big)^{\frac{1}{2}} < \infty \Big\},$$

where we always set $v(\infty) = 0$. In particular, the constant function $1 \in V^2$ with norm 1. For any finite sequence $\{\phi_j\}_{j=1}^{N-1}$ with $\sum_{j=1}^{N-1} |\phi_j|^2 = 1$, the step function $\phi = \sum_{j=1}^{N-1} \phi_j 1_{[t_j,t_{j+1})}$ with $-\infty < t_1 < \cdots < t_N = \infty$ is called U^2 atom. We define the space U^2 by

$$U^2 = \Big\{ u = \sum_{k=1}^{\infty} c_k \psi_k \,\Big|\, (c_k)_k \in \ell^1(\mathbb{N}) \text{ and } \psi_k \text{ is } U^2 \text{ atom} \Big\},$$

endowed with the U^2-norm:

$$\|u\|_{U^2} = \inf \Big\{ \sum_{k=1}^{\infty} |c_k| \,\Big|\, u = \sum_{k=1}^{\infty} c_k \psi_k, \quad c_k \in \mathbb{C}, \quad \psi_k \text{ is } U^2 \text{ atom} \Big\}.$$

We define the space DU^2 via the distributional derivatives as

$$DU^2 = \{u' \mid u \in U^2\},$$

with the norm $\|u'\|_{DU^2} = \|u\|_{U^2}$. Then DU^2 function is a distribution function with the following finite norm:

$$\|f\|_{DU^2} = \sup\Big\{\int_{\mathbb{R}} f\varphi dt \,\Big|\, \|\varphi\|_{V^2} \le 1, \varphi \in C_0^\infty(\mathbb{R})\Big\}.$$

We have the following pleasant estimates which will be used frequently:

$$\begin{aligned} &\|f\|_{L^\infty} \le \|f\|_{V^2} \le \|f\|_{L^\infty} + 2\|f'\|_{DU^2}, \quad \|f\|_{V^2} \le 2\|f\|_{U^2}, \\ &\|fg\|_{V^2} \le \|f\|_{L^\infty}\|g\|_{V^2} + \|f\|_{V^2}\|g\|_{L^\infty}, \quad \|fg\|_{DU^2} \le 2\|f\|_{V^2}\|g\|_{DU^2}, \\ &\|f(u)\|_{V^2} \le C(f', \|u\|_{L^\infty})\|u\|_{V^2}. \end{aligned} \tag{4.1}$$

4.1.2. *Space $l^p_\tau DU^2$.* We take the localised version of U^2, V^2, DU^2-norms

$$\|u\|_{l^p_\tau U} = \big\|\|\chi_{\tau,k} u\|_U\big\|_{\ell^p_k(\mathbb{Z})}, \quad U = V^2, U^2 \text{ or } DU^2,$$

where $\tau \ge 2$ is the frequency scale and χ is a smooth function compactly supported on $[-\frac{2}{3}, \frac{2}{3}]$ with value 1 on the interval $[-\frac{1}{3}, \frac{1}{3}]$, such that $\chi_{\tau,k}$ form a partition of unity:

$$1 = \sum_{k\in\mathbb{Z}} \chi_{\tau,k}, \quad \chi_{\tau,k} = \chi\big(\tau(\cdot - \frac{k}{\tau})\big) = \chi(\tau \cdot - k). \tag{4.2}$$

For any fixed real positive number $a > 0$, $\|u\|_{l^p_\tau U}$ is equivalent to $\big\|\|\tilde\chi_{\tau,k} u\|_U\big\|_{\ell^p_k(\mathbb{Z})}$ with $\tilde\chi = \chi(\cdot/a)$.

Proposition 4.1 ([31])**.** *We have the following properties of the norm $\|\cdot\|_{l^p_\tau DU^2}$:*

- *The following inequality describes the effect of the phase shift*

$$\|e^{i\xi x} u\|_{l^2_\tau DU^2} \le C\sqrt{(\tau + |\xi|)/\tau}\,\|u\|_{l^2_\tau DU^2}. \tag{4.3}$$

- *The following inequality describes the effect of taking the derivative*

$$\|f\|_{l^p_\tau U^2} \lesssim \tau\|f\|_{l^p_\tau DU^2} + \|f'\|_{l^p_\tau DU^2}. \tag{4.4}$$

- *The $l^p_\tau DU^2$, $p \ge 2$-norm and $\dot H^{\frac12 - \frac1p}$-norm is related by*

$$\|u\|_{l^p_\tau DU^2} \le C\tau^{\frac1p - 1}\|u\|_{\dot H^{\frac12 - \frac1p}}.$$

4.2. **The renormalised transmission coefficient.** Let $q \in X^s$, $s > \frac12$ and $(\lambda, z) \in \mathcal{R}$ with $\zeta = \lambda + z \in \mathcal{U}$. Recall the renormalised Lax equation (ODE) in the proof of Proposition 3.1:

$$w_x = \begin{pmatrix} 0 & 0 \\ 0 & 2iz \end{pmatrix} w + \begin{pmatrix} 0 & q_2 \\ q_3 & q_4 \end{pmatrix} w, \quad \lim_{x\to-\infty} w(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{ODE}$$

and its formally equivalent iterative version (3.13):

$$w = \sum_{n=0}^\infty w_n,\ w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ w_n(t) = \begin{pmatrix} \int_{-\infty}^t q_2(x) w^2_{n-1}(x)\,\mathrm{d}x \\ \int_{-\infty}^t e^{\varphi(t)-\varphi(x)} q_3(x) w^1_{n-1}(x)\,\mathrm{d}x \end{pmatrix}, \tag{4.5}$$

$n \ge 1$, such that if $q - 1 \in \mathcal{S}$ then

$$e^{\int_{-\infty}^\infty q_1\,\mathrm{d}m} T^{-1}(\lambda) = \lim_{t\to\infty} w^1(t) = \sum_{n=0}^\infty \lim_{t\to\infty} w^1_n(t),$$

where $q_1, q_2, q_3, q_4, \varphi$ are given in $(*)$. In particular, noticing $w^1_{2j+1} = w^2_{2j} = 0$, we can rewrite (4.5) as

$$w^1(t) = \sum_{j=0}^{\infty} w^1_{2j}(t) = \sum_{j=0}^{\infty} (S^j 1)(t), \quad w^2(t) = \sum_{j=0}^{\infty} w^2_{2j+1}(t) = \sum_{j=0}^{\infty} (S_1 S^j 1)(t), \tag{4.6}$$

where we recall the definition of the operator S in (3.12):

$$(Sf)(t) = \int_{x<y<t} e^{\varphi(y)-\varphi(x)} q_2(y)(q_3 f)(x)\,\mathrm{d}x\ \mathrm{d}y\,,$$

and we define the operator S_1:

$$(S_1 f)(t) = \int_{-\infty}^{t} e^{\varphi(t)-\varphi(x)} (q_3 f)(x)\,\mathrm{d}x\,.$$

We are going to solve (ODE) rigorously and define the renormalised transmission coefficient T_{c}^{-1} in Subsection 4.2.2. Before that we give some preliminary estimates for q, q_2, q_3, q_4 and w_n.

4.2.1. *Preliminary estimates.* We first claim the following L^∞-estimate:

$$\begin{aligned} &\|q\|_{L^\infty} \lesssim 1 + \tau^{\frac12} \||q|^2 - 1\|^{\frac12}_{l^\infty_\tau DU^2} + \|q'\|_{l^\infty_\tau DU^2}, \quad \forall \tau > 0, \\ &\text{and in particular, } \|q\|_{L^\infty} \lesssim 1 + \|\,\mathbf{q}\,\|_{l^\infty_2 DU^2} \lesssim 1 + E^s(q). \end{aligned} \tag{4.7}$$

Indeed, we notice that by the partition of unit (4.2), for any $\tau > 0$, at each point $x \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such that the function $\chi_{\tau,k}$ or $\bar\chi_{\tau,k}$ with $\bar\chi = \chi(\cdot/2)$ taking value 1 at point x, and thus $\|q\|_{L^\infty} \le \sup_k \|\chi_{\tau,k} q\|_{L^\infty} + \sup_k \|\bar\chi_{\tau,k} q\|_{L^\infty}$. By fundamental theorem of calculus: $(\chi_{\tau,k} q)(x) = \int_{\frac{k-1}{\tau}}^{x} \chi'_{\tau,k}(y) q(y)\,\mathrm{d}y + \int_{\frac{k-1}{\tau}}^{x} \chi_{\tau,k} q'\,\mathrm{d}y\,,$ we derive (4.7) from Hölder's inequality and (4.1) as follows:

$$\begin{aligned} |(\chi_{\tau,k} q)(x)| &\le \Big(\int_{\frac{k-1}{\tau}}^{x} |\chi'_{\tau,k}||q|^2\,\mathrm{d}y\Big)^{\frac12} \Big(\int_{\frac{k-1}{\tau}}^{x} |\chi'_{\tau,k}|\,\mathrm{d}y\Big)^{\frac12} + \Big|\int_{\frac{k-1}{\tau}}^{x} \chi_{\tau,k} q'\,\mathrm{d}y\,\Big| \\ &\lesssim \Big(\int_{\frac{k-1}{\tau}}^{x} |\chi'_{\tau,k}|\,(|q|^2 - 1)\,\mathrm{d}y + \int_{\frac{k-1}{\tau}}^{x} |\chi'_{\tau,k}|\,\mathrm{d}y\Big)^{\frac12} + \|q'\|_{l^\infty_\tau DU^2} \\ &\lesssim (\tau \||q|^2 - 1\|_{l^\infty_\tau DU^2} + 1)^{\frac12} + \|q'\|_{l^\infty_\tau DU^2}. \end{aligned}$$

By use of the fact (3.11) and (4.7) above, we derive immediately from the estimates in (4.1) the following estimates for q and q_2, q_3, q_4 defined in $(*)$:

Lemma 4.1. *Let $(\lambda, z) \in \mathcal{R}$, $\zeta = \lambda + z \in \mathcal{U}$, with $\tau = 2\mathrm{Im}\, z > 0$, $\omega = \mathrm{Im}\,\zeta > 0$. Let $q \in X^s$, $s > \frac12$, $\mathbf{q} = (|q|^2 - 1, q') \in H^{s-1} \hookrightarrow l^2_1 DU^2$ and q_2, q_3, q_4 be defined in $(*)$. Then there exists a constant $\mathcal{C}_0$, depending only on (for any fixed $\tau_1 > 0$)*

$$\omega^{-1},\ \tau_1 \||q|^2 - 1\|_{l^\infty_{\tau_1} DU^2},\ \|q'\|_{l^\infty_{\tau_1} DU^2},\ \|q'\|_{l^\infty_\tau DU^2} \tag{4.8}$$

such that

$$\|q\|_{l^\infty_\tau V^2} \le \mathcal{C}_0, \quad \|q_\kappa\|_{l^p_\tau DU^2} \le \mathcal{C}_0 \|\,\mathbf{q}\,\|_{l^p_\tau DU^2}, \quad \kappa = 2, 3, 4.$$

We are going to study the functions $w^1_{2j}(t), w^2_{2j+1}(t)$ in (4.6). For notational simplicity we first introduce the function

$$\tilde\varphi = \varphi - 2i\mathrm{Re}\, z x = -\tau x + \int_0^x q_4, \quad \tau = 2\mathrm{Im}\, z > 0.$$

If $\mathbf{q} \in l^\infty_\tau DU^2$, then by Lemma 4.1 (with a possibly larger $\mathcal{C}_0$):

$$\Big\|\chi_{\tau,k} e^{\int_{\frac{k}{\tau}}^y q_4 \,\mathrm{d}x}\Big\|_{V^2} \lesssim e^{\mathcal{C}_0 \|\mathbf{q}\|_{l^\infty_\tau DU^2}}, \qquad \Big| e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4} \Big| \lesssim e^{\mathcal{C}_0 \|\mathbf{q}\|_{l^\infty_\tau DU^2}(k-k')}. \tag{4.9}$$

Therefore we have the following properties for the functions $w_{2j}(t), w_{2j+1}(t)$:

Lemma 4.2. *Assume the same hypothesis as in Lemma 4.1 and (with a possibly larger $\mathcal{C}_0$ which depends only on (4.8))*

$$\|\mathbf{q}\|_{l^\infty_\tau DU^2} \le \frac{1}{2\mathcal{C}_0}. \tag{4.10}$$

Then the functions w_{2j}, w_{2j+1} given in (4.6) are well-defined, depending analytically on $\mathbf{q} \in l^2_\tau DU^2$ and satisfying the following estimates (with an universal constant C):

$$\|w_{2j}\|_{U^2} \le \big(C\frac{|\mathrm{Re}\, z| + \tau}{\tau}\big)^j \big(\|q_2\|_{l^2_\tau DU^2}\|q_3\|_{l^2_\tau DU^2}\big)^j,$$
$$\|w_{2j+1}\|_{L^\infty} \le \big(C\frac{|\mathrm{Re}\, z| + \tau}{\tau}\big)^{j+3/2} \big(\|q_2\|_{l^2_\tau DU^2}\|q_3\|_{l^2_\tau DU^2}\big)^j \|q_3\|_{l^\infty_\tau DU^2}.$$

Proof. As $w^1_{2j}(t) = (S^j 1)(t)$, we consider

$$\begin{aligned}
\|Sf\|_{U^2} &= \Big\| \int_{x<y<t} \big(e^{2i\mathrm{Re}\, zy} q_2(y)\big) e^{\tilde\varphi(y)-\tilde\varphi(x)} \big(e^{-2i\mathrm{Re}\, zx}(q_3 f)(x)\big) \,\mathrm{d}x \,\mathrm{d}y \Big\|_{U^2_t} \\
&= \Big\| \big(e^{2i\mathrm{Re}\, zy} q_2(y)\big) \int_{-\infty}^y e^{\tilde\varphi(y)-\tilde\varphi(x)} \big(e^{-2i\mathrm{Re}\, zx}(q_3 f)(x)\big) \,\mathrm{d}x \Big\|_{DU^2_y} \\
&\le \sum_k \Big\| \big(e^{2i\mathrm{Re}\, zy} \chi_{\tau,k}(y) q_2(y)\big) \int_{y-\frac{3}{\tau}}^y e^{\tilde\varphi(y)-\tilde\varphi(x)} \big(e^{-2i\mathrm{Re}\, zx}\tilde\chi_{\tau,k}(x)(q_3 f)(x)\big) \,\mathrm{d}x \Big\|_{DU^2_y} \\
&+ \sum_k \Big\| \big(e^{2i\mathrm{Re}\, zy} \chi_{\tau,k}(y) q_2(y)\big) \int_{-\infty}^{y-\frac{3}{\tau}} \sum_{k' \le k-1} e^{\tilde\varphi(y)-\tilde\varphi(x)} \big(e^{-2i\mathrm{Re}\, zx}\chi_{\tau,k'}(x)(q_3 f)(x)\big) \,\mathrm{d}x \Big\|_{DU^2_y}.
\end{aligned}$$

In the above, the first part on the righthand side can be bounded by (recalling (4.1))

$$\begin{aligned}
&\sum_k \Big\| e^{2i\mathrm{Re}\, zy} \chi_{\tau,k}(y) q_2(y) e^{\tilde\varphi(y)-\tilde\varphi(\frac{k}{\tau})} \Big\|_{DU^2_y} \Big\| \int_{y-\frac{3}{\tau}}^y e^{\tilde\varphi(\frac{k}{\tau})-\tilde\varphi(x)} \big(e^{-2i\mathrm{Re}\, zx}\tilde\chi_{\tau,k}(x)(q_3 f)(x)\big) \,\mathrm{d}x \Big\|_{V^2_y} \\
&\lesssim \sum_k \big\| e^{2i\mathrm{Re}\, z\cdot} \chi_{\tau,k} q_2 \big\|_{DU^2} \big\| \tilde\chi_{\tau,k} e^{\tilde\varphi(y)-\tilde\varphi(\frac{k}{\tau})} \big\|_{V^2} \big\| \tilde\chi_{\tau,k} e^{\tilde\varphi(\frac{k}{\tau})-\tilde\varphi(x)} \big\|_{V^2} \big\| e^{-2i\mathrm{Re}\, z\cdot}\tilde\chi_{\tau,k}(q_3 f) \big\|_{DU^2} \\
&\lesssim \big(\frac{|\mathrm{Re}\, z| + \tau}{\tau}\big) \|\chi_{\tau,k} q_2\|_{\ell^2_k DU^2} \|\tilde\chi_{\tau,k}(q_3 f)\|_{\ell^2_k DU^2} \Big\| \tilde\chi_{\tau,k} e^{\pm\int_{\frac{k}{\tau}}^x q_4} \Big\|^2_{\ell^\infty_k V^2},
\end{aligned}$$

where we have used (4.3) and $\|\tilde\chi_{\tau,k} e^{-\tau(\cdot - \frac{k}{\tau})}\|_{V^2} \lesssim 1$, and the second part on the righthand side can be bounded by

$$\begin{aligned}
&\sum_k \Big\| e^{2i\mathrm{Re}\, zy} \chi_{\tau,k}(y) q_2(y) e^{\tilde\varphi(y)-\tilde\varphi(\frac{k}{\tau})} \Big\|_{DU^2_y} \\
&\quad \times \sum_{k' \le k-1} e^{\tilde\varphi(\frac{k}{\tau})-\tilde\varphi(\frac{k'}{\tau})} \Big\| \int_{-\infty}^{y-\frac{3}{\tau}} e^{\tilde\varphi(\frac{k'}{\tau})-\tilde\varphi(x)} \big(e^{-2i\mathrm{Re}\, zx}(\chi_{\tau,k'} q_3 f)(x)\big) \,\mathrm{d}x \Big\|_{V^2_y} \\
&\lesssim \sum_k \sum_{k' \le k-1} \big\| e^{2i\mathrm{Re}\, z\cdot} \chi_{\tau,k} q_2 \big\|_{DU^2} \big\| e^{-2i\mathrm{Re}\, z\cdot} \chi_{\tau,k'} q_3 f \big\|_{DU^2} \Big\| \tilde\chi_{\tau,k} e^{\pm\int_{\frac{k}{\tau}}^x q_4} \Big\|^2_{V^2} e^{-(k-k')} \Big| e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4} \Big|.
\end{aligned}$$

Under the smallness condition (4.10), by use of the inequality (4.9), we derive

$$\|S\|_{V^2 \mapsto U^2} \lesssim \frac{|\mathrm{Re}\, z| + \tau}{\tau} \|q_2\|_{l^2_\tau DU^2} \|q_3\|_{l^2_\tau DU^2} e^{\mathcal{C}_0 \|\mathbf{q}\|_{l^\infty_\tau DU^2}} \le \frac{2(|\mathrm{Re}\, z| + \tau)}{\tau} \|q_2\|_{l^2_\tau DU^2} \|q_3\|_{l^2_\tau DU^2},$$

and hence Lemma 4.2 holds for $w_{2j} = S^j 1$.

Similarly we consider the operator S_1 (noticing $\|\chi_{\tau,k}e^{2i\mathrm{Re}\, zt}\|_{V^2}\lesssim 1+|\mathrm{Re}\, z|/\tau$):

$$\|S_1 f\|_{L^\infty}\lesssim \sup{}_k \|\chi_{\tau,k}e^{2i\mathrm{Re}\, zt}\|_{V^2}\|\tilde{\chi}_{\tau,k}e^{-2i\mathrm{Re}\, zx}q_3 f\|_{DU^2}\Big\|\tilde{\chi}_{\tau,k}e^{\pm\int_{\frac{k}{\tau}}^x q_4}\Big\|_{V^2}^2$$

$$+\sup{}_k\sum_{k'\le k-1}\|\chi_{\tau,k}e^{2i\mathrm{Re}\, zt}\|_{V^2}\|\chi_{\tau,k}e^{-2i\mathrm{Re}\, zx}q_3 f\|_{DU^2}\Big\|\tilde{\chi}_{\tau,k}e^{\pm\int_{\frac{k}{\tau}}^x q_4}\Big\|_{V^2}^2 e^{-(k-k')}\Big|e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4}\Big|$$

$$\lesssim (\frac{|\mathrm{Re}\, z|+\tau}{\tau})^{\frac{3}{2}}\|q_3\|_{l^\infty_\tau DU^2}\|f\|_{V^2}e^{\mathcal{C}_0\|\mathbf{q}\|_{l^\infty_\tau DU^2}},$$

and hence Lemma 4.2 holds for $w_{2j+1}=S_1S^j 1$. □

4.2.2. *The renormalised transmission coefficient T_c^{-1} for $q\in X^s$.* In order to emphasize the dependence of the renormalised transmission coefficient T_c^{-1} on q, we will denote $T_c^{-1}(\lambda)=T_c^{-1}(\lambda;q)$ in this subsection.

Proposition 4.2. *Let $q\in X^s$, $s>\frac{1}{2}$, then the renormalised Lax equation* (ODE) *has a unique solution $w\in L^\infty$.*

We define the renormalised transmission coefficient as

$$T_c^{-1}(\lambda)=e^{\Phi}\lim_{x\to\infty}w^1(x),$$

where $\Phi=-\frac{i}{2z}\int_{\mathbb{R}}\frac{(|q|^2-1)^2}{|q|^2-\zeta^2}\,\mathrm{d}x+\frac{1}{2z\zeta}\int_{\mathbb{R}}\frac{\bar{q}q'(|q|^2-1)}{(|q|^2-\zeta^2)}\,\mathrm{d}x$ *is given in* (3.23)*. Then*

- *$T_c^{-1}(\lambda;q)$ is a well-defined holomorphic function in $(\lambda,z)\in\mathcal{R}$ and depends analytically on $q\in X^s$ with respect to the analytic structure given in Theorem 6.2 below;*
- *When $q-1\in\mathcal{S}$, the relation* (3.19)*: $T_c^{-1}(\lambda)=e^{-i\mathcal{M}(2z)^{-1}-i\mathcal{P}(2z\zeta)^{-1}}T^{-1}(\lambda)$, the properties* (3.20)*: $|T_c^{-1}(\lambda)|\ge 1$ if $\lambda\in\mathcal{I}_{cut}=(-\infty,-1]\cup[1,\infty)$, $T_c^{-1}\to 1$ as $|\lambda|\to\infty$, and the asymptotic expansions* (3.21), (3.22) *all hold true;*
- *Let $q(t,x)\in\mathcal{C}(I;X^s)$ be a solution of the Gross-Pitaevskii equation (in Definition 1.1), then $T_c^{-1}(\lambda;q(t))$ is conserved on the existence time interval I.*
- $\mathrm{Re}\, T_c{}^{-1}(\lambda)=\mathrm{Re}\, T_c^{-1}(\bar{\lambda})$ *if $(\lambda,z)=(i\sigma,\pm i\frac{\tau}{2})\in\mathcal{R}$, $\tau\ge 2$, $\sigma=\sqrt{\frac{\tau^2}{4}-1}$.*

Proof. **Step 1. Resolution of** (ODE)**.**
If $q\in X^s$, $s>\frac{1}{2}$ and there are the points $(\lambda,z)\in\mathcal{R}$ such that $\tau=2\mathrm{Im}\, z\ge 2$ and the following smallness condition holds (with a possibly larger $\mathcal{C}_0$ than the ones in Lemmas 4.1 and 4.2)

$$\mathcal{C}_0(\frac{|\mathrm{Re}\, z|+\tau}{\tau})^{\frac{1}{2}}\|\mathbf{q}\|_{l^2_\tau DU^2}\le\frac{1}{2},\tag{4.11}$$

we have by Lemmas 4.1 and 4.2 that

$$\|w_n\|_{L^\infty}\le\frac{|\mathrm{Re}\, z|+\tau}{\tau}\big(\mathcal{C}_0(\frac{|\mathrm{Re}\, z|+\tau}{\tau})^{\frac{1}{2}}\|\mathbf{q}\|_{l^2_\tau DU^2}\big)^n.$$

Hence (ODE) (or equivalently (4.6)) has a unique solution $w=\sum_{n\ge 0}w_n\in L^\infty$, depending analytically on $\mathbf{q}\in l^2_\tau DU^2$.

For general $q\in X^s$, $s>\frac{1}{2}$ with $E^s(q)<\infty$, for any fixed point $(\lambda,z)\in\mathcal{R}$, we can take two points $a_0,a_1\in\mathbb{R}$ such that the smallness condition (4.11) holds for $q|_{(-\infty,a_0]\cup[a_1,\infty)}$, by virtue of the embedding $H^{s-1}\hookrightarrow l^2_1DU^2$, $s>\frac{1}{2}$. We solve (ODE) as follows:

- The above analysis implies that we can solve (ODE) until the point a_0: $w(a_0)=b_0$;
- Recalling the change of variables $u\mapsto w$ in (3.16) which renormalises the Lax equation (3.1) to (ODE), we do the change of variables $w\mapsto\underline{u}$ to solve the Lax equation (3.1) with the following initial data at a_0

$$\underline{u}(a_0)=e^{iz(a_1-a_0)+\int_{a_0}^{a_1}q_1\,\mathrm{d}x}\frac{1}{|q(a_0)|^2-\zeta^2}\begin{pmatrix}-i\zeta & q(a_0)\\ \bar{q}(a_0) & i\zeta\end{pmatrix}b_0$$

until the point a_1: $\underline{u}(a_1)$. This is possible since $q \in H^s([a_0,a_1])$, $s > \frac{1}{2}$;

- We finally solve again (ODE) with the initial data $b_1 = \begin{pmatrix} -i\zeta & q(a_1) \\ \bar{q}(a_1) & i\zeta \end{pmatrix} \underline{u}(a_1)$ on the semiline $[a_1,\infty)$. More precisely, we take $\tilde{w} = \begin{pmatrix} 0 \\ e^{2iz(x-a_1)} b_1^2 \end{pmatrix} \Big|_{[a_1,\infty)}$ such that $\dot{w} = w - \tilde{w}$ satisfies
$$\dot{w}_x = \begin{pmatrix} 0 & q_2 \\ q_3 & q_4 + 2iz \end{pmatrix} \dot{w} + b_1^2 e^{2iz(x-a_1)} \begin{pmatrix} q_2|_{[a_1,\infty)} \\ q_4|_{[a_1,\infty)} \end{pmatrix}, \quad \dot{w}|_{a_1} = \begin{pmatrix} b_1^1 \\ 0 \end{pmatrix}.$$
Then under the smallness condition (4.11) for $q|_{[a_1,\infty)}$, we follow exactly Lemma 4.2 [7] to derive the unique solution $\dot{w}$. Hence the solution $w = \dot{w} + \tilde{w} \in L^\infty$ with $w^1 \in V^2$ exists.

Step 2. Well-definedness of T_c^{-1}.
When $q \in X^s$, $s > \frac{1}{2}$, then $\Phi < \infty$ by Lemma 4.1. We can define the renormalised transmission coefficient $T_c^{-1}(\lambda) = e^{\Phi} \lim_{x\to\infty} w^1(x)$, which is holomorphic on the Riemann surface $(\lambda, z) \in \mathcal{R}$.

If $q - 1 \in \mathcal{S}$, then by virtue of $e^{\int_{-\infty}^{\infty} q_1} T^{-1}(\lambda) = \lim_{x\to\infty} w^1(x)$ in Subsection 3.4 and the equality (3.18): $\int_{-\infty}^{\infty} q_1 = -\frac{i}{2z}\mathcal{M} - \frac{i}{2z\zeta}\mathcal{P} - \Phi$, we have the relation (3.19): $T_c^{-1}(\lambda) = e^{-i\mathcal{M}(2z)^{-1} - i\mathcal{P}(2z\zeta)^{-1}} T^{-1}(\lambda)$ and hence the asymptotic expansions (3.21) and (3.22) follow from Proposition 3.1. The properties (3.20): $|T_c^{-1}(\lambda)| \geq 1$ if $\lambda \in \mathcal{I}_{\text{cut}} = (-\infty,-1] \cup [1,\infty)$, $T_c^{-1} \to 1$ as $|\lambda| \to \infty$, and the symmetry $\operatorname{Re} T_c^{-1}(\lambda) = \operatorname{Re} T_c^{-1}(\bar{\lambda})$ for $(\lambda, z) = (i\sigma, \pm i\frac{\tau}{2}) \in \mathcal{R}$ follow from the results in Subsection 3.2.

Fix $(\lambda, z) \in \mathcal{R}$. Provided with the analytic structure of X^s in Theorem 6.2 below, for any neighbourhood $B_r^s(q)$ of q, we can choose a_0, a_1 (depending on $E^s(q), r$) such that the smallness condition (4.11) holds for $p|_{(-\infty,a_0]\cup[a_1,\infty)}$ for all $p \in B_r^s(q)$. Therefore the corresponding solution w_p for (ODE) depends analytically on $\big(\mathbf{p}|_{(-\infty,a_0]}, \mathbf{p}|_{[a_0,a_1]}, \mathbf{p}|_{[a_1,\infty)}\big) \in l^2_\tau DU^2 \times H^s([a_0,a_1]) \times l^2_\tau DU^2$ in $B_r^s(q)$ and hence $T_c^{-1}(\lambda; \cdot)$ depends analytically on $q \in X^s$.

Step 3. Conservation of T_c^{-1} by the Gross-Pitaevskii flow.
If initially $q_0 - 1 \in \mathcal{S}$, then the Gross-Pitaevskii equation (1.1) has a unique global-in-time solution $q \in \mathcal{C}(\mathbb{R}; Z^1)$ (see (1.3) for the definition of Zhidkov's space Z^1) by Zhidkov's well-posedness result. By Faddeev-Takhtajan [14], $(q-1)(t,\cdot) \in \mathcal{S}$ and $\mathcal{M}, \mathcal{P}, T^{-1}(\lambda)$ are all conserved by the Gross-Pitaevskii flow, and hence $T_c^{-1}(\lambda) = e^{-i\mathcal{M}(2z)^{-1} - i\mathcal{P}(2z\zeta)^{-1}} T^{-1}(\lambda)$ is also conserved.

Now let $q \in \mathcal{C}(I; X^s)$, $s > \frac{1}{2}$ be the solution of the Gross-Pitaevskii equation (1.1) with the initial data $q_0 \in X^s$ on the time interval I. Then by the density result in Theorem 1.1, we take $\{q_{0,n}\} \subset 1 + \mathcal{S}$ such that $d^s(q_{0,n}, q_0) \to 0$ as $n \to \infty$. By the continuity of the Gross-Pitaevskii flow in Theorem 2.1, for all $t \in I$, the corresponding solutions q_n, q satisfy $d^s(q_n(t), q(t)) \to 0$ and hence $T_c^{-1}(q_n(t)) \to T_c^{-1}(q(t))$ as $n \to \infty$ by the analyticity

[7] We can solve the ODE for $\dot{w}$ by the following iterative procedure:
$$\dot{w} = \sum_{n=0}^{\infty} \dot{w}_n, \quad \dot{w}_{n+1} = \begin{pmatrix} \int_{a_1}^t q_2 \dot{w}_n^2 \,\mathrm{d}x \\ \int_{a_1}^t e^{\varphi(t)-\varphi(x)} q_3 \dot{w}_n^1 \,\mathrm{d}x \end{pmatrix}, \quad \dot{w}_0 = \begin{pmatrix} b_1^1 + b_1^2 \int_{a_1}^t e^{2iz(x-a_1)} q_2 \,\mathrm{d}x \\ b_1^2 \int_{a_1}^t e^{\varphi(t)-\varphi(x)+2iz(x-a_1)} q_4 \,\mathrm{d}x \end{pmatrix}.$$
Although $\dot{w}_0^2 \neq 0$, there is an exponential decay in the integrand and the same estimates as in Lemma 4.2 imply the well-definedness of $\dot{w}_n$ and $\dot{w}$. For example, we can control straightforward
$$\dot{w}_0 = \begin{pmatrix} b_1^1 + b_1^2 \int_{a_1<x_1<t} e^{2iz(x_1-a_1)} q_2 dx_1 \\ b_1^2 \int_{a_1<x_1<t} e^{2iz(t-a_1) + \int_{x_1}^t q_4} q_4 dx_1 \end{pmatrix},$$
$$\dot{w}_1 = \begin{pmatrix} b_1^2 \int_{a_1<x_1<y_1<t} e^{2iz(y_1-a_1) + \int_{x_1}^{y_1} q_4} q_2(y_1) q_4(x_1) dx_1 dy_1 \\ b_1^1 \int_{a_1<x_1<t} e^{\varphi(t)-\varphi(x_1)} q_3 + b_1^2 \int_{a_1<x_1<y_1<t} e^{\varphi(t)-\varphi(y_1)+2iz(x_1-a_1)} q_2(x_1) q_3(y_1) dx_1 dy_1 \end{pmatrix}$$
in terms of $\| \mathbf{q} \|_{l^2_\tau DU^2}$.

of $T_c^{-1}(\lambda;\cdot)$ above. The conservation of $T_c^{-1}(q_n)$ along the Gross-Pitaevskii flow implies the conservation of $T_c^{-1}(q(t))$ on the existence time interval I. □

4.3. **Superharmonic function G on the upper half-plane.** If $q-1\in\mathcal{S}$, then $G(z)$ defined in (3.25) is a well-defined nonnegative superharmonic function on the z-upper half-plane. Indeed, as $|T_c^{-1}(\lambda)|\geq 1$ if $\lambda\in\mathcal{I}_{\text{cut}}=(-\infty,-1]\cup[1,\infty)$, the trace of $G(z)$ on the real line is non negative:

$$\mu(\xi)=\frac{1}{2}\sum_{\pm}(2\xi)^2\ln\big|T_c^{-1}\big|\big(\pm\sqrt{\xi^2+1}\big)\geq 0,\quad \xi\in\mathbb{R}.$$

On the other hand, since the meromorphic function T_c has only simple poles $\lambda_m\in(-1,1)$, we can take a small enough neighborhood $\mathcal{V}_m$ of λ_m such that $T_c(\lambda)=A_0(\lambda)+\frac{A_1(\lambda)}{\lambda-\lambda_m}$ on $\mathcal{V}_m$, with $A_1\neq 0, A_0$ holomorphic functions on $\mathcal{V}$. For $\lambda_m\neq 0$, $\lambda\in\mathcal{V}_m$ and correspondingly for $z\in\mathcal{U}_m=\{z\in\mathcal{U}:(\lambda,z)\in\mathcal{R},\ \lambda\in\mathcal{V}_m\}$, we can write (noticing $\lambda^2-\lambda_m^2=z^2-z_m^2$, $z_m=i\sqrt{1-\lambda_m^2}\in i(0,1]$)

$$\ln T_c(\lambda)+\ln T_c(-\lambda)=\ln\big(A_0(\lambda)\,(\lambda-\lambda_m)+A_1(\lambda)\big)+\ln\frac{\lambda+\lambda_m}{z+z_m}+\ln\frac{1}{z-z_m}+\ln T_c(-\lambda).$$

For $\lambda_m=0$, $z_m=i$, we can still write $\ln T_c(\lambda)+\ln T_c(-\lambda)$ in $\mathcal{V}_0$ as

$$\ln\big(A_0(\lambda)\lambda+A_1(\lambda)\big)+\ln\frac{1}{z+z_m}+\ln\frac{1}{z-z_m}+\ln\big(A_0(-\lambda)\,\lambda-A_1(-\lambda)\big).$$

Hence $-\Delta_z G=\Delta_z\mathrm{Re}\,\frac{1}{2}\sum_{\lambda=\pm\sqrt{1+z^2}}\big(4z^2\ln T_c(\lambda)\big)$ is a nonnegative measure (3.26) on $\mathcal{U}$. As $T_c^{-1}\to 1$ as $|\lambda|\to\infty$, we derive $G\geq 0$ on the whole upper half plane by maximum principle.

Let $q\in X^s$, then by the density argument as in the last part of last subsection, we deduce that G defined in (3.25) still satisfies $G\geq 0$, $-\Delta G\geq 0$ on the upper half plane. Since the meromorphic function $T_c(q)$ has countably many simple poles $\{\lambda_m\}\subset\mathbb{C}\setminus\mathcal{I}_{\text{cut}}$, the fact $-\Delta G=-\pi\sum(2z)^2\delta_{z=z_m}\geq 0$ implies $z_m\in i(0,1]$ and hence $\lambda_m\in(-1,1)$. This completes the proof of (3.24) for general $q\in X^s$. Theorem 3.1 follows from Proposition 4.2.

5. The energies

In this section we will formulate the energies $\mathcal{E}^s_\tau(q)$, $\tau\geq 2$ for $q\in X^s$, $s>\frac{1}{2}$ in terms of the renormalised transmission coefficient $T_c^{-1}(\lambda)$ defined in Theorem 3.1:

Theorem 5.1. *Let $q\in X^s$, $s>\frac{1}{2}$. Let $T_c^{-1}(\lambda)$ be the renormalised transmission coefficient which is a holomorphic function on the Riemann surface $\mathcal{R}\ni(\lambda,z)$ and has countably many simple zeros $\{\lambda_m\}\subset(-1,1)$ given in Theorem 3.1. Let $G(z)=\frac{1}{2}\sum_{\pm}\mathrm{Re}\left(4z^2\ln T_c^{-1}(\pm\sqrt{z^2+1})\right)$ be the nonnegative superharmonic function on the upper half plane $\mathcal{U}$, with $-\Delta_z G=-\pi\sum_m(2z)^2\delta_{z=z_m}\geq 0$, $z_m=i\sqrt{1-\lambda_m^2}\in i(0,1]$, given by Theorem 3.1.*

Then for $N=[s-1]$, $G(i\tau/2)$ has the following finite expansion as $\tau\to\infty$ (recalling the notations $\mathcal{H}^l$ in the asymptotic expansion (1.13)*):*

$$G(\frac{i\tau}{2})=\sum_{l=0}^{N}(-1)^l\mathcal{H}^{2l+2}\tau^{-2l-1}+\mathcal{H}^{>2N+2}(\frac{i\tau}{2}),\quad \mathcal{H}^{>2N+2}=o(\tau^{-2s+1}),\tag{5.1}$$

such that the trace of G on the real line μ exists, the measure $(1+\xi^2)^N\mu$ is finite and the following trace formula for $\mathcal{H}^{2l+2}$, $0\leq l\leq N$ holds:

$$\mathcal{H}^{2l+2}=\frac{1}{\pi}\int_{\mathbb{R}}\xi^{2l+2}\frac{1}{2}\sum_{\pm}\ln|T_c^{-1}(\pm\sqrt{\xi^2/4+1})|d\xi-\frac{1}{2l+3}\sum_m\mathrm{Im}\,(2z_m)^{2l+3}.\tag{5.2}$$

We define a family of energy functionals $(\mathcal{E}^s_{\tau'})_{\tau'\geq 2} : X^s \mapsto [0,\infty)$ *as follows:*

$$\mathcal{E}^s_{\tau'}(q) = -\frac{2}{\pi}\sin(\pi(s-1))\int_{\tau'}^{\infty}(\tau^2-\tau'^2)^{s-1}\Big(G(i\frac{\tau}{2}) - \sum_{l=0}^{N}(-1)^l\mathcal{H}^{2l+2}\tau^{-2l-1}\Big)\,\mathrm{d}\tau$$
$$+\sum_{l=0}^{N}\tau'^{2(s-1-l)}\binom{s-1}{l}\mathcal{H}^{2l+2},\ \text{with } N\leq s-1<N+1, \tag{5.3}$$

such that $\mathcal{E}^s_{\tau'}$ *is analytic in* $q\in X^s$ *and we have the following trace formula for* $\mathcal{E}^s_{\tau'}$*:*

$$\mathcal{E}^s_{\tau'} = \frac{1}{\pi}\int_{\mathbb{R}}(\xi^2+\tau'^2)^{s-1}d\mu_{G(\frac{\cdot}{2})} - \sum_m \operatorname{Im}\int_0^{2z_m} w^2(w^2+\tau'^2)^{s-1}dw. \tag{5.4}$$

Then there exists a universal constant $C\geq 2$ *(depending only on* s*) such that whenever* $(q,\tau_0)\in X^s\times[C,\infty)$ *satisfying*

$$\frac{1}{\tau_0}\|\,\mathbf{q}\,\|_{l^2_{\tau_0}DU^2} < \frac{1}{2C},\ \text{with } \|\,\mathbf{q}\,\|^2_{l^2_{\tau_0}DU^2} = \||q|^2-1\|^2_{l^2_{\tau_0}DU^2} + \|q'\|^2_{l^2_{\tau_0}DU^2}, \tag{5.5}$$

$\mathcal{E}^s_{\tau_0}(q)$ *is equivalent to the square of the energy norm* $(E^s_{\tau_0}(q))^2$ *in the sense of* (1.19)*:*

$$|\mathcal{E}^s_{\tau_0} - (E^s_{\tau_0})^2| \leq \frac{C}{\tau_0}\|\,\mathbf{q}\,\|_{l^2_{\tau_0}DU^2}(E^s_{\tau_0})^2. \tag{5.6}$$

5.1. **The framework.** We are going to introduce the assumptions and the notations which will be used throughout this section.

5.1.1. *Assumption.* We restrict ourselves on the imaginary axis in this section:

$$\begin{aligned}&(\lambda,z)=(i\sigma,i\tau/2)\in\mathcal{R},\quad \tau\geq\tau_0\geq 2,\quad \sigma=\sqrt{\tau^2/4-1}\in[\tau/2-1,\tau/2),\\ &\zeta=\lambda+z=i\omega,\quad \omega=\sigma+\tau/2\in[\tau/2,\tau).\end{aligned} \tag{5.7}$$

Here $\tau_0\geq 2$ is a constant (to be chosen sufficiently large later), such that the following assumption holds:

$$\big||q|^2-1\big| \leq \frac{1}{64}{\tau_0}^2,\quad q\in X^s,\ s>\frac{1}{2}. \tag{5.8}$$

5.1.2. *Functions* q_2,q_3,q_4. We evaluate the functions q_2,q_3,q_4 defined in $(*)$ on the imaginary axis to arrive at

$$\begin{aligned}
q_2 &= -\frac{1}{\omega^{-2}|q|^2+1}\big(\frac{1}{\omega}q'\big) + \frac{1}{\omega}\frac{q}{\omega^{-2}|q|^2+1}\big(\frac{1}{\omega}(|q|^2-1)\big),\\
q_3 &= \frac{1}{\omega^{-2}|q|^2+1}\big(\frac{1}{\omega}\bar{q}'\big) + \frac{1}{\omega}\frac{\bar{q}}{\omega^{-2}|q|^2+1}\big(\frac{1}{\omega}(|q|^2-1)\big),\\
q_4 &= \frac{-2}{\omega^{-2}|q|^2+1}\big(\frac{1}{\omega}(|q|^2-1)\big) + \Big(\frac{\frac{1}{\omega}q}{\omega^{-2}|q|^2+1}\big(\frac{1}{\omega}\bar{q}'\big) - \frac{\frac{1}{\omega}\bar{q}}{\omega^{-2}|q|^2+1}\big(\frac{1}{\omega}q'\big)\Big).
\end{aligned}$$

Notice that q_2,q_3,q_4 are all linear combinations of the components in

$$Q=\frac{1}{\tau}(|q|^2-1,q',\bar{q}'), \tag{5.9}$$

up to coefficients being some polynomials of the following form

$$P = P\Big(\frac{1}{\omega^{-2}|q|^2+1},\frac{1}{\tau}q,\ \frac{1}{\tau}\bar{q},\ \frac{1}{\tau^2}(|q|^2-1)\Big), \tag{5.10}$$

and for notational simplicity, we denote O to be the following set:

$$O := \Big\{P\cdot\frac{1}{\tau}(|q|^2-1),\ P\cdot\frac{1}{\tau}q',\ P\cdot\frac{1}{\tau}\bar{q}'\ \Big|\ P \text{ is any polynomial of form (5.10)}\Big\}. \tag{5.11}$$

Under the assumption (5.8), we are going to estimate $\|q_\kappa\|_{l^2_\tau DU^2}$ as follows (similar but more accurate as the estimates in Lemma 4.1):

$$\begin{aligned} &\|q_\kappa\|_{l^2_\tau DU^2} \lesssim C_\tau c_\tau, \quad \kappa = 2,3,4, \qquad \text{with } C_\tau = 1 + \frac{1}{\tau}\|q'\|_{l^\infty_\tau DU^2}, \\ &c_\tau(q) = \frac{1}{\tau}\|\,\mathbf{q}\,\|_{l^2_\tau DU^2(\mathbb{R})} = \frac{1}{\tau}\big(\||q|^2-1\|^2_{l^2_\tau DU^2} + \|\partial_x q\|^2_{l^2_\tau DU^2}\big)^{\frac{1}{2}} := \|Q\|_{l^2_\tau DU^2}. \end{aligned} \tag{5.12}$$

5.1.3. *Asymptotic expansion of* $\ln T_c^{-1}$ *on the imaginary axis.* Recall the aymptotic expansions of $T_c^{-1}(\lambda)$ and $\ln T_c^{-1}$ in Theorem 3.1:

$$T_c^{-1}(\lambda) = e^{\Phi}\Big(1 + \sum_{j=1}^\infty T_{2j}\Big), \quad T_{2j} = \wedge^j,$$

$$\ln T_c^{-1}(\lambda) = \Phi + T_2 + \sum_{j=2}^\infty \tilde{T}_{2j},\ \Phi(\lambda) := -\frac{i}{2z}\int_{\mathbb{R}} \frac{(|q|^2-1)^2}{|q|^2-\zeta^2}\,\mathrm{d}x + \frac{1}{2z\zeta}\int_{\mathbb{R}} \frac{\bar{q}q'(|q|^2-1)}{(|q|^2-\zeta^2)}\,\mathrm{d}x\,,$$

where $\tilde{T}_{2j}$ is linear combination of *connected* symbols $_{2j}$ of degree $2j$. Recall the symbols in Subsection 3.3:

$$\wedge^j = \int_{x_1<y_1<\cdots<x_j<y_j} \prod_{n=1}^j e^{\varphi(y_n)-\varphi(x_n)} q_3(x_n)q_2(y_n)\,\mathrm{d}x\,\mathrm{d}y\,,\ \varphi(x) = -\tau x + \int_0^x q_4, \tag{5.13}$$

$$_{2j} = \int_{t_1<\cdots<t_{2j}} \prod_{n=1}^{2j-1} e^{\delta_n(\varphi(t_{n+1})-\varphi(t_n))} q_{\kappa_1}(t_1)\cdots q_{\kappa_{2j}}(t_{2j})dt, \tag{5.14}$$

for some $\delta_n \in \{1,\cdots,j\}$ with $\delta_1 = \delta_{2j-1} = 1$ and $\kappa_n \in \{2,3\}$ with $\kappa_1 = 3$, $\kappa_{2j} = 2$. For notational simplicity, we also introduce the following symbol

$$= \int_{x<y} (e^{\varphi(y)-\varphi(x)} - e^{-\tau(y-x)}) q_2(y)q_3(x)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{5.15}$$

We will rewrite $(\Phi + T_2)$ in the asymptotic expansion of $\ln T_c^{-1}(\lambda)$ above in Appendix A as:

$$\Phi + T_2 = \tilde{T}_2 + \tilde{T}_3,$$

where $\tilde{T}_2, \tilde{T}_3$ identify the *quadratic* and *cubic* terms (in terms of elements in the set O) in the expansion of $\ln T_c^{-1}$ respectively. More precisely we will prove in Appendix A that

Lemma 5.1. *The asymptotic expansion* (3.22) *for the logarithm of the transmission coefficient* $\ln T_c^{-1}$ *reads on the imaginary axis* (5.7) *as follows*

$$\ln T_c^{-1}(i\sigma) = \tilde{T}_2(i\sigma) + \tilde{T}_3(i\sigma) + \sum_{j\ge 2} \tilde{T}_{2j}(i\sigma), \tag{5.16}$$

where $\tilde{T}_2$ *identifies the quadratic part in the expansion of* $\ln T_c^{-1}(i\sigma)$*:*

$$\begin{aligned} \tilde{T}_2(i\sigma) =& \frac{-1}{\tau^2}\int_{x<y} e^{-\tau(y-x)}\Big((|q|^2-1)(y)(|q|^2-1)(x) + q'(y)\bar{q}'(x)\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &- i\frac{\tau+2\omega}{\tau^3\omega^2}\int_{\mathbb{R}} \operatorname{Im}(q'\bar{q})(|q|^2-1)\,\mathrm{d}x \\ &+ \frac{1}{\tau^3\omega}\int_{x<y} e^{-\tau(y-x)}\Big(q'(y)\bar{q}'(x) - \bar{q}'(y)q'(x)\Big)\,\mathrm{d}x\,\mathrm{d}y\,, \end{aligned} \tag{5.17}$$

$\tilde{T}_3(i\sigma)$ identifies the cubic part and reads as linear combination of finite integrals of the following type

$$\begin{aligned}&\int_{\mathbb{R}}\big(\frac{|q|^2-1}{\tau^2}\big)^2 h_1\,\mathrm{d}x\,, \quad , \quad \int_{x<y \text{ or } x>y} e^{-\tau|y-x|}\big(\frac{|q|^2-1}{\tau^2}h_1\big)(y)h_2(x)\,\mathrm{d}x\;\mathrm{d}y\,,\\ &\int_{x<y} e^{-\tau(y-x)}h_1(y)\int_x^y\big(\frac{q' \text{ or } \bar{q}'}{\tau}\big)(m)\,\mathrm{d}m\,h_2(x)\,\mathrm{d}x\;\mathrm{d}y\,, \quad h_1,h_2\in O,\end{aligned} \tag{5.18}$$

and $\tilde{T}_{2j}$, $j\geq 2$ remains the same linear combination of integrals $_{2j}$.

Remark 5.1. *If $q-1\in\mathcal{S}$, then by integration by parts $\int_{x<y} e^{-\tau(y-x)}f(y)g(x)\,\mathrm{d}x\;\mathrm{d}y = \frac{1}{\tau}\int_{\mathbb{R}} fg - \frac{1}{\tau}\int_{x<y} e^{-\tau(y-x)}f(y)g'(x)\,\mathrm{d}x\;\mathrm{d}y$ we can expand $\tilde{T}_2(i\sigma)$ as*

$$\tilde{T}_2(i\sigma) = -\frac{1}{\tau^3}\int_{\mathbb{R}}\big((|q|^2-1)^2+|q'|^2\big)\,\mathrm{d}x\,+\frac{1}{\tau^4}\int_{\mathbb{R}}\big(q'\bar{q}''-3i\mathrm{Im}\,(q'\bar{q})(|q|^2-1)\big)\,\mathrm{d}x\,+O(\frac{1}{\tau^5}),$$

while $\tilde{T}_3(i\sigma) = O(\frac{1}{\tau^5})$, $T_{2j}(i\sigma) = O(\frac{1}{\tau^{3j}})$, $\tilde{T}_{2j}(i\sigma) = O(\frac{1}{\tau^{4j-1}})$, as $\tau\to\infty$. Recalling $\ln T^{-1}(\lambda) = i\mathcal{M}(2z)^{-1}+i\mathcal{P}(2z\zeta)^{-1}+\ln T_c^{-1}(\lambda)$, we derive the finite expansion until the fourth-order for $\ln T^{-1}(i\sigma) = \frac{1}{\tau}\mathcal{M} - i\frac{1}{\tau^2}\mathcal{P} - \frac{1}{\tau^3}\mathcal{H}^2 + i\frac{1}{\tau^4}\mathcal{H}^3 + O(\frac{1}{\tau^5})$, as $\tau\to\infty$, which can be compared with (1.13).

5.2. **Trace formula and the organisation of the section.** In this section we will recall the trace formula (from e.g. [2, 31]) for the nonnegative superharmonic functions G on the upper half-plane $\mathcal{U}$ given in Theorem 3.1. As a consequence we derive the formulation of the energy norm $E^s_{\tau_0}(q)$ in terms of the quadratic term $\tilde{T}_2(i\sigma)$, as well as the equivalence relation (5.6) between $E^s_{\tau_0}(q)$ and the energy $\mathcal{E}^s_{\tau_0}(q)$ (defined in (5.3)).

5.2.1. *Trace formula.* Recall the superharmonic function

$$G(z) = \frac{1}{2}\sum_{\pm}\mathrm{Re}\,\big(4z^2\ln T_{\mathrm{c}}^{-1}(\pm\sqrt{z^2+1})\big)$$

defined on the upper half-plane $\mathcal{U}$ in Theorem 3.1, with the nonnegative Radon measure μ as the trace of G on the real line $\mathbb{R}$ and the nonnegative Radon measure $\nu = -\Delta_z G = -\pi\sum_m (2z)^2\delta_{z=z_m}$ on the upper half-plane $\mathcal{U}$, $z_m\in i(0,1]$. We define

$$\Xi_s(z) = \mathrm{Im}\int_0^z w^2(w^2+\tau_0)^s dw$$

for z in the upper half plane.

Lemma 5.2. *The followings hold true:*

- *Representation of G through μ,ν.*
  The function G can be represented in terms of the Poisson kernel and the fundamental solution of the Laplace equation as follows:

$$G(z) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{\mathrm{Im}\,z}{|z-\xi|^2}d\mu(\xi) + \frac{1}{2\pi}\int_{\mathcal{U}}\ln\Big|\frac{z-\bar{\zeta}}{z-\zeta}\Big|d\nu(\zeta). \tag{5.19}$$

- *Expansion of G at $+i\infty$.*
  If two measures $(1+|\xi|^2)^N\mu$, $\mathrm{Im}\,z(1+|z|^2)^N\nu$ are finite, then we have the following precise expansion of G at $+i\infty$:

$$G(\frac{i\tau}{2}) = \sum_{l=0}^N(-1)^l\mathcal{H}^{2l+2}\tau^{-2l-1}+\mathcal{H}^{>2N+2},\ \mathcal{H}^{>2N+2} = o(\tau^{-2N-1}), \tag{5.20}$$

  where $\mathcal{H}^{2l+2}$ is given in (5.2) *and*

$$\mathcal{H}^{>2N+2} = (-1)^{N+1}\Big(\frac{1}{\pi}\int_{\mathbb{R}}\frac{\xi^{2N+2}}{\tau^2+\xi^2}d\mu_{G(\frac{\cdot}{2})} - \sum_m\mathrm{Im}\int_0^{2z_m}\frac{w^{2N+4}}{\tau^2+w^2}dw\Big)\tau^{-2N-1}.$$

- *Trace formula of G.*
  Let $N=[s-1]$. Then the following trace formula holds

$$
\begin{aligned}
&-\frac{2\sin(\pi(s-1))}{\pi}\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{s-1}\Big(G(\frac{i\tau}{2})-\sum_{l=0}^{N}(-1)^l\mathcal{H}^{2l+2}\tau^{-2l-1}\Big)d\tau \\
&+\sum_{l=0}^{N}\tau_0^{2(s-1-l)}\binom{s-1}{l}\mathcal{H}^{2l+2}=\frac{1}{\pi}\int_{\mathbb{R}}(\xi^2+\tau_0^2)^{s-1}d\mu_{G(\cdot/2)}-\sum_m \Xi_{s-1}(2z_m),
\end{aligned}
\tag{5.21}
$$

whenever either side is finite.

We have the following description of the energy norm by the trace formula:

Lemma 5.3. *Let $q\in X^s$, $s>\frac{1}{2}$, then the energy norm defined in (1.16):*

$$(E^s_{\tau_0}(q))^2=\|\mathbf{q}\|^2_{H^{s-1}_{\tau_0}}=\int_{\mathbb{R}}(\xi^2+\tau_0^2)^{s-1}\big(|\widehat{|q|^2-1}|^2+|\widehat{q'}|^2\big)(\xi)\,\mathrm{d}\xi$$

can be identified as the integral of the quadratic term $\tilde{T}_2(i\sigma)$ in (5.17) as

$$
\begin{aligned}
(E^s_{\tau_0}(q))^2=&-\frac{2}{\pi}\sin(\pi(s-1))\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{s-1}\mathcal{H}_2^{>2N+2}(i\sigma)\,\mathrm{d}\tau \\
&+\sum_{l=0}^{N}\binom{s-1}{l}\tau_0^{2(s-1-l)}\mathcal{H}_2^{2l+2},\quad N=[s-1],
\end{aligned}
\tag{5.22}
$$

where $\mathcal{H}_2^{2l+2}=\int_{\mathbb{R}}\xi^{2l}\big(\big|\widehat{|q|^2-1}\big|^2+|\widehat{q'}|^2\big)(\xi)\,\mathrm{d}\xi\lesssim\tau_0^{-2(s-1-l)}(E^s_{\tau_0}(q))^2$ and

$$\mathcal{H}_2^{>2N+2}(i\sigma)=\mathrm{Re}\,(4z^2\tilde{T}_2)(i\sigma)-\sum_{l=0}^{N}(-1)^l\mathcal{H}_2^{2l+2}\,\tau^{-2l-1}=o(\tau^{1-2s}),\quad \tau\to\infty.$$

Proof. The proof (with $\tau_0=1$) in [31] works here and we give here the proof for readers' convenience. We make use of the unitary Fourier transform and inverse Fourier transform

$$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-ix\xi}f(x)\,\mathrm{d}x\,,\quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{ix\xi}\hat{f}(\xi)\,d\xi,$$

to write for any function $f\in H^s$,

$$
\begin{aligned}
\mathrm{Re}\int_{x<y}e^{-\tau(y-x)}\bar{f}(x)f(y)\,\mathrm{d}x\ \mathrm{d}y\ &=\frac{1}{2\pi}\mathrm{Re}\int_{\mathbb{R}^3}\frac{1}{\tau-i\xi}e^{iy\eta-iy\xi}\hat{f}(\eta)\overline{\hat{f}(\xi)}\,\mathrm{d}y\ d\xi\,d\eta \\
&=\mathrm{Re}\int_{\mathbb{R}}\frac{1}{\tau-i\xi}\hat{f}(\xi)\overline{\hat{f}(\xi)}\,d\xi=\int_{\mathbb{R}}\frac{\tau}{\tau^2+\xi^2}|\hat{f}(\xi)|^2\,d\xi,
\end{aligned}
$$

where the righthand side is the value at the point $i\tau$ of the harmonic function on the upper half plane with the trace $\pi|\hat{f}(\xi)|^2$ on the real axis. Therefore noticing from the definition (5.17) that

$$\mathrm{Re}\,(4z^2\tilde{T}_2)(i\sigma)=\int_{x<y}e^{-\tau(y-x)}\Big((|q|^2-1)(y)(|q|^2-1)(x)+q'(y)\bar{q}'(x)\Big)\,\mathrm{d}x\,\mathrm{d}y\,,$$

let $f=|q|^2-1$ or q' above, then (5.22) follows from the trace formula (5.21). ☐

5.2.2. *Ideas and the organisation of the rest of this section.* Recall the notations on the imaginary axis (5.7) and the expansion (5.16):

$$\ln T_c^{-1}(i\sigma)=\tilde{T}_2(i\sigma)+\tilde{T}_3(i\sigma)+\sum_{j\geq 2}\tilde{T}_{2j}(i\sigma).$$

Recall the energy $\mathcal{E}^s_{\tau_0}$ defined in (5.3) (noticing the symmetry $\mathrm{Re}\,\ln T_c^{-1}(i\sigma) = \mathrm{Re}\,\ln T_c^{-1}(-i\sigma)$ in (3.24) and hence $G(i\frac{\tau}{2}) = \mathrm{Re}\left(4z^2\ln T_c^{-1}\right)(i\sigma)$):

$$\mathcal{E}^s_{\tau_0} = -\frac{2}{\pi}\sin(\pi(s-1))\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{s-1}\Big(\mathrm{Re}\left(4z^2\ln T_c^{-1}\right)(i\sigma) - \sum_{l=0}^{N}(-1)^l\mathcal{H}^{2l+2}\tau^{-2l-1}\Big)\,\mathrm{d}\tau + \sum_{l=0}^{N}\tau_0^{2(s-1-l)}\binom{s-1}{l}\mathcal{H}^{2l+2}.$$

Recall the formulation of the energy norm $(E^s_{\tau_0})^2$ in (5.22).

In order to show the equivalence (5.6) between $\mathcal{E}^s_{\tau_0}$ and $(E^s_{\tau_0})^2$, it suffices to estimate, if $s \in (\frac{1}{2}, \frac{3}{2})$ such that $[s-1] < 1$, their difference $|\mathcal{E}^s_{\tau_0} - (E^s_{\tau_0})^2|$ which concerns cubic or higher order terms in the expansion of $\ln T_c^{-1}(i\sigma)$:

$$|\mathcal{E}^s_{\tau_0} - (E^s_{\tau_0})^2| = \Big|\frac{2}{\pi}\sin(\pi(s-1))\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{s-1}\mathrm{Re}\Big(4z^2\big(\tilde{T}_3 + \sum_{j\geq 2}\tilde{T}_{2j}\big)\Big)(i\sigma)\,\mathrm{d}\tau\,\Big| \tag{5.23}$$

by $Cc_\tau(E^s_{\tau_0})^2$ whenever $c_\tau < \frac{1}{2C}$.

If $s \geq \frac{3}{2}$ is large enough [8], then we also have to do finite expansions for $\tilde{T}_3(i\sigma)$ and $\tilde{T}_{2j}(i\sigma)$, $2 \leq j \leq s$ until k-th order, $k = [2s]$:

$$\tau^2\tilde{T}_3(i\sigma) = \sum_{l=3}^{k}\mathcal{H}_3^l\tau^{-l+1} + \mathcal{H}_3^{>k}(i\sigma),\ \ \tau^2\tilde{T}_{2j}(i\sigma) = \sum_{l=2j}^{k}\mathcal{H}_{2j}^l\tau^{-l+1} + \mathcal{H}_{2j}^{>k}(i\sigma)\ \text{as}\ \sigma\to\infty,$$

such that the difference above (5.23) is replaced by

$$\begin{aligned}|\mathcal{E}^s_{\tau_0} - (E^s_{\tau_0})^2| = \Big|&\frac{2}{\pi}\sin(\pi(s-1))\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{s-1}\mathrm{Re}\Big(\mathcal{H}_3^{>k} + \sum_{2\leq j\leq s}\mathcal{H}_{2j}^{>k}\\ &+\tau^2\sum_{j>s}\tilde{T}_{2j}\Big)(i\sigma)\,\mathrm{d}\tau - \sum_{l=1}^{[s-1]}\tau_0^{2(s-1-l)}\binom{s-1}{l}\mathrm{Re}\big(\mathcal{H}_3^{2l+2} + \sum_{2\leq j\leq s}\mathcal{H}_{2j}^{2l+2}\big)\Big|.\end{aligned} \tag{5.24}$$

We are going to follow the strategy and make use of the estimates in Section 6, [31] to control the differences (5.23) and (5.24). The rest of this section is organised as follows:

- We establish the estimates for high order terms $\tilde{T}_{2j}$, $j > s$ (and $\tilde{T}_3$ if $s \in (\frac{1}{2}, \frac{3}{2})$) and low order terms $\tilde{T}_3, \tilde{T}_{2j}$, $2 \leq j < s$ for $s > \frac{3}{2}$ in Subsections 5.3 and 5.4 respectively;
- We complete the proof of Theorem 5.1 in Subsection 5.5.

5.3. **High order terms.** We are going to derive the estimates for high order terms in this section, which can be viewed as a more accurate version of the estimates in Section 4.2 on the imaginary axis setting:

- We derive first some preliminary estimates for q_2, q_3, q_4, P and then the estimates for the integrals $\wedge^j{}_{,2j}$, in Subsection 5.3.1;
- We make use of these estimates to control the high order terms in Subsection 5.3.2, which can control the difference (5.23) when $s \in (\frac{1}{2}, \frac{3}{2})$.

We will use the estimates (4.1) and (4.4) freely through this section.

[8] Recalling Remark 5.1: $\mathcal{H}_3^2 = 0$, we indeed have to do finite expansions only when $s \geq 2$.

5.3.1. *Preliminary estimates.* Recall the notations in Subsection 5.1 and we are going to estimate q_2, q_3, q_4, P, $\wedge^j$, ${}_{2j}$, , in terms of

$$C_\tau = 1 + \frac{1}{\tau}\|q'\|_{l^\infty_\tau DU^2} \text{ and } \|\,\mathbf{q}\,\|_{l^p_\tau DU^2} = \big\|\big(\||q|^2-1\|_{l^p_\tau DU^2},\, \|q'\|_{l^p_\tau DU^2}\big)\big\|_{\ell^p}.$$

Lemma 5.4. *Assume* (5.7) *and* (5.8)*:* $(\lambda, z) = (i\sigma, i\tau/2) \in \mathcal{R}$*,* $\tau \geq \tau_0 \geq 2$*,* $\zeta = \lambda + z = i\omega$*,* $\omega \in [\frac{\tau}{2}, \tau)$ *and* $q \in X^s$*,* $s > \frac{1}{2}$ *such that* $\big||q|^2 - 1\big| \leq \frac{1}{64}{\tau_0}^2$*.*

Then the following estimates hold for P *defined in* (5.10) *and for* $p \in [1, \infty]$*:*

$$\|P\|_{l^\infty_\tau V^2} \lesssim C_\tau,\ \|q_\kappa\|_{l^p_\tau DU^2} + \|\frac{1}{\tau^2}(|q|^2-1)\|_{l^p_\tau U^2} \lesssim \frac{C_\tau}{\tau}\|\,\mathbf{q}\,\|_{l^p_\tau DU^2},\ \kappa = 2,3,4.$$

Proof. We firstly derive straightforward from the assumption (5.8) that

$$\|q\|_{l^\infty_\tau V^2} \lesssim \|q\|_{L^\infty} + \|q'\|_{l^\infty_\tau DU^2} \lesssim \tau(1+\tau^{-1}\|q'\|_{l^\infty_\tau DU^2}) = \tau C_\tau.$$

Recalling the partition of unity (4.2), we have

$$\begin{aligned}
\Big\|\frac{1}{\omega^{-2}|q|^2+1}\Big\|_{l^\infty_\tau V^2} &\leq 1 + \sup_k \Big\|\frac{\chi_{\tau,k}}{\omega^{-2}|q|^2+1}\Big\|_{V^2} \\
&\leq 1 + 2\sup_k \sup_{\frac{k-1}{\tau}=t_0<t_1<\dots<t_N=\frac{k+1}{\tau}} \Big(\sum_{j=0}^{N-1}\big((\chi_{\tau,k}(t_{j+1}) - \chi_{\tau,k}(t_j)\big)^2 \\
&\qquad + \big(\omega^{-2}(|q(t_{j+1})|^2 - |q(t_j)|^2)\big)^2\Big)^{\frac{1}{2}} \lesssim 1 + \tau^{-1}\|q'\|_{l^\infty_\tau DU^2} = C_\tau,
\end{aligned}$$

where we used $\big||q|^2-1\big| \leq \frac{1}{64}{\tau_0}^2$ and $|q(t_{j+1}) - q(t_j)| \lesssim \|q'\|_{l^\infty_\tau DU^2}$. Similarly we have the same estimate for $\frac{1}{\tau^2}(|q|^2-1)$ and hence for the polynomial P. Therefore the estimates for q_κ, $\kappa = 2,3,4$ and $\|\frac{1}{\tau^2}(|q|^2-1)\|_{l^p_\tau U^2}$ follow from (4.1) and (4.4). □

We claim the following estimates similar as (4.9) (recalling $\chi_{\tau,k}$ in the partition of unity (4.2) and the assumption (5.8)):

$$\begin{aligned}
&\Big\|\chi_{\tau,k}\Big(e^{\int_{\frac{k}{\tau}}^x q_4\,dx'} - 1\Big)\Big\|_{V^2} \lesssim C_\tau \tilde{c}_{\tau,k}, \quad \Big\|\chi_{\tau,k} e^{\int_{\frac{k}{\tau}}^x q_4\,dx'}\Big\|_{V^2} \lesssim 1, \\
&\Big|e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4\,dx} - 1\Big| \lesssim e^{\frac{1}{2}(k-k')} C_\tau \sum_{k''=k'}^{k} \tilde{c}_{\tau,k''}, \quad \Big|e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4\,dx}\Big| \lesssim e^{\frac{1}{4}(k-k')}, \quad k' \leq k-1,
\end{aligned} \tag{5.25}$$

with $\tilde{c}_{\tau,k} = \frac{1}{\tau}\big(\|\tilde{\chi}_{\tau,k}(|q|^2-1)\|_{DU^2} + \|\tilde{\chi}_{\tau,k} q'\|_{DU^2}\big)$, $\tilde{\chi} = \chi(\frac{1}{12}\cdot)$. Indeed, we write

$$q_4 = a + ib, \quad a = \mathrm{Re}\, q_4 = \frac{-2(|q|^2-1)/\omega}{\omega^{-2}|q|^2+1}, \quad b = \mathrm{Im}\, q_4 = \frac{2\mathrm{Im}\,(q\bar{q}')/(\omega^2)}{\omega^{-2}|q|^2+1}.$$

Since we derive from (5.8) that $\big\|\bar{\chi}_{\tau,k}(x)\int_{\frac{k}{\tau}}^x a\,\mathrm{d}x'\big\|_{V^2} \leq \frac{8}{\tau}\|a\|_{L^\infty} \leq \frac{1}{2}$ with $\bar{\chi} = \chi(\frac{1}{2}\cdot)$ supported on $[-\frac{8}{3}, \frac{8}{3}]$ such that $\bar{\chi}\chi = \chi$, we have the following estimate from Lemma 5.4:

$$\big\|\chi_{\tau,k}(x)\big(e^{\int_{\frac{k}{\tau}}^x a\,\mathrm{d}x'} - 1\big)\big\|_{V^2} \lesssim \Big\|\bar{\chi}_{\tau,k}(x)\int_{\frac{k}{\tau}}^x a\,\mathrm{d}x'\Big\|_{V^2} \lesssim \frac{C_\tau}{\tau}\big\|\tilde{\chi}_{\tau,k}(|q|^2-1)\big\|_{DU^2}.$$

On the other hand, since $|e^{ic}| \leq 1$, $c \in \mathbb{R}$, we have

$$\Big\|\chi_{\tau,k}(x)\Big(e^{i\int_{\frac{k}{\tau}}^x b\,\mathrm{d}x'} - 1\Big)\Big\|_{V^2} \lesssim \Big\|\bar{\chi}_{\tau,k}(x)\int_{\frac{k}{\tau}}^x b\,\mathrm{d}x'\Big\|_{V^2} \lesssim \frac{C_\tau}{\tau}\big\|\tilde{\chi}_{\tau,k} q'\big\|_{DU^2}.$$

Then the first line of the estimate (5.25) follows. Similarly we have the second line of (5.25) for $k' \leq k-1$: We derive straightforward from (5.8) that $\Big|e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4\,\mathrm{d}x}\Big| \leq \Big|e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} a\,\mathrm{d}x}\Big| \leq$

$e^{\frac{1}{4}(k-k')}$, and hence by Lemma 5.4

$$\Big| e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4 \, \mathrm{d}x} - 1 \Big| \le e^{\frac{1}{4}(k-k')} \Big| \int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4 \, \mathrm{d}x \Big| \lesssim \big(e^{\frac{1}{4}(k-k')} (k-k') \big) C_\tau \sum_{k''=k'}^{k} \tilde{c}_{\tau,k''}.$$

Therefore we have the following estimates for $\cap^j, {}_{2j}$,:

Lemma 5.5. *Assume the same hypothesis as in Lemma 5.4. Then we have the following estimates:*

$$\begin{aligned} &|\cap^j| \lesssim (\|q_2\|_{l^2_\tau DU^2} \|q_3\|_{l^2_\tau DU^2})^j, \quad |{}_{2j}| \lesssim \max\{\|q_2\|_{l^{2j}_\tau DU^2}, \|q_3\|_{l^{2j}_\tau DU^2}\}^{2j}, \\ &|| \lesssim \frac{C_\tau}{\tau} \| \mathbf{q} \|_{l^3_\tau DU^2} \|q_2\|_{l^3_\tau DU^2} \|q_3\|_{l^3_\tau DU^2}. \end{aligned} \tag{5.26}$$

Proof. We follow the decomposition idea in the proof of Lemma 4.2 to derive that

$$\begin{aligned} &\Big\| \int_{x<y<t} (e^{\varphi(y)-\varphi(x)} - e^{-\tau(y-x)}) g(y) h(x) \, \mathrm{d}x \, \mathrm{d}y \Big\|_{U^2_t} \\ &\lesssim \sum_k \Big\| \int_{y-\frac{3}{\tau}}^{y} (\chi_{\tau,k} g)(y) e^{-\tau(y-x)} \Big(e^{\int_{\frac{k}{\tau}}^{y} q_4} - 1 + e^{\int_{\frac{k}{\tau}}^{y} q_4} (e^{\int_x^{\frac{k}{\tau}} q_4} - 1) \Big) (\tilde{\chi}_{\tau,k} h)(x) \, \mathrm{d}x \Big\|_{DU^2_y} \\ &\quad + \sum_{k' \le k-1} e^{-(k-k')} \Big\| \int_{-\infty}^{y-\frac{3}{\tau}} (\chi_{\tau,k} g)(y) \Big(e^{\int_{\frac{k}{\tau}}^{y} q_4} - 1 + e^{\int_{\frac{k}{\tau}}^{y} q_4} (e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4} - 1) \\ &\qquad + e^{\int_{\frac{k}{\tau}}^{y} q_4} e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4} (e^{\int_x^{\frac{k'}{\tau}} q_4} - 1) \Big) (\chi_{\tau,k'} h)(x) \, \mathrm{d}x \Big\|_{DU^2_y}, \end{aligned}$$

and hence we take $g = q_2$ and $h = q_3$ to arrive at the estimate for $||$ in (5.26) by virtue of the claim (5.25).

Similarly, since $e^{\varphi(y)-\varphi(x)} = e^{-\tau(y-x)} e^{\int_{\frac{k}{\tau}}^{y} q_4} e^{\int_{\frac{k'}{\tau}}^{\frac{k}{\tau}} q_4} e^{\int_x^{\frac{k'}{\tau}} q_4}$, we have the following estimate for the operator S (defined in (3.12)) from the claim (5.25):

$$\|S\|_{V^2 \mapsto U^2} \lesssim \|q_2\|_{l^2_\tau DU^2} \|q_3\|_{l^2_\tau DU^2}, \quad S(f)(t) = \int_{x<y<t} e^{\varphi(y)-\varphi(x)} q_2(y) (q_3 f)(x) \, \mathrm{d}x \, \mathrm{d}y,$$

and hence the estimate for $\cap^j$ in (5.26) follows:

$$|\cap^j| = |\lim_{t\to\infty} (S^j 1)(t)| \le \|S^j 1\|_{V^2} \le (C \|q_2\|_{l^2_\tau DU^2} \|q_3\|_{l^2_\tau DU^2})^j.$$

We apply the above estimate for the operator S iteratively to arrive at

$\|S_{2j}\|_{V^2 \mapsto U^2} \lesssim \max\{\|q_2\|_{l^{2j}_\tau DU^2}, \|q_3\|_{l^{2j}_\tau DU^2}\}^{2j}$, for the operator

$$S_{2j}(f)(t) = \int_{t_1 < \cdots < t_{2j} < t} \prod_{n=1}^{2j-1} e^{\delta_n (\varphi(t_{n+1}) - \varphi(t_n))} q_{\kappa_1}(t_{2j}) \cdots (q_{\kappa_{2j}} f)(t_1) dt_1 \cdots dt_{2j},$$

where $\kappa_n \in \{2,3\}$, $\delta_n \in \{1, \cdots, j\}$, and hence the estimate for ${}_{2j}$ in (5.26) follows. □

5.3.2. *Estimates for high order terms.* Recall (1.15), (1.16), (5.9) and (5.12):

$$\begin{aligned} &\mathbf{q} = (|q|^2 - 1, q'), \quad E^s_\tau = \| \mathbf{q} \|_{H^{s-1}_\tau}, \quad C_\tau = 1 + \frac{1}{\tau} \|q'\|_{l^\infty_\tau DU^2}, \\ &Q = \frac{1}{\tau}(|q|^2 - 1, q', \bar{q}'), \quad c_\tau = \frac{1}{\tau} \| \mathbf{q} \|_{l^2_\tau DU^2} = \|Q\|_{l^2_\tau DU^2}, \end{aligned}$$

and the scaling invariance property (1.18):

$$E^s_{\tau_0} = \tau_0^{s-\frac{1}{2}} \| \mathbf{q}_{\tau_0} \|_{H^{s-1}}, \quad \|f\|_{l^p_\tau DU^2} = \|f_{\tau_0}\|_{l^p_{\tilde{\tau}} DU^2}, \quad f_\tau = \frac{1}{\tau} f(\frac{\cdot}{\tau}), \quad \tilde{\tau} := \frac{\tau}{\tau_0}. \tag{5.27}$$

We are going to give the estimates for high order terms in the expansion of $\ln T_c^{-1}$ in Lemma 5.1: $\tilde T_{2j}$, $j>s$ if $s>\frac{3}{2}$ or $\tilde T_3$ and $\tilde T_{2j}$, $j>s$ if $s\in(\frac12,\frac32)$, which will be used to control the energy difference $|\mathcal{E}^s_{\tau_0}-(E^s_{\tau_0})^2|$ in (5.23) and (5.24). Indeed, after rescaling, the estimates in Sections 5, 6 in [31] work well here and we simply make use of them to derive the estimates. We refer the interested readers there for more detailed analysis.

Proposition 5.1. *Assume* (5.7) *and* (5.8)*. For $j\geq 1$, $T_{2j}=\cap^j$ and for $j\geq 2$, $\tilde T_{2j}$ is finite linear combination of integrals ${}_{2j}$. $\tilde T_3$ is finite linear combination of cubic terms in* (5.18)*.*

Then there exist a constant C and a constant C_j depending on $j\geq 2$ such that

$$\begin{aligned}&|\mathbf{1}_{j\geq1}T_{2j}(i\sigma)|+|\mathbf{1}_{j\geq2}\tilde T_{2j}(i\sigma)|\leq (CC_\tau\|Q\|_{l^2_\tau DU^2})^{2j},\\ &|\mathbf{1}_{j\geq2}\tilde T_{2j}(i\sigma)|\leq C_j\big(C_\tau\|Q\|_{l^{2j}_\tau DU^2}\big)^{2j},\\ &|\tilde T_3(i\sigma)|\leq \big(CC_\tau\|Q\|_{l^3_\tau DU^2}\big)^3.\end{aligned}\tag{5.28}$$

If we assume furthermore (with a possibly larger C depending on s)

$$c_{\tau_0}=\frac{1}{\tau_0}\|\,\mathbf{q}\,\|_{l^2_{\tau_0}DU^2}\leq\frac{1}{C},\tag{5.29}$$

we have for non-half integer $s>\frac12$ that

$$\int_{\tau_0}^\infty(\tau^2-\tau_0^2)^{s-1}\tau^2\sum_{j\geq 2s-1}\Big(\big|T_{2j}(i\sigma)\big|+\mathbf{1}_{j\geq2}\big|\tilde T_{2j}(i\sigma)\big|\Big)\,\mathrm{d}\tau\leq\frac{Cc_{\tau_0}^{2[2s]-2}}{[2s]+1-2s}(E^s_{\tau_0})^2,\tag{5.30}$$

$$\int_{\tau_0}^\infty(\tau^2-\tau_0^2)^{s-1}\tau^2\sum_{s\leq j<2s-1}\big|\mathbf{1}_{j\geq2}\tilde T_{2j}(i\sigma)\big|\,\mathrm{d}\tau\leq\frac{Cc_{\tau_0}^{2[s]}}{[s]+1-s}(E^s_{\tau_0})^2,\tag{5.31}$$

and in particular when $s\in(\frac12,\frac32)$, we have

$$\int_{\tau_0}^\infty(\tau^2-\tau_0^2)^{s-1}\tau^2\Big(\big|\tilde T_3(i\sigma)\big|+\sum_{j\geq2}\big|\tilde T_{2j}(i\sigma)\big|\Big)\,\mathrm{d}\tau\leq\frac{Cc_{\tau_0}}{(s-\frac12)(\frac32-s)}(E^s_{\tau_0})^2.\tag{5.32}$$

Proof. The estimates for $T_{2j}(i\sigma),\tilde T_{2j}(i\sigma)$ in (5.28) follow directly from Lemma 5.4 and Lemma 5.5 [9]. By virtue of the integrals in (5.18), we derive from Lemmas 5.4 and 5.5 that

$$\begin{aligned}|\tilde T_3(i\sigma)|\lesssim C_\tau\Big\|\frac{|q|^2-1}{\tau^2}\Big\|^2_{l^3_\tau U^2}\Big\|\frac{|q|^2-1}{\tau}\Big\|_{l^3_\tau DU^2}+\|\,\|\\ +\Big(\Big\|\frac{|q|^2-1}{\tau^2}\Big\|_{l^3_\tau U^2}+\Big\|\frac{q'}{\tau}\Big\|_{l^3_\tau DU^2}\Big)\sum_{h\in O}\|h\|^2_{l^3_\tau DU^2}\leq\big(CC_\tau\|Q\|_{l^3_\tau DU^2}\big)^3.\end{aligned}$$

If $C_\tau\leq 2$, by change of variables $\tau\to\tilde\tau=\frac{\tau}{\tau_0}$ we bound the integral in terms of $|T_{2j}(i\sigma)|$ in (5.30) as follows:

$$\begin{aligned}&\int_{\tau_0}^\infty(\tau^2-\tau_0^2)^{s-1}\tau^2\big(\big|T_{2j}(i\sigma)\big|+\mathbf{1}_{j\geq2}\big|\tilde T_{2j}(i\sigma)\big|\big)\,\mathrm{d}\tau\leq(2C)^{2j}\int_{\tau_0}^\infty(\tau^2-\tau_0^2)^{s-1}\tau^2\|Q\|^{2j}_{l^2_\tau DU^2}\,\mathrm{d}\tau\\ &=(2C)^{2j}\tau_0^{2s-1}\int_1^\infty(\tilde\tau^2-1)^{s-1}\|Q_{\tau_0}\|^{2j-2}_{l^2_{\tilde\tau}DU^2}\|\,\mathbf{q}_{\tau_0}\,\|^2_{l^2_{\tilde\tau}DU^2}d\tilde\tau,\end{aligned}$$

[9] If $|T_{2j}|\leq A$ then $|\tilde T_{2j}|\leq(2A)^j$, $j\geq2$. Indeed, following the proof of Proposition 5.10 in [31], by multiplying q_2,q_3 by η, we arrive from the expansions (3.21) and (3.22) at $\ln(1+\sum_{j\geq1}\eta^{2j}T_{2j})=\eta^2T_2+\sum_{j\geq2}\eta^{2j}\tilde T_{2j}$. We introduce a partial order $\preceq$ in the set of holomorphic functions near zero, where $g\preceq h$ means that the absolute value of each coefficient in the Taylor series of g at zero is bounded by the corresponding coefficient in the Taylor series of h. In particular, $\ln(1+\zeta)\preceq\frac{\zeta}{1-\zeta}:=f(\zeta)$ and hence $\sum_{j=2}^\infty\tilde T_{2j}\eta^{2j}\preceq f\circ f(A\eta)\preceq\sum_{j=1}^\infty 2^{j-1}A^j\eta^j$, $|T_{2j}|\leq A=CC_\tau c_\tau$ such that $|\tilde T_{2j}|\leq(2A)^j$, $j\geq2$.

with the equality ensured by (5.27). Proposition 5.13 in [31] implies that in the regime $-\frac{1}{2} < s-1 \le \frac{j-1}{2} \le j-1$, the above integral is bounded by

$$\frac{1}{j+1-2s}(2C)^{2j}\tau_0^{2s-1}\|\frac{1}{\tau_0}\,\mathbf{q}_{\tau_0}\|^{2j-2}_{l^2_1 DU^2}\|\,\mathbf{q}_{\tau_0}\,\|^2_{H^{s-1}} = \frac{1}{j+1-2s}(2C)^{2j}c_{\tau_0}^{2j-2}(E^s_{\tau_0})^2,$$

with the equality ensured by (5.27).

Therefore under the smallness assumption (5.29) such that $C_\tau \le 1 + c_\tau \le 1 + c_{\tau_0} \le 2$, (5.30) holds.

Similarly, if $C_\tau \le 2$, we bound the integral in terms of $|\tilde{T}_{2j}(i\sigma)|$ in (5.31) as

$$\int_{\tau_0}^\infty (\tau^2-\tau_0^2)^{s-1}\tau^2\big|\tilde{T}_{2j}(i\sigma)\big|\,\mathrm{d}\tau \le C_j 2^{2j}\tau_0^{2s-1}\int_1^\infty (\tilde{\tau}^2-1)^{s-1}\|Q_{\tau_0}\|^{2j-2}_{l^{2j}_{\tilde{\tau}}DU^2}\|\,\mathbf{q}_{\tau_0}\,\|^2_{l^{2j}_{\tilde{\tau}}DU^2}d\tilde{\tau}.$$

Proposition 6.2 in [31] and (5.27) implies (5.31).

Finally we can do the same as for $\tilde{T}_{2j}$ above to $\tilde{T}_3$ (with $2j$ replaced by 3 in the above) for $s \in (\frac{1}{2}, \frac{3}{2})$. Hence (5.32) follows from (5.30) and (5.31). □

5.4. **Low order terms.** Let $s > \frac{3}{2}$ and we aim to get the estimates for the low order terms $\tilde{T}_3$, $\tilde{T}_{2j}$, $2 \le j < s$ in this section, which will be used to control the energy difference $|\mathcal{E}^s_{\tau_0} - (E^s_{\tau_0})^2|$ in (5.24). Indeed, as in Section 6 [31], we have to do integration by parts to expand low order terms until k-th order, $k = [2s]-1$.

We will first derive the general formula for finite expansions in Subsection 5.4.1 and then apply it to the low order terms in Subsection 5.4.2.

5.4.1. *Finite expansion.* Let us do integration by parts in the following integral:

$$\begin{aligned}\int_{x<y} e^{\varphi(y)-\varphi(x)}g(y)h(x)\,\mathrm{d}x\;\mathrm{d}y &= \int_{x<y} e^{-\tau(y-x)}e^{\int_x^y q_4}g(y)h(x)\,\mathrm{d}x\;\mathrm{d}y\\ &= \frac{1}{\tau}\int_{\mathbb{R}} gh\,\mathrm{d}x + \frac{1}{\tau}\int_{x<y} e^{\varphi(y)-\varphi(x)}\big(g(y)(q_4h)(x) - g(y)h'(x)\big)\,\mathrm{d}x\;\mathrm{d}y\\ &= \frac{1}{\tau}\int_{\mathbb{R}} gh\,\mathrm{d}x + \frac{1}{\tau}\int_{x<y} e^{\varphi(y)-\varphi(x)}\big((q_4g)(y)h(x) + g'(y)h(x)\big)\,\mathrm{d}x\;\mathrm{d}y\,,\end{aligned}$$

which leads us to define the two operators $D_\pm$:

$$D_\pm(g) = (q_4 \pm \partial_x)g. \tag{5.33}$$

Then we have

Lemma 5.6. *For any $k \in \mathbb{N}$, we have the following formal finite-order expansion:*

$$\int_{x<y} e^{\varphi(y)-\varphi(x)}g(y)h(x)\,\mathrm{d}x\;\mathrm{d}y = \sum_{\ell=1}^k b_\ell + b^{\ge k+1},$$

where

$$\begin{aligned}b_\ell &= \frac{1}{\tau^\ell}\int_{\mathbb{R}}(D_+^{m_\ell}g)(D_-^{n_\ell}h)\,\mathrm{d}y = \int_{\mathbb{R}}\Big((\frac{D_+}{\tau})^{m_\ell}\frac{g}{\tau}\Big)\Big((\frac{D_-}{\tau})^{n_\ell}h\Big)\,\mathrm{d}y\,,\ m_\ell+n_\ell+1=\ell,\\ b^{\ge k+1}(z) &= \frac{1}{\tau^k}\int_{x<y} e^{\varphi(y)-\varphi(x)}(D_+^m g)(y)(D_-^n h)(x)\,\mathrm{d}x\;\mathrm{d}y\\ &= \int_{x<y} e^{\varphi(y)-\varphi(x)}\Big((\frac{D_+}{\tau})^m g\Big)(y)\Big((\frac{D_-}{\tau})^n h\Big)(x)\,\mathrm{d}x\;\mathrm{d}y\,,\ m+n=k.\end{aligned}$$

In order to study the applications of the operators $D_\pm$ on q_2, q_3, we make use of the structures and the preliminary estimates for q_2, q_3, q_4, P derived in Lemma 5.4. Recall $\mathbf{q} = (|q|^2-1, q')$ and $Q = \frac{1}{\tau}(|q|^2-1, q', \bar{q}')$. In the following we will be flexible in the notations concerning Q in the sense that the notation $Q\cdots Q$ will be understood as one element in the set $\{Q^{\alpha_1}\cdots Q^{\alpha_n} \mid \alpha_\beta = 1,2,3\}$.

Notice that q_2, q_3, q_4 are, up to the multiplication by $\frac{1}{\omega}q$, $\frac{1}{\omega}\bar{q}$, $\frac{1}{\omega^{-2}|q|^2+1}$, $\omega \in [\frac{\tau}{2}, \tau)$, linear combinations of components of Q, and the applications (perhaps several times) of the operators D_+, ∂_x or of the multiplication operators $\mathcal{M}_{q_\kappa}$ on $q_{\kappa'}$, $\kappa, \kappa' = 2, 3, 4$ are, up to the multiplication by the polynomials in $\frac{1}{\omega}q$, $\frac{1}{\omega}\bar{q}$, $\frac{1}{\omega^{-2}|q|^2+1}$,

applications of the derivative ∂_x on $Q, Q', Q'', \cdots$ or the multiplication of Q.

For notational simplicity, we introduce the set O_M, $M \geq 1$, which concerns $(M-1)$-times applications of the operators $\frac{1}{\tau}D_\pm$ or $\frac{1}{\tau}\partial_x$ or $\mathcal{M}_{\frac{1}{\tau}Q}$ on Q, as follows (noticing that $O_1 = O$ defined in (5.11)):

$$O_M = \Big\{ h_{(M,\alpha)} = P \cdot \Big(\prod_{\gamma=1}^{\alpha-1} \frac{Q^{(\ell_\gamma)}}{\tau^{1+\ell_\gamma}} \Big) \frac{Q^{(\ell_\alpha)}}{\tau^{\ell_\alpha}} \,\Big|\, P \text{ is any polynomial of form (5.10)}, \quad \alpha = 1, \cdots, M, \quad \ell_1 + \cdots + \ell_\alpha + \alpha = M \Big\}. \tag{5.34}$$

In the following $h_{(M,\alpha)}$ will always denote an element in O_M which is homogeneous of degree α in Q and sometimes we will denote simply $h_{(M)} \in O_M$ without pointing out the precise homogeneity. Then the operators $\frac{1}{\tau}D_\pm, \frac{1}{\tau}\partial_x, \mathcal{M}_{\frac{1}{\tau}Q}$ map O_M to O_{M+1} and $\mathcal{M}_{\frac{Q^{(m)}}{\tau^{1+m}}}$ maps O_M to O_{M+m}.

We can rewrite the finite expansion in Lemma 5.6 with $\bar{g} = h = q'$ by use of the notations $h_{(m)} \in O_m$ as follows:

$$\begin{aligned}
&\frac{1}{\tau^2}\int_{x<y} e^{\varphi(y)-\varphi(x)} \bar{q}'(y) q'(x)\,\mathrm{d}x\,\mathrm{d}y = \sum_{\ell=1}^{k} a_\ell + a^{\geq k+1}, \text{where} \\
&a_\ell = \int_{\mathbb{R}} h_{(\ell+1,\alpha)}\,\mathrm{d}x\,, \quad \alpha \geq 2, \\
&a^{\geq k+1} = \int_{x<y} e^{\varphi(y)-\varphi(x)} h_{(m)}(y) h_{(n)}(x)\,\mathrm{d}x\,\mathrm{d}y\,,\; m+n = k+2.
\end{aligned}$$

Motivated by this finite expansion formula, we derive the following estimates.

Lemma 5.7. *Assume the same hypothesis in Lemma 5.4. Recall c_τ, C_τ, E^s_τ defined in (5.12), (1.16). Let O_m be the set defined in (5.34).*

Then for any $M \geq 1$, there exists a constant $C(M)$ such that the following holds for any $h_{(M,\alpha)} \in O_M$ homogeneous of degree α, $\alpha \in [1, M]$ in Q:

$$\|h_{(M,\alpha)}\|_{l^p_\tau DU^2} \leq C(M) C_\tau \sum_{\ell_1+\cdots+\ell_\alpha = M-\alpha,\ \sup_\beta \ell_\beta \leq [M/2]}^{M-1} \prod_{\gamma=1}^{\alpha} \Big\| \frac{Q^{(\ell_\gamma)}}{\tau^{\ell_\gamma}} \Big\|_{l^{p\alpha}_\tau DU^2}. \tag{5.35}$$

Furthermore, the following estimates hold when $k = [2s] \leq 2s$, $\alpha \geq 2$:

$$\tau^{\ell+2} \Big| \int_{\mathbb{R}} h_{(\ell+1,\alpha)}\,\mathrm{d}x \Big| \lesssim \tau_0^{\ell-2s+1} C_{\tau_0} c_{\tau_0}^{\alpha-2} (E^s_{\tau_0})^2,\ \tau \geq \tau_0,\ 1 \leq \ell \leq k-1, \tag{5.36}$$

$$\begin{aligned}
&\int_{\tau_0}^{\infty} (\tau^2 - \tau_0^2)^{s-1} \Big| \tau^2 \int_{x<y} \big(e^{\varphi(y)-\varphi(x)} - e^{-\tau(y-x)} \big) h_{(m,\alpha_1)}(y) h_{(n,\alpha_2)}(x)\,\mathrm{d}x\,\mathrm{d}y \Big|\,\mathrm{d}\tau \\
&\lesssim \frac{1}{|\sin(2\pi s)|} C_{\tau_0}^2 \sum_{\alpha=\alpha_1+\alpha_2-1}^{k-1} c_{\tau_0}^{\alpha} (E^s_{\tau_0})^2, \quad m+n = k,
\end{aligned} \tag{5.37}$$

(5.38)

$$\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{s-1}\Big|\tau^2\int_{t_1<\cdots<t_n}\prod_{i=1}^{n-1}e^{\delta_i(\varphi(t_{i+1})-\varphi(t_i))}h_{(m_1,\alpha_1)}(t_1)\cdots h_{(m_n,\alpha_n)}(t_n)dt\Big|\,\mathrm{d}\tau$$
$$\lesssim\frac{1}{|\sin(2\pi s)|}C_{\tau_0}^n\sum_{\alpha=\alpha_1+\cdots+\alpha_n-2}^{k+n-3}c_{\tau_0}^{\alpha}(E_{\tau_0}^s)^2,\quad 1\le\delta_i\le n/2,\quad m_1+\cdots+m_n=k+1.$$

Proof. The estimate (5.35) comes from the estimates in (4.1), (4.4) and the estimate for P in Lemma 5.4:

$$\|h_{(M,\alpha)}\|_{l_\tau^p DU^2}\lesssim C_\tau\prod_{\gamma=1}^{\alpha-1}\Big\|\frac{Q^{(\ell_\gamma)}}{\tau^{1+\ell_\gamma}}\Big\|_{l_\tau^{\frac{p\alpha}{\alpha-1}}U^2}\Big\|\frac{Q^{(\ell_\alpha)}}{\tau^{\ell_\alpha}}\Big\|_{l_\tau^{p\alpha}DU^2},\quad \ell_1+\cdots+\ell_\alpha=M-\alpha$$
$$\lesssim C_\tau\sum_{M-\alpha\le\ell_1+\cdots+\ell_\alpha\le M-1}\prod_{\gamma=1}^{\alpha}\Big\|\frac{Q^{(\ell_\gamma)}}{\tau^{\ell_\gamma}}\Big\|_{l_\tau^{p\alpha}DU^2}\text{ by use of (4.4).}$$

In the above, by integration by parts we can always choose $\sup_{1\le\beta\le\alpha}\ell_\beta\le[M/2]$.

We now turn to the proof of (5.36). We first bound $\tau^{\ell+2}\Big|\int_{\mathbb{R}}h_{(\ell+1,\alpha)}\,\mathrm{d}x\Big|$, $\tau\ge\tau_0$, $\alpha\ge 2$ by

$$\tau_0^{\ell+2-\alpha}\Big|\int_{\mathbb{R}}P\cdot\Big(\prod_{\gamma=1}^{\alpha-1}\frac{\mathbf{q}^{(\ell_\gamma)}}{\tau_0^{1+\ell_\gamma}}\Big)\frac{\mathbf{q}^{(\ell_\alpha)}}{\tau_0^{\ell_\alpha}}\,\mathrm{d}x\Big|=\tau_0^{\ell+2-\alpha}\Big|\int_{\mathbb{R}}P(\frac{\cdot}{\tau_0})\cdot\prod_{\gamma=1}^{\alpha}\mathbf{q}_{\tau_0}^{(\ell_\gamma)}\,\mathrm{d}x\Big|,\quad f_\tau=\frac{1}{\tau}f(\frac{\cdot}{\tau}),$$

for some $\ell_1+\cdots+\ell_\alpha=\ell+1-\alpha$. We can do as above for $h_{(M,\alpha)}$ to derive that

$$\tau^{\ell+2}\Big|\int_{\mathbb{R}}h_{(\ell+1,\alpha)}\,\mathrm{d}x\Big|\lesssim C_{\tau_0}\tau_0^{\ell+2-\alpha}\sum_{\ell+1-\alpha\le\ell_1'+\cdots+\ell_\alpha'\le\ell,\ \ell_\beta'\le[(\ell+1)/2]}\prod_{\gamma=1}^{\alpha}\|\mathbf{q}_{\tau_0}^{(\ell_\gamma')}\|_{l_1^\alpha DU^2}.$$

By the proof of Proposition 6.5 [31], the pointwise bound (5.36) holds:

$$\tau^{\ell+2}\Big|\int_{\mathbb{R}}h_{(\ell+1,\alpha)}\,\mathrm{d}x\Big|\lesssim C_{\tau_0}\tau_0^{\ell+2-\alpha}\|\,\mathbf{q}_{\tau_0}\|_{l_1^2DU^2}^{\alpha-2}\|\,\mathbf{q}_{\tau_0}\|_{H^{s-1}}^2,\quad 1\le\ell\le 2s-1$$
$$\sim\tau_0^{\ell-2s+1}C_{\tau_0}c_{\tau_0}^{\alpha-2}(E_{\tau_0}^s)^2,\text{ by the scaling property (5.27).}$$

We now consider the following integral

$$\tau^2\int_{x<y}e^{\varphi(y)-\varphi(x)}h_{(m)}(y)h_{(n)}(x)\,\mathrm{d}x\,\mathrm{d}y\,,\ m+n=k+1,$$

which, by the estimate for $\cap$ in Lemma 5.5 and Lemma 5.7, is bounded by

$$\tau^2C_\tau^2\sum_{\alpha=2}^{k+1}\sum_{\ell_1+\cdots+\ell_\alpha=k+1-\alpha}^{k-1}\prod_{\gamma=1}^{\alpha}\|\frac{Q^{(\ell_\gamma)}}{\tau^{\ell_\gamma}}\|_{l_\tau^\alpha DU^2}.$$

Therefore, exactly as in the proof of Proposition 5.1, we do change of variable $\tau\to\tilde\tau=\frac{\tau}{\tau_0}$ and make use of the scaling property (5.27), such that

$$\int_{\tau_0}^{\infty}(\tau^2-\tau_0^2)^{s-1}\Big|\tau^2\int_{x<y}e^{\varphi(y)-\varphi(x)}h_{(m)}(y)h_{(n)}(x)\,\mathrm{d}x\,\mathrm{d}y\Big|\,\mathrm{d}\tau\lesssim\tau_0^{2s-1}\int_1^\infty(\tilde\tau^2-1)^{s-1}$$
$$\cdot\,C_{\tau_0}^2\sum_{\alpha=2}^{k+1}\sum_{\ell_1+\cdots+\ell_\alpha=k+1-\alpha,\ \ell_\beta\le[k/2]}^{k-1}\Big(\prod_{\gamma=1}^{\alpha-2}\Big\|\frac{Q_{\tau_0}^{(\ell_\gamma)}}{\tilde\tau^{\ell_\gamma}}\Big\|_{l_{\tilde\tau}^\alpha DU^2}\Big)\Big\|\frac{\mathbf{q}_{\tau_0}^{(\ell_{\alpha-1})}}{\tilde\tau^{\ell_{\alpha-1}}}\Big\|_{l_{\tilde\tau}^\alpha DU^2}\Big\|\frac{\mathbf{q}_{\tau_0}^{(\ell_\alpha)}}{\tilde\tau^{\ell_\alpha}}\Big\|_{l_{\tilde\tau}^\alpha DU^2}d\tilde\tau,$$

which, by Propositions 6.4 and 6.5 in [31], is bounded by

$$\frac{\tau_0^{2s-1}C_{\tau_0}^2}{|\sin(2\pi s)|}\sum_{\alpha=2}^{k+1}\|\frac{1}{\tau_0}\,\mathbf{q}_{\tau_0}\|_{l_1^\alpha DU^2}^{\alpha-2}\|\,\mathbf{q}_{\tau_0}\|_{H^{s-1}}^2=\frac{C_{\tau_0}^2\sum_{\alpha=0}^k c_{\tau_0}^\alpha}{2s-[2s]}(E_{\tau_0}^s)^2,\quad\text{if }\frac{k}{2}<s<\frac{k+1}{2}.$$

Similarly, by use of the estimates for $,_{2j}$ in Lemma 5.5 and Lemma 5.7 we derive the estimates (5.37) and (5.38) respectively. □

5.4.2. *Estimates for low order terms* $\tilde{T}_3, \tilde{T}_{2j}$. In this section we are going to do finite expansions in Lemma 5.6 to the low order terms $\tilde{T}_3, \tilde{T}_{2j}$, $2 \leq j < s$, $s > \frac{3}{2}$, keeping in mind the estimates in Lemma 5.7.

Proposition 5.2. *Assume* (5.7) *and* (5.8) *and let* $s > \frac{3}{2}$ *be away from half integers such that* $k := [2s] \geq 3$. *Then*

- *we can expand* $\tau^2 \tilde{T}_3(i\sigma)$ *until* k*th-order:*

$$\tau^2 \tilde{T}_3(i\sigma) = \sum_{\ell=3}^{k} \mathcal{H}_3^\ell \tau^{-\ell+1} + \mathcal{H}_3^{>k}(i\sigma), \quad \mathcal{H}_3^\ell \textit{ independent of } \tau, \quad \sigma \to \infty,$$

$$\textit{such that } \tau_0^{2s-\ell} |\mathcal{H}_3^\ell| \leq C C_{\tau_0} \sum_{\alpha=1}^{k-2} c_{\tau_0}^\alpha (E_{\tau_0}^s)^2, \tag{5.39}$$

$$\textit{and } \int_{\tau_0}^\infty (\tau^2 - \tau_0^2)^{s-1} \big| \mathcal{H}_3^{>k}(i\sigma) \big| \, \mathrm{d}\tau \leq \frac{C C_{\tau_0}^3 \sum_{\alpha=1}^{k-1} c_{\tau_0}^\alpha}{|\sin(2\pi s)|} (E_{\tau_0}^s)^2;$$

- *for* $2 \leq j < s$, *we can expand* $\tau^2 \tilde{T}_{2j}(i\sigma)$ *until* k*th-order:*

$$\tau^2 \tilde{T}_{2j}(i\sigma) = \sum_{\ell=2j}^{k} \mathcal{H}_{2j}^\ell \tau^{-\ell+1} + \mathcal{H}_{2j}^{>k}(i\sigma), \quad \mathcal{H}_{2j}^\ell \textit{ independent of } \tau, \textit{ as } \sigma \to \infty,$$

$$\textit{such that } \tau_0^{2s-\ell} |\mathcal{H}_{2j}^\ell| \leq C C_{\tau_0} \sum_{\alpha=2j-2}^{k-2} c_{\tau_0}^\alpha (E_{\tau_0}^s)^2, \tag{5.40}$$

$$\textit{and } \int_{\tau_0}^\infty (\tau^2 - \tau_0^2)^{s-1} \big| \mathcal{H}_{2j}^{>k}(i\sigma) \big| \, \mathrm{d}\tau \leq \frac{C C_{\tau_0}^{2j} \sum_{\alpha=2j-2}^{k-1} c_{\tau_0}^\alpha}{|\sin(2\pi s)|} (E_{\tau_0}^s)^2.$$

Proof. We will follow exactly the procedure in Section 6 [31] and hence be sketchy.

Recall that $\tilde{T}_3$ is linear combinations of integrals of type in (5.18). We do integration by parts as in Lemma 5.6 to the integrals in (5.18) from the left and right sides alternatively to expand $\tau^2 \tilde{T}_3(i\sigma)$ until $(k-1)$th-order: We do integration by parts $(k-1)$-times to the integral while $(k-2)$-times to the last two integrals in (5.18), and we also notice the expansion $\operatorname{Im} \zeta = \omega = \tau + O(\tau^{-1})$ as $\tau \to \infty$, to arrive at the expansion in (5.39).

Here, $\mathcal{H}_3^\ell$, $3 \leq \ell \leq k$ is the leading order (in terms of τ^{-1}) of a linear combination of integrals of the following forms with bounded coefficients:

$$\tau^2 \int_{\mathbb{R}} \Big((D_+^{m_\ell} q_2)(D_-^{n_\ell} q_3) - (\partial_x^{m_\ell} q_2)\big((-\partial_x)^{n_\ell} q_3\big) \Big) \, \mathrm{d}x \,, \quad m_\ell + n_\ell \leq \ell - 2,$$

$$\tau^3 \int_{\mathbb{R}} \big(\frac{(|q|^2 - 1) \text{ or } q' \text{ or } \bar{q}'}{\tau^2} h_1 \big)^{(m_\ell - 1)} h_2^{(n_\ell)} \, \mathrm{d}x \,, \quad h_1, h_2 \in O \text{ defined in (5.11)}.$$

Then $\mathcal{H}_3^\ell$ is linear combination of integrals of type $\tau^{m+1} \int_{\mathbb{R}} h_{(m,\alpha)} \, \mathrm{d}x$, $3 \leq m \leq \ell$ with $h_{(m,\alpha)} \in O_m$ homogeneous of degree $\alpha \geq 3$ in Q. Hence the estimate for $\mathcal{H}_3^\ell$ in (5.39) follows from (5.36) in Lemma 5.7.

Here $\mathcal{H}_3^{>k}$ is linear combination of integrals such as $\tau \int_{\mathbb{R}} h_{(k,\alpha)} \, \mathrm{d}x$, $\alpha \geq 3$ (which appears still because of the expansion of ω in terms of τ) and the following integrals with bounded coefficients, $m+n=k-1$, $m_1+m_2+m_3=k+1$ and $h_{(m)}, h_{(m,\alpha)} \in O_m$:

$$\tau^2 \int_{x<y} \Big((e^{\varphi(y)-\varphi(x)} - e^{-\tau(y-x)}) (\frac{1}{\tau^m} D_+^m q_2)(y) (\frac{1}{\tau^n} D_-^n q_3)(x) \, \mathrm{d}x \, \mathrm{d}y \,,$$

$$\tau^2 \int_{x<y} e^{-\tau(y-x)} \Big((\frac{1}{\tau^m} D_+^m q_2)(y) (\frac{1}{\tau^n} D_-^n q_3)(x) - (\frac{1}{\tau^m} \partial_y^m q_2)(y) \big(\frac{1}{\tau^n} (-\partial_x)^n q_3\big)(x) \Big) \, \mathrm{d}x \, \mathrm{d}y \,,$$

$$\tau^2 \int_{x<y} e^{-\tau(y-x)} h_{(m+1,\alpha_m)}(y) h_{(n+1,\alpha_n)}(x) \, \mathrm{d}x \, \mathrm{d}y \,, \quad \alpha_m + \alpha_n \geq 3,$$

$$\tau^2 \int_{x<t<y} e^{-\tau(y-x)} h_{(m_1)}(y) h_{(m_2)}(t) h_{(m_3)}(x) \, \mathrm{d}x \, dt \, \mathrm{d}y \,.$$

The estimate for $\mathcal{H}_3^{>k}$ in (5.39) then follows from Lemma 5.7.

Similarly, since $\tilde{T}_{2j}$ is a linear combination of integrals ${}_{2j}$ reading as (5.14), (5.40) follows from Lemmas 5.6 and 5.7. ☐

5.5. **The energies.** We restrict ourselves on the imaginary axis (5.7): $(\lambda, z) = (i\sigma, i\tau/2) \in \mathcal{R}$. For any $q \in X^s$, $s > \frac{1}{2}$, by (1.21), there exists $\tau_0 \geq C$ such that the smallness assumption (5.5): $c_{\tau_0} = \frac{1}{\tau_0} \| \mathbf{q} \|_{l^2_{\tau_0} DU^2} < \frac{1}{2C}$ holds. Consequently the condition (5.8): $||q|^2 - 1| \leq \frac{1}{64} {\tau_0}^2$ holds if τ_0, C above have been chosen large enough. Indeed, by view of (4.7) such that $\|q\|_{L^\infty} \lesssim 1 + \tau_0 c_{\tau_0}^{1/2} + \tau_0 c_{\tau_0}$, (5.8) follows from (5.5) for large C and τ_0. From now on we fix τ_0.

5.5.1. *The expansion of* $G(i\tau/2)$. Let us first consider rigorously the expansion of the real part of the expansion of $\ln T_{\mathrm{c}}^{-1}(i\sigma)$ in Lemma 5.1:

$$\begin{aligned} \mathrm{Re}\,(4z^2 \ln T_{\mathrm{c}}^{-1})|_{\lambda=i\sigma} &= \mathrm{Re}\,(4z^2 \tilde{T}_2)|_{\lambda=i\sigma} + \mathrm{Re}\,(4z^2 \tilde{T}_f)|_{\lambda=i\sigma} \\ &\quad + \sum_{j>s} \mathbf{1}_{j\geq 2} \mathrm{Re}\,(4z^2 \tilde{T}_{2j})|_{\lambda=i\sigma}, \qquad \tilde{T}_f := \tilde{T}_3 + \sum_{j=2}^{[s]} \tilde{T}_{2j}. \end{aligned} \tag{5.41}$$

Recall the expansion for $\mathrm{Re}\,(4z^2 \tilde{T}_2)|_{\lambda=i\sigma}$ in (5.22) with $N = [s-1]$:

$$\mathrm{Re}\,(4z^2 \tilde{T}_2)(i\sigma) = \sum_{l=0}^{N} (-1)^l \mathcal{H}_2^{2l+2} \, \tau^{-2l-1} + \mathcal{H}_2^{>2N+2}(i\sigma), \text{ as } \tau \to \infty,$$

$$\text{with } \tau_0^{2(s-1-l)} \mathcal{H}_2^{2l+2} \leq (E_{\tau_0}^s)^2, \quad \mathcal{H}_2^{>2N+2}(i\sigma) = o(\tau^{1-2s}).$$

If $s > \frac{3}{2}$ is away from half integers, then by Proposition 5.1 and Proposition 5.2 with $k = [2s]$ under the smallness assumption (5.5),

- We can expand $\tilde{T}_f = \tilde{T}_3 + \sum_{j=2}^{[s]} \tilde{T}_{2j}$ as

$$\mathrm{Re}\,(4z^2 \tilde{T}_f)(i\sigma) = \sum_{\ell=3}^{k} \mathcal{H}_f^\ell \tau^{-\ell+1} + \mathcal{H}_f^{>k}(i\sigma) \text{ as } \sigma \to \infty. \tag{5.42}$$

  Here $\mathcal{H}_f^\ell = -\mathrm{Re}\,(\mathcal{H}_3^\ell + \sum_{j=2}^{[s]} \mathcal{H}_{2j}^\ell)$ with $\mathcal{H}_f^{2l+1} = 0$ and $\tau_0^{2(s-1-l)} |\mathcal{H}_f^{2l+2}| \leq C c_{\tau_0} (E_{\tau_0}^s)^2$, and $|\mathcal{H}_f^{>k}(i\sigma)| \leq |\mathcal{H}_3^{>k}(i\sigma)| + \sum_{j=2}^{[s]} |\mathcal{H}_{2j}^{>k}(i\sigma)| = o(\tau^{1-2s})$;
- $\sum_{j>s} \big| 4z^2 \tilde{T}_{2j}(i\sigma) \big| = o(\tau^{1-2s})$.

To conclude, noticing that $k \geq 2N+2$ and $k > 2N+2$ only if $k \in 2\mathbb{Z}+1$, by (5.41) and (5.42) above, we have the following expansion for $G(i\frac{\tau}{2}) = \mathrm{Re}\,(4z^2 \ln T_{\mathrm{c}}^{-1})(i\sigma)$ when $s > \frac{1}{2}$ away from half integers:

$$G(i\frac{\tau}{2}) = \mathrm{Re}\,(4z^2 \ln T_{\mathrm{c}}^{-1})(i\sigma) = \sum_{l=0}^{N} (-1)^l \mathcal{H}^{2l+2} \tau^{-2l-1} + \mathcal{H}^{>2N+2}(i\frac{\tau}{2}), \text{ as } \tau \to \infty, \tag{5.43}$$

where (noticing that $\mathcal{H}^k = 0$ if $k > 2N+2$ such that $k \in 2\mathbb{Z}+1$) [10]

$$\begin{aligned} &\mathcal{H}^{2l+1} = \mathcal{H}_2^{2l+1} + (-1)^l \mathcal{H}_f^{2l+1} \text{ with } |\tau_0^{2(s-1-l)}\mathcal{H}^{2l+1}| \le C(E_{\tau_0}^s)^2, \\ &\mathcal{H}^{>2N+1}(i\frac{\tau}{2}) = \mathcal{H}_2^{>2N+1} + \mathcal{H}_f^{>k} + \sum_{j>s} \mathrm{Re}\,(4z^2\tilde{T}_{2j})(i\sigma) = o(\tau^{1-2s}). \end{aligned} \tag{5.45}$$

By Proposition 5.2 and the above expansion (5.43) for the non negative superharmonic function G on the upper half plane, the trace of G on the real line exists as a finite Radon measure μ such that the measure $(1+\xi^2)^N \mu$ is finite. Furthermore, by view of $G = \mathrm{Re}\,(z^2 \ln \frac{1}{z-z_m})+$ harmonic function in a small enough neighborhood of z_m (with λ_m the zeros of $T_c^{-1}(\lambda)$), $\mathcal{H}^{2l+1}$ indeed reads as in (5.2).

5.5.2. *The energies.* If $s \in (\frac{1}{2}, \frac{3}{2})$, by view of the difference (5.23) for $\left|\mathcal{E}_{\tau_0}^s - (E_{\tau_0}^s)^2\right|$, under the smallness condition (5.5), we derive the equivalence relation (5.6) from Proposition 5.1.

Similarly for $s \ge \frac{3}{2}$, by the above finite expansion (5.43) for $G(i\tau/2)$ and Propositions 5.1 and 5.2, we also have the inequality (5.6) since we derive from (5.24) that

$$\begin{aligned} &|\mathcal{E}_{\tau_0}^s - (E_{\tau_0}^s)^2| \le \sum_{l=0}^{N} \tau_0^{2(s-1-l)} \binom{s-1}{l} |\mathcal{H}_f^{2l+2}| \\ &+ \Big| \frac{2}{\pi} \sin(\pi(s-1)) \int_{\tau_0}^{\infty} (\tau^2 - \tau_0^2)^{s-1} \big(\mathcal{H}_f^{>k} + \sum_{j>s} \mathrm{Re}\,(4z^2\tilde{T}_{2j})\big)(i\sigma)\,\mathrm{d}\tau \Big| \le C c_{\tau_0} (E_{\tau_0}^s)^2. \end{aligned}$$

Here we noticed that for the estimates in Propositions 5.1 and 5.2, if $s = m$ is an integer then the singularity $(2s - [2s])^{-1}$ is compensated by the coefficient $\sin(\pi(s-1))$, while if $s = m + \frac{1}{2}$ we can do the frequency decomposition as in (5.44). Therefore for any $q \in X^s$, $s > \frac{1}{2}$ there exists $\tau_0 \ge C \ge 2$ such that the energy $\mathcal{E}_{\tau_0}^s$ is well-defined in (5.3) satisfying (5.6).

Furthermore we derive the trace formula (5.4) by Proposition 5.2. For general $\tau' \ge 2$, we can still define our energy $\mathcal{E}_{\tau'}^s$ as in (5.3) since $T_c^{-1}(\lambda)$ is holomorphic on $\mathcal{R} = \{(\lambda, z) \,|\, \lambda^2 = z^2 + 1,\, \lambda \notin \mathcal{I}_{\mathrm{cut}},\, \mathrm{Im}\, z > 0\}$ and $G(i\frac{\tau}{2})$ is integrable on the finite interval $\tau \in [\tau', \tau_0]$ if $\tau' \le \tau_0$. The analyticity of $\mathcal{E}_{\tau'}^s(q)$ in $q \in X^s$ follows from the analyticity of the renormalised transmission coefficient $T_c^{-1}(\lambda; q)$ in Theorem 3.1.

6. The metric space

In this section we study the metric space X^s defined in (1.4):

$$X^s = \{q \in H^s_{\mathrm{loc}}(\mathbb{R}) \,:\, |q|^2 - 1 \in H^{s-1}(\mathbb{R}), \quad q' \in H^{s-1}(\mathbb{R})\}/\mathbb{S}^1, \quad s \ge 0,$$

and its endowed metric defined in (1.5):

$$d^s(q,p) = \Big(\int_{\mathbb{R}} \inf_{|\lambda|=1} \|\,\mathrm{sech}\,(\cdot - y)(\lambda q - p)\|^2_{H^s(\mathbb{R})}\,\mathrm{d}y \Big)^{\frac{1}{2}}.$$

---

[10]If $s = m$ an integer then (5.45) follows from the proof of Proposition 5.2; if $s = m + \frac{1}{2}$, then we can replace $\mathcal{H}_f^{>k}(i\sigma)$ in $\mathcal{H}^{>2N+2}(i\tau/2)$ by (noticing $\mathcal{H}_f^{>k} = \mathcal{H}_f^{>k-1}$, $\mathcal{H}_f^k = 0$ when $k \in 2\mathbb{Z}+1$)

$$\Big((\mathcal{H}_f^{>k})(i\sigma; Q) - (\mathcal{H}_f^{>k})(i\sigma; Q_{<\tau}) \Big) + \big(\mathcal{H}_f^{>k-1}(i\sigma; Q_{<\tau}) \big) \tag{5.44}$$

such that $\mathcal{H}^{>2N+2}(i\tau/2) = o(\tau^{1-2s})$ holds: Here $Q_{<\tau} = \frac{1}{\tau}((|q|^2-1)_{<\tau}, q'_{<\tau}, \bar{q}'_{<\tau})$ denotes the low frequency part of $Q = \frac{1}{\tau}(|q|^2 - 1, q', \bar{q}')$ and hence there exists at least one high frequency $Q_{\ge\tau}$ in the first part of the decomposition (5.44) while there is only low frequency part $Q_{<\tau}$ in the second part of (5.44), from which we derive $\mathcal{H}^{>2N+2}(i\tau/2) = o(\tau^{1-2s})$ from the proof of Proposition 5.2 (see also Section 6 [31]).

Recall the energy $E^s(q)$ associated to $q \in X^s$ given by (1.17) and (1.14):

$$E^s(q) = \Big(\|q'\|^2_{H_2^{s-1}(\mathbb{R})} + \||q|^2 - 1\|^2_{H_2^{s-1}(\mathbb{R})} \Big)^{1/2}.$$

This section is devoted to the proof of Theorem 1.1 and will be divided into two subsections, with the first subsection devoted to the study of the metric structure (see Theorem 6.1 below), and the second one to the analytic structure (see Theorem 6.2 below).

6.1. **The metric structure.** We study in this subsection the metric structure of the metric space (X^s, d^s).

Theorem 6.1. *Suppose that $s \ge 0$. Then (X^s, d^s) is a separable complete metric space. Moreover, there exists a constant c depending on $\Lambda > 0$ such that for any $q, p \in X^s$ with $E^s(q), E^s(p) \le \Lambda$,*

$$|E^s(q) - E^s(p)| \le c d^s(q,p).$$

$1 + C_0^\infty(\mathbb{R})$ is a dense subset. Every metric ball is contractible. If $s > 0$ then every closed metric ball is weakly sequentially compact.

By weakly sequentially compact we mean that if (q_j) is a sequence in $B = \overline{B_r^s(q)}$ with $B_r^s(q) = \{p \in X^s \,|\, d^s(p,q) < r\}$, then there is a subsequence and $p \in B$ so that $q_{j_k} \to p$ and $|q_{j_k}|^2 - 1 \to |p|^2 - 1$ as distributions. If $s > 0$ then the boundedness and the convergence $q_{j_k} \to p$ as a distribution imply that $q_{j_k} \to p$ in $L^2(K)$ for every compact interval K and hence $|q_{j_k}|^2 - 1 \to |p|^2 - 1$ in $L^1(K)$ for every compact interval K, and hence as distribution. Thus only the convergence of $q_{j,k} \to p$ as a distribution and the weakly compactness of closed balls has to be proven.

Before proving Theorem 6.1, we claim the following lemma stating the relation between the energy and the metric, whose proof is postponed to the end of this subsection.

Lemma 6.1. *If $q \in X^s$ then with an absolute constant c we have*

$$d^s(1,q) \le c E^s(q). \tag{6.1}$$

If $p \in X^s$ and $q \in H^s_{loc}$ so that $d^s(p,q) < \infty$, then $q \in X^s$ and

$$E^s(q) \le E^s(p) + c(1 + E^s(p))^{\frac12} d^s(p,q) + c(d^s(p,q))^2. \tag{6.2}$$

Proof of Theorem 6.1. We organize the proof into a series of steps. The cutoff functions η and ρ will be chosen appropriately in each step and may vary from step to step.

Step 1. Suppose that $q, p \in X^s$ with $E^s(q), E^s(p) < \infty$. Then $d^s(q,p) = 0$ iff there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $p = \lambda q$. Since $p = \lambda q$ implies $d^s(q,p) = 0$ trivially, we assume that $d^s(q,p) = 0$. Then as $\|\operatorname{sech}(\cdot - y) f\|_{H^s} \ge C(y,a,b)\|f\|_{H^s(I)}$ for any $y \in \mathbb{R}$ and any interval $I = [a,b]$, there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ so that

$$\|\lambda q - p\|_{H^s(I)} = 0.$$

Hence $p = \lambda q$.

Step 2. Triangle inequality. If $q, p, r \in H^s_{\rm loc}$ with $d^s(q,p) < \infty$ and $d^s(p,r) < \infty$, then we simply integrate the square of the following triangle inequality

$$\inf_{|\mu|=1} \|\operatorname{sech}(\cdot - y)(q - \mu r)\|_{H^s} \le \inf_{|\lambda_1|=1} \|\operatorname{sech}(\cdot - y)(q - \lambda_1 p)\|_{H^s} + \inf_{|\lambda_2|=1} \|\operatorname{sech}(\cdot - y)(p - \lambda_2 r)\|_{H^s},$$

to derive the triangle inequality

$$d^s(q,r) \le d^s(q,p) + d^s(p,r).$$

Step 3. (X^s, d^s) **is a metric space.**

We deduce from the triangle inequality in Step 2 and (6.1) that whenever $p, q \in X^s$ then $d^s(p,q) < \infty$ and (X^s, d^s) is a metric space.

Step 4. (X^s, d^s) **is a complete metric space.**

Let (q_n) be representatives of a Cauchy sequence in (X^s, d^s) and let $y \in \mathbb{R}$. There exists $q \in H^s_{\rm loc}$ and a sequence $(\lambda_n(y))$ with $|\lambda_n(y)| = 1$ such that

$$\operatorname{sech}(\cdot - y)\lambda_n(y) q_n \to \operatorname{sech}(\cdot - y) q \qquad \text{in } H^s.$$

Clearly q does not depend on y. This implies pointwise convergence of the integrant with respect to y in the definition of the distance function and

$$d^s(q_n, q) \le \sup{}_{m\ge n} d^s(q_n, q_m) \to 0 \qquad \text{with } n \to \infty.$$

Lemma 6.1 implies $q \in X^s$.

Step 5. A dense subset.

We claim that $1 + C_0^\infty(\mathbb{R}) \subset X^s$ is dense. Let q satisfy $E^s(q) < \infty$. We fix a monoton function $\eta \in C^\infty$ with $\eta(x) = 1$ for $x > \frac12$ and $\eta(x) = 0$ for $x \le -\frac12$. Since

$$\|(1-\eta(R\pm x))(|u|^2-1)\|_{H^{s-1}} + \|(1-\eta(R\pm x))u_x\|_{H^{s-1}} \to 0$$

as $R \to \infty$, given $\varepsilon > 0$ there exists R_0 so that all these quantities above are at most of size ε for $R > R_0$.

Let $\hat u_\pm = \int_{\mathbb{R}} \eta'(x \mp R) u \, dx$ with $R > R_0$. We claim that there exists an absolute constant c such that

$$||\hat u_\pm| - 1| \le c\varepsilon.$$

This estimate follows from Lemma 6.3 below. Multiplying by a complex constant of modulus 1 we may assume that $\hat u_- \in [\frac12, 2]$. We choose $\omega \in [-\pi, \pi)$ so that $\hat u_+ = e^{i\omega}|\hat u_+|$. We define

$$u_R = \eta(R-x)\eta(R+x)u(x) + (1-\eta(R-x))e^{i\omega(\ln(3)/\ln(2+|x/R|^2))} + (1-\eta(R+x)).$$

It is not hard to see that

$$\lim_{R\to\infty} d^s(u_R, u) = 0.$$

Clearly $u_R - 1$ vanishes for $x < -2R$ and it decays as $x \to \infty$. After convolving $u_R - 1$ with a Dirac sequence we may assume that (without changing the notation) that in addition $u_R \in C^\infty$. By a standard cutoff argument we may assume $u_R - 1 \in C_0^\infty(\mathbb{R})$.

Step 6. Weak compactness.

Let $s \ge 0$, $q \in X^s$, $r < \infty$ and $q_n \in X^s$ so that $d^s(q, q_n) \le r$. We claim that there exists a weakly convergent subsequence with a limit p in the same closed ball. This follows by an easy modification of Step 5. Lemma 6.1 implies that the weak limit is in X^s. If $s > 0$ we obtain

$$E^s(p) \le \liminf_{n\to\infty} E^s(q_n). \tag{6.3}$$

Indeed, if q_n converges weakly to p then up to choosing λ_n

$$\eta(|q_n|^2 - 1) \to \eta(|p|^2 - 1)$$

in H^{s-1} by compactness. We easily deduce (6.3). □

In the remainder of this subsection we will give two technical lemmas (Lemma 6.2 and Lemma 6.3) and their proofs, as well as the proof of Lemma 6.1.

Lemma 6.2. *Let $0 < \delta < \frac14$ and suppose that $\eta^{(k)} \le C\eta$ for all $k \ge 0$, $|\eta'| \le \delta\eta$. Let $s \in \mathbb{R}$, then*

$$\|\eta\langle D\rangle^s f\|_{L^2} \le c\|\langle D\rangle^s(\eta f)\|_{L^2},$$

where the operator $\langle D\rangle^s$ is defined by the Fourier multiplier $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}$.

Proof. The operator $\langle D\rangle^s$ is defined by convolution with (up to a multiplication by a power of 2π)

$$g(x) = \int_{\mathbb{R}} e^{i\xi x}\langle\xi\rangle^s d\xi,$$

which is understood as inverse Fourier transform *resp.* as oscillatory integral if $x \neq 0$. To be more specific, let us assume $x > 0$. Then we move the contour of the integration to $\{\xi + i\tau \mid \xi \in \mathbb{R}, -1 < \tau < 1\}$ in the complex space:

$$g(x) = \int_{\mathbb{R}} e^{i(\xi+i\tau)x}(1+(\xi+i\tau)^2)^{s/2}d\xi = e^{-\tau x}\int_{\mathbb{R}} e^{i\xi x}(\xi^2+2i\tau\xi+1-\tau^2)^{s/2}d\xi.$$

We take $\tau = 1$. We take the smooth cutoff function ρ with $\rho = 1$ around 0 to decompose the integration into the part close to 0 and the part away from 0:

$$e^x g(x) = \int_{\mathbb{R}} e^{i\xi x}\xi^{s/2}[\rho(\xi+2i)^{s/2}]d\xi + \int_{\mathbb{R}} e^{i\xi x}\xi^s[(1-\rho)(1+2i/\xi)^{s/2}]d\xi.$$

Similarly for $x < 0$ we take $\tau = -1$. Then by the theory of oscillatory integrals,

$$|g(x)| \le Ce^{-|x|}\Big((1+|x|)^{-1-s/2} \; + \; \big(1+\chi_{\{|x|\le 1\}}|x|^{-1-s}\big)\Big),$$

and the exponential decay holds:

$$|\partial_x^k g(x)| \le Ce^{-|x|}(1+|x|^{-1-s/2}), \quad \forall |x| \ge 1.$$

We denote $g = g^s$ to emphasize the dependence of the above function $g(x)$ on s and we decompose $g^s(x)$ into

$$g^s(x) = g_1^s(x) + g_2^s(x), \quad g_1^s(x) = \rho(x)g^s(x),$$

such that

$$\begin{aligned} &|g_1^s| \le C\chi_{\{|x|\le 1\}}(1+|x|^{-1-s}), \\ &|\partial_x^k g_2^s(x)| \le C_k e^{-\frac{|x|}{2}}, \text{ and hence } \|g_2^s * f\|_{H^N} \le C_N\|f\|_{-N}, \quad \forall N \in \mathbb{N}. \end{aligned} \tag{6.4}$$

The claimed inequality is equivalent to

$$\|\eta\langle D\rangle^s\eta^{-1}\langle D\rangle^{-s}f\|_{L^2} \le c\|f\|_{L^2}, \quad \text{i.e.} \quad \|\eta g^s * (\eta^{-1}g^{-s} * f)\|_{L^2} \le c\|f\|_{L^2},$$

and by duality it suffices to consider the case $s \ge 0$. We do the above decomposition for g^s and it remains to show

$$\|\eta g_j^s * (\eta^{-1}g_l^{-s} * f)\|_{L^2} \le c\|f\|_{L^2}, \quad j,l = 1,2.$$

When $j = l = 2$, then the integral kernel of the operator on the LHS reads as

$$k_2^s(x,y) = \eta(x)\int_{\mathbb{R}} g_2^s(x-z)\eta^{-1}(z)g_2^{-s}(z-y)dz.$$

By $|\partial_x^k\eta(x)| \le c_k e^{\delta|x-z|}\eta(z)$, the estimate follows from (6.4):

$$\int_{\mathbb{R}} e^{-\frac{1}{2}|x-z|}e^{\delta|x-z|}e^{-\frac{1}{2}|z-y|}dz \le Ce^{-\frac{1}{4}|x-y|}.$$

It is also straightforward to check the other cases by use of (6.4):

$$\begin{aligned} \|\eta g_1^s * (\eta^{-1}g_2^{-s} * f)\|_{L^2} \le c\|g_2^{-s} * f\|_{H^s} \le c\|f\|_{L^2}, \\ \|\eta g_2^s * (\eta^{-1}g_1^{-s} * f)\|_{L^2} \le c\|g_1^{-s} * f\|_{H^s} \le c\|f\|_{L^2}, \\ \|\eta g_1^s * (\eta^{-1}g_1^{-s} * f)\|_{L^2} \le c\|g_1^{-s} * f\|_{H^s} \le c\|f\|_{L^2}. \end{aligned}$$

□

We turn to another technical lemma. Let $\eta(x) = (1+x^2)^{-1}$. Then

$$\left|\int_{\mathbb{R}} \operatorname{sech}^2(x-y)(|q|^2-1)(x)\,\mathrm{d}x\right| \le C\|\eta(\cdot-y)\langle D\rangle^{-1}(|q|^2-1)\|_{L^2}, \tag{6.5}$$

since

$$\eta^{-1}(\cdot-y)\operatorname{sech}^2(\cdot-y) \in \mathcal{S}(\mathbb{R}) \subset H^1(\mathbb{R}).$$

Lemma 6.3. *Let η_0 be a nonnegative Schwartz function. Then*

$$\int_{\mathbb{R}\times\mathbb{R}} \eta_0(x)\eta_0(y)|q(x)-q(y)|^2 \,\mathrm{d}x\ \mathrm{d}y \le c\|\eta\langle D\rangle^{-1}q_x\|^2_{L^2}, \tag{6.6}$$

and with $\kappa = \int_{\mathbb{R}} \operatorname{sech}^2(x)\,\mathrm{d}x$,

$$\left|\left|\frac{1}{\kappa}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)q(x)\,\mathrm{d}x\right| - 1\right| \le c\left(\|\eta(\cdot-y)\langle D\rangle^{-1}q_x\|_{L^2} + \|\eta(\cdot-y)\langle D\rangle^{-1}(|q|^2-1)\|_{L^2}\right). \tag{6.7}$$

Proof. Straightforward calculation yields

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} \eta_0(x)\eta_0(y)|q(x)-q(y)|^2 dxdy &= 2\mathrm{Re}\int_{x<y}\eta_0(x)\eta_0(y)\int_{x<z_1,z_2<y} q'(z_1)\bar{q}'(z_2)dz_1dz_2dxdy\\ &= 2\mathrm{Re}\int_{\mathbb{R}\times\mathbb{R}}\int_{-\infty}^{\min\{z_1,z_2\}}\eta_0(x)dx\int_{\max\{z_1,z_2\}}^{\infty}\eta_0(y)dyq'(z_1)\bar{q}'(z_2)dz_1dz_2.\end{aligned}$$

Let

$$\rho(z_1,z_2) = 2\int_{-\infty}^{\min\{z_1,z_2\}}\eta_0(x)dx\int_{\max\{z_1,z_2\}}^{\infty}\eta_0(y)dy.$$

The assertion (6.6) follows once we prove with $\kappa_0 = 2\int_{\mathbb{R}}\eta_0$,

$$\|\rho(.,.)\|_{L^2} + \|\partial_{z_1}\rho\|_{L^2} + \|\partial_{z_2}\rho\|_{L^2} + \|\partial^2_{z_1z_2}\rho - \kappa_0\delta_{z_1-z_2}\eta_0(z_1)\|_{L^2} \le c.$$

Indeed,

$$\partial_{z_1}\rho(z_1,z_2) = \begin{cases} 2\eta_0(z_1)\int_{z_2}^{\infty}\eta_0(y)dy & \text{if } z_1<z_2\\ -2\int_{-\infty}^{z_2}\eta_0(x)dx\eta_0(z_1) & \text{if } z_2<z_1\end{cases}$$

and

$$\partial^2_{z_1z_2}\rho(z_1,z_2) = -2\eta_0(z_1)\eta_0(z_2) \text{ if } z_1\ne z_2.$$

At the diagonal $\{(z_1,z_2)\,|\,z_1=z_2\}$ we have

$$\partial_{z_1}\rho(z,z_+) - \partial_{z_1}\rho(z,z_-) = 2\eta_0(z)\int_{z_+}^{\infty}\eta_0 + 2\eta_0(z)\int_{-\infty}^{z_-}\eta_0 = 2\eta_0(z)\int_{\mathbb{R}}\eta_0$$

and hence

$$\partial^2_{z_1z_2}\rho(z_1,z_2) = -2\eta_0(z_1)\eta_0(z_2) + 2\delta_{z_1-z_2}\eta_0(z_1)\int_{\mathbb{R}}\eta_0.$$

We turn to the proof of (6.7). Let $\kappa = \int_{\mathbb{R}}\operatorname{sech}^2(x)\,\mathrm{d}x$. Then

$$\begin{aligned}\left|\int_{\mathbb{R}}\operatorname{sech}^2(x-y)q(x)\,\mathrm{d}x\right| \le& \int_{\mathbb{R}}\operatorname{sech}^2(x-y)|q(x)|\,\mathrm{d}x\\ \le&\kappa^{1/2}\left(\int_{\mathbb{R}}\operatorname{sech}^2(x-y)|q|^2(x)\,\mathrm{d}x\right)^{\frac12}\\ \le&\kappa\Big(1+\kappa^{-1}\int_{\mathbb{R}}\operatorname{sech}^2(x-y)(|q|^2-1)(x)\,\mathrm{d}x\Big)\\ \le&\kappa + C\|\eta(x-y)\langle D\rangle^{-1}(|q|^2-1)\|^{\frac12}_{L^2},\end{aligned}$$

where in the last step we used (6.5). This implies the desired estimate (6.7) if for some $\varepsilon>0$, $\left|\frac{1}{\kappa}\int_{\mathbb{R}}\operatorname{sech}^2(x-y)q(x)dx\right| \ge 1+\varepsilon$.

We hence fix ε ($\varepsilon = \frac12$ being legitimate) and consider the case

$$\left|\frac{1}{\kappa}\int_{\mathbb{R}}\operatorname{sech}^2(x-y)q(x)\,\mathrm{d}x\right| \le (1+\varepsilon). \tag{6.8}$$

Using Fubini and (6.6) we have

$$
\begin{aligned}
&\Big\| \frac{1}{\kappa}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)q\,\mathrm{d}x \,-q \Big\|^2_{L^2(\operatorname{sech}^2(.-y))} \\
&= \int_{\mathbb{R}} \operatorname{sech}^2(x'-y)\Big|\frac{1}{\kappa}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)q(x)\,\mathrm{d}x \,-q(x')\Big|^2 \mathrm{d}x' \\
&\le \frac{1}{\kappa}\int_{\mathbb{R}^2} \operatorname{sech}^2(x'-y)\operatorname{sech}^2(x-y)|q(x)-q(x')|^2\,\mathrm{d}x\,\mathrm{d}x' \\
&\le c\|\eta(\cdot-y)\langle D\rangle^{-1}q_x\|^2_{L^2},
\end{aligned} \tag{6.9}
$$

and hence by triangle inequality,

$$
\begin{aligned}
\Big|\Big|\frac{1}{\kappa}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)q(x)\,\mathrm{d}x\Big|^2 - 1\Big| &\le \Big|\frac{1}{\kappa}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)(|q|^2-1)(x)\,\mathrm{d}x\Big| \\
&+ \Big|\frac{1}{\kappa}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)|q|^2(x)\,\mathrm{d}x \,- \Big|\frac{1}{\kappa}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)q(x)\,\mathrm{d}x\Big|^2\Big|,
\end{aligned}
$$

where the last term is estimated using (6.9) and by writing it as $|A^2-B^2| = |A+B||A-B|$. In this last step we made use of (6.8). □

We complete this subsection with verifying the relation between metric and energy stated in Lemma 6.1.

Proof of Lemma 6.1. We claim that there exists a constant $c>0$ so that

$$d^s(q,1) \le cE^s(q).$$

This is the first claim of the lemma. We begin with the most difficult case $s=0$ and fix $y\in\mathbb{R}$. Then, with $\kappa = \int_{\mathbb{R}} \operatorname{sech}^2(x)\,\mathrm{d}x$,

$$
\begin{aligned}
&\inf_{|\lambda|=1}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)|q-\lambda|^2(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \operatorname{sech}^2(x-y)(|q|^2+1)\,\mathrm{d}x \,- 2\sup_{|\lambda|=1}\operatorname{Re}\lambda\overline{\int_{\mathbb{R}} \operatorname{sech}^2(x-y)q(x)\,\mathrm{d}x} \\
&= \int_{\mathbb{R}} \operatorname{sech}^2(x-y)(|q|^2-1)\,\mathrm{d}x \,- 2\Big(\Big|\int_{\mathbb{R}} \operatorname{sech}^2(x-y)q(x)\,\mathrm{d}x\Big| - \kappa\Big) \\
&\le c\left(\|\eta(x-y)\langle D\rangle^{-1}(|q|^2-1)\|_{L^2} + \|\eta(x-y)\langle D\rangle^{-1}q_x\|_{L^2}\right),
\end{aligned}
$$

where the bound on the first term follows by (6.5), and the second term by (6.7). Let $\delta\in(0,1)$ be a small constant to be determined later and we define the set

$$Y_\delta = \{y\in\mathbb{R} \mid \|\eta(\cdot-y)\langle D\rangle^{-1}(|q|^2-1)\|_{L^2} + \|\eta(\cdot-y)\langle D\rangle^{-1}q_x\|_{L^2} \ge \delta > 0\}.$$

Then

$$
\begin{aligned}
d^0(q,1) &= \Big(\int_{\mathbb{R}}\inf_{|\lambda|=1}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)|q-\lambda|^2(x)\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac12} \\
&\le \Big(\int_{Y_\delta^C}\inf_{|\lambda|=1}\int_{\mathbb{R}} \operatorname{sech}^2(x-y)|q(x)-\lambda|^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac12} \\
&\quad + c\Big(\int_{Y_\delta}\left(\|\eta(x-y)\langle D\rangle^{-1}(|q|^2-1)\|_{L^2} + \|\eta(x-y)\langle D\rangle^{-1}q_x\|_{L^2}\right)\mathrm{d}y\Big)^{\frac12},
\end{aligned}
$$

where the last term is bounded by $c_\delta E^0(q)$ for some constant c_δ depending on δ.

It remains to consider those $y\in Y_\delta^C$. We regularize q by taking the convolution with the Schwartz function $\frac{1}{\kappa}\operatorname{sech}^2(x)$, and we can always decompose (as in Section 2)

$$q = b + q^1, \text{ with } b\in L^2, \quad q^1 = q * (\frac{1}{\kappa}\operatorname{sech}^2) \in X^\sigma,\ \forall\sigma\ge 0,$$

such that

$$\|\operatorname{sech}^2(x-y)\, b\|_{L^2} + \|\operatorname{sech}^2(x-y)\,(q^1)_x\|_{L^2} \le c\|\eta(x-y)\langle D\rangle^{-1} q_x\|_{L^2}.$$

Fix $y \in Y_\delta^C$ and in the following we will simply denote $\eta(x-y)$ by η. We use Lemma 6.3 and then choose δ small enough such that

$$||q^1(y)| - 1| \le c\left(\|\eta\langle D\rangle^{-1}(|q|^2-1)\|_{L^2} + \|\eta\langle D\rangle^{-1} q_x\|_{L^2}\right) \le c\delta < \frac{1}{2},$$

and thus $|q^1(y)| \neq 0$. Moreover (as in (2.3)),

$$\|q^1\|_{L^\infty([y-R,y+R])} \le c(1 + \|\eta\langle D\rangle^{-1}(|q|^2-1)\|_{L^2} + \|\eta\langle D\rangle^{-1} q_x\|_{L^2}) \le c,$$

and

$$\Big|\int_{\mathbb{R}} \eta(|q^1|^2 - 1)\,\mathrm{d}x\Big| \le \left|\int \eta(|q|^2-1)\,\mathrm{d}x\right| + \int \eta(|q|^2 - |q^1|^2)\,\mathrm{d}x\,,$$

where the second term on the righthand side above is bounded by

$$\left(\|\eta^{1/2} q\|_{L^2} + \|\eta^{1/2} q^1\|_{L^2}\right)\|\eta^{1/2} b\|_{L^2} \le c\|\eta^{1/2} b\|_{L^2}.$$

Since

$$\begin{aligned}\Big\|\operatorname{sech}^2(x-y)\big(q(x) - \frac{q^1(y)}{|q^1(y)|}\big)\Big\|_{L^2_x} &\le \|\operatorname{sech}^2(x-y)(q-q^1)(x)\|_{L^2_x} \\ &\quad + \|\operatorname{sech}^2(x-y)(q^1(x) - q^1(y))\|_{L^2_x} + \|\operatorname{sech}^2(x-y)(|q^1(y)|-1)\|_{L^2_x},\end{aligned}$$

we take the square and then integrate with respect to y in the set Y_δ^C, to complete the proof of $d^0(q,1) \le cE^0(q)$. To deal with general $s>0$ we slightly modify the steps.

To prove the second claim (6.2) we consider the case when the right hand side is finite. By the triangle inequality,

$$\||q|^2 - 1\|_{H^{s-1}} \le \||p|^2-1\|_{H^{s-1}} + \||q|^2 - |p|^2\|_{H^{s-1}}.$$

We now verify

$$\||q|^2 - |p|^2\|_{H^{s-1}} \le c\sqrt{1 + E^s(p) + E^s(q)}\, d^s(p,q). \tag{6.10}$$

Since for any $|\lambda| = 1$,

$$|q|^2 - |p|^2 = \operatorname{Re}\left((\lambda q + p)(\overline{\lambda q - p})\right) = \operatorname{Re}\Big(\big(\lambda(b_q + q^1) + (b_p + p^1)\big)(\overline{\lambda q - p})\Big),$$

where $q = b_q + q^1$ and $p = b_p + p^1$ are decompositions above, we have for $s \ge 0$

$$\||q|^2 - |p|^2\|_{H^{s-1}} \lesssim (\|(b_q, b_p)\|_{H^s} + \|((q^1)_x, (p^1)_x)\|_{H^{s-1}} + \|(q^1, p^1)\|_{L^\infty})\|\lambda q - p\|_{H^s},$$

and hence (6.10) follows:

$$\begin{aligned}\||q|^2 - |p|^2\|^2_{H^{s-1}} &\le c\int_{\mathbb{R}} \|\operatorname{sech}(x-y)(|q|^2 - |p|^2)(x)\|^2_{H^{s-1}_x}\, dy \\ &\le c(1 + E^s(p) + E^s(q))\int_{\mathbb{R}} \inf_{|\lambda|=1} \|\operatorname{sech}(x-y)(\lambda q - p)\|^2_{H^s}\, dy \\ &= c(1 + E^s(p) + E^s(q))(d^s(q,p))^2.\end{aligned}$$

Then for any $\epsilon > 0$ small enough, there exists $c_\epsilon > 0$ such that

$$\begin{aligned}\||q|^2 - 1\|_{H^{s-1}} &\le E^s(p) + c(1 + E^s(p) + E^s(q))^{\frac{1}{2}} d^s(p,q) \\ &\le \epsilon E^s(q) + E^s(p) + c(1 + E^s(p))^{\frac{1}{2}} d^s(p,q) + c_\epsilon (d^s(p,q))^2.\end{aligned} \tag{6.11}$$

Let $\eta \in \mathcal{S}(\mathbb{R})$. We calculate

$$\int_{\mathbb{R}} \|\eta(x-y) f(x)\|^2_{L^2_x}\, dy = \int_{\mathbb{R}}\int_{\mathbb{R}} |\eta(x-y) f(x)|^2 dx\, dy = \|\eta\|^2_{L^2}\|f\|^2_{L^2}.$$

We take $\|\eta\|_{L^2}=1$, such that

$$\begin{aligned}\|q'\|_{H^{s-1}} &= \Big(\int_{\mathbb{R}} \|\eta(x-y)\langle D\rangle^{s-1} q'(x)\|^2_{L^2_x}dy\Big)^{1/2}\\ &\le \Big(\int_{\mathbb{R}} \|\eta(x-y)\langle D\rangle^{s-1} p'\|^2_{L^2}dy\Big)^{1/2} + \Big(\int_{\mathbb{R}} \inf_\lambda \|\eta(x-y)\langle D\rangle^{s-1}(q'-\lambda p')\|^2_{L^2}dy\Big)^{1/2}\\ &= \|p'\|_{H^{s-1}} + \Big(\int_{\mathbb{R}} \inf_\lambda \|\eta(x-y)\langle D\rangle^{s-1}(q'-\lambda p')\|^2_{L^2}dy\Big)^{1/2}.\end{aligned}$$

Then by choosing ϵ sufficiently small in (6.11), (6.2) follows from Lemma 6.2 (taking $\eta = C\operatorname{sech}(\delta x)$ with $C>0$ such that $\|\eta\|_{L^2}=1$). □

6.2. **The analytic structure.** In this subsection we focus on the analytic structure of the metric space (X^s, d^s).

Theorem 6.2. *Let $\eta \in C_0^\infty([-1,1])$ with $\eta = 1$ on $[-1/2,1/2]$. Let $E^s(q) < \infty$. There exist r and L depending only on $E^s(q)$ such that the map*

$$B^s_r(q) \ni p \mapsto ((a_n)_n, b) \in l^2_d \times \tilde H^s, \text{ with} \tag{6.12}$$

$$\|(a_n)_n\|_{l^2_d} = \Big(\sum_n |a_n - a_{n-1}|^2\Big)^{\frac12},$$

$$\tilde H^s = \{ b \in H^s \mid \langle \eta((x-4Ln)/L)b, \eta((x-4Ln)/L)q\rangle_{H^s} \in \mathbb{R}, \quad \forall n \in \mathbb{Z}\}$$

is a biLipschitz map to its image. If $d^s(q,q_1) < r$ then the coordinate change in the intersection is an analytic diffeomorphism with uniformly bounded derivatives.

Proof. We define

$$\|f\|_{H^{-1}(I)} = \sup\Big\{ \int_{\mathbb{R}} fg\,\mathrm{d}x \,\Big|\, g\in C_0^\infty(I),\, \|g\|_{H^1}=1\Big\}.$$

We denote

$$f^a(x) = \int_{-\frac12}^{\frac12} f(x+y)\,\mathrm{d}y\,, \quad \forall f \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

Proposition 6.1. *There exists $\varepsilon>0$ such that*

$$\frac12 \le |q^a(x)| \le 2 \text{ and } \|q\|_{L^2([x-\frac12,x+\frac12])} \ge \frac12,$$

if

$$\|q_x\|_{H^{-1}([x-\frac12,x+\frac12])} + \||q|^2-1\|_{H^{-1}([x-1,x+1])} \le \varepsilon.$$

In particular, if the interval I satisfies

$$|I| \ge 6((E^s(q))^2/\varepsilon^2+1),$$

then

$$\|q\|_{H^s(I)} \ge E^s(q)\varepsilon^{-1}.$$

Proof. Without loss of generality we take $x=0$ and we consider

$$\|q-q^a(0)\|^2_{L^2([-\frac12,\frac12])} = \int_{-\frac12}^{\frac12} |q(y)-q^a(0)|^2\,\mathrm{d}y = \int_{-\frac12}^{\frac12} \Big|q(y) - \int_{-\frac12}^{\frac12} q(x)\,\mathrm{d}x\Big|^2\,\mathrm{d}y\,,$$

which reads as

$$\begin{aligned}&\int_{-\frac12}^{\frac12}\Big(\int_{-\frac12}^{\frac12}\int_x^y q_x(z)\,\mathrm{d}z\,\mathrm{d}x\Big)\Big(\int_{-\frac12}^{\frac12}\int_{x'}^y \bar q_x(z')\,\mathrm{d}z'\,\mathrm{d}x'\Big)\,\mathrm{d}y\\ &:= \int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12} k(z,z')q_x(z)\bar q_x(z')\,\mathrm{d}z\,\mathrm{d}z'.\end{aligned}$$

In the above,

$$
\begin{aligned}
k(z,z') &= \int_{A(z,z')} \operatorname{sign}(y-x)\operatorname{sign}(y-x')\,\mathrm{d}y\ \mathrm{d}x\ \mathrm{d}x',\\
A(z,z') &= \Big(\Big\{-\frac{1}{2}<x<z<y<\frac{1}{2}\Big\}\cap\Big\{-\frac{1}{2}<x'<z'<y<\frac{1}{2}\Big\}\Big)\\
&\quad \cup\Big\{-\frac{1}{2}<x<z<y<z'<x'<\frac{1}{2}\Big\}\cup\Big\{-\frac{1}{2}<x'<z'<y<z<x<\frac{1}{2}\Big\}\\
&\quad \cup\Big(\Big\{-\frac{1}{2}<y<z<x<\frac{1}{2}\Big\}\cap\Big\{-\frac{1}{2}<y<z'<x'<\frac{1}{2}\Big\}\Big),
\end{aligned}
$$

such that

$$
\begin{aligned}
k(z,z') &= \mathbf{1}_{z<z'}(z+\frac{1}{2})(\frac{1}{2}-z')+\mathbf{1}_{z'<z}(z'+\frac{1}{2})(\frac{1}{2}-z)\\
&= \frac{1}{4}-zz'-\frac{1}{2}|z-z'|,\quad \forall (z,z')\in[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}],
\end{aligned}
$$

which is symmetric and Lipschitz continuous with

$$
k(\pm\frac{1}{2},z') = k(z,\pm\frac{1}{2}) = 0,
$$

and smooth away from the diagonal with uniformly bounded derivatives of all orders.

We claim that

$$
\|q-q^a(0)\|_{L^2([-\frac{1}{2},\frac{1}{2}])} \le c\|q_x\|_{H^{-1}([-\frac{1}{2},\frac{1}{2}])}, \tag{6.13}
$$

and it is equivalent to say that the integral operator with the integral kernel $k(z,z')$ maps from H^{-1} to H_0^1. That is, the integral operator with the integral kernel

$$
\partial_z k(z,z') = -z' + \frac{1}{2}(\mathbf{1}_{z<z'} - \mathbf{1}_{z'<z})
$$

maps from H^{-1} to L^2. This is equivalent to the adjoint operator mapping from L^2 to H_0^1, which, by $\partial_z k(z,\pm\frac{1}{2})=0$, is equivalent to the fact that

$$
\partial_{zz'}k(z,z') = -1+\delta_{z-z'}
$$

is the integral kernel of an operator mapping from L^2 to L^2: This is obvious.

Let $\eta\in C_0^\infty([-1,1])$ such that $0\le\eta\le 1$, $\eta=\frac{1}{2}$ on $[-\frac{1}{2},\frac{1}{2}]$ and $\int\eta=1$. Then

$$
\Big|\int_{\mathbb{R}}\eta(y)\big(|q(x+y)|^2-1\big)\,\mathrm{d}y\Big| \le c\||q|^2-1\|_{H^{-1}([x-1,x+1])},
$$

and hence by $\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}|q(x+y)|^2\,\mathrm{d}y \le \int\eta|q(x+\cdot)|^2 = \int\eta(|q(x+\cdot)|^2-1)+\int\eta$,

$$
|q^a(x)| + \|q\|_{L^2([x-\frac{1}{2},x+\frac{1}{2}])} \le c\big(1+\||q|^2-1\|_{H^{-1}([x-1,x+1])}\big)^{\frac{1}{2}}.
$$

Thus with a different test function, still denoted by η, with $\eta\in C_0^\infty([-\frac{1}{2},\frac{1}{2}])$ and $\eta=1$ on $[-\frac{1}{4},\frac{1}{4}]$, $\int\eta=1$, $0\le\eta\le1$, we have for any $x\in\mathbb{R}$,

$$
\begin{aligned}
\big||q^a(x)|^2-1\big| &\le \Big|\int_{\mathbb{R}}\eta(y)\big(|q(x+y)|^2-1\big)\,\mathrm{d}y\Big| + \int_{\mathbb{R}}\eta(y)\big|q^a(x)+q(x+y)\big|\big|q^a(x)-q(x+y)\big|\,\mathrm{d}y\\
&\le c\||q|^2-1\|_{H^{-1}([x-\frac{1}{2},x+\frac{1}{2}])} + c\big(1+\||q|^2-1\|_{H^{-1}([x-1,x+1])}\big)\|q_x\|_{H^{-1}([x-\frac{1}{2},x+\frac{1}{2}])},
\end{aligned}
$$

where we used (6.13) to control $\|q^a(x)-q(x+\cdot)\|_{L^2([-\frac{1}{2},\frac{1}{2}])}$. Therefore there exists $\varepsilon<0$ such that if $\|q_x\|_{H^{-1}([x-\frac{1}{2},x+\frac{1}{2}])}+\||q|^2-1\|_{H^{-1}([x-1,x+1])}\le\varepsilon$ then

$$
\frac{1}{2}\le|q^a(x)|\le 2 \text{ and } \|q\|_{L^2([x-\frac{1}{2},x+\frac{1}{2}])}\ge\frac{1}{2}.
$$

By Tschebycheff's inequality we have

$$
\#\{n:\|q_x\|_{H^{-1}([2n-1,2n+1])}+\||q|^2-1\|_{H^{-1}([2n-1,2n+1])}\ge\varepsilon\}\le\varepsilon^{-2}(E^s(q))^2,
$$

and hence if $|I| \geq 6(\varepsilon^{-2}(E^s(q))^2+1)$ then for all $s \geq 0$,

$$\|q\|^2_{H^s(I)} \geq \|q\|^2_{L^2(I)} \geq \frac{1}{4}\cdot\frac{2}{3}|I| \geq \varepsilon^{-2}(E^s(q))^2+1.$$

□

After these preparation we turn to the crucial construction. Let

$$L = 6(\varepsilon^{-2}(E^s(q)^2+1))$$

in the sequel. We replace $\operatorname{sech}(x)$ by $\operatorname{sech}_{\mathrm{L}}(x) = \frac{e^L}{2}\varphi * (\operatorname{sech}(\max\{L,|x|\}))$ for some fixed smooth compactly supported function φ. This function is close to 1 on an interval of length $2L$. For $q \in X^s$, by Proposition 6.1, we have

$$\|\operatorname{sech}_{\mathrm{L}}(x-y)q(x)\|_{H^s_x}(y) \geq \frac{1}{2}E^s(q)/\varepsilon, \quad \forall y \in \mathbb{R}\,. \tag{6.14}$$

We replace sech by sech_L in the definition of the distance. This leads to an equivalent metric with constants of size $e^L/2$. Expand the quantity in the integrand in the definition of $d^s(q,p)$ as

$$\begin{aligned}\|\operatorname{sech}_{\mathrm{L}}(x-y)(\lambda q-p)\|^2_{H^s_x} = {}& \|\operatorname{sech}_{\mathrm{L}}(x-y)q\|^2_{H^s_x} + \|\operatorname{sech}_{\mathrm{L}}(x-y)p\|^2_{H^s_x}\\ &- 2\mathrm{Re}\left[\bar\lambda\langle \operatorname{sech}_{\mathrm{L}}(x-y)p, \operatorname{sech}_{\mathrm{L}}(x-y)q\rangle_{H^s_x}\right].\end{aligned}$$

Let

$$\mu = \mu(y) := \langle \operatorname{sech}_{\mathrm{L}}(x-y)p, \operatorname{sech}_{\mathrm{L}}(x-y)q\rangle_{H^s_x}.$$

If $\mu \neq 0$, we take $\lambda = \lambda(y) = \frac{\mu(y)}{|\mu(y)|}$ such that

$$\inf_{|\lambda|=1}\|\operatorname{sech}_{\mathrm{L}}(x-y)(\lambda q-p)\|^2_{H^s_x} = \|\operatorname{sech}_{\mathrm{L}}(x-y)q\|^2_{H^s_x} + \|\operatorname{sech}_{\mathrm{L}}(x-y)p\|^2_{H^s_x} - 2|\mu(y)|.$$

Suppose that $d^s(q,p) \leq \frac{1}{32}E^s(q)/\varepsilon$. Then for any $y_0 \in \mathbb{R}$ and the interval $I_0 = [y_0-\frac{1}{2}, y_0+\frac{1}{2}]$, we have

$$\begin{aligned}&e^{-\frac{1}{2}}\inf_{|\lambda|=1}\|\operatorname{sech}_{\mathrm{L}}(x-y_0)(\lambda q(x)-p(x))\|^2_{H^s_x}\\ &\leq \int_{I_0}\inf_{|\lambda|=1} e^{-|y-y_0|}\|\operatorname{sech}_{\mathrm{L}}(x-y_0)(\lambda q(x)-p(x))\|^2_{H^s_x}\,dy\\ &\leq \int_{I_0}\inf_{|\lambda|=1}\|\operatorname{sech}_{\mathrm{L}}(x-y)(\lambda q(x)-p(x))\|^2_{H^s_x}\,dy \leq d^s(q,p) \leq \frac{1}{32}E^s(q)/\varepsilon.\end{aligned}$$

This together with (6.14) implies that given y there exists λ with modulus 1 so that

$$\|\operatorname{sech}_{\mathrm{L}}(\lambda q-p)\|_{H^s} \leq \frac{1}{4}E^s(q)/\varepsilon \leq \frac{1}{2}\|\operatorname{sech}_{\mathrm{L}}(x-y)q\|_{H^s}, \tag{6.15}$$

and hence

$$\begin{aligned}|\mu| &= |\langle \operatorname{sech}_{\mathrm{L}}(x-y)p, \operatorname{sech}_{\mathrm{L}}(x-y)q\rangle_{H^s_x}|\\ &\geq \|\operatorname{sech}_{\mathrm{L}}(x-y)q\|^2_{H^s} - \left|\langle \operatorname{sech}_{\mathrm{L}}(x-y)(p-\lambda q), \operatorname{sech}_{\mathrm{L}}(x-y)q\rangle_{H^s_x}\right|\\ &\geq \|\operatorname{sech}_{\mathrm{L}}(x-y)q\|^2_{H^s} - \frac{1}{2}\|\operatorname{sech}_{\mathrm{L}}(x-y)q\|^2_{H^s_x} = \frac{1}{2}\|\operatorname{sech}_{\mathrm{L}}(x-y)q\|^2_{H^s}.\end{aligned} \tag{6.16}$$

We are going to study the map

$$p(x) \mapsto \lambda(y) = \frac{\mu(y)}{|\mu(y)|},$$

in a small ball around q with the radius depending only on $E^s(q)$.

Lemma 6.4. *If $d^s(q,p) < \frac{1}{32}E^s(q)/\varepsilon$, then*

$$\|\lambda_y\|_{H^N} \leq c_N d^s(q,p), \quad \forall N \in \mathbb{N}\,.$$

Proof. We calculate

$$\lambda_y = \frac{\mu_y}{|\mu|} - \frac{1}{2}\frac{|\mu|^2\mu_y + \mu^2\bar\mu_y}{|\mu|^3} = \frac{1}{2}\frac{\mu(\bar\mu\mu_y - \mu\bar\mu_y)}{|\mu|^3}.$$

If $p = \lambda q$, then $\lambda_y = 0$.

We differentiate μ to get

$$\begin{aligned}\mu' &= \frac{d}{dy}\langle \operatorname{sech}_{\mathrm{L}}(x-y)p, \operatorname{sech}_{\mathrm{L}}(x-y)q\rangle \\ &= -\langle \operatorname{sech}'_{\mathrm{L}}(x-y)p, \operatorname{sech}_{\mathrm{L}}(x-y)q\rangle - \langle \operatorname{sech}_{\mathrm{L}}(x-y)p, \operatorname{sech}'_{\mathrm{L}}(x-y)q\rangle,\end{aligned}$$

and for notational simplicity we denote $\operatorname{sech}_{\mathrm{L}}(x-y) = \rho$ and $\operatorname{sech}'_{\mathrm{L}}(x-y) = \rho'$ such that $\mu = \langle \rho p, \rho q\rangle$ and $\mu' = -\langle \rho' p, \rho q\rangle - \langle \rho p, \rho' q\rangle$ in the following of the proof. We expand $p = \lambda q + (p - \lambda q)$ with $|\lambda(y)| = 1$ to see that the difference $\bar\mu\mu' - \mu\bar\mu'$ is the summation of

$$\begin{aligned}&\langle \rho(\overline{p-\lambda q}), \rho\bar q\rangle\mu' - \bar\lambda\langle\rho\bar q, \rho\bar q\rangle\big(\langle\rho'(p-\lambda q), \rho q\rangle + \langle\rho(p-\lambda q), \rho' q\rangle\big) \\ &- \langle\rho(p-\lambda q), \rho q\rangle\bar\mu' + \lambda\langle\rho q, \rho q\rangle\big(\langle\rho'(\overline{p-\lambda q}), \rho\bar q\rangle + \langle\rho(\overline{p-\lambda q}), \rho'\bar q\rangle\big)\end{aligned}$$

and

$$\begin{aligned}&-\bar\lambda\langle\rho\bar q, \rho\bar q\rangle\big(\lambda\langle\rho' q, \rho q\rangle + \lambda\langle\rho' q, \rho q\rangle\big) + \lambda\langle\rho q, \rho q\rangle\big(\bar\lambda\langle\rho'\bar q, \rho\bar q\rangle + \langle\rho\bar q, \rho'\bar q\rangle\big) \\ &= -\|\rho q\|^2_{H^s} 2\mathrm{Re}\,\langle\rho' q, \rho q\rangle + \|\rho q\|^2_{H^s} 2\mathrm{Re}\,\langle\rho' q, \rho q\rangle = 0.\end{aligned}$$

Therefore

$$|\lambda_y| \le \frac{1}{2}\sum_{\rho_1,\rho_2,\rho_3,\rho_4\in\{\rho,\rho'\}} \frac{\|\rho_1(p-\lambda q)\|_{H^s}\|\rho_2 q\|_{H^s}\big(|\langle\rho_3 p, \rho_4 q\rangle| + |\langle\rho q, \rho q\rangle|\big)}{|\mu|^2},$$

and by (6.15), (6.16),

$$|\lambda_y| \le c\|\rho(p-\lambda q)\|_{H^s}, \text{ and hence } \|\lambda_y\|_{L^2} \le c d^s(q,p).$$

We easily obtain the claimed estimates for higher order derivatives. □

Next we study what happens when we modify the weight. Let $\eta \in C_0^\infty([-1,1])$ with $\eta = 1$ on $[-1/2, 1/2]$ and we define

$$\tilde\mu = \tilde\mu(y) = \langle \eta((\cdot - y)/L)p, \eta((\cdot - y)/L)q\rangle_{H^s}, \quad \tilde\lambda = \tilde\lambda(y) = \frac{\tilde\mu(y)}{|\tilde\mu(y)|}.$$

Lemma 6.5. *Assume the same hypotheses as in Lemma 6.4, then*

$$\|\tilde\lambda - \lambda\|_{H^N} \le c_N d^0(q,p), \quad \forall N \in \mathbb{N},$$

if the righthand side is bounded by a constant depending on the energy.

Proof. Since

$$|\tilde\lambda - \lambda| = \left|\frac{|\tilde\mu(y)|\mu(y) - |\mu(y)|\tilde\mu(y)}{|\mu(y)||\tilde\mu(y)|}\right|,$$

this amount to bounding the difference

$$A = |\tilde\mu(y)|\mu(y) - |\mu(y)|\tilde\mu(y).$$

This vanishes again for λq and we can continue as in the proof of Lemma 6.4 since

$$|A| \le A_1 + A_2$$

where

$$\begin{aligned}A_1 =& \Big|\big|\langle\eta(x-y)p, \eta(x-y)q\rangle\big|\langle \operatorname{sech}_L(x-y)p, \operatorname{sech}_L(x-y)q\rangle \\ &- \big|\langle\eta(x-y)\lambda q, \eta(x-y)q\rangle\big|\langle \operatorname{sech}_L(x-y)\lambda q, \operatorname{sech}_L(x-y)q\rangle\Big| \\ \le& \big|\langle\eta(x-y)(p-\lambda q), \eta(x-y)q\rangle\big|\big|\langle \operatorname{sech}_L(x-y)q, \operatorname{sech}_L(x-y)q\rangle\big| \\ &+ \big|\langle\eta(x-y)q, \eta(x-y)q\rangle\big|\big|\langle \operatorname{sech}_L(x-y)(p-\lambda q), \operatorname{sech}_L(x-y)q\rangle\big|\end{aligned}$$

$$A_2 = \Big| |\langle \operatorname{sech}_L(x-y)p, \operatorname{sech}_L(x-y)q\rangle| \langle \eta(x-y)p, \eta(x-y)q\rangle - |\langle \operatorname{sech}_L(x-y)\lambda q, \operatorname{sech}_L(x-y)q\rangle| \langle \eta(x-y)\lambda q, \eta(x-y)q\rangle \Big|.$$

Bounding the derivatives is done in the same fashion as for λ.

□

There exists $r < \frac{1}{32}E^s(q)/\varepsilon$ small enough such that for any $p \in B^s_r(q)$, we can construct a function $\theta = \theta(x)$ using the function $\tilde\lambda = \tilde\lambda(y)$ as follows:

(1) We choose a sequence $(a_n)_{n\in\mathbb{Z}}$ so that
$$e^{ia_n} = \tilde\lambda(4nL)$$
and
$$\sum_n |a_{n-1} - a_n|^2 < CLd^s(q,p) < \frac{1}{2}$$
where the latter is satisfied for small enough r. The sequence is unique up to the addition of a multiple of 2π.

(2) We fix a smooth partition of unity
$$\sum_n \rho((x-4Ln)/L) = 1 \text{ with } \rho = 1 \text{ on } [-1,1] = \operatorname{Supp}(\eta)$$
and define
$$\theta(x) = \sum a_n \rho((x-4Ln)/L).$$

(3) We define the map
$$p \to e^{-i\theta}p - q =: b.$$

This defines the map (6.12) in Theorem 6.2:
$$B^s_r(q) \ni p \mapsto ((a_n)_n, b) \in l^2_d \times \tilde H^s.$$
Indeed, it suffices to show $b \in \tilde H^s$. Since
$$\begin{aligned}\|\operatorname{sech}_{\mathrm L}(x-y)b(x)\|_{H^s_x} \le{}& \|\operatorname{sech}_{\mathrm L}(x-y)((\chi(y))^{-1}p(x) - q(x))\|_{H^s_x} \\ &+ \|\operatorname{sech}_{\mathrm L}(x-y)((\chi(y))^{-1} - (\tilde\chi(x))^{-1})p(x)\|_{H^s_x} \\ &+ \|\operatorname{sech}_{\mathrm L}(x-y)((\tilde\chi(x))^{-1} - e^{-i\theta(x)})p(x)\|_{H^s_x},\end{aligned}$$
by Lemma 6.2, Lemma 6.4 and Lemma 6.5, we derive that
$$\|b\|^2_{H^s} \lesssim \int \|\operatorname{sech}_{\mathrm L}(x-y)b\|^2_{H^s_x}\,\mathrm{d}y \lesssim d^s(q,p).$$
Furthermore, it is straightforward to calculate
$$\begin{aligned}&\langle \eta((x-4nL)/L)b, \eta((x-4nL)/L)q\rangle \\ =&e^{-ia_n}\langle \eta((x-4nL)/L)p, \eta((x-4nL)/L)q\rangle - \langle \eta((x-4nL)/L)q, \eta((x-4nL)/L)q\rangle \\ =&|\tilde\mu(4nL)| - \|\eta((\cdot-4nL)/L)q\|^2_{H^s} \in \mathbb{R}.\end{aligned}$$

Let us consider the map
$$l^2_d \times \tilde H^s \ni ((a_n)_n, b) \mapsto e^{i\theta}(q+b),$$
where the function θ is constructed as above from $\theta(4nL) = a_n$. Since
$$|e^{i\theta}(q+b)|^2 - |e^{i\tilde\theta}(q+\tilde b)|^2 = |b|^2 - |\tilde b|^2 + 2\mathrm{Re}\,[\bar q(b-\tilde b)],$$
we derive that
$$\big\| |e^{i\theta}(q+b)|^2 - |e^{i\tilde\theta}(q+\tilde b)|^2 \big\|_{H^{s-1}} \le c\|b-\tilde b\|_{H^s}.$$

Furthermore, we also derive (expanding the norm)

$$\int_{\mathbb{R}} \Big\| \operatorname{sech}(x-y)\Big(e^{i\tilde\theta(y)-i\theta(y)}e^{i\theta(x)}(q+b) - e^{i\tilde\theta(x)}(q+\tilde b)\Big)\Big\|^2_{H^s_x} \,\mathrm{d}y$$
$$\le c\big(\|b-\tilde b\|^2_{H^s} + \|\big((a_n)-(\tilde a_n)\big)_n\|^2_{l^2_d}\big).$$

On the other side, the map

$$B^s_r(q) \ni p \mapsto \tilde\lambda(y) = \frac{\langle \eta((x-y)/L)p, \eta((x-y)/L)q\rangle_{H^s_x}}{|\cdot|} \in |D|^{-1}H^N$$

and the map

$$p \mapsto b = e^{-i\theta}p - q$$

are Lipschitz continuous. This proves the biLipschitz continuity.

Finally, the following two maps describing the coordinate changes are smooth:

$$((a_n)_n, b) \mapsto \tilde\lambda_1(y) = \frac{\langle \eta((x-y)/L)(e^{i\theta(x)}(q+b)), \eta((x-y)/L)q_1\rangle_{H^s_x}}{|\cdot|} \in |D|^{-1}H^N,$$
$$((a_n)_n, b) \mapsto b_1 = e^{i\theta - i\theta_1}(q+b) - q_1.$$

□

Appendix A. Calculation of the quadratic term

We prove Lemma 5.1 here: We derive the expansion (5.16) of $\ln T_c^{-1}$ from the expansion (3.22) on the imaginary axis (5.7). It suffices to show

$$\Phi + T_2 = \tilde T_2 + \tilde T_3, \quad \text{when } (\lambda, z) = (i\sqrt{\tau^2/4-1}, i\tau/2),\ \zeta = \lambda + z,\ \tau \ge 2, \tag{A.1}$$

where Φ, $T_2 = \cap$, $\tilde T_2$ are given in (3.23), (5.13), (5.17) respectively:

$$\Phi := -\frac{i}{2z}\int_{\mathbb{R}} \frac{(|q|^2-1)^2}{|q|^2-\zeta^2}\,\mathrm{d}x + \frac{1}{2z\zeta}\int_{\mathbb{R}} \frac{q'\bar q(|q|^2-1)}{(|q|^2-\zeta^2)}\,\mathrm{d}x\,,$$
$$\cap = \int_{x<y} e^{\varphi(y)-\varphi(x)} q_3(x) q_2(y)\,\mathrm{d}x\,\mathrm{d}y\,,$$
$$\tilde T_2(i\sigma) = \frac{1}{4z^2}\int_{x<y} e^{2iz(y-x)}\Big((|q|^2-1)(y)(|q|^2-1)(x) + q'(y)\bar q'(x)\Big)\,\mathrm{d}x\,\mathrm{d}y$$
$$-\, i\frac{z+\zeta}{4z^3\zeta^2}\int_{\mathbb{R}} \operatorname{Im}(q'\bar q)(|q|^2-1)\,\mathrm{d}x + \frac{1}{8z^3\zeta}\int_{x<y} e^{2iz(y-x)}\Big(q'(y)\bar q'(x) - \bar q'(y)q'(x)\Big)dxdy.$$

Here, $\tilde T_3$ identifies the summation of the cubic terms in $\ln T_c^{-1}$ and reads as the finite linear combination of the integrals of type (5.18), that is, of the cubic terms from the following set:

$$\Big\{ \int_{\mathbb{R}}\Big(\frac{|q|^2-1}{\tau^2}\Big)^2 h\,\mathrm{d}x\,,\,,\ \int_{x<y} e^{2iz(y-x)} h_1(y)\Big(\frac{|q|^2-1}{\tau^2}h_2\Big)(x)\,\mathrm{d}x\,\mathrm{d}y\,,$$
$$\int_{x<y} e^{2iz(y-x)}\Big(\frac{|q|^2-1}{\tau^2}h_1\Big)(y) h_2(x)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{H}$$
$$\int_{x<y} e^{2iz(y-x)} h_1(y)\int_x^y \frac{q' \text{ or } \bar q'}{\tau}\,\mathrm{d}m\, h_2(x)\,\mathrm{d}x\,\mathrm{d}y\ \Big|\ h, h_1, h_2 \in O\Big\}.$$

Here $= \int_{x<y}(e^{\varphi(y)-\varphi(x)} - e^{2iz(y-x)})q_2(y)q_3(x)\,\mathrm{d}x\,\mathrm{d}y$ is defined in (5.15) and the set $O = \{P\cdot\frac{1}{\tau}(|q|^2-1),\ P\cdot\frac{1}{\tau}q',\ P\cdot\frac{1}{\tau}\bar q'\}$ (defined in (5.11)) where P is polynomial of form (5.10): $P = P(\frac{1}{\omega^{-2}|q|^2+1}, \frac{1}{\tau}q, \frac{1}{\tau}\bar q, \frac{1}{\tau^2}(|q|^2-1))$.

For notational simplicity, we will always denote H to be the finite summation of some cubic terms from the above set (H), which may change from line to line. Thus the goal equality (A.1) reads as

$$\Phi + \int_{x<y} e^{2iz(y-x)} q_2(y) q_3(x)\,\mathrm{d}x\,\mathrm{d}y \;= \tilde{T}_2 + H, \tag{A.2}$$

and we are going to decompose the quantity $\Phi + \int_{x<y} e^{2iz(y-x)} q_2(y) q_3(x)\,\mathrm{d}x\,\mathrm{d}y$ into the quadratic and cubic terms in the following. We will use freely the equality in (3.17): $\frac{1}{|q|^2-\zeta^2} = -\frac{1}{2z\zeta} + \frac{|q|^2-1}{2z\zeta(|q|^2-\zeta^2)}$.

We can first rewrite Φ as the finite summation of quadratic and cubic terms:

$$\begin{aligned}\Phi &= \frac{i}{4z^2\zeta}\int_{\mathbb{R}} (|q|^2-1)^2\,\mathrm{d}x \;-\; \frac{i}{4z^2\zeta}\int_{\mathbb{R}} \frac{(|q|^2-1)^3}{|q|^2-\zeta^2}\,\mathrm{d}x \\ &\quad - \frac{1}{4z^2\zeta^2}\int_{\mathbb{R}} q'\bar{q}(|q|^2-1)\,\mathrm{d}x \;+\; \frac{1}{4z^2\zeta^2}\int_{\mathbb{R}} \frac{q'\bar{q}(|q|^2-1)^2}{|q|^2-\zeta^2}\,\mathrm{d}x \\ &= \frac{i}{4z^2\zeta}\int_{\mathbb{R}} (|q|^2-1)^2\,\mathrm{d}x \;-\; \frac{1}{4z^2\zeta^2}\int_{\mathbb{R}} q'\bar{q}(|q|^2-1)\,\mathrm{d}x \;+ H.\end{aligned} \tag{A.3}$$

Recall q_2, q_3 defined in $(*)$ such that the product $q_2(y)q_3(x)$ reads as

$$\begin{aligned} q_2(y)q_3(x) =& \frac{q'(y)\bar{q}'(x)}{4z^2} - \frac{q'(y)}{4z^2}\frac{(|q|^2-1)\bar{q}'}{|q|^2-\zeta^2}(x) + \frac{\zeta}{2z}\frac{(|q|^2-1)q'}{|q|^2-\zeta^2}(y)\frac{\bar{q}'}{|q|^2-\zeta^2}(x) \\ &+ i\zeta\Big(\frac{q'}{|q|^2-\zeta^2}(y)\frac{(|q|^2-1)\bar{q}}{|q|^2-\zeta^2}(x) - \frac{(|q|^2-1)q}{|q|^2-\zeta^2}(y)\frac{\bar{q}'}{|q|^2-\zeta^2}(x)\Big) \\ &+ \frac{(|q|^2-1)q}{|q|^2-\zeta^2}(y)\frac{(|q|^2-1)\bar{q}}{|q|^2-\zeta^2}(x). \end{aligned}$$

We decompose $\int_{x<y} e^{2iz(y-x)} q_2(y) q_3(x)\,\mathrm{d}x\,\mathrm{d}y$ into

$$\int_{x<y} e^{2iz(y-x)} q_2(y) q_3(x)\,\mathrm{d}x\,\mathrm{d}y \;= \frac{1}{4z^2}\int_{x<y} e^{2iz(y-x)} q'(y)\bar{q}'(x)\,\mathrm{d}x\,\mathrm{d}y \;+ G + H, \tag{A.4}$$

$$\begin{aligned}\text{with } G = \frac{i}{4z^2\zeta}\int_{x<y} & e^{2iz(y-x)}\Big(q'(y)\big((|q|^2-1)\bar{q}\big)(x) - \big((|q|^2-1)q\big)(y)\bar{q}'(x)\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{4z^2\zeta^2}\int_{x<y} e^{2iz(y-x)}\big((|q|^2-1)q\big)(y)\big((|q|^2-1)\bar{q}\big)(x)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

By virtue of $q(y) - q(x) = \int_x^y q'(m)dm$,

$$\begin{aligned} G =& \frac{i}{4z^2\zeta}\int_{x<y} e^{2iz(y-x)}\Big((q'\bar{q})(y)(|q|^2-1)(x) - (|q|^2-1)(y)(q\bar{q}')(x)\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{4z^2\zeta^2}\int_{x<y} e^{2iz(y-x)}(|q|^2-1)(y)\big((|q|^2-1)|q|^2\big)(x)\,\mathrm{d}x\,\mathrm{d}y \;+ H, \end{aligned}$$

and hence

$$\begin{aligned} G =& \frac{i}{4z^2\zeta}\int_{x<y} e^{2iz(y-x)}\Big(\mathrm{Re}\,(q'\bar{q})(y)(|q|^2-1)(x) - (|q|^2-1)(y)\mathrm{Re}\,(q'\bar{q})(x)\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &- \frac{1}{4z^2\zeta}\int_{x<y} e^{2iz(y-x)}\Big(\mathrm{Im}\,(q'\bar{q})(y)(|q|^2-1)(x) + (|q|^2-1)(y)\mathrm{Im}\,(q'\bar{q})(x)\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{4z^2\zeta^2}\int_{x<y} e^{2iz(y-x)}(|q|^2-1)(y)(|q|^2-1)(x)\,\mathrm{d}x\,\mathrm{d}y \;+ H. \end{aligned}$$

Noticing $2\mathrm{Re}\,(q'\bar{q}) = (|q|^2-1)'$ and the integration by parts

$$\begin{aligned}\int_{x<y} e^{2iz(y-x)}g(y)h(x)\,\mathrm{d}x\,\mathrm{d}y &= \frac{i}{2z}\Big(\int_{\mathbb{R}} gh\,\mathrm{d}x - \int_{x<y} e^{2iz(y-x)}g(y)h'(x)\,\mathrm{d}x\,\mathrm{d}y\Big)\\ &= \frac{i}{2z}\Big(\int_{\mathbb{R}} gh\,\mathrm{d}x + \int_{x<y} e^{2iz(y-x)}g'(y)h(x)\,\mathrm{d}x\,\mathrm{d}y\Big),\end{aligned}$$

we derive that

$$\begin{aligned}G = &\frac{1}{2z\zeta}\int_{x<y} e^{2iz(y-x)}(|q|^2-1)(y)(|q|^2-1)(x)\,\mathrm{d}x\,\mathrm{d}y - \frac{i}{4z^2\zeta}\int_{\mathbb{R}}(|q|^2-1)^2\,\mathrm{d}x\\ &+ \frac{i}{4z^3\zeta}\int_{x<y} e^{2iz(y-x)}\Big(\mathrm{Im}\,(q'\bar{q})(y)\mathrm{Re}\,(q'\bar{q})(x) - \mathrm{Re}\,(q'\bar{q})(y)\mathrm{Im}\,(q'\bar{q})(x)\Big)\,\mathrm{d}x\,\mathrm{d}y\\ &- \frac{i}{4z^3\zeta}\int_{\mathbb{R}}\mathrm{Im}\,(q'\bar{q})(|q|^2-1)\,\mathrm{d}x + \frac{1}{4z^2\zeta^2}\int_{x<y} e^{2iz(y-x)}(|q|^2-1)(y)(|q|^2-1)(x)\,\mathrm{d}x\,\mathrm{d}y + H.\end{aligned}$$

Hence by $\frac{1}{2z\zeta} + \frac{1}{4z^2\zeta^2} = \frac{1}{4z^2}$

$$\begin{aligned}G =&\frac{1}{4z^2}\int_{x<y} e^{2iz(y-x)}(|q|^2-1)(y)(|q|^2-1)(x)\,\mathrm{d}x\,\mathrm{d}y - \frac{i}{4z^2\zeta}\int_{\mathbb{R}}(|q|^2-1)^2\,\mathrm{d}x\\ &+ \frac{1}{8z^3\zeta}\int_{x<y} e^{2iz(y-x)}\Big((q'\bar{q})(y)(q\bar{q}')(x) - (q\bar{q}')(y)(q'\bar{q})(x)\Big)\,\mathrm{d}x\,\mathrm{d}y\\ &- \frac{i}{4z^3\zeta}\int_{\mathbb{R}}\mathrm{Im}\,(q'\bar{q})(|q|^2-1)\,\mathrm{d}x + H,\end{aligned}$$

which, by virtue of $q(y)-q(x) = \int_x^y q'(m)dm$ and $|q|^2 = (|q|^2-1)+1$ again, reads as

$$\begin{aligned}G = &\frac{1}{4z^2}\int_{x<y} e^{2iz(y-x)}(|q|^2-1)(y)(|q|^2-1)(x)\,\mathrm{d}x\,\mathrm{d}y - \frac{i}{4z^2\zeta}\int_{\mathbb{R}}(|q|^2-1)^2\,\mathrm{d}x\\ &+ \frac{1}{8z^3\zeta}\int_{x<y} e^{2iz(y-x)}\Big(q'(y)\bar{q}'(x) - \bar{q}'(y)q'(x)\Big)\,\mathrm{d}x\,\mathrm{d}y - \frac{i}{4z^3\zeta}\int_{\mathbb{R}}\mathrm{Im}\,(q'\bar{q})(|q|^2-1)\,\mathrm{d}x + H.\end{aligned}$$

To conclude, we arrive at (A.2) and hence (A.1) by summing up (A.3) and (A.4) (noticing again $\int_{\mathbb{R}}\mathrm{Re}\,(q'\bar{q})(|q|^2-1)\,\mathrm{d}x = 0$).

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