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Instantaneous Frequency Estimation and Signal Separation Using Fractional Continuous Wavelet Transform

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Abstract

In the signal processing field, time-frequency representations (TFR's) have intensively been improved to provide effective and powerful tools for reliable signal analysis. One of the most valuable and frequently used tools is Fourier transform (FT) which has been used to study the frequency content of stationary signals in the Fourier domain (FD). However, FT is not sufficient to study the frequency of non-stationary signals. For this particular type of signals to be best analyzed, some transforms such as the short time Fourier transform (STFT) and the continuous wavelet transform (CWT) have been introduced to provide us with a signal representation in the time-frequency plane. Another transform based on STFT and CWT; namely, the synchrosqueezing transform (SST), was introduced to improve the sharpness of the TFR's by assigning the coefficient value to a different point in the TF plane. Also, TFR's with satisfactory energy concentration and the corresponding SST's involving both time and frequency variables were introduced; namely, the instantaneous frequency-embedded STFT (CWT) (IFE-STFT/IFE-CWT), where a rough estimation of the IF of a targeted component was used to achieve an accurate IF estimation. Recently, the STFT, the CWT and the corresponding SST's with a time-varying window width are proposed and studied. These transforms have shown the confidence in the accuracy of both sharpening the TFR and separating the components of a multicomponent non-stationary signal, which then led to obtain a more accurate component retrieval formula at any local time. In order to improve the time-frequency resolutions, the concept of fractional Fourier transform (FrFT) was introduced as a potent tool to analyze time-varying signals; however, it fails in locating the frequency content in the fractional Fourier domain (FrFD). To this regard, the short time fractional FT (STFrFT) and the fractional CWT (FrCWT) were proposed to solve this issue by displaying the time and FrFD-frequency contents jointly in the time-FrFD-frequency plane. In this dissertation, we provide a component retrieval formula for a multicomponent signal from its FrCWT with integral involving only the scale variable and then introducing the corresponding SST (FrWSST). We also introduce the first and second order SST based on the IFE-CWT (IFE-WSST) and then propose time-FrFD-frequency representations with satisfactory energy concentration; namely, IFE-FrCWT and the corresponding SST (IFE-FrWSST). Lastly, we consider the FrCWT with a time-varying window width; namely, the adaptive FrCWT (AFr-CWT) and the corresponding SST (AFrWSST). We propose these TFR's in the FrFD for the purpose of not only improving the accuracy of the IF estimation and the energy concentration of these transforms, but also enhancing the separation conditions for the components of a multicomponent signal to be retrieved more accurately.

Keywords: Fractional wavelet transform, Instantaneous frequency-embedded fractional wavelet transform, Adaptive fractional wavelet transform, Synchrosqueezing transform

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Abbreviations

FT: Fourier transform
FrFT: Fractional Fourier transform
IF: Instantaneous frequency
IA: Instantaneous amplitude
$PC(\mathbb{R})$: Set of piecewise - continuous functions
IMT(F): Intrinsic mode type (function)
AHM: Adaptive harmonic model
LFM: Linear frequency modulation
LFT: Localized Fourier transform
FD: Fourier domain
FrFD: Fractional Fourier domain
TF: Time - frequency
TFR: Time - frequency representation
ITFR: Ideal time - frequency representation
TFA: Time - frequency analysis
STFT: Short time Fourier transform
CWT: Continuous wavelet transform
FrCWT: Fractional continuous wavelet transform
IFE – CWT: Instantaneous frequency-embedded CWT
RAM: Reassignment method
SST: Synchrosqueezing transform
FSST: STFT - based synchrosqueezing transform
WSST: CWT - based synchrosqueezing transform
IFE – WSST: IFE-CWT - based synchrosqueezing transform
FrWSST: FrCWT - based synchrosqueezing transform
IFE – FrCWT: Instantaneous frequency-embedded fractional CWT
IFE – FrWSST: IFE-FrCWT - based synchrosqueezing transform
ACWT: Adaptive continuous wavelet transform
AWSST: ACWT - based synchrosqueezing transform
AFrCWT: Adaptive fractional continuous wavelet transform
AFrWSST: AFrCWT - based synchrosqueezing transform

CHAPTER 1

Introduction

Time-frequency analysis has been seen as a significant and powerful tool in the field of signal processing and analysis. It is used to facilitate and understand the oscillatory features of signals whose their frequencies may or may not change with time.

By considering an integrable function $x(t)$ on $\mathbb{R}$ as a function in PC($\mathbb{R}$), applying Fourier transform (FT) to $x(t)$ aims to take it from the time domain $\mathbb{R}$ to $\widehat{x}(\xi)$ in the frequency domain $\mathbb{R}$. FT is used to study the frequency contents of time dependent signals, where the analog signal can be reconstructed back from the frequency content by using the inverse Fourier transform.

The purpose of localizing signals $x \in L_2(\mathbb{R})$ before applying FT is apparent through the aim of using a suitable real-valued time-window function $u \in (L_1 \cap L_2)(\mathbb{R})$ in the short-time Fourier transform (STFT). This window function allows to move along the t-axis without partitioning it into disjoint intervals. Therefore with this window function, the STFT takes $x(t)$ from the time domain $\mathbb{R}$ to a quantity $\mathbb{V}_x(t,\xi)$ in the time-frequency domain $\mathbb{R}^2$, where we can see that x can easily be reconstructed back from its localized Fourier transform as it will be seen later in Section 2.3.

For signals $x \in L_1(\mathbb{R})$ (or PC$^*([a,b])$, $a,b \in \mathbb{R}$), the frequency contents are investigated in some desirable neighborhood of any t by adopting the Fourier basis functions $e^{i2\pi\xi t}$, $\xi \in \mathbb{R}$. Instead of that, a general wavelet $\psi \in L_2(\mathbb{R})$ is used to generate a whole family of wavelets through

$$\psi_{a,b}(t) = \frac{1}{a}\,\psi\Big(\frac{t-b}{a}\Big),$$

where the factor $a > 0$ is for adjusting the scale and the length of the wavelet and the parameter $b \in \mathbb{R}$ is for shifting the support interval of $\psi_{a,b}$ along the whole real axis. Thus, for a function $x \in L_2(\mathbb{R})$, the continuous wavelet transform (CWT) of x is defined as the inner product of x with the family wavelet $\{\psi_{a,b}\}$. It aims to analyze the time and frequency contents of x depending on the width of the window function $\psi_{a,b}(t)$. If $\psi \in L_2(\mathbb{R})$ satisfies the admissibility condition; that is,

$$\int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi}\,d\xi < \infty,$$

then the reconstruction operations for x from the CWT are allowed.

For a real value $\xi_k > 0$ and $a_k \in \mathbb{R}$, a signal $x(t)$ defined by

$$x(t) = a_0 + \sum_{k=1}^{N} x_k(t) = a_0 + \sum_{k=1}^{N} a_k \cos(2\pi\xi_k t)$$

is a *superposition* of the sub-signals $x_k(t) = a_k \cos(2\pi\xi_k t)$ for $k = 1, 2, ..., N$, where ξ_k are their constant frequencies. These frequencies are independent of the time variable t, so $x(t)$ is classified as a *stationary* signal. However, signals with time dependent frequencies are classified to be *non-stationary*, and they are defined to be superpositions of the sub-signals $x_k(t) = A_k(t)\cos(2\pi\phi_k(t))$ for $k = 1, 2, ..., N$, by

$$x(t) = A_0(t) + \sum_{k=1}^{N} x_k(t) = A_0(t) + \sum_{k=1}^{N} A_k(t)\cos(2\pi\phi_k(t)),$$

where $A_k > 0$, $\phi_k \in C^2(\mathbb{R})$ such that $\phi_k' > 0$ is the instantaneous frequency (IF) of $x_k(t)$ for $k = 1, 2, ..., N$, and $A_0(t)$ is some polynomial. These types of signals seek for best analysis into the time-frequency plane since FT is ineffective to study their frequency contents. Therefore, a powerful time-frequency method based on the CWT; namely, synchrosqueezing transform (SST), was recently developed. The SST, also named the CWT-based synchrosqueezing transform (WSST) that was introduced by Daubechies and Maes in [24] with a further development by Daubechies, Lu and Wu in [23] (also see [25]) and comprehensively studied by Wu in his Ph.D. dissertation [64] considering both the CWT and the STFT to compute some reference frequency from the source signal for the SST operation to squeeze out the IF's of the components of multicomponent signal, is a special type of the reassignment method which is considered as a post processing technique to sharpen the TF representation of a signal by allocating the coefficient value to a different point in the TF plane, and with a further study by Thakur and Wu in [60]. Chapter 3 is prepared to present the SST in slightly more details. This idea was also modified with the STFT; namely, STFT-based synchrosqueezing transform (FSST) which was proposed by Oberlin, Meignen and Perrier in [49] using a different well-separated condition. Later on, They proposed and studied the second-order SST based on both STFT and CWT in [5], [47] and [50] (also see [3]).

Li and Liang in [40] introduced the generalized SST that aims to transform a signal $x(t) = A(t)\cos(2\pi\,\phi(t))$ or $x(t) = A(t)\,e^{i2\pi\phi(t)}$ to a signal with a constant frequency by

$$x(t) \longrightarrow x(t)\,e^{-i2\pi(\varphi(t)+\xi_0 t)},$$

where ξ_0 is the target frequency. However, this method has limitations that in practice estimating $\phi'(t)$ is needed since it is unknown, and only one variable is involved as conventional SST [40]. Later on, a transform that involves both variables of STFT was introduced by Wang, Chen, etc., in [62]; namely, the demodulation-transform based SST, and the idea was motivated to be with CWT by Jiang and Suter in [30].

A time-varying window width was recently adapted to the SST based on STFT and discussed in [55], and later on in [35] the authors proposed and studied the adaptive CWT and the corresponding SST for IF estimation and multicomponent signal separation (also see [7, 13, 14, 34, 37, 43, 44]).

As known, Fourier analysis is one of the most powerful and frequently used tools in the field of signal processing, and to visualize the FT operator; a change in representation of the signal corresponding to a counter-clockwise rotation of the axis by an angle $\frac{\pi}{2}$ is required. The fractional Fourier transform (FrFT), which provides a generalization of the conventional FT, was introduced in mathematics literature by V. Namias in 1980 [46] where it can be considered as a rotation by an angle α in the time-frequency plane (also see [52, 54, 58, 67]). The FrFT is an effective tool to analyze the chirp signal. In some applications; however, it fails in locating the fractional Fourier domain (FrFD)-frequency contents. Many authors have proposed some works to adapt the FrFT to the STFT; namely, short time fractional Fourier transform (STFrFT), to solve this issue and improve the performance in concentration of the traditional time-frequency representations (TFR's) [1, 9]. The STFrFT is to display the time and FrFD-frequency information jointly in the time-FrFD-frequency plane and provide the signal with a 2-D support; namely, the short-time fractional Fourier domain (STFrFD)-support [59].

In 1997, Mendlovic, Zalevsky, etc., introduced the fractional continuous wavelet transform (FrCWT), which takes advantage of the localization existing in the FrFT to improve the reconstruction performance of the CWT [45]. Recently, the FrCWT has been developed to be more general and has elegant mathematical properties by Dai, Zheng, etc. It is to display the time and FrFD-frequency information jointly in the time-FrFD-frequency plane [22] (also see [56]).

This dissertation is organized as follows: Preliminaries in Chapter 2, where the fundamental concepts of the TFR's for analyzing stationary and non-stationary signals are provided. In Chapter 3, the concept of the SST is introduced to show the improvement in the sharpness of TFR's. In Chapter 4, we review the FrFT and the FrCWT, and we then establish a retrieved formula for a signal from the FrCWT with integral involving only one variable and define a new SST based on the FrCWT. In Chapter 5, we introduce the IFE-SST and the second-order IFE-SST based on CWT where the derivation of the phase transformation comes directly from the transform not like that was based on reassignment operators [50], [47]. Then we propose the IFE fractional continuous wavelet transform (IFE-FrCWT) and the corresponding SST (IFE-FrWSST). In Chapter 6, we briefly review the CWT with a time-varying parameter (ACWT) and the corresponding SST (AWSST) [35], and we then propose the FrCWT with a time-varying parameter (AFrCWT) and the corresponding SST (AFrWSST). Finally, we end this dissertation with a conclusion and future work in Chapter 7.

CHAPTER 2

Preliminaries

2.1 Fourier transform

Fourier transform (FT) is a mathematical transform which decomposes a function of time, $x(t)$, into its constituent frequency. It may be used to study the frequency content of stationary functions, where this type of functions can be considered as functions in the set of *piecewise continuous* functions on $\mathbb{R}$, $\mathrm{PC}(\mathbb{R})$. Thus applying FT to such functions with time-domain $\mathbb{R}$ reveals its entire frequency content. In this brief section, FT concept and some useful properties will be introduced.

Definition 2.1.1. Let $x(t)$ be a function in the space of all integrable functions on $\mathbb{R}$, denoted $L_1(\mathbb{R})$. Then the FT of $x(t)$, $\widehat{x}(\xi)$, is defined by

$$\widehat{x}(\xi) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi\xi t}\, dt, \quad \xi \in \mathbb{R}.^{1} \tag{2.1.1}$$

Notice that since $e^{-i2\pi\xi t} = \cos(2\pi\xi t) - i\sin(2\pi\xi t)$, the FT of $x(t)$ reveals its frequency content in terms of the oscillation of *cosine* and *sine* functions as follows

$$\widehat{x}(\xi) = \int_{-\infty}^{\infty} x(t)\cos(2\pi\xi t)dt - i\int_{-\infty}^{\infty} x(t)\sin(2\pi\xi t)\, dt,$$

where $\xi - Hz$ (i.e., ξ radian per second) is the frequency variable.

Furthermore, for $x(t) \in L_2(\mathbb{R})$ such that its FT $\widehat{x}(\xi) \in L_1(\mathbb{R})$, $x(t)$ can be retrieved back from its frequency content (or FT) by the *inverse* Fourier transform (IFT), which is defined by: (see [[12], section 7.2, pp. 329 − 332 and Theorem 4, p 332]).

$$x(t) = \int_{-\infty}^{\infty} \widehat{x}(\xi)\, e^{i2\pi t\xi}\, d\xi.$$

The Parseval identity of FT, which can be used to write some transformations in the frequency-domain (FD), is defined by

$$\int_{-\infty}^{\infty} x(t)\,\overline{y(t)}\, dt = \int_{-\infty}^{\infty} \widehat{x}(\xi)\,\overline{\widehat{y}(\xi)}\, d\xi, \tag{2.1.2}$$

for $x,\, y \in L_2(\mathbb{R})$.

Remark: For $x(t) \in L_2(\mathbb{R})$, the FT of $x'(t)$ is given to be

$$\widehat{x'}(\xi) = i\,\xi\,\widehat{x}(\xi), \tag{2.1.3}$$

while the FT of a function $y(t) = t\,x(t)$ is defined by

$$\widehat{y}(\xi) = i\,\frac{d}{d\xi}\,\widehat{x}(\xi). \tag{2.1.4}$$

[1]For given ξ, $\widehat{x}(\xi)$ represents the part of x that oscillates at *frequency* ξ on the whole *time-domain.*

Now, from equations (2.1.3) and (2.1.4), one can obtain the FT of a function $z(t) = t\,x'(t)$ as follows

$$\widehat{z}(\xi) = i\,\frac{d}{d\xi}\,\widehat{x'}(\xi) = i\,\frac{d}{d\xi}\,(i\,\xi\,\widehat{x}(\xi)) = -\widehat{x}(\xi) - \xi\,\widehat{x}^{'}(\xi). \tag{2.1.5}$$

2.2 Stationary and non-stationary signals

As known by considering the Fourier *series* of an even function extensions, every finite-energy signal $x(t)$ on $[0, \frac{L}{2}]$ has a Fourier *cosine* series representation, denoted S^c_x, defined as

$$x(t) = (S^c_x)(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\big(\frac{2\pi k t}{L}\big),$$

which converges to x in $L_2([0, \frac{L}{2}])$, where cosine coefficients are given by

$$a_k = \frac{4}{L}\int_0^{\frac{L}{2}} x(t)\cos\big(\frac{2\pi k t}{L}\big)\,dt, \quad k = 0, 1, 2, \ldots.$$

Thus, the L-periodic signal x has an instantaneous frequency $\xi_k = \frac{k}{L} - Hz$ for all $k \in \mathbb{Z}^+$ where $a_k \neq 0$. Notice that the unit $Hertz$ (Hz) used to measure the number of cycles of oscillation per second when t represents the time variable. Under the following subsections, we will focus on presenting signals with frequencies that may or may not change with time.

2.2.1 Stationary signals

Signals with frequencies that do not change with time are called stationary signals, which are best analyzed by using FS methods or FT. To study the frequency content of a stationary signal $x(t)$, we first consider the standard signal model given by

$$x(t) = a_0 + \sum_{k=1}^{N} a_k \cos(2\pi\xi_k t), \tag{2.2.1.1}$$

for arbitrary frequency values $\xi_k > 0$ and $a_k \in \mathbb{R}$ for $k = 1, 2, ..., N$. Notice that this signal defined above in (2.2.1.1), which is a finite-energy signal with time-domain $\mathbb{R}$, has frequencies ξ_k for $k = 1, 2, ..., N$, that are independent of the time variable $t \in \mathbb{R}$. Thus by applying FT to (2.2.1.1), we have

$$\widehat{x}(\xi) = a_0\,\delta(\xi) + \frac{1}{2}\sum_{k=1}^{N} a_k\big(\delta(\xi - \xi_k) + \delta(\xi + \xi_k)\big), \tag{2.2.1.2}$$

where the frequencies can easily be determined. This stationary signal defined in (2.2.1.1) is a special case of the general stationary signal model

$$x(t) = a_0 + \sum_{k=1}^{N} a_k \cos 2\pi(\xi_k t + e_k), \tag{2.2.1.3}$$

with $e_k = 0$. In this general model, *sine* functions and negative amplitudes for stationary signals are allowed to be used, and of course every x defined in (2.2.1.3) and the corresponding x defined in (2.2.1.1) have the same frequency since the FT of the general model x is given by

$$\widehat{x}(\xi) = a_0\, \delta(\xi) + \frac{1}{2}\sum_{k=1}^{N} a_k\, e^{\frac{i2\pi\xi e_k}{\xi_k}} \big(\delta(\xi - \xi_k) + \delta(\xi + \xi_k)\big).$$

From this overview, we see that the FT is useful to discover the frequency contents of stationary signals whose frequencies do not change with time. However, when specific frequency values are assumed, the Fourier transform does not display the time instants. Because of this, it becomes hard to analyze their frequency contents.

2.2.2 Non-stationary signals

In this subsection, we will focus on signals that their frequencies change with time. These types of signals are called non-stationary signals, and they can be defined as follows

$$x(t) = T(t) + \sum_{k=1}^{N} A_k(t)\, \cos(2\pi\phi_k(t)), \tag{2.2.2.1}$$

where $A_k(t) > 0$, $\phi_k(t) \in C^2(\mathbb{R})$ such that $\phi_k'(t) > 0$, and $T(t)$ is some polynomial that may possibly be embedded with noise. In other words, $x(t)$ is a superposition of signal components; that is,

$$x_k(t) = A_k(t)\, \cos(2\pi\phi_k(t)), \quad k = 1, 2, ..., N. \tag{2.2.2.2}$$

In this regard, the question is to know when is x said to be a superposition? To answer this question, we will first have the following definition

Definition 2.2.1. Let $x : \mathbb{R} \to \mathbb{C}$ be a continuous function, and $x \in L_\infty(\mathbb{R})$. Then x is said to be *intrinsic* mode type (IMT) with accuracy $\epsilon > 0$ if $x(t) = A(t)\, e^{i2\pi\phi(t)}$ with some properties that $A(t)$ and $\phi(t)$ satisfy:

- $A(t) \in (C^1 \cap L_\infty)(\mathbb{R}), \quad \phi(t) \in C^2(\mathbb{R});$
- $\inf_t \phi'(t) > 0, \quad \sup_t \phi'(t) < \infty, \quad t \in \mathbb{R};$
- $|A'(t)| \le \epsilon|\phi'(t)|, \quad |\phi''(t)| \le \epsilon|\phi'(t)| \quad \forall\, t \in \mathbb{R}.$

After that, one can define a superposition function as follows:

Definition 2.2.2. A function $x : \mathbb{R} \to \mathbb{C}$ is said to be a *superposition* of well-separated intrinsic mode components with separation $0 < d < 1$ and up to accuracy $\epsilon > 0$, if there exists a finite N, such that

$$x(t) = \sum_{k=1}^{N} x_k(t) = \sum_{k=1}^{N} A_k(t)\, e^{i2\pi\phi_k(t)}, \tag{2.2.2.3}$$

where all $x_k(t)$ are IMT functions, and their phase functions $\phi_k(t)$ satisfy, for some $0 < d < 1$, that

$$\phi_k'(t) > \phi_{k-1}' \quad \text{and} \quad |\phi_k'(t) - \phi_{k-1}'| \geq d(\phi_k'(t) + \phi_{k-1}') \quad \forall\, t \in \mathbb{R}. \tag{2.2.2.4}$$

From equation (2.2.2.2), the functions $A_k(t)$ and $\phi_k(t)$ are called *amplitude* and *phase* functions that generalizing the constants a_k and the linear functions $\xi_k\, t$ in (2.2.1.1) respectively. The derivative function of $\phi_k(t)$, $\phi_k'(t)$, for $k = 1, 2, ..., N$, is the extension of the frequency ξ_k in (2.2.1.1), where each $\phi_k'(t)$ is called the instantaneous frequency of $x_k(t)$. The trend $T(t)$ is a generalization of the constant factor a_0 in the *stationary* signal model (2.2.1.1). Notice that $x(t)$, defined in (2.2.2.1), is said to be $non-linear$ if the amplitude functions $A_k(t)$ are allowed to be non-constants and $non-stationary$ if the phase functions $\phi_k(t)$ are non-linear functions. Thus, such signal model is called the *adaptive harmonic* model (AHM).

For a signal $x(t)$ defined in (2.2.2.1), if $x(t)$ is a blind source signal, it is definitely not feasible to determine its specific signal components $x_k(t)$ for $k = 1, 2, ..., N$, by any decomposition scheme, without knowing prior knowledge of these components and/or specifying appropriate restrictions on the AHM. Those restrictions, in the signal processing literature, are described as follows:

$$\begin{cases} A_k \in (C^1 \cap L_\infty)(\mathbb{R}),\ \ \phi_k \in C^2(\mathbb{R}); \\ \\ \inf_t A_k(t) > c_1,\ \ \sup_t A_k(t) < c_2; \\ \\ \inf_t \phi_k'(t) > c_1,\ \ \sup_t \phi_k'(t) < c_2; \\ \\ |A_k'(t)| \leq \epsilon\, \phi_k'(t),\ \ |\phi_k''(t)| \leq \epsilon\, \phi_k'(t), \end{cases} \tag{2.2.2.5}$$

for all $t \in \mathbb{R}$, where $0 < \epsilon << 1$ and $\epsilon << c_1 < c_2 < \infty$. In this model, we also assume that all components are well-separated, which means their respective phase functions $\phi_k(t)$ satisfy (2.2.2.4). In this subsection and later sections, we will denote the class of functions $x(t)$ satisfying the AHM conditions (2.2.2.4) and (2.2.2.5) by $\mathbf{A}_{\epsilon,d}^{c_1,c_2}$.

Notice that the representation of a signal component $x_k(t)$ defined in (2.2.2.2) is not unique in general. That means there exist smooth functions $\alpha(t)$ and $\beta(t)$ such that $\cos(t) = (1+\alpha(t))\,\cos(t+\beta(t))$. Thus, in the case of defining the signal $x(t)$, we have

$$x(t) = A(t)\,\cos(2\pi\phi(t)) = (A(t)+\alpha(t))\,\cos(2\pi(\phi(t)+\beta(t))).$$

If this $x(t)$ satisfies the conditions in (2.2.2.5), then one can show for some constant C depending only on c_1 and c_2 that $|\alpha(t)| \leq C\,\epsilon$ and $|\beta'(t)| \leq C\,\epsilon$ [23]. In other words, when ϵ is small enough, we see that the definition of the instantaneous frequency (IF) and the instantaneous amplitude (IA) are unique up to a negligible error because of their rigorous definitions [16].

2.3 Short-time Fourier transform

Let $x(t)$ be a signal that truncated by some characteristic function $\chi_{(a,b)}(t)$, then computing $(\widehat{x\chi_{(a,b)}})(\xi)$ is more simple than computing $\widehat{x}(\xi)$ which needs all $x(t)$-values on the entire real axis. Instead of using a characteristic function $\chi_{(a,b)}(t)$, we can consider a *real-valued time-window* function $u(t)$ that is allowed to move (continuously) along the t-axis without having any partitions of the t-axis into disjoint intervals. This is the main idea of the so-called short-time Fourier transform (STFT), where this window function $u(t)$ in this transform is used to localize the signal $x(t)$ before applying FT to it. Because of this it will also be called the *localized* Fourier transform (LFT).

Definition 2.3.1. Let $u \in (L_1 \cap L_2)(\mathbb{R})$ and $t \in \mathbb{R}$. Then for any $x \in L_2(\mathbb{R})$, the short-time Fourier transform (STFT) of x, denoted $\mathbb{V}_x(t,\eta)$, at the time-frequency (or *space*-frequency) point $(t,\eta) \in \mathbb{R}^2$ is defined by

$$\mathbb{V}_x(t,\eta) = \int_{-\infty}^{\infty} x(\tau)\,u(\tau-t)\,e^{-i2\pi\eta(\tau-t)}\,d\tau. \tag{2.3.1}$$

As known, the FT takes a function $x(t)$ from the time domain $\mathbb{R}$ to $\widehat{x}(\xi)$ in the frequency domain $\mathbb{R}$, but the STFT, with the window function u, will take $x(t)$ from the time domain $\mathbb{R}$ to the time-frequency domain $\mathbb{R}^2$. With $u \in (L_1 \cap L_2)(\mathbb{R})$ and $\widehat{u} \in (L_1 \cap L_2)(\mathbb{R})$ such that $u(0) \neq 0$, $x(t)$ can be retrieved back from the STFT, $\mathbb{V}_x(t,\eta)$, called the *inverse* short-time Fourier transform, which is defined by

$$x(t) = \frac{1}{||u(0)||_2^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{V}_x(t,\eta)\,\overline{u(\tau-t)}\,e^{-i2\pi\eta(\tau-t)} d\tau d\eta, \tag{2.3.2}$$

where $x \in (L_1 \cap L_2)(\mathbb{R})$ with $\widehat{x} \in L_1(\mathbb{R})$.

By using Parseval identity of FT, the STFT can be written as follows

$$\mathbb{V}_x(t,\eta) = \int_{-\infty}^{\infty} \widehat{x}(\xi)\, \widehat{u}(\eta-\xi)\, e^{i2\pi t\xi}\, d\xi, \tag{2.3.3}$$

and then one may verify that $x(t)$ can be retrieved back from the STFT with integral involving only the frequency variable η as follows

$$x(t) = \frac{1}{u(0)} \int_{-\infty}^{\infty} \mathbb{V}_x(t,\eta)\, d\eta. \tag{2.3.4}$$

Note that for a real-valued signal $x(t)$ and a real window function $u(t)$ where $\widehat{x}(-\eta) = \overline{\widehat{x}(\eta)}$ and $\widehat{u}(-\eta) = \overline{\widehat{u}(\eta)}$, equation (2.3.4) becomes

$$x(t) = \frac{2}{u(0)} \operatorname{Re}\Big\{ \int_{0}^{\infty} \mathbb{V}_x(t,\eta)\, d\eta \Big\}. \tag{2.3.5}$$

2.4 Continuous wavelet transform

In the previous section, a suitable window function is used to introduce the *localized* Fourier transform (LFT), where the Fourier *basis* functions $e^{i2\pi\xi t}$, $\xi \in \mathbb{R}$ are adopted to investigate the frequency contents of functions $x(t)$, for $x \in L_1(\mathbb{R})$ (or $x \in \mathrm{PC}^*([a,b])$), in some desirable neighborhood of any t. Instead of adapting Fourier basis functions, we will consider a general function $\psi \in L_2(\mathbb{R})$ with

$$\text{P.V.} \int_{-\infty}^{\infty} \psi(t)\, dt = \lim_{A\to\infty} \int_{-A}^{A} \psi(t)\, dt = 0, \tag{2.4.1}$$

and

$$\psi(t) \to 0 \;\; as \;\; t \to \pm\infty, \tag{2.4.2}$$

where P.V. denotes the *Cauchy* principal value. Then ψ is called a *wavelet*.
For any wavelet ψ, one can generate a whole family of wavelets, small waves, through

$$\psi_{a,b}(t) = \frac{1}{a}\, \psi\Big(\frac{t-b}{a}\Big), \tag{2.4.3}$$

where $a > 0$ and $b \in \mathbb{R}$. Notice that using the factor a is to adjust the scale and length of the wavelet, and the translation operator with the parameter $b \in \mathbb{R}$, $\psi(t-b)$, is used to allow shifting the support interval of $\psi_{a,b}$ along the entire real axis (i.e., by changing the values of b). Also, the normalization by $\frac{1}{a}$ multiplication used in (2.3.4) is to present $L_1 - norm$ on $\mathbb{R}$ such that $\|\psi_{a,b}\|_1 = \|\psi\|_1$ for all $a > 0$ and $b \in \mathbb{R}$. The two-parameter family, $\{\psi_{a,b}\}$, of functions $\psi_{a,b}(t)$ is called the family of wavelets that generated by a signal wavelet function $\psi \in L_2(\mathbb{R})$ and used as the integration *kernel*. From (2.4.1), we see that the graph of ψ oscillates (i.e., ψ has a *wavy* shape); and (2.4.2) tells that this wave dies down as $t \to \pm\infty$.

The graphs of $\psi_{a,b}(t)$ may be small or large waves, and that depends on how small or large the values of the factor $a > 0$ are. In particular, when a tends to 0, we observe that $\psi_{a,b}(t)$ zooms in to a smaller region near $t = b$, which is the time location.

Definition 2.4.1. For a function $x(t) \in L_2(\mathbb{R})$, the continuous wavelet transform (CWT), denoted $W_x^{\psi}(a,b)$, of $x(t)$ at the *time-scale* point (a,b) is defined as the inner product of $x(t)$ with the family of wavelets $\psi_{a,b}$ by

$$W_x^{\psi}(a,b) = \langle x(t), \psi_{a,b}(t)\rangle = \frac{1}{a}\int_{-\infty}^{\infty} x(t)\,\overline{\psi\Big(\frac{t-b}{a}\Big)}\,dt. \tag{2.4.4}$$

Notice that the CWT is a time-frequency method that can be used to analyze the time and frequency contents of a function $x(t) \in L_2(\mathbb{R})$, or it is for analyzing the oscillation behavior of $x(t)$. To be more precise on the specifications of a window function, we indeed want the two terminologies of *width* and *center* of the window function. Let $u(t)$ and $\widehat{u}(\xi)$ have a nice localization, having small window widths. Then, one can define the *time* window and the *frequency* window respectively as

$$[t^* - \Delta_u,\, t^* + \Delta_u] \quad \textit{and} \quad [\xi^* - \Delta_{\widehat{u}},\, \xi^* + \Delta_{\widehat{u}}],$$

where t^* and ξ^* are called, as they will be defined in the next definition, the centers of $u(t)$ and $\widehat{u}(\xi)$ respectively.

Definition 2.4.2. (Time-frequency window width)
Assume that $u \in (L_1 \cap L_2)(\mathbb{R})$ be a non-trivial window function such that $t\,u(t) \in L_2(\mathbb{R})$. Then the center of the localization window function $u(t)$ is defined by

$$t^* = \frac{\int_{\mathbb{R}} t\,|u(t)|^2\,dt}{\int_{\mathbb{R}} |u(t)|^2\,dt}, \tag{2.4.5}$$

and the radius of the window function $u(t)$ is defined by

$$\Delta_u = \left(\frac{\int_{\mathbb{R}} (t-t^*)^2\,|u(t)|^2\,dt}{\int_{\mathbb{R}} |u(t)|^2\,dt}\right)^{\frac{1}{2}}. \tag{2.4.6}$$

Thus, the window width of $u(t)$ is defined to be $2\Delta_u$. A similar definition is for both the center ξ^* and the radius $\Delta_{\widehat{u}}$ of $\widehat{u}(\xi)$. One can easily use *Hölder* inequality to show that if $t\,u(t) \in L_2(\mathbb{R})$ for $u(t) \in L_2(\mathbb{R})$, then $t\,u^2(t) \in L_1(\mathbb{R})$. That is,

$$\int_{-\infty}^{\infty} |t\,u^2(t)|\,dx \le \|tu\|_2 \|u\|_2.$$

Therefore, the center t^* is well-defined. Now, if $\xi\,\widehat{u}(\xi) \in L_2(\mathbb{R})$, then the time-frequency localization window is defined by

$$[t^* - \Delta_u,\, t^* + \Delta_u] \times [\xi^* - \Delta_{\widehat{u}},\, \xi^* + \Delta_{\widehat{u}}].$$

In this regard, the wavelet $\psi_{a,b}(t)$, defined in (2.4.3), is called a window function that used to localize a function $x(t)$ in order to test its time and frequency contents. This localization depends on the width of the window function which can then be calculated after computing the center and radius of $\psi_{a,b}(x)$ using (2.4.5) and (2.4.6), as follows:

$$t^*_{\psi_{a,b}} = a\, t^*_\psi + b \tag{2.4.7}$$

while

$$\Delta_{\psi_{b,a}} = a\Delta_\psi. \tag{2.4.8}$$

Thus, the width of the window function $\psi_{a,b}(t)$ is $2a\Delta_\psi$, and the time-frequency window of $\psi_{a,b}$ is given by

$$\big[b + at^* - a\Delta_\psi,\, b + at^* + a\Delta_\psi\big] \times \big[\frac{\xi^*}{a} - \frac{1}{a}\Delta_{\widehat{\psi}},\, \frac{\xi^*}{a} + \frac{1}{a}\Delta_{\widehat{\psi}}\big].$$

The formula in (2.1.1) is the FT of a function $x(t) \in L_1(\mathbb{R})$, so we can consider $\xi^*_{\widehat{\psi}_{a,b}}$ and $\Delta_{\widehat{\psi}_{b,a}}$ to describe the center and radius of the window function $\widehat{\psi}_{a,b}$ in the frequency domain. Thus, we first see that the FT of $\psi_{a,b}(t)$ is $\widehat{\psi}_{a,b}(\xi) = e^{-i2\pi\xi b}\,\widehat{\psi}(a\xi)$, and then from equations (2.4.3), (2.4.5) and (2.4.6), we have

$$\xi^*_{\widehat{\psi}_{a,b}} = \frac{1}{a}\,\xi^*_{\widehat{\psi}}, \tag{2.4.9}$$

and

$$\Delta_{\widehat{\psi}_{a,b}} = \frac{1}{a}\Delta_{\widehat{\psi}}. \tag{2.4.10}$$

From equations (2.4.9) and (2.4.10), we notice that the localization window $\psi_{a,b}$ in the CWT has a nice feature which is the window width is not fixed, varying with the scaling variable a. Hence, the CWT of $x(t)$ zooms in, as the time-window width, $\Delta_{\psi_{a,b}} = a\Delta_\psi$, narrows when the value of a is smaller, that means providing a higher resolution in the time domain where the frequency-window width, $\Delta_{\widehat{\psi}_{a,b}} = \frac{1}{a}\Delta_{\widehat{\psi}}$, widens. However, when the value of a is larger, the CWT zooms out as the time-window width, $\Delta_{\psi_{a,b}} = a\Delta_\psi$, widens, while the frequency-window width, $\Delta_{\widehat{\psi}_{a,b}} = \frac{1}{a}\Delta_{\widehat{\psi}}$, narrows, which means facilitating the analysis of high-frequency contents.

Since the window function $\psi_{a,b}(t)$ slides along the real axis as the value of $b \in \mathbb{R}$ changes, this window function facilitates the analysis of $x(t)$ for different time and frequency detail over the time axis.

Definition 2.4.3. Let $\psi \in L_2(\mathbb{R})$ be a wavelet. If the FT of ψ, $\widehat{\psi}$, satisfies

$$c_\psi = \int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi} \, d\xi < \infty, \tag{2.4.11}$$

then ψ is called an admissible wavelet.

Thus it is clear that the signal $x(t)$ can be retrieved back from the CWT by the *inverse* wavelet transform, if $c_\psi \neq 0$, as (see [[12], Theorem 3, p 389]).

$$x(t) = c_\psi^{-1} \int_0^\infty \int_{-\infty}^\infty W_x^\psi(a,b) \psi_{a,b}(t) \, db \, \frac{da}{a}, \tag{2.4.12}$$

for any admissible wavelet $\psi \in L_2(\mathbb{R})$ that satisfies (2.4.11), and for all $x \in (L_1 \cap L_\infty)(\mathbb{R})$, where $L_\infty(\mathbb{R})$ denotes the space of all bounded functions.

Note that the Parseval identity of FT can also be used to rewrite the CWT as follows:

$$W_x^\psi(a,b) = \int_{-\infty}^\infty \widehat{x}(\xi) \, \overline{\widehat{\psi}(a\xi)} \, e^{i2\pi b\xi} \, d\xi. \tag{2.4.13}$$

A function $x(t)$ is said to be analytic if $x(t)$ satisfies that $\widehat{x}(\xi) = 0$ for $\xi < 0$. Then by considering analytic continuous wavelets, we assume that ψ satisfies

$$0 \neq c_\psi = \int_0^\infty \overline{\widehat{\psi}(\xi)} \, \frac{d\xi}{\xi} < \infty. \tag{2.4.14}$$

Therefore, $x(t) \in L_2(\mathbb{R})$ can be retrieved back from the CWT as

$$x(b) = c_\psi^{-1} \int_0^\infty W_x^\psi(a,b) \, \frac{da}{a} \tag{2.4.15}$$

in the case that $x(t)$ is an analytic signal and c_ψ is as defined in (2.4.14). Furthermore, for a real-valued signal $x(t) \in L_2(\mathbb{R})$, equation (2.4.15) becomes

$$x(b) = \mathrm{Re}\left(2\, c_\psi^{-1} \int_0^\infty W_x^\psi(a,b) \, \frac{da}{a} \right). \tag{2.4.16}$$

2.5 Instantaneous frequency-embedded CWT

A time-frequency representation with satisfactory energy concentration was first introduced by Wang, Chen, etc., in [62] with STFT involving both time and frequency variables, and then motivated by Jiang and Suter in [30] with CWT. Our focus in more details here will be on the motivated one with CWT, namely; the instantaneous frequency-embedded CWT (IFE-CWT), where a differentiable function, $\varphi(t)$, is used with $\varphi'(t) > 0$. Now by assuming that $x(t) \in L_2(\mathbb{R})$, we consider the generalized signal form given by

$$x_{\varphi,b,\xi_0}(t) = x(t)\ e^{-i2\pi(\varphi(t)-\varphi(b)-\varphi'(b)(t-b)-\xi_0 t)}, \tag{2.5.1}$$

where $\xi_0 > 0$, and we come up with the following definition.

Definition 2.5.1. Suppose $\varphi(t)$ is a differentiable function with $\varphi'(t) > 0$. The IFE-CWT, denoted $W_x^{\mathrm{E},\psi}(a,b)$, of $x(t) \in L_2(\mathbb{R})$ with $\varphi(t)$ and a continuous wavelet ψ is defined by

$$W_x^{\mathrm{E},\psi}(a,b) = < x_{\varphi,b,\xi_0}(t), \psi_{a,b}(t) >= \int_{-\infty}^{\infty} x_{\varphi,b,\xi_0}(t)\ \overline{\psi_{a,b}(t)}\, dt. \tag{2.5.2}$$

In fact, the above definition of the IFE-CWT can be extended to slowly growing functions. Now by using Parseval identity of FT, the IFE-CWT will be written as follows

$$W_x^{\mathrm{E},\psi}(a,b) = e^{i2\pi\varphi(b)} \int_{-\infty}^{\infty} \widehat{\widetilde{x}}(\xi)\ \overline{\widehat{\psi}(a\xi + a\varphi'(b))}\ e^{i2\pi\xi b}\, d\xi, \tag{2.5.3}$$

where

$$\widetilde{x}(t) = x(t)\, e^{-i2\pi(\varphi(t)-\xi_0 t)}.$$

Consequently, a function $x(t) \in L_2(\mathbb{R})$ can be retrieved back from the IFE-CWT as

$$x(b) = c_\psi^{-1}\, e^{-i2\pi\xi_0 b} \int_{-\infty}^{\infty} W_x^{\mathrm{E},\psi}(a,b)\, \frac{da}{|a|}. \tag{2.5.4}$$

However, if the scale variable a is restricted to $a > 0$, $x(t)$ can be retrieved back from the IFE-CWT as follows

$$x(b) = c_\psi^{-1}\, e^{-i2\pi\xi_0 b} \int_{0}^{\infty} W_x^{\mathrm{E},\psi}(a,b)\, \frac{da}{a}, \tag{2.5.5}$$

where c_ψ is defined by (2.4.14).

CHAPTER 3

Synchrosqueezing Transform

At the beginning of this chapter, we will briefly present an overview of time-frequency analysis method; namely, the *reassignment* (RAM). Before that, we know that the FT of a signal $x(t)$ in the time domain, $\widehat{x}(\xi)$, provides us with a frequency domain representation. Thus the FT is not sufficient to study the frequency content of non-stationary signals defined in (2.2.2.1) because their IF's $\phi'_k(t)$ for $k = 1, 2, ..., N$, change with time. To best analyze these types of signals, we need methods that provide us with a signal representation in the time-frequency plane.

In the late 1970's, Kodera, Gendrin and De Villedary introduced the reassignment method [32, 33]. Then it was generalized by Auger and Flandrin [2]. The RAM is a general way to sharpen the time-frequency representation (TFR) towards its ideal time-frequency representation (ITFR). That is by creating a modified version of a time-frequency representation; for instance, STFT and CWT, by moving its time-frequency values away from where they are computed. This is in order to produce a better localization of the signal components. substantially, the time-frequency values (t, ξ) are reassigned to the center of gravity or the local centroid $(\tilde{t}, \tilde{\xi})$ of the energy contributions of the TFR [10, 12]. This improves classic time-frequency representations by providing an obvious graphical display of the oscillatory features of a signal, facilitating signals interpretation. This method is very effective, but it is not straightforward to reconstruct the signal components.

Recently, time-frequency analysis methods have seen significant developments including the Fourier-based synchrosqueezing transform (FSST) and the wavelet-based synchrosqueezing transform (WSST), where the synchrosqueezing transform (SST); the first signal resolution approach for non-stationary signals, was originally introduced by Daubechies and Maes in [24] as a special type of reassignment methods and comprehensively studied by Wu in his Ph.D. dissertation [64] considering both the CWT and the STFT with a further and full development of SST based on STFT by Thakur and Wu in [60]. Later on, many other studies related to the SST have been done to improve the sharpness of TFR's by assigning the coefficient value to different point in the TF plane, such as in [5], [47], [49], [50].

3.1 CWT-based synchrosqueezing transform

In Section 2.4, the CWT of a signal $x(t)$, $W_x^{\psi}(b, a)$, was introduced and defined as the inner product of $x(t)$ with the mother wavelet $\psi_{a,b}(t)$ in (2.4.3), which we again display its formula here

$$W_x^{\psi}(a, b) = \langle x(t), \psi_{a,b}(t) \rangle = \frac{1}{a} \int_{-\infty}^{\infty} x(t)\, \overline{\psi\Big(\frac{t-b}{a}\Big)}\, dt. \tag{3.1.1}$$

The representation of the positive quantity $|W_x^\psi(a,b)|^2$ in the TF plane is called the *scalogram* of $x(t)$. By using equations (2.2.2.3) and (3.2.2), the corresponding approximation for the CWT of $x(t)$ is defined as follows:

$$W_x^\psi(a,b) \approx W_{\tilde{x}_t}^\psi(a,b) = \sum_{k=1}^{N} x_k(t)\, \overline{\widehat{\psi}(a\phi_k'(t))}. \tag{3.1.2}$$

Notice that the representation of x in the TF plane is concentrated around ridges corresponding to their instantaneous frequencies defined by $a = \xi_\psi/\phi_k'(t)$, where $\xi_\psi = \arg\max_\xi |\widehat{\psi}(\xi)|$ is the center frequency of the wavelet. Also, if $\phi_k'(t)$, for $k \in \{1,2,...,N\}$, are separated enough compared to the support of $\widehat{\psi}$, we see each mode occupies a distinct domain of the TF plane, allowing their detection, separation and reconstruction. This requires the frequency separation condition described as:

- When $\operatorname{supp}\widehat{\psi} \subset [1-\Delta, 1+\Delta]$, a multicomponent signal x is separated if the instantaneous frequencies satisfy, for each $k \in \{1,2,...,N\}$, that

$$\frac{\phi_{k+1}'(t) - \phi_k'(t)}{\phi_{k+1}'(t) + \phi_k'(t)} > \Delta, \quad t \in \mathbb{R}.$$

Remark: The scalogram of a multicomponent signal $x(t)$ defined in (2.2.2.3) is given to be of the form

$$|W_x^\psi(a,b)|^2 = \sum_{k=1}^{N} |W_{x_k}^\psi(a,b)|^2$$

when the CWT's of signal components, $W_{x_k}^\psi(a,b)$ for $k = 1,2,...,N$, do not overlap in the TF plane. i.e., for all (a,b)

$$W_{x_k}^\psi(a,b)\, {W^*}_{x_l}^\psi(a,b) = 0, \quad k \neq l.$$

However, in general, the scalogram of $x(t)$ is given by

$$|W_x^\psi(a,b)|^2 = \sum_{k=1}^{N} |W_{x_k}^\psi(a,b)|^2 + \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l\neq k}}^{N} W_{x_k}^\psi(a,b)\, {W^*}_{x_l}^\psi(a,b).$$

Observe that the second term on the right hand-side of this equation represents the cross-terms resulting from the interaction between two different signal components. These terms are usually undesirable components because of non-linear structure of the scalogram. It is clear here to see that the cross-terms appear only at the TF points where the auto-terms overlap; however, they might appear even if the components do not overlap in any other quadratic TF representations [57].

3.1.1 First-Order WSST

It is an approach in the literature to the non-stationary signal analysis that introduced by I. Daubechies and others in [23]. This approach is first to estimate the IF's of signal components before reconstructing the sub-signals, where the signal satisfies the properties of the AHM in (2.2.2.4) and (2.2.2.5) (i.e., when the signal $x \in \mathbf{A}_{\epsilon,d}^{c_1,c_2}$). It is a special case of reallocation methods that aims to sharpen TFR's which means to reassign the scale variable a to the frequency variable. It mainly works through *squeezing* the CWT defined in (2.4.4), where the analysis wavelet ψ satisfies the *admissibility* condition in the sense that its FT vanishes on the negative frequency axis, i.e., $\widehat{\psi}(\xi) = 0$ for $\xi < 0$. To extract the IF, we consider the chirp signal $x(t) = e^{i2\pi ct}$ as it was used in Section 3.2. Then the CWT of $x(t)$, $W_x^{\psi}(a,b)$, is given by

$$W_x^{\psi}(a,b) = e^{i2\pi bc}\, \overline{\widehat{\psi}(ac)}. \tag{3.1.1.1}$$

By taking the first-order partial derivative of both sides of this above equation (3.1.1.1) with respect to b, the exact IF , c, of $x(t)$ can be obtained by

$$c = \frac{\frac{\partial}{\partial b} W_x^{\psi}(a,b)}{i2\pi\, W_x^{\psi}(a,b)}.$$

Based on this, it can be concluded that for a general signal $x(t) \in L_2(\mathbb{R})$, at (a,b) on which $W_x^{\psi}(a,b) \neq 0$, the first-order phase transformation, which considered to be the best candidate to estimate the IF, is defined by

$$\Omega_x^{1^{\rm st}}(a,b) = \frac{\frac{\partial}{\partial b} W_x^{\psi}(a,b)}{i2\pi\, W_x^{\psi}(a,b)}. \tag{3.1.1.2}$$

The synchrosqueezing transform based on the CWT (WSST) is to reallocate the values $W_x^{\psi}(a,b)$ according to the map $(a,b) \longrightarrow (\Omega_x^{1^{\rm st}}(a,b), b)$. In other words, it is to reallocate the scale variable a by transforming the CWT of x, $W_x^{\psi}(a,b)$, to a quantity on the time-frequency plane:

$$S_x^{\rm CWT}(\xi,b) = \int\limits_{\{a\in R_+ : W_x^{\psi}(a,b)\neq 0\}} W_x^{\psi}(a,b)\ \delta(\Omega_x^{1^{\rm st}}(a,b) - \xi)\, \frac{da}{a}, \tag{3.1.1.3}$$

where ξ is the frequency variable. Notice that for stability purpose, if x has been contaminated by noise, the determination of those pairs (a,b) on which $W_x^{\psi}(a,b) = 0$ is rather unstable. Because of this we consider a threshold Γ for $|W_x^{\psi}(a,b)|$, below which $\Omega_x^{1^{\rm st}}(a,b)$ is not defined, this is just by replacing $\{a : W_x^{\psi}(a,b) \neq 0\}$ defined in (3.1.1.3) by a smaller region $\{a : |W_x^{\psi}(a,b)| \geq \Gamma\}$. Then (3.1.1.3) becomes

$$S_x^{\rm CWT}(\xi,b) = \int\limits_{\{a\in R_+ :\, |W_x^{\psi}(a,b)| \geq \Gamma\}} W_x^{\psi}(a,b)\ \delta(\Omega_x^{1^{\rm st}}(a,b) - \xi)\, \frac{da}{a}. \tag{3.1.1.4}$$

Thus, considering $c_\psi \neq 0$ to be the constant defined in (2.4.14), a mono-component analytic signal $x(t) \in L_2(\mathbb{R})$, by (2.4.15), can be retrieved back from the WSST as:

$$x(b) = c_\psi^{-1} \int_0^\infty S_x^{\text{CWT}}(\xi, b)\, d\xi. \tag{3.1.1.5}$$

Notice that when $x(t)$ is a real-valued signal, then, by (2.4.16),

$$x(b) = \text{Re}\bigg(2\, c_\psi^{-1} \int_0^\infty S_x^{\text{CWT}}(\xi, b) d\xi\bigg). \tag{3.1.1.6}$$

However, for a multicomponent signal $x(t)$ in (2.2.2.3), when $A_k(t)$ and $\phi_k(t)$ satisfy certain conditions as in definition 2.2.1, each component $x_k(t)$ can be retrieved back from the WSST, i.e., for some $\Gamma > 0$

$$x_k(t) \approx \text{Re}\bigg(2\, c_\psi^{-1} \int_{|\xi - \phi_k'(b)| < \Gamma} S_x^{\text{CWT}}(\xi, b)\, d\xi\bigg). \tag{3.1.1.7}$$

3.1.2 Second-Order WSST

The second-order SST means to adapt the SST to superpositions of perturbed linear chirps. It aims to define a new approximation of the phase transformation that is associated with the second order partial derivatives of the CWT of a given signal $x(t) \in L_2(\mathbb{R})$, which means obtaining an invertible sharpened TF representation of the same quality as that obtained by the RAM. Similar to the second-order FSST derivation, let $x(t)$ be a linear chirp defined in (3.2.2.1), and without using the reassignment operators, the second-order phase transformation, denoted $\Omega_x^{2^{\text{nd}}}(a, b)$, can be defined as:

$$\Omega_x^{2^{\text{nd}}}(a,b) = \begin{cases} \text{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^\psi(a,b)}{i2\pi\, W_x^\psi(a,b)} - a\, \frac{W_x^{\mathcal{T}\psi}(a,b)}{W_x^\psi(a,b) \times J_W(a,b)} \times \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^\psi(a,b)}{i2\pi\, W_x^\psi(a,b)} \right) \right\}; \\ \quad \text{when } W_x^\psi(a,b) \neq 0 \text{ and } J_W(a,b) \neq 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^\psi(a,b)}{i2\pi\, W_x^\psi(a,b)} \right\}; \\ \quad \text{when } W_x^\psi(b,a) \neq 0 \text{ and } J_W(a,b) = 0, \end{cases} \tag{3.1.2.1}$$

where $\mathcal{T}\psi := t\psi(t)$ and $J_W(a,b) = \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^\psi(a,b)}{i2\pi\, W_x^\psi(a,b)}\right)$. Now for a signal $x(t) \in L_2(\mathbb{R})$, the second-order WSST of $x(t)$ is defined by

$$S_x^{\text{CWT}}(\xi, b) = \int_{\{a \in R_+ : W_x^\psi(a,b) \neq 0\}} W_x^\psi(a,b)\ \delta(\Omega_x^{2^{\text{nd}}}(a,b) - \xi)\, \frac{da}{a}.$$

3.2 STFT-based synchrosqueezing transform

In Chapter 2, we briefly presented the Fourier transform of a signal $x(t)$, $\widehat{x}(\xi)$, defined in (2.1.1). Then, in Section 2.3, the STFT of $x(t)$ can locally be obtained by sliding the window function u as defined in (2.3.1), which we again display its formula here

$$\mathbb{V}_x(t,\eta) = \int_{-\infty}^{\infty} x(\tau)\, u(\tau - t)\, e^{-i2\pi\eta(\tau-t)}\, d\tau. \tag{3.2.1}$$

The representation of the positive quantity $|\mathbb{V}_x(t,\eta)|^2$ in the TF plane is called the *spectrogram* of x. Note that if the window function $u(t)$ is in the **Schwartz class** - $\mathbb{S}$, the set of all functions $x \in C^{\infty}(\mathbb{R})$ with rapidly decreasing derivatives, then the STFT, $\mathbb{V}_x(t,\eta)$, of a slowly growing function $x(t)$ with $u(t)$ is well defined. A signal $x(t)$ is said to be rapidly decreasing if for any integer $N \geq 0$, there exists a constant C_N such that $|t|^N |x(t)| \leq C_N$ for all $t \in \mathbb{R}$. It is clear that $\mathbb{S}$ is closed under differentiation and multiplication by polynomials. Also, since $x \in \mathbb{S}$ are bounded and decay faster than any polynomial as $|t| \to \infty$, they are integrable, which makes sense to take their Fourier transform. When $x(t)$ has the form defined in (2.2.2.3) with some assumed slow variations on $A_k(t)$ and $\phi_k'(t)$, $x(t)$ can then be written in the following approximated form

$$x(\tau) \approx \widetilde{x}_t(\tau) = \sum_{k=1}^{N} A_k(t)\, e^{i2\pi\left(\phi_k(t)+\phi_k'(t)(\tau-t)\right)}, \tag{3.2.2}$$

for τ close to a fixed time t. Therefore, the corresponding approximation for the STFT of $x(t)$ is defined as follows

$$\mathbb{V}_x(t,\eta) \approx \mathbb{V}_{\widetilde{x}_t}(t,\eta) = \sum_{k=1}^{N} x_k(t)\, \widehat{u}(\eta - \phi_k'(t)). \tag{3.2.3}$$

Notice that the representation of $x(t)$ in the TF plane is concentrated around ridges corresponding to their instantaneous frequencies defined by $\eta = \phi_k'(t)$. Also, if $\phi_k'(t)$ for $k \in \{1, 2, ..., N\}$, are separated enough compared to the support of $\widehat{u}$, we see each mode occupies a distinct domain of the TF plane, allowing their detection, separation and reconstruction. This requires the frequency separation condition described as

- When $\operatorname{supp}\widehat{u} \subset [-\Delta, \Delta]$, a multicomponent signal x is separated if the instantaneous frequencies satisfy, for each $k \in \{1, 2, ..., N\}$, that

$$\phi_{k+1}'(t) - \phi_k'(t) > 2\Delta, \quad t \in \mathbb{R}.$$

Remark: For a multicomponent signal $x(t)$ defined in (2.2.2.3), the spectrogram of $x(t)$ is given to be in the form

$$|\mathbb{V}_x(t,\eta)|^2 = \sum_{k=1}^{N} |\mathbb{V}_{x_k}(t,\eta)|^2,$$

when the STFT's of signal components, $\mathbb{V}_{x_k}(t,\eta)$ for $k = 1, 2, ..., N$, do not overlap in the TF plane. i.e., for all (t, η)

$$\mathbb{V}_{x_k}(t,\eta)\,\mathbb{V}^*_{x_l}(t,\eta) = 0, \quad k \neq l.$$

However, in general, the spectrogram of $x(t)$ is given by

$$|\mathbb{V}_x(t,\eta)|^2 = \sum_{k=1}^{N} |\mathbb{V}_{x_k}(t,\eta)|^2 + \sum_{k=1}^{N}\sum_{\substack{l=1\\ l\neq k}}^{N} \mathbb{V}_{x_k}(t,\eta)\,\mathbb{V}^*_{x_l}(t,\eta).$$

3.2.1 First-Order FSST

First, by considering the chirp signal $x(t) = e^{i2\pi ct}$ with a constant instantaneous frequency (IF), $c > 0$, the STFT of $x(t)$, $\mathbb{V}_x(t,\eta)$, is given by

$$\mathbb{V}_x(t,\eta) = e^{i2\pi ct}\,\widehat{u}(\eta - c). \tag{3.2.1.1}$$

Now, by taking the first-order partial derivative of both sides of equation (3.2.1.1) with respect to t, the exact IF, c, of $x(t)$ can be obtained by

$$c = \frac{\frac{\partial}{\partial t}\mathbb{V}_x(t,\eta)}{i2\pi\,\mathbb{V}_x(t,\eta)}.$$

From that, we conclude that for a general signal $x(t) \in L_2(\mathbb{R})$, at (t,η) on which $\mathbb{V}_x(t,\eta) \neq 0$, the first-order phase transformation, which is the best candidate to estimate the IF, is defined by

$$\Omega_x^{1^{\text{st}}}(t,\eta) = \frac{\frac{\partial}{\partial t}\mathbb{V}_x(t,\eta)}{i2\pi\,\mathbb{V}_x(t,\eta)}. \tag{3.2.1.2}$$

The synchrosqueezing transform based on STFT (FSST) is to reassign the (complex) coefficients $\mathbb{V}_x(t,\eta)$ according to the map $(t,\eta) \longrightarrow (t, \Omega_x^{1^{\text{st}}}(t,\eta))$, which means reassigning the frequency variable η by transforming the STFT of $x(t)$, $\mathbb{V}_x(t,\eta)$, to a quantity on the time-frequency plane:

$$S_x^{\text{STFT}}(t,\xi) = \int_{\{\zeta:\,\mathbb{V}_x(t,\zeta)\neq 0\}} \mathbb{V}_x(t,\zeta)\;\delta(\Omega_x^{1^{\text{st}}}(t,\zeta) - \xi)\,d\zeta, \tag{3.2.1.3}$$

where ξ is the frequency variable. Thus, considering $u(t) \in L_2(\mathbb{R})$ with $u(0) \neq 0$, a mono-component signal $x(t) \in L_2(\mathbb{R})$ can be retrieved back by

$$x(t) = \frac{1}{u(0)}\int_{-\infty}^{\infty} S_x^{\text{STFT}}(t,\xi)\,d\xi. \tag{3.2.1.4}$$

Note that when $x(t)$ and $u(t)$ are real-valued functions, then

$$x(t) = \frac{2}{u(0)} \operatorname{Re}\left(\int_0^{\infty} S_x^{\mathrm{STFT}}(t,\xi) d\xi \right). \tag{3.2.1.5}$$

However, for a multicomponent signal $x(t)$ defined in (2.2.2.3), when $A_k(t)$ and $\phi_k(t)$ satisfy certain conditions as in definition 2.2.1, each component $x_k(t)$ can be retrieved back from the FSST, i.e., for some $\Gamma > 0$

$$x_k(t) \approx \frac{1}{u(0)} \int\limits_{|\xi - \phi_k'(t)| < \Gamma} S_x^{\mathrm{STFT}}(t,\xi) d\xi. \tag{3.2.1.6}$$

In figure 3.1, a multicomponent test signal given by

$$x(t) = x_1(t) + x_2(t) = \cos(2\pi(2t + 0.2\cos(t))) + \cos(2\pi(3t + 0.02t^2))$$

is used and sampled uniformly with 512 sample points, where $0 \le t \le 30$. The STFT of $x(t)$ and the corresponding SST are implemented with the Gaussian window function defined by $g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$, or in the FD by $\widehat{g}(\xi) = e^{-2\pi^2\sigma^2\xi^2}$, where the window width σ used with this example is selected to be 0.025.

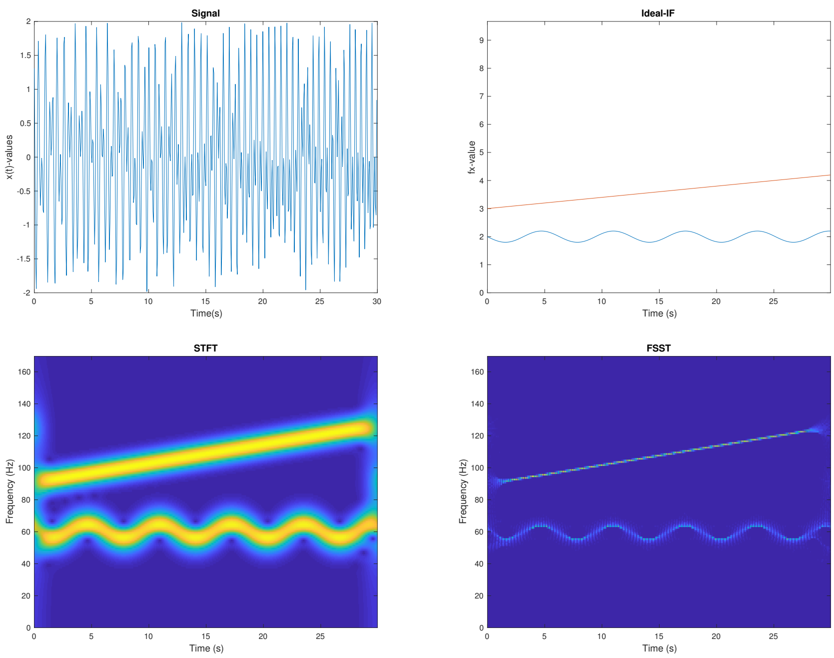

Figure 3.1: Top row: The signal $x(t)$ and the ideal IFs of $x_1(t)$ (blue) and $x_2(t)$ (red); Bottom row: The STFT and the corresponding SST (FSST).

3.2.2 Second-Order FSST

The second-order SST was introduced in [49]. It aims on how to adapt SST to a signal $x(t)$ that is a linear frequency modulation (LFM) signal (i.e., linear chirp) and defined by

$$x(t) = A(t)\, e^{i2\pi\phi(t)} = e^{pt+\frac{q}{2}t^2}\ e^{i2\pi(ct+\frac{r}{2}t^2)}, \tag{3.2.2.1}$$

with phase function $\phi(t) = c\,t + \frac{r}{2}\,t^2$, the IF $\phi'(t) = c + rt$, chirp rate $\phi''(t) = r$ and the IA $A(t) = e^{pt+\frac{q}{2}t^2}$, where $p, q \in \mathbb{R}$ and $|p|$ and $|q|$ are much smaller than c, which is positive. Note that, in [50] the reassignment operators are used to get the derivation of the second-order phase transformation of a signal $x(t) \in L_2(\mathbb{R})$, denoted $\Omega_x^{2^{\text{nd}}}(t,\eta)$, where a compact TF representation can be achieved. However, it is possible to derive the second-order phase transformation, $\Omega_x^{2^{\text{nd}}}(t,\eta)$, without using the reassignment operators, which can be defined as

$$\Omega_x^{2^{\text{nd}}}(t,\eta) = \begin{cases} \text{Re}\left\{ \frac{\frac{\partial}{\partial t}\mathbb{V}_x(t,\eta)}{i2\pi\mathbb{V}_x(t,\eta)} - J_{_F}(t,\eta) \times \frac{\mathbb{V}_x^{\mathcal{T}u}(t,\eta)}{i2\pi\mathbb{V}_x(t,\eta)} \right\}; \\ \qquad \text{when } \frac{\partial}{\partial\eta}\Big(\frac{\mathbb{V}_x^{\mathcal{T}u}(t,\eta)}{\mathbb{V}_x(t,\eta)}\Big) \neq 0 \text{ and } \mathbb{V}_x(t,\eta) \neq 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{Re}\left\{ \frac{\frac{\partial}{\partial t}\mathbb{V}_x(t,\eta)}{i2\pi\mathbb{V}_x(t,\eta)} \right\}; \\ \qquad \text{when } \frac{\partial}{\partial\eta}\Big(\frac{\mathbb{V}_x^{\mathcal{T}u}(t,\eta)}{\mathbb{V}_x(t,\eta)}\Big) = 0 \text{ and } \mathbb{V}_x(t,\eta) \neq 0, \end{cases} \tag{3.2.2.2}$$

where $\mathcal{T}u := tu(t)$ and

$$J_{_F}(t,\eta) = \frac{1}{\frac{\partial}{\partial\eta}\Big(\frac{\mathbb{V}_x^{\mathcal{T}u}(t,\eta)}{\mathbb{V}_x(t,\eta)}\Big)} \times \frac{\partial}{\partial\eta}\left(\frac{\frac{\partial}{\partial t}\mathbb{V}_x(t,\eta)}{\mathbb{V}_x(t,\eta)}\right).$$

Now for a signal $x(t) \in L_2(\mathbb{R})$, the second-order FSST of $x(t)$ is defined by

$$S_x^{\text{STFT}}(t,\xi) = \int_{\{\zeta:\mathbb{V}_x(t,\zeta)\neq 0\}} \mathbb{V}_x(t,\zeta)\ \delta(\Omega_x^{2^{\text{nd}}}(t,\zeta) - \xi)\, d\zeta.$$

CHAPTER 4

Fractional CWT-based Synchrosqueezing transform

4.1 Fractional Fourier transform

The fractional Fourier transform (FrFT) was reintroduced in the signal processing by Almeida in 1994 as a generalization of the traditional Fourier transform, which represents a given signal in the fractional Fourier domain (FrFD) [1].

Definition 4.1.1. For a signal $x(t) \in L_2(\mathbb{R})$, the α-order FrFT of $x(t)$ is defined by

$$X_\alpha(\xi) = \int_{-\infty}^{\infty} x(t)\, K_\alpha(t,\xi)\, dt. \tag{4.1.1}$$

$K_\alpha(t,\xi)$ is called the transform kernel which is given by

$$K_\alpha(t,\xi) = \begin{cases} B_\alpha\, e^{i\pi[t^2+\xi^2]\cot(\alpha) - it\xi\csc(\alpha)}, & \alpha \neq j\pi; \\ \delta(t-\xi), & \alpha = 2j\pi; \\ \delta(t+\xi), & \alpha = (2j+1)\pi, \end{cases} \tag{4.1.2}$$

where $B_\alpha = \sqrt{(1 - i\cot(\alpha))/2\pi}$ and j is an integer number. The α-order FrFT can be considered as a rotation of signal in the TF plane for an angle α. Now, by letting that $x \in L_2(\mathbb{R})$ such that its α-order FrFT $X_\alpha \in L_1(\mathbb{R})$, $x(t)$ can be retrieved back from its α-order FrFT, $X_\alpha(\xi)$. For that, we indeed want to introduce the inverse of the α-order FrFT which can be considered as a rotation for an angle $-\alpha$ and defined by

$$x(t) = \int_{-\infty}^{\infty} X_\alpha(\xi)\, \overline{K_\alpha(t,\xi)}\, d\xi, \tag{4.1.3}$$

where $\overline{K_\alpha(t,\xi)} = K_{-\alpha}(t,\xi)$. Notice that the argument ξ used here is termed the FrFD-frequency, and it is to represent a new physical quantity that is extended from the frequency concept to the FrFD-frequency where we can interpret the FrFT as the FrFD-spectrum. In short briefing, the relationship between FrFT and FT can clearly be seen by setting $a = \cot(\alpha)$, $b = \csc(\alpha)$ and $c = \sqrt{1 - i\cot(\alpha)}$ in equations (4.1.1) and (4.1.2) to have

$$X_\alpha(\xi) = e^{i\pi a\xi^2}\, \widehat{y}(b\xi), \text{ where } \widehat{y}(b\xi) \text{ is the FT of } y(t) = c\, x(t)\, e^{i\pi a t^2}.$$

Likewise, the well-known Parseval identity for FT can be extended to the FrFT as well, which will be used to write some transformations in the fractional frequency-domain and is defined by

$$\int_{-\infty}^{\infty} x(t)\, \overline{y(t)}\, dt = \int_{-\infty}^{\infty} X_\alpha(\xi)\, \overline{Y_\alpha(\xi)}\, d\xi. \tag{4.1.4}$$

4.2 Fractional continuous wavelet transform

For a signal $x(t) \in L_2(\mathbb{R})$, the α-order fractional continuous wavelet transform (FrCWT) of $x(t)$ is defined by

$$W_x^{\alpha,\psi}(a,b) = < x(t), \psi_{\alpha,a,b}(t) > = \int_{-\infty}^{\infty} x(t)\, \overline{\psi_{\alpha,a,b}(t)}\, dt, \tag{4.2.1}$$

where the α-order fractional family of wavelets $\psi_{\alpha,a,b}(t)$ is defined by multiplying the conventional family of wavelets $\psi_{a,b}(t)$ with a chirp as follows [8]

$$\psi_{\alpha,a,b}(t) = e^{-i\pi\left(t^2 - b^2 - (\frac{t-b}{a})^2\right)\,\cot(\alpha)}\, \psi_{a,b}(t). \tag{4.2.2}$$

Thus by using Parseval identity defined above in (4.1.4), the α-order FrCWT in the FrFD is given by [22]

$$W_x^{\alpha,\psi}(a,b) = A_\alpha \times \int_{-\infty}^{\infty} e^{i\pi(a\xi)^2\cot(\alpha)}\, X_\alpha(\xi)\, \overline{\Psi_\alpha(a\xi)}\, K_{-\alpha}(\xi,b)\, d\xi, \tag{4.2.3}$$

where $A_\alpha = \sqrt{2\pi/(1+i\cot(\alpha))}$. Now from equations (4.2.1) and (4.2.2), the α-order FrCWT becomes

$$W_x^{\alpha,\psi}(a,b) = \int_{-\infty}^{\infty} x(t)\, e^{i\pi\left(t^2 - b^2 - (\frac{t-b}{a})^2\right)\cot(\alpha)}\, \overline{\psi_{a,b}(t)}\, dt, \tag{4.2.4}$$

or it can be simplified and defined as

$$W_x^{\alpha,\psi}(a,b) = W_{\tilde{x}}^{\psi}(a,b)\; e^{-i\pi(a^2+1)\frac{b^2}{a^2}\cot(\alpha)}, \tag{4.2.5}$$

where

$$\widetilde{x}(t) := x(t)\; e^{\frac{i\pi}{a^2}\left((a^2-1)t^2 + 2bt\right)\cot(\alpha)}. \tag{4.2.6}$$

From equation (4.2.3), the analyzed signal $x(t)$ can be retrieved back from the α-order FrCWT as follows

$$x(t) = \frac{c_{\Psi_\alpha}^{-1}}{2\pi\sin(\alpha)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x^{\alpha,\psi}(a,b)\, \psi_{\alpha,a,b}(t)\, \frac{da}{a^2}\, db, \tag{4.2.7}$$

where

$$c_{\Psi_\alpha} = \int_0^{\infty} |\Psi_\alpha(\xi)|^2\, \frac{d\xi}{\xi} < \infty. \tag{4.2.8}$$

In the following proposition, on the other hand, it will be shown that we can verify that the original signal $x(b)$ can be retrieved back from the α-order FrCWT with integral involving only the scale variable a:

Proposition 4.2.1. *Let $W_x^{\alpha,\psi}(a,b)$ be the α-order FrCWT of a signal $x(t)$ defined in (4.2.3). Then $x(t)$ can be retrieved back by*

$$x(b) = c_{\Psi_\alpha}^{-1} \int_0^\infty W_x^{\alpha,\psi}(a,b)\, \frac{da}{a}, \tag{4.2.9}$$

where $c_{\Psi_\alpha} \neq 0$ is defined to be

$$c_{\Psi_\alpha} = A_\alpha \times \int_0^\infty e^{i\pi\eta^2 \cot(\alpha)}\, \overline{\Psi_\alpha(\eta)}\, \frac{d\eta}{\eta}. \tag{4.2.10}$$

Proof. By integrating both sides of equation (4.2.3) with integral involving only the scale variable a, we can have

$$\begin{aligned}\int_0^\infty W_x^{\alpha,\psi}(a,b)\, \frac{da}{a} &= A_\alpha \times \int_0^\infty \int_{-\infty}^\infty e^{i\pi(a\xi)^2\cot(\alpha)}\, X_\alpha(\xi)\, \overline{\Psi_\alpha(a\xi)}\, K_{-\alpha}(\xi,b)\, d\xi\, \frac{da}{a} \\ &= A_\alpha \times \int_{-\infty}^\infty X_\alpha(\xi)\, K_{-\alpha}(\xi,b) \int_0^\infty e^{i\pi(a\xi)^2\cot(\alpha)}\, \overline{\Psi_\alpha(a\xi)}\, \frac{da}{a}\, d\xi.\end{aligned}$$

Since $\overline{K_\alpha(b,\xi)} = K_{-\alpha}(\xi,b)$ and with this setting

$$\int_0^\infty e^{i\pi(a\xi)^2\cot(\alpha)}\, \overline{\Psi_\alpha(a\xi)}\, \frac{da}{a} = \int_0^\infty e^{i\pi\eta^2\cot(\alpha)}\, \overline{\Psi_\alpha(\eta)}\, \frac{d\eta}{\eta},$$

we will have

$$\begin{aligned}\int_0^\infty W_x^{\alpha,\psi}(a,b)\, \frac{da}{a} &= \int_{-\infty}^\infty X_\alpha(\xi)\, \overline{K_\alpha(b,\xi)} \left(A_\alpha \times \int_0^\infty e^{i\pi\eta^2\cot(\alpha)}\, \overline{\Psi_\alpha(\eta)}\, \frac{d\eta}{\eta} \right) d\xi \\ &= c_{\Psi_\alpha} \int_{-\infty}^\infty X_\alpha(\xi)\, \overline{K_\alpha(b,\xi)}\, d\xi \\ &= c_{\Psi_\alpha}\, x(b),\end{aligned}$$

which completes the proof of (4.2.9). □

Now, equation (4.2.4) can be rewritten as follows

$$W_x^{\alpha,\psi}(a,b) = e^{-i\pi(a^2+1)\frac{b^2}{a^2}\cot(\alpha)} \int_{-\infty}^\infty x(t)\, e^{i\pi\left(\frac{a^2-1}{a^2}\right)\cot(\alpha)\, t^2}\; e^{i2\pi\frac{b}{a^2}\cot(\alpha)\, t}\; \overline{\psi_{a,b}(t)}\, dt, \tag{4.2.11}$$

and by setting

$$\widetilde{x}(t) = x(t)\, e^{i\pi\left(\frac{a^2-1}{a^2}\right)\cot(\alpha)\, t^2} \qquad \text{and} \qquad \psi_1(t) = e^{-i2\pi\frac{b}{a^2}\cot(\alpha) t}\; \psi_{a,b}(t),$$

it becomes

$$W_x^{\alpha,\psi}(a,b) = e^{-i\pi(a^2+1)\frac{b^2}{a^2}\cot(\alpha)} \int_{-\infty}^\infty \widetilde{x}(t)\, \overline{\psi_1(t)}\, dt. \tag{4.2.12}$$

Using Parseval identity of FT will help to write (4.2.12) as follows

$$W_x^{\alpha,\psi}(a,b) = e^{-i\pi(a^2+1)\frac{b^2}{a^2}\cot(\alpha)} \int_{-\infty}^{\infty} \widehat{\widetilde{x}}(\xi)\, \overline{\widehat{\psi}_1(\xi)} d\xi, \tag{4.2.13}$$

where FT of $\psi_1(t)$, $\widehat{\psi}_1(\xi)$, is defined by

$$\widehat{\psi}_1(\xi) = e^{-i2\pi\left(\frac{b^2}{a^2}\cot(\alpha)+\xi b\right)}\, \widehat{\psi}(\frac{b}{a}\cot(\alpha) + a\xi). \tag{4.2.14}$$

Thus, equation (4.2.13) in its definitive form becomes

$$W_x^{\alpha,\psi}(a,b) = e^{-i\pi(a^2-1)\frac{b^2}{a^2}\cot(\alpha)} \int_{-\infty}^{\infty} \widehat{\widetilde{x}}(\xi)\, \overline{\widehat{\psi}(\frac{b}{a}\cot(\alpha) + a\xi)\, e^{i2\pi\xi b}}\, d\xi. \tag{4.2.15}$$

In the figure below, we use the following test signal with two components

$$x(t) = x_1(t) + x_2(t) = e^{2i\pi(30t+10t^2)} + e^{2i\pi(50t+30t^2)} \quad \text{for} \quad 0 \le t \le 1,$$

which is sampled uniformly with 512 sample points. Observe that the α-order fractional CWT of $x(t)$ defined in (4.2.15) is implemented with Morlet's wavelet using a fixed positive window width $\sigma = 0.9$, where $\mu = 1$ and the rotation angle $\alpha = \frac{9\pi}{16}$; however, by setting $\alpha = \frac{\pi}{2}$, the FrCWT will be the conventional CWT. One can clearly see that the α-order FrCWT provides more energy concentration than the conventional CWT.

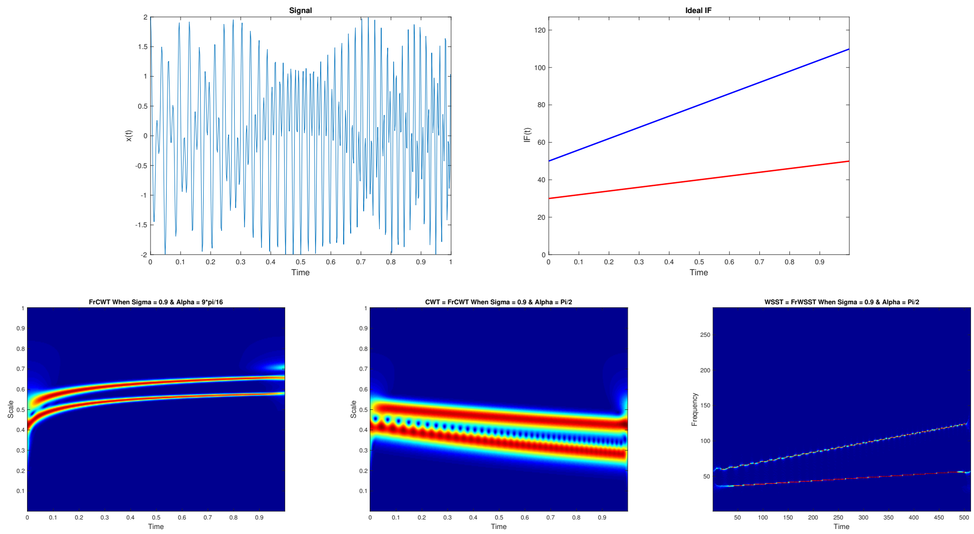

Figure 4.1: Top row: The signal $x(t)$ and the ideal IFs of $x_1(t)$ (red) and $x_2(t)$ (blue); Bottom row: FrCWT of $x(t)$ with $\sigma = 0.9$, $\mu = 1$ and $\alpha = \frac{9\pi}{16}$ (Left); Conventional CWT when $\alpha = \frac{\pi}{2}$ (Middle); SST (WSST) when $\alpha = \frac{\pi}{2}$ (Right).

4.3 FrCWT-based synchrosqueezing transform

4.3.1 First-Order FrWSST

The derivation of the first-order phase transformation can be obtained by rewriting equation (4.2.11) as follows

$$W_x^{\alpha,\psi}(a,b) = \int_{-\infty}^{\infty} x(at+b)\, e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\,\cot(\alpha)}\, \overline{\psi(t)}\, dt, \tag{4.3.1.1}$$

and by considering the chirp signal $x(t) = e^{i2\pi ct}$ with a constant frequency $c > 0$, the α-order FCrWT of $x(t)$, $W_x^{\alpha,\psi}(a,b)$, becomes

$$W_x^{\alpha,\psi}(a,b) = \int_{-\infty}^{\infty} e^{i2\pi\left((a^2-1)\cot(\alpha)\,\frac{t^2}{2}+(ac+ab\cot(\alpha))\,t+bc\right)}\, \overline{\psi(t)}\, dt. \tag{4.3.1.2}$$

Now, taking the first-order partial derivative of both sides of equation (4.3.1.2) with respect to b will lead to have

$$\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b) = i2\pi c\, W_x^{\alpha,\psi}(a,b) + i2\pi a\ \cot(\alpha)\, W_x^{\alpha,\mathcal{T}\psi}(a,b), \tag{4.3.1.3}$$

where $\mathcal{T}\psi := t\psi(t)$ and

$$W_x^{\alpha,\mathcal{T}\psi}(a,b) = \int_{-\infty}^{\infty} e^{i2\pi\left((a^2-1)\cot(\alpha)\,\frac{t^2}{2}+(ac+ab\cot(\alpha))\,t+bc\right)}\, t\,\overline{\psi(t)}\, dt. \tag{4.3.1.4}$$

Thus, by dividing both sides of equation (4.3.1.3) by $i2\pi\, W_x^{\alpha,\psi}(a,b)$, we will obtain the exact IF, c, of the chirp signal $x(t)$ which is given by

$$c = \mathrm{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{i2\pi\, W_x^{\alpha,\psi}(a,b)} - a\ \cot(\alpha)\, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} \right\}.$$

In conclusion, the first-order phase transformation, which is the best candidate to estimate the IF, of a signal $x(t) \in L_2(\mathbb{R})$ at (a,b) on which $W_x^{\alpha,\psi}(a,b) \neq 0$ is given by

$$\Omega_x^{1^{\text{st}}}(a,b) = \mathrm{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{i2\pi\, W_x^{\alpha,\psi}(a,b)} - a\ \cot(\alpha)\, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} \right\}. \tag{4.3.1.5}$$

The synchrosqueezed FrCWT (FrWSST) will reassign the scale variable a to the FrFD frequency variable for getting more sharpened time-frequency representation of a signal $x(t) \in L_2(\mathbb{R})$. Thus, the FrWSST is to transform the FrCWT, $W_x^{\alpha,\psi}(a,b)$, of $x(t)$ to a quantity, denoted $S_x^{\text{FrCWT}}(\xi,b)$, on the time-FrFD-frequency plane, and it is defined by

$$S_x^{\text{FrCWT}}(\xi,b) = \int_{\{a\in\mathbb{R}_+ : W_x^{\alpha,\psi}(a,b)\neq 0\}} W_x^{\alpha,\psi}(a,b)\ \delta(\Omega_x^{1^{\text{st}}}(a,b)-\xi)\, \frac{da}{a}, \tag{4.3.1.6}$$

where ξ is the frequency variable.

The analyzed signal $x(t) \in L_2(\mathbb{R})$ can be retrieved back from FrWSST and, therefore, for a mono-component signal and $c_{\Psi_\alpha} \neq 0$, by (4.2.9), we have

$$x(b) = c_{\Psi_\alpha}^{-1} \int_0^\infty S_x^{\text{FrCWT}}(\xi, b)\, d\xi. \tag{4.3.1.7}$$

However, for a multicomponent signal $x(t)$ in (2.2.2.3), when $A_k(t)$ and $\phi_k(t)$ satisfy certain conditions as in definition 2.2.1, each component $x_k(t)$ can be retrieved back from FrWSST, i.e., for $\Gamma > 0$

$$x_k(b) \approx c_{\Psi_\alpha}^{-1} \int_{|\xi - \phi_k'(b)| < \Gamma} S_x^{\text{FrCWT}}(\xi, b)\, d\xi. \tag{4.3.1.8}$$

4.3.2 Second-Order FrWSST

Here in this subsection, equation (4.3.1.1) will be used to derive the second-order phase transformation of a signal $x(t) \in L_2(\mathbb{R})$. This new phase transformation is associated with the second-order partial derivative of the FrCWT of a linear frequency modulation signal $x(t)$ defined in (3.2.2.1), which again be displayed here

$$x(t) = A(t)\, e^{i2\pi\phi(t)} = e^{pt + \frac{q}{2}t^2}\; e^{i2\pi(ct + \frac{r}{2}t^2)}, \tag{4.3.2.1}$$

where

$$x'(t) = \big(p + qt + i2\pi(c + rt)\big)\, x(t). \tag{4.3.2.2}$$

Therefore, by taking the first-order partial derivative of both sides of equation (4.3.1.1) with respect to b and then using equations (4.3.2.1) and (4.3.2.2), we will have

$$\begin{aligned} \frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b) &= \big((p + i2\pi c) + (q + i2\pi r)\, b\big)\; W_x^{\alpha,\psi}(a,b) \\ &\quad + a\,\big(q + i2\pi(r + \cot(\alpha))\big)\; W_x^{\alpha,\mathcal{T}\psi}(a,b). \end{aligned} \tag{4.3.2.3}$$

By dividing both sides of this above equation (4.3.2.3) by $W_x^{\alpha,\psi}(a,b)$, we have

$$\begin{aligned} \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} &= p + i2\pi c + (q + i2\pi r)\, b \\ &\quad + a\,\big(q + i2\pi(r + \cot(\alpha))\big)\; \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}, \end{aligned} \tag{4.3.2.4}$$

and then taking the first-order partial derivative of both sides with respect to a, we will have

$$\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} \right) = \big(q + i2\pi(r + \cot(\alpha))\big)\; \frac{\partial}{\partial a}\left(a\; \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} \right). \tag{4.3.2.5}$$

Thus, for $J_\alpha(a,b) = \frac{\partial}{\partial a}\big(a\ \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\big) \neq 0$, we have

$$q + i2\pi(r + cot(\alpha)) = \frac{1}{J_\alpha(a,b)} \times \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right). \tag{4.3.2.6}$$

Now, by substituting (4.3.2.6) into (4.3.2.4), we have

$$\begin{aligned}\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} &= p + qb + i2\pi(c + rb) \\ &\quad + a\ \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b) \times\ J_\alpha(a,b)}\ \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right),\end{aligned} \tag{4.3.2.7}$$

and, therefore, the exact IF, $\phi'(t) = c + rb$, of $x(t)$ given in (4.3.2.1) is defined by

$$\phi'(b) = \mathrm{Re}\left\{\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{i2\pi\, W_x^{\alpha,\psi}(a,b)} - \left(a\ \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b) \times\ J_\alpha(a,b)}\right) \times \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right)\right\}. \tag{4.3.2.8}$$

For a general signal $x(t)$, the phase transformation of $x(t)$ is defined as

$$\Omega_x^{2^{\text{nd}}}(a,b) = \begin{cases} \mathrm{Re}\left\{\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{i2\pi\, W_x^{\alpha,\psi}(a,b)} - a\ \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)\times J_\alpha(a,b)}\ \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right)\right\}; \\ \quad \text{when} \quad W_x^{\alpha,\psi}(a,b) \neq 0 \quad \text{and} \quad J_\alpha(a,b) \neq 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{Re}\left\{\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{i2\pi\ W_x^{\alpha,\psi}(a,b)} - a\ \cot(\alpha)\ \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right\}; \\ \quad \text{when} \quad W_x^{\alpha,\psi}(a,b) \neq 0 \quad \text{and} \quad J_\alpha(a,b) = 0. \end{cases} \tag{4.3.2.9}$$

Based on this derivation, we provide the following theorem where its proof is given in Appendix.

Theorem 4.3.1. *If $x(t)$ is an LFM signal given by (4.3.2.1), then at (a,b) on which $W_x^{\alpha,\psi}(a,b) \neq 0$ and $\frac{\partial}{\partial a}\big(a\ \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\big) \neq 0$, $\Omega_x^{2^{\text{nd}}}(a,b)$ defined by (4.3.2.9) is the IF of $x(t)$; namely, $\Omega_x^{2^{\text{nd}}}(a,b) = \phi'(b) = c + rb$.*

Ultimately, with this phase transformation $\Omega_x^{2^{\text{nd}}}(a,b)$ given above in (4.3.2.9), the second-order FrWSST, $S_x^{\text{FrCWT}}(\xi,b)$, of a signal $x(t) \in L_2(\mathbb{R})$ is defined by

$$S_x^{\text{FrCWT}}(\xi, b) = \int\limits_{\{a \in \mathbb{R}_+ : W_x^{\alpha,\psi}(a,b) \neq 0\}} W_x^{\alpha,\psi}(a,b)\, \delta(\Omega_x^{2^{\text{nd}}}(a,b) - \xi)\, \frac{da}{a}, \qquad (4.3.2.10)$$

where ξ is the frequency variable. For reconstructing a mono-component signal $x(t)$ or a multicomponent signal $x(t) = \sum_k x_k(t)$ from the second-order FrWSST, it can similarly be defined as that with $\Omega_x^{1^{\text{st}}}(a,b)$ for FrWSST.

CHAPTER 5

IFE-Fractional CWT-based SST

5.1 IFE-CWT-based SST

First, we open Chapter 5 by introducing new approximations of the phase transformation that are associated with the first- and the second-order partial derivatives of the IFE-CWT of a signal $x(t) \in L_2(\mathbb{R})$. They aim at achieving an accurate IF estimation when a rough estimation of the IF of a targeted component is used. Afterward, we describe another proposed work to generate a time-FrFD-frequency representation with satisfactory energy concentration; namely, the instantaneous frequency embedded fractional CWT (IFE-FrCWT), and then introducing new synchrosqueezing transforms based on the IFE-FrCWT that will be used for the purpose of enhancing the concentration of the TF representation.

5.1.1 First-Order IFE-WSST

For $x(t) \in L_2(\mathbb{R})$, we can rewrite equation (2.5.2) as follows

$$W_x^{\mathrm{E},\psi}(a,b) = \int_{-\infty}^{\infty} x(at+b)\, e^{-i2\pi\left(\varphi(at+b)-\varphi(b)-(\varphi'(b)+\xi_0)\,at-\xi_0 b\right)}\,\overline{\psi(t)}\,dt, \tag{5.1.1.1}$$

and by considering the chirp signal $x(t) = e^{i2\pi ct}$ with a constant frequency $c > 0$, the IFE-CWT of $x(t)$ becomes

$$W_x^{\mathrm{E},\psi}(a,b) = \int_{-\infty}^{\infty} e^{-i2\pi\varphi(at+b)+i2\pi(\varphi'(b)+\xi_0+c)\,at+i2\pi(\varphi(b)+\xi_0 b+cb)}\,\overline{\psi(t)}\,dt. \tag{5.1.1.2}$$

Thus from (5.1.1.2), we will have

$$\begin{aligned}\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b) = {}& -i2\pi\, W_{x\varphi'}^{\mathrm{E},\psi}(a,b) + i2\pi a\,\varphi''(b)\; W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)\\ &+ i2\pi(\varphi'(b)+\xi_0+c)\;\; W_x^{\mathrm{E},\psi}(a,b).\end{aligned} \tag{5.1.1.3}$$

Therefore, at (a,b) on which $W_x^{\mathrm{E},\psi}(a,b) \neq 0$, the exact IF, c, of the chirp signal $x(t)$ is given by

$$c = \mathrm{Re}\left\{\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} - a\,\varphi''(b)\,\frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right\} - \varphi'(b) - \xi_0.$$

Thus, for a general signal $x(t)$, the first-order phase transformation at (a,b) on which $W_x^{\mathrm{E},\psi}(a,b) \neq 0$ is defined to be

$$\Omega_x^{1^{\mathrm{st}}}(a,b) = \mathrm{Re}\left\{\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{i2\pi W_x^{\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} - a\,\varphi''(b)\,\frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right\} - \varphi'(b) - \xi_0. \tag{5.1.1.4}$$

Eventually, the IFE synchrosqueezing transform (IFE-WSST) of $x(t)$, denoted $S_x^{\mathrm{E-CWT}}(\xi, b)$, is defined by

$$S_x^{\mathrm{E-CWT}}(\xi, b) = \int\limits_{\{a\in\mathbb{R}_+ : W_x^{\mathrm{E},\psi}(a,b)\neq 0\}} W_x^{\mathrm{E},\psi}(a, b)\ \delta(\Omega_x^{1^{\mathrm{st}}}(a, b) - \xi)\ \frac{da}{a}, \tag{5.1.1.5}$$

where ξ is the frequency variable.

Now, with equation (2.5.5), the input signal $x(t)$ can be retrieved back from the IFE-WSST as follows

$$x(b) = c_\psi^{-1}\ e^{-i2\pi\xi_0 b} \int_0^\infty S_x^{\mathrm{E-CWT}}(\xi, b)\, d\xi. \tag{5.1.1.6}$$

In addition, the k$^{\text{th}}$ component $x_k(t)$ of $x(t)$ defined in (2.2.2.3), which satisfying certain conditions, can be retrieved back from the IFE-WSST as follows

$$x_k(b) \approx c_\psi^{-1}\ e^{-i2\pi\xi_0 b} \int\limits_{|\xi-\phi_k'(b)|<\Gamma} S_x^{\mathrm{E-CWT}}(\xi, b)\, d\xi, \tag{5.1.1.7}$$

for certain $\Gamma > 0$.

5.1.2 Second-Order IFE-WSST

The major idea is to define a new phase transformation $\Omega_x^{2^{\mathrm{nd}}}$ which is associated with the second-order partial derivative of the IFE-CWT of a signal $x(t) \in L_2(\mathbb{R})$. Notice that the derivation of the phase transformation and the provided formulation for $\Omega_x^{2^{\mathrm{nd}}}$ are slightly different not only from that was introduced in [47], where it was based on reassignment operators, but also from that was presented in [30]. Now, for a linear frequency modulation signal $x(t)$ defined in (3.2.2.1), taking the first-order partial derivative of both sides of equation (5.1.1.1) with respect to b gives that

$$\begin{aligned}\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a, b) &= \big(p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb)\big)\, W_x^{\mathrm{E},\psi}(a, b)\\ &\quad + \big(q + i2\pi(r + \varphi''(b))\big)\, a\, W_x^{\mathrm{E},\mathcal{T}\psi}(a, b)\\ &\quad - i2\pi\, W_{x\varphi'}^{\mathrm{E},\psi}(a, b).\end{aligned} \tag{5.1.2.1}$$

Thus at (a, b) on which $W_x^{\mathrm{E},\psi}(a, b) \neq 0$, we have

$$\begin{aligned}\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} &= p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb) \\ &+ (q + i2\pi(r + \varphi''(b)))\, a\, \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} \\ &- i2\pi\, \frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\end{aligned} \tag{5.1.2.2}$$

Now, taking partial derivative $\frac{\partial}{\partial a}$ to both sides of this above equation leads to have

$$\begin{aligned}\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) &= \big(q + i2\pi(r + \varphi''(b))\big)\, \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) \\ &- i2\pi\, \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right).\end{aligned} \tag{5.1.2.3}$$

Thus we see that

$$\begin{aligned}q + i2\pi(r + \varphi''(b)) = \frac{1}{J^{\mathrm{E}}(a,b)} \times \Bigg[&\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) \\ &+ i2\pi\, \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right)\Bigg],\end{aligned} \tag{5.1.2.4}$$

where $J^{\mathrm{E}}(a,b) = \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) \neq 0$. Now by substituting (5.1.2.4) into (5.1.2.2), we will have

$$\begin{aligned}\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} &= p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb) \\ &- i2\pi\, \frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} + a\, \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)\, \cdot\, J^{\mathrm{E}}(a,b)} \times \\ &\left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) + i2\pi \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right)\right]\end{aligned} \tag{5.1.2.5}$$

Therefore, the exact IF, $\phi'(b) = c + rb$, of the linear frequency modulation $x(t)$ given in (3.2.2.1) is defined by

$$\begin{aligned}\phi'(b) = c + rb = \mathrm{Re}\Bigg\{&\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{i2\pi\ W_x^{\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} \\ &- \left(\left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\mathrm{E},\psi}(a,b)}\right) + \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right)\right]\right. \\ &\left.\times\, a\, \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)\ \times\ J^{\mathrm{E}}(a,b)}\right)\Bigg\} - \varphi'(b) - \xi_0.\end{aligned} \tag{5.1.2.6}$$

For a general signal $x(t)$, the phase transformation of $x(t)$ is defined by

$$\Omega_x^{2^{\text{nd}}}(a,b) = \begin{cases} \text{Re}\Bigg\{ \frac{\frac{\partial}{\partial b} W_x^{\text{E},\psi}(a,b)}{i2\pi\, W_x^{\text{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\text{E},\psi}(a,b)}{W_x^{\text{E},\psi}(a,b)} - \\ \left(\left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\text{E},\psi}(a,b)}{i2\pi W_x^{\text{E},\psi}(a,b)} \right) + \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\text{E},\psi}(a,b)}{W_x^{\text{E},\psi}(a,b)} \right) \right] \right. \\ \left. \times a\, \frac{W_x^{\text{E},\mathcal{T}\psi}(a,b)}{W_x^{\text{E},\psi}(a,b) \times J^{\text{E}}(a,b)} \right) \Bigg\} - \varphi'(b) - \xi_0; \\ \quad \text{when } W_x^{\text{E},\psi}(a,b) \neq 0 \text{ and } J^{\text{E}}(a,b) \neq 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{Re}\Bigg\{ \frac{\frac{\partial}{\partial b} W_x^{\text{E},\psi}(a,b)}{i2\pi W_x^{\text{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\text{E},\psi}(a,b)}{W_x^{\text{E},\psi}(a,b)} - a\varphi''(b) \\ \times \frac{W_x^{\text{E},\mathcal{T}\psi}(a,b)}{W_x^{\text{E},\psi}(a,b)} \Bigg\} - \varphi'(b) - \xi_0; \\ \quad \text{when } W_x^{\text{E},\psi}(a,b) \neq 0 \text{ and } J^{\text{E}}(a,b) = 0. \end{cases} \tag{5.1.2.7}$$

Thus, we provide the following theorem where its proof is given in Appendix.

Theorem 5.1.1. *If $x(t)$ is an LFM signal given by (4.3.2.1), then at (a,b) on which $W_x^{E,\psi}(a,b) \neq 0$ and $\frac{\partial}{\partial a}\big(a\, \frac{W_x^{E,\mathcal{T}\psi}(a,b)}{W_x^{E,\psi}(a,b)}\big) \neq 0$, $\Omega_x^{2^{\text{nd}}}(a,b)$ defined by (5.1.2.7) is the IF of $x(t)$; namely, $\Omega_x^{2^{\text{nd}}}(a,b) = \phi'(b) = c + rb$.*

Ultimately, with this phase transformation $\Omega_x^{2^{\text{nd}}}(a,b)$ defined above in (5.1.2.7), the second-order IEF-WSST, $S_x^{\text{E}-\text{CWT}}(\xi,b)$, of a signal $x(t) \in L_2(\mathbb{R})$ is defined as in (5.1.1.5)

$$S_x^{\text{E}-\text{CWT}}(\xi,b) = \int_{\{a\in\mathbb{R}_+ : W_x^{\text{E},\psi}(a,b)\neq 0\}} W_x^{\text{E},\psi}(a,b)\ \delta(\Omega_x^{2^{\text{nd}}}(a,b) - \xi)\, \frac{da}{a}, \tag{5.1.2.8}$$

where ξ is the frequency variable. Reconstructing a mono-component signal $x(t)$ or multicomponent signal $x(t) = \sum_k x_k(t)$ from the second-order IEF-WSST will be similar to that were defined in (5.1.1.6) and (5.1.1.7) respectively with $\Omega_x^{1^{\text{st}}}(a,b)$ for IEF-WSST.

5.2 Instantaneous frequency-embedded FrCWT

For a signal $x(t) \in L_2(\mathbb{R})$, we consider the generalized signal form given by

$$x_{\varphi,b,\xi_0}(t) = x(t)\ e^{-i2\pi(\varphi(t)-\varphi(b)-\varphi'(b)(t-b)-\xi_0 t)}, \tag{5.2.1}$$

where $\varphi(t)$ is a differentiable function with $\varphi' > 0$ and $\xi_0 > 0$. Then the α-order instantaneous frequency embedded fractional continuous wavelet transform (IFE-FrCWT), denoted $W_x^{\alpha,\mathrm{E},\psi}(a,b)$, is defined as follows

$$W_x^{\alpha,\mathrm{E},\psi}(a,b) = < x_{\varphi,b,\xi_0}(t), \psi_{\alpha,a,b}(t) >= \int_{-\infty}^{\infty} x_{\varphi,b,\xi_0}(t)\, \overline{\psi_{\alpha,a,b}(t)}\, dt, \tag{5.2.2}$$

where $\psi_{\alpha,a,b}$ is defined in (4.2.2). Thus, from equations (5.2.1) and (5.2.2), the α-order IFE-FrCWT of $x(t)$ can be written in the following form

$$W_x^{\alpha,\mathrm{E},\psi}(a,b) = e^{i2\pi\varphi(b)-i\pi(1+a^2)\frac{b^2}{a^2}\cot(\alpha)-i2\pi b\,\varphi'(b)} \int_{-\infty}^{\infty} \widetilde{x}(t)\, \overline{\widetilde{\psi}(t)}\, dt, \tag{5.2.3}$$

where

$$\widetilde{x}(t) = x(t)\, e^{-i2\pi\varphi(t)+i\pi\left(1-\frac{1}{a^2}\right)\cot(\alpha)\, t^2}, \tag{5.2.4}$$

and

$$\widetilde{\psi}(t) = e^{-i2\pi(\xi_0+\frac{b}{a^2}\cot(\alpha)+\varphi'(b))\, t}\ \psi_{a,b}(t). \tag{5.2.5}$$

Now, the FT of $\widetilde{\psi}(t)$ is given by

$$\widehat{\widetilde{\psi}}(\xi) = e^{-i2\pi(\xi_0+\xi+\frac{b}{a^2}\cot(\alpha)+\varphi'(b))\, b}\ \widehat{\psi}(a\xi + a\xi_0 + a\varphi'(b) + \frac{b}{a}\cot(\alpha)), \tag{5.2.6}$$

and by using Parseval identity of FT, the α-order IFE-FrCWT is defined by

$$\begin{aligned} W_x^{\alpha,\mathrm{E},\psi}(a,b) =& e^{i2\pi(\varphi(b)+\xi_0 b+\left(\frac{1-a^2}{2a^2}\right) b^2\cot(\alpha))} \\ &\times \int_{-\infty}^{\infty} \widehat{\widetilde{x}}(\xi)\, \overline{\widehat{\psi}(a\xi + a\xi_0 + a\varphi'(b) + \frac{b}{a}\cot(\alpha))}\, e^{i2\pi\xi b}\, d\xi. \end{aligned} \tag{5.2.7}$$

Since the α-order FrFT of $x_{\varphi,b,\xi_0}(t)$ given by (5.2.1) can be defined as

$$X_{\alpha;\varphi,b,\xi_0}(\xi) = \int_{-\infty}^{\infty} x_{\varphi,b,\xi_0}(t)\, K_\alpha(t,\xi)dt, \tag{5.2.8}$$

and by using Parseval identity of FrFT defined in (4.1.4), we provide the following proposition to define the α-order IFE-FrCWT in the FrFD.

Proposition 5.2.1. *Let $W_x^{\alpha,E,\psi}(a,b)$ be the α-order IFE-FrCWT of a signal $x(t)$ defined in (5.2.2). Then, for $A_\alpha = \sqrt{2\pi/(1+i\cot(\alpha))}$, the α-order IFE-FrCWT in the FrFD is given by*

$$W_x^{\alpha,E,\psi}(a,b) = A_\alpha \times \int_{-\infty}^{\infty} e^{i\pi(a\xi)^2\cot(\alpha)}\, X_{\alpha;\varphi,b,\xi_0}(\xi)\, \overline{\Psi_\alpha(a\xi)}\, K_{-\alpha}(\xi,b)\, d\xi. \tag{5.2.9}$$

Proof. By taking the α-order FrFT on both sides of equation (4.2.2), we have

$$\begin{aligned}\Psi_{\alpha,a,b}(\xi) &= \int_{-\infty}^{\infty} \psi_{\alpha,a,b}(t)\, K_{\alpha}(t,\xi)\, dt \\ &= \int_{-\infty}^{\infty} e^{-i\pi(t^2-b^2-(\frac{t-b}{a})^2)\cot(\alpha)}\ \psi_{a,b}(t)\ B_\alpha\ e^{i\pi(t^2+\xi^2)\cot(\alpha)-it\xi\csc(\alpha)}\, dt \\ &= \int_{-\infty}^{\infty} \frac{1}{a}\,\psi\big(\frac{t-b}{a}\big)\ B_\alpha\ e^{i\pi(b^2+\xi^2)\cot(\alpha)-it\xi\csc(\alpha)}\ e^{i\pi(\frac{t-b}{a})^2\cot(\alpha)}\, dt,\end{aligned}$$

which we can rewrite it as follows

$$\begin{aligned}\Psi_{\alpha,a,b}(\xi) &= \frac{1}{B_\alpha}\int_{-\infty}^{\infty}\Bigg(\frac{1}{a}\,\psi\big(\frac{t-b}{a}\big)\ B_\alpha\ e^{i\pi(b^2+\xi^2)\cot(\alpha)-i\left(\frac{t-b}{a}+\frac{b}{a}\right)a\xi\csc(\alpha)} \\ &\qquad\qquad B_\alpha\ e^{i\pi\left((\frac{t-b}{a})^2+(a\xi)^2-(a\xi)^2\right)\cot(\alpha)}\Bigg)\, dt \\ &= \frac{e^{-i\pi(a\xi)^2\cot(\alpha)}}{B_\alpha}\int_{-\infty}^{\infty}\Bigg(\frac{1}{a}\,\psi\big(\frac{t-b}{a}\big)\ B_\alpha\ e^{i\pi\left((\frac{t-b}{a})^2+(a\xi)^2\right)\cot(\alpha)-i(\frac{t-b}{a})\,a\xi\csc(\alpha)} \\ &\qquad\qquad B_\alpha\ e^{i\pi(b^2+\xi^2)\cot(\alpha)-ib\xi\csc(\alpha)}\Bigg)\, dt \\ &= \frac{e^{-i\pi(a\xi)^2\cot(\alpha)}}{B_\alpha}\int_{-\infty}^{\infty}\psi\big(\frac{t-b}{a}\big)\ B_\alpha\ e^{i\pi\left((\frac{t-b}{a})^2+(a\xi)^2\right)\cot(\alpha)-i(\frac{t-b}{a})\,a\xi\csc(\alpha)} \\ &\qquad\qquad K_\alpha(b,\xi)\, d\big(\frac{t-b}{a}\big) \\ &= \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)}\int_{-\infty}^{\infty}\psi(t)\ B_\alpha\ e^{i\pi\left(t^2+(a\xi)^2\right)\cot(\alpha)-it\,a\xi\csc(\alpha)}\, dt \\ &= \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)}\int_{-\infty}^{\infty}\psi(t)\ K_\alpha(t,a\xi)\, dt \\ &= \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\,\Psi_\alpha(a\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)}.\end{aligned}$$

From the Parseval identity of the FrFT defined in (4.1.4) and then substituting $\Psi_{\alpha,a,b}(\xi) = \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\,\Psi_\alpha(a\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)}$ into (5.2.2), we will have

$$\begin{aligned}W_x^{\alpha,E,\psi}(a,b) &= < X_{\alpha;\varphi,b,\xi_0}(\xi),\, \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\,\Psi_{\sigma(b);\alpha}(a\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)} > \\ &= \frac{1}{\overline{B}_\alpha}\times\int_{-\infty}^{\infty} e^{i\pi(a\xi)^2\cot(\alpha)}\ X_{\alpha;\varphi,b,\xi_0}(\xi)\,\overline{\Psi_\alpha(a\xi)}\ \overline{K_\alpha(b,\xi)}\, d\xi \\ &= A_\alpha\times\int_{-\infty}^{\infty} e^{i\pi(a\xi)^2\cot(\alpha)}\ X_{\alpha;\varphi,b,\xi_0}(\xi)\,\overline{\Psi_\alpha(a\xi)}\, K_{-\alpha}(b,\xi)\, d\xi.\end{aligned}$$

This proves (5.2.9). □

In the following proposition, we will see that the original signal $x(b)$ can be retrieved back from its α-order IFE-FrCWT with integral involving only the scale variable a.

Proposition 5.2.2. *Let $W_x^{\alpha,E,\psi}(a,b)$ be the α-order IFE-FrCWT of a signal $x(t)$ defined in (5.2.9). Then $x(t)$ can be retrieved back by*

$$x(b) = c_{\Psi_\alpha}^{-1}\, e^{-i2\pi\xi_0 b} \int_0^\infty W_x^{\alpha,\mathrm{E},\psi}(a,b)\, \frac{da}{a}, \tag{5.2.10}$$

where $c_{\Psi_\alpha} \neq 0$ is as defined in (4.2.10).

Proof. Integrating both sides of equation (5.2.9) with integral involving only the scale variable a leads to have

$$\begin{aligned}
\int_0^\infty W_x^{\alpha,E,\psi}(a,b)\,\frac{da}{a} &= A_\alpha \times \int_0^\infty \int_{-\infty}^\infty e^{i\pi(a\xi)^2\cot(\alpha)}\, X_{\alpha;\varphi,b,\xi_0}(\xi)\, \overline{\Psi_\alpha(a\xi)}\, K_{-\alpha}(\xi,b)\, d\xi\, \frac{da}{a} \\
&= A_\alpha \times \int_{-\infty}^\infty X_{\alpha;\varphi,b,\xi_0}(\xi)\, K_{-\alpha}(\xi,b) \int_0^\infty e^{i\pi(a\xi)^2\cot(\alpha)}\, \overline{\Psi_\alpha(a\xi)}\, \frac{da}{a}\, d\xi \\
&= \int_{-\infty}^\infty X_{\alpha;\varphi,b,\xi_0}(\xi)\, \overline{K_\alpha(b,\xi)} \left(A_\alpha \times \int_0^\infty e^{i\pi\eta^2\cot(\alpha)}\, \overline{\Psi_\alpha(\eta)}\, \frac{d\eta}{\eta} \right) d\xi \\
&= c_{\Psi_\alpha} \int_{-\infty}^\infty X_{\alpha;\varphi,b,\xi_0}(\xi)\, \overline{K_\alpha(b,\xi)}\, d\xi \\
&= c_{\Psi_\alpha}\, x_{\varphi,b,\xi_0}(b).
\end{aligned}$$

From equation (5.2.1), $x_{\varphi,b,\xi_0}(b) = x(b)\, e^{i2\pi\xi_0 b}$ and then we have

$$\int_0^\infty W_x^{\alpha,E,\psi}(a,b)\, \frac{da}{a} = c_{\Psi_\alpha}\, x(b)\, e^{i2\pi\xi_0 b},$$

which completes the proof of (5.2.10). □

5.3 IFE-FrCWT-based SST

5.3.1 First-Order IFE-FrWSST

We can rewrite the α-order IFE-FrCWT of $x(t)$, $W^{\alpha,\mathrm{E},\psi}(a,b)$, defined in (5.2.2) as follows

$$\begin{aligned}
W_x^{\alpha,\mathrm{E},\psi}(a,b) = \int_{-\infty}^\infty &\{e^{i2\pi(\varphi(b)+\xi_0 b) - i2\pi\varphi(at+b) + i2\pi(\xi_0 + b\cot(\alpha) + \varphi'(b))\,at}\} \\
&\times e^{i\pi(a^2-1)\cot(\alpha)\,t^2} x(at+b)\, \overline{\psi(t)}\, dt,
\end{aligned} \tag{5.3.1.1}$$

and by considering the chirp signal $x(t) = e^{i2\pi ct}$ with a constant frequency $c > 0$, the α-order IFE-FrCWT of $x(t)$ becomes

$$W_x^{\alpha,\mathrm{E},\psi}(a,b) = \int_{-\infty}^{\infty} \{e^{-i2\pi(\varphi(at+b)-(\xi_0+b\cot(\alpha)+\varphi'(b)+c)at-\varphi'(b)-\xi_0 b-cb)}\} \times e^{i\pi(a^2-1)\cot(\alpha)\,t^2}\,\overline{\psi(t)}\,dt. \tag{5.3.1.2}$$

Now, the first-order phase transformation derivation can be found by taking the first-order partial derivative of both sides of this above equation (5.3.1.2) with respect to b; i.e., we can get

$$\begin{aligned}\frac{\partial}{\partial b}\,W_x^{\alpha,\mathrm{E},\psi}(a,b) = &-i2\pi\,W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)\\ &+ i2\pi\,a(\varphi''(b)+\cot(\alpha))\,W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)\\ &+ i2\pi(\varphi'(b)+\xi_0+c)\,W_x^{\alpha,\mathrm{E},\psi}(a,b),\end{aligned} \tag{5.3.1.3}$$

where $\mathcal{T}\psi := t\psi(t)$. Thus, if we divide both sides of this above equation (5.3.1.3) by $W_x^{\alpha,\mathrm{E},\psi}(a,b) \neq 0$, the exact IF, c, of the chirp signal $x(t)$ can be extracted as follows

$$\begin{aligned}c = \mathrm{Re}\Bigg\{&\frac{\frac{\partial}{\partial b}\,W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\,W_x^{\alpha,\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\\ &- a(\varphi''(b)+\cot(\alpha))\,\frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\Bigg\} - \varphi'(b) - \xi_0.\end{aligned} \tag{5.3.1.4}$$

Ultimately, the first-order phase transformation of a signal $x(t) \in L_2(\mathbb{R})$ at (a,b) on which $W_x^{\alpha,\mathrm{E},\psi}(a,b) \neq 0$ is given by

$$\begin{aligned}\Omega_x^{1^{\mathrm{st}}}(a,b) = \mathrm{Re}\Bigg\{&\frac{\frac{\partial}{\partial b}\,W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\,W_x^{\alpha,\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\\ &- a(\varphi''(b)+\cot(\alpha))\,\frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\Bigg\} - \varphi'(b) - \xi_0.\end{aligned} \tag{5.3.1.5}$$

Thus the first-order IFE-FrWSST, denoted $S_x^{\mathrm{E-FrCWT}}(\xi,b)$, on the time-FrFD-frequency plane is defined by

$$S_x^{\mathrm{E-FrCWT}}(\xi,b) = \int\limits_{\{a\in\mathbb{R}_+ : W_x^{\alpha,\mathrm{E},\psi}(a,b)\neq 0\}} W_x^{\alpha,\mathrm{E},\psi}(a,b)\;\delta(\Omega_x^{1^{\mathrm{st}}}(a,b)-\xi)\,\frac{da}{a}, \tag{5.3.1.6}$$

where ξ is the frequency variable.

It is very clear now to see that the analyzed signal $x(t) \in L_2(\mathbb{R})$ can be retrieved back from the first-order IFE-FrWSST. Thus for a mono-component signal and $c_{\Psi_\alpha} \neq 0$, by (5.2.10), we have

$$x(b) = c_{\Psi_\alpha}^{-1}\; e^{-i2\pi\xi_0 b} \int_0^\infty S_x^{\mathrm{E-FrCWT}}(\xi, b)\, d\xi. \tag{5.3.1.7}$$

However, for a multicomponent signal $x(t)$ defined in (2.2.2.3), when $A_k(t)$ and $\phi_k(t)$ satisfy certain conditions as in definition 2.2.1, each component $x_k(t)$ can be retrieved back from the first-order IFE-FrWSST, i.e., for $\Gamma > 0$

$$x_k(b) \approx c_{\Psi_\alpha}^{-1}\; e^{-i2\pi\xi_0 b} \int\limits_{|\xi-\phi_k'(b)|<\Gamma} S_x^{\mathrm{E-FrCWT}}(\xi, b)\, d\xi. \tag{5.3.1.8}$$

5.3.2 Second-Order IFE-FrWSST

The second-order IFE-FrWSST is considered in this subsection, where equation (5.3.1.1) is displayed here

$$\begin{aligned} W_x^{\alpha,\mathrm{E},\psi}(a,b) = \int_{-\infty}^{\infty} &\{ e^{i2\pi(\varphi(b)+\xi_0 b)-i2\pi\varphi(at+b)+i2\pi(\xi_0+b\,\cot(\alpha)+\varphi'(b))\,at} \} \\ &\times e^{i\pi(a^2-1)\,\cot(\alpha)\,t^2}\, x(at+b)\,\overline{\psi(t)}\,dt, \end{aligned} \tag{5.3.2.1}$$

and then used to derive the second-order phase transformation of a signal $x(t) \in L_2(\mathbb{R})$. This new phase transformation is associated with the second-order partial derivative of the α-order IFE-FrCWT of a linear frequency modulation signal $x(t)$ defined in (3.2.2.1), which also displayed here

$$x(t) = A(t)\, e^{i2\pi\phi(t)} = e^{pt+\frac{q}{2}t^2}\; e^{i2\pi(ct+\frac{r}{2}t^2)}, \tag{5.3.2.2}$$

where

$$x'(t) = \big(p + qt + i2\pi(c+rt)\big)\, x(t). \tag{5.3.2.3}$$

Now, by taking the first-order partial derivative of both sides of the equation (5.3.2.1) with respect to b and then using equations (5.3.2.2) and (5.3.2.3), we will have

$$\begin{aligned} \frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b) = &\big(p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb)\big)\, W_x^{\alpha,\mathrm{E},\psi}(a,b) \\ &+ \big((q + i2\pi r) + i2\pi(\cot(\alpha) + \varphi''(b))\big)\, a\, W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b) \\ &- i2\pi\, W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b). \end{aligned} \tag{5.3.2.4}$$

Dividing both sides of this above equation (5.3.2.4) by $W_x^{\alpha,\mathrm{E},\psi}(a,b)$ will lead to have

$$\begin{aligned}\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} &= p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb) \\ &+ \big((q + i2\pi r) + i2\pi(\cot(\alpha) + \varphi''(b))\big)\, a\, \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \\ &- i2\pi\, \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)},\end{aligned} \tag{5.3.2.5}$$

and then by taking the first-order partial derivative of both sides with respect to a, we get

$$\begin{aligned}\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right) &= \big((q + i2\pi r) + i2\pi(\cot(\alpha) + \varphi''(b))\big)\, \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right) \\ &- i2\pi\, \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right)\end{aligned} \tag{5.3.2.6}$$

Thus we see that

$$\begin{aligned}q + i2\pi r + i2\pi(\cot(\alpha) + \varphi''(b)) = \frac{1}{J_\alpha^{\mathrm{E}}(a,b)} \times \Bigg[&\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right) \\ &+ i2\pi\, \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right)\Bigg]\end{aligned} \tag{5.3.2.7}$$

where $J_\alpha^{\mathrm{E}}(a,b) = \frac{\partial}{\partial a}\big(a\, \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\big) \neq 0$. Now by substituting (5.3.2.7) into (5.3.2.5), we will have

$$\begin{aligned}\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} &= p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb) \\ &+ \left(\left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right) + i2\pi\, \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right)\right]\right. \\ &\left.\times\, a\, \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b) \times J_\alpha^{\mathrm{E}}(a,b)}\right) + i2\pi\, \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)},\end{aligned} \tag{5.3.2.8}$$

and then the exact IF, $\phi'(t)$, of $x(t)$ given in (5.3.2.2) is defined by

$$\begin{aligned}\phi'(b) = c + rb = \mathrm{Re}\Bigg\{&\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \\ &- \left(\left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right) + \frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right)\right]\right. \\ &\left.\times\, a\, \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b) \times J_\alpha^{\mathrm{E}}(a,b)}\right)\Bigg\} - \varphi'(b) - \xi_0.\end{aligned} \tag{5.3.2.9}$$

For a general signal $x(t)$, the phase transformation of $x(t)$ is defined as

$$\Omega_x^{2^{\mathrm{nd}}}(a,b) = \begin{cases} \mathrm{Re}\Bigg\{ \frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} - \\ \quad \Bigg(\Bigg[\frac{\partial}{\partial a}\Bigg(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)} \Bigg) + \frac{\partial}{\partial a}\Bigg(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \Bigg) \Bigg] \\ \quad \times a\, \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b) \times J_\alpha^{\mathrm{E}}(a,b)} \Bigg) \Bigg\} - \varphi'(b) - \xi_0; \\ \quad \text{when } W_x^{\alpha,\mathrm{E},\psi}(a,b) \neq 0 \text{ and } J_\alpha^{\mathrm{E}}(a,b) \neq 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{Re}\Bigg\{ \frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} - a(\varphi''(b) + \cot(\alpha)) \\ \quad \times \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \Bigg\} - \varphi'(b) - \xi_0; \\ \quad \text{when } W_x^{\alpha,\mathrm{E},\psi}(a,b) \neq 0 \text{ and } J_\alpha^{\mathrm{E}}(a,b) = 0. \end{cases} \tag{5.3.2.10}$$

Thus, from this above derivation we provide the following theorem where its proof is given in Appendix.

Theorem 5.3.1. *If x(t) is an LFM signal given by (4.3.2.1), then at (a,b) on which* $W_x^{\alpha,\mathrm{E},\psi}(a,b) \neq 0$ *and* $\frac{\partial}{\partial a}\big(a\, \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\big) \neq 0$, $\Omega_x^{2^{\mathrm{nd}}}(a,b)$ *defined by (5.3.2.10) is the IF of x(t); namely,* $\Omega_x^{2^{\mathrm{nd}}}(a,b) = \phi'(b) = c + rb$.

Eventually, with this phase transformation $\Omega_x^{2^{\mathrm{nd}}}(a,b)$ defined above in (5.3.2.10), the second-order IEF-FrWSST, $S_x^{\mathrm{E-FrCWT}}(\xi,b)$, of a signal $x(t) \in L_2(\mathbb{R})$ is defined as in (5.3.1.6)

$$S_x^{\mathrm{E-FrCWT}}(\xi,b) = \int\limits_{\{a\in\mathbb{R}_+ : W_x^{\alpha,\mathrm{E},\psi}(a,b)\neq 0\}} W_x^{\alpha,\mathrm{E},\psi}(a,b)\ \delta(\Omega_x^{2^{\mathrm{nd}}}(a,b) - \xi)\ \frac{da}{a}, \tag{5.3.2.11}$$

where ξ is the frequency variable. Reconstructing a mono-component signal $x(t)$ or multicomponent signal $x(t) = \sum_k x_k(t)$ back from the second-order IEF-FrWSST is similar to those were defined in (5.3.1.7) and (5.3.1.8) respectively with $\Omega_x^{1^{\mathrm{st}}}(a,b)$ for IEF-FrWSST.

CHAPTER 6

Adaptive FrCWT - Based SST

In a recent study, a time-varying window width was adapted to the SST based on the STFT, where minimizing the Rényi entropy of the SST is the used way to select the width of the window [55]. Later on, Li, Cai and Jiang in [35] proposed the adaptive CWT and its corresponding SST's with a time-varying parameter. As known in the literature on SST, the common used continuous wavelets for the SST based on the CWT are the bump wavelet $\psi_{\text{bump}}(t)$ defined by

$$\widehat{\psi}_{\text{bump}}(\xi) = e^{1-\frac{1}{1-\sigma^2(\xi-\mu)^2}} \chi_{(\mu-\frac{1}{\sigma},\mu+\frac{1}{\sigma})}(\xi) \quad \text{with} \quad \sigma\mu > 1, \tag{6.1}$$

and the Morlet's wavelet $\psi_{\text{Morlet}}(t)$ defined by

$$\widehat{\psi}_{\text{Morlet}}(\xi) = e^{-2\sigma^2\pi^2(\xi-\mu)^2} - e^{-2\sigma^2\pi^2(\xi+\mu)^2}, \tag{6.2}$$

where $\sigma > 0$ and $\mu > 0$. In these wavelets, the parameter, σ, controls the width of the TF localization window, and it is constantly selected to be a fixed positive constant. However, the authors in [35] considered a time-varying parameter, $\sigma(t)$, in their study to define the CWT (called the adaptive CWT) and its corresponding SST's for IF estimation and multicomponent signal separation.

6.1 Adaptive CWT

We briefly review the CWT with a time-varying parameter in this section by considering the continuous wavelets of the form

$$\psi_\sigma(t) = \frac{1}{\sigma}\,\overline{\vartheta(\frac{t}{\sigma})}\, e^{i2\pi\mu t} - \frac{1}{\sigma}\,\overline{\vartheta(\frac{t}{\sigma})}\, c_\sigma(\mu), \tag{6.1.1}$$

or, in the frequency domain,

$$\widehat{\psi}_\sigma(\xi) = \overline{\widehat{\vartheta}(\sigma(\mu-\xi))} - c_\sigma(\mu)\,\overline{\widehat{\vartheta}(\sigma\xi)}, \tag{6.1.2}$$

where $\mu > 0$, $\vartheta(t) \in L_2(\mathbb{R})$ with certain decaying order as $t \to \infty$ and $c_\sigma(\mu)$ is a constant such that $\widehat{\psi}_\sigma(0) = 0$. One can let $c_\sigma(\mu) = 0$ if $\widehat{\vartheta}(-\sigma\mu) = 0$; in another respect, one can let $c_\sigma(\mu) = \overline{\widehat{\vartheta}(-\sigma\mu)}/\overline{\widehat{\vartheta}(0)}$ if $\widehat{\vartheta}(-\sigma\mu) \neq 0$. For instance, if $\vartheta(t)$ is given by $\widehat{\vartheta}(\xi) = e^{1-\frac{1}{1-\xi^2}} \chi_{(-1,1)}(\xi)$, then $\psi_\sigma(t) = \frac{1}{\sigma}\,\overline{\vartheta(\frac{t}{\sigma})}\, e^{i2\pi\mu t}$ is the bump wavelet defined in (6.1). However, if $\vartheta(t)$ is the Gaussian function $\vartheta(t) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}$, then $\psi_\sigma(t)$ is the Morlet's wavelet defined in (6.2).

It is noticeable that the representation of the CWT is effected by the choice of the parameter σ for the continuous wavelet $\psi_\sigma(t)$. Now, by using the time-varying parameter, $\sigma(t)$, the adaptive CWT of $x(t)$ (a slowly increasing function) with a continuous wavelet ψ_σ given in (6.1.1) is defined by

$$W_x^\psi(a,b;\sigma(b)) = \langle x(t), \psi_{\sigma(b);a,b}(t)\rangle = \int_{-\infty}^{\infty} x(t)\, \overline{\psi_{\sigma(b);a,b}(t)}\, dt, \tag{6.1.3}$$

or, in the frequency domain,

$$W_x^\psi(a,b;\sigma(b)) = \int_{-\infty}^{\infty} \widehat{x}(\xi)\, \overline{\widehat{\psi}_{\sigma(b)}(a\xi)}\; e^{i2\pi\xi b} d\xi, \tag{6.1.4}$$

where $\psi_{\sigma(b);a,b}(t) = \frac{1}{a}\psi_{\sigma(b)}(\frac{t-b}{a})$ is the adaptive family of wavelets and $\sigma(b)$ is a positive function. For restricted $a > 0$, the above equation (6.1.4) becomes

$$W_x^\psi(a,b;\sigma(b)) = \int_{0}^{\infty} \widehat{x}(\xi)\, \overline{\widehat{\psi}_{\sigma(b)}(a\xi)}\; e^{i2\pi\xi b} d\xi \tag{6.1.5}$$

if ψ_σ or $x(t)$ is analytic. Consequently, an analytic signal $x(t)$ can be retrieved back from the adaptive CWT as

$$x(b) = c_\psi^{-1}(b) \int_0^\infty W_x^\psi(a,b;\sigma(b))\, \frac{da}{a}, \tag{6.1.6}$$

where

$$c_\psi(b) = \int_0^\infty \overline{\widehat{\psi}_{\sigma(b)}(a\xi)} \frac{d\xi}{\xi}. \tag{6.1.7}$$

In addition, if ψ_σ is analytic, then a real-valued signal $x(t)$ can be retrieved back as

$$x(b) = \mathrm{Re}\left(2\, c_\psi^{-1}(b) \int_0^\infty W_x^\psi(a,b;\sigma(b))\, \frac{da}{a}\right). \tag{6.1.8}$$

In practice, ϑ is chosen to be a fast decaying function and therefore the second term in (6.1.2), $\overline{c_\sigma(\mu)\,\widehat{\vartheta}(-\sigma\xi)}$, is extremely small. For instance, when ψ_σ is the Morlet's wavelet, the second term in (6.1.2) equals $e^{-2\sigma^2\pi^2(\xi^2+\mu^2)}$ and therefore it is a negligible quantity if $\sigma = \mu = 1$, i.e., $e^{-2\sigma^2\pi^2(\xi^2+\mu^2)} \le e^{-2\pi^2} = 2.6753 \times 10^{-9}$. Now, it is just for simplification that ψ_σ is considered as follows

$$\psi_\sigma(t) = \frac{1}{\sigma}\, \overline{\vartheta(\frac{t}{\sigma})}\, e^{i2\pi\,\mu t}, \tag{6.1.9}$$

or, in the frequency domain,

$$\widehat{\psi}_\sigma(\xi) = \overline{\widehat{\vartheta}(\sigma(\mu-\xi))}. \tag{6.1.10}$$

Thus, the associated adaptive CWT is defined by

$$W_x^\psi(a,b;\sigma(b)) = \int_{-\infty}^{\infty} x(at+b)\, \frac{1}{\sigma(b)}\, \vartheta(\frac{t}{\sigma(b)})\, e^{-i2\pi\,\mu t}\, dt. \tag{6.1.11}$$

If ϑ is the Gaussian function, $\vartheta(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$, then

$$\psi_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{t}{\sigma})^2} e^{i2\pi\mu t} \tag{6.1.12}$$

or, in the frequency domain,

$$\widehat{\psi}_\sigma(\xi) = e^{-2\pi^2\sigma^2(\xi-\mu)^2} \tag{6.1.13}$$

is the Morlet's wavelet in the simplified version.

6.2 Adaptive CWT-based SST

6.2.1 First-Order AWSST

To derive the first-order phase transformation, we consider the chirp signal $x(t) = e^{i2\pi ct}$ with a constant frequency $c > 0$. Then from equation (6.1.11), we have

$$W_x^\psi(a, b; \sigma(b)) = \int_{-\infty}^{\infty} e^{i2\pi c(at+b)} \frac{1}{\sigma(b)} \vartheta(\frac{t}{\sigma(b)}) e^{-i2\pi\mu t} dt. \tag{6.2.1.1}$$

Taking the first-order partial derivative of this above equation (6.2.1.1) with respect to b gives that

$$\frac{\partial}{\partial b} W_x^\psi(a, b; \sigma(b)) = \big(i2\pi c - \frac{\sigma'(b)}{\sigma(b)}\big) W_x^\psi(a, b; \sigma(b)) - \frac{\sigma'(b)}{\sigma(b)} W_x^{\psi_2}(a, b; \sigma(b)), \tag{6.2.1.2}$$

where

$$\psi_2(t) = \frac{t}{\sigma^2(b)} \vartheta'(\frac{t}{\sigma(b)}) e^{i2\pi\mu t}, \tag{6.2.1.3}$$

or, in the frequency domain, from (2.1.5)

$$\widehat{\psi_2}(\xi) = -\widehat{\vartheta}(\sigma(b)(\xi-\mu)) - \sigma(b)(\mu+\xi)\,\widehat{\vartheta}'(\sigma(b)(\xi-\mu)), \tag{6.2.1.4}$$

and

$$W_x^{\psi_2}(a, b; \sigma(b)) = \int_{-\infty}^{\infty} x(at+b) \frac{t}{\sigma^2(b)} \overline{\vartheta'(\frac{t}{\sigma(b)})} e^{-i2\pi\mu t} dt. \tag{6.2.1.5}$$

Thus, if $W_x^\psi(a, b; \sigma(b)) \neq 0$, the exact IF, c, of $x(t)$ can be obtained by

$$c = \frac{\frac{\partial}{\partial b} W_x^\psi(a, b; \sigma(b))}{i2\pi W_x^\psi(a, b; \sigma(b))} + \frac{\sigma'(b)}{\sigma(b)} \frac{W_x^{\psi_2}(a, b; \sigma(b))}{i2\pi W_x^\psi(a, b; \sigma(b))} + \frac{\sigma'(b)}{i2\pi\,\sigma(b)}. \tag{6.2.1.6}$$

From that, for a general signal $x(t) \in L_2(\mathbb{R})$, the first-order phase transformation of $x(t)$ at (a,b) on which $W_x^{\psi}(a,b;\sigma(b)) \neq 0$ is given to be the real part of the quantity on the right-hand side of (6.2.1.6)

$$\Omega_x^{1^{st}}(a,b;\sigma(b)) = \mathrm{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} + \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} \right\}. \quad (6.2.1.7)$$

Thus the synchrosqueezed ACWT (AWSST), denoted $S_x^{\mathrm{ACWT}}(\xi,b;\sigma(b))$, on the time-FD-frequency plane is defined by

$$S_x^{\mathrm{ACWT}}(\xi,b;\sigma(b)) = \int_{\mathrm{I}} W_x^{\psi}(a,b;\sigma(b))\ \delta(\Omega_x^{1^{\mathrm{st}}}(a,b;\sigma(b)) - \xi)\, \frac{da}{a}, \quad (6.2.1.8)$$

where ξ is the frequency variable and $\mathrm{I} = \{a \in \mathbb{R}_+ : W_x^{\psi}(a,b;\sigma(b)) \neq 0\}$. Now, for an analytic mono-component signal $x(t) \in L_2(\mathbb{R})$, by (6.1.6), it can be retrieved back from the AWSST as

$$x(b) = c_{\psi}^{-1}(b) \int_0^{\infty} S_x^{\mathrm{ACWT}}(\xi,b;\sigma(b))\, d\xi, \quad (6.2.1.9)$$

and for a real-valued mono-component signal $x(t) \in L_2(\mathbb{R})$, by (6.1.8), it can be retrieved back as

$$x(b) = \mathrm{Re}\Big(2\, c_{\psi}^{-1}(b) \int_0^{\infty} S_x^{\mathrm{ACWT}}(\xi,b;\sigma(b))\, d\xi \Big), \quad (6.2.1.10)$$

where $c_{\psi}(b)$ is defined by (6.1.7). In addition, for multicomponent signal $x(t)$ in (2.2.2.3), the k^{th} component $x_k(b)$ can be retrieved back from the AWSST for certain $\Gamma > 0$ as follows

$$x_k(b) \approx \mathrm{Re}\Big(2\, c_{\psi}^{-1}(b) \int_{|\xi - \phi_k'(b)| < \Gamma} S_x^{\mathrm{ACWT}}(\xi,b;\sigma(b))\, d\xi \Big). \quad (6.2.1.11)$$

If ψ_{σ} is as defined by (6.1.13), then

$$\mathrm{Re}\left\{ \frac{W_x^{\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} \right\} = \mathrm{Re}\left\{ \frac{1}{i2\pi}\, (2\pi^2\sigma^2(ac-\mu)^2 - 1) \right\} = 0,$$

and therefore the first-order phase transformation of $x(t)$ at (a,b) on which $W_x^{\psi}(a,b;\sigma(b)) \neq 0$ may be defined by

$$\Omega_x^{1^{\mathrm{st}}}(a,b;\sigma(b)) = \mathrm{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} \right\}. \quad (6.2.1.12)$$

6.2.2 Second-Order AWSST

By considering the linear frequency modulation signal $x(t)$ defined in (3.2.2.1), where $x'(t) = (p + qt + i2\pi(c + rt))\, x(t)$, and from equation (6.1.11) we have

$$\begin{aligned}\frac{\partial}{\partial b} W_x^{\psi}(a, b; \sigma(b)) = &\int_{-\infty}^{\infty} x'(at + b)\, \frac{1}{\sigma(b)}\, \vartheta(\frac{t}{\sigma(b)})\, e^{-i2\pi\,\mu t}\, dt \\ &- \int_{-\infty}^{\infty} x(at + b)\, \frac{\sigma'(b)}{\sigma^2(b)}\, \vartheta(\frac{t}{\sigma(b)})\, e^{-i2\pi\,\mu t}\, dt \\ &- \int_{-\infty}^{\infty} x(at + b)\, \frac{t\,\sigma'(b)}{\sigma^3(b)}\, \vartheta'(\frac{t}{\sigma(b)})\, e^{-i2\pi\,\mu t}\, dt,\end{aligned} \tag{6.2.2.1}$$

or as in the following abbreviated form

$$\begin{aligned}\frac{\partial}{\partial b} W_x^{\psi}(a, b; \sigma(b)) = &(p + qb + i2\pi(c + rb) - \frac{\sigma'(b)}{\sigma(b)})\, W_x^{\psi}(a, b; \sigma(b)) \\ &+ (q + i2\pi r)\, a\, \sigma(b)\, W_x^{\psi_1}(a, b; \sigma(b)) \\ &- \frac{\sigma'(b)}{\sigma(b)}\, W_x^{\psi_2}(a, b; \sigma(b)),\end{aligned} \tag{6.2.2.2}$$

where

$$\psi_1(t) = \frac{t}{\sigma^2(b)} \vartheta(\frac{t}{\sigma(b)})\, e^{i2\pi\mu t}, \tag{6.2.2.3}$$

or, in the frequency domain, from (2.1.4)

$$\widehat{\psi_1}(\xi) = -i\,\sigma(b)\,(\widehat{\vartheta})'(\sigma(b)(\xi - \mu)), \tag{6.2.2.4}$$

and

$$W_x^{\psi_1}(a, b; \sigma(b)) = \int_{-\infty}^{\infty} x(at + b)\, \frac{t}{\sigma^2(b)}\, \overline{\vartheta(\frac{t}{\sigma(b)})}\, e^{-i2\pi\,\mu t}\, dt. \tag{6.2.2.5}$$

Thus at (a, b) on which $W_x^{\psi}(a, b; \sigma(b)) \neq 0$ we get

$$\begin{aligned}\frac{\frac{\partial}{\partial b} W_x^{\psi}(a, b; \sigma(b))}{W_x^{\psi}(a, b; \sigma(b))} = &\, p + qb + i2\pi(c + rb) - \frac{\sigma'(b)}{\sigma(b)} \\ &+ (q + i2\pi r)\, a\, \sigma(b)\, \frac{W_x^{\psi_1}(a, b; \sigma(b))}{W_x^{\psi}(a, b; \sigma(b))} \\ &- \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\psi_2}(a, b; \sigma(b))}{W_x^{\psi}(a, b; \sigma(b))}.\end{aligned} \tag{6.2.2.6}$$

Now, taking partial derivative $\frac{\partial}{\partial a}$ to both sides of this above equation (6.2.2.6) leads to have

$$\begin{aligned}\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\psi}(a, b; \sigma(b))}{W_x^{\psi}(a, b; \sigma(b))}\right) = &\,(q + i2\pi r)\, \sigma(b)\, \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\psi_1}(a, b; \sigma(b))}{W_x^{\psi}(a, b; \sigma(b))}\right) \\ &- \frac{\sigma'(b)}{\sigma(b)}\, \frac{\partial}{\partial a}\left(\frac{W_x^{\psi_2}(a, b; \sigma(b))}{W_x^{\psi}(a, b; \sigma(b))}\right).\end{aligned} \tag{6.2.2.7}$$

Thus, if $\frac{\partial}{\partial a}\big(a\ \frac{W_x^{\psi_1}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))}\big) \neq 0$, then $(q+i2\pi r)\,\sigma(b) = J_{\sigma(b)}(a,b)$ where $J_{\sigma(b)}(a,b)$ is defined by

$$J_{\sigma(b)}(a,b) = \frac{1}{\frac{\partial}{\partial a}\left(a\ \frac{W_x^{\psi_1}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))}\right)} \times \left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b}W_x^{\psi}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))}\right) + \frac{\sigma'(b)}{\sigma(b)}\ \frac{\partial}{\partial a}\left(\frac{W_x^{\psi_2}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))}\right)\right] \tag{6.2.2.8}$$

By substituting (6.2.2.8) into (6.2.2.6), we have

$$\begin{aligned}\frac{\frac{\partial}{\partial b}W_x^{\psi}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))} &= p+qb+i2\pi(c+rb) - \frac{\sigma'(b)}{\sigma(b)} \\ &\quad + J_{\sigma(b)}(a,b)\times\ a\ \frac{W_x^{\psi_1}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))} - \frac{\sigma'(b)}{\sigma(b)}\ \frac{W_x^{\psi_2}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))}.\end{aligned} \tag{6.2.2.9}$$

Therefore, the exact IF, $\phi'(b) = c+rb$, of the linear frequency modulation $x(b)$ given in (3.2.2.1) is defined by

$$\begin{aligned}\phi'(b) &= \frac{\frac{\partial}{\partial b}W_x^{\psi}(a,b;\sigma(b))}{i2\pi W_x^{\psi}(a,b;\sigma(b))} - J_{\sigma(b)}(a,b)\times\ a\ \frac{W_x^{\psi_1}(a,b;\sigma(b))}{i2\pi W_x^{\psi}(a,b;\sigma(b))} \\ &\quad + \frac{\sigma'(b)}{\sigma(b)}\ \frac{W_x^{\psi_2}(a,b;\sigma(b))}{i2\pi W_x^{\psi}(a,b;\sigma(b))} - \frac{1}{i2\pi}\big(p+qb-\frac{\sigma'(b)}{\sigma(b)}\big).\end{aligned} \tag{6.2.2.10}$$

Since $\phi'(b)$ is real, i.e., the real part of $\frac{1}{i2\pi}(p+qb-\frac{\sigma'(b)}{\sigma(b)})$ is zero, we have

$$\begin{aligned}\phi'(b) = \mathrm{Re}\bigg\{&\frac{\frac{\partial}{\partial b}W_x^{\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} - J_{\sigma(b)}(a,b)\times a\ \frac{W_x^{\psi_1}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} \\ &+ \frac{\sigma'(b)}{\sigma(b)}\ \frac{W_x^{\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))}\bigg\}.\end{aligned} \tag{6.2.2.11}$$

Eventually, for a general signal $x(t)$, the second-order phase transformation of $x(t)$ is defined by

$$\Omega_x^{2^{\text{nd}}}(a,b;\sigma(b)) = \begin{cases} \text{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} - J_{\sigma(b)}(a,b) \times a\, \frac{W_x^{\psi_1}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} \right. \\ \qquad \left. + \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} \right\}; \\ \\ \text{when } W_x^{\psi}(a,b;\sigma(b)) \neq 0 \quad \text{and} \quad \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\psi_1}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))} \right) \neq 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} + \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\psi}(a,b;\sigma(b))} \right\}; \\ \\ \text{when } W_x^{\psi}(a,b;\sigma(b)) \neq 0 \quad \text{and} \quad \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\psi_1}(a,b;\sigma(b))}{W_x^{\psi}(a,b;\sigma(b))} \right) = 0. \end{cases} \tag{6.2.2.12}$$

Therefore, with this phase transformation $\Omega_x^{2^{\text{nd}}}(a,b;\sigma(b))$, the AWSST of $x(t)$ is defined by

$$S_x^{\text{ACWT}}(\xi,b;\sigma(b)) = \int_{\text{I}} W_x^{\psi}(a,b;\sigma(b))\; \delta(\Omega_x^{2^{\text{nd}}}(a,b;\sigma(b)) - \xi)\, \frac{da}{a}, \tag{6.2.2.13}$$

where ξ is the frequency variable and $\text{I} = \{a \in \mathbb{R}_+ : W_x^{\psi}(a,b;\sigma(b)) \neq 0\}$. The reconstruction formulas for a mono-component signal $x(t)$ or multicomponent signal $x(t) = \sum_k x_k(t)$ from the second-order AWSST are similar to those with $\Omega_x^{1^{\text{st}}}(a,b;\sigma(b))$ for AWSST.

The rest of this chapter is left to propose a new work that aims to adapt the concept of the time-varying parameter, $\sigma = \sigma(t)$, to the time-FrFD-frequency representation; namely, the adaptive α-order FrCWT (AFrCWT), and then introducing new SST's based on the AFrCWT for the purpose of enhancing the concentration of TF representations.

6.3 Adaptive FrCWT

We consider the adaptive α-order fractional family of wavelets of the form

$$\psi_{\sigma(b);\alpha,a,b}(t) = e^{-i\pi(t^2-b^2-(\frac{t-b}{a})^2)\cot(\alpha)}\, \psi_{\sigma(b);a,b}(t), \tag{6.3.1}$$

where $\psi_{\sigma(b);a,b}(t) = \frac{1}{a}\psi_{\sigma(b)}(\frac{t-b}{a})$. Then we define the α-order FrCWT of a signal

$x(t)$ with a time-varying parameter as follows

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = \langle x(t), \psi_{\sigma(b);\alpha,a,b}(t)\rangle = \int_{-\infty}^{\infty} x(t)\, \overline{\psi_{\sigma(b);\alpha,a,b}(t)}\, dt, \qquad (6.3.2)$$

and, by using Parseval identity defined in (4.1.4), the following proposition presents the definition of the α-order AFrCWT in the FrFD where its proof is similar to the proof of (5.2.9) and given in Appendix for self-containedness.

Proposition 6.3.1. *Let $W_x^{\alpha,\psi}(a,b;\sigma(b))$ be the time-varying α-order FrCWT of a signal $x(t)$ defined in (6.3.2). Then, for $A_\alpha = \sqrt{2\pi/(1+i\cot(\alpha))}$, we define the α-order AFrCWT in the FrFD by*

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = A_\alpha \times \int_{-\infty}^{\infty} e^{i\pi(a\xi)^2\cot(\alpha)}\, X_\alpha(\xi)\, \overline{\Psi_{\sigma(b);\alpha}(a\xi)}\, K_{-\alpha}(\xi,b)\, d\xi. \quad (6.3.3)$$

Now from equations (6.3.1) and (6.3.2), the AFrCWT becomes

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = \int_{-\infty}^{\infty} x(t)\, e^{i\pi\left(t^2-b^2-(\frac{t-b}{a})^2\right)\cot(\alpha)}\, \overline{\psi_{\sigma(b);a,b}(t)}\, dt, \qquad (6.3.4)$$

or in the simplified form as

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = W_{\widetilde{x}}^{\psi}(a,b;\sigma(b))\, e^{-i\pi(a^2+1)\frac{b^2}{a^2}\cot(\alpha)}, \qquad (6.3.5)$$

where $\widetilde{x}(t)$ is defined in (4.2.6). The following proposition shows that the original signal $x(b)$ can be retrieved back from the α-order AFrCWT with integral involving only the scale variable a.

Proposition 6.3.2. *Let $W_x^{\alpha,\psi}(a,b;\sigma(b))$ be the time-varying α-order FrCWT of a signal $x(t)$ defined in (6.3.3). Then $x(t)$ can be retrieved back by*

$$x(b) = c^{-1}_{\Psi_{\sigma(b);\alpha}}(b) \int_0^{\infty} W_x^{\alpha,\psi}(a,b;\sigma(b))\, \frac{da}{a}, \qquad (6.3.6)$$

where $c_{\Psi_{\sigma(b);\alpha}}(b) \neq 0$ is defined by

$$c_{\Psi_{\sigma(b);\alpha}}(b) = A_\alpha \times \int_0^{\infty} e^{i\pi\eta^2\cot(\alpha)}\, \overline{\Psi_{\sigma(b);\alpha}(\eta)}\, \frac{d\eta}{\eta}. \qquad (6.3.7)$$

Obtaining equation (6.3.6) is straightforward in the sense that it can directly be followed the way obtaining equation (4.2.9) for the conventional FrCWT. For self-containedness, the proof of this proposition is provided in Appendix. Now, equation (6.3.4) can be rewritten as follows

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = e^{-i\pi(a^2+1)\frac{b^2}{a^2}\cot(\alpha)} \int_{-\infty}^{\infty} \widetilde{x}(t)\, \overline{\psi_{\sigma(b);1}(t)}\, dt, \qquad (6.3.8)$$

where $\widetilde{x}(t) = x(t)\, e^{i\pi(\frac{a^2-1}{a^2}) cot(\alpha)\, t^2}$ and $\psi_{\sigma(b);1}(t) = \psi_{\sigma(b);a,b}(t)\, e^{-i2\pi \frac{b}{a^2} cot(\alpha)\, t}$.

Thus, using Parseval identity of FT helps to write equation (6.3.8) as

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = e^{-i\pi(a^2+1)\frac{b^2}{a^2}\cot(\alpha)} \int_{-\infty}^{\infty} \widehat{\widetilde{x}}(\xi)\, \overline{\widehat{\psi}_{\sigma(b);1}(\xi)}\, d\xi, \tag{6.3.9}$$

where FT of $\psi_{\sigma(b);1}(t)$, $\widehat{\psi}_{\sigma(b);1}(\xi)$, is given by

$$\widehat{\psi}_{\sigma(b);1}(\xi) = e^{-i2\pi(\frac{b^2}{a^2}\cot(\alpha)+b\xi)}\ \widehat{\psi}_{\sigma(b)}(\frac{b}{a}\cot(\alpha) + a\xi), \tag{6.3.10}$$

and then equation (6.3.9) in its definitive form becomes

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = e^{-i\pi(a^2-1)\frac{b^2}{a^2}\cot(\alpha)} \int_{-\infty}^{\infty} \widehat{\widetilde{x}}(\xi)\, \overline{\widehat{\psi}_{\sigma(b)}\big(\frac{b}{a}\cot(\alpha) + a\xi\big)}\, e^{i2\pi b\xi} d\xi. \tag{6.3.11}$$

Note that if we set $\alpha = \frac{\pi}{2}$ or 90^{o}, then the AFrCWT, $W_x^{\alpha,\psi}(a,b;\sigma(b))$, is the ACWT, $W_x^{\psi}(a,b;\sigma(b))$. Also, if we assume $\psi_{\sigma(b)}$ to be as defined in (6.1.9) or (6.1.10), then $\widehat{\psi}_{\sigma(b)}(\frac{b}{a}\cot(\alpha) + a\xi) = \overline{\widehat{\vartheta}\big(\sigma(b)(\mu - a\xi - \frac{b}{a}\cot(\alpha))\big)}$ and thus the associated AFrCWT $W_x^{\alpha,\psi}(a,b;\sigma(b))$ is defined by

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = \int_{-\infty}^{\infty} x(at+b)\, e^{i\pi\big((a^2-1)\,t^2+2ab\,t\big)\cot(\alpha)}\, \frac{1}{\sigma(b)}\, \vartheta(\frac{t}{\sigma(b)})\, e^{-i2\pi\,\mu t}\, dt. \tag{6.3.12}$$

6.4 Adaptive FrCWT-based SST

6.4.1 First-Order AFrWSST

We again consider the chirp signal $x(t) = e^{i2\pi ct}$ with a constant frequency $c > 0$ as an example to derive the first-order phase transformation. By using equation (6.3.12), we have

$$W_x^{\alpha,\psi}(a,b;\sigma(b)) = \int_{-\infty}^{\infty} e^{i2\pi\big((a^2-1)\cot(\alpha)\,\frac{t^2}{2}+(ac+ab\cot(\alpha))\,t+cb\big)}\, \frac{1}{\sigma(b)}\, \vartheta(\frac{t}{\sigma(b)})\, e^{-i2\pi\,\mu t}\, dt, \tag{6.4.1.1}$$

Taking the first-order partial derivative of this above equation with respect to b will give us that

$$\begin{aligned} \frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b)) &= (i2\pi\, c - \frac{\sigma'(b)}{\sigma(b)})\ W_x^{\alpha,\psi}(a,b;\sigma(b)) \\ &\quad + i2\pi\, a\,\sigma(b)\cot(\alpha)\ W_x^{\alpha,\psi_1}(a,b;\sigma(b)) \\ &\quad - \frac{\sigma'(b)}{\sigma(b)}\ W_x^{\alpha,\psi_2}(a,b;\sigma(b)), \end{aligned} \tag{6.4.1.2}$$

where $\psi_1(t)$ and $\psi_2(t)$ are defined in (6.2.2.3) and (6.2.1.3) respectively, and $W_x^{\alpha,\psi_1}(a,b;\sigma(b))$ and $W_x^{\alpha,\psi_2}(a,b;\sigma(b))$ are defined by

$$W_x^{\alpha,\psi_1}(a,b;\sigma(b)) = \int_{-\infty}^{\infty} x(at+b)\, e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\, \frac{t}{\sigma^2(b)}\, \vartheta(\frac{t}{\sigma(b)})\, e^{-i2\pi\,\mu t}\, dt,$$

and

$$W_x^{\alpha,\psi_2}(a,b;\sigma(b)) = \int_{-\infty}^{\infty} x(at+b)\, e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\, \frac{t}{\sigma^2(b)}\, \vartheta'(\frac{t}{\sigma(b)})\, e^{-i2\pi\,\mu t}\, dt.$$

Thus, if $W_x^{\alpha,\psi}(a,b;\sigma(b)) \neq 0$, the exact IF, c, of $x(t)$ can be obtained by

$$\begin{aligned} c = & \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} + \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} \\ & - a\,\sigma(b)\cot(\alpha)\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} + \frac{1}{i2\pi}\,\frac{\sigma'(b)}{\sigma(b)}. \end{aligned} \tag{6.4.1.3}$$

From that, for a general signal $x(t) \in L_2(\mathbb{R})$, the first-order phase transformation of $x(t)$ at (a,b) on which $W_x^{\alpha,\psi}(a,b;\sigma(b)) \neq 0$ is given to be the real part of the quantity on the right-hand side of (6.4.1.3)

$$\begin{aligned} \Omega_x^{1^{\text{st}}}(a,b;\sigma(b)) = \text{Re}\Bigg\{ & \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} \\ & + \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} \\ & - a\,\sigma(b)\cot(\alpha)\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} \Bigg\} \end{aligned} \tag{6.4.1.4}$$

Thus the synchrosqueezed AFrCWT (AFrWSST), denoted $S_x^{\text{AFrCWT}}(\xi,b;\sigma(b))$, on the time-FrFD-frequency plane is defined by

$$S_x^{\text{AFrCWT}}(\xi,b;\sigma(b)) = \int_{\text{I}} W_x^{\alpha,\psi}(a,b;\sigma(b))\;\delta(\Omega_x^{1^{\text{st}}}(a,b;\sigma(b)) - \xi)\;\frac{da}{a}, \tag{6.4.1.5}$$

where ξ is the frequency variable and $\text{I} = \{a \in \mathbb{R}_+ : W_x^{\psi}(a,b;\sigma(b)) \neq 0\}$. Now, for an analytic mono-component signal $x(t) \in L_2(\mathbb{R})$, by (6.3.6), it can be retrieved back from the AFrWSST as

$$x(b) = c^{-1}_{\Psi_{\sigma(b);\alpha}}(b) \int_0^{\infty} S_x^{\text{AFrCWT}}(\xi,b;\sigma(b))\, d\xi, \tag{6.4.1.6}$$

and for a real-valued mono-component signal $x(t) \in L_2(\mathbb{R})$, it can be retrieved back as

$$x(b) = \text{Re}\Bigg(2\, c^{-1}_{\Psi_{\sigma(b);\alpha}}(b) \int_0^{\infty} S_x^{\text{AFrCWT}}(\xi,b;\sigma(b))\, d\xi\Bigg), \tag{6.4.1.7}$$

where $c_{\Psi_{\sigma(b);\alpha}}(b)$ is defined by (6.3.7). In addition, for multicomponent signal $x(t)$ in (2.2.2.3), the k^{th} component $x_k(b)$ can be retrieved back from the AFrWSST for certain $\Gamma > 0$ as follows

$$x_k(b) \approx \mathrm{Re}\bigg(2\, c^{-1}_{\Psi_{\sigma(b);\alpha}}(b) \int\limits_{|\xi-\phi'_k(b)|<\Gamma} S_x^{\mathrm{AFrCWT}}(\xi, b; \sigma(b))\, d\xi\bigg). \tag{6.4.1.8}$$

6.4.2 Second-Order AFrWSST

For LFM signal $x(t)$ defined in (3.2.2.1) and by taking $\frac{\partial}{\partial b}$ to both sides of equation (6.3.12), we will have

$$\begin{aligned} \frac{\partial}{\partial b} W_x^{\alpha,\psi}(a, b; \sigma(b)) &= (p + qb + i2\pi(c + rb) - \frac{\sigma'(b)}{\sigma(b)})\, W_x^{\alpha,\psi}(a, b; \sigma(b)) \\ &+ \big(q + i2\pi(r + \cot(\alpha))\big)\, a\,\sigma(b)\;\; W_x^{\alpha,\psi_1}(a, b; \sigma(b)) \\ &- \frac{\sigma'(b)}{\sigma(b)}\, W_x^{\alpha,\psi_2}(a, b; \sigma(b)). \end{aligned} \tag{6.4.2.1}$$

Thus at (a, b) on which $W_x^{\alpha,\psi}(a, b; \sigma(b)) \neq 0$ we have

$$\begin{aligned} \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a, b; \sigma(b))}{W_x^{\alpha,\psi}(a, b; \sigma(b))} &= p + qb + i2\pi(c + rb) - \frac{\sigma'(b)}{\sigma(b)} \\ &+ \big(q + i2\pi(r + \cot(\alpha))\big)\, a\,\sigma(b)\, \frac{W_x^{\alpha,\psi_1}(a, b; \sigma(b))}{W_x^{\alpha,\psi}(a, b; \sigma(b))} \\ &- \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\alpha,\psi_2}(a, b; \sigma(b))}{W_x^{\alpha,\psi}(a, b; \sigma(b))}, \end{aligned} \tag{6.4.2.2}$$

and then by taking $\frac{\partial}{\partial a}$ to both sides of this above equation (6.4.2.2) we get

$$\begin{aligned} \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a, b; \sigma(b))}{W_x^{\alpha,\psi}(a, b; \sigma(b))}\right) &= \big(q + i2\pi(r + \cot(\alpha))\big)\,\sigma(b)\, \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\alpha,\psi_1}(a, b; \sigma(b))}{W_x^{\alpha,\psi}(a, b; \sigma(b))}\right) \\ &- \frac{\sigma'(b)}{\sigma(b)}\, \frac{\partial}{\partial a}\left(\frac{W_x^{\alpha,\psi_2}(a, b; \sigma(b))}{W_x^{\alpha,\psi}(a, b; \sigma(b))}\right). \end{aligned} \tag{6.4.2.3}$$

Thus, if $\frac{\partial}{\partial a}\left(a\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\right) \neq 0$, then

$$\big(q + i2\pi(r + \cot(\alpha))\big)\,\sigma(b) = J^{\alpha}_{\sigma(b)}(a, b), \tag{6.4.2.4}$$

where $J^{\alpha}_{\sigma(b)}(a,b)$ is defined by

$$J^{\alpha}_{\sigma(b)}(a,b) = \frac{1}{\frac{\partial}{\partial a}\left(a\,\frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\right)} \times \left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b}W_x^{\alpha,\psi}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\right) + \frac{\sigma'(b)}{\sigma(b)}\,\frac{\partial}{\partial a}\left(\frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\right)\right]. \tag{6.4.2.5}$$

Now by substituting (6.4.2.4) into (6.4.2.2), we have

$$\begin{aligned}\frac{\frac{\partial}{\partial b}W_x^{\alpha,\psi}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} &= p + qb + i2\pi(c+rb) - \frac{\sigma'(b)}{\sigma(b)} \\ &\quad + J^{\alpha}_{\sigma(b)}(a,b) \times \; a\,\frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} \\ &\quad - \frac{\sigma'(b)}{\sigma(b)}\,\frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}.\end{aligned} \tag{6.4.2.6}$$

Therefore, the exact IF, $\phi'(b) = c + rb$, of the linear frequency modulation $x(b)$ given in (3.2.2.1) is defined by

$$\begin{aligned}\phi'(b) &= \frac{\frac{\partial}{\partial b}W_x^{\alpha,\psi}(a,b;\sigma(b))}{i2\pi W_x^{\alpha,\psi}(a,b;\sigma(b))} - J^{\alpha}_{\sigma(b)}(a,b) \times \; a\,\frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{i2\pi W_x^{\alpha,\psi}(a,b;\sigma(b))} \\ &\quad + \frac{\sigma'(b)}{\sigma(b)}\,\frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{i2\pi W_x^{\alpha,\psi}(a,b;\sigma(b))} - \frac{1}{i2\pi}\left(p + qb - \frac{\sigma'(b)}{\sigma(b)}\right).\end{aligned} \tag{6.4.2.7}$$

Since $\phi'(b)$ is real, i.e., the real part of $\frac{1}{i2\pi}(p + qb - \frac{\sigma'(b)}{\sigma(b)})$ is zero, then we have

$$\begin{aligned}\phi'(b) = \mathrm{Re}\Bigg\{&\frac{\frac{\partial}{\partial b}W_x^{\alpha,\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} \\ &- J^{\alpha}_{\sigma(b)}(a,b) \times \; a\,\frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} \\ &+ \frac{\sigma'(b)}{\sigma(b)}\,\frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))}\Bigg\}.\end{aligned} \tag{6.4.2.8}$$

Then, for a general signal $x(t)$, the phase transformation is defined by

$$\Omega_x^{2^{\text{nd}}}(a,b;\sigma(b)) = \begin{cases} \text{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} - J^{\alpha}_{\sigma(b)}(a,b) \times \; a \, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} \right. \\ \qquad \left. + \frac{\sigma'(b)}{\sigma(b)} \, \frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} \right\}; \\ \text{when } W_x^{\psi}(a,b;\sigma(b)) \neq 0 \;\; \text{and} \;\; \frac{\partial}{\partial a}\left(a \, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} \right) \neq 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} - a\,\sigma(b)\cot(\alpha) \, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} \right. \\ \qquad \left. + \frac{\sigma'(b)}{\sigma(b)} \, \frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} \right\}; \\ \text{when } W_x^{\psi}(a,b;\sigma(b)) \neq 0 \;\; \text{and} \;\; \frac{\partial}{\partial a}\left(a \, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} \right) = 0. \end{cases} \tag{6.4.2.9}$$

Based on this derivation, we now provide the following theorem where its proof is given in Appendix.

Theorem 6.4.1. *If $x(t)$ is an LFM signal given by (4.3.2.1), then at (a,b) on which $W_x^{\alpha,\psi}(a,b;\sigma(b)) \neq 0$ and $\frac{\partial}{\partial a}\big(a \, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\big) \neq 0$, $\Omega_x^{2^{\text{nd}}}(a,b;\sigma(b))$ defined by (6.4.2.9) is the IF of $x(t)$; namely, $\Omega_x^{2^{\text{nd}}}(a,b;\sigma(b)) = \phi'(b) = c + rb$.*

Lastly, with this phase transformation $\Omega_x^{2^{\text{nd}}}(a,b;\sigma(b))$ defined above in (6.4.2.9), the second-order AFrWSST, $S_x^{\text{AFrCWT}}(\xi,b;\sigma(b))$, of a signal $x(t) \in L_2(\mathbb{R})$ is defined as in (6.4.1.5)

$$S_x^{\text{AFrCWT}}(\xi,b;\sigma(b)) = \int_{\text{I}} W_x^{\alpha,\psi}(a,b;\sigma(b)) \; \delta(\Omega_x^{2^{\text{nd}}}(a,b;\sigma(b)) - \xi) \, \frac{da}{a}, \tag{6.4.2.10}$$

where ξ is the frequency variable and $\text{I} = \{a \in \mathbb{R}_+ : W_x^{\alpha,\psi}(a,b;\sigma(b)) \neq 0\}$. For reconstructing a mono-component signal $x(t)$ or a multicomponent signal $x(t) = \sum_k x_k(t)$ from the second-order AFrWSST, it can similarly be defined as that with $\Omega_x^{1^{\text{st}}}(a,b;\sigma(b))$ for AFrWSST.

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CHAPTER 7

Conclusion and Future Work

In this chapter, we make the primary conclusion of this dissertation and describe our future research problems. First, a formula with integral involving only the scale variable is established to reconstruct the original signal from the FrCWT. Then, two phase transformations, that are associated with the first and the second order partial derivatives of the FrCWT of a time-varying signal, are derived. Based on those transformations, we introduce the fractional synchrosqueezed wavelet transform (FrWSST) that transforms the FrCWT value from a time-scale point to a quantity on the time-FrFD-frequency plane.

Based on IFE-CWT, we introduce the IFE synchrosqueezing transform (IFE-WSST) with the first two orders of the phase transformation for further enhancing the concentration and reducing the diffusion for the curved IF profile. We also propose a time-FrFD-frequency representation with satisfactory energy concentration for both monocomponent signals and multicomponent signals; namely, the instantaneous frequency-embedded FrCWT (IFE-FrCWT) and then the corresponding SST (IFE-FrWSST). Lastly, we propose the adaptive FrCWT (AFrCWT) and the corresponding adaptive SST (AFrWSST) with a time-varying parameter $\sigma = \sigma(t)$ to close this dissertation.

There are several research problems related to this topic. Other than the problems we studied in this dissertation, we have the following list of problems for our future research:

1. (*Numerical experiments: Including real data experiments*)

   Performing some experimental studies is to show the potential outperformance; obtained from time-frequency representations in the FrFD, in the estimation accuracy of the instantaneous frequency and the improvement in the energy concentration of the time-frequency distribution for multicomponent strong frequency modulation signals.

2. (*Instantaneous frequency-embedded* ACWT/AFrCWT)

   Study TF and TFrF representations with satisfactory energy concentration that involve both variables of adaptive CWT (ACWT) and adaptive FrCWT (AFrCWT); namely, the instantaneous frequency-embedded ACWT/AFrCWT (IFE-AFrCWT/IFE-AFrCWT).

3. (STFrFT-*based synchrosqueezing transform*)

   Extend the idea of SST in the time-FrFD-frequency plane to the short-time fractional Fourier transform (STFrFT), namely; the STFrFT-based synchrosqueezing transform (FrFSST).

4. (*Instantaneous frequency-embedded* STFrFT)

Generate a time-FrFD-frequency representation with satisfactory energy concentration that involves both variables of STFrFT; namely, the instantaneous frequency-embedded STFrFT (IFE-STFrFT).

5. (STFrFT *with a time-varying parameter*)

   Study the time-FrFD-frequency analysis by adapting time-varying window width to the STFrFT, namely; the adaptive STFrFT (ASTFrFT).

6. (*Instantaneous frequency-embedded* ASTFT/ASTFrFT)

   Study TF and TFrF representations with satisfactory energy concentration that involve both variables of adaptive STFT (ASTFT) and adaptive STFrFT (ASTFrFT); namely, the instantaneous frequency-embedded ASTFT/ASTFrFT (IFE-ASTFTT/IFE-ASTFrFTT).

7. Present several *Applications* by applying these time-FrFD-frequency representations to multicomponent signals that are commonly seen in nature and engineering problems, where the frequencies of these signals usually change with the time.

Appendix

Proof of Theorem 4.3.1. For $x(t)$ given by (4.3.2.1) and from equations (4.3.2.2); $x'(t) = \big(p + qt + i2\pi(c + rt)\big)\, x(t)$, and (4.3.1.1), we have

$$\begin{aligned}
\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b) &= \int_{-\infty}^{\infty} x'(at+b)\, e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\, \overline{\psi(t)}\, dt \\
&\quad + i2\pi\, a\, \cot(\alpha) \times \int_{-\infty}^{\infty} x(at+b)\, e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\, t\,\overline{\psi(t)}\, dt \\
&= \big((p + i2\pi\, c) + (q + i2\pi r)\, b\big)\times \\
&\qquad \int_{-\infty}^{\infty} x(at+b)\, e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\, \overline{\psi(t)}\, dt \\
&\quad + a\big(q + i2\pi(r + \cot(\alpha))\big)\times \\
&\qquad \int_{-\infty}^{\infty} x(at+b)\, e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\, t\,\overline{\psi(t)}\, dt \\
&= \big((p + i2\pi c) + (q + i2\pi r)\, b\big)\; W_x^{\alpha,\psi}(a,b) \\
&\qquad + a\,\big(q + i2\pi(r + \cot(\alpha))\big)\; W_x^{\alpha,\mathcal{T}\psi}(a,b).
\end{aligned}$$

Thus if $W_x^{\alpha,\psi}(a,b) \neq 0$, we have

$$\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} = p + i2\pi c + (q + i2\pi r)\, b + a\,\big(q + i2\pi(r + \cot(\alpha))\big)\, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}. \qquad (\dagger)$$

Taking the first-order partial derivative $\frac{\partial}{\partial a}$ to both sides of this above equation (†) leads to have

$$\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right) = \big(q + i2\pi(r + \cot(\alpha))\big)\, \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right).$$

Therefore, if in addition, $\frac{\partial}{\partial a}\big(a\, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\big) \neq 0$, then

$$q + i2\pi(r + cot(\alpha)) = \frac{1}{\frac{\partial}{\partial a}\left(a\, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right)} \times \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right).$$

Back to (†), we have

$$\begin{aligned}
\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} &= p + qb + i2\pi(c + rb) \\
&\quad + a\, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b) \times \frac{\partial}{\partial a}\left(a\, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right)} \times \frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\right).
\end{aligned}$$

Hence,

$$\phi'(b) = c + rb = \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{i2\pi\, W_x^{\alpha,\psi}(a,b)} - \frac{1}{i2\pi}(p+qb)$$
$$- \left(a \, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b) \times \frac{\partial}{\partial a}\big(a \, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\big)} \right) \times \frac{\partial}{\partial a} \left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} \right).$$

Since $\phi'(b)$ is real, we conclude that

$$\phi'(b) = c + rb = \operatorname{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{i2\pi\, W_x^{\alpha,\psi}(a,b)} - \left(a \, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b) \times \frac{\partial}{\partial a}\big(a \, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\big)} \right) \times \frac{\partial}{\partial a} \left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)} \right) \right\}.$$

Therefore for an LFM signal $x(t)$ given by (4.3.2.1), at (a,b) on which $W_x^{\alpha,\psi}(a,b) \neq 0$ and $\frac{\partial}{\partial a}\big(a \, \frac{W_x^{\alpha,\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\psi}(a,b)}\big) \neq 0$, the phase transformation $\Omega_x^{2^{\text{nd}}}(a,b)$ defined by (4.3.2.9) is the exact IF of $x(t)$; namely, $\Omega_x^{2^{\text{nd}}}(a,b) = \phi'(b) = c + rb$. This completes the proof of Theorem 4.3.1. ■

Proof of Theorem 5.1.1. For $x(t)$ given by (4.3.2.1) and from equations (4.3.2.2); $x'(t) = \big(p + qt + i2\pi(c+rt)\big)\, x(t)$, and (5.1.1.1), we have

$$\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b) = \int_{-\infty}^{\infty} x'(at+b)\, e^{-i2\pi\big(\varphi(at+b)-\varphi(b)-(\varphi'(b)+\xi_0)\, at - \xi_0 b\big)}\, \overline{\psi(t)}\, dt$$
$$+ \int_{-\infty}^{\infty} \big(-i2\pi\varphi'(at+b) + i2\pi\varphi''(b)\, at + i2\pi(\varphi'(b)+\xi_0)\big) \times$$
$$x(at+b)\, e^{-i2\pi\big(\varphi(at+b)-\varphi(b)-(\varphi'(b)+\xi_0)\, at - \xi_0 b\big)}\, \overline{\psi(t)}\, dt.$$
$$= \int_{-\infty}^{\infty} \big(p + qb + i2\pi(c+rb) + (q + i2\pi\, r)\, at\big) \times$$
$$x(at+b)\, e^{-i2\pi\big(\varphi(at+b)-\varphi(b)-(\varphi'(b)+\xi_0)\, at - \xi_0 b\big)}\, \overline{\psi(t)}\, dt$$
$$+ \int_{-\infty}^{\infty} \big(-i2\pi\varphi'(at+b) + i2\pi\varphi''(b)\, at + i2\pi(\varphi'(b)+\xi_0)\big) \times$$
$$x(at+b)\, e^{-i2\pi\big(\varphi(at+b)-\varphi(b)-(\varphi'(b)+\xi_0)\, at - \xi_0 b\big)}\, \overline{\psi(t)}\, dt$$

which can be reformulated as follows

$$
\begin{aligned}
\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b) = \big(p+qb+i2\pi(\varphi'(b)+\xi_0+c+rb)\big)\times \\
\int_{-\infty}^{\infty} x(at+b)\, e^{-i2\pi\left(\varphi(at+b)-\varphi(b)-(\varphi'(b)+\xi_0)\,at-\xi_0 b\right)}\ \overline{\psi(t)}\, dt \\
+\big(q+i2\pi(r+\varphi''(b))\big)\, a\times \\
\int_{-\infty}^{\infty} x(at+b)\, e^{-i2\pi\left(\varphi(at+b)-\varphi(b)-(\varphi'(b)+\xi_0)\,at-\xi_0 b\right)}\ \overline{\psi(t)}\, dt \\
-\, i2\pi \int_{-\infty}^{\infty} x(at+b)\, \varphi'(at+b)\, e^{-i2\pi\left(\varphi(at+b)-\varphi(b)-(\varphi'(b)+\xi_0)\,at-\xi_0 b\right)}\ \overline{\psi(t)}\, dt \\
= \big(p+qb+i2\pi(\varphi'(b)+\xi_0+c+rb)\big)\, W_x^{\mathrm{E},\psi}(a,b) \\
+\big(q+i2\pi(r+\varphi''(b))\big)\, a\, W_x^{\mathrm{E},\mathcal{T}\psi}(a,b) - i2\pi\, W_{x\varphi'}^{\mathrm{E},\psi}(a,b).
\end{aligned}
$$

Thus if $W_x^{\mathrm{E},\psi}(a,b)\neq 0$, we have

$$
\begin{aligned}
\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} &= p+qb+i2\pi(\varphi'(b)+\xi_0+c+rb) \\
&\quad + (q+i2\pi(r+\varphi''(b)))\, a\, \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} - i2\pi\, \frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}. \qquad (\ddagger)
\end{aligned}
$$

By taking the first-order partial derivative $\frac{\partial}{\partial a}$ to both sides of (‡) , we have

$$
\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) = \big(q+i2\pi(r+\varphi''(b))\big)\,\frac{\partial}{\partial a}\left(a\,\frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) - i2\pi\,\frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right).
$$

Therefore, if in addition, $\frac{\partial}{\partial a}\big(a\,\frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\big)\neq 0$, then

$$
q+i2\pi(r+\varphi''(b)) = \frac{1}{\frac{\partial}{\partial a}\left(a\,\frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right)} \times \left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) + i2\pi\,\frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right)\right].
$$

Back to (‡), we have

$$
\begin{aligned}
\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} &= p+qb+i2\pi(\varphi'(b)+\xi_0+c+rb) + \Bigg(a\,\frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)\times\frac{\partial}{\partial a}\big(a\,\frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\big)} \\
&\quad \times \left[\frac{\partial}{\partial a}\left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right) + i2\pi\frac{\partial}{\partial a}\left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right)\right]\Bigg) - i2\pi\,\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}.
\end{aligned}
$$

Hence,

$$\phi'(b) = c + rb = \frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{i2\pi\ W_x^{\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} - \left(\left[\frac{\partial}{\partial a} \left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\mathrm{E},\psi}(a,b)} \right) + \frac{\partial}{\partial a} \left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} \right) \right] \times a \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b) \times \frac{\partial}{\partial a}\big(a \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\big)} \right) - \varphi'(b) - \xi_0 - \frac{1}{i2\pi}(p + qb).$$

Since $\phi'(b)$ is real, we conclude that

$$\phi'(b) = c + rb = \mathrm{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{i2\pi\ W_x^{\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} - a \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b) \times \frac{\partial}{\partial a}\left(a \frac{W_x^{\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)}\right)} \times \left[\frac{\partial}{\partial a} \left(\frac{\frac{\partial}{\partial b} W_x^{\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\mathrm{E},\psi}(a,b)} \right) + \frac{\partial}{\partial a} \left(\frac{W_{x\varphi'}^{\mathrm{E},\psi}(a,b)}{W_x^{\mathrm{E},\psi}(a,b)} \right) \right] \right\} - \varphi'(b) - \xi_0.$$

Therefore for an LFM signal $x(t)$ given by (4.3.2.1), at (a,b) on which $W_x^{\mathrm{E},\psi}(a,b) \neq 0$ and $\frac{\partial}{\partial a}\big(a \frac{W_x^{E,\mathcal{T}\psi}(a,b)}{W_x^{E,\psi}(a,b)}\big) \neq 0$, the second order phase transformation $\Omega_x^{2^{\mathrm{nd}}}(a,b)$ defined by (5.1.2.7) is the exact IF of $x(t)$; namely, $\Omega_x^{2^{\mathrm{nd}}}(a,b) = \phi'(b) = c + rb$. This completes the proof of Theorem 5.1.1. ■

Proof of Theorem 5.3.1. For $x(t)$ given by (4.3.2.1) and from equations (4.3.2.2); $x'(t) = \big(p + qt + i2\pi(c + rt)\big)\, x(t)$, and (5.3.1.1), we have

$$\begin{aligned}
\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b) &= \int_{-\infty}^{\infty} \left(e^{i2\pi(\varphi(b)+\xi_0 b) - i2\pi\varphi(at+b) + i2\pi(\xi_0 + b\,\cot(\alpha) + \varphi'(b))\,at + i\pi(a^2-1)\,\cot(\alpha)\,t^2}\right) \times \\
&\qquad x'(at+b)\,\overline{\psi(t)}\,dt \\
&+ \int_{-\infty}^{\infty} \big(i2\pi(\varphi'(b)+\xi_0) - i2\pi\varphi'(at+b) + i2\pi(\cot(\alpha) + \varphi''(b))\,at\big) \times x(at+b) \\
&\qquad \times \left(e^{i2\pi(\varphi(b)+\xi_0 b) - i2\pi\varphi(at+b) + i2\pi(\xi_0 + b\,\cot(\alpha) + \varphi'(b))\,at + i\pi(a^2-1)\,\cot(\alpha)\,t^2}\right) \overline{\psi(t)}\,dt \\
&= \int_{-\infty}^{\infty} \left(e^{i2\pi(\varphi(b)+\xi_0 b) - i2\pi\varphi(at+b) + i2\pi(\xi_0 + b\,\cot(\alpha) + \varphi'(b))\,at + i\pi(a^2-1)\,\cot(\alpha)\,t^2}\right) \times \\
&\qquad \big(p + q(at+b) + i2\pi(c + r(at+b))\big) \times x(at+b)\,\overline{\psi(t)}\,dt \\
&+ \int_{-\infty}^{\infty} \big(i2\pi(\varphi'(b)+\xi_0) - i2\pi\varphi'(at+b) + i2\pi(\cot(\alpha) + \varphi''(b))\,at\big) \times x(at+b) \\
&\qquad \times \left(e^{i2\pi(\varphi(b)+\xi_0 b) - i2\pi\varphi(at+b) + i2\pi(\xi_0 + b\,\cot(\alpha) + \varphi'(b))\,at + i\pi(a^2-1)\,\cot(\alpha)\,t^2}\right) \overline{\psi(t)}\,dt
\end{aligned}$$

which can be reformulated as follows

$$
\begin{aligned}
\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b) &= \big(p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb)\big)\times \\
&\qquad \int_{-\infty}^{\infty} \big(e^{i2\pi(\varphi(b)+\xi_0 b)-i2\pi\varphi(at+b)+i2\pi(\xi_0+b\,\cot(\alpha)+\varphi'(b))\,at+i\pi(a^2-1)\,\cot(\alpha)\,t^2}\big)\times \\
&\qquad\qquad x(at+b)\,\overline{\psi(t)}\,dt \\
&\quad - i2\pi \int_{-\infty}^{\infty} \big(e^{i2\pi(\varphi(b)+\xi_0 b)-i2\pi\varphi(at+b)+i2\pi(\xi_0+b\,\cot(\alpha)+\varphi'(b))\,at+i\pi(a^2-1)\,\cot(\alpha)\,t^2}\big)\times \\
&\qquad\qquad x(at+b)\,\varphi'(at+b)\,\overline{\psi(t)}\,dt \\
&\quad + \big(q + i2\pi(r + \cot(\alpha) + \varphi''(b))\big)\,a\,\times \\
&\qquad \int_{-\infty}^{\infty} \big(e^{i2\pi(\varphi(b)+\xi_0 b)-i2\pi\varphi(at+b)+i2\pi(\xi_0+b\,\cot(\alpha)+\varphi'(b))\,at+i\pi(a^2-1)\,\cot(\alpha)\,t^2}\big)\times \\
&\qquad\qquad x(at+b)\,t\,\overline{\psi(t)}\,dt \\
&= \big(p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb)\big)\,W_x^{\alpha,\mathrm{E},\psi}(a,b) - i2\pi\,W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b) \\
&\quad + \big(q + i2\pi(r + \cot(\alpha) + \varphi''(b))\big)\,a\,W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b).
\end{aligned}
$$

Thus if $W_x^{\alpha,\mathrm{E},\psi}(a,b) \neq 0$, we have

$$
\begin{aligned}
\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} &= p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb) - i2\pi\,\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \\
&\quad + \big(q + i2\pi(r + \cot(\alpha) + \varphi''(b))\big)\,a\,\frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}. \qquad (\dagger\dagger)
\end{aligned}
$$

By taking the first-order partial derivative $\frac{\partial}{\partial a}$ to both sides of ($\dagger\dagger$) , we have

$$
\begin{aligned}
\frac{\partial}{\partial a}\bigg(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\bigg) &= -i2\pi\,\frac{\partial}{\partial a}\bigg(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\bigg) \\
&\quad + \big(q + i2\pi(r + \cot(\alpha) + \varphi''(b))\big)\,\frac{\partial}{\partial a}\bigg(a\,\frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\bigg).
\end{aligned}
$$

Therefore, if in addition, $\frac{\partial}{\partial a}\big(a\,\frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\big) \neq 0$, then

$$
\begin{aligned}
q + i2\pi(r + \cot(\alpha) + \varphi''(b)) &= \frac{1}{\frac{\partial}{\partial a}\big(a\,\frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\big)} \times \bigg[\frac{\partial}{\partial a}\bigg(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\bigg) \\
&\qquad + i2\pi\,\frac{\partial}{\partial a}\bigg(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\bigg)\bigg].
\end{aligned}
$$

Back to (††), we have

$$\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} = p + qb + i2\pi(\varphi'(b) + \xi_0 + c + rb)$$
$$+ \left(\left[\frac{\partial}{\partial a} \left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \right) + i2\pi \frac{\partial}{\partial a} \left(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \right) \right] \times \right.$$
$$\left. a \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b) \times \frac{\partial}{\partial a}\left(a \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right)} \right) + i2\pi \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}$$

Hence

$$\phi'(b) = c + rb = \frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}$$
$$- \left(\left[\frac{\partial}{\partial a} \left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)} \right) + \frac{\partial}{\partial a} \left(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \right) \right] \times \right.$$
$$\left. a \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b) \times \frac{\partial}{\partial a}\left(a \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right)} \right) - \varphi'(b) - \xi_0 - \frac{1}{i2\pi}(p + qb).$$

Since $\phi'(b)$ is real, we conclude that

$$\phi'(b) = c + rb = \mathrm{Re}\left\{ \frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)} + \frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \right.$$
$$- \left(\left[\frac{\partial}{\partial a} \left(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\mathrm{E},\psi}(a,b)}{i2\pi\, W_x^{\alpha,\mathrm{E},\psi}(a,b)} \right) + \frac{\partial}{\partial a} \left(\frac{W_{x\varphi'}^{\alpha,\mathrm{E},\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)} \right) \right] \times \right.$$
$$\left.\left. a \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b) \times \frac{\partial}{\partial a}\left(a \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right)} \right) \right\} - \varphi'(b) - \xi_0.$$

Therefore for an LFM signal $x(t)$ given by (4.3.2.1), at (a,b) on which $W_x^{\alpha,\mathrm{E},\psi}(a,b) \neq 0$ and $\frac{\partial}{\partial a}\left(a \frac{W_x^{\alpha,\mathrm{E},\mathcal{T}\psi}(a,b)}{W_x^{\alpha,\mathrm{E},\psi}(a,b)}\right) \neq 0$, the phase transformation defined by (5.3.2.10) is the exact IF of $x(t)$; namely, $\Omega_x^{2^{\mathrm{nd}}}(a,b) = \phi'(b) = c + rb$. This completes the proof of Theorem 5.3.1. ■

Proof of Proposition 6.3.1. By taking the α-order FrFT on both sides of equation (6.3.1), we have

$$
\begin{aligned}
\Psi_{\sigma(b);\alpha,a,b}(\xi) &= \int_{-\infty}^{\infty} \psi_{\sigma(b);\alpha,a,b}(t)\, K_{\alpha}(t,\xi)\, dt \\
&= \int_{-\infty}^{\infty} e^{-i\pi(t^2-b^2-(\frac{t-b}{a})^2)\cot(\alpha)}\ \psi_{\sigma(b);a,b}(t)\ B_\alpha\ e^{i\pi(t^2+\xi^2)\cot(\alpha)-it\xi\csc(\alpha)}\, dt \\
&= \int_{-\infty}^{\infty} \frac{1}{a}\, \psi_{\sigma(b)}\big(\frac{t-b}{a}\big)\ B_\alpha\ e^{i\pi(b^2+\xi^2)\cot(\alpha)-it\xi\csc(\alpha)}\ e^{i\pi(\frac{t-b}{a})^2\cot(\alpha)}\, dt,
\end{aligned}
$$

or it can be written as follows

$$
\begin{aligned}
\Psi_{\sigma(b);\alpha,a,b}(\xi) &= \frac{1}{B_\alpha} \int_{-\infty}^{\infty} \Big(\frac{1}{a}\, \psi_{\sigma(b)}\big(\frac{t-b}{a}\big)\ B_\alpha\ e^{i\pi(b^2+\xi^2)\cot(\alpha)-i\left(\frac{t-b}{a}+\frac{b}{a}\right)a\xi\csc(\alpha)} \\
&\qquad\qquad B_\alpha\ e^{i\pi\left((\frac{t-b}{a})^2+(a\xi)^2-(a\xi)^2\right)\cot(\alpha)}\Big)\, dt \\
&= \frac{e^{-i\pi(a\xi)^2\cot(\alpha)}}{B_\alpha} \int_{-\infty}^{\infty} \Big(\frac{1}{a}\, \psi_{\sigma(b)}\big(\frac{t-b}{a}\big)\ B_\alpha\ e^{i\pi\left((\frac{t-b}{a})^2+(a\xi)^2\right)\cot(\alpha)-i(\frac{t-b}{a})\,a\xi\csc(\alpha)} \\
&\qquad\qquad B_\alpha\ e^{i\pi(b^2+\xi^2)\cot(\alpha)-ib\xi\csc(\alpha)}\Big)\, dt \\
&= \frac{e^{-i\pi(a\xi)^2\cot(\alpha)}}{B_\alpha} \int_{-\infty}^{\infty} \psi_{\sigma(b)}\big(\frac{t-b}{a}\big)\ B_\alpha\ e^{i\pi\left((\frac{t-b}{a})^2+(a\xi)^2\right)\cot(\alpha)-i(\frac{t-b}{a})\,a\xi\csc(\alpha)} \\
&\qquad\qquad K_\alpha(b,\xi)\, d\big(\frac{t-b}{a}\big) \\
&= \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)} \int_{-\infty}^{\infty} \psi_{\sigma(b)}(t)\ B_\alpha\ e^{i\pi\left(t^2+(a\xi)^2\right)\cot(\alpha)-it\,a\xi\csc(\alpha)}\, dt \\
&= \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)} \int_{-\infty}^{\infty} \psi_{\sigma(b)}(t)\ K_\alpha(t,a\xi)\, dt \\
&= \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\, \Psi_{\sigma(b);\alpha}(a\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)}. \qquad (\bigstar)
\end{aligned}
$$

From the Parseval identity of the FrFT defined in (4.1.4) and then substituting ($\bigstar$) into (6.3.2), we have

$$
\begin{aligned}
W_x^{\alpha,\psi}(a,b;\sigma(b)) &= < X_{\alpha}(\xi), \frac{1}{B_\alpha}\ K_\alpha(b,\xi)\, \Psi_{\sigma(b);\alpha}(a\xi)\, e^{-i\pi(a\xi)^2\cot(\alpha)} > \\
&= \frac{1}{\overline{B}_\alpha} \times \int_{-\infty}^{\infty} e^{i\pi(a\xi)^2\cot(\alpha)}\ X_{\alpha}(\xi)\, \overline{\Psi_{\sigma(b);\alpha}(a\xi)}\, \overline{K_\alpha(b,\xi)}\, d\xi \\
&= A_\alpha \times \int_{-\infty}^{\infty} e^{i\pi(a\xi)^2\cot(\alpha)}\ X_{\alpha}(\xi)\, \overline{\Psi_{\sigma(b);\alpha}(a\xi)}\, K_{-\alpha}(b,\xi)\, d\xi.
\end{aligned}
$$

This proves (6.3.3). ■

Proof of Proposition 6.3.2. Integrating both sides of equation (6.3.3) with integral involving only the scale variable a leads to have

$$\begin{aligned}\int_0^{\infty} W_x^{\alpha,\psi}(a,b;\sigma(b))\,\frac{da}{a} &= A_\alpha \times \int_0^{\infty}\int_{-\infty}^{\infty} e^{i\pi (a\xi)^2\cot(\alpha)}\, X_\alpha(\xi)\,\overline{\Psi_{\sigma(b);\alpha}(a\xi)}\, K_{-\alpha}(\xi,b)\,d\xi\,\frac{da}{a}\\ &= A_\alpha \times \int_{-\infty}^{\infty} X_\alpha(\xi)\,K_{-\alpha}(\xi,b)\int_0^{\infty} e^{i\pi (a\xi)^2\cot(\alpha)}\,\overline{\Psi_{\sigma(b);\alpha}(a\xi)}\,\frac{da}{a}\,d\xi\\ &= \int_{-\infty}^{\infty} X_\alpha(\xi)\,\overline{K_\alpha(b,\xi)}\left(A_\alpha \times \int_0^{\infty} e^{i\pi\eta^2\cot(\alpha)}\,\overline{\Psi_{\sigma(b);\alpha}(\eta)}\,\frac{d\eta}{\eta}\right)d\xi\\ &= c_{\Psi_{\sigma(b);\alpha}}(b)\times\int_{-\infty}^{\infty} X_\alpha(\xi)\,\overline{K_\alpha(b,\xi)}\,d\xi\\ &= c_{\Psi_{\sigma(b);\alpha}}(b)\;x(b),\end{aligned}$$

which shows (6.3.6) as it is desired. ∎

Proof of Theorem 6.4.1. For $x(t)$ given by (4.3.2.1) and from equations (4.3.2.2); $x'(t) = \big(p + qt + i2\pi(c + rt)\big)\,x(t)$, and (6.3.12), we have

$$\begin{aligned}\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b)) &= \int_{-\infty}^{\infty} x'(at+b)\,e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\,\frac{1}{\sigma(b)}\,\vartheta(\frac{t}{\sigma(b)})\,e^{-i2\pi\,\mu t}\,dt\\ &+\int_{-\infty}^{\infty} x(at+b)\times\Bigg(i2\pi a\cot(\alpha)\,t\,e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\,\frac{1}{\sigma(b)}\,\vartheta(\frac{t}{\sigma(b)})\,e^{-i2\pi\,\mu t}\\ &\quad + e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\Big(-\frac{\sigma'(b)}{\sigma^2(b)}\,\vartheta(\frac{t}{\sigma(b)}) - \frac{\sigma'(b)}{\sigma^3(b)}\,t\,\vartheta'(\frac{t}{\sigma(b)})\Big)e^{-i2\pi\,\mu t}\Bigg)\,dt\\ &= \big(p+qb+i2\pi(c+rb) - \frac{\sigma'(b)}{\sigma(b)}\big)\times\\ &\qquad \int_{-\infty}^{\infty} x(at+b)\,e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\,\frac{1}{\sigma(b)}\,\vartheta(\frac{t}{\sigma(b)})\,e^{-i2\pi\,\mu t}\,dt\\ &+ a\,(q+i2\pi(r+\cot(\alpha)))\times\\ &\qquad \int_{-\infty}^{\infty} x(at+b)\,e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\,\frac{t}{\sigma(b)}\,\vartheta(\frac{t}{\sigma(b)})\,e^{-i2\pi\,\mu t}\,dt\\ &- \frac{\sigma'(b)}{\sigma(b)}\int_{-\infty}^{\infty} x(at+b)\,e^{i\pi\left((a^2-1)\,t^2+2ab\,t\right)\cot(\alpha)}\,\frac{t}{\sigma^2(b)}\,\vartheta'(\frac{t}{\sigma(b)})\,e^{-i2\pi\,\mu t}\,dt,\\ &= (p+qb+i2\pi(c+rb) - \frac{\sigma'(b)}{\sigma(b)})\;W_x^{\alpha,\psi}(a,b;\sigma(b))\\ &+ \big(q+i2\pi(r+\cot(\alpha))\big)\,a\,\sigma(b)\;\;W_x^{\alpha,\psi_1}(a,b;\sigma(b))\\ &- \frac{\sigma'(b)}{\sigma(b)}\;W_x^{\alpha,\psi_2}(a,b;\sigma(b)).\end{aligned}$$

Thus if $W_x^{\alpha,\psi}(a,b;\sigma(b)) \neq 0$, we have

$$\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} = p + qb + i2\pi(c+rb) - \frac{\sigma'(b)}{\sigma(b)} + \big(q + i2\pi(r+\cot(\alpha))\big)\, a\, \sigma(b)\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} - \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}. \tag{‡‡}$$

Taking the first-order partial derivative $\frac{\partial}{\partial a}$ to both sides of this above equation (‡‡) leads to have

$$\frac{\partial}{\partial a}\Bigg(\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\Bigg) = \big(q + i2\pi(r+\cot(\alpha))\big)\, \sigma(b)\, \frac{\partial}{\partial a}\Bigg(a\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\Bigg) - \frac{\sigma'(b)}{\sigma(b)}\, \frac{\partial}{\partial a}\Bigg(\frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\Bigg).$$

Therefore, if in addition, $\frac{\partial}{\partial a}\big(a\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\big) \neq 0$, then

$$J^{\alpha}_{\sigma(b)}(a,b) = \big(q + i2\pi(r+\cot(\alpha))\big)\, \sigma(b),$$

where $J^{\alpha}_{\sigma(b)}(a,b)$ is defined by (6.4.2.5). Back to (‡‡), we have

$$\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} = p + qb + i2\pi(c+rb) - \frac{\sigma'(b)}{\sigma(b)} + J^{\alpha}_{\sigma(b)}(a,b) \times\ a\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))} - \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}.$$

Hence,

$$\phi'(b) = c + rb = \frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} - J^{\alpha}_{\sigma(b)}(a,b) \times\ a\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} + \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} - \frac{1}{i2\pi}\Big(p + qb - \frac{\sigma'(b)}{\sigma(b)}\Big).$$

Since $\phi'(b)$ is real, we conclude that

$$\phi'(b) = c + rb = \mathrm{Re}\Bigg\{\frac{\frac{\partial}{\partial b} W_x^{\alpha,\psi}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} - J^{\alpha}_{\sigma(b)}(a,b) \times\ a\, \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))} + \frac{\sigma'(b)}{\sigma(b)}\, \frac{W_x^{\alpha,\psi_2}(a,b;\sigma(b))}{i2\pi\, W_x^{\alpha,\psi}(a,b;\sigma(b))}\Bigg\}.$$

Therefore, for an LFM signal $x(t)$ given by (4.3.2.1), at (a,b) on which $W_x^{\alpha,\psi}(a,b;\sigma(b)) \neq 0$ and $\frac{\partial}{\partial a}\big(a\ \frac{W_x^{\alpha,\psi_1}(a,b;\sigma(b))}{W_x^{\alpha,\psi}(a,b;\sigma(b))}\big) \neq 0$, the second order phase transformation $\Omega_x^{2^{\mathrm{nd}}}(a,b;\sigma(b))$ defined by (6.4.2.9) is the exact IF of $x(t)$; namely, $\Omega_x^{2^{\mathrm{nd}}}(a,b;\sigma(b)) = \phi'(b) = c+rb$. This completes the proof of Theorem 6.4.1. ■

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