ℤ-graded supergeometry: Differential graded modules, higher algebroid representations, and linear structures

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To my family and Julia...
...for their encouragement and support

Abstract

The purpose of this thesis is to present a self-standing review of $\mathbb{Z}$- (respectively $\mathbb{N}$-)graded supergeometry with emphasis in the development and study of two particular structures therein. Namely, representation theory and linear structures of $\mathcal{Q}$-manifolds and higher Lie algebroids (also known in the mathematics and physics literature as $\mathbb{Z}\mathcal{Q}$- and $\mathbb{N}\mathcal{Q}$-manifolds, respectively).

Regarding the first notion, we introduce differential graded modules (or for short DG-modules) of $\mathcal{Q}$-manifolds and the equivalent notion of representations up to homotopy in the case of Lie n-algebroids ($n \in \mathbb{N}$). These are generalisations of the homonymous structures of [93, 42, 4] that exist already in the case of ordinary Lie algebroids, i.e. when $n = 1$. The adjoint and coadjoint modules are described, and the corresponding split versions of the adjoint and coadjoint representations up to homotopy of Lie n-algebroids are explained. In particular, the case of Lie 2-algebroids is analysed in detail. The compatibility of a graded Poisson bracket with the homological vector field on a $\mathbb{Z}$-graded manifold is shown to be equivalent to an (anti-)morphism from the coadjoint module to the adjoint module, leading to an alternative characterisation of non-degeneracy of graded Poisson structures. Applying this result to symplectic Lie 2-algebroids, gives another algebraic characterisation of Courant algebroids in terms of their adjoint and coadjoint representations. In addition, the Weil algebra of a general $\mathcal{Q}$-manifold is defined and is computed explicitly in the case of Lie n-algebroids over a base (smooth) manifold M together with a choice of a splitting and linear TM-connections. Similarly to the work of Abad and Crainic in [4], our computation involves the coadjoint representation of the Lie n-algebroid and the induced 2-term representations up to homotopy of the tangent bundle TM on the vector bundles of the underlying complex of the Lie n-algebroid given by the choice of the linear connections.

The second object that we define and explore in this work is the linear structures on $\mathbb{Z}$-graded manifolds, for which we see the connection with DG-modules and representations up to homotopy. In the world of split Lie n-algebroids, this leads to the notion of higher VB-algebroids, which we call VB-Lie n-algebroids; that is, Lie n-algebroids that are in some sense linear over another Lie n-algebroid. We prove that there is an equivalence between the category of VB-Lie n-algebroids over a Lie n-algebroid $\underline{A}$ and the category of $(n + 1)$-term representations up to homotopy of $\underline{A}$, generalising a well-known result from the theory of ordinary VB-algebroids over Lie algebroids, i.e., in our setting, VB-Lie 1-algebroids over Lie 1-algebroids.

Keywords: graded manifold, supermanifold, $\mathcal{Q}$-manifold, Lie n-algebroid, VB-algebroid, representation up to homotopy, differential graded module, adjoint representation, adjoint module, Weil algebra.

Περίληψη

Ο σκοπός της παρούσας διδακτορικής διατριβής είναι να παρουσιάσει μια αυτάρκη ανασκόπηση των $\mathbb{Z}$- και $\mathbb{N}$-βαθμωτών υπεργεωμετριών ($\mathbb{Z}$- και $\mathbb{N}$-graded supergeometry) με έμφαση στην ανάπτυξη και μελέτη δύο συγκεκριμένων μαθηματικών κατασκευών. Πιο συγκεκριμένα, στην ανάπτυξη και μελέτη θεωρίας αναπαραστάσεων και γραμμικών δομών σε $\mathcal{Q}$-πολλαπλότητες ($\mathcal{Q}$-manifolds), και ανώτερα αλγεβροειδή Lie (higher Lie algebroids), γνωστά στη μαθηματική και φυσική βιβλιογραφία και ως $\mathbb{Z}\mathcal{Q}$- και $\mathbb{N}\mathcal{Q}$-πολλαπλότητες αντίστοιχα ($\mathbb{Z}\mathcal{Q}$- και $\mathbb{N}\mathcal{Q}$-manifolds).

Όσον αφορά την πρώτη κατασκευή, αυτή της θεωρίας αναπαραστάσεων, εισάγουμε την έννοια των βαθμωτών διαφορικών προτύπων σε $\mathcal{Q}$-πολλαπλότητες (differential graded modules ή DG-modules) και την ισοδύναμη έννοια των αναπαραστάσεων ως προς ομοτοπία (representations up to homotopy) στην περίπτωση των n-αλγεβροειδών Lie για $n \in \mathbb{N}$. Οι ορισμοί αυτοί είναι γενικεύσεις των ήδη υπαρχουσών ομώνυμων εννοιών από την περίπτωση των συνήθων αλγεβροειδών Lie, δηλ. όταν ισχύει $n = 1$, οι οποίοι εμφανίζονται και μελετώνται στα άρθρα [93, 42, 4]. Ορίζουμε και περιγράφουμε τα συζυγές (adjoint) και συν-συζυγές (coadjoint) πρότυπα όπως επίσης και τις αντίστοιχες συζυγείς και συν-συζυγείς αναπαραστάσεις ως προς ομοτοπία στην περίπτωση των n-αλγεβροειδών Lie. Συγκεκριμένα η περίπτωση των 2-αλγεβροειδών Lie αναλύεται ενδελεχώς. Αποδεικνύουμε πως η συμβιβαστότητα ενός βαθμωτού γινομένου Poisson (graded Poisson bracket) και ενός ομολογικού διανυσματικού πεδίου[1] σε μία $\mathbb{Z}$-βαθμωτή πολλαπλότητα είναι ισοδύναμη με την ύπαρξη ενός (αντι-)μορφισμού από το συν-συζυγές στο συζυγές πρότυπο, του οποίου επακόλουθο είναι ενάς εναλακτικός χαρακτηρισμός μη-εκφυλισιμότητας (non-degeneracy) βαθμωτών δομών Poisson. Εφαρμόζοντας αυτό το αποτέλεσμα στην περίπτωση των συμπλεκτικών 2-αλγεβροειδών Lie, παίρνουμε έναν άλλον αλγεβρικό χαρακτηρισμό αλγεβροειδών Courant βάση των συζυγών και συν-συζυγών τους προτύπων. Επιπλέον, ορίζουμε την άλγεβρα Weil μιας γενικής $\mathbb{Z}$-βαθμωτής πολλαπλότητας και την περιγράφουμε λεπτομερώς στην περίπτωση των n-αλγεβροειδών Lie πάνω από μία λεία πολλαπλότητα M μαζί με μία επιλογή διάσπασης (splitting) και γραμμικών TM-συνοχών. Όπως και στην εργασία των Abad και Crainic [4], ο υπολογισμός μας περιλαμβάνει την συν-συζυγή αναπαράσταση ως προς ομοτοπία του n-αλγεβροειδούς Lie και τις επαγόμενες αναπαραστάσεις ως προς ομοτοπία 2ου βαθμού της εφαπτομενικής δέσμης TM στις διανυσματικές δέσμες υπεράνω του M που καταστευάζονται απο την επιλογή των γραμμικών συνοχών και σχηματίζουν το αλυσιδωτό σύμπλοκο (chain complex) του n-αλγεβροειδούς Lie.

Το δεύτερο μαθηματικό αντικείμενο που ορίζουμε και μελετάμε σε αυτή την εργασία είναι οι γραμμικές δομές σε $\mathbb{Z}$-βαθμωτές πολλαπλότητες, όπου βλέπουμε τη σύνδεση τους με τα διαφορικά βαθμωτά πρότυπα και τις αναπαραστάσεις ως προς ομοτοπία. Στην περίπτωση των διασπασμένων (split) n-αλγεβροειδών Lie, η κατασκευή αυτή οδηγεί στον ορισμό ανώτερων VB-αλγεβροειδών, τα οποία ονομάζουμε ανώτερα VB-n-αλγεβροειδή Lie. Δηλαδή n-αλγεβροειδή Lie τα οποία κατά μία έννοια είναι 'γραμμικά' πάνω άπο ένα άλλο n-αλγεβροειδές Lie. Αποδεικνύουμε την ύπαρξη μιας ισοδυναμίας κατηγοριών μεταξύ της κατηγορίας των VB-n-αλγεβροειδών Lie πάνω από το n-αλγεβροειδές Lie $\underline{A} \to M$ και της κατηγορίας των αναπαραστάσεων ως προς ομοτοπία $(n+1)$-ου βαθμού του $\underline{A} \to M$, γενικεύοντας έτσι ένα γνωστό αποτέλεσμα από τη θεωρία των συνήθων VB-αλγεβροειδών Lie, ήτοι στην δική μας ορολογία των VB-1-αλγεβροειδών Lie πάνω από 1-αλγεβροειδή Lie.

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[1] Ήτοι το ομολογικό διανυσματικό πεδίο είναι Poisson.

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Contents

Chapter 1

Introduction

Throughout history mathematics and physics have always been complementary to each other in our way of understanding and explaining our universe. They have constantly influenced one another, with the exchanging of ideas and results going both ways: On the one hand physicists have applied theoretical results of mathematics to obtain physical and practical outcomes, while on the other hand, many physical ideas have served as inspiration for mathematicians to develop abstract mathematical theories. Their notable interaction in modern research can be seen by two of the most elegant and closely related branches of modern mathematics and theoretical physics, namely *Differential Geometry* and *Mathematical Physics*; in short, the former studies *(differentiable) manifolds* which is the idea of "space" in its most general form and the latter deals with the mathematical methods suitable for physical theories.

There are two fundamental concepts of Differential Geometry that are needed in physics and are relevant for this thesis. The first is the notion of *Poisson* and *symplectic* structures on manifolds which serve as phase spaces in the Hamiltonian description of classical mechanics. The second comes from the spaces known as *Lie groups* together with their infinitesimal counterparts called *Lie algebras*, or more generally *Lie groupoids* and *Lie algebroids*, which provide a rigorous mathematical description of the concept of *symmetry*. A symmetry of a physical system is a characteristic of the system that is preserved under some transformation and its physical importance can be deduced from the famous Noether's Theorem which states that each physical symmetry has a corresponding conservation law. Some remarkable examples of symmetry groups in physics are the Poincaré group $\mathbb{R}^{1,3} \rtimes O(1,3)$ in Special Relativity, the unitary group $U(1)$ in quantum electrodynamics and the product group $U(1) \times SU(2) \times SU(3)$ in the Standard Model of particle physics. Roughly, the symmetries of the Poincaré group correspond to conservation of mass-energy, conservation of linear momentum, and conservation of angular momentum; the conservation law of the unitary group is the conservation of electric charge; the conservation laws of $U(1) \times SU(2) \times SU(3)$ are the conservation of electric charge, the conservation of weak isospin and the conservation of color charge.

The present work studies a common subfield of Differential Geometry and Mathematical Physics

known as $\mathbb{Z}$-*graded supergeometry*. Our main mathematical objects will be a generalisation of manifolds called *graded manifolds* as well as *higher Poisson structures* and *$\mathcal{Q}$-manifolds*, with emphasis on the special case of *higher (split) Lie algebroids* for which we will develop a well-behaved *Representation Theory*. These terms will be explained briefly in the rest of the introduction where we provide some historical background and sketch of what will be analysed later in great detail.

Poisson and symplectic structures – A historical overview   The first appearance of the Poisson bracket was considered already by Siméon Denis Poisson himself more than 200 years ago in his pioneering improvement of Lagrangian mechanics [78] as an operation on physical observables, i.e. functions of the phase space of classical mechanical systems. In this work, Poisson introduced the bracket

$$\{f,g\} := \sum_{i=1}^{m} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p^i} \right),$$

where f and g are functions of the variables q^i and $p^i := \frac{\partial L}{\partial \dot{q}^i}$ for a mechanical system with Lagrangian $L = L(q^i, \dot{q}^i)$, and proved that if the functions f and g are first integrals of the system, i.e. they are preserved under the dynamics of the system, then so is the function $\{f,g\}$. The Poisson bracket was then used by William Rowan Hamilton in 1835 [44] to express his equations of motion

$$\dot{p}^i = -\frac{\partial H}{\partial q^i} \qquad \text{and} \qquad \dot{q}^i = \frac{\partial H}{\partial p^i},$$

with the function H given by $H = \sum_{i=1}^{m} \dot{q}^i p^i - L$ and called *Hamiltonian* function of the system. Thus the theory that is now called Hamiltonian mechanics was established as an improvement of Newtonian mechanics. Seven to eight years later, Carl Jacobi [47] proved that the Poisson bracket satisfies what is now known as the *Jacobi equation*:

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}.$$

In a modern terminology, a *Poisson structure* on a smooth manifold M is a bivector field $\pi \in \mathfrak{X}^2(M) := \Gamma(\wedge^2 TM)$ such that the corresponding bracket $\{f,g\} := \langle \mathrm{d}f \wedge \mathrm{d}g, \pi \rangle$ satisfies the Jacobi identity. The prototype example is $M = \mathbb{R}^{2m}$ with coordinates $(q^1, \ldots, q^m, p^1, \ldots, p^m)$ and

$$\pi = \sum_{i=1}^{m} \frac{\partial}{\partial p^i} \wedge \frac{\partial}{\partial q^i} \tag{1.1}$$

with the corresponding bracket being the one discovered by Poisson. Given a Poisson structure on the manifold M, there is an induced $C^\infty(M)$-linear map $\pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ which sends the 1-form α to the vector field $i_\alpha \pi$, and in particular, the exact 1-form $\mathrm{d}f$ to the vector field $X_f := \{f, \cdot\}$ called the *Hamiltonian vector field of f*. If the map $\pi^\sharp$ is non-degenerate (i.e. it is an isomorphism), the Poisson structure is called *symplectic* and is equivalent to a 2-form $\omega \in \Omega^2(M)$ (called *symplectic form*) which is closed and non-degenerate, i.e. $\mathrm{d}\omega = 0$ and the map $(\pi^\sharp)^{-1} = \omega^\flat : \mathfrak{X}(M) \to \Omega^1(M), X \mapsto i_X \omega = \omega(X, \cdot)$ is an isomorphism. The 2-form that corresponds to

the Poisson bracket on $M = \mathbb{R}^{2m}$ is defined by $\omega = \sum_{i=1}^{m} \mathrm{d}q^i \wedge \mathrm{d}p^i$. Using the above formalism, Hamilton's equations of motion on a symplectic manifold (M, ω) with Hamiltonian H are just the integral curves of the vector field $\{\cdot, H\} = -X_H$.

Although not all Poisson structures come from a symplectic form and thus cannot be written in the form of equation (1.1), symplectic structures are still present in their local structure. This breakthrough was achieved in 80's with the famous paper of Weinstein [104] which proved the *local splitting of Poisson manifolds*: Locally around every point, a Poisson structure is a product of a symplectic structure with a degenerate Poisson structure that vanishes at that point. The theorem is essentially a generalisation of Darboux's theorem in symplectic geometry. In the next years, many splitting theorems were proved for other similar structures such as Lie algebroids [36, 39, 105], generalised complex manifolds [1], Dirac structures [13], Courant algebroids and L_∞-algebroids [12], etc.

Nowadays, Poisson geometry is a very active field of research in mathematics with connections to numerous areas, such as non-commutative geometry, topological field theories, representation theory, etc. The interested reader may find many graduate texts written on the topic, for example [37, 64, 94, 31, 103].

Lie groupoids and Lie algebroids   The reader who has taken a standard undergraduate course in differential geometry may be familiar with Lie groups and Lie algebras. A Lie group is a group (in the algebraic sense) which is also a smooth manifold and its elements can be smoothly inverted and multiplied in pairs. A Lie algebra is a vector space equipped with a bracket operation $[\cdot, \cdot]$ that is bilinear, anti-symmetric and satisfies the Jacobi identity. As it was mentioned earlier, the importance of Lie groups and Lie algebras is that they offer a mathematical treatment for simple symmetries in physics. A Lie groupoid, denoted $\mathcal{G} \rightrightarrows M$, is in some sense a "smooth" collection of Lie groups and hence provides a systematic way of describing more complicated symmetries that a single group would fail to capture [106]. As a first short definition we give the following: a *Lie groupoid* is a small category such that every morphism is an isomorphism with all the objects being smooth. The detailed description of its definition is long and so we postpone it to Chapter 9. The situation gets a little bit easier if one considers Lie algebroids, which can be thought of as a "linear approximation" of Lie groupoids and so may reflect some of their properties. They are in some sense infinite dimensional Lie algebras over a possible "curved" space, and serve as a common generalisation of Lie algebras and tangent vector bundles. Their close relation to Poisson manifolds comes from the fact that a Lie algebroid structure on a vector bundle $A \to M$ is equivalent to a Poisson bracket on the manifold A^* that is fibre-wise linear. The precise mathematical definition is as follows.

Definition. A *Lie algebroid* is a smooth vector bundle $A \to M$ together with a smooth vector bundle map $\rho : A \to TM$ over the identity on M, called *anchor*, and a Lie bracket $[\cdot, \cdot]$ on its space of sections $\Gamma(A)$ which are compatible: $\rho([a, b]) = [\rho(a), \rho(b)]$ and satisfy the *Leibniz identity*

$$[a, fb] = \rho(a)f \cdot b + f[a, b]$$

for all $f \in C^\infty(M), a, b \in \Gamma(A)$. Equivalently, a Lie algebroid can be defined as a vector bundle $A \to M$ together with an operator $\mathrm{d}_A : \Omega^\bullet(A) := \Gamma(\wedge^\bullet A^*) \to \Omega^{\bullet+1}(A)$ that squares to zero: $\mathrm{d}_A^2 = 0$ and satisfies

$$\mathrm{d}_A(\alpha \wedge \beta) = \mathrm{d}_A\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \mathrm{d}_A\beta.$$

One obtains a Lie algebroid from a Lie groupoid pretty much the same way that a Lie algebra is obtained by a Lie group, i.e. by differentiation. The interesting problem was the integration of Lie algebroids, i.e. under what circumstances a Lie algebroid comes in this manner from a Lie groupoid. This was solved in 2001 in the work of Cattaneo and Felder [22], and in 2003 in the paper by Crainic and Fernandes [26].

Courant algebroids   In the 90's the works of Courant [25] and Dorfman [34] in mechanical systems with constraints led them to the discovery of what is now known as the *Courant bracket* on sections of the vector bundle $\mathbb{T}M := TM \oplus T^*M$:

$$[\![X + \omega, Y + \eta]\!] = [X, Y] + \pounds_X\eta - i_Y\mathrm{d}\omega$$

for all $X, Y \in \mathfrak{X}(M), \omega, \eta \in \Omega^1(M)$. The abstraction of this idea was done a few years later by Liu, Weinstein and Xu [66], and gave rise to Lie bialgebroids and consequently to Courant algebroids.

Definition. A *Courant algebroid* is a smooth vector bundle $E \to M$ equipped with a fibre-wise non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : E \times_M E \to \mathbb{R}$, a bilinear bracket $[\![\cdot, \cdot]\!]$ on the smooth sections $\Gamma(E)$, and an anchor map $\rho : E \to TM$, which satisfy the following conditions: for $f \in C^\infty(M), e_1, e_2, e_3 \in \Gamma(E)$

1. $\rho([\![e_1, e_2]\!]) = [\rho(e_1), \rho(e_2)]$

2. $[\![e_1, fe_2]\!] = \rho(e_1)f \cdot e_2 + f[\![e_1, e_2]\!]$

3. $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$

4. $\rho(e_1)\langle e_2, e_3 \rangle = \langle[\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$

5. $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2 \rangle$.

Here, we use the notation $\mathcal{D} := \rho^* \circ \mathrm{d}$ and identify E with E^* via the pairing: $\langle\mathcal{D}(f), e\rangle = \rho(e)f$ for all $f \in C^\infty(M), e \in \Gamma(E)$. Later it was shown that the first two axioms above can be deduced from the other three.

Introducing *Dirac structures*, which were also the original insight of Courant and Weinstein's work, leads one to view Courant algebroids as a suitable framework for simultaneous treatment of pre-symplectic and Poisson structures. A Dirac structure of a Courant algebroid $E \to M$ is a vector subbundle $D \to M$ which is maximally isotropic with respect to $\langle \cdot, \cdot \rangle$ and integrable:

$$\langle D, D \rangle = 0, \qquad \mathrm{rank}(D) = \frac{1}{2}\,\mathrm{rank}(E), \qquad [\![\Gamma(D), \Gamma(D)]\!] \subset \Gamma(D).$$

Then the graphs of pre-symplectic forms $\omega \in \Omega^2(M)$ (i.e. $\mathrm{d}\omega = 0$) and Poisson tensors $\pi \in \mathfrak{X}^2(M)$ are Dirac structures of the standard Courant algebroid $\mathbb{T}M$:

$$\mathrm{Graph}(\omega) := \{v + i_v\omega \mid v \in TM\} \qquad \text{and} \qquad \mathrm{Graph}(\pi) := \{i_\alpha\pi + \alpha \mid \alpha \in T^*M\}.$$

In addition, Courant algebroids and Dirac structures are important in modern mathematical physics due to the fact that they are the fundamental objects in generalised complex geometry and hence in mirror symmetry.

Origin of supergeometry Supermanifolds with a $\mathbb{Z}_2$-grading were introduced already in the 60's in the pioneering work of Berezin [10]. They appear often in the physics literature, as they have applications in superstring theory due to their success in providing a formal mathematical description of supersymmetric field theories [84]. Intuitively, one should think of them as spaces with two kinds of coordinate functions: *even* and *odd*, or in physicists' language *bosonic* and *fermionic*. The characteristic difference of these two coordinates is that the former are commutative, while the latter are anti-commutative; that is, a $\mathbb{Z}_2$-*graded supermanifold* is a space which locally admits a coordinate representation of the form (x^i, θ^j), where x^i are ordinary real-valued spacetime coordinates and θ^j are "formal" Grassmann-valued coordinates satisfying

$$x^i x^j = x^j x^i \qquad \text{and} \qquad \theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha.$$

This setting encodes the quantum viewpoint of the cosmos in which particles are divided into two categories depending on their spin: *bosons* and *fermions*; the former commute, while the latter anti-commute.

In a more mathematical terminology, the definition of a $\mathbb{Z}_2$-graded supermanifold of dimension $p|q$ is sheaf-theoretic, defining it as an ordinary p-dimensional manifold equipped with a sheaf of $\mathbb{Z}_2$-graded algebras that locally looks like the algebra of functions of the $p|q$-dimensional *superspace*: $C^\infty(\mathbb{R}^p) \otimes \Lambda^\bullet(\xi^1, \ldots, \xi^q)$; here, $\Lambda^\bullet(\xi^1, \ldots, \xi^q)$ is the Grassmann algebra on q generators. Locally, the functions (or physical observables) on supermanifolds take the form of a "power series" in the odd coordinates θ^j with coefficients in the ring of smooth functions over $\mathbb{R}^p$ with coordinates $x = (x^1, \ldots x^p)$:

$$f(x, \theta) = f_0(x) + \sum_{r=1}^{q} \sum_{i_1 < \ldots < i_r} f_{i_1 \ldots i_r}(x)\, \theta^{i_1} \ldots \theta^{i_r}.$$

Some mathematical works on this subject were written by Berezin [11], Kostant [60], Deligne and Morgan [33], Manin [70], Tuynman [92], Varadarajan [97], and Carmeli, Caston and Fioresi [20].

Introduction of $\mathbb{Z}$- and $\mathbb{N}$-gradings The possibility of more general gradings was mentioned already in the works of Kostant [60] and Tuynman [92] but was not studied there. The use of a $\mathbb{Z}$-grading was treated properly in the PhD thesis of Mehta [71], who was inspired by previous works on graded geometry from Kontsevich [57], Roytenberg [83], Ševera [87], and Voronov [100]. Similarly to the $\mathbb{Z}_2$-case, a $\mathbb{Z}$- (respectively $\mathbb{N}$-)graded manifold can be thought of as a space whose

coordinates admit a $\mathbb{Z}$- (respectively $\mathbb{N}$-)grading and may commute or anti-commute depending on their degree:

$$\xi_1\xi_2 = (-1)^{\deg(\xi_1)\deg(\xi_2)}\xi_2\xi_1.$$

In sheaf-theoretic terminology, the structure sheaf of the graded manifold is now a sheaf of $\mathbb{Z}$-(respectively $\mathbb{N}$-)graded algebras and locally the functions look like a sum of the form

$$f(x,\xi) = f_0(x) + \sum_{r=1}^{\infty}\sum_{i_1 \leq \ldots \leq i_r} f_{i_1\ldots i_r}(x)\,\xi^{i_1}\ldots\xi^{i_r}.$$

The important result of the works mentioned above was the realisation that many complicated definitions of classical differential geometric objects can be equivalently described in very simple terms in the language of graded geometry. The table at the end of this chapter offers a summary of the equivalences between graded and classical geometries.

$\mathcal{Q}$-structures, higher (split) Lie algebroids and Poisson brackets   A $\mathcal{Q}$-*manifold* (also known in the literature as differential graded manifold, or DG-manifold) is a $\mathbb{Z}$-graded manifold equipped with a degree 1 vector field $\mathcal{Q}$ that squares to zero: $\mathcal{Q}^2 = 0$. In the special case where the underlying manifold is $\mathbb{N}$-graded of degree $n \in \mathbb{N}$ (i.e. the highest degree of its coordinates is n), it is also called $\mathbb{N}\mathcal{Q}$-manifold of degree n or Lie n-algebroid. Alternatively, a Lie n-algebroid can be described by means of a graded vector bundle together with a family of higher brackets on its space of sections [88]. In this case, we refer to them as *split* Lie n-algebroids. The name comes from the equivalence of Lie 1-algebroids and ordinary Lie algebroids as in the definition above [93]. Other correspondences of graded structures with classical objects can be realised by extending Poisson geometry to the graded world. A Poisson structure on a graded manifold, known as $\mathcal{P}$-manifold, is a bracket on its space of (graded) functions which satisfies graded versions of anti-symmetry, Leibniz and Jacobi identities. The compatibility of the vector field $\mathcal{Q}$ and the Poisson bracket is known as a $\mathcal{PQ}$-structure or Poisson Lie n-algebroid in the case of an $\mathbb{N}$-grading.

Representations of Lie groupoids and Lie algebroids   The goal of Representation Theory in mathematics is to study general abstract structures by "representing" them as simpler and more concrete objects. Informally, a representation is an action of a mathematical structure which is in some sense "linear". A crucial representation theory is that of Lie groups and Lie algebras, whose objects are represented as matrices over a linear vector space. Formally, a representation of a Lie group G or a Lie algebra $\mathfrak{g}$ is a vector space V together with a Lie group homomorphism $G \to \mathrm{GL}(V)$, respectively a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathrm{GL}(V)$ is the general linear group of all invertible linear transformations of V under their composition and $\mathfrak{gl}(V)$ is the space of endomorphisms of V equipped with the Lie bracket given by the commutator of endomorphisms $[\phi,\psi] := \phi \circ \psi - \psi \circ \phi$, for $\phi,\psi \in \mathfrak{gl}(V)$. The basic example of a representation of a Lie group or a Lie algebra is the *adjoint representation*: The adjoint representation of a Lie algebra $\mathfrak{g}$ is given by the vector space $\mathfrak{g}$ together with the Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $x \mapsto \mathrm{ad}_x := [x,\cdot]$.

Given a Lie group G with corresponding Lie algebra $\mathfrak{g}$, its adjoint representation is the vector space $\mathfrak{g}$ together with the Lie group homomorphism $G \to \mathfrak{gl}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g := \mathrm{d}\Psi_g|_e$, where $e \in G$ is the identity element of G and $\Psi_g : G \to G, \Psi_g(h) := ghg^{-1}$ is the conjugation with respect to the element $g \in G$. In fact, one can differentiate a Lie group representation to obtain a representation of its Lie algebra and this process links the two adjoint representations defined above. The converse procedure of integrating a Lie algebra representation to obtain a representation of its Lie group is also possible under the extra condition that G is a simply connected Lie group.

Passing to representations of Lie groupoids and Lie algebroids, the role of the vector space V is played by a smooth vector bundle $E \to M$. Precisely, a representation of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a vector bundle $E \to M$ together with a groupoid homomorphism $\mathcal{G} \to \mathrm{GL}(E)$ covering the identity map on M; here, $\mathrm{GL}(E) \rightrightarrows M$ denotes the *general linear groupoid* of E. Similarly, a representation of a Lie algebroid $A \to M$ is a vector bundle $E \to M$ together with a Lie algebroid homomorphism $A \to \mathrm{Der}(E)$, where $\mathrm{Der}(E)$ is the *derivation Lie algebroid* of E equipped with the commutator bracket; explicitly, $\mathrm{Der}(E)$ consists of $\mathbb{R}$-linear operators $\delta : \Gamma(E) \to \Gamma(E)$ such that there exists a vector field $X \in \mathfrak{X}(M)$ with the property that $\delta(f\sigma) = X(f)\sigma + f\delta(\sigma)$ for all $f \in C^\infty(M), \sigma \in \Gamma(E)$. That is, a representation of a Lie algebroid $A \to M$ can be defined as a vector bundle $E \to M$ together with an A-connection ∇ on the sections of E which is *flat*: $R_\nabla = 0$, i.e. $\nabla_{[a,b]} = \nabla_a\nabla_b - \nabla_b\nabla_a =: [\nabla_a, \nabla_b]$ for all $a, b \in \Gamma(A)$. Equivalently, it can be described by an exterior derivative $\mathrm{d}_\nabla : \Omega^\bullet(A, E) \to \Omega^{\bullet+1}(A, E)$ that squares to zero, called the *(action) differential*. From a more geometric point of view, a Lie algebroid representation of $A \to M$ on $E \to M$ is a Lie algebra map from the sections of A to the vector fields over the manifold E that are fibre-wise linear, i.e. their (local) flows consist of vector bundle automorphisms of E.

Representations up to homotopy – DG-modules[1]. Although the above notion of representations of a Lie algebroid seems to generalise well the notion of a Lie algebra representation, it suffers from a non-trivial problem: It does not include a well-defined notion of the adjoint representation of the Lie algebroid. Some early attempts to define the adjoint representation of a Lie algebroid can be traced back to Evens, Lu and Weinstein [38]. The nowadays accepted solution to this problem was done by Gracia-Saz and Mehta [42], and independently by Abad and Crainic [4], who showed that the notion of *representation up to homotopy of a Lie algebroid* is a good notion of representations of Lie algebroids which includes the adjoint representation. Roughly, the idea is to let the Lie algebroid A act via a differential on Lie algebroid forms $\Omega^\bullet(A, \underline{E})$ which take values on a cochain complex of vector bundles $\underline{E} = \bigoplus_i E_i[i]$ instead of just a single vector bundle E. This notion is essentially a $\mathbb{Z}$-graded analogue of Quillen's super-representations [81].

After their discovery, representations up to homotopy have been extensively studied in other works, see e.g. [73, 6, 7, 8, 35, 76, 90, 41, 19, 50, 16, 98]. In particular, the adjoint representation up to homotopy of a Lie algebroid is proving to be as fundamental in the study of Lie algebroids, as the adjoint representation of a Lie algebra is in the study of Lie algebras. In that context, the adjoint representation controls deformations of Lie algebras (see e.g. [27] and references therein),

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[1]This paragraph coincides with the introduction written for [53]

symmetries of a Lie algebra, and it is a key to the classification and the algebraic integration of Lie algebras [95, 96]. The adjoint representation up to homotopy of a Lie algebroid appears naturally in each attempt to understand a feature of Lie algebras in the more general context of Lie algebroids. In particular, the deformations of a Lie algebroid are controlled by the cohomology with values in its adjoint representation up to homotopy [93, 75], and an ideal in a Lie algebroid is a subrepresentation of the adjoint representation up to homotopy [35]. While a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is a pair of Lie algebras on dual vector spaces whose coadjoint representations form a matched pair, a Lie bialgebroid is a pair of Lie algebroids on vector bundles in duality, whose coadjoint 2-representations form a matched pair (see Theorem 3.11 in [41]). From another point of view, 2-term representations up to homotopy, which are equivalent to decompositions of VB-algebroids [42], have proved to be a powerful tool in the study of multiplicative structures on Lie groupoids (se e.g. [54, 35, 18, 3]), which, at the infinitesimal level, can be described as linear structures on algebroids. An algebraic viewpoint of representations up to homotopy of Lie algebroids was achieved by Mehta in [73], where he showed that representations up to homotopy of a Lie algebroid $A \to M$ are equivalent, up to isomorphism, to *Lie algebroid modules over A* in the sense of [93]; that is, differential graded modules over the differential graded algebra $(\Omega(A), \mathrm{d}_A)$.

In this work, we extend the above notions of modules, and consequently of representations up to homotopy, to the context of higher Lie algebroids. The definition is the natural generalisation of the case of usual Lie algebroids explained above, i.e. differential graded modules over the space of smooth functions of the underlying graded manifold. The obtained notion is analysed in detail, including the two most important examples of representations, namely, the adjoint and the coadjoint representations (up to homotopy).

In addition to the impact of representations up to homotopy in the study of Lie algebroids in the last ten years, our general motivation for studying representations up to homotopy of higher Lie n-algebroids comes from the case $n = 2$, and in particular from Courant algebroids. As it was pointed out by Mehta, in light of AKSZ theory, it seems reasonable to expect that the category of representations (up to homotopy) of Courant algebroids might have interesting connections to 3-dimensional topology. The results in this thesis should be useful in the study of such representations. The first step is the search for a good notion not only of the adjoint representation of a Courant algebroid, but also of its ideals, similar to the work done in [54]. Since Courant algebroids are equivalent to Lie 2-algebroids with a compatible symplectic structure [83, 87], the natural question that arises is the following:

Question. Is a compatible Poisson or symplectic structure on a Lie n-algebroid encoded in its adjoint representation?

The answer to this question is positive, since it turns out that a Poisson bracket on a Lie n-algebroid gives rise to a natural map from the coadjoint to the adjoint representation which is an anti-morphism of left representations and a morphism of right representations (see Theorem 5.3.1, Corollary 5.3.2 and Section 6.8), i.e. it anti-commutes with the differentials of their structure as left representations and commutes with the differentials of their structure as right representations.

Further, the Poisson structure is symplectic if and only if this map is in fact a left anti-isomorphism and right isomorphism. This result is already known in some special cases, including Poisson Lie 0-algebroids, i.e. ordinary Poisson manifolds $(M, \{\cdot, \cdot\})$, and Courant algebroids over a point, i.e. quadratic Lie algebras $(\mathfrak{g}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$. In the former case the map reduces to the natural map $\sharp : T^*M \to TM$ obtained from the Poisson bracket on M, and in the latter case it is the inverse of the map defined by the non-degenerate pairing $\mathfrak{g} \to \mathfrak{g}^*, x \mapsto \langle x, \cdot \rangle$.

Linear structures Similarly to the case of ordinary manifolds, one can consider vector bundles over $\mathbb{Z}$-manifolds [71]. Geometrically, they can be viewed as graded manifolds that admit two kinds of (graded) local coordinates: smooth and linear. The structures explained before may be defined to be fibre-wise linear leading to *linear $\mathcal{Q}$-manifolds*, *linear $\mathcal{P}$-manifolds*, etc. In this setting, a linear split Lie n-algebroid gives rise to the notion of *split VB-Lie n-algebroids* (Chapter 8); that is, double vector bundles in the sense of Mackenzie [67] with a linear split Lie n-algebroid structure over a split Lie n-algebroid side bundle. In fact, a linear $\mathcal{Q}$-structure on a vector bundle in the category of graded manifolds is the same as a differential graded module in the sense described above. From a classical geometric point of view, a representation up to homotopy of a split Lie n-algebroid is equivalent to a split VB-Lie n-algebroid (Theorem 8.5.2, Proposition 8.5.3 and Theorem 8.5.4).

Remark on collaboration Parts of the present work (differential graded modules, representations up to homotopy of Lie n-algebroids, and some results on VB-Lie n-algebroids) were done jointly with the author's PhD supervisor Madeleine Jotz Lean and Rajan Mehta. The idea of studying representations up to homotopy of higher Lie n-algebroids was proposed by Mehta, whose approach was more algebraic via module theory, while Jotz Lean's approach was differential geometric via split structures and representations up to homotopy. For example, the two view points of the adjoint module/representation of a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ are given by $(\mathfrak{X}(\mathcal{M}), \pounds_{\mathcal{Q}})$ (algebraic definition proposed by Mehta – Chapter 5) and the complex $TM[0] \oplus A_1[1] \oplus \ldots \oplus A_n[n]$ (geometric, or split, definition proposed by Jotz Lean – Chapter 6), where $A_1[1] \oplus \ldots \oplus A_n[n] \simeq \mathcal{M}$ is a splitting of the underlying [2]-manifold over the base smooth manifold M. As it was claimed by Mehta, it should be possible that the two approaches can be unified. The present thesis achieves this result by giving the precise isomorphism connecting the adjoint module and the adjoint representation up to homotopy (Section 6.4 for the case of Lie 2-algebroids and Section 6.6 for the general case of Lie n-algebroids). At the time of submission of this thesis, the results of the joint work with Madeleine Jotz Lean and Rajan Mehta are submitted for publication and can also been found online in the following preprint [53]:

"Modules and representations up to homotopy of Lie n-algebroids", arXiv:2001.01101, 2020

A more detailed list of the results found (in a more concise form) in the above preprint is the following:

- Differential graded modules (Chapter 5): Basic definitions, adjoint and coadjoint modules, map between coadjoint and adjoint modules of $\mathcal{PQ}$-manifolds (sections 5.1, 5.2, and 5.3).

- Representations up to homotopy (Chapter 6): Basic definitions, detailed analysis of the case of a Lie 2-algebroid and its adjoint representation, comparison of adjoint module with adjoint representation (sections 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8).

- Higher VB-algebroid structures (Chapter 8): Differential geometric definition of VB-Lie n-algebroids and correspondence of VB-Lie n-algebroids with $(n+1)$-representations (parts of sections 8.1 and 8.5).

New results and main achievements of the thesis

- The development of a well-behaved representation theory of Lie n-algebroids for general $n \in \mathbb{N}$ – Chapter 5 and Chapter 6.

- The precise connection between the adjoint module and the adjoint representation of Lie n-algebroids. In particular, a recipe for computing explicitly the structure objects of the adjoint representation of any Lie n-algebroid – Chapter 6 and more precisely Section 6.4 and Section 6.6.

- The explicit formulae for the structure objects that make up 3-term representations of a split Lie 2-algebroid and in particular the thorough analysis of its adjoint representation – Chapter 6 and more precisely Section 6.2, Section 6.3, and Section 6.5.

- An alternative description of Courant algebroids in terms of the adjoint and coadjoint modules of the underlying Lie 2-algebroid – Chapter 5 and more precisely Section 5.3.

- The description of linear structures on vector bundles in the category of graded manifolds and, in particular, the introduction of the notion of higher VB-Lie algebroids in the split setting – Chapter 7 and Chapter 8.

- The equivalence between the category of representations up to homotopy of higher Lie algebroids and the category of higher VB-Lie algebroids – Chapter 8 and more precisely Section 8.5 and Section 8.6.

A list of important propositions/theorems of the thesis grouped by chapter is the following:

- <u>Chapter 5</u>: Theorem 5.3.1, Theorem 5.4.1.

- <u>Chapter 6</u>: Proposition 6.3.1 (together with Remark 6.3.2), Proposition 6.5.2 and Proposition 6.5.4.

- <u>Chapter 8</u>: Theorem 8.5.2, Proposition 8.5.3 and Theorem 8.5.4.

Outline of the thesis

- Chapter 2: This chapter sets some notation and conventions, and introduces the fundamental constructions needed for the rest of the work. In particular, it recalls graded vector bundles and complexes, different notions of algebroids, double vector bundles, and the language of sheaves.

- Chapter 3: In this chapter, we recall the basic notions from graded geometry. Specifically, we define $\mathbb{Z}$- and $\mathbb{N}$-graded manifolds, graded vector fields and $\mathcal{Q}$-structures, Lie n-algebroids, and the graded generalisation of Poisson brackets.

- Chapter 4: This chapter is devoted to the category of vector bundles over graded manifolds and in particular to the tangent and cotangent bundles. We explain various geometric constructions such as (pseudo)differential forms and (pseudo)multivector fields, the Weil and the Poisson-Weil algebras of a $\mathcal{Q}$-manifold, homotopy Poisson structures, and graded symplectic forms. Moreover, we recall the correspondence of Poisson manifolds and Courant algebroids with degree 1 and 2 symplectic $\mathcal{Q}$-manifolds.

- Chapter 5: Here we introduce the notion of differential graded modules for $\mathcal{Q}$-manifolds and construct some important examples including the adjoint and coadjoint modules. We investigate the relation between adjoint and coadjoint modules, and Weil and Poisson-Weil algebras for the case of $\mathcal{PQ}$-manifolds and then the result is applied to Courant algebroids yielding the description in terms of its adjoint and coadjoint modules.

- Chapter 6: In this chapter, we generalise the notion of representations up to homotopy to higher Lie n-algebroids and spell out the structure objects for the case of a split Lie 2-algebroid. We give an explicit description of the adjoint and coadjoint representations of split Lie 2-algebroids, and explain how to compute the adjoint representation of any split Lie n-algebroid using its identification with the adjoint module. In addition, we describe the Weil algebra of a split Lie n-algebroid and compute the map between the coadjoint and the adjoint representation of a Poisson Lie n-algebroid for $n = 0, 1, 2$.

- Chapter 7: This chapter studies linear structures on vector bundles over graded manifolds. In particular, it recalls the space of linear multivector fields and defines linear $\mathcal{Q}$-manifolds and higher linear Poisson structures.

- Chapter 8: In this chapter, we introduce higher split VB-Lie algebroids and define their Weil algebra and the induced higher fat Lie algebroid. Additionally, we prove the equivalence between higher split VB-Lie algebroids and representations up to homotopy.

- Chapter 9: This chapter summarises the results of the thesis and suggests some further open research topics that are related to this work.

- Appendices: In Appendix A, there are some extra minor results on graded geometry, namely the geometrisation of $\mathbb{N}$-graded vector bundles over $\mathbb{N}$-manifolds, characteristic classes of 1st

order for general Lie n-algebroids, and a brief reminder of the Cartan calculus for $\mathbb{Z}$-manifolds. In Appendix B, we provide some long computations that were skipped in the main body of the thesis.

Relation to other work   During the preparation of this work, the author learnt that Caseiro and Laurent-Gengoux also consider representations up to homotopy of Lie n-algebroids, in particular the adjoint representation, in a work in preparation. In [98], Vitagliano considers representations of strongly homotopy of Lie-Rinehart algebras. This kind of algebras are the purely algebraic versions of graded vector bundles over graded manifolds equipped with a homological vector field that is tangent to the zero section. If the base manifold has grading concentrated in degree 0 and the vector bundle is negatively graded, the notion recovers the one of split Lie n-algebroids. In that case, Vitagliano's representations correspond to the representations up to homotopy considered in this work. In addition, since the DG $\mathcal{M}$-modules considered in this thesis are the sheaves of sections of $\mathcal{Q}$-vector bundles, they are themselves also special cases of Vitagliano's strongly homotopy Lie-Rinehart algebras.

Our study of linear structures is related to the great detailed works of La Pastina and Vitagliano [61, 62, 63], where they consider linear structures of Lie algebroids and Lie groupoids (called there vector bundles over Lie algebroids and Lie groupoids) and study their deformations. Most of the language they use is differential geometric, in contrast to our approach which is algebraic and more general since it covers Lie n-algebroids for general $n \in \mathbb{N}$.

Notation and conventions   Ordinary letters such as $M, N, \ldots, A, E, V, F, \ldots, X, Y, Z, \ldots$ denote classical differential geometric objects, i.e. (smooth) manifolds, (smooth) vector bundles, vector fields over smooth manifolds, etc. Underlined ordinary symbols denote graded objects of classical differential geometry, e.g., $\underline{E} \to M$ is a smooth graded vector bundle over the smooth manifold M, $\underline{\mathrm{Hom}}(\underline{E}, \underline{F})$ denotes the space of graded homomorphisms between $\underline{E} \to M$ and $\underline{F} \to M$, $\underline{S}(\underline{E})$ denotes the graded symmetric algebra of $\underline{E} \to M$, etc. Calligraphic letters such as $\mathcal{M}, \mathcal{N}, \ldots, \mathcal{A}, \mathcal{E}, \mathcal{F}, \ldots, \mathcal{Q}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ are used for objects of graded geometry, e.g., graded manifolds, graded vector bundles and modules, graded vector fields on graded manifolds, etc.

We work in the smooth category, unless otherwise stated. Our bundles are always assumed to be vector bundles and of finite rank, even if it is not stated explicitly. For many of our formulae, we adopt Einstein's summation convention, i.e. repeated upper and lower indices are assumed to be summed over all possible values.

Summary of equivalences between graded and classical geometries

Graded geometry	Classical geometry	Source
Degree 2 $\mathbb{N}$-manifolds	metric VB-algebroids	[32, 49]
Degree n $\mathbb{N}\mathcal{Q}$-manifolds/Lie n-algebroids	split Lie n-algebroids	[14, 88]
Lie 1-algebroids	Lie algebroids	[93]
Lie 2-algebroids	VB-Courant algebroids	[65, 50]
Poisson Lie 1-algebroids	Lie bialgebroids	[100]
Poisson Lie 2-algebroids	LA-Courant algebroids	[65, 52]
Symplectic Lie 1-algebroids (Lagrangian submanifolds)	Poisson manifolds (coisotropic submanifolds)	[87] ([23])
Symplectic Lie 2-algebroids (Lagrangian submanifolds)	Courant algebroids (Dirac structures)	[83, 87] ([83, 65])
Symplectic $\mathcal{Q}$-manifolds $T^*[k]A[1]$	higher Courant algebroids $A \oplus \wedge^{k-1}A^*$	[28, 29]
Degree 1 $\mathcal{Q}$-groupoids	LA-groupoids	[72]

Chapter 2

Preliminaries

In this chapter, we provide all the necessary notation and background that is required for understanding the mathematical content of this thesis. In what follows here and in the next chapters, objects like manifolds, maps, vector bundles, etc., are considered to be smooth even if it is not explicitly mentioned.

2.1 Graded vector bundles and complexes

Given two ordinary vector bundles $E \to M$ and $F \to N$, there is a bijection between vector bundle morphisms $\phi \colon E \to F$ covering $\phi_0 \colon M \to N$ and morphisms of modules $\phi^\star \colon \Gamma(F^*) \to \Gamma(E^*)$ over the pull-back $\phi_0^* \colon C^\infty(N) \to C^\infty(M)$. Explicitly, the map $\phi^\star$ is defined by $\phi^\star(f)(m) = \phi_m^* f_{\phi_0(m)}$ for $f \in \Gamma(F)$ and $m \in M$, where $\phi_m \colon E_m \to F_{\phi(m)}$ is the induced linear map between the fibres over the points $m \in M$ and $\phi_0(m) \in N$.

Similarly as before, underlined symbols denote graded objects also in the geometric setting. For instance, a **graded vector bundle over a smooth manifold** M is a vector bundle $q \colon \underline{E} \to M$ together with a direct sum decomposition

$$\underline{E} = \bigoplus_{n \in \mathbb{Z}} E_n[n]$$

of ordinary (non-graded) vector bundles E_n over M. The finiteness assumption for the rank of $\underline{E}$ that we will generally assume implies that $\underline{E}$ is both upper and lower bounded, i.e. there exists an $n_0 \in \mathbb{N}$ such that $E_n = 0$ for all $|n| > n_0$. Here, an element $e \in E_n$ is *(degree-)homogeneous* of degree $|e| = -n$. That is, for $k \in \mathbb{Z}$, the degree k component of $\underline{E}$ (denoted with upper index $\underline{E}^k$) equals E_{-k}.

All the usual algebraic constructions from the theory of ordinary vector bundles extend to the graded setting by applying pointwise the constructions of linear algebra. More precisely, for graded vector bundles $\underline{E}, \underline{F}$, the dual $\underline{E}^*$, the direct sum $\underline{E} \oplus \underline{F}$, the space of graded homomorphisms $\underline{\mathrm{Hom}}(\underline{E}, \underline{F})$, the tensor product $\underline{E} \otimes \underline{F}$, and the symmetric and antisymmetric powers $\underline{S}(\underline{E})$ and

$\underline{A}(\underline{E})$ are all well-defined in the realm of graded vector bundles.

A **(cochain) complex of vector bundles** is a graded vector bundle $\underline{E}$ over M equipped with a degree one endomorphism (called the **differential**)

$$\ldots \xrightarrow{\partial} E_{i+1} \xrightarrow{\partial} E_i \xrightarrow{\partial} E_{i-1} \xrightarrow{\partial} \ldots$$

which squares to zero; $\partial^2 = 0$.

Given two complexes $(\underline{E}, \partial)$ and $(\underline{F}, \partial)$, one may construct new complexes by considering all the constructions discussed above. Namely, the bundles $\underline{S}(\underline{E})$, $\underline{A}(\underline{E})$, $\underline{E}^*$, $\underline{\operatorname{Hom}}(\underline{E}, \underline{F})$ and $\underline{E} \otimes \underline{F}$ inherit a degree one operator that squares to 0. The basic principle for all the constructions is formal the graded derivation rule. For example, for $\phi \in \underline{\operatorname{Hom}}(\underline{E}, \underline{F})$ and $e \in \underline{E}, f \in \underline{F}$:

$$\partial_{\underline{F}}(\phi(e)) = \partial_{\underline{\operatorname{Hom}}}(\phi)(e) + (-1)^{|\phi|}\phi(\partial_{\underline{E}}(e))$$

$$\partial_{\underline{E} \otimes \underline{F}}(e \otimes f) = \partial_{\underline{E}}(e) \otimes f + (-1)^{|e|}e \otimes \partial_{\underline{F}}(f).$$

Using the language of (graded) commutators, the differential of $\underline{\operatorname{Hom}}(\underline{E}, \underline{F})$ can be expressed formally as

$$\partial(\phi) := \partial \circ \phi - (-1)^{|\phi|}\phi \circ \partial = \partial \circ \phi - (-1)^{|\phi| \cdot |\partial|}\phi \circ \partial =: [\partial, \phi].$$

The **shift functor** $[k]$, for $k \in \mathbb{Z}$, yields a new complex $(\underline{E}[k], \partial[k])$ whose i-th component is $\underline{E}[k]^i = \underline{E}^{i+k} = E_{-i-k}$ with differential $\partial[k] = \partial$. Formally, $\underline{E}[k]$ is obtained by tensoring with $(\mathbb{R}[k], 0)$ from the right[1]:

$$\partial[k](e \otimes 1) = \partial(e) \otimes 1 + (-1)^{|e|}e \otimes 0 = \partial(e) \otimes 1$$

for all homogeneous $e \in \underline{E}$. A **degree k morphism** between two complexes $(\underline{E}, \partial)$ and $(\underline{F}, \partial)$ over M, or simply k-**morphism**, is, by definition, a degree preserving morphism $\phi \colon \underline{E} \to \underline{F}[k]$ over the identity on M; that is, a family of vector bundle maps $\phi_i \colon \underline{E}^i \to F[k]^i$ over the identity on M that commutes with the differentials:[2] $\phi \circ \partial = \partial \circ \phi$.

2.2   Algebroids and basic connections

In this section, we give the basic definitions and constructions regarding the different notions of algebroids which will be used in the rest of this work.

A **dull algebroid** [48] is a vector bundle $Q \to M$ endowed with an **anchor map** $\rho_Q \colon Q \to TM$ and a **bracket** (i.e. an $\mathbb{R}$-bilinear map) $[\cdot, \cdot] \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ on its space of sections $\Gamma(Q)$, such that

$$\rho_Q[q_1, q_2] = [\rho_Q(q_1), \rho_Q(q_2)] \tag{2.1}$$

---

[1]If one chose to tensor from the left, the resulting complex would still have i-th component $\underline{E}[k]^i = \underline{E}^{i+k}$, but the Leibniz rule would give the differential $\partial[k] = (-1)^k\partial$.

[2]This becomes $\phi \circ \partial = (-1)^k\partial \circ \phi$ for the other convention.

and the Leibniz identity is satisfied in both entries:

$$[f_1 q_1, f_2 q_2] = f_1 f_2 [q_1, q_2] + f_1 \rho_Q(q_1) f_2 \cdot q_2 - f_2 \rho_Q(q_2) f_1 \cdot q_1,$$

for all $q_1, q_2 \in \Gamma(Q)$ and all $f_1, f_2 \in C^\infty(M)$. If the bracket is in addition skew-symmetric, dull algebroids are also found in the literature as **almost Lie algebroids** [79].

A dull algebroid gives the usual notion of a **Lie algebroid** if its bracket is also skew-symmetric and satisfies the Jacobi identity

$$\mathrm{Jac}_{[\cdot, \cdot]}(q_1, q_2, q_3) := [q_1, [q_2, q_3]] - [[q_1, q_2], q_3] - [q_2, [q_1, q_3]] = 0,$$

for all $q_1, q_2, q_3 \in \Gamma(Q)$.

Given a skew-symmetric dull algebroid Q, there is an associated operator d_Q of degree 1 on the **space of Q-forms** $\Omega^\bullet(Q) = \Gamma(\wedge^\bullet Q^*)$, defined by the formula

$$\mathrm{d}_Q \tau(q_1, \ldots, q_{k+1}) = \sum_{i < j} (-1)^{i+j} \tau([q_i, q_j], q_1, \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots, q_{k+1})$$
$$+ \sum_i (-1)^{i+1} \rho_Q(q_i)(\tau(q_1, \ldots, \hat{q}_i, \ldots, q_{k+1})),$$

for $\tau \in \Omega^k(Q)$ and $q_1, \ldots, q_{k+1} \in \Gamma(Q)$; the notation $\hat{q}$ means that q is omitted. The operator d_Q satisfies as usual

$$\mathrm{d}_Q(\tau_1 \wedge \tau_2) = (\mathrm{d}_Q \tau_1) \wedge \tau_2 + (-1)^{|\tau_1|} \tau_1 \wedge \mathrm{d}_Q \tau_2,$$

for $\tau_1, \tau_2 \in \Omega^\bullet(Q)$. In general, the operator d_Q squares to zero only on 0-forms $f \in \Omega^0(M) = C^\infty(M)$, since $\mathrm{d}_Q^2 f = 0$ for all $f \in C^\infty(M)$ is equivalent to the compatibility of the anchor with the bracket (2.1). The identity $\mathrm{d}_Q^2 = 0$ on $\Omega^1(Q) = \Gamma(Q^*)$ and consequently to all forms in $\Omega^\bullet(Q)$ due to $\Omega^1(Q)$-linearity of d_Q^2 is equivalent to $(Q, \rho_Q, [\cdot, \cdot])$ being a Lie algebroid.

Let now $Q \to M$ be a skew-symmetric dull algebroid and suppose that $E \to M$ is another vector bundle. A **Q-connection on E** is defined similarly as the usual case of Lie algebroids, as a map $\nabla \colon \Gamma(Q) \times \Gamma(E) \to \Gamma(E), (q, e) \mapsto \nabla_q e$ that is $C^\infty(M)$-linear in the first argument and satisfies

$$\nabla_q(fe) = \pounds_{\rho_Q(q)} f \cdot e + f \nabla_q e,$$

for all $q \in \Gamma(Q), e \in \Gamma(E)$ and $f \in C^\infty(M)$. The **dual connection ∇^*** is the Q-connection on E^* defined by the formula

$$\langle \nabla_q^* \varepsilon, e \rangle = \pounds_{\rho_Q(q)} \langle \varepsilon, e \rangle - \langle \varepsilon, \nabla_q e \rangle,$$

for all $\varepsilon \in \Gamma(E^*), e \in \Gamma(E)$ and $q \in \Gamma(Q)$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between E and its dual E^*.

A **Q-connection on a graded vector bundle** $(\underline{E} = \bigoplus_{n \in \mathbb{Z}} E_n[n], \partial)$ is a family of Q-connections ∇^n, $n \in \mathbb{Z}$, on each of the bundles E_n. If $\underline{E}$ is a complex with differential ∂, then the Q-connection is a **connection on the complex** $(\underline{E}, \partial)$ if it commutes with ∂, i.e. $\partial(\nabla_q^n e) = \nabla_q^{n-1}(\partial e)$

for $q \in \Gamma(Q)$ and $e \in \Gamma(E_n)$.

The **curvature of a Q-connection** on a vector bundle E is defined by

$$R_\nabla(q_1, q_2)e = \nabla_{q_1}\nabla_{q_2}e - \nabla_{q_2}\nabla_{q_1}e - \nabla_{[q_1,q_2]}e,$$

for all $q_1, q_2 \in \Gamma(Q)$ and $e \in \Gamma(E)$, and generally, it is an element of $\Gamma(Q^* \otimes Q^* \otimes E^* \otimes E)$. If the dull bracket of Q is skew-symmetric, then the curvature is a 2-form with values in the endomorphism bundle $\operatorname{End}(E) = E^* \otimes E$: $R_\nabla \in \Omega^2(Q, \operatorname{End}(E))$. A connection is called as usual **flat** if its curvature R_∇ vanishes identically.

Given a Q-connection ∇ on E, and assuming that $[\cdot, \cdot]$ is skew-symmetric, there is an induced operator d_∇ on the space of E-valued Q-forms $\Omega^\bullet(Q, E) = \Omega^\bullet(Q) \otimes_{C^\infty(M)} \Gamma(E)$ given by the usual Koszul formula

$$\mathrm{d}_\nabla\tau(q_1, \ldots, q_{k+1}) = \sum_{i<j}(-1)^{i+j}\tau([q_i, q_j], q_1, \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots, q_{k+1})$$
$$+ \sum_i(-1)^{i+1}\nabla_{q_i}(\tau(q_1, \ldots, \hat{q}_i, \ldots, q_{k+1})),$$

for all $\tau \in \Omega^k(Q, E)$ and $q_1, \ldots, q_{k+1} \in \Gamma(Q)$. It satisfies

$$\mathrm{d}_\nabla(\tau_1 \wedge \tau_2) = \mathrm{d}_Q\tau_1 \wedge \tau_2 + (-1)^k\tau_1 \wedge \mathrm{d}_\nabla\tau_2,$$

for all $\tau_1 \in \Omega^k(Q)$ and $\tau_2 \in \Omega^\bullet(Q, E)$, and squares to zero if and only if Q is a Lie algebroid and ∇ is flat.

Suppose that $\nabla \colon \mathfrak{X}(M) \times \Gamma(Q) \to \Gamma(Q)$ is a TM-connection on the vector bundle Q. The induced **basic connections** on Q and TM are defined similarly as the ones associated to Lie algebroids [42, 4]:

$$\nabla^{\mathrm{bas}} = \nabla^{\mathrm{bas},Q} \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q), \ \ \nabla^{\mathrm{bas}}_{q_1}q_2 = [q_1, q_2] + \nabla_{\rho_Q(q_2)}q_1$$

and

$$\nabla^{\mathrm{bas}} = \nabla^{\mathrm{bas},TM} \colon \Gamma(Q) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \ \ \nabla^{\mathrm{bas}}_q X = [\rho_Q(q), X] + \rho_Q(\nabla_X q).$$

The **basic curvature** is the 2-form $R^{\mathrm{bas}}_\nabla \in \Omega^2(Q, \operatorname{Hom}(TM, Q))$ defined by

$$R^{\mathrm{bas}}_\nabla(q_1, q_2)X = -\nabla_X[q_1, q_2] + [q_1, \nabla_X q_2] + [\nabla_X q_1, q_2] + \nabla_{\nabla^{\mathrm{bas}}_{q_2}X}q_1 - \nabla_{\nabla^{\mathrm{bas}}_{q_1}X}q_2.$$

The basic connections and the basic curvature satisfy

$$\nabla^{\mathrm{bas},TM} \circ \rho_Q = \rho_Q \circ \nabla^{\mathrm{bas},Q}, \tag{2.2}$$

$$\rho_Q \circ R^{\mathrm{bas}}_\nabla = R_{\nabla^{\mathrm{bas},TM}}, \tag{2.3}$$

$$R^{\mathrm{bas}}_\nabla \circ \rho_Q + \operatorname{Jac}_{[\cdot,\cdot]} = R_{\nabla^{\mathrm{bas},Q}}. \tag{2.4}$$

2.3 Double vector bundles, linear splittings and duals

Recall that a **double vector bundle** (D, V, F, M) is a commutative diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\ \pi_F\ } & F \\
{\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle q_F} \\
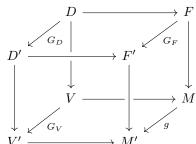V & \xrightarrow{\ q_V\ } & M
\end{array}
$$

such that all the arrows are vector bundle projections and the structure maps of the bundle $D \to V$ are bundle morphisms over the corresponding structure maps of $F \to M$. More precisely, a commutative square (D, V, F, M) as above is a double vector bundle if the following three conditions hold:

1. all four arrows are vector bundles;

2. the bundle projection π_V is a vector bundle morphism over the bundle projection q_F;

3. the addition map $+_V \colon D \times_V D \to D$ of the vector bundle $D \to V$ is a vector bundle morphism over the addition map $+ \colon F \times_M F \to F$ of the vector bundle $F \to M$.

We refer the reader to [67] for a more detailed treatment of double vector bundles. The above conditions for the structure maps of $D \to V$ are equivalent to the same conditions holding for the structure maps of $D \to F$ over $V \to M$. The bundles V and F are called **the side bundles of** D. The intersection of the kernels $C := \pi_V^{-1}(0^V) \cap \pi_F^{-1}(0^F)$ is the **core of** D and is naturally a vector bundle over M, with projection denoted by $q_C \colon C \to M$. The inclusion $C \hookrightarrow D$ is denoted by $C_m \ni c_m \mapsto \overline{c_m} \in \pi_V^{-1}(0_m^V) \cap \pi_F^{-1}(0_m^F)$.

A **morphism** (G_D, G_V, G_F, g) **of two double vector bundles** (D, V, F, M) and (D', V', F', M') is a commutative cube

$$
\begin{array}{ccccc}
 & D & \longrightarrow & F & \\
 & \swarrow{\scriptstyle G_D} & & \swarrow{\scriptstyle G_F} & \\
D' & \longrightarrow & F' & & \\
\downarrow & & \downarrow & & \\
 & V & \longrightarrow & M & \\
 & & & \searrow{\scriptstyle g} & \\
\downarrow & \swarrow{\scriptstyle G_V} & & \downarrow & \\
V' & \longrightarrow & M' &
\end{array}
$$

such that all the faces are vector bundle maps.

Given a double vector bundle (D, V, F, M), the space of sections of D over V, denoted by $\Gamma_V(D)$, is generated as a $C^\infty(V)$-module by two special types of sections, called **core** and **linear** sections and denoted by $\Gamma_V^c(D)$ and $\Gamma_V^l(D)$, respectively (see [67]). The core sections are induced from sections of C and more precisely the core section $c^\dagger \in \Gamma_V^c(D)$ corresponding to $c \in \Gamma(C)$ is defined as

$$
c^\dagger(v_m) = 0_{v_m}^D +_F \overline{c(m)}, \quad \text{for } m \in M \text{ and } v_m \in V_m.
$$

A section $\delta \in \Gamma_V(D)$ is linear over $f \in \Gamma(F)$, if $\delta\colon V \to D$ is a vector bundle morphism $V \to D$ over $f\colon M \to F$.

Finally, a section $\psi \in \Gamma(V^* \otimes C)$ defines a linear section $\psi^\wedge\colon V \to D$ over the zero section $0^F\colon M \to F$ by

$$\psi^\wedge(v_m) = 0^D_{v_m} +_F \overline{\psi(v_m)}$$

for all $m \in M$ and $v_m \in V_m$. This type of linear section is called a **core-linear** section. In terms of the generators $\theta \otimes c \in \Gamma(V^* \otimes C)$, the correspondence above reads $(\theta \otimes c)^\wedge = \ell_\theta \cdot c^\dagger$, where ℓ_θ is the linear function on V associated to $\theta \in \Gamma(V^*)$.

Example 2.3.1 (Decomposed double vector bundle). Let V, F, C be vector bundles over the same manifold M. Set $D := V \times_M F \times_M C$ with vector bundle structures $D = q_V^!(F \oplus C) \to V$ and $D = q_F^!(V \oplus C) \to F$. Then (D, V, F, M) is a double vector bundle, called the **decomposed double vector bundle** with sides V and F and with core C. Its core sections over F have the form $c^\dagger\colon f_m \mapsto (0^V_m, f_m, c(m))$, for $m \in M, f_m \in F_m$ and $c \in \Gamma(C)$, and the space of linear sections $\Gamma^l_V(D)$ is naturally identified with $\Gamma(F) \oplus \Gamma(V^* \otimes C)$ via $(f, \psi)\colon v_m \mapsto (f(m), v_m, \psi(v_m))$ where $\psi \in \Gamma(V^* \otimes C)$ and $f \in \Gamma(F)$. This yields the canonical **linear horizontal lift** $h\colon \Gamma(F) \hookrightarrow \Gamma^l_V(D)$. Given two decomposed double vector bundles $D := V \times_M F \times_M C$ and $D' := V' \times_M F' \times_M C'$ over the same base manifold M, any double vector bundle morphism from D to D' is of the form

$$(v, f, c) \mapsto (G_V(v), G_F(f), G_C(c) + \Phi_v(f))$$

for all sections $v \in \Gamma(V), f \in \Gamma(F), \Gamma(C)$ [35], where $G_C\colon C \to C'$ is a vector bundle morphism and $\Phi \in \Gamma(V^* \otimes F^* \otimes C')$.

Example 2.3.2 (Tangent bundle of a vector bundle). Given a vector bundle $q\colon E \to M$, its tangent bundle TE is naturally a vector bundle over the manifold E. In addition, applying the tangent functor to all the structure maps of $E \to M$ yields a vector bundle structure on $Tq\colon TE \to TM$ which is called the **tangent prolongation** of E. Hence, (TE, TM, E, M) has a natural double vector bundle structure with sides TM and E:

$$
\begin{array}{ccc}
TE & \xrightarrow{\ p_E\ } & E \\
{\scriptstyle Tq}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
TM & \xrightarrow{\ p_M\ } & M
\end{array}
$$

Its core is naturally identified with $E \to M$ and the inclusion $E \hookrightarrow TE$ is given by $E_m \ni e_m \mapsto \frac{d}{dt}\big|_{t=0} te_m \in T^q_{0_m} E$. For $e \in \Gamma(E)$, the section $Te \in \Gamma^l_{TM}(TE)$ is linear over e. The core vector field $e^\dagger \in \Gamma_{TM}(TE)$ is defined by $e^\dagger(v_m) = T_m 0^E(v_M) +_E \frac{d}{dt}\big|_{t=0} te(m)$ for $m \in M$ and $v_m \in T_M M$ and the **vertical lift** $e^\uparrow \in \Gamma_E(TE) = \mathfrak{X}(E)$ is the (core) vector field defined by the flow $\mathbb{R} \times E \to E, (t, e'_m) \mapsto e'_m + te(m)$. Elements of $\Gamma^l_E(TE) =: \mathfrak{X}^l(E)$ are called **linear vector fields** and are equivalent to derivations $\delta\colon \Gamma(E) \to \Gamma(E)$ over some element in $\mathfrak{X}(M)$ [67]. We denote by X_δ the linear vector field which corresponds to the derivation δ. The construction goes as follows: First

note that derivations of the vector bundle $q : E \to M$ are in correspondence with derivations of the dual vector bundle $q' : E^* \to M$ via the map that sends δ over $X \in \mathfrak{X}(M)$ to the derivation δ^* over X defined by $X\langle \varepsilon, e \rangle = \langle \delta^* \varepsilon, e \rangle + \langle \varepsilon, \delta e \rangle$ for $e \in \Gamma(E)$ and $\varepsilon \in \Gamma(E^*)$. Then the correspondence $\delta \longleftrightarrow X_\delta$ is obtained via the formulae

$$X_\delta(\ell_\varepsilon) = \ell_{\delta^* \varepsilon} \qquad \text{and} \qquad X_\delta(q^* f) = q^*(X(f))$$

for $f \in C^\infty(M)$ and $\varepsilon \in \Gamma(E^*)$.

Example 2.3.3 (Double tangent bundle). Applying the construction of the previous example to the vector bundle $q_M : TM \to M$, we obtain the double vector bundle

$$
\begin{array}{ccc}
TTM & \xrightarrow{\;q_{TM}\;} & TM \\
{\scriptstyle Tq_M}\downarrow & & \downarrow{\scriptstyle q_M} \\
TM & \xrightarrow{\;q_M\;} & M
\end{array}
$$

called the **double tangent bundle** of the manifold M. There is a canonical isomorphism of double vector bundles $J_M : TTM \to TTM$ over the identities on the sides TM and M which interchanges the sides, called the **canonical involution** of TTM [67]. To describe it, we write elements $(d, X^h, X^v, \gamma) \in TTM$ as second derivatives

$$d = \frac{\partial^2 \gamma}{\partial t \, \partial s}(0,0)$$

where $\gamma : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to M$ is smooth, and the notation means that γ is first differentiated with respect to the variable s, yielding a smooth curve of tangent vectors $X_t^v = \frac{\partial \gamma}{\partial s}(t, 0)$ in TM whose velocity at 0 gives d: $\frac{d}{dt}|_{t=0} X_t^v = d$. That is, we have that

$$\frac{\partial \gamma}{\partial s}(0,0) = q_{TM}(d) \qquad \text{and} \qquad \frac{\partial \gamma}{\partial t}(0,0) = Tq_M(d).$$

The map $J_M : TTM \to TTM$ is then given as

$$d = \frac{\partial^2 \gamma}{\partial t \, \partial s}(0,0) \mapsto J_M(d) := \frac{\partial^2 \gamma}{\partial s \, \partial t}(0,0).$$

A **linear splitting** of a double vector bundle (D, V, F, M) with core C is a double vector bundle embedding Σ of the decomposed double vector bundle $V \times_M F$ into D over the identities on V and F. It is well-known that every double vector bundle admits a linear splitting, see [40, 32, 80] or [46] for the general case. Moreover, a linear splitting is equivalent to a **decomposition** of D, i.e. to an isomorphism of double vector bundles $S : V \times_M F \times_M C \to D$ over the identity on V, F and C. Given Σ, the decomposition is obtained by setting $S(v_m, f_m, c_m) = \Sigma(v_m, f_m) +_F (0_{f_m} +_V \overline{c_m})$, and conversely, given S, the splitting is defined by $\Sigma(v_m, f_m) = S(v_m, f, 0_m^C)$.

A linear splitting of D, and consequently a decomposition of D, is also equivalent to a **(linear)**

horizontal lift, i.e. a right splitting of the short exact sequence

$$0 \longrightarrow \Gamma(V^* \otimes C) \longrightarrow \Gamma^l_V(D) \longrightarrow \Gamma(F) \longrightarrow 0 \tag{2.5}$$

of $C^\infty(M)$-modules. The correspondence is given by $\sigma_F(f)(v_m) = \Sigma(v_m, f_m)$ for $f \in \Gamma(F)$, $m \in M$ and $v_m \in V_m$. Note that all the previous constructions can be done similarly if one interchanges the roles of V and F.

Example 2.3.4. For the tangent bundle TE of a vector bundle $E \to M$, a linear splitting is equivalent to a choice of a TM-connection on E. Specifically, given a horizontal lift $\sigma \colon \mathfrak{X}(M) \to \mathfrak{X}^l(E)$, the corresponding connection ∇ is defined by formula

$$\sigma(Y) = X_{\nabla_Y} \in \mathfrak{X}^l(E) \qquad \text{for all } Y \in \mathfrak{X}(M).$$

Double vector bundles can be dualised in two ways, namely, as the dual of D either over V or over F [67]. Precisely, from a double vector bundle (D, V, F, M) with core C, one obtains the **dual double vector bundles**

$$
\begin{array}{ccc}
D^*_V & \xrightarrow{\pi_{C^*}} & C^* \\
{\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle q_{C^*}} \\
V & \xrightarrow{q_V} & M
\end{array}
\qquad
\begin{array}{ccc}
D^*_F & \xrightarrow{\pi_F} & F \\
{\scriptstyle \pi_{C^*}}\downarrow & & \downarrow{\scriptstyle q_F} \\
C^* & \xrightarrow{q_{C^*}} & M
\end{array}
$$

with cores F^* and V^*, respectively; here, there is a little abuse of notation for the names of the bundle projections. The vertical bundles structures of (D^*_V, V, C^*, M) are the usual duals of the vector bundles $D \to V$ and $C \to M$. The projection $\pi_{C^*} \colon D^*_V \to C^*$ is defined by

$$\langle \pi_{C^*}(\Phi), c_m \rangle = \langle \Phi, 0^D_{v_m} +_F \overline{c_m} \rangle,$$

where $m \in M, c_m \in C_m, \Phi \colon D_V|_{v_m} = \pi_V^{-1}(v_m) \to \mathbb{R}$ and $v_m \in V_m$, and the addition $+_{C^*}$ of D^*_V over C^* is defined by

$$\langle \Phi_1 +_{C^*} \Phi_2, d_1 +_F d_2 \rangle = \langle \Phi_1, d_1 \rangle + \langle \Phi_2, d_2 \rangle,$$

where Φ_1, Φ_2 are elements of (D^*_V, V, C^*, M) of the form (Φ_1, v_1, γ, m) and (Φ_2, v_2, γ, m), respectively (i.e. $v_1 = \pi_V(\Phi_1), \gamma = \phi_{C^*}(\Phi_1), m = q_V(\pi_V(\Phi_1)) = q_{C^*}(\pi_{C^*}(\Phi_1))$ and similarly for Φ_2). In this definition we use that every element $(d, v_1 + v_2, \gamma, m)$ can be written as a sum $d = d_1 +_F d_2$ with (Φ_1, v_1, γ, m) being arbitrary and $d_2 := d -_F d_1$. It can be shown that the definition is independent of the choice of d_1 and d_2 (see [67] for more details).

Given a linear splitting $\Sigma \colon V \times_M F \to D$, the **dual (linear) splitting** $\Sigma^* \colon V \times_M C^* \to D^*_V$ [41, 35, 49] is defined by

$$\langle \Sigma^*(v_m, \gamma_m), \Sigma(v_m, f_m) \rangle_{D_V} = 0 \qquad \text{and} \qquad \langle \Sigma^*(v_m, \gamma_m), c^\dagger(v_m) \rangle_{D_V} = \langle \gamma_m, c(m) \rangle_C,$$

for all $m \in M$ and $v_m \in V_m$, $f_m \in F_m$, $\gamma_m \in C^*_m$, $c \in \Gamma(C)$, where $\langle \cdot, \cdot \rangle_{D_V}$ denotes the pairing of

$D \to V$ with $D_V^* \to V$ and $\langle \cdot, \cdot \rangle_C$ denotes the pairing of $C \to M$ with $C^* \to M$.

2.4 Sheaves on topological spaces

Since graded manifolds, which are the main object of this thesis, are defined in terms of sheaves on ordinary smooth manifolds, it is worth recalling some basic definitions. Generally, there are many sources on this topic but what follows in this section is based on [45]. Intuitively, the language of sheaf theory is a formal way of expressing local algebraic data on a general topological space (not necessarily a smooth manifold).

Let X be a topological space. We define the category $\mathrm{Top}(X)$ whose objects are the open subsets of X and the set of morphisms between two objects is given by the inclusion of open sets, i.e. if $U, V \subset X$ are open then

$$\mathrm{Hom}(V, U) := \begin{cases} \{*\}, & \text{if } V \subset U \\ \emptyset, & \text{if } V \not\subset U, \end{cases}$$
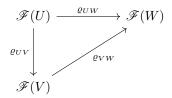
where $\{*\}$ denotes the one-element set.

Definition 2.4.1. A **presheaf** $\mathscr{F}$ **on** X **with values in a (fixed) category** $\mathcal{C}$ is a contravariant functor from $\mathrm{Top}(X)$ to $\mathcal{C}$.

Remark 2.4.2. Equivalently, the presheaf $\mathscr{F}$ can be defined as a covariant functor $\mathrm{Top}(X)^{\mathrm{op}} \to \mathcal{C}$; here, the notation means the opposite category, i.e. for a category $\mathcal{D}$, $\mathcal{D}^{\mathrm{op}}$ has the same objects with $\mathcal{D}$ but its morphisms are given by $\mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{D}}(B, A)$ for all objects A, B of $\mathcal{D}$.

Unravelling the above definition yields that a presheaf $\mathscr{F}$ on X with values in the category $\mathcal{C}$ consists of the following data:

- an object $\mathscr{F}(U)$ of $\mathcal{C}$ for every open subset $U \subset X$,

- a morphism $\varrho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ for every inclusion of open sets $V \subset U$ such that

  1. ϱ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$,

  2. if $W \subset V \subset U$ are three open subsets of X, then $\varrho_{UW} = \varrho_{VW} \circ \varrho_{UV}$, i.e. the following diagram is commutative

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varrho_{UW}} & \mathscr{F}(W) \\ {\scriptstyle \varrho_{UV}} \downarrow & \nearrow {\scriptstyle \varrho_{VW}} & \\ \mathscr{F}(V) & & \end{array}$$

Remark 2.4.3. 1. Most frequently, the category $\mathcal{C}$ is taken to be the category of sets, abelian groups, rings, etc. For our purposes, the base space will be a smooth manifold M and the target category $\mathcal{C}$ will be the category of $\mathbb{Z}$-graded, graded commutative, associative, unital $C^\infty(M)$-algebras with degree-preserving morphisms.

2. Usually, if the category $\mathcal{C}$ has a terminal object 0, then $\mathscr{F}(\emptyset) = 0$ is also required, e.g. $\mathscr{F}(\emptyset) = \{*\}$ for sets, $\mathscr{F}(\emptyset) = \{0\}$ for abelian groups, etc.

The elements of $\mathscr{F}(U)$ are called **sections over** U and the maps ϱ_{UV} are called **restrictions**. The restriction $\varrho_{UV}(\sigma), \sigma \in \mathscr{F}(U)$, for $V \subset U$ is also written as $\sigma|_V$.

Definition 2.4.4. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two sheaves on X with values in $\mathcal{C}$. A **morphism of presheaves** is a natural transformation of functors. $\phi : \mathscr{F}_1 \to \mathscr{F}_2$. An **isomorphism of sheaves** is a natural equivalence of functors.

The above definition implies that a morphism of presheaves $\phi : \mathscr{F}_1 \to \mathscr{F}_2$ consists of morphisms $\phi_U : \mathscr{F}_1(U) \to \mathscr{F}_2(U)$ of $\mathcal{C}$ which are compatible with the restriction maps: $\phi_V \circ \varrho^1_{UV} = \varrho^2_{UV} \circ \phi_U$, i.e the following diagram commutes for all open subset $V \subset U$ of X:

$$
\begin{array}{ccc}
\mathscr{F}_1(U) & \xrightarrow{\ \phi_U\ } & \mathscr{F}_2(U) \\
{\scriptstyle \varrho^1_{UV}}\big\downarrow & & \big\downarrow{\scriptstyle \varrho^2_{UV}} \\
\mathscr{F}_1(V) & \xrightarrow[\ \phi_V\ ]{} & \mathscr{F}_2(V)
\end{array}
$$

The morphism $\phi : \mathscr{F}_1 \to \mathscr{F}_2$ is an isomorphism if the map $\phi_U : \mathscr{F}_1(U) \to \mathscr{F}_2(U)$ is an isomorphism in $\mathcal{C}$ for all open subsets $U \subset X$.

From the discussion above, it follows that presheaves on X with values in the category $\mathcal{C}$ form a category which is the category of functors $\mathrm{PreSh}_\mathcal{C}(X) := \mathrm{Fun}(\mathrm{Top}^{\mathrm{op}}(X), \mathcal{C})$.

Example 2.4.5 (Continuous functions). The presheaf of continuous functions C^0 on X

$$
C^0(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}
$$

with $\varrho_{UV}(f) = f|_V$ is a presheaf of sets, abelian groups, rings and $\mathbb{R}$-algebras. More generally, given another topological space Y, one obtains the sheaf of sets

$$
C^0_U(X, Y) = \{f : U \to Y \mid f \text{ continuous}\}
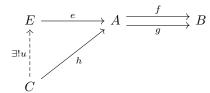$$

with $\varrho_{UV}(f) = f|_V$.

Example 2.4.6 (Smooth functions). If M is a smooth manifold, then C^∞ defines the presheaf of smooth functions on M which again is a presheaf of sets, abelian groups, rings and $\mathbb{R}$-algebras. Moreover, the inclusion $C^\infty(U) \subset C^0(U)$ for every open $U \subset M$ defines a morphism of presheaves $C^\infty \to C^0$.

Definition 2.4.7. Given two objects A, B of a category $\mathcal{C}$ together with two morphisms $f, g : A \to B$, the **equaliser of** $f, g : A \to B$ consists of an object E in $\mathcal{C}$ and a morphism $e : E \to A$ satisfying

1. $f \circ e = g \circ e$, and such that

2. given any object C in $\mathcal{C}$ and morphism $h : C \to A$, then there exists a unique $u : C \to E$ such that $e \circ u = h$.

That is, the equaliser of $f, g : A \to B$ fits in the following commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ e\ } & A \underset{g}{\overset{f}{\rightrightarrows}} B \\
\uparrow & & \\
\exists! u \; \vdots & \nearrow{\scriptstyle h} & \\
C & &
\end{array}
$$

Assume now that all products $\prod_{i \in I}$ exist in the category $\mathcal{C}$.

Definition 2.4.8. A **sheaf** is a presheaf $\mathscr{F}$ such that for all open covers $U = \bigcup_{i \in I} U_i$, the map

$$
e_I : \mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i), \qquad \sigma \mapsto (\sigma|_{U_i})_{i \in I}
$$

is an equaliser for

$$
f_I, g_I : \prod_{i \in I} \mathscr{F}(U_i) \to \prod_{i,j \in I} \mathscr{F}(U_{ij}), \qquad f_I((\sigma_i)_{i \in I}) = \left(\sigma_i|_{U_{ij}}\right)_{i,j \in I}, g_I((\sigma_i)_{i \in I}) = \left(\sigma_j|_{U_{ij}}\right)_{i,j \in I}
$$

where $U_{ij} := U_i \cap U_j$.

Explicitly, the above definition formalises the following statements:

1. If U is an open set of X together with an open cover $U = \bigcup_{i \in I} U_i$ and $\sigma, \tau \in \mathscr{F}(U)$ are sections over U such that $\sigma|_{U_i} = \tau|_{U_i}$ for all i, then $\sigma = \tau$. In particular, if the category $\mathcal{C}$ is the category of abelian groups, rings, etc. then $\sigma|_{U_i} = 0$ for all i implies $\sigma = 0$.

2. If U is an open set of X together with an open cover $U = \bigcup_{i \in I} U_i$ and we have elements $\sigma_i \in \mathscr{F}(U_i)$ for each i with the property that for each i, j, $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$, then there is a (unique due to the first statement) element $\sigma \in \mathscr{F}(U)$ such that $\sigma|_{U_i} = \sigma_i$ for all i.

Remark 2.4.9. Note that not all presheaves are automatically sheaves. One such example is the sheaf of continuous and bounded functions on $\mathbb{R}$.

Definition 2.4.10. A **morphism of sheaves** is just a morphism of the underline presheaves.

Hence, one obtains the category of sheaves $\mathrm{Sh}_{\mathcal{C}}(X)$ as a full subcategory of $\mathrm{PreSh}_{\mathcal{C}}(X)$.

Example 2.4.11. The presheaves of continuous and smooth functions C^0 and C^∞ given in Example 2.4.5 and Example 2.4.6 are also examples of sheaves.

Example 2.4.12. Other examples of sheaves coming from differential geometry are the sheaf of vector fields over a smooth manifold, the sheaf of differential forms, and more generally the sheaf of sections of any fibre bundle.

Remark 2.4.13. Note that if $\mathcal{B}$ is a basis for the topology on X, then one can define a sheaf $\mathscr{F}$ on X by defining only objects $\mathscr{F}(U)$ for each $U \in \mathcal{B}$, which are compatible with the restrictions.
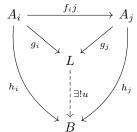
Let now $(I, \leq)$ be a directed set, i.e. $\leq$ is a reflexive and transitive relation on the set I, and suppose that $(A_i)_{i \in I}$ if a family of objects in $\mathcal{C}$ together with a collection of morphisms $f_{ij} : A_i \to A_j$ for all $i \leq j$. The pair (A_i, f_{ij}) is called **direct system** over I if $f_{ii} = \mathrm{id}_{A_i}$ for all $i \in I$, and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$. In a more categorical language, I is an index category and direct system is a covariant functor $f : I \to \mathcal{C}, i \mapsto A_i, (i \leq j) \mapsto (f_{ij} : A_i \to A_j)$.

Definition 2.4.14. Let (A_i, f_{ij}) be a direct system of objects and morphisms in $\mathcal{C}$. A **direct limit** of (A_i, f_{ij}) is a pair (L, g_i) with L an object in $\mathcal{C}$ and $g_i : A_i \to L$ a morphism for all $i \in I$, such that

1. $g_i = g_j \circ f_{ij}$ for all $i \leq j$,

2. given another pair (B, h_i) for all $i \in I$ with B an object in $\mathcal{C}$ and $h_i : A_i \to B$ a morphism which has the property $h_i = h_j \circ f_{ij}$ for all $i \leq j$, then there is a unique morphism $u : L \to B$ such that $u \circ g_i = h_i$ for all $i \in I$.

The object of the direct limit is denoted by $L = \varinjlim_{i \in I} A_i = \varinjlim A_i$.

In the language of diagrams, the direct limit of the direct system (A_i, f_{ij}) fits in the following commutative diagram

$$
\begin{array}{ccc}
A_i & \xrightarrow{\ \ f_{ij}\ \ } & A_j \\
\end{array}
$$
with morphisms g_i, g_j to L, h_i, h_j to B, and $\exists! u : L \to B$.

If the category $\mathcal{C}$ is the category of the usual algebraic structures such as groups, rings, algebras, etc. then the direct limit of (A_i, f_{ij}) is given by

$$
\varinjlim A_i = \left. \bigsqcup_{i \in I} A_i \middle/ \sim \right.,
$$

where by definition $A_i \ni a_i \sim a_j \in A_j$ if and only if there is $k \in I$ such that $i, j \leq k$ and $f_{ik}(a_i) = f_{jk}(a_j)$.

Definition 2.4.15. Suppose that $\mathscr{F}$ is a presheaf on X with values in $\mathcal{C}$ and suppose also that direct limits exist in $\mathcal{C}$. Given a point $x \in X$, the **stalk** of $\mathscr{F}$ at x is defined as

$$
\mathscr{F}_x := \varinjlim_{x \in U} \mathscr{F}(U),
$$

where the direct limit is taken over all open sets U containing x together with the restriction maps.
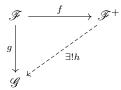
Remark 2.4.16. For all $x \in U$, there is a natural map $\mathscr{F}(U) \to \mathscr{F}_x$ which sends $\sigma \in \mathscr{F}(U)$ to σ_x called the **germ of σ at x**.

The importance of the last constructions is shown in the following definitions.

Definition 2.4.17. Let $\mathscr{F}$ be a presheaf on X. The **sheaf $\mathscr{F}^+$ associated to** $\mathscr{F}$ assigns to an open $U \subset X$ the set of functions $\sigma : U \to \bigsqcup_{x \in U} \mathscr{F}_x$ such that

1. $\sigma(x) \in \mathscr{F}_x$ for all $x \in U$,

2. for each $x \in U$, there is an open $V \subset U$ containing x and a section $\tau \in$ such that $\sigma(y) = \tau_y$ for all $y \in V$.

The sheaf $\mathscr{F}^+$ comes together with a natural morphism (of presheaves) $f : \mathscr{F} \to \mathscr{F}^+$ with the property that for any sheaf $\mathscr{G}$ and any morphism $g : \mathscr{F} \to \mathscr{G}$, there is a unique morphism $h : \mathscr{F} \to \mathscr{G}$ such that $g = h \circ f$:

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\;\;f\;\;} & \mathscr{F}^+ \\
\downarrow{\scriptstyle g} & \swarrow{\scriptstyle \exists! h} & \\
\mathscr{G} & &
\end{array}
$$

Remark 2.4.18. Note that in case $\mathscr{F}$ is already a sheaf, then $\mathscr{F}^+ = \mathscr{F}$.

Example 2.4.19. It was mentioned above that an example of a presheaf that is not a sheaf is given by the sheaf of continuous and bounded functions on the real line:

$$B^0(\mathbb{R}) := \{ f \in C^0(\mathbb{R}) \mid f \text{ is bounded} \}.$$

The sheaf associated to the presheaf B^0 is the sheaf of continuous functions on $\mathbb{R}$:

$$(B^0)^+ = C^0(\mathbb{R}).$$

Using the sheaf associated to a presheaf, one can construct many sheaves from the usual algebraic constructions such as tensors, inverse images, etc. Here we give the examples that will be relevant to us later.

Definition 2.4.20. Let $f : X \to Y$ be a continuous map and $\mathscr{F}_1, \mathscr{F}_2$ be a sheaves on X and Y, respectively. We define the **direct image sheaf** $f_* \mathscr{F}_1$ on Y as the sheaf which sends an open $V \subset Y$ to $f_* \mathscr{F}_1(V) = \mathscr{F}_1(f^{-1}(V))$, and the **inverse image sheaf** $f^{-1} \mathscr{F}_2$ on X as the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{V \supset f(U)} \mathscr{F}(V),$$

where the limit is taken over all open sets $V \subset Y$ containing $f(U)$.
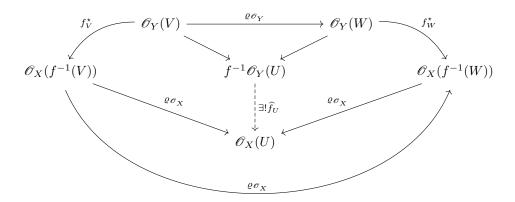
Definition 2.4.21. A **ringed space** is a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings. A **morphism of ringed spaces** $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ consists of a continuous map $f : X \to Y$ and a map of sheaves $f^\star : \mathscr{O}_Y \to f_* \mathscr{O}_X$. A **sheaf of $\mathscr{O}_X$-modules**, or $\mathscr{O}_X$-**module**, is a sheaf (of groups) $\mathscr{F}$ on X, such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module for all open subsets $U \subset X$,

and such that for all open subsets $V \subset U \subset X$ the restriction $\mathscr{F}(U) \to \mathscr{F}(V)$ is compatible with the ring homomorphism $\mathscr{O}(U) \to \mathscr{O}(V)$, i.e. $\mathscr{F}(U) \to \mathscr{F}(V)$ is a map of modules. A **morphism** $\mathscr{F}_1 \to \mathscr{F}_2$ **of** $\mathscr{O}_X$**-modules** is a morphism of sheaves such that for each open subset $U \subset X$, the map $\mathscr{F}_1(U) \to \mathscr{F}_2(U)$ is a morphism of $\mathscr{O}_X(U)$-modules.

Definition 2.4.22. Let $(X, \mathscr{O}_X)$ be a ringed space. Given two sheaves $\mathscr{F}_1, \mathscr{F}_2$ of $\mathscr{O}_X$-modules, we define the **tensor product sheaf** $\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$ as the sheaf associated to the presheaf defined by the following assignment:
$$U \mapsto \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U).$$

Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces and $\mathscr{F}$ be a sheaf of $\mathscr{O}_Y$-modules. Then $f^{-1}\mathscr{F}$ is a sheaf of $f^{-1}\mathscr{O}_Y$-modules. Moreover, by the properties of f and the universal property of direct limits, we obtain the following commutative diagram for all open subsets $U \subset X$ and all open subsets $W \subset V \subset Y$ containing $f(U)$:

That is, the sheaf $\mathscr{O}_X$ is also a sheaf of $f^{-1}\mathscr{O}_Y$-modules via a natural morphism $\widehat{f} : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ and thus we may give the following definition.

Definition 2.4.23. Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces and $\mathscr{F}$ a sheaf of $\mathscr{O}_Y$-modules. We define the **pull-back sheaf** $f^!\mathscr{F}$ as the $\mathscr{O}_X$-module
$$\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{F}.$$

Chapter 3

$\mathbb{Z}$- and $\mathbb{N}$-graded supergeometry

The goal of this chapter is to cover the basics of graded geometry. In particular, it offers quick but sufficient introduction to the theory of $\mathbb{Z}$- and $\mathbb{N}$-manifolds together with various geometric constructions on them.

3.1 The categories of $\mathbb{Z}$- and $\mathbb{N}$-graded manifolds

A **$\mathbb{Z}$-graded manifold** $\mathcal{M}$ **of dimension** $(m; (r_i)_{i \in \mathbb{Z}})$ with $m, r_i \in \mathbb{N}^1$, for short **$\mathbb{Z}$-manifold**, is a ringed space $\mathcal{M} = (M, \mathcal{C}^\infty(\mathcal{M}))$, where M is an ordinary m-dimensional smooth manifold and $\mathcal{C}^\infty(\mathcal{M})$ is a sheaf of $\mathbb{Z}$-graded, graded commutative, associative, unital $C^\infty(M)$-algebras, such that every point $p \in M$ has an open neighbourhood U which satisfies

$$\mathcal{C}_U^\infty(\mathcal{M}) \cong C^\infty(U) \otimes \underline{S}(\underline{V}) \tag{3.1}$$

as sheaves of graded commutative algebras, where $\underline{V} = \bigoplus_{i \in \mathbb{Z}} V_i[i]$ is a fixed $\mathbb{Z}$-graded vector space of finite dimension and such that $\dim(V_i) = r_i$ for all $i \in \mathbb{Z}$; here $\underline{S}(\underline{V})$ denotes the graded symmetric algebra of the graded vector space $\underline{V}$. The smooth manifold M is sometimes called **base manifold** or **body** of $\mathcal{M}$. An **$\mathbb{N}$-graded manifold**, or simply **$\mathbb{N}$-manifold** is a $\mathbb{Z}$-graded manifold $\mathcal{M}$, such that the vector space $\underline{V}$ is only positively graded, i.e. $V_i = \{0\}$ for all $i \geq 0^2$; we write $[n]$-**manifold** for an $\mathbb{N}$-graded manifold $\mathcal{M}$ if the model space $\underline{V}$ has $V_n \neq \{0\}$ and $V_i = \{0\}$ for all $i > n$. **A morphism of $\mathbb{Z}$-** (respectively $\mathbb{N}$-)**manifolds** $\mu \colon \mathcal{N} \to \mathcal{M}$ over a smooth map $\mu_0 \colon N \to M$ of the underlying smooth manifolds is a (degree 0) morphism of sheaves of graded algebras $\mu^\star \colon \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{N})$ over $\mu_0^* \colon C^\infty(M) \to C^\infty(N)$. As usual, the degree of a (degree-)homogeneous element $\xi \in \mathcal{C}^\infty(\mathcal{M})$ will be denoted $|\xi|$.

Given a coordinate chart $(U, x^1, \ldots, x^m)$ of M as in (3.1) and a choice of a basis $\xi_i^1, \ldots, \xi_i^{r_i}$ for each vector space V_i, one obtains the $m + \sum_i r_i$ (graded) "local coordinates" $(x^1, \ldots, x^m, \xi_i^1, \ldots, \xi_i^{r_i})$

---

[1] Only finitely many of the variables r_i are assumed to be non-zero.

[2] This means that only positively graded summands of $\underline{V}$ survive as the space V_i is in the $-i$ position of the graded sum.

of the graded manifold $\mathcal{M}$ equipped with the following degrees:

- the coordinates $(x^1, \ldots, x^m)$ have degree 0,

- the coordinates $(\xi_i^1, \ldots, \xi_i^{r_i})$ have degree $-i$, for $i \in \mathbb{Z}$ and $j \in \{1, \ldots, r_i\}$.

Remark 3.1.1. Note that the degree 0 coordinates of an $\mathbb{N}$-manifold consist entirely of the coordinates of the base manifold M. This is not the case for the degree 0 coordinates of a general $\mathbb{Z}$-manifold $\mathcal{M}$, unless the vector space V_0 is trivial. Even in this case, the degree 0 functions of the $\mathbb{Z}$-manifold may have a "polynomial" part that comes from the graded coordinates of non-zero degree, since, for instance, the local functions $\xi_i^j \xi_{-i}^k$ have degree 0 for all $i \in \mathbb{Z} \setminus \{0\}$. On the contrary, by definition of the structure sheaf of a graded manifold, the degree 0 functions of an $\mathbb{N}$-manifold $\mathcal{M}$ are given by $\mathcal{C}^\infty(\mathcal{M})^0 = C^\infty(M)$.

Many examples of $\mathbb{Z}$- (respectively $\mathbb{N}$-)graded manifolds come from finite graded vector bundles $\underline{E} = \bigoplus_{i \in \mathbb{Z}} E_i[i] \to M$ as follows: The construction of a graded manifold from $\underline{E}$ is done by assigning the degree i to the fibre coordinates of the vector bundle E_i^*, for each $i \in \mathbb{Z}$. The resulting graded manifold has dimension $(\dim(M); (\mathrm{rank}(E_{-i}))_{i \in \mathbb{Z}})$ and is denoted by the same notation $\mathcal{M} := \underline{E}$. The structure sheaf of $\mathcal{M} = \underline{E}$ is given by the sections of the (graded) symmetric algebra of $\underline{E}^*$:

$$\mathcal{C}^\infty(\mathcal{M}) = \Gamma(\underline{S}(\underline{E}^*)) = \Gamma\left(\wedge E_{\mathrm{odd}}^* \otimes S E_{\mathrm{even}}^*\right), \tag{3.2}$$

where

$$E_{\mathrm{odd}}^* = \bigoplus_{i \in 2\mathbb{Z}+1} E_i^*[-i] \qquad \text{and} \qquad E_{\mathrm{even}}^* = \bigoplus_{i \in 2\mathbb{Z}} E_i^*[-i].$$

It is clear from the construction that $\mathcal{M} = \underline{E}$ is an $\mathbb{N}$-manifold if E_i is zero for all $i \geq 0$. In particular, as the next proposition shows, every $\mathbb{N}$-manifold is (non-canonically) isomorphic to an $\mathbb{N}$-manifold obtained in that way.

Proposition 3.1.2 ([83, 14])**.** Any $[n]$-manifold over M is isomorphic to an $[n]$-manifold corresponding to a graded vector bundle $\underline{E} = \bigoplus_{i=1}^n E_i[i] \to M$.

Proof. The sketch of the proof can be found in [83, 14]. Here we follow [28], where a complete proof using induction over n is given.

First observe that for each $[n]$-manifold $\mathcal{M}$, there is a tower of $[i]$-manifolds

$$\mathcal{M}_n = \mathcal{M} \longrightarrow \mathcal{M}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_0 = M$$

for $i = 0, 1, \ldots, n$, whose sheaf of functions is given by:

$$\mathcal{C}^\infty(\mathcal{M}_i) := \mathrm{span}\{\xi \in \mathcal{C}^\infty(\mathcal{M}) \mid |\xi| \leq i\}.$$

We will now prove the result by induction on the degree $n \geq 1$ of the $\mathbb{N}$-manifold $\mathcal{M}$.

For $n = 1$, the local description of the $[1]$-manifold in (3.1) implies that $\mathcal{C}^\infty(\mathcal{M})$ is generated as an algebra by $\mathcal{C}^\infty(\mathcal{M})^0 = C^\infty(M)$ and $\mathcal{C}^\infty(\mathcal{M})^1$. In other words, we have that $\mathcal{C}^\infty(\mathcal{M})^k =$

$\underline{S}^k(\mathcal{C}^\infty(\mathcal{M})^1)$. Moreover, the sheaf $\mathcal{C}^\infty(\mathcal{M})^1$ is a locally free and finitely generated sheaf of $C^\infty(M)$-modules, and thus there exists a vector bundle $E \to M$ whose sheaf of sections is given by $\Gamma(E) = \mathcal{C}^\infty(\mathcal{M})^1$. That is, $\mathcal{M} \simeq E$.

Suppose the result holds for some degree $n - 1 \geq 1$ and let $\mathcal{M}$ be an $[n]$-manifold. This implies that there is a graded vector bundle $\underline{E}_0 = \bigoplus_{i=1}^{n-1} E_i[i] \to M$ such that $\mathcal{M}_{n-1} \simeq \underline{E}_0$ and consequently $\mathcal{C}^\infty(\mathcal{M}_{n-1})^k \simeq \Gamma(\underline{S}^k(\underline{E}_0^*))$. The graded algebra $\mathcal{C}^\infty(\mathcal{M})$ is generated by $\mathcal{C}^\infty(\mathcal{M}_{n-1})$ and $\mathcal{C}^\infty(\mathcal{M})^n$, and the latter fits in the short exact sequence

$$0 \longrightarrow \mathcal{C}^\infty(\mathcal{M}_{n-1})^n \longrightarrow \mathcal{C}^\infty(\mathcal{M})^n \longrightarrow \mathcal{C}^\infty(\mathcal{M})^n/\mathcal{C}^\infty(\mathcal{M}_{n-1})^n \longrightarrow 0. \tag{3.3}$$

It follows from $C^\infty(M) \cdot \mathcal{C}^\infty(\mathcal{M})^n \subset \mathcal{C}^\infty(\mathcal{M})^n$ that $\mathcal{C}^\infty(\mathcal{M})^n$ is a $C^\infty(M)$-module and from equation (3.1) that it is locally freely generated. Therefore, there exists a vector bundle $E \to M$ with the property that $\Gamma(E) = \mathcal{C}^\infty(\mathcal{M})^n$. Setting $E_n^* := E/\underline{S}^n(\underline{E}_0^*) \to M$ and choosing a splitting of the short exact sequence of vector bundles

$$0 \longrightarrow \underline{S}^n(\underline{E}_0^*) \longrightarrow E \longrightarrow E/\underline{S}^n(\underline{E}_0^*) \longrightarrow 0,$$

we obtain that $E \cong E_n^* \oplus \underline{S}^n(\underline{E}_0^*)$. Hence, we conclude that $\mathcal{M} \simeq \underline{E}$, where we define the graded vector bundle $\underline{E} \to M$ to be the direct sum $\underline{E} := \underline{E}_0 \oplus E[n]$. $\qquad\square$

Definition 3.1.3. A graded manifold associated to a graded vector bundle $\underline{E} \to M$ as above is called **split $\mathbb{Z}$-** (respectively $\mathbb{N}$-)**graded manifold**.

Remark 3.1.4.   1. Note that different choices of splittings of the $\mathbb{N}$-graded manifold $\mathcal{M}$, i.e. isomorphisms $\mathcal{M} \cong \underline{E} = \bigoplus_i E_i[i]$, lead to isomorphic vector bundles E_i. What changes is the way to view the sections of $\Gamma(E_i^*)$ as functions of the graded manifold via the splittings of the short exact sequence (3.3) (see also Example 3.1.9).

2. Suppose that $U \subset M$ is a trivialising neighbourhood for all the vector bundles $E_i \to M$, i.e. $E_i|_U \cong U \times \mathbb{R}^{k_i}$ for all $i \in \mathbb{Z}$, where $k_i = \mathrm{rank}(E_i)$. Then the graded coordinates of the manifold $\mathcal{M}$ are given by the following:

- a set of coordinates $(x^1, \dots, x^m)$ of M over U have degree 0,
- the sections of a local frame $(\varepsilon_i^1, \dots, \varepsilon_i^{k_i})$ of E_i^* over U have degree i.

Example 3.1.5 (Linear $\mathbb{Z}$-manifolds). Consider a finite collection of natural numbers $k_i \in \mathbb{N}$, for $i \in \mathbb{Z}$ (necessarily including 0). Then one can define the trivial vector bundles

$$E_i := \mathbb{R}^{k_0} \times \mathbb{R}^{k_i} \to \mathbb{R}^{k_0}$$

for all $i \in \mathbb{Z}$, and sum over $\mathbb{R}^{k_0}$ to obtain the graded vector bundle

$$\underline{E} := \bigoplus_{i \in \mathbb{Z}} E_i[i] \to \mathbb{R}^{k_0}.$$

The split $\mathbb{Z}$-manifold obtained by $\underline{E}$ with body $\mathbb{R}^{k_0}$, denoted $\mathbb{R}^{\{k_i\}}$, is a called **linear $\mathbb{Z}$-manifold**. Its structure sheaf is given locally by

$$C^\infty(U) \otimes \underline{S}\left(\bigoplus_{i \neq 0} \mathbb{R}^{k_i}[-i]\right),$$

where $U \subset \mathbb{R}^{k_0}$ is open.

Example 3.1.6 (Product $\mathbb{Z}$-manifold). Given two $\mathbb{Z}$-manifolds $\mathcal{M}$ and $\mathcal{N}$ with bodies M and N, respectively, one can construct the **product $\mathbb{Z}$-manifold** $\mathcal{M} \times \mathcal{N}$ with body $M \times N$ by defining its sheaf of functions as

$$\mathcal{C}^\infty(\mathcal{M} \times \mathcal{N}) = \mathrm{pr}_M^! \, \mathcal{C}^\infty(\mathcal{M}) \,\widehat{\otimes}\, \mathrm{pr}_N^! \, \mathcal{C}^\infty(\mathcal{N}),$$

where $\widehat{\otimes}$ denotes the completion of the projective tensor topology; that is, the unique topology on $C^\infty(U) \otimes C^\infty(U')$ such that its completion is isomorphic to $C^\infty(U \times U')$ for all open subsets $U \subset \mathbb{R}^m$ and $U' \subset \mathbb{R}^n$ (see [20] and references therein for the case of $\mathbb{Z}_2$-graded supermanifolds). Explicitly, given two open sets $U \subset M, U' \subset N$ with

$$\mathcal{C}^\infty_U(\mathcal{M}) \cong C^\infty(U) \otimes \underline{S}(\underline{V}) \qquad \text{and} \qquad \mathcal{C}^\infty_{U'}(\mathcal{N}) \cong C^\infty(U') \otimes \underline{S}(\underline{W}),$$

the open set $U \times U' \subset M \times N$ is sent to

$$\mathcal{C}^\infty_U(\mathcal{M}) \,\widehat{\otimes}\, \mathcal{C}^\infty_{U'}(\mathcal{N}) \cong C^\infty(U \times U') \otimes \underline{S}(\underline{V} \oplus \underline{W}).$$

In particular, the sets of coordinates $\{\xi^j\}$ and $\{\zeta^i\}$ on $\mathcal{M}$ and $\mathcal{N}$, respectively, induce the set of coordinates $\{\xi^j, \zeta^i\}$ on $\mathcal{M} \times \mathcal{N}$ and hence the homogeneous functions $\xi \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{N})^k$ can be written as a finite sum

$$\xi = \sum_j \eta_j \theta_j,$$

where $\eta_j \in \mathcal{C}^\infty(\mathcal{M})^{p_j}, \theta_j \in \mathcal{C}^\infty(\mathcal{M})^{q_j}$ and $p_j + q_j = k$ for all j.

Example 3.1.7. From the discussion above, it is clear that every graded vector space

$$\underline{V} = \bigoplus_{i \in \mathbb{Z}} V_i[i]$$

gives a $\mathbb{Z}$-manifold over the trivial base manifold $M = \{*\}$. In particular, applying this to the [1]-shift of a non-graded vector space $\mathfrak{g}$, one obtains the [1]-manifold $\mathfrak{g}[1]$ with structure sheaf $\mathcal{C}^\infty(\mathfrak{g}[1]) = \wedge \mathfrak{g}^*$. The interesting scenario comes from the case where $\mathfrak{g}$ is a Lie algebra, as it is the motivation for the term "Lie n-algebra" (or more generally "Lie n-algebroid") that we will see later.

Example 3.1.8. Again from the discussion above, it follows that for every (non-graded) vector bundle $E \to M$, one obtains the [1]-manifold $E[1]$ whose sheaf of functions is given by $\mathcal{C}^\infty(E[1]) =$

$\Gamma(\wedge E^*)$. Applying this to the tangent and cotangent bundles of the manifold M yields the [1]-manifolds $T[1]M$ and $T^*[1]M$ whose function sheaves are given by

$$\mathcal{C}^\infty(T[1]M) = \Omega(M) \qquad \text{and} \qquad \mathcal{C}^\infty(T^*[1]M) = \wedge \mathfrak{X}(M).$$

That is, the spaces of differential forms and multivector fields on a smooth manifold M can be realised as graded functions of [1]-manifolds with base M.

In the split setting, a morphism $\mu \colon \underline{F} \to \underline{E}$ between two split [n]-manifolds $\underline{F} = F_1[1] \oplus \ldots \oplus F_n[n]$ and $\underline{E} = E_1[1] \oplus \ldots \oplus E_n[n]$ over N and M, respectively, is a map $\mu^\star \colon \Gamma(\underline{S}(\underline{E}^*)) \to \Gamma(\underline{S}(\underline{F}^*))$ which is a morphism of sheaves of graded algebras over $\mu_0^* \colon C^\infty(M) \to C^\infty(N)$, where $\mu_0 \colon N \to M$ is a smooth map on the base manifolds. The map $\mu^\star$ decomposes into a family of maps $\mu_i^\star \colon \Gamma(E_i^*) \to \Gamma(\underline{S}^i(\underline{F}^*)), i = 1, \ldots, n$ and since we have that

$$\Gamma(\underline{S}^i(\underline{F}^*)) = \bigoplus_{j_1 + 2j_2 + \ldots + nj_n = i} \Gamma\left((F_1^*)^{\otimes j_1} \otimes (F_2^*)^{\otimes j_2} \otimes \ldots \otimes (F_n^*)^{\otimes j_n}\right),$$

the components $\mu_i^\star$ admit a decomposition

$$\mu_i^\star = \sum_{j_1 + 2j_2 + \ldots + nj_n = i} (\mu_i^{j_1, j_2 \cdots, j_n})^\star,$$

where all $(\mu_i^{j_1, j_2 \cdots, j_n})^\star \colon \Gamma(E_i^*) \to \Gamma\left((F_1^*)^{\otimes j_1} \otimes (F_2^*)^{\otimes j_2} \otimes \ldots \otimes (F_n^*)^{\otimes j_n}\right)$ are morphisms over the pull-back map $\mu_0^* \colon C^\infty(M) \to C^\infty(N)$. Note that by abuse of notation, $\otimes$ in the above equation denotes the multiplication of the graded symmetric algebra. Hence, we obtain the vector bundle maps

$$\mu_i^{j_1, j_2 \cdots, j_n} \colon F_1^{\otimes j_1} \otimes F_2^{\otimes j_2} \otimes \ldots \otimes F_n^{\otimes j_n} \to E_i$$

covering $\mu_0 \colon N \to M$. In particular, we have the component maps $\mu_i^{0, \ldots, 1, \ldots, 0} \colon F_i \to E_i$ (1 in the i-th slot) covering the map μ_0 which we call **the principal part of** μ.

Example 3.1.9 (Change of splitting for [2]-manifolds). By construction, splittings of a [2]-manifold $\mathcal{M}$ are given by splittings of the short exact sequence

$$0 \longrightarrow \Gamma(\wedge^2 E_1) \longrightarrow \mathcal{C}^\infty(\mathcal{M})^2 \longrightarrow \Gamma(E_2) \longrightarrow 0,$$

where $E_1 \to M$ and $E_2 \to M$ are the vector bundles whose sheaves of sections are defined by

$$\Gamma(E_1) \simeq \mathcal{C}^\infty(\mathcal{M})^1 \qquad \text{and} \qquad \Gamma(E_2) \simeq \frac{\mathcal{C}^\infty(\mathcal{M})^2}{\mathcal{C}^\infty(\mathcal{M})^1 \cdot \mathcal{C}^\infty(\mathcal{M})^1}.$$

A change of splitting for the exact sequence above is equivalent to a section $\sigma \in \text{Hom}(\wedge^2 E_1, E_2)$. Then the induced isomorphism of [2]-manifolds $\mu_\sigma \colon E_1[1] \oplus E_2[2] \to E_1[1] \oplus E_2[2]$ over the identity on M is given by the components $\mu_\sigma^\star(\varepsilon_1) = \varepsilon_1$ and $\mu_\sigma^\star(\varepsilon_2) = \varepsilon_2 + \sigma^\star \varepsilon_2$ for all $\varepsilon_1 \in \Gamma(E_1^*), \varepsilon_2 \in \Gamma(E_2^*)$. The principal part of μ_σ consists of id_{E_1} and id_{E_2}.

3.2   Vector fields on graded manifolds

Using the language of graded derivations, the usual notion of vector field can be generalised to a notion of vector field on a $\mathbb{Z}$-manifold $\mathcal{M}$.

Definition 3.2.1. A **vector field of degree** j on $\mathcal{M}$ is a degree j (graded) derivation of $\mathcal{C}^\infty(\mathcal{M})$, i.e. a map $\mathcal{X} \colon \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ such that $|\mathcal{X}(\xi)| = j + |\xi|$ and $\mathcal{X}(\xi\zeta) = \mathcal{X}(\xi)\zeta + (-1)^{j|\xi|}\xi\mathcal{X}(\zeta)$, for homogeneous elements $\xi, \zeta \in C^\infty(\mathcal{M})$.

As usual, $|\mathcal{X}|$ denotes the degree of a homogeneous vector field $\mathcal{X}$. The **Lie bracket of two vector fields** $\mathcal{X}, \mathcal{Y}$ on $\mathcal{M}$ is the graded commutator

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - (-1)^{|\mathcal{X}||\mathcal{Y}|}\mathcal{Y}\mathcal{X} \tag{3.4}$$

which is again a vector field of degree $|\mathcal{X}| + |\mathcal{Y}|$. A straightforward computation shows that the following relations hold:

(i) $[\mathcal{X}, \mathcal{Y}] = -(-1)^{|\mathcal{X}||\mathcal{Y}|}[\mathcal{Y}, \mathcal{X}]$,

(ii) $[\mathcal{X}, \xi\mathcal{Y}] = \mathcal{X}(\xi)\mathcal{Y} + (-1)^{|\mathcal{X}||\xi|}\xi[\mathcal{X}, \mathcal{Y}]$,

(iii) $(-1)^{|\mathcal{X}||\mathcal{Z}|}[\mathcal{X}, [\mathcal{Y}, \mathcal{Z}]] + (-1)^{|\mathcal{Y}||\mathcal{X}|}[\mathcal{Y}, [\mathcal{Z}, \mathcal{X}]] + (-1)^{|\mathcal{Z}||\mathcal{Y}|}[\mathcal{Z}, [\mathcal{X}, \mathcal{Y}]] = 0$,

for $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ homogeneous vector fields on $\mathcal{M}$, and ξ a homogeneous element of $\mathcal{C}^\infty(\mathcal{M})$.

Example 3.2.2 (Coordinate vector fields). The local generators ξ_i^j for $i \in \mathbb{Z}, j = 1, \dots, r_i$ of $\mathcal{C}^\infty(\mathcal{M})$ over an open coordinate chart $U \subseteq M$ given by the definition of the $\mathbb{Z}$-manifold $\mathcal{M}$ define the (local) vector fields $\partial_{\xi_i^j}$ of degree $-|\xi_i^j|$ (alternatively denoted $\frac{\partial}{\partial \xi_i^j}$), which send ξ_i^j to 1 and the other local generators to 0. The sheaf $\underline{\mathrm{Der}}_U(\mathcal{C}^\infty(\mathcal{M}))$ of graded derivations of $\mathcal{C}_U^\infty(\mathcal{M})$ is locally freely generated as a $C_U^\infty(\mathcal{M})$-module by all ∂_{x^k} and all $\partial_{\xi_i^j}$, where $x^1, \dots, x^m$ are coordinates of M defined on U.

Note that in the case of a split $\mathbb{Z}$-manifold $\underline{E} = \bigoplus_i E_i[i]$, each section $e \in \Gamma(E_j)$ defines a derivation $\hat{e}$ of degree $-j$ on $\mathcal{M}$ by the relations: $\hat{e}(f) = 0$ for $f \in \mathcal{C}^\infty(M)$, $\hat{e}(\varepsilon) = \langle \varepsilon, e \rangle$ for $\varepsilon \in \Gamma(E_j^*)$ and $\hat{e}(\varepsilon) = 0$ for $\varepsilon \in \Gamma(\underline{E}^*)$ with $|\varepsilon| \neq j$. In particular, $\widehat{e_j^i} = \partial_{\varepsilon_j^i}$ for $\{e_j^i\}$ a local basis of E_j and $\{\varepsilon_j^i\}$ the dual basis of E_j^*.

Given now an $\mathbb{N}$-manifold $\mathcal{M}$ of degree n together with a choice of splitting $\mathcal{M} \cong \underline{E}$ for some graded vector bundle

$$\underline{E} = \bigoplus_{i=1}^n E_i[i],$$

the space of vector fields over $\mathcal{M}$ can be explicitly described in terms of ordinary vector fields over the body M and connections on the bundles E_i. More precisely, given TM-connections $\nabla^i \colon \mathfrak{X}(M) \to \mathrm{Der}(E_i)$ on the vector bundles E_i for each i, the space of vector fields on the split $\mathbb{N}$-manifold $\mathcal{M} \cong \underline{E}$ is generated as a $\mathcal{C}^\infty(\mathcal{M})$-module by

$$\left\{ \bigoplus_{i=1}^n \nabla_X^i \mid X \in \mathfrak{X}(M) \right\} \cup \{\hat{e} \mid e \in \Gamma(E_i) \text{ for some } i\}. \tag{3.5}$$

The vector fields of the form $\bigoplus_i \nabla^i_X$ have degree 0 and are understood to send $f \in C^\infty(M)$ to $X(f) \in C^\infty(M)$, and $\varepsilon \in \Gamma(E_i^*)$ to $\nabla^{i,*}_X \varepsilon \in \Gamma(E_i^*)$. The non-zero degree vector fields are generated by those of the form $\hat{e}$.

3.3 $\mathcal{Q}$-structures and Lie n-algebroids

Due to the sign convention for the Lie bracket in (3.4), it follows that if a vector field $\mathcal{X}$ on $\mathcal{M}$ has odd degree, then the equation

$$[\mathcal{X}, \mathcal{X}] = 2\mathcal{X}^2 = 0 \tag{3.6}$$

is not necessarily satisfied and thus the search for such a vector field is not trivial. If in particular the degree of the vector field is 1, then the structure sheaf $C^\infty(\mathcal{M})$ becomes a complex. This leads to the following definition.

Definition 3.3.1. A **homological vector field** $\mathcal{Q}$ on a $\mathbb{Z}$-manifold $\mathcal{M}$ is a degree 1 derivation of $C^\infty(\mathcal{M})$ such that $\mathcal{Q}^2 = \frac{1}{2}[\mathcal{Q}, \mathcal{Q}] = 0$.

Definition 3.3.2. A $\mathbb{Z}$-manifold $\mathcal{M}$ equipped with a homological vector field $\mathcal{Q}$ is called a $\mathcal{Q}$-**manifold**. A **morphism between two $\mathcal{Q}$-manifolds** $(\mathcal{N}, \mathcal{Q}_\mathcal{N})$ and $(\mathcal{M}, \mathcal{Q}_\mathcal{M})$ is a morphism $\mu : \mathcal{N} \to \mathcal{M}$ of the underlying $\mathbb{Z}$-manifolds that commutes with the homological vector fields:

$$\mu^\star \circ \mathcal{Q}_\mathcal{M} = \mathcal{Q}_\mathcal{N} \circ \mu^\star.$$

Remark 3.3.3. A homological vector field on an [1]-manifold $\mathcal{M} = A[1]$ is the differential d_A associated to a Lie algebroid structure on the vector bundle A over M [93]: The functions of $A[1]$ are given by the sheaf of A-forms $\Omega^\bullet(A)$, and the bracket and the anchor for the Lie algebroid structure on the vector bundle A are connected to the homological vector field d_A via

$$\langle \mathrm{d}_A f, a \rangle = \rho(a)f \qquad \text{and} \qquad \mathrm{d}_A \alpha(a_1, a_2) = -\alpha(a_1, a_2) + \rho(a_1)\left(\alpha(a_2)\right) - \rho(a_2)\left(\alpha(a_1)\right)$$

for all $a, a_1, a_2 \in \Gamma(A), \alpha \in \Gamma(A^*)$ and $f \in C^\infty(M)$. The following definition generalises this to arbitrary degrees.

Definition 3.3.4. A **Lie n-algebroid** is an $[n]$-manifold $\mathcal{M}$ endowed with a homological vector field $\mathcal{Q}$ – the pair $(\mathcal{M}, \mathcal{Q})$ is also called $\mathbb{N}\mathcal{Q}$-**manifold of degree** n. A **split Lie n-algebroid** is a split $[n]$-manifold $\mathcal{M}$ endowed with a homological vector field $\mathcal{Q}$. A **morphism of (split) Lie n-algebroids** is a morphism μ of the underlying $[n]$-manifolds such that $\mu^\star$ commutes with the homological vector fields.

Example 3.3.5 (Product $\mathcal{Q}$-manifold). Given two $\mathcal{Q}$-manifolds $(\mathcal{M}, \mathcal{Q}_\mathcal{M})$ and $(\mathcal{N}, \mathcal{Q}_\mathcal{N})$ with bodies M and N, respectively, the product $\mathcal{M} \times \mathcal{N}$ inherits a $\mathcal{Q}$-manifold structure called **product $\mathcal{Q}$-manifold**. Its homological vector field $\mathcal{Q}_{\mathcal{M} \times \mathcal{N}}$ is given on the generating functions by

$$\mathcal{Q}_{\mathcal{M} \times \mathcal{N}}(\xi\zeta) = \mathcal{Q}_\mathcal{M}(\xi)\zeta + (-1)^{|\xi|}\xi\mathcal{Q}_\mathcal{N}(\zeta)$$

for all homogeneous $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\zeta \in \mathcal{C}^\infty(\mathcal{N})$.

The homological vector field associated to a split Lie n-algebroid $\underline{A} = A_1[1] \oplus \ldots \oplus A_n[n] \to M$ can be equivalently described by a family of brackets which satisfy some Leibniz and higher Jacobi identities [88]. More precisely, a homological vector field on $\underline{A}$ is equivalent to an L_∞-algebra structure[1] on $\Gamma(\underline{A})$ that is anchored by a vector bundle morphism $\rho \colon A_1 \to TM$. Such a structure is given by multibrackets $[\![\cdot, \ldots, \cdot]\!]_i \colon \Gamma(\underline{A})^i \to \Gamma(\underline{A})$ of degree 1 for $1 \leq i \leq n+1$ such that

1. $[\![\cdot, \cdot]\!]_2$ satisfies the Leibniz identity with respect to ρ,

2. $[\![\cdot, \ldots, \cdot]\!]_i$ is $C^\infty(M)$-linear in each entry for all $i \neq 2$,

3. **(graded skew symmetry)** each $[\![\cdot, \ldots, \cdot]\!]_i$ is graded alternating: for a permutation $\sigma \in S_i$ and for all $a_1, \ldots, a_i \in \Gamma(\underline{A})$ degree-homogeneous sections

$$[\![a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(i)}]\!]_i = \mathrm{Ksgn}(\sigma, a_1, \ldots, a_i) \cdot [\![a_1, a_2, \ldots, a_i]\!]_i,$$

and

4. **(strong homotopy Jacobi identity)** for $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \Gamma(\underline{A})$ sections of homogeneous degree:

$$\sum_{i+j=k+1} (-1)^{i(j-1)} \sum_{\sigma \in \mathrm{Sh}_{i,k-i}} \mathrm{Ksgn}(\sigma, a_1, \ldots, a_k) [\![[\![a_{\sigma(1)}, \ldots, a_{\sigma(i)}]\!]_i, a_{\sigma(i+1)}, \ldots, a_{\sigma(k)}]\!]_j = 0.$$

Here, $\mathrm{Sh}_{i,k-i}$ is the set of all $(i, k-i)$-shuffles[2] and $\mathrm{Ksgn}(\sigma, a_1, \ldots, a_k)$ is the $(a_1, \ldots, a_k)$-graded signature of the permutation $\sigma \in S_k$, i.e.

$$a_1 \wedge \ldots \wedge a_k = \mathrm{Ksgn}(\sigma, a_1, \ldots, a_k) a_{\sigma(1)} \wedge \ldots \wedge a_{\sigma(k)}.$$

The explicit correspondence between the homological vector field $\mathcal{Q}$ and the brackets $[\![\cdot, \ldots, \cdot]\!]_i$ is given by the following equations:

1. $\mathcal{Q}(f) = \rho^*(\mathrm{d}f)$;

2. $\mathcal{Q}(\alpha_k)(a_1^{i_1}, \ldots, a_j^{i_j}) = -\langle \alpha_k, [\![a_1^{i_1}, \ldots, a_j^{i_j}]\!]_j \rangle,\ j \neq 2$;

3. $\mathcal{Q}(\alpha_k)(a_1^1, a_2^k) = \rho(a_1^1)\langle \alpha_k, a_2^k \rangle - \langle \alpha_k, [\![a_1^1, a_2^k]\!]_2 \rangle,\ k \neq 1$;

4. $\mathcal{Q}(\alpha_1)(a_1^1, a_2^1) = \rho(a_1^1)\langle \alpha_1, a_2^1 \rangle - \rho(a_2^1)\langle \alpha_1, a_1^1 \rangle - \langle \alpha_1, [\![a_1^1, a_2^1]\!]_2 \rangle$

for all $f \in C^\infty(M), \alpha_k \in \Gamma(A_k^*), a_p^{i_p} \in \Gamma(A_{i_p}), i_1 + \ldots + i_j = k + 2 - j$.

In particular, the vector bundle $A_1 \to M$ of a split Lie n-algebroid is always endowed with skew-symmetric dull bracket on its space of sections $\Gamma(A_1)$. The following alternative geometric

---

[1]We note that the sign convention agrees with, e.g. [55, 99]. In [85], the term "$L_\infty[1]$-algebra" was used for brackets with this sign convention.

[2]A $(i, k-i)$-shuffle is an element $\sigma \in S_k$ such that $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(k)$.

description in the special case of a split Lie 2-algebroid $(\mathcal{M} = A_1[1] \oplus A_2[2], \mathcal{Q})$ will be used extensively in the rest of the work (see [50]). For consistency with the notation in [50], set $A_1 := Q$ and $A_2^* =: B$.

Definition 3.3.6. A **split Lie 2-algebroid** $Q[1] \oplus B^*[2]$ is given by a pair of an anchored vector bundle $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with a vector bundle map $\ell \colon B^* \to Q$, a skew-symmetric dull bracket $[\cdot, \cdot] \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$, a linear Q-connection ∇ on B, and a vector valued 3-form $\omega \in \Omega^3(Q, B^*)$ such that

(i) $\nabla^*_{\ell(\beta_1)}\beta_2 + \nabla^*_{\ell(\beta_2)}\beta_1 = 0$, for all $\beta_1, \beta_2 \in \Gamma(B^*)$,

(ii) $[q, \ell(\beta)] = \ell(\nabla^*_q \beta)$ for all $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,

(iii) $\mathrm{Jac}_{[\cdot,\cdot]} = \ell \circ \omega \in \Omega^3(Q, Q)$,

(iv) $R_{\nabla^*}(q_1, q_2)\beta = -\omega(q_1, q_2, \ell(\beta))$ for $q_1, q_2 \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,

(v) $\mathrm{d}_{\nabla^*}\omega = 0$.

To pass from the definition above to the homological vector field $\mathcal{Q}$, set $\mathcal{Q}(f) = \rho^* \mathrm{d}f \in \Gamma(Q^*)$, $\mathcal{Q}(\tau) = \mathrm{d}_Q \tau + \partial_B \tau \in \Omega^2(Q) \oplus \Gamma(B)$, and $\mathcal{Q}(b) = \mathrm{d}_\nabla b - \langle \omega, b \rangle \in \Omega^1(Q, B) \oplus \Omega^3(Q)$ for $f \in C^\infty(M), \tau \in \Omega(Q)$ and $b \in \Gamma(B)$, where $\partial_B := \ell^*$. For a detailed proof of this equivalence see Appendix B.1 or the original source [50].

On the other hand we may obtain as follows the data of Definition 3.3.6 from a given homological vector field $\mathcal{Q}$. Define the vector bundle map ℓ to be the 1-bracket and ρ to be the anchor. The 2-bracket induces the skew-symmetric dull bracket on Q and the Q-connection on B^* via the formula

$$\llbracket q_1 \oplus \beta_1, q_2 \oplus \beta_2 \rrbracket_2 = [q_1, q_2]_Q \oplus (\nabla^*_{q_1}\beta_2 - \nabla^*_{q_2}\beta_1).$$

Finally, the 3-bracket induces the 3-form ω via the formula

$$\llbracket q_1 \oplus 0, q_2 \oplus 0, q_3 \oplus 0 \rrbracket_3 = 0 \oplus \omega(q_1, q_2, q_3).$$

Example 3.3.7 (Lie 2-algebras). If we consider a Lie 2-algebroid over a point, then we recover the notion of **Lie 2-algebra** [9]. Specifically, a Lie 2-algebroid over a point consists of a pair of vector spaces $\mathfrak{g}_0, \mathfrak{g}_1$, a linear map $\ell \colon \mathfrak{g}_0 \to \mathfrak{g}_1$, a skew-symmetric bilinear bracket $[\cdot, \cdot] \colon \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$, a bilinear action bracket $[\cdot, \cdot] \colon \mathfrak{g}_1 \times \mathfrak{g}_0 \to \mathfrak{g}_0$, and an alternating trilinear bracket $[\cdot, \cdot, \cdot] \colon \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ such that

1. $[\ell(x), y] + [\ell(y), x] = 0$ for $x, y \in \mathfrak{g}_0$,

2. $[x, \ell(y)] = \ell([x, y])$ for $x \in \mathfrak{g}_1$ and $y \in \mathfrak{g}_0$,

3. $\mathrm{Jac}_{[\cdot,\cdot]}(x, y, z) = \ell([x, y, z])$ for $x, y, z \in \mathfrak{g}_1$,

4. $[[x, y], z] + [y, [x, z]] - [x, [y, z]] = [x, y, \ell(z)]$ for $x, y \in \mathfrak{g}_1$ and $z \in \mathfrak{g}_0$,

5. and the higher Jacobi identity

$$
\begin{aligned}
0 =& [x,[y,z,w]] - [y,[x,z,w]] + [z,[x,y,w]] - [w,[x,y,z]] \\
& - [[x,y],z,w] + [[x,z],y,w] - [[x,w],y,z] - [[y,z],x,w] + [[y,w],x,z] - [[z,w],x,y].
\end{aligned}
$$

holds for $x,y,z,w \in \mathfrak{g}_1$.

Example 3.3.8 (Derivation Lie 2-algebr(oid)). For any Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$, the **derivation Lie 2-algebra** is defined as the complex

$$
\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})
$$

with brackets given by $[\delta_1,\delta_2] = \delta_1\delta_2 - \delta_2\delta_1$, $[\delta,x] = \delta x$, $[\delta_1,\delta_2,\delta_3] = 0$ for all $\delta,\delta_i \in \mathrm{Der}(\mathfrak{g}), i = 1,2,3$, and $x \in \mathfrak{g}$.

A global analogue of this construction can be achieved only under strong assumptions on the Lie algebroid $A \to M$. Precisely, let $A \to M$ be a Lie algebra bundle. Then the space of all derivations D of the vector bundle A which preserve the bracket

$$
D[a_1,a_2] = [Da_1,a_2] + [a_1,Da_2] \qquad \text{for all } a_1,a_2 \in \Gamma(A)
$$

is the module of sections of a vector bundle over M, denoted $\mathrm{Der}_{[\cdot,\cdot]}(A) \to M$. Together with the usual commutator bracket and the anchor $\rho'(D) = X$, where D is a derivation of $\Gamma(A)$ covering $X \in \mathfrak{X}(M)$, the vector bundle $\mathrm{Der}_{[\cdot,\cdot]}(A)$ is a Lie algebroid over M [67]. Since the anchor of A is trivial, the complex

$$
A \stackrel{\mathrm{ad}}{\to} \mathrm{Der}_{[\cdot,\cdot]}(A) \stackrel{\rho'}{\to} TM
$$

becomes a Lie 2-algebroid with the $\mathrm{Der}_{[\cdot,\cdot]}(A)$-connection on A given by $\nabla_D a = Da$ and $\omega = 0$.

Example 3.3.9 (Courant algebroids). Let $E \to M$ be a Courant algebroid with pairing $\langle\cdot,\cdot\rangle\colon E \times_M E \to E$, anchor ρ and bracket $[\![\cdot,\cdot]\!]$, and choose a metric linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then $E[1] \oplus T^*M[2]$ becomes as follows a split Lie 2-algebroid. The skew-symmetric dull bracket is given by $[e,e'] = [\![e,e']\!] - \rho^*\langle\nabla.e,e'\rangle$ for all $e,e' \in \Gamma(E)$. The basic connection is $\nabla^{\mathrm{bas}}\colon \Gamma(E) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ and the basic curvature is given by $\omega_\nabla \in \Omega^2(E,\mathrm{Hom}(TM,E))$

$$
\omega_\nabla(e,e')X = -\nabla_X[\![e,e']\!] + [\![\nabla_X e,e']\!] + [\![e,\nabla_X e']\!] + \nabla_{\nabla^{\mathrm{bas}}_{e'}X}e - \nabla_{\nabla^{\mathrm{bas}}_e X}e' - P^{-1}\langle\nabla_{\nabla^{\mathrm{bas}}_{\cdot}X}e,e'\rangle,
$$

where $P\colon E \to E^*$ is the isomorphism defined by the pairing, for all $e,e' \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. The map ℓ is $\rho^*\colon T^*M \to E$, the E-connection on T^*M is $\nabla^{\mathrm{bas},*}$ and the form $\omega \in \Omega^3(E,T^*M)$ is given by $\omega(e_1,e_2,e_3) = \langle\omega_\nabla(e_1,e_2)(\cdot),e_3\rangle$. The kind of split Lie 2-algebroids that arise in this way are the **split symplectic Lie 2-algebroids** [83]. They are splittings of the symplectic Lie 2-algebroid which is equivalent to the tangent prolongation of E. The tangent prolongation of E is an LA-Courant algebroid [50, 52].

3.4 Generalised functions of a Lie n-algebroid

We now turn to the space of generalised functions (or alternatively the vector valued functions) of a Lie n-algebroid. In the following, $(\mathcal{M}, \mathcal{Q})$ is a Lie n-algebroid with underlying manifold M. Consider the space $\mathcal{C}^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(\underline{E})$ for a graded vector bundle $\underline{E} \to M$ of finite rank. For simplicity, the tensor product $\mathcal{C}^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(\underline{E})$ is sometimes written as $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$. That is, in the rest of work this particular kind of tensor products is always considered as a tensor product of $C^\infty(M)$-modules.

First suppose that $(\mathcal{M}, \mathcal{Q}) = (A[1], \mathrm{d}_A)$ is a Lie algebroid. The space of $\underline{E}$-**valued differential forms** $\Omega(A; \underline{E}) := \Omega(A) \otimes_{C^\infty(M)} \Gamma(\underline{E}) = \mathcal{C}^\infty(A[1]) \otimes_{C^\infty(M)} \Gamma(\underline{E})$ has a natural grading given by

$$\Omega(A; \underline{E})_p = \bigoplus_{i-j=p} \Omega^i(A; E_j).$$

It is well-known (see [4]) that any degree preserving vector bundle map $h \colon \underline{E} \otimes \underline{F} \to \underline{G}$ induces a wedge product operation

$$(\cdot \wedge_h \cdot) \colon \Omega(A; \underline{E}) \times \Omega(A; \underline{F}) \to \Omega(A; \underline{G})$$

which is defined on $\omega \in \Omega^p(A; E_i)$ and $\eta \in \Omega^q(A; F_j)$ by

$$(\omega \wedge_h \eta)(a_1, \ldots, a_{p+q}) = \sum_{\sigma \in \mathrm{Sh}_{p,q}} (-1)^{qi} \mathrm{sgn}(\sigma) h\left(\omega(a_{\sigma(1)}, \ldots, a_{\sigma(p)}), \eta(a_{\sigma(p+1)}, \ldots, a_{\sigma(p+q)})\right),$$

for all $a_1, \ldots, a_{p+q} \in \Gamma(A)$.

In particular, the above rule reads

$$\theta \wedge_h \zeta = (-1)^{qi} (\omega \wedge \eta) \otimes h(e, f),$$

for all $\theta = \omega \otimes e$ and $\zeta = \eta \otimes f$ where ω is a p-form, η is a q-form, and e and f are homogeneous sections of $\underline{E}$ and $\underline{F}$ of degree i and j, respectively.

Some notable cases for special choices of the map h are given by the identity, the composition of endomorphisms, the evaluation and the 'twisted' evaluation maps, the graded commutator of endomorphisms and the natural pairing of a graded vector bundle with its dual. In particular, the evaluation $(\Phi, e) \mapsto \Phi(e)$ and the twisted evaluation $(e, \Phi) \mapsto (-1)^{|\Phi||e|}\Phi(e)$ make $\Omega(A; \underline{E})$ a graded $\Omega(A; \underline{\mathrm{End}}(\underline{E}))$-bimodule.

In the general case of a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$, the space $\Omega(A)$ is replaced by the generalised smooth functions $\mathcal{C}^\infty(\mathcal{M})$ of $\mathcal{M}$. The space $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$ has a natural grading, where the homogeneous elements of degree p are given by

$$\bigoplus_{i-j=p} \mathcal{C}^\infty(\mathcal{M})^i \otimes \Gamma(E_j).$$

Similarly as in the case of a Lie algebroid, given a degree preserving map

$$h\colon \underline{E} \otimes \underline{F} \to \underline{G},$$

one obtains the multiplication

$$(\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})) \times (\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{F})) \to \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{G})$$

$$(\omega, \eta) \mapsto \omega \wedge_h \eta.$$

In particular, for elements of the form $\xi \otimes e \in \mathcal{C}^\infty(\mathcal{M})^i \otimes \Gamma(E_j), \zeta \otimes f \in \mathcal{C}^\infty(\mathcal{M})^k \otimes \Gamma(F_\ell)$ the above rule reads

$$(\xi \otimes e) \wedge_h (\zeta \otimes f) = (-1)^{(-j)k}\xi\zeta \otimes h(e, f),$$

where on the right hand side the multiplication $\xi\zeta$ is the one in $\mathcal{C}^\infty(\mathcal{M})$. The special cases above are defined similarly for the n-algebroid case. Moreover, $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$ is endowed with the structure of a graded $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{End}}(E))$-bimodule.

Finally, the following lemma will be useful later as it is a generalisation of [4, Lemma A.1], and gives the connection between the space $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{Hom}}(E, F))$ and the homomorphisms from $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$ to $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{F})$.

Lemma 3.4.1. There is a one-to-one correspondence between the degree n elements of $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{Hom}}(E, F))$ and the operators $\Psi\colon \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E}) \to \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{F})$ of degree n which are $\mathcal{C}^\infty(\mathcal{M})$-linear in the graded sense:

$$\Psi(\xi \wedge \eta) = (-1)^{nk}\xi \wedge \Psi(\eta),$$

for all $\xi \in \mathcal{C}^\infty(\mathcal{M})^k$, and all $\eta \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$.

Proof. The element $\Phi \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{End}}(E))$ induces the operator $\widehat{\Phi}$ given by left multiplication by Φ:

$$\widehat{\Phi}(\eta) = \Phi \wedge \eta.$$

This clearly satisfies $\widehat{\Phi}(\xi \wedge \eta) = (-1)^{nk}\xi \wedge \widehat{\Phi}(\eta)$, for all $\xi \in \mathcal{C}^\infty(\mathcal{M})^k$, $\eta \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$. Conversely, an operator Ψ of degree n must send a section $e \in \Gamma(E_k)$ into the sum

$$\Gamma(F_{k-n}) \oplus \left(\mathcal{C}^\infty(\mathcal{M})^1 \otimes \Gamma(F_{k-n+1})\right) \oplus \left(\mathcal{C}^\infty(\mathcal{M})^2 \otimes \Gamma(F_{k-n+2})\right) \oplus \ldots,$$

defining the elements

$$\Psi_i \in C^\infty(\mathcal{M})^i \otimes \Gamma(\underline{\mathrm{Hom}}^{n-i}(E, F)).$$

Thus, this yields the (finite) sum $\widetilde{\Psi} = \sum_i \Psi_i \in \left(\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{Hom}}(E, F))\right)^n$. Clearly,

$$\widetilde{\widehat{\Phi}} = \Phi \text{ and } \widehat{\widetilde{\Psi}} = \Psi. \qquad \square$$

Schematically, for a Lie n-algebroid $\mathcal{M}$, the above lemma gives the following diagram:

$$\left(\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{Hom}}(\underline{E}, \underline{F}))\right)^n \xleftrightarrow{1\text{-}1} \left\{ \begin{array}{c} \text{Degree } n \text{ operators } \Psi \\ \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E}) \to \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{F}) \\ \text{which are } \mathcal{C}^\infty(\mathcal{M})\text{-linear in the graded sense} \end{array} \right\}.$$

In particular, if $\underline{E} = \underline{F}$, then

$$\left(\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{End}}(\underline{E}))\right)^n \xleftrightarrow{1\text{-}1} \left\{ \begin{array}{c} \text{Degree } n \text{ operators } \Psi \text{ on } \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E}) \text{ which} \\ \text{are } \mathcal{C}^\infty(\mathcal{M})\text{-linear in the graded sense} \end{array} \right\}.$$

3.5 Graded Poisson structures

Poisson structures can be defined on $\mathbb{Z}$-manifolds for any $k \in \mathbb{Z}$. More precisely, a **degree k Poisson bracket** on a $\mathbb{Z}$-manifold $\mathcal{M}$ is an $\mathbb{R}$-bilinear map $\{\cdot, \cdot\}_k \colon \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ of degree k, i.e. $|\{\xi_1, \xi_2\}_k| = |\xi_1| + |\xi_2| + k$, such that it satisfies the following conditions:

1. $\{\xi_1, \xi_2\}_k = -(-1)^{(|\xi_1|+k)(|\xi_2|+k)}\{\xi_2, \xi_1\}_k$,

2. $\{\xi_1, \xi_2\xi_3\}_k = \{\xi_1, \xi_2\}_k\xi_3 + (-1)^{(|\xi_1|+k)|\xi_2|}\xi_2\{\xi_1, \xi_3\}_k$,

3. $\{\xi_1, \{\xi_2, \xi_3\}\}_k = \{\{\xi_1, \xi_2\}_k, \xi_3\}_k + (-1)_k^{(|\xi_1|+k)(|\xi_2|+k)}\{\xi_2, \{\xi_1, \xi_3\}\}_k$,

for homogeneous elements $\xi_1, \xi_2, \xi_3 \in \mathcal{C}^\infty(\mathcal{M})$. $\mathbb{Z}$-graded manifolds endowed with a Poisson bracket of degree k are called $\mathcal{P}_k$**-manifolds**, or simply $\mathcal{P}$**-manifolds**. An $\mathbb{N}$-manifold together with a Poisson bracket of degree k is called $\mathbb{N}\mathcal{P}_k$**-manifold**, or simply $\mathbb{N}\mathcal{P}$**-manifold**. A **morphism of two $\mathcal{P}$-manifolds** (respectively $\mathbb{N}\mathcal{P}$-manifolds) $(\mathcal{N}, \{\cdot, \cdot\}_\mathcal{N})$ and $(\mathcal{M}, \{\cdot, \cdot\}_\mathcal{M})$ with both brackets of degree k is a morphism of graded manifolds $\mathcal{F} \colon \mathcal{N} \to \mathcal{M}$ which respects the Poisson brackets: $\mathcal{F}^\star\{\xi_1, \xi_2\}_\mathcal{M} = \{\mathcal{F}^\star\xi_1, \mathcal{F}^\star\xi_2\}_\mathcal{N}$ for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$.

As in the ordinary case, every degree k Poisson bracket on $\mathcal{M}$ induces a degree k map

$$\sharp \colon \mathcal{C}^\infty(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

which sends ξ to its **Hamiltonian vector field** $\mathcal{X}_\xi = \{\xi, \cdot\}_k$ (of degree $|\xi|+k$). A graded manifold is called **symplectic** if it is equipped with a degree k Poisson bracket whose Hamiltonian vector fields generate all of $\mathfrak{X}(\mathcal{M})$. Note that the Hamiltonian map characterises the morphisms between graded Poisson manifolds as morphisms $\mathcal{F} \colon \mathcal{N} \to \mathcal{M}$ such that the Hamiltonian vector field $\mathcal{X}_\xi$ on $\mathcal{M}$ is $\mathcal{F}$-related to the Hamiltonian vector field $\mathcal{X}_{\mathcal{F}^\star\xi}$ on $\mathcal{N}$ for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$.

A homogeneous vector field $\mathcal{X}$ of a graded Poisson manifold $\mathcal{M}$ is called **Poisson vector field** if it is a derivation of the Poisson bracket:

$$\mathcal{X}\{\xi_1, \xi_2\}_k = \{\mathcal{X}(\xi_1), \xi_2\}_k + (-1)^{|\mathcal{X}|(|\xi_1|+k)}\{\xi_1, \mathcal{X}(\xi_2)\},$$

for all homogeneous $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$. Equivalently, using the Hamiltonian vector field notation the defining formula of Poisson vector fields can be expressed as $\mathcal{X}_{\mathcal{X}(\xi)} = [\mathcal{X}, \mathcal{X}_\xi]$ for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$.

Yet another equivalent form of the compatibility condition of $\mathcal{X}$ with $\{\cdot,\cdot\}_k$ will be given later using the generalisation of the Schouten bracket on the space of (pseudo)multivector fields on $\mathcal{M}$. Note that similarly to the case of ordinary Poisson manifolds, the Jacobi identity of a Poisson bracket can be restated as the commutativity of the Hamiltonian map $\sharp$ with the brackets $\{\cdot,\cdot\}_k$ and $[\cdot,\cdot]$:

$$\mathcal{X}_{\{\xi_1,\xi_2\}_k} = [\mathcal{X}_{\xi_1}, \mathcal{X}_{\xi_2}],$$

for all $\xi_1,\xi_2 \in \mathcal{C}^\infty(\mathcal{M})$, or equivalently, every Hamiltonian vector field is Poisson:

$$\mathcal{X}_{\xi_1}\{\xi_2,\xi_3\}_k = \{\mathcal{X}_{\xi_1}(\xi_2),\xi_3\}_k + (-1)^{(|\xi_1|+k)(|\xi_2|+k)}\{\xi_2,\mathcal{X}_{\xi_1}(\xi_3)\}_k,$$

for all $\xi_1,\xi_2,\xi_3 \in \mathcal{C}^\infty(\mathcal{M})$.

A $\mathcal{PQ}$-**manifold** $(\mathcal{M},\mathcal{Q},\{\cdot,\cdot\}_k)$ is a graded manifold $\mathcal{M}$ endowed with a compatible pair of a homological vector field $\mathcal{Q}$ and a Poisson bracket $\{\cdot,\cdot\}_k$, i.e a **Poisson homological vector field**:

$$\mathcal{Q}\{\xi_1,\xi_2\}_k = \{\mathcal{Q}(\xi_1),\xi_2\}_k + (-1)^{|\xi_1|+k}\{\xi_1,\mathcal{Q}(\xi_2)\}_k$$

for all $\xi_1,\xi_2 \in \mathcal{C}^\infty(\mathcal{M})$. If the underlying manifold $\mathcal{M}$ is $\mathbb{N}$-graded of degree $n \in \mathbb{N}$, then the $(\mathcal{M},\mathcal{Q},\mathcal{P})$ is called **Poisson Lie n-algebroid**, or **symplectic Lie n-algebroid** if the Poisson bracket is symplectic. A **morphism of $\mathcal{PQ}$-manifolds** (similarly of Poisson and symplectic Lie n-algebroids) is a morphism of the underlying graded manifolds which is a morphism of both $\mathcal{Q}$- and $\mathcal{P}$-manifolds.

Clearly, a Poisson (symplectic) Lie 0-algebroid is a usual Poisson (symplectic) manifold M. A Poisson Lie 1-algebroid is a Lie bialgebroid (A,A^*) and a symplectic Lie 1-algebroid is again in correspondence with a usual Poisson manifold as it becomes the Lie bialgebroid (T^*M,TM) of a Poisson manifold M. In [83], it was shown by Roytenberg that split symplectic Lie 2-algebroids correspond to Courant algebroids. More details about these correspondences are given in Chapter 4 and in particular in Section 4.5 and Section 4.7.

Chapter 4

Graded tangent and cotangent bundles

This chapter defines (graded) vector bundles in the category of graded manifolds and analyses, in particular, two fundamental examples of vector bundles accompanying any graded manifold. Namely, the tangent and cotangent bundles of a graded manifold. As the tangent and cotangent bundles are our main objects for the theory of modules and representations in the next chapters, we provide here many details about their structure. Moreover, we explain the equivalence of Poisson manifolds and Courant algebroids with symplectic Lie 1- and 2-algebroids, respectively. The following is based on [71].

4.1 Vector bundles over graded manifolds

Suppose that $\{k_i\}$ is a finite collection of natural numbers. As usual, a **vector bundle of rank** $\{k_i\}$ [71, 73] in the category of $\mathbb{Z}$-graded manifolds is defined as a surjection of graded manifolds $q\colon \mathcal{E} \to \mathcal{M}$ endowed with an atlas of local trivialisations $\mathcal{E}|_{q^{-1}(U)} \cong \mathcal{M}|_U \times \mathbb{R}^{\{k_i\}}$ over the open set $U \subset M$, such that the transition functions between two local trivialisations are linear in the fibre coordinates.

Remark 4.1.1. The standard algebraic constructions such as tensor products, direct sums, duals, etc., can be applied in this category yielding new vector bundles. In particular, one may apply the degree shifting functor $[j]$ to $\mathcal{E}$ to produce another vector bundle $q[j]\colon \mathcal{E}[j] \to \mathcal{M}$ obtained by $\mathcal{E}$ after shifting the fibre coordinates by j, i.e. the fibre coordinates $\{e^i\}$ of $\mathcal{E}$ induce the fibre coordinates $\{e^i\}$ of degree $|e^i| - j$ for $\mathcal{E}[j]$.

The **space of sections of** $\mathcal{E}$ is defined as $\Gamma(\mathcal{E}) := \bigoplus_{j \in \mathbb{Z}} \Gamma^j(\mathcal{E})$, where $\Gamma^j(\mathcal{E})$ consists of the (homogeneous) **degree** j **sections** of $\mathcal{E}$, i.e. morphisms of graded manifolds $\sigma\colon \mathcal{M} \to \mathcal{E}[j]$ such that $q \circ \sigma = \mathrm{id}_{\mathcal{M}}$. There is another equivalent definition for a vector bundle given by its sheaf of sections $\Gamma(\mathcal{E})$ as an analogue of the Serre-Swan theorem of ordinary vector bundles over smooth

manifolds [71]. Explicitly, a vector bundle over $\mathcal{M}$ can be defined as a sheaf $\mathcal{E}$ of locally freely generated (left) graded $\mathcal{C}^\infty(\mathcal{M})$-modules over the smooth manifold M.

Note that there are two special classes of functions associated to every vector bundle $\mathcal{E}$ called **basic** and **linear** functions and denoted $\mathcal{C}^\infty_{\text{bas}}(\mathcal{E})$ and $\mathcal{C}^\infty_{\text{lin}}(\mathcal{E})$, respectively. The former consists of the pull-back of functions of $\mathcal{C}^\infty(\mathcal{M})$ via the projection q while the latter is given by the functions in $\mathcal{C}^\infty(\mathcal{E})$ which are linear in the fibre coordinates. In other words, any linear function $\ell \in \mathcal{C}^\infty(\mathcal{E})$ can be written locally as $\ell = e^j \xi_j$, where e^j is a set of fibre coordinates of $\mathcal{E}$ and ξ_j are local smooth functions on $\mathcal{M}$. In fact, the linear functions of a vector bundle $\mathcal{E}$ are identified with $\Gamma(\mathcal{E}^*)$, the $\mathcal{C}^\infty(\mathcal{M})$-module of sections of the dual bundle $\mathcal{E}^* \to \mathcal{M}$. We define the bigrading (p, q) on the space of functions $\mathcal{C}^\infty(\mathcal{E})$ of the total space of $q : \mathcal{E} \to \mathcal{M}$ so that it satisfies

$$\mathcal{C}^\infty(\mathcal{E})^i = \bigoplus_{p+k=i} \mathcal{C}^\infty(\mathcal{E})^{p,k},$$

for all $i \in \mathbb{Z}$; that is, the horizontal degree p is the "polynomial" degree obtained from the linear coordinates. This implies in particular that a basic function $q^\star(\xi) \in \mathcal{C}^\infty_{\text{bas}}(\mathcal{E})^i$ has bidegree $(0, i)$ for all $\xi \in \mathcal{C}^\infty(\mathcal{M})^i$ and a linear function $\ell \in \mathcal{C}^\infty_{\text{lin}}(\mathcal{E})^i$ has bidegree $(1, i-1)$.

Example 4.1.2 (Split vector bundles). Suppose $\mathcal{M}$ is a $\mathbb{Z}$-manifold over M and $\underline{E} = \bigoplus_{i \in \mathbb{Z}} E[i] \to M$ is a graded vector bundle over M. Then the tensor product $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$ defines the sheaf of sections of a vector bundle $\mathcal{E} \to \mathcal{M}$. In the case of a split $\mathbb{Z}$-manifold $\mathcal{M} = \underline{A} = \bigoplus_i A_i[i]$, this implies in particular that the coordinates of the $\mathbb{Z}$-manifold $\mathcal{E}$ are given by

1. **Basic:** coordinates x^i of M of degree 0 (bidegree $(0,0)$) and coordinates α^i_k of degree i (bidegree $(0, i)$), where $\{\alpha^i_k\}$ is a local frame of A^*_i.

2. **Linear:** coordinates ε^i_k of degree i (bidegree $(1, i-1)$), where $\{\varepsilon^i_k\}$ is a local frame of E^*_i.

Remark 4.1.3. Mehta proved in [75] that all the vector bundles over $\mathbb{N}$-graded manifolds come non-canonically from this construction. Briefly, the proof goes as follows: Let $\mathcal{E} \to \mathcal{M}$ be a vector bundle over the $\mathbb{N}$-manifold $\mathcal{M}$. The pull-back[1] $0^!_\mathcal{M}\mathcal{E}$ with respect to the zero embedding $0_\mathcal{M}: M \to \mathcal{M}$ is an ordinary graded vector bundle $\underline{E}$ over M and hence splits as a direct sum $\underline{E} = \bigoplus_i E_i[i]$. The double pull-back $\pi^!_\mathcal{M} 0^!_\mathcal{M}\mathcal{E}$, where $\pi_\mathcal{M}: \mathcal{M} \to M$ is the projection map, is non-canonically isomorphic to $\mathcal{E}$ as vector bundles over $\mathcal{M}$. Then, as a sheaf over M, $\Gamma(\mathcal{E})$ is identified with $\Gamma(\pi^!_\mathcal{M} 0^!_\mathcal{M}\mathcal{E}) = \Gamma(\pi^!_\mathcal{M}\underline{E})$, which in turn is canonically isomorphic to $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$.

Example 4.1.4 (Tangent and cotangent bundles). For every graded manifold $\mathcal{M}$, the **tangent bundle** $T\mathcal{M}$ is a vector bundle over $\mathcal{M}$ whose sheaf of sections is given by the vector fields $\mathfrak{X}(\mathcal{M})$. For a description of $T\mathcal{M}$ in terms of local trivialisations and transition maps, we proceed as follows [71, 73]: Given a local chart $U \subset M$ of $\mathcal{M}$ with coordinates $\xi^j_i, i \in \mathbb{Z}, j = 1, \ldots, r_i$, of degree $|\xi^j_i| = -i$ as in (3.1), we form the product manifold $\mathrm{T}\,\mathcal{M}|_U := \mathcal{M}|_U \times \mathbb{R}^{\{r_i\}}$ with coordinates $\xi^j_i, \dot\xi^j_i$ of degree $|\xi^j_i| = -i = |\dot\xi^j_i|$. Given two charts $U, V \subset M$ of $\mathcal{M}$ with coordinates ξ^j_i and ζ^j_i, respectively, and the coordinate transformation $\mu_{UV} : \mathcal{M}|_U \to \mathcal{M}|_V$ from ξ^j_i to ζ^j_i, we obtain the

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[1]Obtained by the induced $C^\infty(M)$-module structure of $\Gamma(\mathcal{E})$

coordinate transformation $T\mu_{UV}: T\mathcal{M}|_U \to T\mathcal{M}|_V$ defined by

$$(T\mu_{UV})^\star \left(\zeta_i^j \right) = \mu_{UV}^\star \left(\zeta_i^j \right) \qquad \text{and} \qquad (T\mu_{UV})^\star \left(\dot{\zeta}_i^j \right) = \sum_{s,t} \dot{\xi}_s^t \frac{\partial}{\partial \xi_s^t} \mu_{UV}^\star \left(\zeta_i^j \right).$$

The graded manifold $T\mathcal{M}$ is obtained by gluing the local charts $T\mathcal{M}|_U$ with the transformations defined above. The dual of $T\mathcal{M}$ is the **cotangent bundle** $T^*\mathcal{M}$ with sheaf of sections the 1-forms $\Omega^1(\mathcal{M})$. If $\mathcal{M}$ is $\mathbb{N}$-graded, then in terms of the isomorphism of Example 4.1.2, the tangent and cotangent bundles are represented by the sheaves

$$\Gamma(T\mathcal{M}) \cong \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{A} \oplus TM[0]) \qquad \text{and} \qquad \Gamma(T^*\mathcal{M}) \cong \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(T^*M[0] \oplus \underline{A}^*),$$

where the vector bundle $\underline{A} = \bigoplus_i A_i[i] \to M$ is a splitting of the base manifold $\mathcal{M}$, i.e. $\mathcal{C}^\infty(\mathcal{M}) \cong \Gamma(\underline{S}(\underline{A}^*))$. We will see more geometric structures on $T\mathcal{M}$ and $T^*\mathcal{M}$ in the coming sections.

4.2 (Pseudo)differential forms and the Weil algebra of a $\mathcal{Q}$-manifold

Let $\mathcal{M}$ be an $\mathbb{Z}$-manifold over a smooth manifold M and $\xi_i^1, \ldots, \xi_i^{r_i}, i \in \mathbb{Z}$ be its local generators over some open $U \subset M$, as in (3.1). By definition, the **(shifted) tangent bundle**[1] $T[1-k]\mathcal{M}$ of $\mathcal{M}$, for $k \in \mathbb{Z}$, is the $\mathbb{Z}$-manifold [71, 73] whose structure sheaf assigns to every coordinate domain $(U, x^1, \ldots, x^m)$ of M that trivialises $\mathcal{M}$, the space

$$\mathcal{C}_U^\infty(T[1-k]\mathcal{M}) = \bigoplus_{i \in \mathbb{Z}} \underbrace{\mathcal{C}_U^\infty(\mathcal{M})^i}_{(0,\, i)} \left\langle \underbrace{(\mathrm{d}x^s)_{s=1}^m}_{(1,\, -k)}; \underbrace{(\mathrm{d}\xi_j^s)_{s=1}^{r_j}}_{(1,\, -k+|\xi_j^s|)}, j \in \mathbb{Z} \right\rangle.$$

It carries the bigrading (p, q), where p is the grading of "(pseudo)differential forms" (or "polynomial degree"[2]) and q comes from the grading of $\mathcal{M}$. Note that as the structure sheaf of a $\mathbb{Z}$-manifold the above algebra of functions is considered with the *total degree*. That is, an element of bidegree (p, q) is a function of degree $p + q$; e.g. the degrees of the functions ξ and $\mathrm{d}\xi$ in $\mathcal{C}^\infty(T[1-k]\mathcal{M})$ are $|\xi|$ and $|\xi| + 1 - k$, respectively, for all homogeneous $\xi \in \mathcal{C}^\infty(\mathcal{M})$.

We are interested in the case of $k = 0$. Suppose that $(\mathcal{M}, \mathcal{Q})$ is a $\mathcal{Q}$-manifold. Then $T[1]\mathcal{M}$ is a $\mathbb{Z}$-manifold, which inherits the two commuting differentials $\mathcal{L}_\mathcal{Q}$ and d defined as follows:

- $\mathrm{d}: \mathcal{C}^\infty(T[1]\mathcal{M})^\bullet \to \mathcal{C}^\infty(T[1]\mathcal{M})^{\bullet+1}$ is the **de Rham differential** defined on generators by $C^\infty(M) \ni f \mapsto \mathrm{d}f$, $\xi_i^j \mapsto \mathrm{d}\xi_i^j$, $\mathrm{d}f \mapsto 0$ and $\mathrm{d}\xi_i^j \mapsto 0$, and is extended to the whole algebra as a derivation of bidegree $(1, 0)$:

$$\mathrm{d}(\xi_1 \mathrm{d}\xi_2) = \mathrm{d}\xi_1 \mathrm{d}\xi_2 + (-1)^{|\xi_1|} \xi_1 \mathrm{d}\xi_2, \qquad \text{for homogeneous } \xi_1 \in \mathcal{C}^\infty(\mathcal{M}) \text{ and all } \xi_2 \in \mathcal{C}^\infty(\mathcal{M}).$$

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[1] Note that here there is a sign difference in the notation with [71] and [73]. $T[1]\mathcal{M}$ here is the same as $T[-1]\mathcal{M}$ in these papers.

[2] This is obtained by counting the "d's" and thus it is non-negative; e.g., the element $\xi \in \mathcal{C}^\infty(\mathcal{M})$ viewed as an element in $\mathcal{C}^\infty(T[1-k]\mathcal{M})$ has polynomial degree 0 and the element $\mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n$ has polynomial degree n.

Note that this does not require the existence of the $\mathcal{Q}$-manifold structure on $\mathcal{M}$ and is hence well-defined for all $\mathbb{Z}$-manifolds.

- $\pounds_{\mathcal{Q}} \colon \mathcal{C}^\infty(T[1]\mathcal{M})^\bullet \to \mathcal{C}^\infty(T[1]\mathcal{M})^{\bullet+1}$ is the **Lie derivative** with respect to the vector field $\mathcal{Q}$, i.e. the graded commutator $\pounds_{\mathcal{Q}} = [i_{\mathcal{Q}}, \mathrm{d}] = i_{\mathcal{Q}} \circ \mathrm{d} - \mathrm{d} \circ i_{\mathcal{Q}}$, and it is a derivation of bidegree $(0, 1)$. Here, $i_{\mathcal{Q}}$ is the bidegree $(-1, 1)$-derivation on $T[1]\mathcal{M}$, which sends $\xi \in \mathcal{C}^\infty(\mathcal{M})$ to 0, $\mathrm{d}\xi$ to $\mathcal{Q}(\xi)$ for $\xi \in \mathcal{C}^\infty(\mathcal{M})$, and is extended to the whole algebra as a derivation of bidegree $(-1, 1)$:

$$i_{\mathcal{Q}}(\xi_1 \mathrm{d}\xi_2) = i_{\mathcal{Q}}(\xi_1)\mathrm{d}\xi_2 + (-1)^{|\xi_1|(1-1)}\xi_1 i_{\mathcal{Q}}(\mathrm{d}(\xi_2)) = \xi_1 \mathcal{Q}(\xi_2).$$

Note that the above operators $\mathrm{d}, i_{\mathcal{Q}}$ and $\pounds_{\mathcal{Q}}$ are vector fields on the manifold $T[1]\mathcal{M}$ of degree $1, 0$ and 1, and bidegree $(1, 0), (-1, 1)$ and $(0, 1)$, respectively. In particular, $i_{\mathcal{Q}}$ and $\pounds_{\mathcal{Q}}$ are special cases of the two vector fields $i_{\mathcal{X}}$ and $\pounds_{\mathcal{X}}$ on $T[1]\mathcal{M}$ obtained for any $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$; the former is of bidegree $(-1, |\mathcal{X}|)$ and is defined on the generators by $i_{\mathcal{X}}(\xi) = 0$ and $i_{\mathcal{X}}(\mathrm{d}\xi) = \mathcal{X}(\xi)$, while the latter is of bidegree $(0, |\mathcal{X}|)$ and is defined as the Lie bracket $\pounds_{\mathcal{X}} = [i_{\mathcal{X}}, \mathrm{d}]$. Locally, they have the following coordinate representations: Given local coordinates $\{\xi^j\}$ on $\mathcal{M}$, the family $\{\xi^j, \dot{\xi}^j\}$ forms a local system of coordinates for the graded manifold $T[1]\mathcal{M}$ with ξ^j and $\dot{\xi}^j$ of degree $|\xi^j|$ and $|\xi^j| + 1$, respectively; here we do not use the letter x^i for the coordinates of the body and we write $\dot{\xi}^j$ for the linear coordinate $\mathrm{d}\xi^j$ for simplicity. Then for a homogeneous $\mathcal{X} = \zeta^j \frac{\partial}{\partial \xi^j} \in \mathfrak{X}(\mathcal{M})$, the vector fields above become

$$\mathrm{d} = \dot{\xi}^j \frac{\partial}{\partial \xi^j}, \qquad i_{\mathcal{X}} = \zeta^j \frac{\partial}{\partial \dot{\xi}^j}, \qquad \pounds_{\mathcal{X}} = \zeta^j \frac{\partial}{\partial \xi^j} + (-1)^{|\mathcal{X}|}\mathrm{d}\zeta^j \frac{\partial}{\partial \dot{\xi}^j}.$$

By checking their values on local generators, it is easy to see that $\pounds_{\mathcal{Q}}^2 = 0, \mathrm{d}^2 = 0$ and $[\pounds_{\mathcal{Q}}, \mathrm{d}] = \pounds_{\mathcal{Q}} \circ \mathrm{d} + \mathrm{d} \circ \pounds_{\mathcal{Q}} = 0$. Hence, the pair

$$W(\mathcal{M}) := \left(\Omega(\mathcal{M}) := \bigoplus_{i \in \mathbb{Z}} \Omega^{\bullet,\bullet}(\mathcal{M}), \pounds_{\mathcal{Q}} + \mathrm{d} \right),$$

where $\Omega^{\bullet,\bullet}(\mathcal{M}) := \mathcal{C}^\infty(T[1]\mathcal{M})^{\bullet,\bullet}$, forms a double complex. This double complex is called the **Weil algebra** of $(\mathcal{M}, \mathcal{Q})$.

Remark 4.2.1. The terminology "Weil algebra" becomes clear from the case of a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$, which is analysed in detail in the next chapter.

Remark 4.2.2. In total, one has three differential complexes, and consequently three cohomologies, associated to any $\mathcal{Q}$-manifold $(\mathcal{M}, \mathcal{Q})$:

1. The **de Rham complex** consisting of the columns of $\Omega^{\bullet,\bullet}(\mathcal{M})$ with differential d and the induced **de Rham cohomology** $H_{dR}^\bullet(\mathcal{M})$ of $\mathcal{M}$.

2. The **Lie complex** of differential forms consisting of the rows of $\Omega^{\bullet,\bullet}(\mathcal{M})$ with differential $\pounds_{\mathcal{Q}}$ and the induced **Lie cohomology** of (pseudo)differential forms $H_{\Omega,L}^\bullet(\mathcal{M})$ of $\mathcal{M}$.

3. The **Weil complex** $W(\mathcal{M})$ and **Weil cohomology** $H_W^\bullet(\mathcal{M})$ of $(\mathcal{M}, \mathcal{Q})$ consisting of the the differential graded algebra given by the Weil algebra defined above, i.e. the total complex of $\Omega^{\bullet,\bullet}(\mathcal{M})$ and its cohomology.

4.3   (Pseudo)multivector fields and the Poisson-Weil algebra of a $\mathcal{PQ}$-manifold

From the theory of ordinary Poisson manifolds, it is well-known that a Poisson bracket $\{\cdot, \cdot\}$ on a smooth manifold M is equivalent to a bivector field $\pi \in \mathfrak{X}^2(M) := \Gamma(\wedge^2 TM)$ that satisfies the differential equation $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket on the space $\mathfrak{X}^\bullet(M)$ of multivector fields on M. The precise correspondence is given by the derived bracket formula

$$\{f, g\} = [[f, \pi], g]$$

for all $f, g \in C^\infty(M)$. Furthermore, the last equality defines a cohomology theory for Poisson manifolds on $\mathfrak{X}^\bullet(M)$ with differential given by $\mathrm{d}_\pi := [\pi, \cdot]$. As we will see now, this is also true in the setting of graded manifolds carrying the graded version of a Poisson bracket as defined above.

Suppose that $\mathcal{M}$ is a $\mathbb{Z}$-graded manifold over M and $k \in \mathbb{Z}$. The algebra of functions of the graded manifold $T^*[1-k]\mathcal{M}$ is, by definition, $\underline{S}(\mathfrak{X}(\mathcal{M})[k-1])$ considered with the total grading, and locally it is given by

$$\mathcal{C}_U^\infty(T^*[1-k]\mathcal{M}) = \bigoplus_{i \in \mathbb{Z}} \underbrace{\mathcal{C}_U^\infty(\mathcal{M})^i}_{(0,i)} \left\langle \underbrace{\left(\frac{\partial}{\partial x^s}\right)_{s=1}^m}_{(1,-k)} ; \underbrace{\left(\frac{\partial}{\partial \xi_j^s}\right)_{s=1}^{r_j}}_{(1,-k-|\xi_j^s|)}, \; j \in \mathbb{Z} \right\rangle.$$

where $U \subset M$ is a local chart with coordinates $\{x^s\}$, and $\{\xi_j^s\}$ are local coordinates of $\mathcal{M}$ as in (3.1). The resulting $\mathbb{Z}$-manifold $T^*[1-k]\mathcal{M}$ is called **(shifted) cotangent bundle** and its structure sheaf will be denoted by $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$. The bigrading (p, q) is such that the horizontal degree p counts the "multivector field" (or "polynomial") degree[1] and the vertical degree q counts the "derivation" degree. These are denoted $|\cdot|_p$ and $|\cdot|_q$, respectively. The total degree is denoted $|\cdot| := |\cdot|_p + |\cdot|_q$. If $k = 0$, the resulting function algebra of the graded manifold $T^*[1]\mathcal{M}$ is given by the space of **(pseudo)multivector fields** $\mathfrak{X}^{\bullet,\bullet}(\mathcal{M}) := \mathfrak{A}_0^{\bullet,\bullet}(\mathcal{M})$, and if in particular $\mathcal{M}$ is an ordinary (non-graded) smooth manifold M then $\mathfrak{A}_0(M) = \Gamma(\wedge TM)$. See [29] for more on graded cotangent bundles over $\mathbb{N}$-manifolds.

The **bidegree** $(-1, k)$ **Schouten bracket** on $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$ is defined inductively by the following conditions:

1. For all $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and all vector fields $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}^\bullet(\mathcal{M})$:

$$[\mathcal{X}, \xi]_k := \mathcal{X}(\xi) \qquad \text{and} \qquad [\mathcal{X}, \mathcal{Y}]_k := [\mathcal{X}, \mathcal{Y}].$$

---

[1] This is obtained by counting the partial derivatives "$\frac{\partial}{\partial \xi_j^s}$" and is always non-negative.

2. For homogeneous elements of higher degree $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathfrak{A}_k^{\bullet, \bullet}(\mathcal{M})$, it is extended by the rules:

   (a) $[\mathcal{X}, \mathcal{Y}]_k = -(-1)^{(|\mathcal{X}|+k-1)(|\mathcal{Y}|+k-1)}[\mathcal{Y}, \mathcal{X}]_k$,

   (b) $[\mathcal{X}, \mathcal{Y} \wedge \mathcal{Z}]_k = [\mathcal{X}, \mathcal{Y}]_k \wedge \mathcal{Z} + (-1)^{(|\mathcal{X}|+k-1)|\mathcal{Y}|}\mathcal{Y} \wedge [\mathcal{X}, \mathcal{Z}]_k$.

Using an induction argument, it can be shown that the Schouten bracket on $\mathfrak{A}_k^{\bullet, \bullet}(\mathcal{M})$ satisfies the following graded version of the Jacobi identity:

$$[\mathcal{X}, [\mathcal{Y}, \mathcal{Z}]_k]_k = [[\mathcal{X}, \mathcal{Y}]_k, \mathcal{Z}]_k + (-1)^{(|\mathcal{X}|+k-1)(|\mathcal{Y}|+k-1)}[\mathcal{Y}, [\mathcal{X}, \mathcal{Z}]_k]_k,$$

for all homogeneous elements $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathfrak{A}_k^{\bullet, \bullet}(\mathcal{M})$.

As in the ordinary case, the antisymmetry and the Leibniz identity of a degree k Poisson bracket on $\mathcal{M}$ imply the existence of a bivector field π of bidegree $(2, -k)$, i.e. $\pi \in \mathfrak{A}_k^{2, -k}(\mathcal{M})$. The correspondence is given again by Voronov's derived bracket formula

$$\{\xi_1, \xi_2\}_k = [[\xi_1, \pi]_k, \xi_2]_k$$

for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$. Note that this implies the equality

$$[\xi, \pi]_k = \mathcal{X}_\xi \qquad \text{for all } \xi \in \mathcal{C}^\infty(\mathcal{M}).$$

A similar formula can be used to determine the component functions of any multivector field over a local coordinate chart of $\mathcal{M}$. Specifically, given a (local) element $\mathcal{X} \in \mathfrak{A}_k^{\ell, \bullet}(\mathcal{M})$ whose coordinate representation is a finite sum of the form

$$\mathcal{X} = \sum \xi_{i_1 \ldots i_\ell} \frac{\partial}{\partial \xi_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial \xi_{i_\ell}}$$

with $\xi_{i_1}, \ldots, \xi_{i_\ell}$ coordinate functions and $\xi_{i_1 \ldots i_\ell}$ a local smooth function, one has

$$[\xi_{i_\ell}, [\xi_{i_{\ell-1}}, [\ldots [\xi_{i_1}, \mathcal{X}]_k]_k]_k]_k = \pm \xi_{i_1 \ldots i_\ell}.$$

In particular, we obtain the following Lemma.

Lemma 4.3.1. An ℓ-vector field $\mathcal{X} \in \mathfrak{A}_k^{\ell, \bullet}(\mathcal{M})$ is zero if and only if

$$[\xi_\ell, [\xi_{\ell-1}, [\ldots [\xi_1, \mathcal{X}]_k]_k]_k]_k = 0$$

for all $\xi_1, \ldots, \xi_\ell \in \mathcal{C}^\infty(\mathcal{M})$.

Proof. If $\mathcal{X} = 0$, then clearly $[\xi_\ell, [\xi_{\ell-1}, [\ldots [\xi_1, \mathcal{X}]_k]_k]_k]_k = 0$ for all $\xi_1, \ldots, \xi_\ell \in \mathcal{C}^\infty(\mathcal{M})$. Conversely, suppose $[\xi_\ell, [\xi_{\ell-1}, [\ldots [\xi_1, \mathcal{X}]_k]_k]_k]_k = 0$ for all $\xi_1, \ldots, \xi_\ell \in \mathcal{C}^\infty(\mathcal{M})$ and write the vector field $\mathcal{X}$ in coordinates as

$$\mathcal{X} = \sum \xi_{i_1 \ldots i_\ell} \frac{\partial}{\partial \xi_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial \xi_{i_\ell}}.$$

For the coordinate functions we have

$$0 = [\xi_{i_\ell}, [\xi_{i_{\ell-1}}, [\dots [\xi_{i_1}, \mathcal{X}]_k]_k]_k]_k = \pm \xi_{i_1 \dots i_\ell}$$

and hence $\mathcal{X} = 0$. □

Proposition 4.3.2. Let $\{\cdot, \cdot\}_k$ be Poisson bracket of even degree k on a graded manifold $\mathcal{M}$ and denote the corresponding bivector field by $\pi \in \mathfrak{A}_k^{2,-k}(\mathcal{M})$. Then the Jacobi identity for $\{\cdot, \cdot\}_k$ is equivalent to the differential equation

$$[\pi, \pi]_k = 0.$$

Proof. For $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$ we compute

$$[\xi_1, [\xi_2, [\pi, \pi]_k]_k]_k = 2(-1)^{|\xi_2|+k-1}[\xi_1, [\pi, [\xi_2, \pi]_k]_k]_k$$
$$= 2(-1)^{|\xi_2|+k-1}\left([\mathcal{X}_{\xi_1}, \mathcal{X}_{\xi_2}]_k - \mathcal{X}_{\{\xi_1,\xi_2\}_k} \right).$$

Applying Lemma 4.3.1 above to the last equation, one sees that $[\pi, \pi]_k = 0$ if and only if the Hamiltonian map commutes with the brackets $\{\cdot, \cdot\}_k$ and $[\cdot, \cdot]$, or equivalently, if and only if $\{\cdot, \cdot\}_k$ satisfies the Jacobi identity. □

In addition to the double complex and the Weil algebra explained above, we will see now that the shifted space of multivector fields $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$ of a $\mathbb{Z}$-graded manifold $\mathcal{M}$ with $\mathcal{Q}$ compatible with $\{\cdot, \cdot\}_k$ carries two commuting differentials, yielding another double complex and a differential graded algebra associated to $\mathcal{M}$.

Suppose first that $(\mathcal{M}, \mathcal{Q})$ is a $\mathcal{Q}$-manifold. Then one may use the Schouten bracket with $\mathcal{Q}$ to obtain a canonical operator of bidegree $(0, 1)$ on $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$ which extends the canonical differential of the adjoint action. Explicitly, one has the **Lie derivative**

$$\mathcal{L}_\mathcal{Q} \colon \mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M}) \to \mathfrak{A}_k^{\bullet,\bullet+1}(\mathcal{M}), \qquad \mathcal{X} \mapsto [\mathcal{Q}, \mathcal{X}]_k.$$

The Jacobi identity of the Schouten bracket on $\mathfrak{A}^{\bullet,\bullet}(\mathcal{M})$ together with $[\mathcal{Q}, \mathcal{Q}]_k = 0$ yield that $\mathcal{L}_\mathcal{Q}^2 = 0$ on the entire $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$ and hence $\mathcal{L}_\mathcal{Q}$ is a differential operator:

$$[\mathcal{Q}, [\mathcal{Q}, \mathcal{X}]_k]_k = [[\mathcal{Q}, \mathcal{Q}]_k, \mathcal{X}]_k + (-1)^{(1+1-k+k-1)^2}[\mathcal{Q}, [\mathcal{Q}, \mathcal{X}]_k]_k = -[\mathcal{Q}, [\mathcal{Q}, \mathcal{X}]_k]_k,$$

for every $\mathcal{X} \in \mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$.

Suppose now that the $\mathbb{Z}$-manifold $\mathcal{M}$ carries a Poisson bracket $\{\cdot, \cdot\}_k$ of degree k with corresponding Poisson bivector field $\pi \in \mathfrak{A}_k^{2,-k}(\mathcal{M})$. Then $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$ is equipped with another canonical differential operator of bidegree $(1, 0)$ given by the Schouten bracket with respect to π:

$$d_\pi \colon \mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M}) \to \mathfrak{A}_k^{\bullet+1,\bullet}(\mathcal{M}), \qquad \mathcal{X} \mapsto [\pi, \mathcal{X}]_k.$$

Due to Jacobi identity and Proposition 4.3.2, one computes that d_π indeed squares to zero:

$$[\pi, [\pi, \mathcal{X}]_k]_k = [[\pi, \pi]_k, \mathcal{X}]_k + (-1)^{(2-k+k-1)^2}[\pi, [\pi, \mathcal{X}]_k]_k = -[\pi, [\pi, \mathcal{X}]_k]_k,$$

for all $\mathcal{X} \in \mathfrak{A}_k^{\bullet, \bullet}(\mathcal{M})$.

Proposition 4.3.3. Let $\mathcal{M}$ be a $\mathbb{Z}$-manifold with a Poisson bivector field $\pi \in \mathfrak{A}_k^{2,-k}(\mathcal{M})$. Then the following are equivalent for any 1-vector field $\mathcal{Y} \in \mathfrak{A}_k^{1,j}(\mathcal{M}) = \mathfrak{X}^{j+k}(\mathcal{M})$:

1. $\mathcal{Y}$ is a Poisson vector field with respect to the bracket induced by π:

$$[\mathcal{Y}, \mathcal{X}_\xi] = \mathcal{X}_{\mathcal{Y}(\xi)}, \qquad \text{for all } \xi \in \mathcal{C}^\infty(\mathcal{M}).$$

2. The vector field $\mathcal{Y}$ commutes with π, i.e. $[\pi, \mathcal{Y}]_k = 0$.

Proof. For any homogeneous function $\xi \in \mathcal{C}^\infty(\mathcal{M})$ we compute

$$\begin{aligned}
[\xi, [\mathcal{Y}, \pi]_k]_k &= [[\xi, \mathcal{Y}]_k, \pi]_k + (-1)^{(|\xi|+k-1)(j+k)}[\mathcal{Y}, [\xi, \pi]_k]_k \\
&= -(-1)^{(|\xi|+k-1)(j+k)}[\mathcal{Y}(\xi), \pi]_k + (-1)^{(|\xi|+k-1)(j+k)}[\mathcal{Y}, \mathcal{X}_\xi] \\
&= -(-1)^{(|\xi|+k-1)(j+k)}\mathcal{X}_{\mathcal{Y}(\xi)} + (-1)^{(|\xi|+k-1)(j+k)}[\mathcal{Y}, \mathcal{X}_\xi] \\
&= (-1)^{(|\xi|+k-1)(j+k)}\left([\mathcal{Y}, \mathcal{X}_\xi] - \mathcal{X}_{\mathcal{Y}(\xi)}\right).
\end{aligned}$$

Since $[\pi, \mathcal{Y}]_k = 0$ if and only if $[\mathcal{Y}, \pi]_k = 0$ the result follows. $\qquad\square$

Suppose now that $(\mathcal{M}, \mathcal{Q})$ carries a Poisson bivector field $\pi \in \mathfrak{A}_k^{2,-k}(\mathcal{M})$. The analysis above implies that $\mathfrak{A}^{\bullet, \bullet}(\mathcal{M})$ is endowed with two canonical differential operators d_π and $\mathcal{L}_\mathcal{Q}$ of bidegree $(1,0)$ and $(0,1)$, respectively. Assuming that $\mathcal{Q}$ is a Poisson vector field with respect to the given Poisson bracket, one computes

$$d_\pi \circ \mathcal{L}_\mathcal{Q}(\mathcal{X}) = [\pi, [\mathcal{Q}, \mathcal{X}]_k]_k = [[\pi, \mathcal{Q}]_k, \mathcal{X}]_k + (-1)^{1^2}[\mathcal{Q}, [\pi, \mathcal{X}]_k]_k = -\mathcal{L}_\mathcal{Q} \circ d_\pi(\mathcal{X}),$$

for all $\mathcal{X} \in \mathfrak{A}^{\bullet, \bullet}(\mathcal{M})$. It follows that $\mathfrak{A}_k^{\bullet, \bullet}(\mathcal{M})$ together with the "twisted" differential $(-1)^{k-1}d_\pi$ and $\mathcal{L}_\mathcal{Q}$ forms a double complex with anticommuting differentials which we call the **Poisson-Weil double complex of** $\mathcal{M}$. Hence, given a graded manifold $\mathcal{M}$ with a homological vector field $\mathcal{Q}$ and a compatible Poisson bracket $\{\cdot, \cdot\}_k$, one obtains the following cohomologies:

1. The **Poisson** or **Lichnerowicz complex** consisting of the columns of $\mathfrak{A}_k^{\bullet, \bullet}(\mathcal{M})$ with differential d_π and the induced Poisson cohomology $H_P^\bullet(\mathcal{M})$ of $\mathcal{M}$.

2. The **Lie complex of (pseudo)multivector fields** consisting of the rows of $\mathfrak{A}_k^{\bullet, \bullet}(\mathcal{M})$ with differential $\mathcal{L}_\mathcal{Q}$ and the induced cohomology $H_{\mathfrak{A}_k, L}^\bullet(\mathcal{M})$ of $\mathcal{M}$.

3. The **Poisson-Weil algebra** consisting of the differential graded algebra given by the total

complex

$$\left(\mathfrak{A}_k(\mathcal{M}) := \bigoplus_{i \in \mathbb{Z}} \bigoplus_{p+q=i} \mathfrak{A}_k^{p,q}(\mathcal{M}), \delta := (-1)^{k-1} \mathrm{d}_\pi + \mathcal{L}_{\mathcal{Q}} \right)$$

and the induced Poisson-Weil cohomology $H_{PW}^\bullet(\mathcal{M})$ of $\mathcal{M}$. Moreover, δ is a degree $k-1$ derivation of the Schouten bracket:

$$\delta[\mathcal{X}, \mathcal{Y}]_k = [\delta(\mathcal{X}), \mathcal{Y}]_k + (-1)^{i+j+k-1}[\mathcal{X}, \delta(\mathcal{Y})]_k$$

for all $\mathcal{X} \in \mathfrak{A}_k^{i,j}(\mathcal{M}), \mathcal{Y} \in \mathfrak{A}_k(\mathcal{M})$, and therefore we obtain the DGLA

$$\left(\widehat{\mathfrak{A}}_k(\mathcal{M}) := \mathfrak{A}_k(\mathcal{M})[1-k], \delta, [\cdot, \cdot]_k \right).$$

Remark 4.3.4. Some special cocycles of the Poisson-Weil cohomology are the following:

1. The cocycles of bidegree $(0, i)$ consist of Casimir functions of degree i on $\mathcal{M}$ which are closed but not exact in the Lie $\mathcal{Q}$-manifold cohomology.

2. The cocycles of bidegree $(1, i)$ consist of degree $i + k$ vector fields on $\mathcal{M}$ which are Poisson but not symplectic and are invariant under $\mathcal{Q}$.

Remark 4.3.5. As we will see in Section 5.4, the three cohomologies defined above are naturally related to the three cohomologies from Remark 4.2.2 via a map of bigraded algebras $\pi^\sharp : \Omega^{\bullet,\bullet}(\mathcal{M}) \to \mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$ induced from the $\mathcal{PQ}$-manifold structure on $\mathcal{M}$. This map is also the reason for the choice of the sign $(-1)^{k-1}$ in the horizontal differential of $\mathfrak{A}_k(\mathcal{M})$.

4.4 Homotopy Poisson structures and their deformations

As DGLA's have connection to deformation theory, it is worth mentioning the deformations that are related to the DGLA obtained by the Poisson-Weil algebra defined in the previous section. For this we need the notion of homotopy Poisson structures, which extend simultaneously the notions of $\mathcal{Q}$-manifolds, $\mathcal{P}$-manifolds and $\mathcal{PQ}$-manifolds. These can be found in [74], in [24, 21, 86] under the name $\mathcal{P}_\infty$-*manifolds*, and in [17, 99, 101] under the name *higher Poisson manifolds*.

A **homotopy Poisson structure of degree** k on a graded manifold $\mathcal{M}$ is, by definition, an element $\Theta \in \mathfrak{A}_k^{2-k}(\mathcal{M})$ such that $[\Theta, \Theta]_k = 0$. The name "homotopy" Poisson is explained by the following remarks: Considering bidegrees, one can write

$$\mathfrak{A}_k^{2-k}(\mathcal{M}) = \mathcal{C}^\infty(\mathcal{M})^{2-k} \oplus \mathfrak{A}_k^{1,1-k}(\mathcal{M}) \oplus \mathfrak{A}_k^{2,-k}(\mathcal{M}) \oplus \mathfrak{A}_k^{3,-1-k}(\mathcal{M}) \oplus \dots$$

and in particular

1. if $\Theta = \mathcal{Q} \in \mathfrak{A}_k^{1,1-k}(\mathcal{M}) = \mathfrak{X}^1(\mathcal{M})$, then $\mathcal{Q}$ is homotopy Poisson if and only if $(\mathcal{M}, \mathcal{Q})$ is a $\mathcal{Q}$-manifold;

2. if $\Theta = \pi \in \mathfrak{A}_k^{2,-k}(\mathcal{M})$, then π is homotopy Poisson if and only if $(\mathcal{M}, \pi)$ is a $\mathcal{P}_k$-manifold;

3. if $\Theta = \mathcal{Q} + \pi \in \mathfrak{A}_k^{1,1-k}(\mathcal{M}) \oplus \mathfrak{A}_k^{2,-k}(\mathcal{M})$, then $\mathcal{Q} + \pi$ is homotopy Poisson if and only if

$$[\mathcal{Q}, \mathcal{Q}]_k = 0, \qquad [\mathcal{Q}, \pi,]_k = 0 \qquad \text{and} \qquad [\pi, \pi]_k = 0,$$

i.e. π is a Poisson bivector field and $\mathcal{Q}$ is a Poisson homological vector field on $\mathcal{M}$. That is, $\mathcal{M}$ is a $\mathcal{P}_k \mathcal{Q}$-manifold;

4. if $\Theta = \sum_{i \geq 0} \Theta_i \in \bigoplus_{i \geq 0} \mathfrak{A}_k^{i,2-k-i}(\mathcal{M})$, then expanding $[\Theta, \Theta]_k = 0$ and comparing degrees, we obtain the following system of equations:

$$\sum_{p+q=j} [\Theta_p, \Theta_q] = 0, \qquad j = 1, 2, \ldots$$

In particular, for $j = 2, 3, 4$ the equation reads

   (a) $[\Theta_1, \Theta_1]_k + 2[\Theta_0, \Theta_2]_k = 0$,

   (b) $[\Theta_1, \Theta_2]_k + [\Theta_0, \Theta_3]_k = 0$,

   (c) $[\Theta_2, \Theta_2]_k + 2[\Theta_1, \Theta_3]_k + 2[\Theta_0, \Theta_4] = 0$,

i.e. the failure of $(\mathcal{M}, \Theta_1, \Theta_2)$ to be a $\mathcal{P}\mathcal{Q}$-manifold is measured by the above higher homotopy terms.

Definition 4.4.1. Let $\Theta \in \mathfrak{A}_k^{2-k}(\mathcal{M})$ be a homotopy Poisson structure on $\mathcal{M}$.

1. An **infinitesimal deformation of** Θ, or **formal deformation of** Θ, is an element $\Theta' \in \mathfrak{A}_k^{2-k}(\mathcal{M})$ for which the infinitesimal change of Θ up to order 2 in the direction of Θ' is a homotopy Poisson structure of degree k, i.e. such that

$$[\Theta + \varepsilon\Theta', \Theta + \varepsilon\Theta']_k = \mathcal{O}(\varepsilon^2) \equiv 0 \mod \varepsilon^2,$$

where ε is a formal infinitesimal parameter.

2. A **deformation of** Θ is an element $\Theta' \in \mathfrak{A}_k^{2-k}(\mathcal{M})$ such that $\Theta + \Theta'$ is a homotopy Poisson structure, i.e.

$$[\Theta + \Theta', \Theta + \Theta']_k = 0.$$

Given $\Theta := \pi + \mathcal{Q}$ with $\mathcal{Q}$ and π a $\mathcal{P}\mathcal{Q}$-structure on $\mathcal{M}$, one computes that for every element $\Theta' \in \mathfrak{A}_k^{2-k}(\mathcal{M})$

$$[\mathcal{Q} + \pi + \varepsilon\Theta', \mathcal{Q} + \pi + \varepsilon\Theta'] = 2\varepsilon\left([\mathcal{Q}, \Theta']_k + [\pi, \Theta']_k\right) + \varepsilon^2[\Theta', \Theta'] \equiv 2\varepsilon\delta(\Theta') \mod \varepsilon^2.$$

In other words, the degree $2 - k$ cocycles of $\mathfrak{A}_k^\bullet(\mathcal{M})$ (or the degree 1 cocycles of $\widehat{\mathfrak{A}}_k^\bullet(\mathcal{M})$) are in one-to-one correspondence with infinitesimal deformations of $\Theta = \pi + \mathcal{Q}$ viewed as a homotopy Poisson

structure on $\mathcal{M}$. In particular, the cocycles of the form $\Lambda + \mathcal{X} \in \mathfrak{A}_k^{2,-k}(\mathcal{M}) \oplus \mathfrak{A}_k^{1,1-k}(\mathcal{M})$ are in one-to-one correspondence with infinitesimal deformations of $\pi + \mathcal{Q}$ as a $\mathcal{PQ}$-manifold structure on $\mathcal{M}$, i.e. $(\pi + \varepsilon\Lambda, \mathcal{Q} + \varepsilon\mathcal{X})$ forms a compatible pair of a degree k Poisson bivector and a homological vector field if and only if $\Lambda + \mathcal{X}$ is a cocycle in $H_{PW}^{2-k}(\mathcal{M})$. Counting degrees, one sees that for $\Theta' = \Lambda + \mathcal{X} \in \mathfrak{A}_k^{2,-k}(\mathcal{M}) \oplus \mathfrak{A}_k^{1,1-k}(\mathcal{M})$:

$$\delta(\Theta') = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \mathcal{L}_{\mathcal{Q}}(\mathcal{X}) = 0 \\ \mathrm{d}_\pi(\Lambda) = 0 \\ \mathrm{d}_\pi(\mathcal{X}) + \mathcal{L}_{\mathcal{Q}}(\Lambda) = 0 \end{cases} \qquad (4.1)$$

Similarly, the **Maurer-Cartan elements** of $\widehat{\mathfrak{A}}_k^\bullet(\mathcal{M})$, i.e. $\Theta' \in \widehat{\mathfrak{A}}_k^1(\mathcal{M}) = \mathfrak{A}_k^{2-k}(\mathcal{M})$ such that

$$\delta(\Theta') + \frac{1}{2}[\Theta', \Theta']_k = 0,$$

are in one-to-one correspondence with deformations of $\Theta = \pi + \mathcal{Q}$ as a homotopy Poisson structure, and the Maurer-Cartan elements of the subspace $\mathfrak{A}_k^{2,-k}(\mathcal{M}) \oplus \mathfrak{A}_k^{1,1-k}(\mathcal{M})$ are in correspondence with deformations of $\mathcal{Q} + \pi$ viewed as a $\mathcal{PQ}$-manifold structure. By considering terms of the same degrees, one obtains the following equivalence

$$\delta(\Theta') + \frac{1}{2}[\Theta', \Theta'] = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \mathcal{L}_{\mathcal{Q}}(\mathcal{X}) + \frac{1}{2}[\mathcal{X}, \mathcal{X}]_k = 0 \\ \mathrm{d}_\pi(\Lambda) + \frac{1}{2}[\Lambda, \Lambda]_k = 0 \\ \mathrm{d}_\pi(\mathcal{X}) + \mathcal{L}_{\mathcal{Q}}(\Lambda) + [\mathcal{X}, \Lambda]_k = 0 \end{cases} \qquad (4.2)$$

where $\Theta' = \Lambda + \mathcal{X} \in \mathfrak{A}_k^{2,-k}(\mathcal{M}) \oplus \mathfrak{A}_k^{1,1-k}(\mathcal{M})$.

4.5 Homotopy Lie bialgebroids

In this section, we analyse some of the constructions of this chapter in the case of [1]-manifolds of the form $A[1]$, where $A \to M$ is a vector bundle.

Recall that a **Lie bialgebroid** is given by a pair of two Lie algebroids in "duality" over M (A, A^*), such that for all $a, b \in \Gamma(A)$

$$\mathrm{d}_{A^*}[a, b] = [\mathrm{d}_{A^*}a, b] + [a, \mathrm{d}_{A^*}b],$$

where $[\cdot, \cdot] \colon \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is the Lie bracket on the space of sections of A and $\mathrm{d}_{A^*} \colon \Omega^\bullet(A^*) = \Gamma(\wedge^\bullet A) \to \Omega^{\bullet+1}(A^*)$ is the Lie algebroid differential of A^*. The definition of Lie bialgebroids is symmetric, in the sense that (A, A^*) is a Lie bialgebroid if and only if (A^*, A) is a Lie bialgebroid [68, 58, 82]. Moreover, it is well-know that Lie bialgebroids structures on (A, A^*) are in one-to-one correspondence with degree -1 Poisson Lie 1-algebroid structures on the [1]-manifold $A[1]$ [69], i.e. with a pair of a homological vector field compatible and a degree -1 Poisson bracket on $A[1]$.

We now briefly recall this correspondence.

Suppose that d_A is a homological vector field on the $[1]$-manifold $A[1]$ giving rise to a Lie algebroid structure on $A \to M$. A degree -1 Poisson bracket $\{\cdot,\cdot\}$ on the function space $\mathcal{C}^\infty(A[1]) = \Omega(A)$ is characterised by two maps $\rho_* : \Gamma(A^*) \to \mathfrak{X}(M)$ and $[\cdot,\cdot]_* : \Gamma(A^*) \times \Gamma(A^*) \to \Gamma(A^*)$ defined via

$$\{\alpha, f\} = \rho_*(\alpha)f \qquad \text{and} \qquad \{\alpha, \beta\} = [\alpha, \beta]_*$$

for all $f \in C^\infty(M)$ and all $\alpha, \beta \in \Gamma(A^*)$. The antisymmetry, Leibniz and Jacobi identities of $\{\cdot,\cdot\}$ are equivalent to $(A^*, \rho_*, [\cdot,\cdot]_*)$ being a Lie algebroid.

For the Lie bialgebroid condition one proceeds as follows: Let $\pi \in \mathfrak{A}^{2;1}_{-1}(A[1])$ be the Poisson bivector field corresponding to $\{\cdot,\cdot\}$. Then $(A[1], d_A, \{\cdot,\cdot\})$ is a Poisson Lie 1-algebroid if and only if d_A is a Poisson vector field, or equivalently, if and only if $[\pi, d_A]_{-1} = 0$, where $[\cdot,\cdot]_{-1}$ is the bidegree $(-1,-1)$ Schouten bracket on $\mathfrak{A}_{-1}(A[1])$. Using the Jacobi identity of $[\cdot,\cdot]_{-1}$, we compute

$$[[\alpha, [\pi, d_A]_{-1}]_{-1}, \beta]_{-1} = [d_A\alpha, \beta]_* - d_A[\alpha, \beta]_* + [\alpha, d_A\beta]_*.$$

Hence, d_A is a Poisson vector field if and only if (A, A^*) is a Lie bialgebroid.

From the above, it follows that $(\widehat{\mathfrak{A}}_{-1}(A[1]), \delta)$ is the DGLA that governs the (infinitesimal) deformations of a Lie bialgebroid (A, A^*), which in particular correspond to (cocycles) Maurer-Cartan elements in $\mathfrak{A}^{2;1}_{-1}(A[1]) \oplus \mathfrak{A}^{1;2}_{-1}(A[1])$. We will express all the operators locally in classical differential geometric language in terms of the splitting, but before doing that, we will use the homotopy Poisson structures on graded manifolds that were explained before to relax the definition of a Lie bialgebroid structure on (A, A^*). A detailed study of the following structures can be found e.g. in [82, 59] under the names of "proto-bialgebroid", "quasi-Lie bialgebroid" and "Lie quasi-bialgebroid".

Suppose that $A \to M$ is a vector bundle with a homotopy Poisson structure Θ of degree -1 on the graded manifold $A[1]$. Then we distinguish the following three cases:

- **Lie bialgebroid:** This case is for $\Theta = \pi + d_A \in \mathfrak{A}^{2;1}_{-1}(A[1]) \oplus \mathfrak{A}^{1;2}_{-1}(A[1])$ and was discussed above.

- **Quasi-Lie bialgebroid:** If $\Theta = \pi + d_A + \omega \in \mathfrak{A}^{2;1}_{-1}(A[1]) \oplus \mathfrak{A}^{1;2}_{-1}(A[1]) \oplus \mathfrak{A}^{0;3}_{-1}(A[1])$, it follows that π is a degree -1 Poisson structure on $A[1]$, and the maps $[[\alpha, \pi]_{-1}, f]_{-1} = \rho_*(\alpha)f$ and $[[\alpha, \pi]_{-1}, \beta] = [\alpha, \beta]_*$ turn the triple $(A^*, \rho_*, [\cdot,\cdot]_*)$ into a Lie algebroid over M. The degree 1 operator $d_A : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A)$ is a quasi-algebroid structure[1] on A whose failure to become a Lie algebroid differential is controlled by the Hamiltonian vector field $\mathcal{X}_\omega = [\omega, \pi]_{-1}$ of the d_A-closed function ω on $A[1]$ of degree 3: $[d_A, d_A]_{-1} + 2\mathcal{X}_\omega = 0$. Moreover, the operator d_A is a derivation of the Lie bracket on sections of A^*. In other words, the failure of (A, A^*) to be a Lie bialgebroid comes from the failure of d_A to be a Lie algebroid differential on A. It is a Lie bialgebroid if and only if ω is a Casimir function for the Poisson structure on $A[1]$,

---

[1] A quasi-algebroid is, by definition, a vector bundle $Q \to M$ together with an anchor $\rho : Q \to TM$ and a skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(Q)$ that satisfies $[a, fb] = f[a, b] + (\rho(a)f)b$, for all $f \in C^\infty(M), a, b \in \Gamma(Q)$.

or equivalently, if and only if $\mathrm{ad}_{[\cdot,\cdot]_*}(\omega) := [\omega, \cdot]_* = 0$.

- **Lie quasi-bialgebroid:** If $\Theta = \Lambda + \pi + \mathrm{d}_A \in \mathfrak{A}^{3;0}_{-1}(A[1]) \oplus \mathfrak{A}^{2;1}_{-1}(A[1]) \oplus \mathfrak{A}^{1;2}_{-1}(A[1])$, it follows that d_A is a Lie algebroid differential on A which is also a derivation of the quasi-algebroid structure on A^* induced by π. Moreover, Λ commutes with π and itself, and the failure of π to be a Lie algebroid structure on A^* is measured by the homotopy term $[\mathrm{d}_A, \Lambda]_{-1}$: $[\pi, \pi]_{-1} + 2[\mathrm{d}_A, \Lambda] = 0$.

Recall that given a TM-connection ∇ on the vector bundle A, one can describe the module of vector fields of the graded manifold $A[1]$ via the $\Omega(A)$-module isomorphism

$$\mathrm{ad}_\nabla(A) := \Omega(A) \otimes \Gamma(TM[0] \oplus A[1]) \to \mathfrak{X}(A[1])$$

which maps the generators $X \in \mathfrak{X}(M)$ and $a \in \Gamma(A)$ to ∇^*_X and $\widehat{a}$, respectively. Locally, given the two dual frames $\{a_j\}$ and $\{\alpha^j\}$ of A and A^*, we have the equality $\widehat{a}_j = \frac{\partial}{\partial \alpha^j}$. That is, we have the isomorphism

$$\mathfrak{X}(A[1]) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{X}^i(A[1]) \cong \bigoplus_{i \in \mathbb{Z}} \left(\Omega^i(A, TM) \oplus \Omega^{i+1}(A, A) \right) = \mathrm{ad}_\nabla(A)$$

and thus

$$\mathfrak{X}(A[1])[-1] = \bigoplus_{i \in \mathbb{Z}} \mathfrak{X}^{i-1}(A[1]) \cong \bigoplus_{i \in \mathbb{Z}} \left(\Omega^{i-1}(A, TM) \oplus \Omega^i(A, A) \right) = \mathrm{ad}_\nabla(A)[-1].$$

The DGLA of deformations of the Lie bialgebroid is given by the (graded) symmetric algebra $(\underline{S}(\mathrm{ad}_\nabla(A)[-1]), \delta)$ considered with the total grading.

Next, we express the differential $\delta = \mathrm{d}_\pi + \mathcal{L}_{\mathrm{d}_A}$ in this setting. As we will see in Chapter 6, the operator $\mathcal{L}_{\mathrm{d}_A}$ is the differential of the adjoint module and thus becomes the differential of the adjoint representation up to homotopy from [4]

$$\mathcal{D}_{\mathrm{ad}_\nabla(A)} = \rho \oplus \nabla^{\mathrm{bas}} \oplus R^{\mathrm{bas}}_\nabla.$$

The horizontal differential d_π acts on the generators as follows: Let $\xi_1, \xi_2 \in \Omega(A)$ be homogeneous functions of $A[1]$. Then we compute

$$[\pi, \xi_1]_{-1}(\xi_2) = [[\pi, \xi_1]_{-1}, \xi_2]_{-1} = -(-1)^{|\xi_1|}[\mathcal{X}_{\xi_1}, \xi_2]_{-1} = -(-1)^{|\xi_1|}\{\xi_1, \xi_2\}.$$

Applying this to functions of $A[1]$ of the form $f, g \in C^\infty(M)$ and $\alpha, \beta \in \Omega^1(A)$, we obtain

$$\mathrm{d}_\pi(f)(g) = 0, \qquad \mathrm{d}_\pi(f)(\beta) = \rho_*(\beta)f, \qquad \mathrm{d}_\pi(\alpha)(g) = \rho_*(\alpha)g, \qquad \mathrm{d}_\pi(\alpha)(\beta) = [\alpha, \beta]_*.$$

Locally, suppose that we have the equations

$$\rho_*(\alpha^j) = (\rho_*)_j^m \frac{\partial}{\partial x^m} \qquad \text{and} \qquad [\alpha^j, \alpha^s]_* = c_t^{js}\alpha^t$$

where $\{x^i\}$ are coordinates in a chart of M over which we have the dual frames $\{a_j\}$ and $\{\alpha^j\}$ of A and A^*, respectively. From the formulas above, we obtain the vector fields

$$\mathrm{d}_\pi(f) = (\rho_*)_j^m \frac{\partial f}{\partial x^m} \frac{\partial}{\partial \alpha^j} \qquad \text{and} \qquad \mathrm{d}_\pi(\alpha) = \rho_*(\alpha)x^i \frac{\partial}{\partial x^i} + [\alpha, \alpha^s]_* \frac{\partial}{\partial \alpha^s}.$$

On the coordinate functions, the vector fields become

$$\mathrm{d}_\pi(x^n) = (\rho_*)_j^n \frac{\partial}{\partial \alpha^j} \qquad \text{and} \qquad \mathrm{d}_\pi(\alpha^j) = (\rho_*)_j^i \frac{\partial}{\partial x^i} + c_t^{js}\alpha^t \frac{\partial}{\partial \alpha^s}.$$

An easy computation shows that a general vector field of the form

$$\Lambda = \zeta \frac{\partial}{\partial \xi^i} \wedge \frac{\partial}{\partial \xi^j} \in \mathfrak{A}_{-1}^{2,s}(A[1])$$

satisfies the equation $[[\xi_j, \Lambda]_{-1}, \xi_i]_{-1} = -(-1)^{|\xi_j|s}\zeta$; that is, the coordinate functions of a 2-vector field Λ can be recovered by computing expressions of the form $[[\xi_i, \Lambda], \xi_j]$, where ξ_i are the coordinates of the graded manifold $A[1]$. Now we use this to obtain the coordinate representations for the 2-vector fields on $A[1]$ of the form $\mathrm{d}_\pi(\nabla_X^*)$ and $\mathrm{d}_\pi(\widehat{a})$, where $X \in \mathfrak{X}(M)$ and $a \in \Gamma(A)$: Using the graded anti-symmetry and the Jacobi identity of $[\cdot, \cdot]_{-1}$, we compute for $f, g \in C^\infty(M)$ and $\alpha, \beta \in \Omega^1(A)$

$$\{0\} = \mathfrak{A}_{-1}^{0,-1}(A[1]) \ni [[f, [\pi, \nabla_X^*]_{-1}]_{-1}, g]_{-1} = 0$$

$$[[\alpha, [\pi, \nabla_X^*]_{-1}]_{-1}, \beta]_{-1} = -\nabla_X^*[\alpha, \beta]_* + [\nabla_X^*\alpha, \beta]_* + [\alpha, \nabla_X^*\beta]_*$$

$$[[f, [\pi, \nabla_X^*]_{-1}]_{-1}, \alpha]_{-1} = [X, \rho_*(\alpha)]f - \rho(\nabla_X^*\alpha)f = \left((\nabla^*)_\alpha^{\mathrm{bas}}X\right)f$$

which imply that

$$\mathrm{d}_\pi(\nabla_X^*) = \left((\nabla^*)_{\alpha^j}^{\mathrm{bas}}X\right)x^i \frac{\partial}{\partial \alpha^j} \wedge \frac{\partial}{\partial x^i}$$
$$+ \left([\nabla_X^*\alpha^n, \alpha^m]_* + [\alpha^n, \nabla_X^*\alpha^m]_* - \nabla_X^*[\alpha^n, \alpha^m]_*\right)\frac{\partial}{\partial \alpha^m} \wedge \frac{\partial}{\partial \alpha^n}.$$

Moreover, we have that

$$\{0\} = \mathfrak{A}_{-1}^{0,-2}(A[1]) \ni [[f, [\pi, \widehat{a}]_{-1}]_{-1}, g]_{-1} = 0$$

$$\{0\} = \mathfrak{A}_{-1}^{0,-1}(A[1]) \ni [[f, [\pi, \widehat{a}]_{-1}]_{-1}, \alpha]_{-1} = 0$$

$$[[\alpha, [\pi, \widehat{a}]_{-1}]_{-1}, \beta]_{-1} = \rho_*(\alpha)(\beta(a)) - \rho_*(\beta)(\alpha(a)) - [\alpha, \beta]_*(a) = \mathrm{d}_{A^*}a(\alpha, \beta),$$

which yield

$$\mathrm{d}_\pi(\widehat{a}) = \mathrm{d}_{A^*} a(\alpha^i, \alpha^j) \frac{\partial}{\partial \alpha^i} \wedge \frac{\partial}{\partial \alpha^j}.$$

Remark 4.5.1. Note that with the same reasoning we can compute the local expression of $\pi \in \mathfrak{A}_{-1}^{2,1}(A[1])$. In particular, from the equations $\{f, g\} = 0, \{\alpha, f\} = \rho_*(\alpha)f$ and $\{\alpha, \beta\} = -[\alpha, \beta]_*$, it follows that

$$\pi = \rho_*(\alpha^j) \wedge \frac{\partial}{\partial \alpha^j} + [\alpha^s, \alpha^t]_* \frac{\partial}{\partial \alpha^s} \wedge \frac{\partial}{\partial \alpha^t} = (\rho_*)_j^i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \alpha^j} + c_m^{st} \alpha^m \frac{\partial}{\partial \alpha^s} \wedge \frac{\partial}{\partial \alpha^t}.$$

4.6 Graded symplectic forms

In ordinary manifolds, it is well-known that a manifold with a symplectic form $\omega \in \Omega^2(M)$ is also Poisson with respect to the induced bracket on functions given by

$$\{f, g\} := \omega(X_f, X_g) = -\langle \mathrm{d}f, X_g \rangle = -X_g(f) = X_f(g),$$

for $f, g \in C^\infty(M)$. However, the converse is not true in general. In this section, we will see that this bridge between non-degenerate Poisson brackets and symplectic forms can be carried over to the world of supergeometry.

Suppose that $\mathcal{M}$ is a $\mathbb{Z}$-manifold and $k \in \mathbb{Z}$. A 2-form $\omega \in \Omega^{2,-k}(\mathcal{M})$ is called **symplectic** (of degree $-k$) if it is closed, i.e. $\mathrm{d}\omega = 0$, and non-degenerate, i.e. the induced map

$$\omega^\flat : T\mathcal{M} \to T^*[-k]\mathcal{M}, \qquad \mathcal{X} \mapsto i_\mathcal{X}\omega,$$

is a vector bundle isomorphism. Given a homogeneous function $\xi \in \mathcal{C}^\infty(\mathcal{M})$, the (unique) vector field $\mathcal{X}_\xi \in \mathfrak{X}^{|\xi|+k}(\mathcal{M})$ which satisfies the equation

$$i_{\mathcal{X}_\xi}\omega = -(-1)^{|\xi|}\mathrm{d}\xi$$

is called **Hamiltonian vector field of** ξ (with respect to ω). A **symplectic vector field** is a vector field $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$ such that $\pounds_\mathcal{X}\omega = 0$. Due to $\mathrm{d}^2 = 0$ and $\mathrm{d}\omega = 0$, it follows that every Hamiltonian vector field is symplectic: For all $\xi \in \mathcal{C}^\infty(\mathcal{M})$

$$\pounds_{\mathcal{X}_\xi}\omega = [i_{\mathcal{X}_\xi}, \mathrm{d}]\omega = i_{\mathcal{X}_\xi}(\mathrm{d}\omega) - (-1)^{|\xi|+k-1}\mathrm{d}(i_{\mathcal{X}_\xi}\omega) = (-1)^{k-1}\mathrm{d}^2\xi = 0.$$

As in the case of ordinary (non-graded) symplectic manifolds, one can define a bracket $\{\cdot, \cdot\}_k$ of degree k on $\mathcal{M}$ as

$$\{\xi_1, \xi_2\}_k := \mathcal{X}_{\xi_1}(\xi_2) = i_{\mathcal{X}_{\xi_1}}(\mathrm{d}\xi_2) = (-1)^{|\xi_2|}i_{\mathcal{X}_{\xi_1}}i_{\mathcal{X}_{\xi_2}}\omega.$$

Proposition 4.6.1. The bracket $\{\cdot, \cdot\}_k$ defined above is a degree k Poisson bracket on $\mathcal{M}$.

Proof. Given two homogeneous functions $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$, we have that $|\{\xi_1, \xi_2\}_k| = |\mathcal{X}_{\xi_1}(\xi_2)| =$

$\xi_1 + \xi_2 + k$. It remains to show that $\{\cdot,\cdot\}_k$ satisfies the graded anti-symmetry, Leibniz, and Jacobi identities. In order to justify these, we need the identities from Appendix A.2. Then we compute

$$\{\xi_1, \xi_2\}_k = -(-1)^{|\xi_2|} i_{\mathcal{X}_{\xi_1}} i_{\mathcal{X}_{\xi_2}} \omega = (-1)^{(|\xi_1|+k)(|\xi_2|+k)+|\xi_1|} i_{\mathcal{X}_{\xi_2}} i_{\mathcal{X}_{\xi_1}} \omega = -(-1)^{(|\xi_1|+k)(|\xi_2|+k)} \{\xi_2, \xi_1\}_k,$$

$$\begin{aligned}
\{\xi_1, \xi_2\xi_3\}_k &= i_{\mathcal{X}_{\xi_1}}(\mathrm{d}(\xi_2\xi_3)) \\
&= i_{\mathcal{X}_{\xi_1}}((\mathrm{d}\xi_2)\xi_3) + (-1)^{|\xi_2|} i_{\mathcal{X}_{\xi_1}}(\xi_2 \mathrm{d}\xi_3) \\
&= \{\xi_1, \xi_2\}_k \xi_3 + (-1)^{(|\xi_1|+k)|\xi_2|} \xi_2 \{\xi_1, \xi_3\}_k.
\end{aligned}$$

Therefore, anti-symmetry and Leibniz rule hold. For the Jacobi identity, we prove the equivalent equation $[\mathcal{X}_{\xi_1}, \mathcal{X}_{\xi_2}] = \mathcal{X}_{\{\xi_1,\xi_2\}_k}$:

$$\begin{aligned}
i_{[\mathcal{X}_{\xi_1}, \mathcal{X}_{\xi_2}]}\omega &= [\pounds_{\mathcal{X}_{\xi_1}}, i_{\mathcal{X}_{\xi_2}}]\omega \\
&= \pounds_{\mathcal{X}_{\xi_1}}(i_{\mathcal{X}_{\xi_2}}\omega) - (-1)^{(|\xi_1|+k)(|\xi_2|+k-1)} i_{\mathcal{X}_{\xi_2}}(\pounds_{\mathcal{X}_{\xi_1}}\omega) \\
&= -(-1)^{|\xi_2|}\pounds_{\mathcal{X}_{\xi_1}}(\mathrm{d}\xi_2) \\
&= (-1)^{|\xi_2|+|\xi_1|+k-1}\mathrm{d}i_{\mathcal{X}_{\xi_1}}(\mathrm{d}\xi_2) \\
&= -(-1)^{|\xi_1|+|\xi_2|+k}\mathrm{d}\{\xi_1, \xi_2\}_k \\
&= i_{\mathcal{X}_{\{\xi_1,\xi_2\}_k}}\omega.
\end{aligned}$$

The result follows due to non-degeneracy of ω. $\qquad\square$

Clearly, the Hamiltonian vector fields with respect to $\{\cdot,\cdot\}_k$ and ω coincide. As the proposition below shows, the same is true also for symplectic and Poisson vector fields. For its proof, we will need the following lemma.

Lemma 4.6.2. Given a p-form $\eta \in \Omega^{p,\bullet}(\mathcal{M})$, the following are equivalent:

1. $\eta = 0$,

2. $i_{\mathcal{X}_1} \ldots i_{\mathcal{X}_p}\eta = 0$ for all $\mathcal{X}_1, \ldots, \mathcal{X}_p \in \mathfrak{X}(\mathcal{M})$.

Proof. One direction is obvious. For the second, consider local coordinates $\{\xi^j\}$ of $\mathcal{M}$ and write

$$\eta = \sum_{j_1 \ldots j_p} \zeta_{j_1 \ldots j_p} \mathrm{d}\xi^{j_1} \wedge \ldots \wedge \mathrm{d}\xi^{j_p}$$

for some local functions $\zeta_{j_1 \ldots j_p}$ of $\mathcal{M}$. Using repeatedly the Leibniz rule for the contraction with respect to the coordinate vector fields $i_{\frac{\partial}{\partial\xi^{j_1}}}, \ldots, i_{\frac{\partial}{\partial\xi^{j_p}}}$ yields

$$0 = i_{\frac{\partial}{\partial\xi^{j_1}}} \ldots i_{\frac{\partial}{\partial\xi^{j_p}}}\eta = i_{\frac{\partial}{\partial\xi^{j_1}}} \ldots i_{\frac{\partial}{\partial\xi^{j_p}}}\left(\sum_{j_1 \ldots j_p} \zeta_{j_1 \ldots j_p} \mathrm{d}\xi^{j_1} \wedge \ldots \wedge \mathrm{d}\xi^{j_p}\right) = \pm\zeta_{j_1 \ldots j_p}. \qquad\square$$

Proposition 4.6.3. A homogeneous vector field $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$ is symplectic for ω if and only if it is Poisson for $\{\cdot, \cdot\}_k$.

Proof. Consider any homogeneous $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$ and any homogeneous function $\xi \in \mathcal{C}^\infty(\mathcal{M})$. Then we compute

$$
\begin{aligned}
i_{[\mathcal{X}, \mathcal{X}_\xi]}\omega &= [\mathcal{L}_\mathcal{X}, i_{\mathcal{X}_\xi}]\omega \\
&= \mathcal{L}_\mathcal{X}(i_{\mathcal{X}_\xi}\omega) - (-1)^{|\mathcal{X}|(|\xi|+k-1)}i_{\mathcal{X}_\xi}(\mathcal{L}_\mathcal{X}\omega) \\
&= -(-1)^{|\xi|}\mathcal{L}_\mathcal{X}(\mathrm{d}\xi) - (-1)^{|\mathcal{X}|(|\xi|+k-1)}i_{\mathcal{X}_\xi}(\mathcal{L}_\mathcal{X}\omega) \\
&= (-1)^{|\xi|+|\mathcal{X}|-1}\mathrm{d}\mathcal{X}(\xi) - (-1)^{|\mathcal{X}|(|\xi|+k-1)}i_{\mathcal{X}_\xi}(\mathcal{L}_\mathcal{X}\omega) \\
&= i_{\mathcal{X}_{\mathcal{X}(\xi)}}\omega - (-1)^{|\mathcal{X}|(|\xi|+k-1)}i_{\mathcal{X}_\xi}(\mathcal{L}_\mathcal{X}\omega)
\end{aligned}
$$

Hence, if $\mathcal{X}$ is symplectic, then $[\mathcal{X}, \mathcal{X}_\xi] = \mathcal{X}_{\mathcal{X}(\xi)}$ for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$, which is equivalent to $\mathcal{X}$ being Poisson. Conversely, suppose $\mathcal{X}$ is Poisson. Then it follows from the computation above that $i_{X_\xi}(\mathcal{L}_\mathcal{X}\omega) = 0$ for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$. Due to non-degeneracy of ω, the Hamiltonian vector fields span all of $\mathfrak{X}(\mathcal{M})$ and thus $i_\mathcal{Y}(\mathcal{L}_\mathcal{X}\omega)$ for all $\mathcal{Y} \in \mathfrak{X}(\mathcal{M})$. An application of Lemma 4.6.2 finishes the proof. $\square$

Example 4.6.4 (Cotangent bundle). Similarly to the non-graded case, all shifted cotangent bundles $T^*[1-k]\mathcal{M}$ carry a canonical symplectic form $\omega_{\mathrm{can}} \in \Omega^{k-1}(T^*[1-k]\mathcal{M})$. Its corresponding Poisson bivector is given locally by

$$
\pi = \sum_i \frac{\partial}{\partial e^i} \wedge \frac{\partial}{\partial \xi^i},
$$

where $\{\xi^j\}$ are local coordinates on $\mathcal{M}$ and $\{e^j\}$ are the linear coordinates of $T^*[1-k]\mathcal{M}$ corresponding to the local frame $\{\frac{\partial}{\partial \xi^j}\}$ of $T[k-1]\mathcal{M}$. That is, $\{\xi^j, e^j\}$ are the canonical **Darboux coordinates** for ω_{can}.

4.7 Poisson manifolds and Courant algebroids in the graded setting

In this section, we recall the correspondence of ordinary Poisson manifolds and Courant algebroids with symplectic Lie 1- and symplectic Lie 2-algebroids, respectively. The results were first described in [83, 87] and we refer the reader to these works for more details. For the Courant algebroid case see also [30]. The following is based on [83] and [30].

Note that on a symplectic $[n]$-manifold $(\mathcal{M}, \omega)$ with $\omega \in \Omega^{2,n}(\mathcal{M})$, every symplectic vector field $\mathcal{X} \in \mathfrak{X}^i(\mathcal{M})$ with $i + n \neq 0$ is Hamiltonian and

$$
i_\mathcal{X}\omega = \mathrm{d}\left(\frac{1}{i+n}i_{\mathcal{X}_E}i_\mathcal{X}\omega\right),
$$

where $\mathcal{X}_E$ is the Euler vector field on $\mathcal{M}$ (Roytenberg [83]). In particular, every vector field of

degree 1 is Hamiltonian, and thus any vector field $\mathcal{Q} = \mathcal{X}_\Theta = \{\Theta, \cdot\} \in \mathfrak{X}^1(\mathcal{M})$ is homological if and only if the function $\Theta \in \mathcal{C}^\infty(\mathcal{M})^{n+1}$ satisfies the **(classical) Master equation**

$$\{\Theta, \Theta\} = 0.$$

Suppose $A[1]$ is a symplectic [1]-manifold over M with corresponding Poisson bracket $\{\cdot, \cdot\}$ of degree -1. Due to the Leibniz rule of the Poisson bracket, we obtain a map $A^* \to TM, \alpha \to \{\alpha, \cdot\}$, which is an isomorphism due to the non-degeneracy of $\{\cdot, \cdot\}$. Finally, due to Jacobi identity, the Poisson bracket on $\mathcal{C}^\infty(A[1])^1 = \Gamma(A^*)$ corresponds under this isomorphism to the commutator of vector fields on the base manifold M. This implies that symplectic [1]-manifolds over M are always of the form $T^*[1]M$. A homological vector field $\mathcal{Q}$ on $T^*[1]M$ is of the form $\mathcal{Q} = \{\Theta, \cdot\}$, where Θ is a degree 2 function on $T^*[1]M$. Since $\mathcal{C}^\infty(T^*[1]M) = \Gamma(\wedge TM)$, it follows that $\mathcal{C}^\infty(T^*[1]M)^2 = \Gamma(\wedge^2 TM)$ and thus the function Θ is a Poisson tensor π on M. Therefore, we have proved the following result.

Theorem 4.7.1 ([83]). Symplectic Lie 1-algebroids with induced Poisson bracket of degree -1 are in one-to-one correspondence with ordinary Poisson manifolds.

Passing to the degree $n = 2$ case, we obtain the graded geometric description of Courant algebroids. The correspondence can be seen as follows: Let $(\mathcal{M}, \mathcal{Q}, \omega)$ be a symplectic Lie 2-algebroid with corresponding degree -2 Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{C}^\infty(\mathcal{M})$, and choose a splitting for the underlying [2]-manifold $\mathcal{M}$ so that $\mathcal{C}^\infty(M)^0 = C^\infty(M)$ and $\mathcal{C}^\infty(\mathcal{M})^1 \cong \Gamma(E)$ for a vector bundle $E \to M$. The restriction of the Poisson bracket on $\Gamma(E) \times \Gamma(E)$ defines the pairing $\langle \cdot, \cdot \rangle$ on E. Suppose now that $\Theta \in \mathcal{C}^\infty(M)^3$ is such that $\mathcal{Q} = \mathcal{X}_\Theta$ and it satisfies the classical Master equation: $\{\Theta, \Theta\} = 0$. Then we define

$$\rho(e)f = \{\{e, \Theta\}, f\} = \{\{\Theta, f\}, e\} \qquad \text{and} \qquad [\![e_1, e_2]\!] = \{\{e_1, \Theta\}, e_2\},$$

for $f \in C^\infty(M), e, e_1, e_2 \in \Gamma(E)$, and hence we have additionally a bundle map $\rho : E \to TM$ and a bracket $[\![\cdot, \cdot]\!]$ on $\Gamma(E)$. The quadruple $(E, \rho, [\![\cdot, \cdot]\!], \langle \cdot, \cdot \rangle)$ has the structure of a Courant algebroid over the base manifold M.

The converse can be found in [83] using local coordinates. Here, we adopt the method from [30] and describe it in an invariant way. Suppose $(E, \rho, [\![\cdot, \cdot]\!], \langle \cdot, \cdot \rangle)$ is a Courant algebroid over the smooth manifold M. We define the [2]-manifold $\mathcal{M}$ whose sheaf of functions is given by

$$\mathcal{C}^\infty(\mathcal{M})^0 = C^\infty(M), \qquad \mathcal{C}^\infty(\mathcal{M})^1 = \Gamma(E), \qquad \mathcal{C}^\infty(\mathcal{M})^2 = \Gamma(\text{Der}_{\langle \cdot, \cdot \rangle}(E)),$$

where $\text{Der}_{\langle \cdot, \cdot \rangle}(E)$ denotes the derivations of the vector bundle $E \to M$ preserving the pairing $\langle \cdot, \cdot \rangle$ on E: $D \in \text{Der}_{\langle \cdot, \cdot \rangle}(E)$ if and only if D is a derivation of E over the vector field $X_D \in \mathfrak{X}(M)$ such that

$$X_D \langle e_1, e_2 \rangle = \langle De_1, e_2 \rangle + \langle e_1, De_2 \rangle$$

for all $e_1, e_2 \in \Gamma(E)$. We now recall the **Keller-Waldmann algebra** [56] $(C^\bullet(E), \bullet)$ of the

Courant algebroid $E \to M$. A k-**cochain** on E, $k \geq 1$, is a map

$$\omega \colon \underbrace{\Gamma(E) \times \ldots \Gamma(E)}_{k\text{-times}} \to C^\infty(M)$$

that is $C^\infty(M)$-linear in the last entry and such that for $k \geq 2$ there exists a map

$$\sigma_\omega \colon \underbrace{\Gamma(E) \times \ldots \Gamma(E)}_{(k-2)\text{-times}} \to \mathfrak{X}(M),$$

called the **symbol**, with

$$\sigma_\omega(e_1, \ldots, \widehat{e}_i, \widehat{e}_{i+1}, \ldots, e_k)\langle e_i, e_{i+1}\rangle = \omega(e_1, \ldots, e_i, e_{i+1}, \ldots, e_k) + \omega(e_1, \ldots, e_{i+1}, e_i, \ldots, e_k)$$

for all $e_j \in \Gamma(E)$ and $1 \leq i \leq k-1$. The space of k-cochains is denoted $C^\bullet(E)$ and by definition $C^0(E) := C^\infty(M)$. The space $C^\bullet(E)$ has the structure of a graded commutative algebra with product $\bullet$ defined by

$$(\omega \bullet \eta)(e_1, \ldots, e_{k+m}) := \sum_{\sigma \in \mathrm{Sh}_{k,m}} \mathrm{sgn}(\sigma)\, \omega(e_{\sigma(1)}, \ldots, e_{\sigma(k)})\, \eta(e_{\sigma(k+1)}, \ldots, e_{\sigma(m)})$$

for $\omega \in C^k(E), \eta \in C^m(E)$ and $e_j \in \Gamma(E)$. Note that in degrees $0, 1$ and 2, $C^\bullet(E)$ is described as follows:

- $C^0(E) = C^\infty(M) = \mathcal{C}^\infty(\mathcal{M})^0$;

- $C^1(E) = \Gamma(E^*) \simeq \Gamma(E) = \mathcal{C}^\infty(\mathcal{M})^1$ via the non-degenerate pairing $\langle \cdot, \cdot \rangle$ on E;

- Given $\omega \in C^2(E)$, let $\widehat{\omega} \colon \Gamma(E) \to \Gamma(E)$ be defined by $\langle \widehat{\omega}(e_1), e_2 \rangle = \omega(e_1, e_2)$, for $e_1, e_2 \in \Gamma(E)$. Then $\widehat{\omega} \in \mathrm{Der}_{\langle \cdot, \cdot \rangle}(E)$ over the vector field $X_{\widehat{\omega}} = \sigma_\omega \in \mathfrak{X}(M)$. That is, $C^2(E) \cong \mathcal{C}^\infty(\mathcal{M})^2$.

Using now the pairing $\langle \cdot, \cdot \rangle$ of E, one can define a symplectic structure on the graded manifold $\mathcal{M}$ whose corresponding degree -2 Poisson bracket $\{\cdot, \cdot\}$ is given in low degrees by

$$\{e_1, e_2\} = \langle e_1, e_2\rangle, \quad \{D, f\} = X_D(f), \quad \{D, e\} = De, \quad \{D_1, D_2\} = D_1 D_2 - D_2 D_1,$$

for $f \in C^\infty(M), e, e_1, e_2 \in \Gamma(E), D, D_1, D_2 \in \mathrm{Der}_{\langle \cdot, \cdot \rangle}(E)$. In particular, since only the pairing of E was used, we have proved the following result.

Proposition 4.7.2 ([83])**.** The above construction establishes a one-to-one correspondence between symplectic [2]-manifolds over M with induced Poisson bracket of degree -2 and vector bundles $E \to M$ equipped with a non-degenerate pairing $\langle \cdot, \cdot \rangle$.

Note that the description of $C^0(E), C^1(E)$ and $C^2(E)$ implies that $(C^\bullet(E), \bullet)$ and $(\mathcal{C}^\infty(\mathcal{M}), \cdot)$ coincide in low degrees. In fact, they are isomorphic as graded commutative algebras via the map

$\Upsilon \colon \mathcal{C}^{\infty}(\mathcal{M})^k \to C^k(E)$ [30, Thm. 2.5] defined by

$$\Upsilon(\xi)(e_1, \ldots, e_k) := \{e_k, \{e_{k-1}, \{\ldots \{e_1, \xi\} \ldots\}\}\}$$

for $e_1, \ldots, e_k \in \Gamma(E)$ and $\xi \in \mathcal{C}^{\infty}(\mathcal{M})^k$. Now we can use this isomorphism to define a function $\Theta \in \mathcal{C}^{\infty}(\mathcal{M})$ of degree 3 satisfying $\{\Theta, \Theta\} = 0$; namely, $\Theta := \Upsilon^{-1}(T)$, where $T \in C^3(E)$ is given by $T(e_1, e_2, e_3) := \langle [\![e_1, e_2]\!], e_3 \rangle$, for $e_1, e_2, e_3 \in \Gamma(E)$. The symplectic homological vector field on the symplectic [2]-manifold $(\mathcal{M}, \{\cdot, \cdot\})$ is given by $\mathcal{Q} := \mathcal{X}_{\Theta}$. Therefore, we obtain the following well-known result.

Theorem 4.7.3 ([83])**.** The above construction gives a one-to-one correspondence between symplectic Lie 2-algebroids with induced Poisson bracket of degree -2 and Courant algebroids.

Chapter 5

Differential graded modules

This chapter defines the notion of a differential graded module over a Lie n-algebroid (or generally a $\mathcal{Q}$-manifold) $(\mathcal{M}, \mathcal{Q})$ and gives the two fundamental examples of modules which come canonically with $(\mathcal{M}, \mathcal{Q})$, namely the adjoint and the coadjoint modules. Moreover, we investigate the influence of a Poisson structure on $(\mathcal{M}, Q)$ on its adjoint and coadjoint modules. Note that the case of differential graded modules over a Lie 1-algebroid $A \to M$ is studied in detail in [75].

5.1 The category of differential graded modules

Let $A \to M$ be a Lie 1-algebroid. A **Lie algebroid module** [93] over A is defined as a sheaf $\mathcal{B}$ of locally freely generated graded $\Omega(A)$-modules over M together with a degree 1 map $\mathcal{D}\colon \mathcal{B} \to \mathcal{B}$ which squares to zero and satisfies the Leibniz rule

$$\mathcal{D}(\alpha\eta) = (\mathrm{d}_A\alpha)\eta + (-1)^{|\alpha|}\alpha\mathcal{D}(\eta),$$

for $\alpha \in \Omega(A)$ and $\eta \in \mathcal{B}$. By definition, the sheaf $\mathcal{B}$ is the sheaf of sections of a vector bundle $\mathcal{E} \to A[1]$ in the category of graded manifolds: $\mathcal{B} = \Gamma(\mathcal{E})$. For a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ over M (or even a general $\mathbb{Z}$-graded $\mathcal{Q}$-manifold), this is generalised to the following definitions.

Definition 5.1.1. 1. A **left differential graded module of** $(\mathcal{M}, \mathcal{Q})$ is a sheaf $\Gamma(\mathcal{E})$ of locally freely generated left graded $\mathcal{C}^\infty(\mathcal{M})$-modules over M (corresponding to a vector bundle $\mathcal{E} \to \mathcal{M}$), together with a map $\mathcal{D}\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ of degree 1, such that $\mathcal{D}^2 = 0$ and

$$\mathcal{D}(\xi\eta) = \mathcal{Q}(\xi)\eta + (-1)^{|\xi|}\xi\mathcal{D}(\eta)$$

for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\eta \in \Gamma(\mathcal{E})$.

2. A **right differential graded module of** $(\mathcal{M}, \mathcal{Q})$ is a sheaf $\Gamma(\mathcal{E})$ of right graded modules

as above together with a map $\mathcal{D}\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ of degree 1, such that $\mathcal{D}^2 = 0$ and

$$\mathcal{D}(\eta\xi) = \mathcal{D}(\eta)\xi + (-1)^{|\eta|}\eta\mathcal{Q}(\xi)$$

for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\eta \in \Gamma(\mathcal{E})$.

3. A **differential graded bimodule of** $(\mathcal{M}, \mathcal{Q})$ is a sheaf $\Gamma(\mathcal{E})$ as above together with left and right differential graded module structures such that the gradings and the differentials coincide, and the two module structures commute: $(\xi_1\eta)\xi_2 = \xi_1(\eta\xi_2)$ for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$ and $\eta \in \Gamma(\mathcal{E})$.

For short we write **left DG** $(\mathcal{M}, \mathcal{Q})$**-module** and **right DG** $(\mathcal{M}, \mathcal{Q})$**-module**, or simply **left DG** $\mathcal{M}$**-module** and **right DG** $\mathcal{M}$**-module**. The cohomology of the induced complexes is denoted by $H_L^\bullet(\mathcal{M}, \mathcal{Q}; \mathcal{E})$ and $H_R^\bullet(\mathcal{M}, \mathcal{Q}; \mathcal{E})$, respectively, or simply by $H_L^\bullet(\mathcal{M}, \mathcal{E})$ and $H_R^\bullet(\mathcal{M}, \mathcal{E})$. If there is no danger of confusion, the prefixes "left" and "right", as well as the subscripts "L" and "R", will be omitted.

Definition 5.1.2. Let $(\mathcal{E}_1, \mathcal{D}_1)$ and $(\mathcal{E}_2, \mathcal{D}_2)$ be two left (right) differential graded modules over the Lie n-algebroids $(\mathcal{M}, \mathcal{Q}_\mathcal{M})$ and $(\mathcal{N}, \mathcal{Q}_\mathcal{N})$, respectively. A **degree** 0 **morphism**, or simply a **morphism**, between $\mathcal{E}_1$ and $\mathcal{E}_2$ consists of a morphism of Lie n-algebroids $\phi\colon \mathcal{N} \to \mathcal{M}$ and a degree preserving map $\mu\colon \Gamma(\mathcal{E}_1) \to \Gamma(\mathcal{E}_2)$ which is left (right) linear: $\mu(\xi\eta) = \phi^\star(\xi)\mu(\eta)$, for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\eta \in \Gamma(\mathcal{E}_1)$, and commutes with the differentials $\mathcal{D}_1$ and $\mathcal{D}_2$.

As in the case of Lie algebroids, new examples of DG $\mathcal{M}$-modules are obtained by considering the usual algebraic constructions. In the following, we describe these constructions only for left DG modules but the case of right DG modules is treated similarly.

Definition 5.1.3 (**Dual module**). Given a DG $\mathcal{M}$-module $\mathcal{E}$ with differential $\mathcal{D}_\mathcal{E}$, one defines a DG $\mathcal{M}$-module structure on the dual sheaf $\mathcal{E}^* := \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{C}^\infty)$ with differential $\mathcal{D}_{\mathcal{E}^*}$ defined via the property

$$\mathcal{Q}(\langle \psi, \eta \rangle) = \langle \mathcal{D}_{\mathcal{E}^*}(\psi), \eta \rangle + (-1)^{|\psi|}\langle \psi, \mathcal{D}_\mathcal{E}(\eta) \rangle,$$

for all $\psi \in \Gamma(\mathcal{E}^*)$ and $\eta \in \Gamma(\mathcal{E})$, where $\langle \cdot, \cdot \rangle$ is the pairing of $\mathcal{E}^*$ and $\mathcal{E}$ [71].

Definition 5.1.4 (**Tensor product module**). For DG $\mathcal{M}$-modules $\mathcal{E}$ and $\mathcal{F}$ with operators $\mathcal{D}_\mathcal{E}$ and $\mathcal{D}_\mathcal{F}$, the corresponding operator $\mathcal{D}_{\mathcal{E} \otimes \mathcal{F}}$ on $\mathcal{E} \otimes \mathcal{F}$ is uniquely characterised by the formula

$$\mathcal{D}_{\mathcal{E} \otimes \mathcal{F}}(\eta \otimes \eta') = \mathcal{D}_\mathcal{E}(\eta) \otimes \eta' + (-1)^{|\eta|}\eta \otimes \mathcal{D}_\mathcal{F}(\eta'),$$

for all $\eta \in \Gamma(\mathcal{E})$ and $\eta' \in \Gamma(\mathcal{F})$.

Definition 5.1.5 (<u>Hom</u>**-module**). For DG $\mathcal{M}$-modules $\mathcal{E}, \mathcal{F}$ with operators $\mathcal{D}_\mathcal{E}$ and $\mathcal{D}_\mathcal{F}$, the differential $\mathcal{D}_{\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F})}$ on $\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F})$ is defined via

$$\mathcal{D}_\mathcal{F}(\psi(\eta)) = \mathcal{D}_{\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F})}(\psi)(\eta) + (-1)^{|\psi|}\psi(\mathcal{D}_\mathcal{E}(\eta)),$$

for all $\psi \in \underline{\mathrm{Hom}}(\Gamma(\mathcal{E}), \Gamma(\mathcal{F}))$ and $\eta \in \Gamma(\mathcal{E})$.

Definition 5.1.6 (**(Anti)symmetric powers module**). For a DG $\mathcal{M}$-module $\mathcal{E}$ with operator $\mathcal{D}_\mathcal{E}$, the corresponding operator $\mathcal{D}_{\underline{S}(\mathcal{E})}$ on $\underline{S}^k(\mathcal{E})$ is uniquely characterised by the formula

$$\mathcal{D}_{\underline{S}(\mathcal{E})}(\eta_1 \eta_2 \ldots \eta_k) = \mathcal{D}_\mathcal{E}(\eta_1)\eta_2 \ldots \eta_k$$
$$+ \eta_1 \sum_{i=2}^{k} (-1)^{|\eta_1|+\ldots+|\eta_{i-1}|} \eta_2 \ldots \mathcal{D}(\eta_i) \ldots \eta_k,$$

for all $\eta_1, \ldots, \eta_k \in \Gamma(\mathcal{E})$. A similar formula gives also the characterisation for the operator $\mathcal{D}_{\underline{A}(\mathcal{E})}$ of the antisymmetric powers $\underline{A}^q(\mathcal{E})$.

Definition 5.1.7 (**Direct sum module**). For DG $\mathcal{M}$-modules $\mathcal{E}, \mathcal{F}$ with operators $\mathcal{D}_\mathcal{E}$ and $\mathcal{D}_\mathcal{F}$, the differential operator $\mathcal{D}_{\mathcal{E}\oplus\mathcal{F}}$ on $\mathcal{E} \oplus \mathcal{F}$ is defined as

$$\mathcal{D}_{\mathcal{E}\oplus\mathcal{F}} = \mathcal{D}_\mathcal{E} \oplus \mathcal{D}_\mathcal{F}.$$

Definition 5.1.8 (**Shifted module**). For $k \in \mathbb{Z}$, the DG $\mathcal{M}$-module $\mathbb{R}[k]$ is defined as $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(M \times \mathbb{R}[k])$ with differential given by $\mathcal{Q}$; here, $M \times \mathbb{R}[k]$ is the $[k]$-shift of the trivial line bundle over M, i.e. $M \times \mathbb{R}$ in degree $-k$ and zero otherwise. Given now a DG-module $\mathcal{E}$ with differential $\mathcal{D}_\mathcal{E}$, we define the shifted module $\mathcal{E}[k] := \mathcal{E} \otimes \mathbb{R}[k]$. Due to the definition of the tensor module, its differential $\mathcal{D}[k]$ acts via

$$\mathcal{D}_\mathcal{E}[k](\eta \otimes 1) = \mathcal{D}_\mathcal{E}(\eta) \otimes 1$$

for all $\eta \in \Gamma(\mathcal{E})$. Abbreviating the element $\eta \otimes 1$ simply as η, the shifted differential $\mathcal{D}_\mathcal{E}[k]$ coincides[1] with $\mathcal{D}_\mathcal{E}$.

Definition 5.1.9. Let $(\mathcal{M}, \mathcal{Q}_\mathcal{M})$ and $(\mathcal{N}, \mathcal{Q}_\mathcal{N})$ be $\mathcal{Q}$-manifolds, and suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are left (right) DG-modules over $\mathcal{M}$ and $\mathcal{N}$, respectively. A **degree k-morphism** or simply k-morphism, for $k \in \mathbb{Z}$, from $\mathcal{E}_1$ to $\mathcal{E}_2$ is defined as a left (right) degree 0 morphism $\mu : \mathcal{E}_1 \to \mathcal{E}_2[k]$; that is, a map sending elements of degree i in $\Gamma(\mathcal{E}_1)$ to elements of degree $i+k$ in $\Gamma(\mathcal{E}_2)$, such that it is linear over a morphism of $\mathcal{Q}$-manifolds $\phi : \mathcal{N} \to \mathcal{M}$ and commutes with the differentials. A k-**isomorphism** is a k-morphism with an inverse.

Remark 5.1.10.   1. The inverse of a k-isomorphism is necessarily a $-k$-isomorphism.

2. For all $k \in \mathbb{Z}$ and all DG $\mathcal{M}$-modules $\mathcal{E}$, there is an obvious k-isomorphism $\mathcal{E} \to \mathcal{E}[k]$ over the identity on $\mathcal{M}$.

Considering the special case of $\mathcal{M} = \mathcal{N}$ in the definition above yields k-morphisms between DG $\mathcal{M}$-modules over the same $\mathcal{Q}$-manifold. The resulting graded categories of left and right DG $\mathcal{M}$-modules are denoted by $\underline{\mathrm{Mod}}_L(\mathcal{M}, \mathcal{Q})$ and $\underline{\mathrm{Mod}}_R(\mathcal{M}, \mathcal{Q})$, or simply by $\underline{\mathrm{Mod}}_L(\mathcal{M})$ and $\underline{\mathrm{Mod}}_R(\mathcal{M})$. The isomorphism classes of these categories are denoted by $\underline{\mathrm{Mod}}_L(\mathcal{M}, \mathcal{Q})$ and $\underline{\mathrm{Mod}}_R(\mathcal{M}, \mathcal{Q})$, or

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[1]Again one could choose to tensor with $\mathbb{R}[k]$ from the left. Then on elements of the form $1 \otimes \eta$, the resulting differential would act as $\mathcal{D}_\mathcal{E}[k] = (-1)^k \mathcal{D}_\mathcal{E}$.

simply by $\underline{\mathrm{Mod}}_L(\mathcal{M})$ and $\underline{\mathrm{Mod}}_R(\mathcal{M})$. Again if there is no danger of confusion, the subscripts "L" and "R" will be omitted.

5.2   Adjoint and coadjoint modules

Recall that every $\mathcal{Q}$-manifold $\mathcal{M}$ comes together with its tangent bundle $T\mathcal{M}$ whose sheaf of sections is the space of vector fields $\mathfrak{X}(\mathcal{M})$ over $\mathcal{M}$. Its left $\mathcal{C}^\infty(\mathcal{M})$-module structure is defined by the property $(\xi_1 \mathcal{X})(\xi_2) = \xi_1 \mathcal{X}(\xi_2)$ for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$ and $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$. In addition to the left module structure, the space of vector fields is also endowed with a right $\mathcal{C}^\infty(\mathcal{M})$-module structure. The right multiplication with functions in $\mathcal{C}^\infty(\mathcal{M})$ is obtained by viewing the elements of $\mathfrak{X}(\mathcal{M})$ of degree i as functions of bidegree $(1, i)$ of the graded manifold $T^*[1]\mathcal{M}$. The two module structures on $\mathfrak{X}(\mathcal{M})$ are related by $\mathcal{X}\xi = (-1)^{|\xi|(|\mathcal{X}|+1)}\xi\mathcal{X}$, for all homogeneous $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$.

Suppose now that $\mathcal{M}$ is endowed with a homological vector field $\mathcal{Q}$, i.e. $(\mathcal{M}, \mathcal{Q})$ is a $\mathcal{Q}$-manifold. Then the Lie derivative on the space of vector fields $\mathcal{L}_\mathcal{Q} := [\mathcal{Q}, \cdot]$ is a degree 1 operator which squares to zero and has both the left and right Leibniz identities with respect to the left and right module structures explained above:

$$\mathcal{L}_\mathcal{Q}(\xi\mathcal{X}) = \mathcal{Q}(\xi)\mathcal{X} + (-1)^{|\xi|}\xi\mathcal{L}_\mathcal{Q}(\mathcal{X}) \quad \text{and} \quad \mathcal{L}_\mathcal{Q}(\mathcal{X}\xi) = \mathcal{L}_\mathcal{Q}(\mathcal{X})\xi + (-1)^{|\mathcal{X}|}\mathcal{X}\mathcal{Q}(\xi)$$

for homogeneous $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$. That is, the sheaf of vector fields over $(\mathcal{M}, \mathcal{Q})$ has a canonical DG $\mathcal{M}$-bimodule structure which we call the **adjoint module** of $\mathcal{M}$:

$$(\mathfrak{X}(\mathcal{M}), \mathcal{L}_\mathcal{Q}).$$

The dual module $\bigoplus_p \mathcal{C}^\infty(T[1]\mathcal{M})_{(p,1)}$ of 1-forms over $\mathcal{M}$ carries the grading obtained from the horizontal grading of the Weil algebra – that is, the elements of $\mathcal{C}^\infty(T[1]\mathcal{M})_{(p,1)}$ have degree p. Its structure operator as a left DG module is given by the Lie derivative $\mathcal{L}_\mathcal{Q} = [i_\mathcal{Q}, \mathrm{d}]$, and as a right DG module is given by $-\mathcal{L}_\mathcal{Q}$. These left and right DG $\mathcal{M}$-modules are called the **coadjoint module** of $(\mathcal{M}, \mathcal{Q})$ and denoted by

$$(\Omega^1(\mathcal{M}), \mathcal{L}_\mathcal{Q}) \qquad \text{and} \qquad (\Omega^1(\mathcal{M}), -\mathcal{L}_\mathcal{Q})$$

5.3 $\mathcal{PQ}$-manifolds: coadjoint vs adjoint modules

In [68], it was shown that a Lie algebroid $A \to M$ with a linear Poisson structure satisfies the Lie bialgebroid compatibility condition if and only if the map $T^*A \to TA$ induced by the Poisson bivector is a Lie algebroid morphism from $T^*A = T^*A^* \to A^*$ to $TA \to TM$ over $\rho* : A^* \to TM$. This is now generalised to give a characterisation of $\mathcal{PQ}$-manifolds in the general setting.

Let $\mathcal{M}$ be a $\mathbb{Z}$-manifold equipped with a homological vector field $\mathcal{Q}$ and a Poisson bivector field $\pi \in \mathfrak{A}_k^{2,-k}(\mathcal{M})$ with corresponding degree k Poisson bracket $\{\cdot, \cdot\}_k$. The Poisson bracket on $\mathcal{M}$

induces a map $\pi^\sharp \colon \Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})[k]$ defined on the generators via the property

$$\pi^\sharp(\mathrm{d}\xi_1)(\xi_2) = \{\xi_1, \xi_2\}_k, \tag{5.1}$$

for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$, and extended odd linearly by the rules[1]

$$\pi^\sharp(\xi_1 \mathrm{d}\xi_2) = (-1)^{|\xi_1|}\xi_1\pi^\sharp(\mathrm{d}\xi_2) \qquad \text{and} \qquad \pi^\sharp((\mathrm{d}\xi_1)\xi_2) = (-1)^{|\xi_2|}\pi^\sharp(\mathrm{d}\xi_1)\,\xi_2.$$

Theorem 5.3.1. Let $\mathcal{M}$ be a graded manifold equipped with a homological vector field $\mathcal{Q}$ and a degree k Poisson bracket $\{\cdot, \cdot\}_k$. Then $(\mathcal{M}, \mathcal{Q}, \{\cdot, \cdot\}_k)$ is a $\mathcal{PQ}$-manifold if and only if $\pi^\sharp \colon \Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ is a degree k anti-morphism[2] of left DG $\mathcal{M}$-modules, or if and only if it is a morphism of right DG $\mathcal{M}$-modules, i.e. $\pi^\sharp \circ \mathcal{L}_\mathcal{Q} = -\mathcal{L}_\mathcal{Q} \circ \pi^\sharp$.

Proof. From (5.1),

$$\Big(\mathcal{L}_\mathcal{Q}(\pi^\sharp(\mathrm{d}\xi_1)) + \pi^\sharp(\mathcal{L}_\mathcal{Q}(\mathrm{d}\xi_1))\Big)\xi_2 = \mathcal{Q}\{\xi_1, \xi_2\} - (-1)^{|\xi_1|+k}\{\xi_1, \mathcal{Q}(\xi_2)\} - \{\mathcal{Q}(\xi_1), \xi_2\}.$$

In other words, the compatibility of $\mathcal{Q}$ with $\{\cdot, \cdot\}_k$ is equivalent to $\mathcal{L}_\mathcal{Q} \circ \pi^\sharp = -\pi^\sharp \circ \mathcal{L}_\mathcal{Q}$. $\square$

A detailed analysis of this map in the cases of Poisson Lie n-algebroids equipped with a degree $-n$ bracket for $n \leq 2$ is given in Section 6.8. The two following corollaries can be realised as obstructions for a $\mathcal{Q}$-manifold endowed with a Poisson bracket to be symplectic. In particular, if the manifold is $\mathbb{N}$-graded of degree $n = 2$ one obtains the corresponding results for Courant algebroids.

Corollary 5.3.2. Let $\mathcal{M}$ be an $\mathbb{Z}$-manifold equipped with a homological vector field $\mathcal{Q}$ and a degree k Poisson bracket $\{\cdot, \cdot\}_k$. Then $(\mathcal{M}, \mathcal{Q}, \{\cdot, \cdot\}_k)$ is symplectic if and only if $\pi^\sharp$ is an anti-isomorphism of left DG $\mathcal{M}$-modules, or if and only if it is an isomorphism of right DG $\mathcal{M}$-modules.

Corollary 5.3.3. For any $\mathcal{PQ}$-manifold $(\mathcal{M}, \mathcal{Q}, \{\cdot, \cdot\}_k)$ there are natural degree k maps in cohomologies $\pi^\sharp \colon H_L^\bullet(\mathcal{M}, \Omega^1) \to H_L^{\bullet+k}(\mathcal{M}, \mathfrak{X})$ and $\pi^\sharp \colon H_R^\bullet(\mathcal{M}, \Omega^1) \to H_R^{\bullet+k}(\mathcal{M}, \mathfrak{X})$ which are isomorphisms if the bracket is symplectic.

5.4 $\mathcal{PQ}$-manifolds: Weil vs Poisson-Weil algebras

In this section, we will see that map $\pi^\sharp$ constructed above can be extended to a natural map between the bigraded algebras $\Omega^{\bullet,\bullet}(\mathcal{M})$ and $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$ which anti-commutes with the horizontal and vertical differentials and consequently with the total differentials. This extension is obtained as follows:

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[1] The reason for this particular choice of signs is given in Appendix B.2.

[2] Using convention of tensoring from the left to obtain the shift, the anti-morphism condition reads $\pi^\sharp \circ \mathcal{L}_\mathcal{Q} = -\mathcal{L}_\mathcal{Q} \circ \pi^\sharp = -(-1)^k \mathcal{L}_\mathcal{Q}[k] \circ \pi^\sharp = -(-1)^{k^2}\mathcal{L}_\mathcal{Q}[k] \circ \pi^\sharp = -(-1)^{k|\pi^\sharp|}\mathcal{L}_\mathcal{Q}[k] \circ \pi^\sharp$.

- On the space $\mathcal{C}^\infty(\mathcal{M})$ it is given, up to a sign, by the identity map

$$\mathcal{C}^\infty(\mathcal{M})^i \to \mathcal{C}^\infty(\mathcal{M})^i, \qquad \xi \mapsto (-1)^i \xi.$$

- On q-forms of the form $\xi \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_q$, where $\xi, \xi_1, \ldots, \xi_q \in \mathcal{C}^\infty(\mathcal{M})$ with ξ homogeneous, it is given by

$$\pi^\sharp(\xi \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_q) = \pi^\sharp(\xi)\pi^\sharp(\mathrm{d}\xi_1) \wedge \ldots \wedge \pi^\sharp(\mathrm{d}\xi_q) = (-1)^{|\xi|}\xi \mathcal{X}_{\xi_1} \wedge \ldots \wedge \mathcal{X}_{\xi_q}.$$

Consider now two homogeneous functions $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$. For the horizontal differential $\mathcal{L}_\mathcal{Q}$, the equality $\mathcal{L}_\mathcal{Q} \circ \pi^\sharp = -\pi^\sharp \circ \mathcal{L}_\mathcal{Q}$ is satisfied on functions because

$$\mathcal{L}_\mathcal{Q}(\pi^\sharp(\xi)) = \mathcal{L}_\mathcal{Q}((-1)^{|\xi|}\xi) = -(-1)^{|\xi|+1}\mathcal{L}_\mathcal{Q}(\xi) = -\pi^\sharp(\mathcal{L}_\mathcal{Q}(\xi))$$

and on 1-forms it is the (anti-)morphism between the coadjoint and the adjoint modules which was explained before.

For the vertical differentials d and $(-1)^{k-1}\mathrm{d}_\pi$, we compute on functions

$$(-1)^{k-1}\mathrm{d}_\pi(\pi^\sharp(\xi)) = (-1)^{k-1+|\xi|}[\pi, \xi] = -[\xi, \pi] = -\mathcal{X}_\xi = -\pi^\sharp(\mathrm{d}\xi),$$

and since all Hamiltonian vector fields are Poisson

$$(-1)^{k-1}\mathrm{d}_\pi(\pi^\sharp(\xi \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_q)) = -\mathcal{X}_\xi \wedge \mathcal{X}_{\xi_1} \wedge \ldots \wedge \mathcal{X}_{\xi_q} = -\sharp(\mathrm{d}(\xi \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_q)).$$

Therefore, we have proved the following result about the map $\pi^\sharp$.

Theorem 5.4.1. Let $(\mathcal{M}, \mathcal{Q}, \{\cdot, \cdot\}_k)$ be a $\mathcal{PQ}$-manifold and consider the bidegree $(0,0)$ map $\pi^\sharp \colon \Omega^{\bullet,\bullet}(\mathcal{M}) \to \mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$ of bigraded algebras constructed above.

1. The map $\pi^\sharp$ is an anti-morphism of complexes and DGA's for the columns of $\Omega^{\bullet,\bullet}(\mathcal{M})$ and $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$.

2. The map $\pi^\sharp$ is an anti-morphism of complexes and DGA's for the rows of $\Omega^{\bullet,\bullet}(\mathcal{M})$ and $\mathfrak{A}_k^{\bullet,\bullet}(\mathcal{M})$.

3. The map $\pi^\sharp$ is an anti-morphism of complexes and DGA's between (the total complexes) $\Omega(\mathcal{M})$ and $\mathfrak{A}_k(\mathcal{M})$.

Consequently, the map $\pi^\sharp$ induces a natural map on the level of the vertical and horizontal cohomologies

$$\pi^\sharp \colon H_{dR}^\bullet(\mathcal{M}) \to H_P^\bullet(\mathcal{M}) \qquad \text{and} \qquad \pi^\sharp \colon H_{\Omega,L}^\bullet(\mathcal{M}) \to H_{\mathfrak{A}_k,L}^\bullet(\mathcal{M}).$$

and also on the level of total cohomologies

$$\pi^\sharp \colon H_W^\bullet(\mathcal{M}) \to H_{PW}^\bullet(\mathcal{M}),$$

Moreover, it is an anti-isomorphism of differential bigraded algebras if and only if the Poisson structure is symplectic, and in this case all the induced maps in the cohomologies are also isomorphisms.

Chapter 6

Representations up to homotopy

This chapter is devoted to the study of representations up to homotopy for general Lie n-algebroids with an extra focus on the case $n = 2$ where we explain thoroughly the adjoint representation. In addition, we analyse the Weil algebra of a split Lie n-algebroid and give the precise map between the coadjoint and adjoint representations of a split Poisson Lie n-algebroid for $n = 0, 1, 2$.

6.1 The category of representations up to homotopy

Recall that a **representation up to homotopy of a Lie algebroid** A [4, 75] is given by an A-module of the form $\Omega(A, \underline{E}) = \Omega(A) \otimes \Gamma(\underline{E})$ for a graded vector bundle $\underline{E}$ over M. In the same manner, a **left representation up to homotopy of a Lie n-algebroid** $(\mathcal{M}, \mathcal{Q})$ is defined as a left DG $\mathcal{M}$-module of the form $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E})$ for a graded vector bundle $\underline{E} \to M$. **Right representations up to homotopy** are defined similarly as right DG $\mathcal{M}$-modules of the form $\Gamma(\underline{E}) \otimes \mathcal{C}^\infty(\mathcal{M})$. In what follows, we will focus only on left representations and the prefix "left" will be omitted for simplicity.

Following the notation from [4], we denote the category of representations up to homotopy of $(\mathcal{M}, \mathcal{Q})$ by $\underline{\mathbb{Rep}}^\infty(\mathcal{M}, Q)$, or simply by $\underline{\mathbb{Rep}}^\infty(\mathcal{M})$. The isomorphism classes of representations up to homotopy of this category are denoted by $\underline{\mathrm{Rep}}^\infty(\mathcal{M}, \mathcal{Q})$, or by $\underline{\mathrm{Rep}}^\infty(\mathcal{M})$. A representation of the form $\underline{E} = E_0 \oplus \ldots \oplus E_{k-1}$ is called k-**term representation**, or simply k-**representation**.

Remark 6.1.1. Since by Remark 4.1.3 every vector bundle over an $\mathbb{N}$-manifold comes from an ordinary graded vector bundle over the base, it follows that the notions of DG-modules and representations up to homotopy are equivalent. However, the equivalence is not canonical.

Example 6.1.2 ($\mathcal{Q}$-closed functions). Let $(\mathcal{M}, \mathcal{Q})$ be a Lie n-algebroid and suppose $\xi \in \mathcal{C}^\infty(\mathcal{M})^k$ such that $\mathcal{Q}(\xi) = 0$. Then one can construct a representation up to homotopy $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{E}_\xi)$ of $\mathcal{M}$ on the graded vector bundle $\underline{E}_\xi = M \times (\mathbb{R}[0] \oplus \mathbb{R}[1-k]) \to M$ (i.e. $\mathbb{R}$ in degrees 0 and $k-1$, and zero otherwise). Its differential $\mathcal{D}_\xi$ is given in components by the map

$$\mathcal{D}_\xi = \sum_i \mathcal{D}_\xi^i,$$

where

$$\mathcal{D}_\xi^i : \mathcal{C}^\infty(\mathcal{M})^i \oplus \mathcal{C}^\infty(\mathcal{M})^{i-k+1} \to \mathcal{C}^\infty(\mathcal{M})^{i+1} \oplus \mathcal{C}^\infty(\mathcal{M})^{i-k+2}$$

is defined by the formula

$$\mathcal{D}_\xi^i(\zeta_1, \zeta_2) = (\mathcal{Q}(\zeta_1) + (-1)^{i-k+1}\zeta_2\xi, \mathcal{Q}(\zeta_2)).$$

If there is an element $\xi' \in \mathcal{C}^\infty(\mathcal{M})^k$ which is $\mathcal{Q}$-cohomologous to ξ, i.e. $\xi - \xi' = \mathcal{Q}(\xi'')$ for some $\xi'' \in \mathcal{C}^\infty(\mathcal{M})^{k-1}$, then the representations $\underline{E}_\xi$ and $\underline{E}_{\xi'}$ are isomorphic via the isomorphism $\mu : \underline{E}_\xi \to \underline{E}_{\xi'}$ defined in components by

$$\mu^i : \mathcal{C}^\infty(\mathcal{M})^i \oplus \mathcal{C}^\infty(\mathcal{M})^{i-k+1} \to \mathcal{C}^\infty(\mathcal{M})^i \oplus \mathcal{C}^\infty(\mathcal{M})^{i-k+1}$$

given by the formula

$$\mu^i(\zeta_1, \zeta_2) = (\zeta_1 + \zeta_2\xi'', \zeta_2).$$

Hence, one obtains a well-defined map $H^\bullet(\mathcal{M}) \to \underline{\text{Rep}}^\infty(\mathcal{M})$. In particular, if $\mathcal{M}$ is a Lie algebroid, the above construction recovers Example 3.5 in [4].

6.2   The case of (split) Lie 2-algebroids

Fix now a split Lie 2-algebroid $\mathcal{M}$, and recall that from the analysis of Section 3.3, $\mathcal{M}$ is given by the sum $Q[1] \oplus B^*[2]$ together with the anchor $\rho_Q : Q \to TM$, the skew-symmetric dull bracket $[\cdot, \cdot] : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ on the vector bundle $Q \to M$, the vector bundle map $\ell : B^* \to Q$, the Q-connection ∇ on the vector bundle $B \to M$, and the vector valued 3-form $\omega \in \Omega^3(Q, B^*)$. In particular, we have obtain the complex which forms the complex

$$B^* \xrightarrow{\ell} Q \xrightarrow{\rho_Q} TM.$$

In order to make the notation lighter, we will write $Q[1] \oplus B^*[2]$ for the split Lie 2-algebroid $(Q[1] \oplus B^*[2], \rho_Q, \ell, [\cdot, \cdot], \nabla, \omega)$.

Unravelling the data of the definition of representations up to homotopy for the special case where E is concentrated only in degree 0 yields the following characterisation.

Proposition 6.2.1. A representation of the Lie 2-algebroid $Q[1] \oplus B^*[2]$ consists of a (non-graded) vector bundle E over M, together with a Q-connection ∇ on E such that[1] :

(i) ∇ is flat, i.e. $R_\nabla = 0$ on $\Gamma(E)$,

(ii) $\nabla_{\ell(\beta)}e = 0$ for all $\beta \in \Gamma(B^*)$ and $e \in \Gamma(E)$, where $\partial_B := \ell^*$. That is, $\partial_B \circ d_\nabla = 0$ on $\Gamma(E)$.

---

[1]Note that all the objects that appear in the following equations act via the generalised wedge products that were discussed before. For example, $\partial(d_\nabla e)$ and $\omega_2(\omega_2(e))$ mean $\partial \wedge d_\nabla e$ and $\omega_2 \wedge \omega_2(e)$, respectively. This is explained in detail in the Appendix of [4].

Proof. Let $(E, \mathcal{D})$ be a representation of the Lie 2-algebroid. Due to the Leibniz rule, $\mathcal{D}$ is completely characterised by what it does on $\Gamma(E)$. By definition, it sends $\Gamma(E)$ into $\Omega^1(Q, E)$. Using the Leibniz rule once more together with the definition of the homological vector field $\mathcal{Q}$ on $\Omega^1(Q)$, for all $f \in C^\infty(M)$ and all $e \in \Gamma(E)$ yields

$$\mathcal{D}(fe) = (\rho_Q^* \mathrm{d}f) \otimes e + f\mathcal{D}(e),$$

which implies that $\mathcal{D} = \mathrm{d}_\nabla$ for a Q-connection ∇ on $\Gamma(E)$. Moreover, by definition of $\mathcal{D}$ one must have $\mathcal{D}^2(e) = 0$ for all $e \in \Gamma(E)$. On the other hand, a straightforward computation yields

$$\mathcal{D}^2(e) = \mathcal{D}(\mathrm{d}_\nabla e) = \mathrm{d}_\nabla^2 e + \partial_B(\mathrm{d}_\nabla e) \in \Omega^2(Q, E) \oplus \Gamma(B \otimes E). \qquad \square$$

Example 6.2.2 (Trivial line bundle). The trivial line bundle $\mathbb{R}[0]$ over M with Q-connection defined by

$$\mathrm{d}_\nabla f = \mathrm{d}_Q f = \rho_Q^* \mathrm{d}f$$

is a representation of the Lie 2-algebroid $Q[1] \oplus B^*[2]$. The operator $\mathcal{D}$ is given by the homological vector field $\mathcal{Q}$ and thus the cohomology induced by the representation is the Lie 2-algebroid cohomology: $H^\bullet(\mathcal{M}, \mathbb{R}) = H^\bullet(\mathcal{M})$. The shifted version of this example was used before to define general shifts of DG $\mathcal{M}$-modules.

Example 6.2.3. More generally, for all $k > 0$, the trivial vector bundle $\mathbb{R}^k$ of rank k over M with Q-connection defined component-wise as in the example above becomes a representation with cohomology $H^\bullet(\mathcal{M}, \mathbb{R}^k) = H^\bullet(\mathcal{M}) \oplus \ldots \oplus H^\bullet(\mathcal{M})$ (k-times).

Remark 6.2.4. Given a split Lie n-algebroid $A_1[1] \oplus \ldots \oplus A_n[n]$ over a smooth manifold M, with $n \geq 2$, the vector bundle $A_1 \to M$ carries a skew-symmetric dull algebroid structure induced by the 2-bracket and the anchor $\rho \colon A_1 \to TM$ given by $\mathcal{Q}(f) = \rho^* \mathrm{d}f$, for $f \in C^\infty(M)$. Hence, Proposition 6.2.1, Example 6.2.2 and Example 6.2.3 can be carried over verbatim to the general case.

A more interesting case is for representations of the form $\underline{E} = E_0 \oplus E_1 \oplus E_2$ concentrated in 3 degrees, say in degrees $-2, -1$ and 0. An explicit description of those representations is given below. The reader should note the similarity of the following proposition with the description of 2-term representations of Lie algebroids from [4].

Proposition 6.2.5. A 3-term representation up to homotopy $(\underline{E} = E_0[2] \oplus E_1[1] \oplus E_2[0], \mathcal{D})$ of $Q[1] \oplus B^*[2]$ is equivalent to the following data:

(i) A degree 1 map $\partial \colon \underline{E} \to \underline{E}$ such that $\partial^2 = 0$,

(ii) a Q-connection ∇ on the complex $\partial \colon E_\bullet \to E_{\bullet+1}$,

(iii) an element $\omega_i \in \Omega^i(Q, \underline{\mathrm{End}}^{1-i}(\underline{E}))$ for $i = 2, 3$,

(iv) an element $\phi_j \in \Gamma(B) \otimes \Omega^j(Q, \underline{\mathrm{End}}^{-j-1}(\underline{E}))$ for $j = 0, 1$

such that[1]

1. $\partial \circ \omega_2 + \mathrm{d}_\nabla^2 + \omega_2 \circ \partial = 0$,

2. $\partial \circ \phi_0 + \partial_B \circ \mathrm{d}_\nabla + \phi_0 \circ \partial = 0$,

3. $\partial \circ \omega_3 + \mathrm{d}_\nabla \circ \omega_2 + \omega_2 \circ \mathrm{d}_\nabla + \omega_3 \circ \partial = \langle \omega, \phi_0 \rangle$,

4. $\mathrm{d}_{\overline{\nabla}} \phi_0 + \partial \circ \phi_1 + \partial_B \circ \omega_2 + \phi_1 \circ \partial = 0$,

5. $\mathrm{d}_\nabla \circ \omega_3 + \omega_2 \circ \omega_2 + \omega_3 \circ \mathrm{d}_\nabla = \langle \omega, \phi_1 \rangle$,

6. $\mathrm{d}_{\overline{\nabla}} \phi_1 + \omega_2 \circ \phi_0 + \partial_B \circ \omega_3 + \phi_0 \circ \omega_2 = 0$,

7. $\phi_0 \circ \phi_0 + \partial_B \circ \phi_1 = 0$,

where $\overline{\nabla}$ is the Q-connection on $B \otimes \underline{\mathrm{End}}^{-j-1}(\underline{E})$ induced by ∇ on B and $\nabla^{\underline{\mathrm{End}}}$ on $\underline{\mathrm{End}}(\underline{E})$.

Remark 6.2.6.   1. If both of the bundles E_1 and E_2 are zero, the equations agree with those of a 1-term representation.

2. The equations in the statement can be summarised as follows:

$$[\partial, \phi_0] + \partial_B \circ \mathrm{d}_\nabla = 0, \qquad \phi_0 \circ \phi_0 + \partial_B \circ \phi_1 = 0,$$

and for all i:

$$[\partial, \omega_i] + [\mathrm{d}_\nabla, \omega_{i-1}] + \omega_2 \circ \omega_{i-2} + \omega_3 \circ \omega_{i-3} + \ldots + \omega_{i-2} \circ \omega_2 = \langle \omega, \phi_{i-3} \rangle,$$

$$\partial_B \circ \omega_{i+2} + [\partial, \phi_{i+1}] + \mathrm{d}_{\overline{\nabla}} \phi_i + \sum_{j \geq 2} [\omega_j, \phi_{i-j+1}] = 0.$$

Proof. It is enough to check how $\mathcal{D}$ acts on $\Gamma(\underline{E})$. Since $\mathcal{D}$ is of degree 1, it maps each $\Gamma(E_i)$ into the direct sum

$$\Gamma(E_{i+1}) \oplus \left(\mathcal{C}^\infty(\mathcal{M})^1 \otimes \Gamma(E_i)\right) \oplus \left(\mathcal{C}^\infty(\mathcal{M})^2 \otimes \Gamma(E_{i-1})\right) \oplus \left(\mathcal{C}^\infty(\mathcal{M})^3 \otimes \Gamma(E_{i-2})\right).$$

Considering the components of $\mathcal{D}$, this translates to the following three equations:

$$\mathcal{D}(e) = \partial(e) + d(e) \in \Gamma(E_1) \oplus \Omega^1(Q, E_0)$$

for $e \in \Gamma(E_0)$,

$$\mathcal{D}(e) = \partial(e) + d(e) + \omega_2(e) + \phi_0(e) \in \Gamma(E_2) \oplus \Omega^1(Q, E_1) \oplus \Omega^2(Q, E_0) \oplus (\Gamma(B) \otimes \Gamma(E_0))$$

for $e \in \Gamma(E_1)$, and

---

[1] In the following equations, the map $\partial_B \colon \Omega^1(Q) \to \Gamma(B)$ extends to $\partial_B \colon \Omega^k(Q) \to \Omega^{k-1}(Q, B)$ by the rule $\partial_B(\tau_1 \wedge \ldots \wedge \tau_k) = \sum_{i=1}^k (-1)^{i+1} \tau_1 \wedge \ldots \wedge \hat{\tau}_i \wedge \ldots \wedge \tau_k \otimes \partial_B \tau_i$, for $\tau_i \in \Omega^1(Q)$.

$$\mathcal{D}(e) = d(e) + \omega_2(e) + \phi_0(e) + \omega_3(e) + \phi_1(e)$$
$$\in \Omega^1(Q, E_2) \oplus \Omega^2(Q, E_1) \oplus (\Gamma(B) \otimes \Gamma(E_1))$$
$$\oplus \Omega^3(Q, E_0) \oplus \left(\Gamma(B) \otimes \Omega^1(Q, E_0)\right)$$

for $e \in \Gamma(E_2)$. Due to Lemma 3.4.1 and the Leibniz rule for $\mathcal{D}$, $\partial \in \underline{\mathrm{End}}^1(\underline{E})$, $d = \mathrm{d}_\nabla$ where ∇ are Q-connections on the vector bundles E_i for $i = 0, 1, 2$, $\omega_i \in \Omega^i(Q, \underline{\mathrm{End}}^{1-i}(\underline{E}))$ for $i = 2, 3$, and $\phi_i \in \Gamma(B) \otimes \Omega^i(Q, \underline{\mathrm{End}}^{-i-1}(\underline{E}))$ for $i = 0, 1$.

A straightforward computation and a degree count in the expansion of the equation $\mathcal{D}^2 = 0$ shows that $(\underline{E}, \partial)$ is a complex, ∇ commutes with ∂, and the equations in the statement hold. Indeed, suppose $e \in \Gamma(E_0)$. Then we have

$$\mathcal{D}^2(e) = \mathcal{D}(\partial(e) + \mathrm{d}_\nabla e) = \partial^2(e) + \mathrm{d}_\nabla(\partial(e)) + \omega_2(\partial(e)) + \phi_0(\partial(e)) + \mathcal{D}(\mathrm{d}_\nabla e).$$

In order to compute $\mathcal{D}(\mathrm{d}_\nabla e)$, we write $\mathrm{d}_\nabla e \in \Omega^1(Q, E_0)$ as a (finite) sum

$$\mathrm{d}_\nabla e = \sum_i \tau_i \otimes e_i',$$

with $\tau_i \in \Gamma(Q^*)$ and $e_i' \in \Gamma(E_0)$. Due to the linearity and the Leibniz rule of $\mathcal{D}$, we have

$$\mathcal{D}(\mathrm{d}_\nabla(e)) = \sum_i \mathcal{D}(\tau_i \otimes e_i')$$
$$= \sum_i \mathcal{Q}(\tau_i) \otimes e_i' - \tau_i \otimes \mathcal{D}(e_i')$$
$$= \sum_i (\mathrm{d}_Q \tau_i + \partial_B \tau_i) \otimes e_i' - \tau_i \otimes (\partial(e_i') + \mathrm{d}_\nabla e_i')$$
$$= \sum_i \mathrm{d}_Q \tau_i \otimes e_i' - \tau_i \otimes \mathrm{d}_\nabla e_i' + \partial_B \tau_i \otimes e_i' - \tau_i \otimes \partial(e_i')$$
$$= \sum_i \mathrm{d}_\nabla(\tau_i \otimes e_i') + \partial_B(\tau_i \otimes e_i') + \partial(\tau_i \otimes e_i')$$
$$= \mathrm{d}_\nabla \left(\sum_i \tau_i \otimes e_i' \right) + \partial_B \left(\sum_i \tau_i \otimes e_i' \right) + \partial \left(\sum_i \tau_i \otimes e_i' \right)$$
$$= \mathrm{d}_\nabla^2 e + \partial_B(\mathrm{d}_\nabla e) + \partial(\mathrm{d}_\nabla e).$$

Coupling bidegrees for $\mathcal{D}^2(e)$, we obtain

$$\mathcal{D}^2(e) = \partial^2(e) + \left(\mathrm{d}_\nabla(\partial(e)) + \partial(\mathrm{d}_\nabla e) \right) + \left(\omega_2(\partial(e)) + \mathrm{d}_\nabla^2 e \right) + \left(\phi_0(\partial(e)) + \partial_B(\mathrm{d}_\nabla e) \right)$$
$$\in \Gamma(E_2) \oplus \Omega^1(Q, E_1) \oplus \Omega^2(Q, E_0) \oplus (\Gamma(B) \otimes \Gamma(E_0)).$$

One proceeds similarly for the cases $e \in \Gamma(E_1)$ and $e \in \Gamma(E_2)$. □

6.3   Adjoint representation of a split Lie 2-algebroid

This section shows that any split Lie 2-algebroid $Q[1] \oplus B^*[2]$ admits an adjoint representation which is a 3-term representation up to homotopy. It is a generalisation of the adjoint representation of a (split) Lie 1-algebroid studied in [4].

Proposition 6.3.1. Any split Lie 2-algebroid $Q[1] \oplus B^*[2]$ admits a 3-term representation up to homotopy as follows: Choose arbitrary TM-connections on Q and B^* and denote them both by ∇. Then the structure objects are[1]

(i) the **adjoint complex** $B^*[2] \to Q[1] \to TM[0]$ with maps $-\ell$ and ρ_Q,

(ii) the two Q-connections ∇^{bas} on Q and TM, and the Q-connection ∇^* on B^* given by the split Lie 2-algebroid,

(iii) the element $\omega_2 \in \Omega^2(Q, \mathrm{Hom}(Q, B^*) \oplus \mathrm{Hom}(TM, Q))$ defined by

$$\omega_2(q_1, q_2)q_3 = -\omega(q_1, q_2, q_3) \in \Gamma(B^*) \text{ and } \omega_2(q_1, q_2)X = -R_\nabla^{\mathrm{bas}}(q_1, q_2)X \in \Gamma(Q)$$

for $q_1, q_2, q_3 \in \Gamma(Q)$ and $X \in \mathfrak{X}(M)$,

(iv) the element $\omega_3 \in \Omega^3(Q, \mathrm{Hom}(TM, B^*))$ defined by

$$\omega_3(q_1, q_2, q_3)X = -(\nabla_X \omega)(q_1, q_2, q_3) \in \Gamma(B^*)$$

for $q_1, q_2, q_3 \in \Gamma(Q)$ and $X \in \mathfrak{X}(M)$,

(v) the element $\phi_0 \in \Gamma(B) \otimes (\mathrm{Hom}(Q, B^*) \oplus \mathrm{Hom}(TM, Q))$ defined by

$$\phi_0(\beta)X = \ell(\nabla_X \beta) - \nabla_X(\ell(\beta)) \in \Gamma(Q) \text{ and } \phi_0(\beta)q = \nabla_{\rho(q)}\beta - \nabla_q^*\beta \in \Gamma(B^*)$$

for $\beta \in \Gamma(B^*), q \in \Gamma(Q), X \in \mathfrak{X}(M)$,

(vi) the element $\phi_1 \in \Gamma(B) \otimes \Omega^1(Q, \mathrm{Hom}(TM, B^*))$ defined by

$$\phi_1(\beta, q)X = \nabla_X \nabla_q^* \beta - \nabla_q^* \nabla_X \beta - \nabla_{\nabla_X q}^* \beta + \nabla_{\nabla_q^{\mathrm{bas}} X}^* \beta \in \Gamma(B^*)$$

for $\beta \in \Gamma(B^*), q \in \Gamma(Q), X \in \mathfrak{X}(M)$.

The proof can be done in two ways. First, one could check explicitly that all the conditions of a 3-representation of $Q[1] \oplus B^*[2]$ are satisfied. This is an easy but long computation and it can be found in Appendix B.3. Instead, the following section shows that given a splitting and TM-connections on the vector bundles Q and B^*, there exists an isomorphism of sheaves of $\mathcal{C}^\infty(\mathcal{M})$-modules between the adjoint module $\mathfrak{X}(\mathcal{M})$ and $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(TM[0] \oplus Q[1] \oplus B^*[2])$, such that the

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[1]Some signs are chosen so that the map given in 6.4 is an isomorphism for the differential of the adjoint module defined earlier.

objects defined above correspond to the differential $\mathcal{L}_\mathcal{Q}$. Another advantage of this approach is that it gives a precise recipe for the definition and the explicit formulas for the components of the adjoint representation of a Lie n-algebroid for general n.

Remark 6.3.2 (Adjoint representation of a Courant algebroid). The adjoint representation of a Courant algebroid $E \to M$ can be deduced from the formulas above and from Example 3.3.9. Choose a linear connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that preserves the metric underlying the Courant algebroid structure on E. As in Example 3.3.9, define the basic connection $\nabla^{\mathrm{bas}} \colon \Gamma(E) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\nabla^{\mathrm{bas}}_e X = [\rho(e), X] + \rho(\nabla_X e)$. Recall that ∇ defines as well the dull bracket $[\cdot, \cdot]$ on sections of E:

$$[e_1, e_2] = [\![e_1, e_2]\!] - \rho^* \langle \nabla. e_1, e_2 \rangle.$$

The dull bracket and the TM-connection on E defines the basic E-connection $\nabla^{\mathrm{bas}} \colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, $\nabla^{\mathrm{bas}}_{e_1} e_2 = [e_1, e_2] + \nabla_{\rho(e_2)} e_1 = [\![e_1, e_2]\!] - \rho^* \langle \nabla. e_1, e_2 \rangle + \nabla_{\rho(e_2)} e_1$. Choose in addition a TM-connection ∇ on TM. The complex is $T^*M[2] \to E[1] \to TM[0]$ with maps $-\rho^*$ and ρ. The E-connections on T^*M, E and TM are ∇^{bas^*}, ∇^{bas} and ∇^{bas} defined as above, respectively. The form $\omega_2 \in \Omega^2(E, \mathrm{Hom}(E, T^*M))$ is given by

$$\langle \omega_2(e, e'), X \rangle = \nabla_X [\![e, e']\!] - [\![\nabla_X e, e']\!] - [\![e, \nabla_X e']\!] - \nabla_{\nabla^{\mathrm{bas}}_{e'} X} e + \nabla_{\nabla^{\mathrm{bas}}_e X} e' + P^{-1} \langle \nabla_{\nabla^{\mathrm{bas}}_. X} e, e' \rangle$$

for $e, e' \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, while the second summand $\omega_2 \in \Omega^2(E, \mathrm{Hom}(TM, E))$ is given by

$$\omega_2(e, e') X = \nabla_X [e, e'] - [e, \nabla_X e'] - [\nabla_X e, e'] - \nabla_{\nabla^{\mathrm{bas}}_{e'} X} e + \nabla_{\nabla^{\mathrm{bas}}_e X} e'.$$

This terms can be compared with the components of the representation up to homotopy of a Lie algebroid $A \to M$, after the choice of a TM-connection on A, see [42]. The remaining terms are given by (iv), (v) and (vi) in Proposition 6.3.1, and as they do not seem more instructive than the general form in the proposition, they are not computed in more detail here.

6.4   Adjoint module vs adjoint representation

Recall that for a split $[n]$-manifold $\mathcal{M} = \bigoplus E_i[i]$, the space of vector fields over $\mathcal{M}$ is generated as a $\mathcal{C}^\infty(\mathcal{M})$-module by two special kinds of vector fields. Namely, the degree $-i$ vector fields $\hat{e}$ for $e \in \Gamma(E_i)$, and the family of vector fields $\nabla^1_X \oplus \ldots \oplus \nabla^n_X$ for $X \in \mathfrak{X}(M)$ and a choice of TM-connections ∇^i on the vector bundles E_i.

   Consider now a Lie 2-algebroid $(\mathcal{M}, \mathcal{Q})$ together with a splitting $\mathcal{M} \simeq Q[1] \oplus B^*[2]$ and a choice of TM-connections ∇^{B^*} and ∇^Q on B^* and Q, respectively. These choices give as follows the adjoint representation ad_∇, whose complex is given by $TM[0] \oplus Q[1] \oplus B^*[2]$. Define a map $\mu_\nabla \colon \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(TM[0] \oplus Q[1] \oplus B^*[2]) \to \mathfrak{X}(\mathcal{M})$ on the generators by

$$\Gamma(B^*) \ni \beta \mapsto \hat{\beta}, \qquad \Gamma(Q) \ni q \mapsto \hat{q}, \qquad \mathfrak{X}(M) \ni X \mapsto \nabla^{B^*}_X \oplus \nabla^Q_X$$

and extend $\mathcal{C}^\infty(\mathcal{M})$-linearly to the whole space to obtain a degree-preserving isomorphism of sheaves of $\mathcal{C}^\infty(\mathcal{M})$-modules. A straightforward computation shows that

$$\mathcal{L}_\mathcal{Q}(\hat{\beta}) = \mu_\nabla\left(-\ell(\beta) + \mathrm{d}_{\nabla^*}\beta\right),$$

$$\mathcal{L}_\mathcal{Q}(\hat{q}) = \mu_\nabla\left(\rho_Q(q) + \mathrm{d}_{\nabla^{\mathrm{bas}}}q + \omega_2(\cdot,\cdot)q + \phi_0(\cdot)q\right),$$

$$\mathcal{L}_\mathcal{Q}(\nabla_X^{B^*}\oplus\nabla_X^Q) = \mu_\nabla\left(\mathrm{d}_{\nabla^{\mathrm{bas}}}X + \phi_0(\cdot)X + \omega_2(\cdot,\cdot)X + \omega_3(\cdot,\cdot,\cdot)X + \phi_1(\cdot,\cdot)X\right)$$

and therefore, the objects in the statement of Proposition 6.3.1 define the differential $\mathcal{D}_{\mathrm{ad}_\nabla} := \mu_\nabla^{-1}\circ\mathcal{L}_\mathcal{Q}\circ\mu_\nabla$ of a 3-representation of $Q[1]\oplus B^*[2]$, called the **adjoint representation of the Lie 2-algebroid** and denoted as $(\mathrm{ad}_\nabla, \mathcal{D}_{\mathrm{ad}_\nabla})$. The adjoint representation is hence, up to isomorphism, independent of the choice of splitting and connections (see the following section for the precise transformations), and gives so a well-defined class $\mathrm{ad}\in\underline{\mathrm{Rep}}_L^\infty(\mathcal{M})$.

Due to the result above, one can also define the **coadjoint representation of a Lie 2-algebroid** $(\mathcal{M}, \mathcal{Q})$ as the isomorphism class $\mathrm{ad}^*\in\underline{\mathrm{Rep}}_L^\infty(\mathcal{M})$. To find an explicit representative of ad^*, suppose that $Q[1]\oplus B^*[2]$ is a splitting of $\mathcal{M}$, and consider its adjoint representation ad_∇ as above for some choice of TM-connections ∇ on B^* and Q. Recall that given a representation up to homotopy $(\underline{E}, \mathcal{D})$ of a general Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$, its dual $\underline{E}^*$ becomes a representation up to homotopy with operator $\mathcal{D}^*$ characterised by the formula

$$\mathcal{Q}(\xi\wedge\xi') = \mathcal{D}^*(\xi)\wedge\xi' + (-1)^{|\xi|}\xi\wedge\mathcal{D}(\xi'),$$

for all $\xi\in\mathcal{C}^\infty(\mathcal{M})\otimes\Gamma(\underline{E}^*)$ and $\xi'\in\mathcal{C}^\infty(\mathcal{M})\otimes\Gamma(\underline{E})$. Here, $\wedge = \wedge_{\langle\cdot,\cdot\rangle}$, where $\langle\cdot,\cdot\rangle$ is the pairing of $\underline{E}$ with $\underline{E}^*$. Unravelling the definition of the dual for the left representation ad_∇, one finds that the structure differential of $\mathrm{ad}_\nabla^* = \mathcal{C}^\infty(\mathcal{M})\otimes\Gamma(B[-2]\oplus Q^*[-1]\oplus TM[0])$ is given by the following objects:

1. the **coadjoint complex** $T^*M\to Q^*\to B$ obtained by $-\rho_Q^*$ and $-\ell^*$,

2. the Q-connections ∇ on B and $\nabla^{\mathrm{bas},*}$ on Q^* and T^*M,

3. the elements

$$\begin{aligned}
\omega_2^*(q_1, q_2)\tau &= \tau\circ\omega_2(q_1, q_2), & \omega_2^*(q_1, q_2)b &= -b\circ\omega_2(q_1, q_2), \\
\phi_0^*(\beta)\tau &= \tau\circ\phi_0(\beta), & \phi_0^*(\beta)b &= -b\circ\phi_0(\beta), \\
\omega_3^*(q_1, q_2, q_3)b &= -b\circ\omega_3(q_1, q_2, q_3), & \phi_1^*(\beta, q)b &= -b\circ\phi_1(\beta, q),
\end{aligned}$$

for all $q, q_1, q_2, q_3\in\Gamma(Q), \tau\in\Gamma(Q^*), b\in\Gamma(B)$ and $\beta\in\Gamma(B^*)$.

Remark 6.4.1. The coadjoint representation can also be obtained from the coadjoint module $\Omega^1(\mathcal{M})$ by the right $\mathcal{C}^\infty(\mathcal{M})$-module isomorphism $\mu_\nabla^\star: \Omega^1(\mathcal{M})\to\Gamma(B[-2]\oplus Q^*[-1]\oplus T^*M[0])\otimes\mathcal{C}^\infty(\mathcal{M})$ which is dual to $\mu_\nabla: \mathcal{C}^\infty(\mathcal{M})\otimes\Gamma(TM[0]\oplus Q[1]\oplus B^*[2])\to\mathfrak{X}(\mathcal{M})$ above. Explicitly, it is defined as the pull-back map $\mu_\nabla^\star(\omega) = \omega\circ\mu_\nabla$ for all $\omega\in\Omega^1(\mathcal{M})$, whose inverse is given on the generators

by $\Gamma(B[-2] \oplus Q^*[-1] \oplus T^*M[0]) \otimes \mathcal{C}^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M})$,

$$\Gamma(B) \ni b \mapsto \mathrm{d}b - \mathrm{d}_{\nabla^*}b, \quad \Gamma(Q^*) \ni \tau \mapsto \mathrm{d}\tau - \mathrm{d}_{\nabla^*}\tau, \quad \text{and} \quad \Omega^1(M) \ni \theta \mapsto \theta.$$

6.5 Coordinate transformation of the adjoint representation

The adjoint representation up to homotopy of a Lie 2-algebroid was constructed after a choice of splitting and TM-connections. This section explains how the adjoint representation transforms under different choices.

First, a morphism of 3-representations of a split Lie 2-algebroid can be described as follows.

Proposition 6.5.1. Let $(\underline{E}, \mathcal{D}_E)$ and $(\underline{F}, \mathcal{D}_F)$ be 3-term representations up to homotopy of the split Lie 2-algebroid $Q[1] \oplus B^*[2]$. A morphism $\mu \colon \underline{E} \to \underline{F}$ is equivalent to the following data:

  (i) For each $i = 0, 1, 2$, an element $\mu_i \in \Omega^i(Q, \underline{\mathrm{Hom}}^{-i}(\underline{E}, \underline{F}))$.

  (ii) An element $\mu_0^b \in \Gamma(B \otimes \underline{\mathrm{Hom}}^{-2}(\underline{E}, \underline{F}))$.

The above objects are subject to the relations

  1. $[\partial, \mu_i] + [\mathrm{d}_\nabla, \mu_{i-1}] + \displaystyle\sum_{j+k=i, i \geq 2} [\omega_j, \mu_k] = \langle \omega, \mu_{i-3}^b \rangle$,

  2. $[\partial, \mu_0^b] + [\phi_0, \mu_0] + \partial_B \circ \mu_1 = 0$,

  3. $\mathrm{d}_{\overline{\nabla}}\mu_0^b + [\phi_0, \mu_1] + [\phi_1, \mu_0] + \partial_B \circ \mu_2 = 0$.

Proof. As before it suffices to check how μ acts on $\Gamma(\underline{E})$, by the same arguments. Then it must be of the type

$$\mu = \mu_0 + \mu_1 + \mu_2 + \mu_0^b,$$

where $\mu_i \in \Omega^i(Q, \underline{\mathrm{Hom}}^{-i}(\underline{E}, \underline{F}))$ and $\mu^b \in \Gamma(B) \otimes \Gamma(\underline{\mathrm{Hom}}^{-2}(\underline{E}, \underline{F}))$. The three equations in the statement come from the expansion of $\mu \circ \mathcal{D}_E = \mathcal{D}_F \circ \mu$ when μ is written in terms of the components defined above. We check here the case $\mu \circ \mathcal{D}_E(e) = \mathcal{D}_F \circ \mu(e)$ for $e \in \Gamma(E_0)$ and omit the rest. One the one hand, we have that

$$\mu(\mathcal{D}_E(e)) = \mu(\partial(e)) + \mu(\mathrm{d}_\nabla e) = \mu_0(\partial(e)) + \mu_1(\partial(e)) + \mu(\mathrm{d}_\nabla e),$$

while on the other hand

$$\mathcal{D}_F(\mu(e)) = \mathcal{D}_F(\mu_0(e)) = \partial(\mu_0(e)) + \mathrm{d}_\nabla(\mu_0(e)).$$

Write $\mathrm{d}_\nabla e \in \Omega^1(Q, E_0)$ as a (finite) sum of the form

$$\mathrm{d}_\nabla e = \sum_i \tau_i \otimes e_i',$$

with $\tau_i \in \Gamma(Q^*)$ and $e_i' \in \Gamma(E_0)$. Then we have

$$\mu(\mathrm{d}_\nabla e) = \mu\left(\sum_i \tau_i \otimes e_i'\right) = \sum_i \tau_i \otimes \mu(e_i') = \sum_i \tau_i \otimes \mu_0(e_i') = \sum_i \mu_0 \wedge (\tau_i \otimes e_i') = \mu_0(\mathrm{d}_\nabla e).$$

Equalising $\mu(\mathcal{D}_{\underline{E}}(e))$ with $\mathcal{D}_{\underline{F}}(\mu(e))$ and comparing degrees, we obtain

$$\mu_0(\partial(e)) = \partial(\mu_0(e)) \qquad \text{and} \qquad \mathrm{d}_\nabla(\mu_0(e)) - \mu_0(\mathrm{d}_\nabla e) = \mu_1(\partial(e)). \qquad \square$$

The transformation of $\mathrm{ad} \in \underline{\mathrm{Rep}}^\infty(\mathcal{M})$ for a fixed splitting $Q[1] \oplus B^*[2]$ of $\mathcal{M}$ and different choices of TM-connections is given by their difference. More precisely, let ∇ and ∇' be the two TM-connections. Then the map $\mu = \mu_{\nabla'}^{-1} \circ \mu_\nabla \colon \mathrm{ad}_\nabla \to \mathrm{ad}_{\nabla'}$ is defined by $\mu = \mu_0 + \mu_1 + \mu_0^b$, where

$$\mu_0 = \mathrm{id}$$
$$\mu_1(q)X = \nabla'_X q - \nabla_X q$$
$$\mu_0^b(\beta)X = \nabla'_X \beta - \nabla_X \beta,$$

for $X \in \mathfrak{X}(M)$, $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$. The equations in Proposition 6.5.1 are automatically satisfied since by construction

$$\mathcal{D}_{\mathrm{ad}_{\nabla'}} \circ \mu = \mathcal{D}_{\mathrm{ad}_{\nabla'}} \circ \mu_{\nabla'}^{-1} \circ \mu_\nabla = \mu_{\nabla'}^{-1} \circ \mathcal{L}_\mathcal{Q} \circ \mu_\nabla = \mu_{\nabla'}^{-1} \circ \mu_\nabla \circ \mathcal{D}_{\mathrm{ad}_\nabla} = \mu \circ \mathcal{D}_{\mathrm{ad}_\nabla}.$$

This yields the following result.

Proposition 6.5.2. Given two pairs of TM-connections on the bundles B^* and Q, the difference of the two TM-connections induces the isomorphism $\mu \colon \mathrm{ad}_\nabla \to \mathrm{ad}_{\nabla'}$ between the corresponding adjoint representations. Explicitly, it is given by $\mu = \mathrm{id} \oplus \left(\nabla' - \nabla\right)$.

The next step is to show how the adjoint representation transforms after a change of splitting of the Lie 2-algebroid. Fix a Lie 2-algebroid $(\mathcal{M}, Q)$ over the smooth manifold M and choose a splitting $Q[1] \oplus B^*[2]$, with structure objects $(\ell, \rho, [\cdot, \cdot]_1, \nabla^1, \omega^1)$ as before. Recall that a change of splitting does not change the vector bundles B^* and Q, and it is equivalent to a section $\sigma \in \Omega^2(Q, B^*)$. The induced isomorphism of [2]-manifolds over the identity on M is given by: $\mathcal{F}_\sigma^\star(\tau) = \tau$ for all $\tau \in \Gamma(Q^*)$ and $\mathcal{F}_\sigma^\star(b) = b + \sigma^\star b \in \Gamma(B) \oplus \Omega^2(Q)$ for all $b \in \Gamma(B)$. If $(\ell, \rho, [\cdot, \cdot]_2, \nabla^2, \omega^2)$ is the structure objects of the second splitting, then the compatibility of σ with the homological vector fields reads the following:

- The skew-symmetric dull brackets are related by $[q_1, q_2]_2 = [q_1, q_2]_1 - \ell(\sigma(q_1, q_2))$.

- The connections are related by $\nabla_q^2 b = \nabla_q^1 b + \partial_B \langle \sigma(q, \cdot), b \rangle$, or equivalently on the dual by $\nabla_q^{2*}\beta = \nabla_q^{1*}\beta - \sigma(q, \ell(\beta))$.

- The curvature terms are related by $\omega^2 = \omega^1 + \mathrm{d}_{2,\nabla^1}\sigma$, where the operator

$$\mathrm{d}_{2,\nabla^1}\sigma \colon \Omega^\bullet(Q, B^*) \to \Omega^{\bullet+1}(Q, B^*)$$

is defined by the usual Koszul formula using the dull bracket $[\cdot,\cdot]_2$ and the connection ∇^{1*}.

The above equations give the following identities between the structure data for the adjoint representations[1] ad^1_∇ and ad^2_∇.

Lemma 6.5.3. *Let* $q, q_1, q_2 \in \Gamma(Q), \beta \in \Gamma(B^*)$ *and* $X \in \mathfrak{X}(M)$. *Then*

(i) $\ell_2 = \ell_1$ *and* $\rho_2 = \rho_1$.

(ii) $\nabla^{2,\mathrm{bas}}_{q_1} q_2 = \nabla^{1,\mathrm{bas}}_{q_1} q_2 - \ell(\sigma(q_1, q_2))$

$\nabla^{2,\mathrm{bas}}_q X = \nabla^{1,\mathrm{bas}}_q X$

$\nabla^{2,*}_q \beta = \nabla^{1,*}_q \beta - \sigma(q, \ell(\beta))$.

(iii) $\omega^2_2(q_1, q_2)q_3 = \omega^1_2(q_1, q_2)q_3 + \mathrm{d}_{2,\nabla^1}\sigma(q_1, q_2, q_3)$

$\omega^2_2(q_1, q_2)X = \omega^1_2(q_1, q_2)X + \nabla_X(\ell(\sigma(q_1, q_2))) - \ell(\sigma(q_1, \nabla_X q_2)) + \ell(\sigma(q_2, \nabla_X q_1))$.

(iv) $\omega^2_3(q_1, q_2, q_3)X = \omega^1_3(q_1, q_2, q_3)X + (\nabla_X(\mathrm{d}_{2,\nabla^1}\sigma))(q_1, q_2, q_3)$.

(v) $\phi^2_0(\beta)q = \phi^1_0(\beta)q + \sigma(q, \ell(\beta))$

$\phi^2_0(\beta)X = \phi^1_0(\beta)X$.

(vi) $\phi^2_1(\beta, q)X = \phi^1_1(\beta, q)X - \sigma(\nabla_X q_1, \ell(\beta)) - \sigma(q, \ell(\nabla_X \beta)) + \nabla_X(\sigma(q, \ell(\beta)))$.

Consider now two Lie n-algebroids $\mathcal{M}_1$ and $\mathcal{M}_2$ over M, and an isomorphism

$$\mathcal{F}\colon (\mathcal{M}_1, \mathcal{Q}_1) \to (\mathcal{M}_2, \mathcal{Q}_2)$$

given by the maps $\mathcal{F}_Q\colon Q_1 \to Q_2$, $\mathcal{F}_B\colon B^*_1 \to B^*_2$, and $\mathcal{F}_0\colon \wedge^2 Q_1 \to B^*_2$. Recall that a 0-morphism between two representations up to homotopy $(\underline{E}_1, \mathcal{D}_1)$ and $(\underline{E}_2, \mathcal{D}_2)$ of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, is given by a degree 0 map

$$\mu\colon \mathcal{C}^\infty(\mathcal{M}_2) \otimes \Gamma(\underline{E}_2) \to \mathcal{C}^\infty(\mathcal{M}_1) \otimes \Gamma(\underline{E}_1),$$

which is $\mathcal{C}^\infty(\mathcal{M}_2)$-linear: $\mu(\xi \otimes e) = \mathcal{F}^\star \xi \otimes \mu(e)$ for all $\xi \in \mathcal{C}^\infty(\mathcal{M}_2)$ and $e \in \Gamma(\underline{E}_2)$, and makes the following diagram commute

$$
\begin{array}{ccc}
\mathcal{C}^\infty(\mathcal{M}_2) \otimes \Gamma(\underline{E}_2) & \xrightarrow{\;\;\mu\;\;} & \mathcal{C}^\infty(\mathcal{M}_1) \otimes \Gamma(\underline{E}_1) \\
\Big\downarrow{\scriptstyle \mathcal{D}_2} & & \Big\downarrow{\scriptstyle \mathcal{D}_1} \\
\mathcal{C}^\infty(\mathcal{M}_2) \otimes \Gamma(\underline{E}_2) & \xrightarrow{\;\;\mu\;\;} & \mathcal{C}^\infty(\mathcal{M}_1) \otimes \Gamma(\underline{E}_1).
\end{array}
$$

The usual analysis as before implies that μ must be given by a morphism of complexes $\mu_0\colon (\underline{E}_2, \partial_2) \to (\underline{E}_1, \partial_1)$ and elements

$$\mu_1 \in \Omega^1(Q_1, \underline{\mathrm{Hom}}^{-1}(\underline{E}_2, \underline{E}_1)),$$

---

[1] Note that the two pairs of TM-connections are identical

$$\mu_2 \in \Omega^2(Q_1, \underline{\mathrm{Hom}}^{-2}(\underline{E}_2, \underline{E}_1)),$$

$$\mu_0^b \in \Gamma(B) \otimes \Gamma(\underline{\mathrm{Hom}}^{-2}(\underline{E}_2, \underline{E}_1)),$$

which satisfy equations similar to the set of equations in Proposition 6.5.1.

A change of splitting of the Lie 2-algebroid transforms as follows the adjoint representation. Since changes of choices of connections are now fully understood, choose the same connection for both splittings $\mathcal{M}_1 \simeq Q[1] \oplus B^*[2] \simeq \mathcal{M}_2$. Suppose that $\sigma \in \Omega^2(Q, B^*)$ is the change of splitting and denote by $\mathcal{F}_\sigma$ the induced isomorphism of the split Lie 2-algebroids whose components are given by $\mathcal{F}^\star_{\sigma,Q} = \mathrm{id}_{Q^*}, \mathcal{F}^\star_{\sigma,B} = \mathrm{id}_B, \mathcal{F}^\star_{\sigma,0} = \sigma^\star$. The composition map $\mu^\sigma : \mathrm{ad}^1_\nabla \to \mathfrak{X}(\mathcal{M}) \to \mathrm{ad}^2_\nabla$ is given in components by

$$\mu_0^\sigma = \mathrm{id}$$
$$\mu_1^\sigma(q_1)q_2 = \sigma(q_1, q_2)$$
$$\mu_2^\sigma(q_1, q_2)X = (\nabla_X \sigma)(q_1, q_2).$$

A similar argument as before implies that μ^σ is a morphism between the two adjoint representations and therefore the following result follows.

Proposition 6.5.4. Given two splittings of a Lie 2-algebroid with induced change of splitting $\sigma \in \Omega^2(Q, B^*)$ and a pair of TM-connections on the vector bundles B^* and Q, the corresponding adjoint representations are isomorphic via $\mu = \mathrm{id} \oplus \sigma \oplus \nabla.\sigma$.

6.6 Adjoint representation of a Lie n-algebroid

The construction of the adjoint representation up to homotopy of a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ for general $n \in \mathbb{N}$ is similar to the case $n = 2$. Specifically, choose a splitting $\mathcal{M} \simeq \bigoplus_{i=1}^n A_i[i]$ and TM-connections ∇^{A_i} on the vector bundles $A_i \to M$ for all $i = 1, \ldots, n$. Then there is an induced isomorphism of $\mathcal{C}^\infty(\mathcal{M})$-modules

$$\mu_\nabla : \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(TM[0] \oplus A_1[1] \oplus \ldots \oplus A_n[n]) \to \mathfrak{X}(\mathcal{M}),$$

which at the level of generators is given by

$$\Gamma(A_i) \ni a \mapsto \hat{a} \quad \text{and} \quad \mathfrak{X}(M) \ni X \mapsto \nabla^{A_n}_X \oplus \ldots \oplus \nabla^{A_1}_X.$$

The isomorphism μ is used to transfer the differential $\pounds_\mathcal{Q}$ of the adjoint module $\mathfrak{X}(\mathcal{M})$ to the differential $\mathcal{D}_{\mathrm{ad}_\nabla} := \mu_\nabla^{-1} \circ \pounds_\mathcal{Q} \circ \mu_\nabla$ on $\mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(TM[0] \oplus A_1[1] \oplus \ldots \oplus A_n[n])$. We obtain thus the **adjoint representation of a Lie n-algebroid** as the equivalence class of $(\mathrm{ad}_\nabla, \mathcal{D}_{\mathrm{ad}_\nabla})$ and the **coadjoint representation** as the equivalence class of its dual $(\mathrm{ad}^*_\nabla, \mathcal{D}^*_{\mathrm{ad}_\nabla})$. Moreover, a formula similar to the transformation of the adjoint representation of a Lie 2-algebroid from Proposition 6.5.2 holds also for general Lie n-algebroids, $n \in \mathbb{N}$, equipped with two choices of TM-connections

on the vector bundles $A_i \to M$, $i = 1, \ldots, n$. This is proved in Proposition 8.6.1 and explained in Remark 8.6.2.

6.7 The Weil algebra of a Lie n-algebroid

In this section, we analyse in detail the Weil algebra of a Lie n-algebroid $(\mathcal{M}, \mathcal{Q})$ using the adjoint representation from the previous section. Suppose $U \subset M$ is a coordinate chart of $\mathcal{M}$ with graded coordinates $\xi_i^1, \ldots, \xi_i^{r_i}$ of degree i, for all $i = 1, \ldots, n$. Recall from Section 4.2, that the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ is an $[n]$-manifold over TM [71, 73], whose local generators over $TU \subset TM$ are given in degree zero by $\mathcal{C}_{TU}^\infty(T\mathcal{M})^0 = C^\infty(TU)$ and in degree i by $\xi_i^1, \ldots, \xi_i^{r_i}, \mathrm{d}\xi_i^1, \ldots, \mathrm{d}\xi_i^{r_i}$. The shifted tangent bundle $T[1]\mathcal{M}$ is an $[n+1]$-manifold over M, with the following local generators over U:

- **degree 0:** $C^\infty(U)$

- **degree 1:** $\xi_1^1, \ldots, \xi_1^{r_1}, \Omega^1(U)$

- **degree 2:** $\xi_2^1, \ldots, \xi_2^{r_2}, \mathrm{d}\xi_1^1, \ldots, \mathrm{d}\xi_1^{r_1}$

   $\vdots$

- **degree n:** $\xi_n^1, \ldots, \xi_n^{r_n}, \mathrm{d}\xi_{n-1}^1, \ldots, \mathrm{d}\xi_{n-1}^{r_{n-1}}$

- **degree $n+1$:** $\mathrm{d}\xi_n^1, \ldots, \mathrm{d}\xi_n^{r_n}$.

In other words, the structure sheaf of $T[1]\mathcal{M}$ assigns to every coordinate domain $(U, x^1, \ldots, x^m)$ of M that trivialises $\mathcal{M}$, the space

$$\mathcal{C}_U^\infty(T[1]\mathcal{M}) = \bigoplus_i \underbrace{\mathcal{C}_U^\infty(\mathcal{M})^i}_{(0,i)} \left\langle \underbrace{(\mathrm{d}x^k)_{k=1}^m}_{(1,0)}, \underbrace{(\mathrm{d}\xi_1^k)_{k=1}^{r_1}}_{(1,1)}, \ldots, \underbrace{(\mathrm{d}\xi_n^k)_{k=1}^{r_n}}_{(1,n)} \right\rangle,$$

as in Section 4.2.

In the case of a Lie 1-algebroid $A \to M$, this is the Weil algebra from [71, 73]. For the 1-algebroid case, see also [4] for an approach with without the language of supergeometry. Here we extend the approach of [4] in the case of split Lie n-algebroids for general n.

Suppose first that $\mathcal{M} = Q[1] \oplus B^*[2]$ is a split Lie 2-algebroid and consider two TM-connections on the vector bundles Q and B^*, both denoted by ∇. Recall from Section 6.4 the (non-canonical) isomorphism of DG $\mathcal{M}$-modules

$$\mathfrak{X}(\mathcal{M}) \cong \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(TM[0] \oplus Q[1] \oplus B^*[2]).$$

This implies that

$$\Omega^1(\mathcal{M}) \cong \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(B[-2] \oplus Q^*[-1] \oplus T^*M[0])$$

as (left) DG $\mathcal{M}$-modules, and thus the generators of the Weil algebra can be identified with

$$\underbrace{\mathcal{C}^\infty(\mathcal{M})^t}_{(0,t)}, \underbrace{\Gamma(\wedge^u T^* M)}_{(u,0)}, \underbrace{\Gamma(S^v Q^*)}_{(v,v)}, \underbrace{\Gamma(\wedge^w B)}_{(w,2w)}.$$

Using also that $\mathcal{C}^\infty(\mathcal{M})^t = \bigoplus_{t=r+2s} \Gamma(\wedge^r Q^*) \otimes \Gamma(S^s B)$, the space of (p,q)-forms is decomposed as

$$\begin{aligned}
W^{p,q}(\mathcal{M}, \nabla) &= \bigoplus_{\substack{q=t+v+2w \\ p=u+w+v}} \mathcal{C}^\infty(\mathcal{M})^t \otimes \Gamma\left(\wedge^u T^* M \otimes S^v Q^* \otimes \wedge^w B\right) \\
&= \bigoplus_{\substack{q=r+2s+v+2w \\ p=u+w+v}} \Gamma\left(\wedge^u T^* M \otimes \wedge^r Q^* \otimes S^v Q^* \otimes \wedge^w B \otimes S^s B\right).
\end{aligned}$$

Therefore, after a choice of splitting and TM-connections ∇ on Q and B^*, the total space of the Weil algebra of $\mathcal{M}$ can be written as

$$W(\mathcal{M}, \nabla) = \bigoplus_{r,s,u,v,w} \Gamma\left(\wedge^u T^* M \otimes \wedge^r Q^* \otimes S^v Q^* \otimes \wedge^w B \otimes S^s B\right).$$

The next step is to express the differentials $\pounds_{\mathcal{Q}}$ and d on $W(\mathcal{M}, \nabla)$ in terms of the two TM-connections ∇. For the vertical differential, recall that by definition the p-th column of the double complex $W(\mathcal{M}, \nabla)$ equals the space of p-forms $\Omega^p(\mathcal{M})$ on $\mathcal{M}$ with differential given by the Lie derivative $\pounds_{\mathcal{Q}}$. Due to the identification of DG $\mathcal{M}$-modules

$$\Omega^p(\mathcal{M}) = \Omega^1(\mathcal{M}) \wedge \ldots \wedge \Omega^1(\mathcal{M}) = \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\mathrm{ad}^*_\nabla \wedge \ldots \wedge \mathrm{ad}^*_\nabla)$$

(p-times) and the Leibniz identity for $\pounds_{\mathcal{Q}}$, it follows that the p-th column of $W(\mathcal{M}, \nabla)$ becomes the p-symmetric power of the coadjoint representation $\underline{S}^p(\mathrm{ad}^*_\nabla)$ and $\pounds_{\mathcal{Q}} = \mathcal{D}_{\underline{S}^p(\mathrm{ad}^*_\nabla)}$.

The horizontal differential d is built from two 2-representations of the tangent Lie algebroid TM, namely the dualisation of the TM-representations on the graded vector bundles $\underline{E}_Q = Q[0] \oplus Q[-1]$ and $\underline{E}_{B^*} = B^*[0] \oplus B^*[-1]$ whose differentials are given by the chosen TM-connections $(\mathrm{id}_Q, \nabla, R_\nabla)$ and $(\mathrm{id}_{B^*}, \nabla, R_\nabla)$, respectively. Indeed, suppose first that $\tau \in \Gamma(Q^*)$ and $b \in \Gamma(B)$ are functions on $\mathcal{M}$, i.e. 0-forms. Then from Remark 6.4.1, it follows that d acts via

$$\mathrm{d}\tau = \tau + \mathrm{d}_{\nabla^*}\tau \qquad \text{and} \qquad \mathrm{d}b = b + \mathrm{d}_{\nabla^*}b. \tag{6.1}$$

If now $\tau \in \Gamma(Q^*), b \in \Gamma(B)$ are 1-forms on $\mathcal{M}$, then

$$\mathrm{d}\tau = \mathrm{d}(\tau + \mathrm{d}_{\nabla^*}\tau - \mathrm{d}_{\nabla^*}\tau) = \mathrm{d}^2\tau - \mathrm{d}(\mathrm{d}_{\nabla^*}\tau) = \mathrm{d}_{\nabla^*}\tau - \mathrm{d}^2_{\nabla^*}\tau,$$

$$\mathrm{d}b = \mathrm{d}(b + \mathrm{d}_{\nabla^*}b - \mathrm{d}_{\nabla^*}b) = \mathrm{d}^2 b - \mathrm{d}(\mathrm{d}_{\nabla^*}b) = \mathrm{d}_{\nabla^*}b - \mathrm{d}^2_{\nabla^*}b,$$

where in both lines the second equation uses the formulae in (6.1) and the last equation can be

seen by a direct computation, e.g., in local coordinates.

Remark 6.7.1. Note that if $B^* = 0$, i.e. $\mathcal{M}$ is an ordinary Lie algebroid $A \to M$, the above construction recovers (up to isomorphism) the connection version of the Weil algebra $W(A, \nabla)$ from [3, 4, 73].

In the general case of a split Lie n-algebroid $\mathcal{M} = A_1[1] \oplus \ldots \oplus A_n[n]$ with a choice of TM-connections on all the bundles A_i, one may apply the same procedure as above to obtain the (non-canonical) DG $\mathcal{M}$-module isomorphisms

$$\mathfrak{X}(\mathcal{M}) \cong \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(TM[0] \oplus A_1[1] \oplus \ldots \oplus A_n[n])$$

$$\Omega^1(\mathcal{M}) \cong \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(A_n^*[-n] \oplus \ldots \oplus A_1^*[-1] \oplus T^*M[0]),$$

and hence the identification of the generators of the Weil algebra with

$$\underbrace{\mathcal{C}^\infty(\mathcal{M})^t}_{(0,t)}, \underbrace{\Gamma(\wedge^u T^*M)}_{(u,0)}, \underbrace{\Gamma(S^{v_1} A_1^*)}_{(v_1,v_1)}, \underbrace{\Gamma(\wedge^{v_2} A_2^*)}_{(v_2,2v_2)}, \ldots, \underbrace{\Gamma(\wedge^{v_n} A_n^*)}_{(v_n,nv_n)}.$$

This then yields

$$W^{p,q}(\mathcal{M}, \nabla) = \bigoplus_{\substack{q=t+v_1+2v_2+\ldots \\ p=u+v_1+v_2+\ldots}} \mathcal{C}^\infty(\mathcal{M})^t \otimes \Gamma\left(\wedge^u T^*M \otimes S^{v_1} A_1^* \otimes \wedge^{v_2} A_2^* \otimes \ldots\right)$$

$$= \bigoplus_{\substack{q=r_1+v_1+2r_2+2v_2+\ldots \\ p=u+v_1+v_2+\ldots}} \Gamma\left(\wedge^u T^*M \otimes \wedge^{r_1} A_1^* \otimes S^{v_1} A_1^* \otimes S^{r_2} A_2^* \otimes \wedge^{v_2} A_2^* \otimes \ldots\right).$$

Similar considerations as before imply that the p-th column of $W(\mathcal{M}, \nabla)$ is given by $\underline{S}^p(\mathrm{ad}_\nabla^*)$ with $\mathcal{L}_Q = \mathcal{D}_{\underline{S}^p(\mathrm{ad}_\nabla^*)}$, and that d is built again by the dualisation of the 2-representations of TM on the graded vector bundles $\underline{E}_{A_i} = A_i[0] \oplus A_i[-1]$, for $i = 1, \ldots, n$, whose differentials are given by $(\mathrm{id}_{A_i}, \nabla, R_\nabla)$.

6.8   Poisson Lie algebroids of low degree

This section describes in detail the degree $-n$ (anti-)morphism $\sharp \colon \mathrm{ad}_\nabla^* \to \mathrm{ad}_\nabla$ of (left) right n-representations in the case of Poisson Lie n-algebroids with a degree $-n$ bracket for $n = 0, 1, 2$. Recall that the map $\sharp$ sends an exact 1-form $\mathrm{d}\xi$ of the graded manifold $\mathcal{M}$ to the vector field $\{\xi, \cdot\}$.

First, consider a Poisson Lie 0-algebroid, i.e. a usual Poisson manifold $(M, \{\cdot, \cdot\})$. Then the Lie 0-algebroid is just M, with a trivial homological vector field – it can be thought of as a trivial Lie algebroid $A = 0 \times M \to M$ with trivial differential $\mathrm{d}_A = 0$, and consequently trivial homological vector field. The coadjoint and adjoint representations are just the vector bundles $T^*M[0]$ and $TM[0]$, respectively, with zero module differentials, and the map $\sharp$ simply becomes the usual vector bundle map induced by the Poisson bivector field that corresponds to the Poisson bracket

$$\sharp \colon T^*M[0] \to TM[0].$$

Consider a Lie algebroid $A \to M$ with anchor $\rho\colon A \to TM$ and a linear Poisson structure $\{\cdot\,,\cdot\}$, i.e. a Lie algebroid structure on the dual $A^* \to M$. As we explained before, the latter is equivalent to a Poisson bracket of degree -1 on the $[1]$-manifold $A[1]$ and $(A[1], \{\cdot\,,\cdot\})$ is a Poisson Lie algebroid if and only if (A, A^*) is a Lie bialgebroid [69]. Recall also that the correspondence is given by $\rho'\colon A^* \to TM$, $\alpha \mapsto \{\alpha, \cdot\}|_{C^\infty(M)}$ and $[\cdot\,,\cdot]_* := \{\cdot\,,\cdot\}|_{\Omega^1(A)\times\Omega^1(A)}$. After a choice of a TM-connection ∇ on the vector bundle A, the map $\sharp\colon \mathrm{ad}_\nabla^* \to \mathrm{ad}_\nabla$ acts via

$$\sharp(\mathrm{d}f) = \sharp_0(\mathrm{d}f) = \{f, \cdot\} = -\rho'^*(\mathrm{d}f) \in \Gamma(A)$$

$$\sharp(\beta) = \sharp_0(\beta) + \sharp_1(\cdot)\beta \in \mathfrak{X}(M) \oplus (\Gamma(A^*) \otimes \Gamma(A))$$

for all $f \in C^\infty(M), \beta \in \Gamma(A)$, where we consider β as an *1-form* on $A[1]$ and identify it with $\mathrm{d}\beta - \mathrm{d}_{\nabla^*}\beta \in \Omega^1(A[1])$ as in Remark 6.4.1, i.e. $\sharp(\beta) = \sharp(\mathrm{d}\beta) - \sharp(\mathrm{d}_{\nabla^*}\beta) = \{\beta, \cdot\} - \sharp(\mathrm{d}_{\nabla^*}\beta)$. Computing how these act on $\alpha \in \Omega^1(M)$ and $g \in C^\infty(M)$, viewed as functions of the graded manifold $A[1]$, gives the components of $\sharp(\beta)$: From the right-hand-side of the equation we obtain

$$\left(\sharp_0(\beta) + \sharp_1(\cdot)\beta\right)g = \sharp_0(\beta)g \in C^\infty(M)$$

while from the left-hand-side we obtain

$$\sharp(\beta)g = \sharp(\mathrm{d}\beta - \mathrm{d}_{\nabla^*}\beta)g = \{\beta, g\} - \sharp(\mathrm{d}_{\nabla^*}\beta)g = \rho'(\beta)g.$$

From this, it follows that $\sharp_0(\beta) = \rho'(\beta)$. Using now this, the right-hand-side gives

$$\left(\sharp_0(\beta) + \sharp_1(\cdot)\beta\right)\alpha = \nabla^*_{\rho'(\beta)}\alpha + \left(\sharp_1(\cdot)\beta\right)\alpha \in \Gamma(A^*) \oplus \Gamma(A^*)$$

while the left-hand-side gives

$$\sharp(\beta)\alpha = \sharp(\mathrm{d}\beta - \mathrm{d}_{\nabla^*}\beta)\alpha = \{\beta, \alpha\} - \sharp(\mathrm{d}_{\nabla^*}\beta)\alpha = [\beta, \alpha]_* + \nabla^*_{\rho'(\alpha)}\beta = (\nabla^*)^{\mathrm{bas}}_\beta\alpha.$$

This implies that $(\sharp_1(\cdot)\beta)\alpha = (\nabla^*)^{\mathrm{bas}}_\beta\alpha - \nabla^*_{\rho'(\alpha)}\beta$ and thus $\sharp$ consists of the (-1)-chain map $\sharp_0$ given by the anti-commutative diagram

$$
\begin{array}{ccc}
T^*M[0] & \xrightarrow{\ -\rho^*\ } & A^*[-1] \\
{\scriptstyle -\rho'^*}\big\downarrow & & \big\downarrow{\scriptstyle \rho'} \\
A[1] & \xrightarrow[\ \ \rho\ \ ]{} & TM[0]
\end{array}
$$

together with $\sharp_1(a)\beta = \langle (\nabla^*)^{\mathrm{bas}}_\beta(\cdot) - \nabla^*_{\rho'(\beta)}(\cdot), a\rangle \in \Gamma(A^{**}) \simeq \Gamma(A)$, for all $\beta \in \Gamma(A^*)$ and $a \in \Gamma(A)$.

By Theorem 5.3.1, $\sharp$ is an (anti-)morphism of 2-representations if and only $(A[1], \mathrm{d}_A, \{\cdot\,,\cdot\})$ is a Poisson Lie 1-algebroid. Hence, $\sharp$ is an (anti-)morphism of 2-representations if and only if (A, A^*) is a Lie bialgebroid. Similarly, [41] shows that ad_∇^* and ad_∇ form a *matched pair* if and only if

(A, A^*) is a Lie bialgebroid.

Note that $(A, \{\cdot, \cdot\})$ is a Poisson Lie algebroid if the induced vector bundle morphism $\sharp \colon T^*A \to TA$ over A is a VB-algebroid morphism over $\rho' \colon A^* \to TM$ [69]. Then the fact that $\sharp \colon \mathrm{ad}^*_\nabla \to \mathrm{ad}_\nabla$ is an (anti-)morphism of 2-representations follows immediately [35], since ad^*_∇ and ad_∇ are equivalent to decompositions of the VB-algebroids $(T^*A \to A^*, A \to M)$ and $(TA \to TM, A \to M)$, respectively.

Now we consider the case of 2-algebroids. First recall that a symplectic Lie 2-algebroid over a point, that is, a Courant algebroid over a point, is a usual Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ together with a non-degenerate pairing $\langle \cdot, \cdot \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that

$$\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0$$

for all $x, y, z \in \mathfrak{g}$. Using the adjoint and coadjoint representations $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $x \mapsto [x, \cdot]$, and $\mathrm{ad}^* \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, $x \mapsto -\mathrm{ad}(x)^*$, and denoting the canonical linear isomorphism induced by the pairing by $P \colon \mathfrak{g} \to \mathfrak{g}^*$, the equation above reads

$$P(\mathrm{ad}(x)y) = \mathrm{ad}^*(x)P(y)$$

for all $x, y \in \mathfrak{g}$. In other words, this condition is precisely what is needed to turn the vector space isomorphism P into an isomorphism of Lie algebra representations between ad and ad^*. In fact, the map of representations up to homotopy $\sharp \colon \mathrm{ad}^* \to \mathrm{ad}$ for Poisson Lie 2-algebroids is a direct generalisation of this construction.

Let $B \to M$ be a usual Lie algebroid with a 2-term representation $(\nabla^Q, \nabla^{Q^*}, R)$ on a complex $\partial_Q \colon Q^* \to Q$. The representation is called **self dual** [49] if it equals its dual, i.e. $\partial_Q = \partial_Q^*$, the connections ∇^Q and ∇^{Q^*} are dual to each other, and $R^* = -R \in \Omega^2(B, \mathrm{Hom}(Q, Q^*))$, i.e. $R \in \Omega^2(B, \wedge^2 Q^*)$. [49] further shows that Poisson brackets $\{\cdot, \cdot\}$ on a split Lie 2-algebroid $Q[1] \oplus B^*[2]$ correspond to self dual 2-representations of B on $Q^*[1] \oplus Q[0]$ as follows: the bundle map $\partial_Q \colon Q^* \to Q$ is $\tau \mapsto \{\tau, \cdot\}|_{\Omega^1(Q)}$, the anchor $\rho_B \colon B \to TM$ is $b \mapsto \{b, \cdot\}|_{C^\infty(M)}$, the B-connection on Q^* is given by $\nabla_b^{Q^*} \tau = \{b, \tau\}$, and the 2-form R and the Lie bracket of B are defined by $\{b_1, b_2\} = [b_1, b_2] - R(b_1, b_2) \in \Gamma(B) \oplus \Omega^2(Q)$.

Fix now a Poisson Lie 2-algebroid $(\mathcal{M}, \mathcal{Q}, \{\cdot, \cdot\})$ together with a choice of a splitting $Q[1] \oplus B^*[2]$ for $\mathcal{M}$, a pair of TM-connections on B^* and Q, and consider the representations ad_∇ and ad^*_∇. Similarly as before, we have that

$$\sharp(\mathrm{d}f) = \sharp_0(\mathrm{d}f) = \{f, \cdot\} = -\rho_B^*(\mathrm{d}f) \in \Gamma(B^*)$$

$$\sharp(\tau) = \sharp_0(\tau) + \sharp_1(\cdot)\tau \in \Gamma(Q) \oplus (\Omega^1(Q) \otimes \Gamma(B^*))$$

$$\sharp(b) = \sharp_0(b) + \sharp_1(\cdot)b + \sharp_2(\cdot, \cdot)b + \sharp^b(\cdot)b \in \mathfrak{X}(M) \oplus \Omega^1(Q, Q) \oplus \Omega^2(Q, B^*) \oplus (\Gamma(B) \otimes \Gamma(B^*))$$

for $f \in C^\infty(M), \tau \in \Gamma(Q^*), b \in \Gamma(B)$, where we identify again τ with $\mathrm{d}\tau - \mathrm{d}_{\nabla^*}\tau$ and b with $\mathrm{d}b - \mathrm{d}_{\nabla^*}b$ as in Remark 6.4.1. Then the map $\sharp \colon \mathrm{ad}^*_\nabla \to \mathrm{ad}_\nabla$ consists of the (-2)-chain map given

by the anti-commutative diagram

$$
\begin{array}{ccccc}
T^*M[0] & \xrightarrow{\;-\rho_Q^*\;} & Q^*[-1] & \xrightarrow{\;-\partial_B\;} & B[-2] \\
{\scriptstyle -\rho_B^*}\big\downarrow & & {\scriptstyle \partial_Q}\big\downarrow & & \big\downarrow{\scriptstyle \rho_B} \\
B^*[2] & \xrightarrow{\;-\partial_B^*\;} & Q[1] & \xrightarrow{\;\rho_Q\;} & TM[0]
\end{array}
$$

and the elements

$$
\sharp_1(q)\tau = \langle \tau, \nabla^Q_\cdot q - \nabla_{\rho_B(\cdot)}q \rangle \in \Gamma(B^*)
$$

$$
\sharp_1(q)b = \nabla^Q_b q - \nabla_{\rho_B(b)}q \in \Gamma(Q)
$$

for $q \in \Gamma(Q), \tau \in \Gamma(Q^*), b \in \Gamma(B)$,

$$
\sharp_2(q_1, q_2)b = -\langle R(b, \cdot)q_1, q_2 \rangle \in \Gamma(B^*)
$$

for $q_1, q_1 \in \Gamma(Q), b \in \Gamma(B)$, where R is the component that comes from the self-dual 2-representation of B from the Poisson structure,

$$
\sharp^b(\beta)b = \langle \beta, \nabla^{\mathrm{bas}}_b(\cdot) - \nabla^*_{\rho_B(b)}(\cdot) \rangle \in \Gamma(B^*)
$$

for $\beta \in \Gamma(B^*), b \in \Gamma(B)$.

Suppose now that the split Lie 2-algebroid is symplectic, i.e. that it is of the form $E[1] \oplus T^*M[2]$ for a Courant algebroid $E \to M$. The only thing that is left from the construction in the Example 3.3.9 is a choice of a TM-connection on TM, and hence on the dual T^*M. The (anti-)isomorphism $\sharp\colon \mathrm{ad}^*_\nabla \to \mathrm{ad}_\nabla$ consists of the (-2)-chain map of the anti-commutative diagram

$$
\begin{array}{ccccc}
T^*M[0] & \xrightarrow{\;-\rho^*\;} & E^*[-1] & \xrightarrow{\;-\rho\;} & TM[-2] \\
{\scriptstyle -\,\mathrm{id}}\big\downarrow & & {\scriptstyle P^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\
T^*M[2] & \xrightarrow{\;-\rho^*\;} & E[1] & \xrightarrow{\;\rho\;} & TM[0]
\end{array}
$$

where $P\colon E \xrightarrow{\sim} E^*$ is the pairing, and the elements $\langle \sharp_2(e_1, e_2)X, Y \rangle = -\langle R_\nabla(X, Y)e_1, e_2 \rangle$ and $\langle \sharp^b(\alpha)X, Y \rangle = -\langle \alpha, T_\nabla(X, Y) \rangle$. Its inverse consists of the 2-chain map given by the anti-commutative diagram

$$
\begin{array}{ccccc}
T^*M[2] & \xrightarrow{\;-\rho^*\;} & E[1] & \xrightarrow{\;\rho\;} & TM[0] \\
{\scriptstyle -\,\mathrm{id}}\big\downarrow & & {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\
T^*M[0] & \xrightarrow{\;-\rho^*\;} & E^*[-1] & \xrightarrow{\;-\rho\;} & TM[-2]
\end{array}
$$

and the elements $\langle \sharp_2^{-1}(e_1, e_2)X, Y \rangle = -\langle R_\nabla(X, Y)e_1, e_2 \rangle$ and $\langle (\sharp^{-1})^b(\alpha)X, Y \rangle = -\langle \alpha, T_\nabla(X, Y) \rangle$.

In other words, $\sharp^2 = \mathrm{id}$. If the connection on TM is torsion-free, then the terms $\sharp^b$ and $(\sharp^{-1})^b$ vanish, as well. In particular, if the base manifold M is just a point, then the bundles TM and T^*M, and the elements $\sharp_2$ and $\sharp_2^{-1}$ are zero. Therefore, the map $\mathrm{ad}^*_\nabla \to \mathrm{ad}_\nabla$ reduces to the linear isomorphism of the pairing and agrees with the one explained above.

Chapter 7

Linear structures on vector bundles

In this chapter, we consider the case where the structures defined before are are considered on vector bundles over graded manifolds and are in some sense "linear". It puts the notions studied before in the context of linear structures on graded vector bundles over graded manifolds and compares with the existing literature. In what follows, suppose that $q : \mathcal{E} \to \mathcal{M}$ is a vector bundle in the category of $\mathbb{Z}$-graded manifolds.

7.1 Linear multivector fields on vector bundles

Recall from Section 4.1 the basic and linear functions of the graded manifold $\mathcal{E}$; the former are elements of $q^\star(\mathcal{C}^\infty(\mathcal{M}))$, while the latter are functions on $\mathcal{E}$ which are linear in the fibre coordinates (i.e. locally of the form $e^j \xi_j$, where e^j are fibre coordinates of $\mathcal{E}$ and ξ_j are local functions on $\mathcal{M}$). Let $\widetilde{\mathcal{X}}$ be a degree j vector field on the graded manifold $\mathcal{E}$. Then $\widetilde{\mathcal{X}}$ is called **linear** if the spaces of basic and linear functions are stable under its action, i.e. if

$$\widetilde{\mathcal{X}}(\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E})) \subset \mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}) \qquad \text{and} \qquad \widetilde{\mathcal{X}}(\mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E})) \subset \mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E}).$$

The graded subspace of **linear vector fields** of the vector bundle $\mathcal{E}$ is denoted $\mathfrak{X}_{\mathrm{lin}}(\mathcal{E}) \subset \mathfrak{X}(\mathcal{E})$. It follows easily from the definition that $\mathfrak{X}_{\mathrm{lin}}(\mathcal{E})$ is closed under the Lie bracket of vector fields, and hence, one obtains the graded Lie subalgebra $(\mathfrak{X}_{\mathrm{lin}}(\mathcal{E}), [\,\cdot\,,\cdot\,])$ of $(\mathfrak{X}(\mathcal{E}), [\,\cdot\,,\cdot\,])$.

Similarly to the ordinary case of vector bundles over smooth manifolds, it follows that a linear vector field $\widetilde{\mathcal{X}}$ induces a degree j vector field on the base manifold $\mathcal{X} : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ and corresponds to a **degree j derivation of $\mathcal{E}^*$ (over $\mathcal{X}$):** $\mathcal{D}^* \in \mathfrak{Der}^j(\mathcal{E}^*)$, i.e. a degree j linear map $\mathcal{D}^* : \Gamma(\mathcal{E}^*) \to \Gamma(\mathcal{E}^*)$ with the following property:

$$\mathcal{D}^*(\xi\theta) = \mathcal{X}(\xi)\theta + (-1)^{j|\xi|}\xi\mathcal{D}^*(\theta),$$

91

for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\theta \in \Gamma(\mathcal{E}^*)$. The correspondence is obtained via the formulae

$$\widetilde{\mathcal{X}}(\ell_\theta) = \ell_{\mathcal{D}^*(\theta)} \qquad \text{and} \qquad \widetilde{\mathcal{X}}(q^\star(\xi)) = q^\star(\mathcal{X}(\xi))$$

for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\theta \in \Gamma(\mathcal{E}^*)$. A straightforward computation shows that given two linear vector fields $\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}} \in \mathfrak{X}_{\mathrm{lin}}(\mathcal{E})$,

$$[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}](\ell_\theta) = \ell_{[\mathcal{D}^*_X, \mathcal{D}^*_Y](\theta)} \qquad \text{and} \qquad [\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}](q^\star \xi) = q^\star\left([\mathcal{X}, \mathcal{Y}](\xi)\right)$$

for all $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\theta \in \Gamma(\mathcal{E}^*)$, and therefore, it follows that the map

$$\mathfrak{X}_{\mathrm{lin}}(\mathcal{E}) \to \mathfrak{Der}(\mathcal{E}^*) := \bigoplus_{j \in \mathbb{Z}} \mathfrak{Der}^j(\mathcal{E}^*), \qquad (\widetilde{\mathcal{X}}, \mathcal{X}) \mapsto \mathcal{D}_{(\widetilde{\mathcal{X}}, \mathcal{X})}$$

is an isomorphism of graded Lie algebras, where the Lie bracket of $\mathfrak{Der}(\mathcal{E}^*)$ is given by the graded commutator.

A degree j derivation $\mathcal{D}^*$ is called **flat** if $\mathcal{D}^* \circ \mathcal{D}^* = 0$. Flat derivations of odd degree correspond to linear vector fields which square to zero. In particular, flat derivations of degree 1 on $\mathcal{E}^*$, i.e. DG-module structures on $\mathcal{E}^*$, correspond to linear homological vector fields on $\mathcal{E}$. This will be analysed extensively in classical differential geometric terms in Chapter 8.

Locally, if $\{\xi^i\}$ are coordinates on $\mathcal{M}$ and $\{e^j\}$ are fibre coordinates of $\mathcal{E}$ with dual coordinates $\{\theta_j\}$ on the fibres of $\mathcal{E}^*$, then

$$\widetilde{\mathcal{X}} = \zeta_i \frac{\partial}{\partial \xi^i} + e^k \eta^j_k \frac{\partial}{\partial e^j}, \qquad \mathcal{X} = \zeta_i \frac{\partial}{\partial \xi^i}, \qquad \text{and} \qquad \mathcal{D}_{(\widetilde{\mathcal{X}}, \mathcal{X})}(\theta_j) = e^k \eta^j_k$$

where $e^k \eta^j_k \in \mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E})$ and $\eta^j_k, \zeta_i \in \mathcal{C}^\infty(\mathcal{M})$.

Let now $k \in \mathbb{Z}$ and consider the space $\mathfrak{A}_k(\mathcal{E})$ together with its the Schouten bracket.

Definition 7.1.1. An element $\widetilde{\mathcal{X}} \in \mathfrak{A}^{s,\bullet}_k(\mathcal{E})$ on a vector bundle $\mathcal{E}$ is called **linear multivector field** if the following conditions hold:

1. $[\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), [\dots [\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), [\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), \widetilde{\mathcal{X}}]_k]_k \dots]_k]_k \subset \mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E})$ (s linear functions),

2. $[\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), [\dots [\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), [\mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E}), \widetilde{\mathcal{X}}]_k]_k \dots]_k]_k \subset \mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E})$ ($s-1$ linear and one basic functions),

3. $[\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), [\dots [\mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E}), [\mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E}), \widetilde{\mathcal{X}}]_k]_k \dots]_k]_k = 0$ (two or more basic functions).

The space of linear multivector fields on the vector bundle $\mathcal{E}$ will be denoted $\mathfrak{A}^{\bullet,\bullet}_{\mathrm{lin},k}(\mathcal{E})$. From the above definition, it follows that the local description of a linear multivector field $\widetilde{\mathcal{X}} \in \mathfrak{A}^{s,\bullet}_{\mathrm{lin},k}(\mathcal{E})$ is given by

$$\widetilde{\mathcal{X}} = \zeta_{i,J} \frac{\partial}{\partial \xi^i} \wedge \frac{\partial}{\partial e^J} + \lambda_K \frac{\partial}{\partial e^K}$$

where J and K are multi-indices of degree $s-1$ and s, respectively, $\zeta_{i,J} \in \mathcal{C}^\infty(\mathcal{M})$ and $\lambda_K \in \mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E})$.

Remark 7.1.2. Note that the space of linear multivector fields is *not* closed under the wedge

product. This is easily seen from the local description because a product of two multivector fields may contain a term of the form $\frac{\partial}{\partial \xi^{i_1}} \wedge \frac{\partial}{\partial \xi^{i_2}}$.

Remark 7.1.3. If $\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}} \in \mathfrak{X}_{\mathrm{lin}}(\mathcal{E})$, then also $[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}] \in \mathfrak{X}_{\mathrm{lin}}(\mathcal{E})$. In particular, the vector fields $[\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}]$ is linear over $[\mathcal{X}, \mathcal{Y}]$.

7.2 Linear $\mathcal{Q}$-manifold structures

A **linear $\mathcal{Q}$-manifold** is a vector bundle $q : \mathcal{E} \to \mathcal{M}$ equipped with a **linear homological vector field**: $\widetilde{\mathcal{Q}} \in \mathfrak{X}_{\mathrm{lin}}^1(\mathcal{E})$ such that $\widetilde{\mathcal{Q}}^2 = 0$. From the discussion above, it follows that given a linear $\mathcal{Q}$-manifold $\mathcal{E}$ with homological vector field $\widetilde{\mathcal{Q}}$, there is an induced homological vector field $\mathcal{Q}$ on the base $\mathcal{M}$. Since the space of linear vector fields on $\mathcal{E}$ is a graded subalgebra of $\mathfrak{X}(\mathcal{E})$, we obtain the cochain complex

$$\pounds_{\widetilde{\mathcal{Q}}} \colon \mathfrak{X}_{\mathrm{lin}}^{\bullet}(\mathcal{E}) \to \mathfrak{X}_{\mathrm{lin}}^{\bullet+1}(\mathcal{E}), \qquad \widetilde{\mathcal{X}} \mapsto [\widetilde{\mathcal{Q}}, \widetilde{\mathcal{X}}].$$

It is a subcomplex of $(\mathfrak{X}(\mathcal{E}), \pounds_{\widetilde{\mathcal{Q}}})$ and, as in Chapter 4, it is the DGLA governing the deformations of the linear $\mathcal{Q}$-manifold structure of $\mathcal{E}$. As usual, the infinitesimal deformations and the deformations of $\widetilde{\mathcal{Q}}$ are in one-to-one correspondence with degree 1 cochains and Maurer-Cartan elements of $(\mathfrak{X}_{\mathrm{lin}}(\mathcal{E}), \pounds_{\widetilde{\mathcal{Q}}})$.

Example 7.2.1 (**Tangent $\mathcal{Q}$-manifold**). It was mentioned before that linear homological vector fields on the vector bundle $\mathcal{E}$ correspond to DG-module structures on $\mathcal{E}^*$, and consequently on $\mathcal{E}$. Hence, the tangent bundle $\mathrm{pr}_{\mathcal{M}} \colon T\mathcal{M} \to \mathcal{M}$ of a $\mathcal{Q}$-manifold $(\mathcal{M}, \mathcal{Q})$ carries a canonical linear $\mathcal{Q}$-manifold structure corresponding to the adjoint module $(\mathfrak{X}(\mathcal{M}), \pounds_{\mathcal{Q}})$. Its linear homological vector field $\mathcal{Q}_T$ is characterised by

$$\mathcal{Q}_T(\mathrm{pr}_{\mathcal{M}}^{\star}(\xi)) = \mathrm{pr}_{\mathcal{M}}^{\star}(\mathcal{Q}(\xi)) \qquad \text{and} \qquad \mathcal{Q}_T(\ell_{\theta}) = \ell_{\pounds_{\mathcal{Q}}(\theta)} \tag{7.1}$$

for all $\xi \in \mathcal{C}^{\infty}(\mathcal{M})$ and all $\theta \in \Gamma(T^*\mathcal{M}) = \Omega^1(\mathcal{M})$; here, ℓ_{θ} is the linear function of $T\mathcal{M}$ that corresponds to the 1-form $\theta \in \Omega^{1,\bullet}(\mathcal{M})$. Locally, suppose that $\{\xi^i\}$ are coordinates in a chart of $\mathcal{M}$ such that $\mathcal{Q}$ takes the form

$$\mathcal{Q} = \zeta^i \frac{\partial}{\partial \xi^i}$$

for some local functions $\zeta^i \in \mathcal{C}^{\infty}(\mathcal{M})$. Denote the linear coordinates of $T\mathcal{M}$ over this chart by $\{\dot{\xi}^i\}$; that is, $\dot{\xi}^i$ is the linear function that corresponds to the 1-form $\mathrm{d}\xi^i$. Then an application of (7.1) for the coordinate functions $\{\xi^i, \dot{\xi}^i\}$ of the graded manifold $T\mathcal{M}$ implies that the vector field $\mathcal{Q}_T$ has the following local expression:

$$\mathcal{Q}_T = \zeta^i \frac{\partial}{\partial \xi^i} - \dot{\xi}^j \frac{\partial \zeta^s}{\partial \xi^j} \frac{\partial}{\partial \dot{\xi}^s}.$$

7.3 Linear homotopy Poisson structures

A **linear Poisson structure of degree** k on a graded vector bundle $\mathcal{E} \to \mathcal{M}$ is a degree k Poisson bracket $\{\cdot, \cdot\}_k$ on the graded manifold $\mathcal{E}$ such that

1. $\{\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), \mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E})\}_k \subset \mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E})$,

2. $\{\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), \mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E})\}_k \subset \mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E})$,

3. $\{\mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E}), \mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E})\}_k = 0$.

Using the corresponding Poisson bivector field $\pi \in \mathfrak{A}^{2,-k}_k(\mathcal{E})$ discussed in Section 4.3, the above axioms may be rewritten as

1. $[[\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), \pi]_k, \mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E})]_k \subset \mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E})$,

2. $[[\mathcal{C}^\infty_{\mathrm{lin}}(\mathcal{E}), \pi]_k, \mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E})]_k \subset \mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E})$

3. $[[\mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E}), \pi]_k, \mathcal{C}^\infty_{\mathrm{bas}}(\mathcal{E})]_k = 0$.

In other words, $\{\cdot, \cdot\}_k$ is a linear Poisson bracket on $\mathcal{E}$ if and only if the corresponding Poisson bivector field is linear: $\pi \in \mathfrak{A}^{2,-k}_{\mathrm{lin},k}(\mathcal{E})$. More generally, a **linear homotopy Poisson structure** on a $\mathbb{Z}$-manifold $\mathcal{M}$ is an element $\Theta \in \mathfrak{A}^{2-k}_{\mathrm{lin},k}(\mathcal{E})$ such that $[\Theta, \Theta]_k = 0$.

Similarly as in Section 4.3, due to the invariance of the space $\mathfrak{A}_{\mathrm{lin},k}(\mathcal{E})$ under the Schouten bracket explained in Remark 7.1.3, one obtains the induced differential operator of bidegree $(1, 0)$

$$\mathrm{d}_\pi : \mathfrak{A}^{\bullet, \bullet}_{\mathrm{lin},k}(\mathcal{E}) \to \mathfrak{A}^{\bullet+1, \bullet}_{\mathrm{lin},k}(\mathcal{E}) \qquad \widetilde{\mathcal{X}} \mapsto [\pi, \widetilde{\mathcal{X}}].$$

It follows from the discussion above that there are two embeddings

$$(\mathfrak{A}^{\bullet, \bullet}_{\mathrm{lin},k}(\mathcal{E}), \pounds_{\widetilde{\mathcal{Q}}}) \hookrightarrow (\mathfrak{A}^{\bullet, \bullet}_k(\mathcal{E}), \pounds_{\widetilde{\mathcal{Q}}}) \qquad \text{and} \qquad (\mathfrak{A}^{\bullet, \bullet}_{\mathrm{lin},k}(\mathcal{E}), \mathrm{d}_\pi) \hookrightarrow (\mathfrak{A}^{\bullet, \bullet}_k(\mathcal{E}), \mathrm{d}_\pi),$$

and hence an embedding on the level of total complexes

$$(\mathfrak{A}^{\bullet, \bullet}_{\mathrm{lin},k}(\mathcal{E}), \pounds_{\widetilde{\mathcal{Q}}} + (-1)^{k-1}\mathrm{d}_\pi) \hookrightarrow (\mathfrak{A}^{\bullet, \bullet}_k(\mathcal{E}), \pounds_{\widetilde{\mathcal{Q}}} + (-1)^{k-1}\mathrm{d}_\pi).$$

This yields the natural question of whether the induced map on cohomologies is also an embedding. Although the obvious answer is "yes", the proof requires some non-trivial work. For instance, consider the case of the Lie complex $\pounds_{\widetilde{\mathcal{Q}}} : \mathfrak{X}^\bullet(\mathcal{E}) \to \mathfrak{X}^{\bullet+1}(\mathcal{E})$ of 1-vector fields of $\mathcal{E}$ endowed with a linear homological vector field $\widetilde{\mathcal{Q}}$. One has to prove the following: Given $\widetilde{\mathcal{X}} \in \mathfrak{X}^\bullet_{\mathrm{lin}}(\mathcal{E})$ with $\pounds_{\widetilde{\mathcal{Q}}}(\mathcal{Y}) = \widetilde{\mathcal{X}}$ for some $\mathcal{Y} \in \mathfrak{X}(\mathcal{E})$ *not necessarily linear*, there is a *linear* vector field $\widetilde{\mathcal{Z}} \in \mathfrak{X}_{\mathrm{lin}}(\mathcal{E})$ such that $\pounds_{\widetilde{\mathcal{Q}}}(\mathcal{Z}) = \widetilde{\mathcal{X}}$. For the case of VB-algebroids, i.e. linear $\mathbb{N}\mathcal{Q}$-manifolds of degree 1, it is solved in [61, 62], using classical differential geometric language and following an analytic approach.

7.4 Graded Lie algebroids and linear $\mathcal{P}$-manifolds

Now we will prove a result which is familiar from ordinary Lie algebroids and linear Poisson structures over smooth manifolds. Namely, there is a correspondence between Lie algebroid structures of degree k on the vector bundle $\mathcal{A} \to \mathcal{M}$ (in the sense of Mehta [71, 73]) and linear Poisson structures of degree k on the dual bundle $\mathcal{A}^*$ viewed as a $\mathbb{Z}$-graded manifold.

Definition 7.4.1. A **(graded) Lie algebroid of degree** k over a graded manifold $\mathcal{M}$ is a graded vector bundle $q \colon \mathcal{A} \to \mathcal{M}$ together with a degree k anchor map ρ and a degree k Lie bracket on $\Gamma(\mathcal{A})$, i.e. a degree 0 vector bundle morphism $\rho \colon \Gamma(\mathcal{A}) \to \mathfrak{X}(\mathcal{M})[k]$ and a degree 0 bilinear operator $[\cdot,\cdot] \colon \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})[k]$ satisfying the following conditions:

1. $[\sigma,\tau] = -(-1)^{(|\sigma|+k)(|\tau|+k)}[\tau,\sigma],$

2. $[\sigma,[\tau,\upsilon]] = [[\sigma,\tau],\upsilon] + (-1)^{(|\sigma|+k)(|\tau|+k)}[\tau,[\sigma,\upsilon]],$

3. $[\sigma,\xi\tau] = (\rho(\sigma)(\xi))\,\tau + (-1)^{(|\sigma|+k)|\xi|}\xi\,[\sigma,\tau],$

for homogeneous elements $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and $\sigma,\tau \in \Gamma(\mathcal{A})$.

Example 7.4.2. Given any graded manifold $\mathcal{M}$, the shifted tangent bundle $T[k]\mathcal{M}$ with anchor $\rho = \mathrm{id}_{\mathfrak{X}(\mathcal{M})[k]}$ and bracket the usual Lie bracket of vector fields $[\cdot,\cdot]$ is a Lie algebroid of degree k over $\mathcal{M}$.

Consider a graded vector bundle $q \colon \mathcal{A} \to \mathcal{M}$ and its dual $p \colon \mathcal{A}^* \to \mathcal{M}$. Suppose that $\{\cdot,\cdot\}_k$ is a linear Poisson structure of degree k on $\mathcal{A}^*$. Define a Lie algebroid structure of degree k on $\mathcal{A}$ by the properties

$$\{\ell_\sigma, p^\star(\xi)\}_k = p^\star(\rho(\sigma)(\xi)), \qquad \{\ell_\sigma, \ell_\tau\}_k = \ell_{[\sigma,\tau]} \qquad \text{and} \qquad \{p^\star(\xi_1), p^\star(\xi_2)\}_k = 0 \qquad (7.2)$$

for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$ and all $\sigma,\tau \in \Gamma(\mathcal{A})$, where ℓ_σ and ℓ_τ are the linear functions of $\mathcal{A}^*$ corresponding to the sections σ and τ, respectively. We show now that the graded anti-symmetry of the Poisson bracket $\{\cdot,\cdot\}_k$ is equivalent to the graded anti-symmetry of $[\cdot,\cdot]$, the Leibniz identity of $\{\cdot,\cdot\}_k$ is equivalent to $\rho(\sigma)$ being a degree $|\sigma| + k$ derivation of $\mathcal{C}^\infty(\mathcal{M})$, ρ being $\mathcal{C}^\infty(\mathcal{M})$-linear and $[\cdot,\cdot]$ having the Leibniz identity, and the Jacobi identity of $\{\cdot,\cdot\}_k$ is equivalent to the Jacobi identity of $[\cdot,\cdot]$. In order to see this, consider homogeneous elements $\xi,\xi_1,\xi_2 \in \mathcal{C}^\infty(\mathcal{M})$ and $\sigma,\tau,\upsilon \in \Gamma(\mathcal{A})$. Then we compute

$$\ell_{[\sigma,\tau]} = \{\ell_\sigma, \ell_\tau\}_k = -(-1)^{(|\sigma|+k)(|\tau|+k)}\{\ell_\tau, \ell_\sigma\}_k = \ell_{-(-1)^{(|\sigma|+k)(|\tau|+k)}[\tau,\sigma]},$$

$$\begin{aligned}
p^\star(\rho(\sigma)(\xi_1\xi_2)) &= \{\ell_\sigma, p^\star(\xi_1)\,p^\star(\xi_2)\}_k \\
&= \{\ell_\sigma, p^\star(\xi_1)\}_k\,p^\star(\xi_2) + (-1)^{(|\sigma|+k)|\xi_1|}p^\star(\xi_1)\{\ell_\sigma, p^\star(\xi_2)\}_k \\
&= p^\star(\rho(\sigma)(\xi_1)\,\xi_2 + (-1)^{(|\sigma|+k)|\xi_1|}\xi_1\rho(\sigma)(\xi_2)),
\end{aligned}$$

$$
\begin{aligned}
p^\star(\rho(\xi_1\sigma)(\xi_2)) &= \{\ell_{\xi_1\sigma}, p^\star(\xi_2)\}_k \\
&= \{p^\star(\xi_1)\,\ell_\sigma, p^\star(\xi_2)\}_k \\
&= -(-1)^{(|\xi_2|+k)(|\xi_1|+|\sigma|+k)}\{p^\star(\xi_2), p^\star(\xi_1)\,\ell_\sigma\}_k \\
&= -(-1)^{(|\xi_2|+k)(|\sigma|+k)}\,p^\star(\xi_1)\,\{p^\star(\xi_2), \ell_\sigma\}_k \\
&= p^\star(\xi_1\rho(\sigma)(\xi_2)),
\end{aligned}
$$

$$
\begin{aligned}
\ell_{[\sigma,\xi\tau]} &= \{\ell_\sigma, p^\star(\xi)\,\ell_\tau\}_k \\
&= \{\ell_\sigma, p^\star(\xi)\}_k\,\ell_\tau + (-1)^{(|\sigma|+k)|\xi|}\,p^\star(\xi)\,\{\ell_\sigma,\ell_\tau\}_k \\
&= p^\star(\rho(\sigma)(\xi))\,\ell_\tau + (-1)^{(|\sigma|+k)|\xi|}\,p^\star(\xi)\,\ell_{[\sigma,\tau]} \\
&= \ell_{(\rho(\sigma)(\xi))\,\tau+(-1)^{(|\sigma|+k)|\xi|}\,\xi\,[\sigma,\tau]},
\end{aligned}
$$

$$
\begin{aligned}
\ell_{[\sigma,[\tau,\upsilon]]} &= \{\ell_\sigma, \ell_{[\tau,\upsilon]}\}_k \\
&= \{\ell_\sigma, \{\ell_\tau, \ell_\upsilon\}_k\}_k \\
&= \{\{\ell_\sigma, \ell_\tau\}_k, \ell_\upsilon\}_k + (-1)^{(|\sigma|+k)(|\tau|+k)}\{\ell_\tau, \{\ell_\sigma, \ell_\upsilon\}_k\}_k \\
&= \{\ell_{[\sigma,\tau]}, \ell_\upsilon\}_k + (-1)^{(|\sigma|+k)(|\tau|+k)}\{\ell_\tau, \ell_{[\sigma,\upsilon]}\}_k \\
&= \ell_{[\sigma,[\tau,\upsilon]]=[[\sigma,\tau],\upsilon]+(-1)^{(|\sigma|+k)(|\tau|+k)}[\tau,[\sigma,\upsilon]]}.
\end{aligned}
$$

Conversely, given a Lie algebroid structure of degree k on $\mathcal{A} \to \mathcal{M}$, the formulae in (7.2) define the action of a linear bivector field $\pi \in \mathfrak{A}^{2,-k}_{\mathrm{lin},k}(\mathcal{A}^*)$ on the coordinate functions of $\mathcal{A}^*$, which characterises uniquely the linear Poisson bracket on the whole $\mathcal{C}^\infty(\mathcal{A}^*)$.

Example 7.4.3 (Symplectic structure on $T^*[1-k]\mathcal{M}$). The natural degree $k-1$ Lie algebroid structure on $T[k-1]\mathcal{M}$ obtained from a graded manifold $\mathcal{M}$ (Example 7.4.2) corresponds to the linear Poisson structure on the graded manifold $T^*[1-k]\mathcal{M}$ given by the canonical symplectic form ω_{can} defined in Example 4.6.4. In particular, recall that the algebra of functions of the graded manifold $T^*[1-k]\mathcal{M}$ is given in degree $i \in \mathbb{Z}$ by

$$
\mathfrak{A}^i_k(\mathcal{M}) = \bigoplus_{p+q=i} \mathfrak{A}^{p,q}_k(\mathcal{M}),
$$

with basic and linear functions

$$
\mathcal{C}^\infty_{\mathrm{bas}}(T^*[1-k]\mathcal{M})^i = \bigoplus_{p+q=i} \mathfrak{A}^{p,q}_{\mathrm{bas},k}(\mathcal{M}) \qquad \text{and} \qquad \mathcal{C}^\infty_{\mathrm{lin}}(T^*[-k]\mathcal{M})^i = \bigoplus_{p+q=i} \mathfrak{A}^{p,q}_{\mathrm{lin},k}(\mathcal{M}).
$$

The linear Poisson bracket coincides with the degree $k-1$ Schouten bracket defined in Section 4.3 and one may recover its local description from the formulae in (7.2). For this, write the Poisson bivector field π in local coordinates as

$$\pi = \kappa^{st} \frac{\partial}{\partial \xi^s} \wedge \frac{\partial}{\partial \xi^t} + \eta^{mn} \frac{\partial}{\partial e^m} \wedge \frac{\partial}{\partial e^n} + \theta^{ji} \frac{\partial}{\partial e^j} \wedge \frac{\partial}{\partial \xi^i}$$

where $\{\xi^j\}$ are local coordinates on $\mathcal{M}$, $\{e^j := \ell_{\partial/\partial \xi^j}\}$ are the linear coordinates of $T^*[1-k]\mathcal{M}$ corresponding to the local frame $\{\frac{\partial}{\partial \xi^j}\}$ of $T[k-1]\mathcal{M}$, and $\kappa^{st}, \eta^{mn}, \theta^{ji}$ are functions of $T^*[1-k]\mathcal{M}$ over this coordinate chart. We recover the functions $\kappa^{st}, \eta^{mn}, \theta^{ji}$ for the coordinate representation of π by plugging the coordinate functions $\{\xi^j, e^j\}$ into $[\cdot, [\cdot, \pi]_k]_k$, as in Section 4.3. The last formula in (7.2) implies $\kappa^{st} = 0$. The second formula in (7.2), together with

$$\left[\frac{\partial}{\partial \xi^m}, \frac{\partial}{\partial \xi^n} \right] = 0 \qquad \Longleftrightarrow \qquad \{e^m, e^n\}_k = 0,$$

yields $\eta^{mn} = 0$. Finally, a straightforward computation using the anti-symmetry and the Leibniz rule of the Schouten bracket $[\cdot, \cdot]_k$, together with the first formula in (7.2), shows that

$$\delta_{ij} = \frac{\partial \xi^i}{\partial \xi^j} = \text{id}\left(\frac{\partial}{\partial \xi^j} \right)(\xi^i) = \{e^j, \xi^i\}_k = -[\xi^i, [e^j, \pi]_k]_k = (-1)^{|\xi^i|(k-1)}\theta^{ji}.$$

The coordinate transformation $\zeta^i := (-1)^{|\xi^i|(k-1)}\xi^i$ brings π into the form of Example 4.6.4:

$$\pi = \sum_i \frac{\partial}{\partial e^i} \wedge \frac{\partial}{\partial \zeta^i}.$$

Proposition 7.4.4. Suppose $(\mathcal{A}, \rho, [\cdot, \cdot])$ is a Lie algebroid of degree k over $\mathcal{M}$ with corresponding linear Poisson bivector π on $\mathcal{A}^*$. Then a linear vector field $\widetilde{\mathcal{X}} \in \mathfrak{X}^i_{\text{lin}}(\mathcal{A}^*)$ over $\mathcal{X} \in \mathfrak{X}(\mathcal{M})$ is Poisson if and only if for all (homogeneous) $\sigma, \tau \in \Gamma(\mathcal{A})$

1. the corresponding operator $\mathcal{D}$ on $\Gamma(\mathcal{A})$ is a derivation of the Lie bracket:

$$\mathcal{D}[\sigma, \tau] = [\mathcal{D}(\sigma), \tau] + (-1)^{i(\sigma+k)}[\sigma, \mathcal{D}(\tau)],$$

2. $\mathcal{D}$ and $\pounds_{\mathcal{X}}$ commute with ρ:

$$\pounds_{\mathcal{X}}(\rho(\sigma)) = [\mathcal{X}, \rho(\sigma)] = \rho(\mathcal{D}(\sigma)).$$

Proof. Both conditions follow easily from the actions of the linear vector field and the linear Poisson bracket on basic and linear functions of $\mathcal{A}^*$. $\square$

Remark 7.4.5. Linear Poisson vector fields on $\mathcal{A}^*$ are related to the "morphic vector fields" on $\mathcal{A}$ defined in [71, 73]. By definition, a vector field $\mathcal{X}$ on $\mathcal{A}$ is morphic if $[\text{d}_{\mathcal{A}}, \mathcal{X}] = 0$, where $\text{d}_{\mathcal{A}}$ is the Lie algebroid differential that encodes the bracket and the anchor of $\mathcal{A}$. As it is shown by Mehta, morphic vector fields can also be described as operators $\mathcal{D}$ on $\Gamma(\mathcal{A})$ with the properties of Proposition 7.4.4.

Chapter 8

Higher split VB-algebroid structures

In this chapter, we introduce the notion of higher split VB-Lie algebroids as a generalisation of VB-algebroids. Intuitively, VB-algebroids may be viewed as vector bundles in the category of Lie algebroids and so in the same spirit higher VB-Lie algebroids may be thought of as vector bundles in the category of higher split Lie algebroids. They are essentially a classical differential geometric description of the linear structures that appeared in the previous chapter and we prove that they are in correspondence with representations up to homotopy.

8.1 Classical interpretation

Suppose that $(\underline{D}, V, \underline{A}, M)$ is a double vector bundle (see Section 2.3) together with graded vector bundle decompositions $\underline{D} = D_1[1] \oplus \ldots \oplus D_n[n]$ and $\underline{A} = A_1[1] \oplus \ldots \oplus A_n[n]$, over V and M, respectively, which are compatible with the projection $\underline{D} \to \underline{A}$. This means that each of the individual squares (D_i, V, A_i, M) also forms a double vector bundle[1]. Schematically, this yields the following sequence of diagrams

---

[1]In supergeometric terminology, the double vector bundle $(\underline{D}, V\underline{A}, M)$ is the splitting of a vector bundle whose total space and base are both $[n]$-manifolds. See Appendix A.1 for more details about this correspondence.

where all the "planes" are double vector bundles. This yields that the core of $(\underline{D}, V, \underline{A}, M)$ is the graded vector bundle $\underline{C} = C_1[1] \oplus \ldots \oplus C_n[n]$, where C_i is the core of (D_i, V, A_i, M), for $i = 1, \ldots, n$.

Definition 8.1.1. The quadruple $(\underline{D}, V, \underline{A}, M)$ is a **(split) VB-Lie n-algebroid** if

1. the graded vector bundle $\underline{D} \to V$ is endowed with a homological vector field $\mathcal{Q}_{\underline{D}}$,

2. the Lie n-algebroid structure of $\underline{D} \to V$ is **linear** over $A \to M$, in the sense that

    (a) the anchor $\rho_D \colon D_1 \to TV$ is a double vector bundle morphism,

    (b) the map ∂_{D_i} fits into a morphism of double vector bundles $(\partial_{D_i}, \mathrm{id}_V, \partial_{A_i}, \mathrm{id}_M)$ between (D_i, V, A_i, M) and (D_{i-1}, V, A_{i-1}, M) for all i,

    (c) the multi-brackets of $\underline{D}$ satisfy the following relations:

        i. the i-bracket of i linear sections is a linear section;

        ii. the i-bracket of $i - 1$ linear sections with a core section is a core section;

        iii. the i-bracket of $i - k$ linear sections with k core sections, $i \geq k \geq 2$, is zero.

Remark 8.1.2.   1. A VB-Lie 1-algebroid as in the definition above is just a VB-algebroid.

2. A VB-Lie n-algebroid structure on $(\underline{D}, V, \underline{A}, M)$ defines uniquely a Lie n-algebroid structure on $\underline{A} \to M$ as follows: The anchor $\rho_D \colon D_1 \to TV$ is linear over the anchor $\rho \colon A_1 \to TM$, and if all $d_j \in \Gamma_V^l(\underline{D})$ are linear sections over $a_j \in \Gamma(\underline{A})$ for $j = 1, 2, \ldots, i$, then the section $[\![d_1, d_2, \ldots, d_i]\!]_{\underline{D}} \in \Gamma_V^l(\underline{D})$ is linear over the section $[\![a_1, a_2, \ldots, a_i]\!]_{\underline{A}} \in \Gamma(\underline{A})$. Therefore, the graded vector bundles $\underline{D} \to V$ and $\underline{A} \to M$ are endowed with homological vector fields $\mathcal{Q}_{\underline{D}}$ and $\mathcal{Q}_{\underline{A}}$ for which the bundle projection $\underline{D} \to \underline{A}$ is a morphism of Lie n-algebroids over the projection $V \to M$. In particular, the homological vector field $\mathcal{Q}_{\underline{A}}$ on the $[n]$-manifold $\underline{A} = A_1[1] \oplus \ldots \oplus A_n[n]$ is determined by the equations

$$\mathcal{Q}_{\underline{D}}(q_V^* f) = \pi_{\underline{A}}^\star(\mathcal{Q}_{\underline{A}}(f)) \qquad \text{and} \qquad \mathcal{Q}_{\underline{D}}(\pi_{\underline{A}}^\star(\alpha)) = \pi_{\underline{A}}^\star(\mathcal{Q}_{\underline{A}}(\alpha))$$

for $f \in C^\infty(M)$ and $\alpha \in \Gamma(\underline{S}(\underline{A}^*))$.

We now need to define the right notion of morphism of VB-Lie n-algebroids. Intuitively, this is given by conditions similar to the bracket relations of a VB-Lie n-algebroid, i.e. a set of linear sections will remain intact (due to the fixed side Lie n-algebroid $\underline{A}$), a set of linear sections and a core section is mapped to a core section, a set of linear sections and two or more core sections is mapped to zero. In order to make it precise, we consider the following bigrading on the functions of the Lie n-algebroid $\underline{D} \to V$:

- basic functions in $q^*(C^\infty(M)) \subset C^\infty(V)$ have bidegree $(0, 0)$;

- basic functions in $\pi_{\underline{A}}^\star(\Gamma(\underline{S}^i(\underline{A}^*)))$ have bidegree $(0, i)$;

- linear functions $\ell_\psi \in C_{\mathrm{lin}}^\infty(V)$, where $\psi \in \Gamma(V^*)$, have bidegree $(1, -1)$;

- linear functions corresponding to elements in $\Gamma(C_i^*)$ have bidegree $(1, i-1)$.

In fact, this is the bigrading of the function space $\mathcal{C}^\infty(\underline{D}_V)$ of the $[n]$-manifold $\underline{D}_V$ viewed as a vector bundle over the $[n]$-manifold $\underline{A}$ in the category of graded manifolds.

Definition 8.1.3. A morphism of VB-Lie n-algebroids from $(\underline{D}, V, \underline{A}, M)$ to $(\underline{D}', V', \underline{A}, M)$ (with fixed side $\underline{A}$) is a bidegree $(0,0)$ morphism of Lie n-algebroids $G_D \colon \underline{D} \to \underline{D}'$ over $G_V \colon V \to V'$ whose action on basic functions is trivial[1]: $G_D^\star(f \circ q_{V'}) = f \circ q_V$ and $G_D^\star(\pi_{\underline{A}}^\star(\alpha)) = \pi_{\underline{A}}^\star(\alpha)$ for all $f \in C^\infty(M)$ and all $\alpha \in \Gamma(\underline{S}(\underline{A}^*))$.

Remark 8.1.4. The condition of the trivial action of a VB-Lie n-algebroid morphism on basic functions is imposed because here we restrict our study to morphisms of representations up to homotopy over a fixed Lie n-algebroid. That is, our aim is to prove a one-to-one correspondence between morphisms of VB-Lie n-algebroids with the same side Lie n-algebroid $\underline{A} \to M$ and morphisms of representations up to homotopy of $\underline{A}$ fixing $\underline{A}$. Dropping this condition yields a degree-preserving morphism $G_{\underline{A}}^\star \colon \Gamma(\underline{S}(\underline{A}^*)) \to \Gamma(\underline{S}(\underline{A}^*))$ satisfying $G_D^\star(\pi_{\underline{A}}^\star(\alpha)) = \pi_{\underline{A}}^\star(G_{\underline{A}}^\star(\alpha))$ and hence a morphism or representations up to homotopy over $G_{\underline{A}}^\star$.

The resulting category is denoted $\mathbb{VB}\text{-Lie}_n(\underline{A})$ and the set of isomorphy classes of this category is denoted $\text{VB-Lie}_n(\underline{A})$. The category whose objects are the **decomposed VB-Lie n-algebroids**, i.e. VB-Lie n-algebroids with decomposed (D_i, V, A_i, M), with morphisms defined as above is denoted $\mathbb{DVB}\text{-Lie}_n(\underline{A})$ and the isomorphy classes $\text{DVB-Lie}_n(\underline{A})$. In fact, since every double vector bundle admits a decomposition, given a VB-Lie n-algebroid $(\underline{D}, V, \underline{A}, M)$, a choice of a decomposition for each double vector bundle (D_i, V, A_i, M) gives a decomposition of $(\underline{D}, V, \underline{A}, M)$. Therefore, we immediately obtain the following proposition.

Proposition 8.1.5. There is an equivalence of categories between $\mathbb{VB}\text{-Lie}_n(\underline{A})$ and $\mathbb{DVB}\text{-Lie}_n(\underline{A})$, and therefore an equivalence between the isomorphy classes $\text{VB-Lie}_n(\underline{A})$ and $\text{DVB-Lie}_n(\underline{A})$.

Example 8.1.6 (**Tangent prolongation of a (split) Lie n-algebroid**). The basic example of a split VB-Lie n-algebroid is obtained by applying the tangent functor to a split Lie n-algebroid $\underline{A} = A_1[1] \oplus \ldots \oplus A_n[n] \to M$. The double vector bundle is given by the diagram

$$
\begin{array}{ccc}
\underline{TA} & \longrightarrow & \underline{A} \\
\downarrow & & \downarrow \\
TM & \longrightarrow & M
\end{array}
$$

where the Lie n-algebroid structure of $\underline{TA} = T\underline{A} = TA_1[1] \oplus \ldots \oplus TA_n[n]$ over the manifold TM is defined by the relations

1. $\rho_{TA} = J_M \circ T\rho_A \colon TA_1 \to TTM$, where $J_M \colon TTM \to TTM$ is the canonical involution from Example 2.3.3.

---

[1] By abuse of notation, we use the same symbol $\pi_{\underline{A}}$ to denote both vector bundle projections $\underline{D} \to \underline{A}$ and $\underline{D}' \to \underline{A}$.

2. $[\![Ta_1, \ldots, Ta_i]\!] = T[\![a_1, \ldots, a_i]\!]$,

3. $[\![Ta_1, \ldots, Ta_{i-1}, a_i^\dagger]\!] = [\![a_{k_1}, \ldots, a_{i-1}, a_i]\!]^\dagger$,

4. $[\![Ta_1, \ldots, Ta_j, a_{j+1}^\dagger, \ldots, a_i^\dagger]\!] = 0$ for all $1 \leq j \leq i-2$,

5. $[\![a_1^\dagger, \ldots, a_i^\dagger]\!] = 0$,

for all sections $a_j \in \Gamma(A_{k_j})$.

Applying the above construction to a split Lie 2-algebroid $Q[1] \oplus B^*[2] \to M$ with structure $(\rho_Q, \ell, \nabla^*, \omega)$ yields as follows the objects $(\rho_{TQ}, T\ell, T\nabla^*, T\omega)$ of the split Lie 2-algebroid structure of $TQ[1] \oplus TB^*[2] \to TM$: The complex $TB^* \to TQ \to TTM$ consists of the anchor of TQ given by $\rho_{TQ} = J_M \circ T\rho_Q$, and the vector bundle map $T\ell \colon TB^* \to TQ$. The bracket of TQ (over TM) is defined by the relations

$$[Tq_1, Tq_2]_{TQ} = T[q_1, q_2]_Q, \qquad [Tq_1, q_2^\dagger]_{TQ} = [q_1, q_2]_Q^\dagger, \qquad [q_1^\dagger, q_2^\dagger]_{TQ} = 0,$$

for $q_1, q_2 \in \Gamma(Q)$. The TQ-connection $T\nabla^* \colon \Gamma_{TM}(TQ) \times \Gamma_{TM}(TB^*) \to \Gamma_{TM}(TB^*)$ is defined by

$$(T\nabla^*)_{Tq}(T\beta) = T(\nabla_q^*\beta), \qquad (T\nabla^*)_{Tq}(\beta^\dagger) = (\nabla_q^*\beta)^\dagger = (T\nabla^*)_{q^\dagger}\beta, \qquad (T\nabla^*)_{q^\dagger}(\beta^\dagger) = 0,$$

for $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$. Finally, the 3-form $T\omega \in \Omega^3(TQ, TB^*)$ is defined by

$$(T\omega)(Tq_1, Tq_2, Tq_3) = T(\omega(q_1, q_2, q_3)), \qquad (T\omega)(Tq_1, Tq_2, q_3^\dagger) = \omega(q_1, q_2, q_3)^\dagger,$$

$$(T\omega)(q_1, q_2^\dagger, q_3^\dagger) = 0 = (T\omega)(q_1^\dagger, q_2^\dagger, q_3^\dagger),$$

for $q_1, q_2, q_3 \in \Gamma(Q)$.

8.2   Supergeometric interpretation

Now we give an equivalent description of a VB-Lie n-algebroid $(\underline{D}, V, \underline{A}, M)$ with decomposed double vector bundles (D_i, V, A_i, M) in terms of its homological vector field.

Proposition 8.2.1. Suppose $(\underline{D}, V, \underline{A}, M)$ is a decomposed double vector bundle. Then a Lie n-algebroid structure on the $[n]$-manifold $\underline{D}_V \to V$ is a VB-Lie n-algebroid structure for $(\underline{D}, V, \underline{A}, M)$ if and only if its corresponding homological vector field $\mathcal{Q}_{\underline{D}}$ satisfies the following relations:

1. $\mathcal{Q}_{\underline{D}}$ maps $q_V^*(C^\infty(M))$ into $\pi_{\underline{A}}^\star(\Gamma(A_1^*))$;

2. $\mathcal{Q}_{\underline{D}}$ maps $C_{\mathrm{lin}}^\infty(V)$ into $\Gamma(A_1^* \otimes V^*) \oplus \Gamma(C_1^*)$;

3. $\mathcal{Q}_{\underline{D}}$ leaves $\pi_{\underline{A}}^\star(\Gamma(\underline{S}(\underline{A})))$ invariant;

4. $\mathcal{Q}_{\underline{D}}$ maps $\Gamma(C_i^*)$ into the degree $i+1$ elements of $\pi_{\underline{A}}^\star(\Gamma(\underline{S}(\underline{A}))) \otimes \Gamma(\underline{C}^*)$.

Proof. The first and third assertions follow from the fact that the projection $\pi_{\underline{A}}$ is a morphism of Lie n-algebroids from $\underline{D}_V \to V$ to $\underline{A} \to M$. To see assertion (2), one proceeds as follows: Recall that the spaces $\Gamma_V(D_i^*)$ are generated as $C^\infty(V)$-modules by core and linear sections, where for the latter we have the equality

$$\Gamma_V^l(D_i^*) = \Gamma(A_i^* \otimes V^*) \oplus \Gamma(C_i^*) \tag{8.1}$$

given by the decomposition, and the (canonical) identification of $\alpha \in \Gamma(\underline{A}^*)$ with $\pi_{\underline{A}}^\star(\alpha) \in \Gamma(D_V^*)$. Moreover, as vector bundles over V, we have the equality

$$D_i = q_V^!(A_i \oplus C_i). \tag{8.2}$$

Note that for $\psi \in \Gamma(V^*)$ the 1-form $\mathrm{d}\ell_\psi$ is a linear section of $T^*V \to V$ over $\psi : M \to V^*$ and $\rho_{D_1} : D_1 \to TV$ is a morphism of double vector bundles. Hence, (2) follows from

$$\mathcal{Q}_{\underline{D}}(\ell_\psi) = \rho_{D_1}^* \mathrm{d}\ell_\psi \in \Gamma_V^l(D_1^*) = \Gamma(A_1^* \otimes V^*) \oplus \Gamma(C_1^*).$$

Suppose now that $\gamma_i \in \Gamma(C_i^*)$. Then using (8.2) we may write the $(i+1)$-degree function $\mathcal{Q}_{\underline{D}}(\gamma)$ in terms of its various components. The components with $\Gamma(\underline{S}^k(\underline{A}))$ together with two or more $\Gamma(C_j^*)$ vanish due to the bracket conditions of a VB-Lie n-algebroid. Recall that from the definition of split Lie n-algebroids in terms of higher brackets (Section 3.3), we have that for terms of the form $\Gamma(C_j^* \wedge C_k^*)$ with $j + k = i + 1$

$$\mathcal{Q}_{\underline{D}}(\gamma_i)(c_j, c_k) = \rho_{D_1}(c_j)\langle\gamma_i, c_k\rangle - \rho_{D_1}(c_k)\langle\gamma_i, c_j\rangle - \langle\gamma_i, [\![c_j, c_k]\!]_2\rangle, \tag{8.3}$$

where $\rho_{D_1}(c_j)$ and $\rho_{D_1}(c_k)$ are zero if $j \neq 1$ and $k \neq 1$, respectively. If $i \neq j$ and $i \neq k$, then the first two terms are again zero, due to the vanishing of the pairing. Observe now that the last term is always zero because it is a bracket with two core sections.

It remains to prove that the terms of the form $\rho_{D_1}(c_1)\langle\gamma_i, c_i\rangle$ are always zero. As a morphism of double vector bundles, the anchor ρ_{D_1} sends the core section c_1 to a vector field on V obtained by a vertical lift. By definition, a vertical lift has flow which lies entirely on the fibre directions. A simple observation of the fact that the sections $\gamma_i \in \Gamma_V(D_i^*), c_i \in \Gamma_V(D_i), c_1 \in \Gamma_V(D_1)$ are obtained by the pull-back under q_V of the corresponding sections $\gamma_i \in \Gamma(C_i^*), c_i \in \Gamma(C_i), c_1 \in \Gamma(C_1)$ implies that the function $\langle\gamma_i, c_i\rangle \in C^\infty(V)$ is invariant under the flow of $\rho_{D_1}(c_1)$. Therefore, all the terms in equation (8.3) are zero and we obtain the result. $\qquad\square$

8.3   The Weil algebra of a split VB-Lie n-algebroid

We now define the Weil algebra of a double vector bundle $(\underline{D}, V, \underline{A}, M)$ as above (with no Lie n-algebroid structure a priori) and see how this can be used to give an alternative description of a VB-Lie n-algebroid structure. The case of $n = 1$, i.e. the ordinary double vector bundles

(D, V, A, M), is studied in great detail in [77].

Definition 8.3.1. The **Weil algebra of** $(\underline{D}, V, \underline{A}, M)$, denoted $W(\underline{D})$, is defined as the space of functions of the graded manifold $\underline{D}[1]$ over M, where the shift functor $[1]$ is applied to the horizontal arrows of $(\underline{D}, V, \underline{A}, M)$.

Remark 8.3.2.   1. The graded manifold $\underline{D}[1]$ whose space of functions defines the Weil algebra of $(\underline{D}, V, \underline{A}, M)$ is the total space of the $[1]$-shifted vector bundle $\underline{D}[1] \to \underline{A}$ in the category of graded manifolds.

   2. Note that in the case of an ordinary (non-graded) double vector bundle (D, V, A, M), our notation for the Weil algebra $W(D)$ is different from the notation in [77], where a double shift $[1, 1]$ for both horizontal and vertical arrows of (D, V, A, M) is used. According to our convention, D is already concentrated in degree -1 and so the induced $[1]$-manifold over V satisfies $\mathcal{C}^\infty(D)^1 = \Gamma_V(D_V^*)$. In [77], the vector bundle D over V is concentrated in degree 0 and thus a (vertical) shift over V is required, in order to give the same $[1]$-manifold.

Unravelling the definition of the Weil algebra of $(\underline{D}, V, \underline{A}, M)$, we find that $W(\underline{D})$ carries a bigrading such that 0-functions of the form $f \circ q_V \in C^\infty(V)$ obtained by pulling-back smooth functions $f \in C^\infty(M)$ are elements of bidegree $(0, 0)$, 1-functions $\ell_\psi \in C^\infty(V)$ for $\psi \in \Gamma(V^*)$ are elements of bidegree $(1, 0)$, i-functions $\alpha \in \Gamma(\underline{S}^i(\underline{A}))$ are elements of bidegree $(0, i)$ and $(i + 1)$-functions $\gamma_i \in \Gamma(C_i^*)$ are elements of bidegree $(1, i)$. In other words, the Weil algebra of $(\underline{D}, V, \underline{A}, M)$ is generated by

$$W^{0,0}(\underline{D}) = C^\infty(M), \qquad W^{0,i}(\underline{D}) = \Gamma(\underline{S}^i(\underline{A}^*)),$$

$$W^{1,0}(\underline{D}) = \Gamma(V^*), \qquad W^{1,i}(\underline{D}) = \bigoplus_{j+k=i} \Gamma(\underline{S}^j(\underline{A}^*)) \otimes \Gamma(C_k^*).$$

Remark 8.3.3. The Weil algebra of a split Lie n-algebroid $\underline{A}$ over M discussed in Section 6.7 can be obtained from the definition above as $W(T\underline{A})$, where $(T\underline{A}, TM, \underline{A}, M)$ is the tangent prolongation of $\underline{A} \to M$.

In this setting, it is clear that Proposition 8.2.1 above may be reformulated as follows.

Theorem 8.3.4. Let $(\underline{D}, V, \underline{A}, M)$ be a decomposed double vector bundle. A VB-Lie n-algebroid structure on $(\underline{D}, V, \underline{A}, M)$ is equivalent to a homological vector field of bidegree $(0, 1)$ on the graded manifold $\underline{D}[1]$.

8.4   The fat Lie n-algebroid

The space of linear functions $\mathcal{C}_{\text{lin}}^\infty(\underline{D})$ of a VB-Lie n-algebroid inherits an obvious structure of an algebra over the ring $C^\infty(M)$ of smooth functions on M. With this structure, $\mathcal{C}_{\text{lin}}^\infty(\underline{D})$ becomes a sheaf of locally free, $\mathbb{N}$-graded, graded commutative, associative, unital $C^\infty(M)$-algebras over M, with local generators given by local frames of $(V^* \otimes \underline{C})^*$ and by pulling-back the local generators of $\underline{A}$ under $\pi_{\underline{A}}$. Hence, it is the space of functions of an $[n]$-manifold over M, denoted $\widehat{\underline{A}}$. In fact,

one has that as a vector bundle over M,

$$\widehat{\underline{A}} = \bigoplus_{i=1}^{n} \widehat{A_i}[i],$$

where $\widehat{A_i}$ is the (non-graded) vector bundle over M whose sheaf of sections corresponds to the locally free $C^\infty(M)$-module structure on $\Gamma_V^l(D_i)$. From the short exact sequence (2.5), it follows that the vector bundle $\widehat{A_i}$ has rank equal to $\mathrm{rank}(A_i) + \mathrm{rank}(V) \cdot \mathrm{rank}(C_i)$, and thus the $[n]$-manifold corresponding to $\widehat{\underline{A}}$ is of dimension

$$(\dim(M); \mathrm{rank}(A_n) + \mathrm{rank}(V) \cdot \mathrm{rank}(C_n), \ldots, \mathrm{rank}(A_1) + \mathrm{rank}(V) \cdot \mathrm{rank}(C_1)).$$

Another interesting fact about the $[n]$-manifold $\widehat{\underline{A}}$ is that it carries a Lie n-algebroid structure which is induced by the VB-Lie n-algebroid structure on $(\underline{D}, V, \underline{A}, M)$. In particular, it is given by the multi-brackets

$$[\![a_1, \ldots, a_i]\!]_{\widehat{\underline{A}}} = [\![a_1, \ldots, a_i]\!]_{\underline{D}}$$

for all $a_1, \ldots, a_i \in \Gamma(\widehat{\underline{A}}) = \Gamma_V^l(\underline{D})$, or equivalently, by the homological vector field on $\widehat{\underline{A}}$ defined by $\mathcal{Q}_{\widehat{\underline{A}}}(\alpha) = \mathcal{Q}_{\underline{D}}(\alpha)$, for all $\alpha \in \Gamma(S(\widehat{\underline{A}}))$. We call $\widehat{\underline{A}}$ the **fat Lie n-algebroid**.

The projection map $\widehat{\underline{A}} \to \underline{A}$ is a Lie n-algebroid morphism whose kernel may be identified with $V^* \otimes \underline{C} = \mathrm{Hom}(V, \underline{C})$. This yields that the graded vector bundle

$$\underline{\mathrm{Hom}(V, \underline{C})} := \bigoplus_{i=1}^{n} (V^* \otimes C_i)[i] = V^* \otimes \underline{C} = \mathrm{Hom}(V, \underline{C}) = \bigoplus_{i=1}^{n} (\mathrm{Hom}(V, C_i))[i]$$

may be equipped with a Lie n-algebroid structure so that the sequence

$$0 \longrightarrow \underline{\mathrm{Hom}(V, \underline{C})} \longrightarrow \widehat{\underline{A}} \longrightarrow \underline{A} \longrightarrow 0 \tag{8.4}$$

is a short exact sequence of Lie n-algebroids over M. Therefore, we obtain the following result.

Theorem 8.4.1. Given a VB-Lie n-algebroid $(\underline{D}, V, \underline{A}, M)$, the construction explained above equips the graded vector bundle $\widehat{\underline{A}}$ with a Lie n-algebroid structure over the manifold M. With this Lie n-algebroid structure, $\widehat{\underline{A}}$ fits in the short exact sequence (8.4) of Lie n-algebroids over M.

8.5 Split VB-Lie n-algebroids and $(n + 1)$-representations

As it is shown in [42], an interesting fact about the tangent prolongation of a Lie algebroid is that it encodes its adjoint representation. The same holds for a split Lie n-algebroid $A_1[1] \oplus \ldots \oplus A_n[n]$, since by definition the adjoint module is exactly the space of sections of the $\mathcal{Q}$-vector bundle $T(A_1[1] \oplus \ldots \oplus A_n[n]) \to A_1[1] \oplus \ldots \oplus A_n[n]$. The next example shows this correspondence explicitly in the case of split Lie 2-algebroids $Q[1] \oplus B^*[2]$.

Example 8.5.1. Choose two TM-connections on Q and B^*, both denoted by ∇. These choices

induce the horizontal lifts $\Gamma(Q) \to \Gamma^l_{TM}(TQ)$ and $\Gamma(B^*) \to \Gamma^l_{TM}(TB^*)$, both denoted by h. More precisely, given a section $q \in \Gamma(Q)$, its lift is defined as $h(q) = Tq - (\nabla.q)^\wedge$. A similar formula holds for $h(\beta)$, as well. Then an easy computation yields the following:

1. $\rho_{TQ}(q^\dagger) = \rho(q)^\uparrow$ and $(T\ell)(\beta^\dagger) = \ell(\beta)^\uparrow$

2. $\rho_{TQ}(h(q)) = X_{\nabla^{\mathrm{bas}}_q}$

3. $(T\ell)(h(\beta)) = h(\ell(\beta)) + (\nabla.(\ell(\beta)) - \ell(\nabla.\beta))^\wedge$

4. $[h(q_1), h(q_2)]_{TQ} = h[q_1, q_2]_Q - R^{\mathrm{bas}}_\nabla(q_1, q_2)^\wedge$

5. $[h(q_1), q_2^\dagger]_{TQ} = (\nabla^{\mathrm{bas}}_{q_1} q_2)^\dagger$

6. $(T\nabla^*)_{h(q)}(\beta^\dagger) = (\nabla^*_q \beta)^\dagger$

7. $(T\nabla^*)_{q^\dagger}(h(\beta)) = (\nabla^*_q \beta - \nabla_{\rho(q)}\beta)^\dagger$

8. $(T\nabla^*)_{h(q)}(h(\beta)) = h(\nabla^*_q \beta) + (\nabla^*_{\nabla.q}\beta - \nabla_{\rho(\nabla.q)}\beta + \nabla^*_q \nabla.\beta - \nabla.\nabla^*_q \beta - \nabla_{[\rho(q),\cdot]}\beta)^\wedge$

9. $(T\omega)(h(q_1), h(q_2), h(q_3)) = h(\omega(q_1, q_2, q_3)) + ((\nabla.\omega)(q_1, q_2, q_3))^\wedge$

10. $(T\omega)(h(q_1), h(q_2), q_3^\dagger) = (\omega(q_1, q_2, q_3))^\dagger$.

The reader should compare the calculations above with the structure objects of the adjoint representation up to homotopy of a split Lie 2-algebroid given in Proposition 6.3.1. In fact, this result is a special case of a correspondence between VB-Lie n-algebroid structures on a decomposed graded double vector bundle $(\underline{D}, V, \underline{A}, M)$ and $(n + 1)$-representations of the Lie n-algebroid $\underline{A}$ on the vector bundle $\underline{E} := V[0] \oplus C_1[1] \oplus \ldots \oplus C_n[n]$ over M. In the general case, it is easier to give the correspondence in terms of the homological vector field on $\underline{D}$ and the dual representation of $\underline{A}$ on $\underline{E}^* = C_n^*[-n] \oplus \ldots \oplus C_1^*[-1] \oplus V^*[0]$.

Suppose that $(\underline{D}, V, \underline{A}, M)$ is a VB-Lie n-algebroid with homological vector fields $\mathcal{Q}_{\underline{D}}$ and $\mathcal{Q}_{\underline{A}}$, and choose a decomposition for each double vector bundle (D_i, V, A_i, M)[1], and consequently for $(\underline{D}, V, \underline{A}, M)$. Consider the dual $\underline{D}^*_V$ and recall that the spaces $\Gamma_V(D^*_i)$ are generated as $C^\infty(V)$-modules by their core and linear sections. For the latter, use the identification $\Gamma^l_V(D^*_i) = \Gamma(A^*_i \otimes V^*) \oplus \Gamma(C^*_i)$ induced by the decomposition. Moreover, the element $\alpha \in \Gamma(A^*_i)$ is identified with the core section $\pi^\star_{\underline{A}}(\alpha) \in \Gamma^c_V(\underline{D}^*)$. Define the representation $\mathcal{D}^*$ of $\underline{A}$ on the dual complex $\underline{E}^*$ by the equations

$$\mathcal{D}^*(\psi) = \mathcal{Q}_{\underline{D}}(\ell_\psi) = \rho^*_{D_1}\mathrm{d}\ell_\psi \qquad \text{and} \qquad \mathcal{D}^*(\gamma) = \mathcal{Q}_{\underline{D}}(\gamma) \tag{8.5}$$

for all $\psi \in \Gamma(V^*)$ and all $\gamma \in \Gamma(C^*_i)$. From Proposition 8.2.1 and the fact that $\mathcal{Q}^2_{\underline{D}} = 0$, it follows that $\mathcal{D}^*$ defined by (8.5) is the differential of a representation up to homotopy of $(\underline{A}, \mathcal{Q}_{\underline{A}})$ on the graded vector bundle $\underline{E}^*$ over M.

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[1]In the case of the tangent Lie n-algebroid, this corresponds to choosing the TM-connections on the vector bundles of the adjoint complex.

Conversely, given a representation $\mathcal{D}^*$ of $(\underline{A}, \mathcal{Q}_{\underline{A}})$ on $\underline{E}^*$, the equations in (8.5) together with

$$\mathcal{Q}_{\underline{D}}(q_V^* f) = \pi_{\underline{A}}^\star(\mathcal{Q}_{\underline{A}}(f)) \qquad \text{and} \qquad \mathcal{Q}_{\underline{D}}(\pi_{\underline{A}}^\star(\alpha)) = \pi_{\underline{A}}^\star(\mathcal{Q}_{\underline{A}}(\alpha)),$$

for all $f \in C^\infty(M)$ and $\alpha \in \Gamma(\underline{A}^*)$, define a VB-Lie n-algebroid structure on the double vector bundle $(\underline{D}, V, \underline{A}, M)$. This yields the following theorem.

Theorem 8.5.2. Let $(\underline{D}, V, \underline{A}, M)$ be a decomposed graded double vector bundle with core $\underline{C}$. There is a one-to-one correspondence between VB-Lie n-algebroid structures on $(\underline{D}, V, \underline{A}, M)$ and $(n+1)$-representations up to homotopy of $\underline{A}$ on the complex $V[0] \oplus C_1[1] \oplus \ldots \oplus C_n[n]$.

We will show now that the correspondence of decomposed VB-Lie n-algebroids with fixed side algebroid $\underline{A} \to M$ and $(n+1)$-representations of $\underline{A} \to M$ can be also achieved on the level of morphisms, and therefore, we obtain an equivalence of categories.

Proposition 8.5.3. Let $(\underline{D}, V, \underline{A}, M)$ and $(\underline{D}', V', \underline{A}, M)$ be two decomposed VB-Lie n-algebroids with cores $\underline{C}$ and $\underline{C}'$, respectively. There exists a one-to-one correspondence between morphisms of VB-Lie n-algebroids from $(\underline{D}, V, \underline{A}, M)$ to $(\underline{D}', V', \underline{A}, M)$ and morphisms of $\underline{A}$-representations from $V[0] \oplus \underline{C}$ to $V'[0] \oplus \underline{C}'$.

Proof. Suppose first that

$$\mu \colon \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{C} \oplus V) \to \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{C}' \oplus V')$$

is a morphism of representations up to homotopy. Denote the differentials of $V[0] \oplus \underline{C}$ and $V'[0] \oplus \underline{C}'$ by $\mathcal{D}$ and $\mathcal{D}'$, respectively, and the dual of μ by $\mu^\star$.

Recall that the space of sections $\Gamma_V(\underline{D}^*)$ of a double vector bundle is generated as $C^\infty(V)$-module by core and linear sections, and that the coordinate functions of the smooth manifold V can be taken of the form ℓ_ψ (fibre-wise linear) and $f \circ q_V$ (basic functions), for $\psi \in \Gamma(V^*)$ and $f \in C^\infty(M)$. We define the map $G_\mu^\star \colon \Gamma(\underline{D}'^*_{V'}) \to \Gamma(\underline{D}^*_V)$ on the generators by

$$G_\mu^\star(f \circ q_{V'}) = f \circ q_V, \quad G_\mu^\star(\ell_\psi) = \ell_{\mu_0^*(\psi)}, \quad G_\mu^\star(\alpha) = \alpha, \quad G_\mu^\star(\gamma) = \mu^\star(\gamma), \tag{8.6}$$

for all $\psi \in \Gamma(V'^*), \alpha \in \Gamma(\underline{S}(\underline{A}^*)), \gamma \in \Gamma(\underline{C}'^*), f \in C^\infty(M)$. A straightforward computation, using $\mathcal{D}' \circ \mu = \mu \circ \mathcal{D}$, shows that $G_\mu^\star \circ \mathcal{Q}_{\underline{D}'} = \mathcal{Q}_{\underline{D}} \circ G_\mu^\star$, and thus $G_\mu \colon \underline{D}_V \to \underline{D}'_{V'}$ is a morphism of Lie n-algebroids over $\mu_0|_V \colon V \to V'$. It is clear from the equations in (8.6) that $G_\mu \colon \underline{D}_V \to \underline{D}'_{V'}$ is of bidegree $(0, 0)$, and hence, together with $\mu_0|_V \colon V \to V'$, they define a morphism of VB-Lie n-algebroids as in Definition 8.1.3. Its principal part $(G_{\mu,0})_i \colon D_i \to D'_i, i = 1, \ldots, n$, with respect to equality (8.2) is given by $\mathrm{id}_{A_i} \oplus \mu_0|_{C_i}$, i.e.

$$G_{\mu,0}^\star = \mathrm{id}_{\underline{A}^*} \oplus (\mu_0|_{\underline{C}})^\star = (\mathrm{id}_{A_1^*} \oplus \mu_0|_{C_1}^*) \oplus \ldots \oplus (\mathrm{id}_{A_n^*} \oplus \mu_0|_{C_n}^*)$$

and therefore the induced core morphism is $\mu_0|_C \colon C \to C'$ over the identity on M.

Conversely, suppose that $(G_D, G_V, \mathrm{id}_{\underline{A}}, \mathrm{id}_M)$ is a morphism between two (decomposed) VB-Lie n-algebroids $(\underline{D}, V, \underline{A}, M)$ and $(\underline{D}', V', \underline{A}, M)$. Observe that, for degree reasons, a morphism of $\underline{A}$-representations from $V'^* \oplus \underline{C}'^*$ to $V^* \oplus \underline{C}^*$ must send the vector bundle V'^* into V^* via a bundle map μ_0^* over the identity on M. Since $G_D^\star$ is a morphism of VB-Lie n-algebroids over the vector bundle map G_V, it has the following properties:

- it maps the basic function $f \circ q_{V'}$ to $f \circ q_V$;

- it maps the fibre-wise linear function ℓ_ψ to a fibre-wise linear function $\ell_{\mu_0^*(\psi)}$, defining thus a $C^\infty(M)$-linear map $\mu_0^*: \Gamma(V'^*) \to \Gamma(V^*)$;

- it maps the section $\gamma_i \in \Gamma_V^l(D_i'^*)$ into the sum

$$\left(\Gamma(\underline{S}^i(\underline{A}^*)) \otimes \Gamma(V^*) \right) \oplus \left(\bigoplus_{j=1}^{i-1} \Gamma(\underline{S}^{i-j}(\underline{A}^*)) \otimes \Gamma(C_j^*) \right) \oplus \Gamma(C_i^*) = (\Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(V^* \oplus \underline{C}^*))^i,$$

Therefore, we may use again the formulae in equation (8.6) to define a $\Gamma(\underline{S}(\underline{A}^*))$-linear map

$$\mu^\star: \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(V'^* \oplus \underline{C}'^*) \to \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(V^* \oplus \underline{C}^*),$$

which commutes with the differentials due to the identity $G_D^\star \circ \mathcal{Q}_{\underline{D}'} = \mathcal{Q}_{\underline{D}} \circ G_D^\star$.   □

Combining now Theorem 8.5.2 with Proposition 8.5.3, we obtain the following result.

Theorem 8.5.4. There is an equivalence of categories between $\mathbb{DVB}\text{-Lie}_n(\underline{A})$ and $\mathbb{Rep}^{n+1}(\underline{A})$, and hence, an equivalence of categories $\mathbb{VB}\text{-Lie}_n(\underline{A})$ and $\mathbb{Rep}^{n+1}(\underline{A})$.

Remark 8.5.5. The theorem above could be used to understand the adjoint representation of a split Lie n-algebroid $\underline{A}$ in classical differential geometric terms. Recall that Section 6.6 gave a method for deriving the structure objects of the adjoint representation, once there is a choice of splitting $\underline{A}$ and TM-connections on the vector bundles A_i for all i. This could be very hard in practice if one were to work with a split Lie n-algebroid and its adjoint representation. Already in the case $n = 2$ the structure objects tend to be long and complicated. The advantage of Theorem 8.5.4 is that it gives a more compact way to define the adjoint representation of a split Lie n-algebroid $\underline{A}$ for any $n \in \mathbb{N}$. It is given by the isomorphy class in VB-Lie$_n(\underline{A})$ of the tangent prolongation $\underline{TA}$ from Example 8.1.6.

8.6   Change of decomposition

In the previous section, we gave the equivalence of VB-Lie n-algebroids and $(n+1)$-representations in terms of a decomposition. Now we will see that two different choices of a decomposition for the VB-Lie n-algebroid $(\underline{D}, V, \underline{A}, M)$ lead to isomorphic representations up to homotopy of $\underline{A}$.

Suppose that $(\underline{D}, V, \underline{A}, M)$ is a VB-Lie n-algebroid with core $\underline{C}$ and consider a pair of decompositions for $\underline{D}$ obtained by the horizontal lifts $h_i, h'_i : \Gamma(A_i) \to \Gamma^l_V(D_i)$ for all the double vector bundles (D_i, V, A_i, M). One then has the isomorphisms of decomposed double vector bundles

$$G_i : V \times_M A_i \times_M C_i \to D_i \to V \times_M A_i \times_M C_i$$

defined by mapping $(v, a_i, c_i) \in V \times_M A_i \times_M C_i$ from the decomposition obtained by h'_i to the element $(v, a_i, c_i + \Delta^i_{a_i}(v))$ in the decomposition of h_i, where $\Delta^i \in \Gamma(A^*_i \otimes V^* \otimes C_i)$ is given by the difference $h_i - h'_i$. Summing up all G_i, one obtains a morphism of decomposed double vector bundles

$$G : V \times_M \underline{A} \times_M \underline{C} \to \mathcal{D} \to V \times_M \underline{A} \times_M \underline{C}$$

inducing the identity on $\underline{A}$. Therefore, it follows from Proposition 8.5.3 that G corresponds to a morphism of $(n+1)$-representations of $\underline{A}$

$$\mu : \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{C} \oplus V) \to \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{C} \oplus V).$$

As usual, we denote the dual morphism by $\mu^\star$. Clearly, the homogeneous part $G^\star_0$ of $G^\star$ with respect to the decomposition $\underline{D}_V = q^!_V(\underline{A} \oplus \underline{C})$ is given by $\mathrm{id}_{\underline{A}^* \oplus \underline{C}^*}$. Recalling then the construction of the correspondence between morphisms of VB-Lie n-algebroids and $(n+1)$-representations, one sees that $\mu_0|_{\underline{C}^*} = \mathrm{id}_{\underline{C}^*} : \underline{C}^* \to \underline{C}^*$. In addition, given a section $\gamma_i \in \Gamma(C^*_i)$, one has that

$$G^\star(\gamma_i) = \gamma_i + (\Delta^i)^*(\gamma_i) = \gamma_i + \left(h^*_i(\gamma_i) - h'^*_i(\gamma_i) \right)$$

Finally, using the second equation in (8.6), we compute

$$\langle \mu^*_0|_{V^*}(\psi(m)), G(v_m, a^i_m, c^i_m) \rangle = G^\star(\ell_\psi)(v_m, a^i_m, c^i_m) = \langle \psi(m), G(v_m, a^i_m, c^i_m) \rangle$$

for a point $(v_m, a^i_m, c^i_m) \in \underline{D}$ in the decomposition of h' and a section $\psi \in \Gamma(V^*)$. Therefore, we also obtain that $\mu^*_0|_{V^*} = \mathrm{id}_{V^*} : V^* \to V^*$ and we conclude that $\mu^\star = \mathrm{id}_{\underline{C}^* \oplus V^*} \oplus \Delta^*$, or equivalently, $\mu = \mathrm{id}_{V \oplus \underline{C}} \oplus \Delta$.

One may use now the same construction for the isomorphism of decomposed double vector bundles $\widetilde{G}$ given by the difference $\widetilde{\Delta} = h' - h$ and obtain another morphism of representations up to homotopy $\widetilde{\mu}$ from the decomposition of h to the decomposition of h'. Observe now that, by construction, the maps $G^\star$ and $\widetilde{G}^\star$ act trivially on the space $\Gamma(\underline{A}^* \otimes V^*)$. Since $(\Delta^i)^*(\gamma_i) \in \Gamma(\underline{A}^* \otimes V^*)$ and $(\widetilde{\Delta}^i)^*(\gamma_i) \in \Gamma(\underline{A}^* \otimes V^*)$, it follows that $\widetilde{G}^\star((\Delta^i)^*(\gamma_i)) = (\Delta^i)^*(\gamma_i)$ and $G^\star((\widetilde{\Delta}^i)^*(\gamma_i)) = (\widetilde{\Delta}^i)^*(\gamma_i)$ for all $\gamma_i \in \Gamma(C^*_i)$, and similarly for h'_i. We conclude that the morphisms $\widetilde{\mu}$ and μ are inverses to each other and therefore we have proved the following result.

Proposition 8.6.1. Suppose $(\underline{D}, V, \underline{A}, M)$ is a VB-Lie n-algebroid. Let h_i and h'_i be two decompositions of the double vector bundle (D_i, V, A_i, M) for all $i = 1, \dots, n$, and denote their difference $\Delta^i = h_i - h'_i$. Then the two corresponding $(n+1)$-representations of $\underline{A}$ on the graded vector bundle $V[0] \oplus \underline{C}$ are isomorphic via the maps $\mu = \mathrm{id}_{V[0] \oplus \underline{C}} \oplus \Delta$ and $\mu^{-1} = \mathrm{id}_{V[0] \oplus \underline{C}} \oplus (-\Delta)$.

Remark 8.6.2. Note that in the case of the double vector bundle $(\underline{T}\underline{A}, TM, \underline{A}, M)$ obtained by the tangent prolongation of a Lie n-algebroid $\underline{A}$ over M, decompositions correspond to TM-connections on the vector bundles $A_i \to M$. The result above then reads that the two representatives of the adjoint representation $\Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\mathrm{ad}_\nabla) = \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(TM[0] \oplus \underline{A}) = \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\mathrm{ad}_{\nabla'})$ of $\underline{A}$ corresponding to the two choices of TM-connections ∇ and ∇' are related by the isomorphisms of $\underline{A}$-representations $\mu \colon \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\mathrm{ad}_\nabla) \to \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\mathrm{ad}_{\nabla'})$ and $\mu^{-1} \colon \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\mathrm{ad}_{\nabla'}) \to \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\mathrm{ad}_\nabla)$ given by $\mu = \mathrm{id}_{\mathrm{ad}_\nabla} \oplus \left(\nabla' - \nabla\right)$ and $\mu^{-1} = \mathrm{id}_{\mathrm{ad}_{\nabla'}} \oplus \left(\nabla - \nabla'\right)$.

Chapter 9

Conclusion: open problems and further research

In this chapter, we summarise the content of the main body of this thesis and present briefly some ideas for future research topics that are related to our results.

Summary   Chapters 2 and 3 laid the mathematical foundations in order for the reader to gain familiarity with our notation and conventions, and understand the structures that are important for the research results lying at the core of this thesis. This included classical differential geometric structures such as graded vector bundles, complexes of vector bundles, various notions of algebroids, double vector bundles, and sheaves on topological spaces, as well as supergeometric notions such as $\mathbb{Z}$- and $\mathbb{N}$-graded manifolds, vector bundles in the category of graded manifolds, $\mathcal{Q}$-structures and Lie n-algebroids, graded Poisson and symplectic structures on graded manifolds.

In Chapter 4, we moved to analysing further vector bundles in the category of graded manifolds, and focused on the two most important examples that are relevant to our work, namely the tangent and the cotangent bundle of a graded manifold. We presented numerous well-known geometric constructions on these vector bundles, such as (pseudo)multivector fields and (pseudo)differential forms, and the Weil and the Poisson-Weil algebras. In addition, we reviewed homotopy Poisson structures explaining how they serve as a unified framework for Poisson brackets and $\mathcal{Q}$-structures on graded manifolds, and recalled the correspondence of bialgebroids with Poisson Lie 1-algebroids, and the correspondence of Poisson manifolds and Courant algebroids with symplectic Lie 1- and symplectic Lie 2-algebroids, respectively.

In Chapter 5, we developed the theory of differential graded modules (DG-modules) for $\mathcal{Q}$-manifolds and Lie n-algebroids, and defined the two fundamental examples, i.e. the adjoint and the coadjoint module (Section 5.2). In particular, Sections 5.3 and 5.4 constructed the morphism relating these two DG-modules (Theorem 5.3.1) and the Weil with the Poisson-Weil algebras (Theorem 5.4.1), in the case of $\mathcal{PQ}$-manifolds. This result was applied to Courant algebroids in order to give an alternative description in terms of its adjoint and coadjoint modules.

Thereafter Chapter 6 defined representations up to homotopy of Lie n-algebroids and, similarly to the case of DG-modules, constructed the adjoint and the coadjoint representations (Section 6.6). In the case of split Lie 2-algebroids, 3-term representations were studied in detail (Proposition 6.2.5) and the explicit formulae for the structure objects of the adjoint and the coadjoint representations were given (Proposition 6.3.1 and and Section 6.4). Additionally, the coordinate transformation of the adjoint representation of a split Lie 2-algebroid was computed (Section 6.5) and the explicit map connecting the adjoint module with the adjoint representation of a general Lie n-algebroid was established (Section 6.4). Section 6.7 analysed the Weil algebra of a split Lie n-algebroid expressing its differentials in terms of the coadjoint representation and n-many double representations of the tangent Lie algebroid of the base smooth manifold, and finally Section 6.8 computed explicitly the map between the coadjoint and adjoint representations of a split Poisson Lie n-algebroid, for $n = 0, 1, 2$.

Chapters 7 and 8 dealt with linear structures of graded manifolds. More precisely, Chapter 7 used supergeometric language to put the notions of vector fields, $\mathcal{Q}$-structures, Poisson brackets, and homotopy Poisson structures in the context of vector bundles in the category of graded manifolds. On the other hand, Chapter 8 analysed linear $\mathcal{Q}$-structures in the language of classical differential geometry. In particular, it developed the notion of (split) VB-Lie n-algebroids as Lie n-algebroids that are linear over another Lie n-algebroid. These were described both classically using higher brackets (Section 8.1) and supergeometrically using a homological vector field (Sections 8.2 and 8.3). We defined the induced fat Lie n-algebroid of VB-Lie n-algebroid in Section 8.4 and proved the equivalence of categories between VB-Lie n-algebroids with fixed side and $(n + 1)$-term representations up to homotopy of its side Lie n-algebroid in Section 8.5 (Theorem 8.5.2, Proposition 8.5.3 and Theorem 8.5.4). Finally, in Section 8.6 we saw how different choices of decompositions of the VB-Lie n-algebroid lead to isomorphic representations up to homotopy of its side (Proposition 8.6.1) and applied this to express the transformation of the adjoint representation of a split Lie n-algebroid under different choices of linear connections (Remark 8.6.2), extending the corresponding result of Proposition 6.5.2 to the general case of $n \in \mathbb{N}$.

DG-modules and representations of Lie n-groupoids   As it was mentioned in the introduction, Lie n-groupoids are the global objects corresponding to Lie n-algebroids [22, 26]. The natural question that arises is whether the constructions developed in this thesis about Lie n-algebroids could be carried over to Lie n-groupoids. In the following, we give more details about this idea.

Definition. A **groupoid** is a small category such that every morphism is an isomorphism. Explicitly, a groupoid $\mathcal{G} \rightrightarrows M$ consists of a set of **objects** M (with no extra structure a priori) and a set of **arrows** $\mathcal{G}$ together with the following structure maps:

1. Two maps $s, t \colon \mathcal{G} \to M$ called **source** and **target**, respectively.

2. A **multiplication map** $m \colon \mathcal{G}_2 \to \mathcal{G}, (g, h) \mapsto gh$, where

$$\mathcal{G}_2 := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\},$$

that satisfies

(a) Compatibility with s and t: for $(g, h) \in \mathcal{G}_2$

$$s(gh) = s(h) \qquad \text{and} \qquad t(gh) = t(g)$$

(b) Associativity: for $(g, h, k) \in G_3$
$$g(hk) = (gh)k,$$

where in general $\mathcal{G}_0 := M, \mathcal{G}_1 := \mathcal{G}$ and for $k \geq 2$

$$\mathcal{G}_k := \{(g_1, \ldots, g_k) \mid g_j \in \mathcal{G}_1 \text{ and } s(g_j) = t(g_{j+1}) \text{ for all } j\}.$$

3. An embedding $1 \colon M \to \mathcal{G}, x \to 1_x$ called **identity map**, such that for all $g \in \mathcal{G}$

$$g1_{s(g)} = g = 1_{t(g)}g$$

and $s(1_x) = x = t(1_x)$ for all $x \in M$.

4. An **inversion map** $i \colon \mathcal{G} \to \mathcal{G}, g \mapsto g^{-1}$ such that for all $g \in \mathcal{G}$

$$s(g)^{-1} = t(g) \qquad t(g)^{-1} = s(g) \qquad g^{-1}g = 1_{s(g)}, \qquad gg^{-1} = 1_{t(g)}.$$

A **Lie groupoid** is a groupoid $\mathcal{G} \rightrightarrows M$ such that M and $\mathcal{G}$ are smooth manifolds (with $\mathcal{G}$ not necessarily Hausdorff), M is a closed submanifold of $\mathcal{G}$, all the structure maps are smooth, and the source and target maps are submersions.

Every groupoid $\mathcal{G} \rightrightarrows M$ comes with the following simplicial manifold (see e.g. [108, 5]) called its **nerve**: the manifold of k-simplices is $\mathcal{G}_k$ with face maps $\mathrm{d}_i^k \colon \mathcal{G}_k \to \mathcal{G}_{k-1}$

$$\mathrm{d}_i^k(g_1, \ldots, g_k) = \begin{cases} (g_2, \ldots, g_k), & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_k), & \text{if } 0 < i < k \\ (g_1, \ldots, g_{k-1}), & \text{if } i = k \end{cases}$$

and degeneracy maps $s_i^k \colon \mathcal{G}_k \to \mathcal{G}_{k+1}$

$$s_i^k(g_1, \ldots, g_k) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_k)$$

for $0 \leq i \leq k$. The spaces $C^k(\mathcal{G}) := C^\infty(\mathcal{G}_k)$ are equipped with the differential $\mathrm{d}_\mathcal{G}$ given by the alternating sum of the face maps d_i, and hence, $(C^k(\mathcal{G}), \mathrm{d}_\mathcal{G})$ is a complex whose cohomology is denoted $H^\bullet(\mathcal{G})$. Moreover, the space $C^\bullet(\mathcal{G})$ carries a natural algebra structure given by the multiplication map

$$(f * h)(g_1, \ldots, g_{k+\ell}) := (-1)^{k\ell} f(g_1, \ldots, g_k) h(g_{k+1}, \ldots, g_{k+\ell})$$

for $f \in C^k(\mathcal{G})$ and $h \in C^\ell(\mathcal{G})$. The triple $(C^k(\mathcal{G}), *, \mathrm{d}_\mathcal{G})$ has the structure of a differential graded algebra, i.e. one has the identity

$$\mathrm{d}_\mathcal{G}(f * h) = \mathrm{d}_\mathcal{G} f * h + (-1)^k f \mathrm{d}_\mathcal{G} h.$$

One uses the construction above to define higher versions of Lie groupoids as follows: A **Lie n-groupoid** is as a simplicial manifold $\mathcal{G}$ that satisfies the Kan conditions (see [107] for details).

Suppose now that $\mathcal{G} \rightrightarrows M$ is a Lie groupoid. Given a vector bundle $E \to M$, we define the graded vector space $C^\bullet(\mathcal{G}, E)$ which in degree k is given by $C^k(\mathcal{G}, E) := \Gamma(\mathcal{G}_k, t^*E)$. This graded space is a (right) graded $C^\bullet(\mathcal{G})$-module whose structure is given by

$$(\eta * f)(g_1, \ldots, g_{k+\ell}) := (-1)^{k\ell} \eta(g_1, \ldots, g_k) f(g_{k+1}, \ldots, g_\ell)$$

for $\eta \in C^k(\mathcal{G}, E)$ and $f \in C^\ell(\mathcal{G})$. Similarly as for Lie algebroids, one defines a **representation** of the Lie groupoid $\mathcal{G} \rightrightarrows M$ on the vector bundle $E \to M$ to be a degree 1 operator $\mathcal{D}$ on $C^\bullet(\mathcal{G}, E)$ that satisfies the Leibniz identity

$$\mathcal{D}(\eta * f) = \mathcal{D}(\eta) * f + (-1)^{|\eta|} \eta * \mathrm{d}_\mathcal{G} f.$$

In the case of a graded vector bundle $\underline{E} = \bigoplus_i E_i[i] \to M$, one considers the $\underline{E}$-valued cochains of $\mathcal{G}$ with respect to the total grading given by

$$C^k(\mathcal{G}, \underline{E}) = \bigoplus_{i-j=k} C^i(\mathcal{G}, E_j).$$

A degree 1 operator on this space which satisfies the above Leibniz rule is called **representation up to homotopy** of the Lie groupoid $\mathcal{G} \rightrightarrows M$ on the graded vector bundle $\underline{E} \to M$ (for more details see, e.g., [5, 2, 43, 89]).

One could generalise the constructions above to the world of Lie n-groupoids and try to prove the results of this thesis in this setting. More precisely, one could define DG-modules over the differential graded algebra $(C^\bullet(\mathcal{G}), \mathrm{d}_\mathcal{G})$ given by a Lie n-groupoid $\mathcal{G}$ and representations up to homotopy of $\mathcal{G}$, and prove that they are, in fact, equivalent. The first step for $n = 1$ is done in [89], where it is proved that representations up to homotopy of a Lie groupoid $\mathcal{G} \rightrightarrows M$ are the same as **cohesive modules** over the differential graded algebra $(C^\bullet(\mathcal{G}), \mathrm{d}_\mathcal{G})$, i.e. (bounded) graded right modules $\mathcal{E}$ over $(C^\bullet(\mathcal{G}), \mathrm{d}_\mathcal{G})$ which are finitely generated, projected, and equipped with a structure differential. One then could focus on the adjoint module and the adjoint representation of $\mathcal{G}$ and describe explicitly the isomorphism between them. Further, it would be interesting to understand when one can find a global analogue of Theorem 5.3.1, i.e. a connection of Poisson structures on the Lie n-groupoid with a natural morphism between its coadjoint and adjoint modules. Lastly, the question that arises is whether and how one can pass from the construction on Lie n-groupoids to the corresponding constructions on Lie n-algebroids and vice versa, i.e. differentiation and integration of modules and representations up to homotopy of Lie n-groupoids.

Double $\mathcal{Q}$-manifolds, ideals and reduction   Recall that in chapters 7 and 8, we developed the notion of linear $\mathcal{Q}$-manifolds, or in the split setting the notion of higher VB-algebroids. These were essentially described as commutative diagrams of the form

$$
\begin{array}{ccc}
\underline{D} & \longrightarrow & \underline{A} \\
\downarrow & & \downarrow \\
V & \longrightarrow & M
\end{array}
$$

together with a homological vector field on bidegree $(0,1)$ on the graded manifold $\underline{D}[1]$ (see Theorem 8.3.4). One could generalise the work of Voronov [102] and consider pairs of commuting homological vector fields $(\mathcal{Q}_1, \mathcal{Q}_2)$ on the graded manifold $\underline{D}[1]$ of bidegree $(1,0)$ and $(0,1)$, respectively; that is, $[\mathcal{Q}_1, \mathcal{Q}_2] = 0$. This would lead to the notion of double structures in graded geometry, i.e. **double $\mathcal{Q}$-manifolds** and **double Lie n-algebroids**.

The basic example of such double structures should be the tangent prolongation of a (split) Lie n-algebroid $\underline{A} \to M$. Next, one could consider subobjects of the tangent double Lie n-algebroid $(T\underline{A}, TM, \underline{A}, M)$ which in turn would form a generalisation of the notion of *infinitesimal ideal systems* of Lie n-algebroids (IIS) from [54]. Infinitesimal ideal systems of a Lie n-algebroid are expected to be linked to subrepresentations of the adjoint representation (up to homotopy) ad, similarly to the work done in [35]. The final step in this direction would be to define the quotient of the Lie n-algebroid $\underline{A} \to M$ with its IIS and obtain a Lie n-algebroid of smaller dimension over a quotient manifold, as it is proved in [54] for Lie algebroids. Note that this base manifold should be the leaf space of M with respect to a foliation given by the IIS. Consequently, one would have a well-defined notion of reduction of Lie n-algebroids for general $n \in \mathbb{N}$.

Appendices

Appendix A

More on graded geometry

A.1 The geometrisation of $\mathbb{N}$-graded vector bundles over $\mathbb{N}$-manifolds

A $\mathbb{Z}$**-graded double vector bundle** is a double vector bundle of the form $(\underline{D} := \bigoplus_{i \in \mathbb{Z}} D_i[i], V, \underline{A} := \bigoplus_{i \in \mathbb{Z}} A_i[i], M)$:

$$
\begin{array}{ccc}
\underline{D} & \longrightarrow & \underline{A} \\
\downarrow & & \downarrow \\
V & \longrightarrow & M
\end{array}
$$

with core $\underline{C} := \bigoplus_{i \in \mathbb{Z}} C_i[i]$, where the (finite) direct sums are taken over the bottom arrow $V \to M$, such that each (D_i, V, A_i, M) is a double vector bundle with core C_i. In fact, since every double vector bundle (D, V, A, M) with core C is non-canonically isomorphic to a decomposed double vector bundle $A \times_M V \times_M C$, it follows that every graded double vector bundle $(\underline{D}, V, \underline{A}, M)$ is non-canonically isomorphic to a **decomposed graded double vector bundle** $\underline{A} \times_M V \times_M \underline{C}$.

Proposition A.1.1. There is an one-to-one correspondence between $\mathbb{N}$-graded double vector bundles of degree n and vector bundles over $[n]$-manifolds whose total space is also an $[n]$-manifold.

Proof. Given an $\mathbb{N}$-graded double vector bundle $\underline{D} = \bigoplus_{i=1}^{n} D_i[i]$ of degree n, one has the associated vector bundle in the category of graded manifolds $\mathcal{E}_{\underline{D}} \to \mathcal{M}$, where the base is given by the split $[n]$-manifold $\mathcal{M}$ with sheaf $\mathcal{C}^{\infty}(\mathcal{M}) = \Gamma(\underline{S}(\underline{A}^*))$ and the sheaf of sections is given by $\Gamma(\mathcal{E}_{\underline{D}}) = \mathcal{C}^{\infty}(\mathcal{M}) \otimes \Gamma(C_n[n] \oplus \ldots \oplus C_1[1] \oplus V[0])$.

Conversely, suppose $\mathcal{E} \to \mathcal{M}$ is a vector bundle in the category of graded manifolds, such that both $\mathcal{E}$ and $\mathcal{M}$ are $[n]$-manifolds. Choose splittings of $\mathcal{M}$ and $\mathcal{E}$ as $\mathcal{C}^{\infty}(\mathcal{M}) = \Gamma(\underline{S}(\underline{A}^*))$ and $\Gamma(\mathcal{E}) = \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{E})$, for some negatively graded vector bundles $\underline{A} := \bigoplus_{i=1}^{n} A_i[i] \to M$ and $\underline{E} := \bigoplus_{i=1}^{n} E_i[i] \to M$, respectively. Then one obtains the family of decomposed double vector bundles $D_i := A_i \times_M E_0 \times_M E_i$, for $i = 1, \ldots, n$. Summing over all such i yields a decomposed

$\mathbb{N}$-graded double vector bundle $\underline{D}_{\mathcal{E}} := \bigoplus_{i=1}^{n} D_i[i]$ of degree n. Since any vector bundle over an $\mathbb{N}$-manifold admits a splitting and any graded double vector bundle admits a decomposition, it follows that the two procedures above are inverse to each other. $\qquad\qquad\square$

A.2   Cartan calculus on $\mathbb{Z}$-graded manifolds

In this appendix, we recall some important formulae from [71] that generalise the known results from the Cartan calculus on ordinary manifolds and are used extensively in this thesis.

In what follows, $\{\xi^j\}$ are local coordinates on the $\mathbb{Z}$-graded manifold $\mathcal{M}$, $\{\xi^j, \dot{\xi}^j\}$ are the corresponding coordinates on $T[1]\mathcal{M}$, and $\xi, \zeta^j \in \mathcal{C}^\infty(\mathcal{M}), \mathcal{X}, \mathcal{Y} \in \mathfrak{X}(\mathcal{M})$ are all homogeneous. Recall that $\mathrm{d} \in \mathfrak{X}^1(T[1]\mathcal{M})$ sends ξ^j to $\mathrm{d}\xi^j = \dot{\xi}^j$ and $\dot{\xi}^j$ to 0, and thus it follows immediately that $\mathrm{d}^2 = 0$ and $[\mathrm{d}, \mathrm{d}] = \mathrm{d}^2 + \mathrm{d}^2 = 0$. Recall also the vector field $i_\mathcal{X} \in \mathfrak{X}^{|\mathcal{X}|-1}(T[1]\mathcal{M})$ which sends ξ^j to 0 and $\dot{\xi}^j$ to $\mathcal{X}(\xi^j)$, and the Lie derivative $\mathcal{L}_\mathcal{X} = [i_\mathcal{X}, \mathrm{d}]$. As we explained in Section 4.2, locally these vector fields have the following form

$$\mathrm{d} = \dot{\xi}^j \frac{\partial}{\partial \xi^j}, \qquad i_\mathcal{X} = \zeta^j \frac{\partial}{\partial \dot{\xi}^j}, \qquad \mathcal{L}_\mathcal{X} = \zeta^j \frac{\partial}{\partial \xi^j} + (-1)^{|\mathcal{X}|}\mathrm{d}\zeta^j \frac{\partial}{\partial \dot{\xi}^j},$$

where $\mathcal{X} = \zeta^j \frac{\partial}{\partial \xi^j}$. This implies that

$$i_{\xi\mathcal{X}} = \xi\, i_\mathcal{X} \qquad \text{and} \qquad \mathcal{L}_{\xi\mathcal{X}} = \xi\, \mathcal{L}_\mathcal{X} + (-1)^{|\xi|+|\mathcal{X}|}\mathrm{d}\xi\, i_\mathcal{X}.$$

Proposition A.2.1. The following identities hold:

1. $[i_\mathcal{X}, i_\mathcal{Y}] = 0$,

2. $[\mathcal{L}_\mathcal{X}, i_\mathcal{Y}] = i_{[\mathcal{X}, \mathcal{Y}]}$,

3. $[\mathcal{L}_\mathcal{X}, \mathcal{L}_\mathcal{Y}] = \mathcal{L}_{[\mathcal{X}, \mathcal{Y}]}$,

4. $[\mathrm{d}, \mathcal{L}_\mathcal{X}] = 0$.

Proof. Write $\mathcal{X} = \zeta^j \frac{\partial}{\partial \xi^j}, \mathcal{Y} = \eta^i \frac{\partial}{\partial \xi^i}$. Using that any two vector fields on $T[1]\mathcal{M}$ are equal if they agree on functions of the form ξ and $\mathrm{d}\xi$, we compute:

$$[i_\mathcal{X}, i_\mathcal{Y}](\xi) = i_\mathcal{X}(i_\mathcal{Y}(\xi)) - (-1)^{(|\mathcal{X}|-1)(|\mathcal{Y}|-1)}i_\mathcal{Y}(i_\mathcal{X}(\xi)) = 0$$

and similarly

$$[i_\mathcal{X}, i_\mathcal{Y}](\mathrm{d}\xi) = i_\mathcal{X}(\mathcal{Y}(\xi)) - (-1)^{(|\mathcal{X}|-1)(|\mathcal{Y}|-1)}i_\mathcal{Y}(\mathcal{X}(\xi)) = 0.$$

This proves the first identity. For the second we compute

$$[\mathcal{L}_\mathcal{X}, i_\mathcal{Y}](\xi) = -(-1)^{|\mathcal{X}|(|\mathcal{Y}|-1)}i_\mathcal{Y}(\mathcal{L}_\mathcal{X}(\xi)) = 0 = i_{[\mathcal{X}, \mathcal{Y}]}(\xi)$$

and

$$[\pounds_{\mathcal{X}}, i_{\mathcal{Y}}](\mathrm{d}\xi) = \pounds_{\mathcal{X}}(\mathcal{Y}(\xi)) - (-1)^{|\mathcal{X}|(|\mathcal{Y}|-1)}i_{\mathcal{Y}}(\pounds_{\mathcal{X}}(\mathrm{d}\xi))$$
$$= \mathcal{X}(\mathcal{Y}(\xi)) - (-1)^{|\mathcal{X}||\mathcal{Y}|}\mathcal{Y}(\mathcal{X}(\xi))$$
$$= [\mathcal{X}, \mathcal{Y}](\xi)$$
$$= i_{[\mathcal{X},\mathcal{Y}]}(\mathrm{d}\xi).$$

For the third identity, clearly we have

$$[\pounds_{\mathcal{X}}, \pounds_{\mathcal{Y}}](\xi) = \mathcal{X}(\mathcal{Y}(\xi)) - (-1)^{|\mathcal{X}||\mathcal{Y}|}\mathcal{Y}(\mathcal{X}(\xi)) = \pounds_{[\mathcal{X},\mathcal{Y}]}(\xi)$$

but also

$$[\pounds_{\mathcal{X}}, \pounds_{\mathcal{Y}}](\mathrm{d}\xi) = -(-1)^{|\mathcal{Y}|-1}\pounds_{\mathcal{X}}(\mathrm{d}\mathcal{Y}(\xi)) + (-1)^{|\mathcal{X}||\mathcal{Y}|+|\mathcal{X}|-1}\pounds_{\mathcal{Y}}(\mathrm{d}\mathcal{X}(\xi))$$
$$= (-1)^{|\mathcal{X}|+|\mathcal{Y}|}\mathrm{d}(\mathcal{X}(\mathcal{Y}(\xi))) - (-1)^{|\mathcal{X}|+|\mathcal{Y}|+|\mathcal{X}||\mathcal{Y}|}\mathrm{d}(\mathcal{Y}(\mathcal{X}(\xi)))$$
$$= (-1)^{|\mathcal{X}|+|\mathcal{Y}|}\mathrm{d}([\mathcal{X}, \mathcal{Y}](\xi))$$
$$= \pounds_{[\mathcal{X},\mathcal{Y}]}(\mathrm{d}\xi).$$

To see the last identity we compute

$$[\mathrm{d}, \pounds_{\mathcal{X}}](\xi) = \mathrm{d}\mathcal{X}(\xi) - (-1)^{|\mathcal{X}|}\pounds_{\mathcal{X}}(\mathrm{d}\xi) = \mathrm{d}\mathcal{X}(\xi) - \mathrm{d}\mathcal{X}(\xi) = 0$$

and

$$[\mathrm{d}, \pounds_{\mathcal{X}}](\mathrm{d}\xi) = \mathrm{d}(\pounds_{\mathcal{X}}(\mathrm{d}\xi)) = -(-1)^{|\mathcal{X}|-1}\mathrm{d}^2\mathcal{X}(\xi) = 0. \qquad \square$$

A.3   Characteristic classes of 1st order

In this appendix, we review some constructions that are relevant to the theory of representations up to homotopy of Lie n-algebroids. Namely, we consider a relaxed version of representations $\mathcal{D}$ forgetting the condition $\mathcal{D}^2 = 0$ (called connections up to homotopy) and use them to define the notion of the Pontryagin algebra (also know as characteristic algebra) of a graded vector bundle with respect to a Lie n-algebroid. The case of ordinary Lie algebroids can be found in [39, 81, 91, 15].

First, we briefly recall some facts from the classical A-Pontryagin algebra of a vector bundle $E \to M$, where $A \to M$ is a Lie algebroid with differential d_A. Given an A-connection ∇ on the vector bundle $E \to M$, one has the curvature operator $R_\nabla \in \Omega^2(A, \mathrm{End}(E))$ and consequently its i-th powers $R_\nabla^i := R_\nabla \circ \ldots \circ R_\nabla \in \Omega^{2i}(A, \mathrm{End}(E))$ for all $i \geq 1$. The trace of the vector-valued $2i$-forms R_∇^i is d_A-closed and therefore we obtain the cohomology classes $[\mathrm{tr}(R_\nabla^i)] \in H^i(A)$. It is a standard result proved in [39], that the cohomology class $[\mathrm{tr}(R_\nabla^i)]$ does not depend on the choice of the A-connection ∇ on E, for all $i \geq 1$. The A-**Pontryagin algebra** of E is the $\mathbb{R}$-subalgebra

$\mathrm{Pont}_A^\bullet(E) \subset H^\bullet(A)$ generated by the cohomology classes $\sigma_A^i(E) := [\mathrm{tr}(R_\nabla^i)]$.

Remark A.3.1. The Pontryagin algebra $\mathrm{Pont}_A^\bullet(E)$ may be viewed as the image of a suitable $\mathbb{R}$-algebra morphism with target $H^{2\bullet}(A)$, as follows: Let $\mathrm{Sym}^\bullet(\mathfrak{gl}_k(\mathbb{R}))^{\mathrm{GL}_k(\mathbb{R})}$ denote the set of $\mathrm{GL}_k(\mathbb{R})$-invariant polynomial functions over $\mathfrak{gl}_k(\mathbb{R})$, i.e. polynomial functions $p\colon \mathfrak{gl}_k(\mathbb{R}) \to \mathbb{R}$ such that $p(gXg^{-1}) = p(X)$ for $g \in \mathrm{GL}_k(\mathbb{R})$ and $X \in \mathfrak{gl}_k(\mathbb{R})$. As an $\mathbb{R}$-algebra, $\mathrm{Sym}^\bullet(\mathfrak{gl}_k(\mathbb{R}))^{\mathrm{GL}_k(\mathbb{R})}$ is generated by the polynomial functions $P_0, P_1, P_2, \ldots$, where $P_i(X) := \mathrm{tr}(X^i)$ for $X \in \mathfrak{gl}_k(\mathbb{R})$ and tr is the usual trace operator (see [15]). Given now a Lie algebroid $A \to M$, a vector bundle $E \to M$ and an A-connection ∇ on E, each polynomial $p \in \mathrm{Sym}^\bullet(\mathfrak{gl}_k(\mathbb{R}))^{\mathrm{GL}_k(\mathbb{R})}$ defines a closed form $p(R_\nabla) \in \Omega^\bullet(A)$ and hence a cohomology class $[p(R_\nabla)] \in H^\bullet(A)$. This yields the **Chern-Weil morphism** of $\mathbb{R}$-algebras

$$\mathrm{Sym}^\bullet(\mathfrak{gl}_k(\mathbb{R}))^{\mathrm{GL}_k(\mathbb{R})} \to H^{2\bullet}(A), \qquad p \mapsto [p(R_\nabla)],$$

whose image in $H^\bullet(A)$ is the Pontryagin algebra $\mathrm{Pont}_A(E)$ of the vector bundle $E \to M$.

Now, we proceed to generalising the construction of the Pontryagin algebra in the context of general Lie n-algebroids, $n \in \mathbb{N}$. In what follows, $\underline{A} = A_1[1] \oplus \ldots \oplus A_n[n]$ will always be a split Lie n-algebroid with homological vector field $\mathcal{Q}$.

Suppose that $\underline{E}$ is a graded vector bundle over M. The **(graded) commutator** of two homogeneous elements $\omega_1, \omega_2 \in \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{End}}(\underline{E}))$ is defined by

$$[\omega_1, \omega_2] = \omega_1 \circ \omega_2 - (-1)^{|\omega_1||\omega_2|}\omega_2 \circ \omega_1.$$

The **graded** or **super trace operator** $\mathrm{str}\colon \Gamma(\underline{\mathrm{End}}^0(\underline{E})) \to C^\infty(M)$ is defined by

$$\mathrm{str}(\Phi) = (-1)^i \mathrm{tr}(\Phi)$$

for $\Phi \in \Gamma(\mathrm{End}(E_i))$; here, tr denotes the usual trace operator. The graded trace operator can be viewed as an element of $\Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{Hom}}(\underline{\mathrm{End}}^0(\underline{E}), \mathbb{R}))$ of degree 0 and hence, due to Lemma 3.4.1, there is a corresponding degree-preserving graded $\Gamma(\underline{S}(\underline{A}^*))$-linear map

$$\mathrm{str}\colon \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{End}}(\underline{E})) \to \Gamma(\underline{S}(\underline{A}^*))$$

which, by definition, vanishes on $\Gamma(\underline{S}^k(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{End}}^i(\underline{E}))$ for all $k \geq 0$ and $i \neq 0$ (i.e. str is non-trivial only on $\Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{End}}^0(\underline{E}))$). The proof of the following proposition can be carried over verbatim from the Lie algebroid case in [51].

Proposition A.3.2. The following identity holds for all $\omega_1, \omega_2 \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\mathrm{End}}(\underline{E}))$:

$$\mathrm{str}[\omega_1, \omega_2] = 0.$$

Definition A.3.3. A **connection up to homotopy** of $(\underline{A}, \mathcal{Q})$ on $\underline{E}$ is a degree 1 operator

$$\mathcal{D} \colon \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{E}) \to \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{E})$$

which satisfies the graded product rule

$$\mathcal{D}(\xi \wedge \omega) = \mathcal{Q}(\xi) \wedge \omega + (-1)^{|\xi|} \xi \wedge \mathcal{D}(\omega)$$

for all homogeneous $\xi \in \Gamma(\underline{S}(\underline{A}^*))$ and all $\omega \in \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{E})$. If $\underline{E}$ is concentrated only in n degrees, i.e. $\underline{E} = E_0[0] \oplus \ldots \oplus E_{n-1}[n-1]$, then a connection up to homotopy on $\underline{E}$ is called an **n-connection**.

Remark A.3.4. Note that if $(\underline{A}, \mathcal{Q})$ is just a Lie algebroid A with differential $\mathrm{d}_A = \mathcal{Q}$ and $\underline{E} = E$ is a usual (non-graded) vector bundle, then the notion of connection up to homotopy agrees with the usual notion of an A-connection on E.

In supergeometric terms, a connection up to homotopy is a degree 1 derivation of the vector bundle over the $[n]$-manifold $\underline{A}$ whose sheaf of sections is given by $\Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{E})$. It follows that a connection up to homotopy $\mathcal{D}$ on $\underline{E}$ which is **flat**, i.e. such that $\mathcal{D}^2 = 0$, is a representation up to homotopy of $\underline{A}$. Moreover, given connections up to homotopy $\mathcal{D}_{\underline{E}}$ on $\underline{E}$ and $\mathcal{D}_{\underline{F}}$ on $\underline{F}$, the same construction as for the representations up to homotopy defines connections up to homotopy on the bundles $\underline{S}(\underline{E}), \underline{\mathrm{Hom}}(\underline{E}, \underline{F})$, etc. In particular, the graded vector bundle $\underline{\mathrm{End}}(\underline{E})$ is endowed with the connection $\mathcal{D}_{\underline{\mathrm{End}}}$ given by the graded commutator

$$\mathcal{D}_{\underline{\mathrm{End}}}(\Phi) = [\mathcal{D}_{\underline{E}}, \Phi] := \mathcal{D}_{\underline{E}} \circ \Phi - (-1)^{|\Phi|} \Phi \circ \mathcal{D}_{\underline{E}}.$$

The following three lemmas are crucial for defining the Pontryagin characters. Their proofs work as the proofs of the similar results in [51], but are repeated here for the convenience of the reader.

Lemma A.3.5. A connection up to homotopy $\mathcal{D}$ of the split Lie n-algebroid $\underline{A}$ on $\underline{E}$ can always be written as a sum

$$\mathcal{D} = \mathrm{d}_\nabla + \Theta,$$

where ∇ is a usual degree-preserving connection of the (skew-symmetric) dull algebroid $A_1 \to M$ and Θ is an element in

$$\bigoplus_{i+j=1, j \neq 0} \Gamma(\underline{S}^i(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{End}}^j(\underline{E})).$$

Proof. Choose a degree-preserving A_1-connection $\nabla' \colon \Gamma(A_1) \times \Gamma(\underline{E}) \to \Gamma(\underline{E})$. The degree 1 operator $\mathcal{D} - \mathrm{d}'_\nabla$ on $\Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{E})$ is graded $\Gamma(\underline{S}(\underline{A}^*))$-linear and hence it is given as the wedge product with a degree 1 element $\Theta' \in \Gamma(\underline{S}(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{End}}(\underline{E}))$. Writing Θ' in terms of its components

$$\Theta' = \sum_{k \in \mathbb{Z}} \Theta'_k \in \bigoplus_{k \in \mathbb{Z}} \Gamma(\underline{S}^k(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{End}}^{1-k}(\underline{E})),$$

one easily checks that $\mathrm{d}_\nabla := \mathrm{d}_{\nabla'} + \Theta'_1$ is a new degree-preserving A_1-connection on $\underline{E}$ and $\mathcal{D} = \mathrm{d}_\nabla + (\Theta' - \Theta'_1)$. The claim is proved with $\Theta = \Theta' - \Theta'_1 \in \bigoplus_{k \in \mathbb{Z}, k \neq 1} \Gamma(\underline{S}^k(\underline{A}^*)) \otimes \Gamma(\underline{\mathrm{End}}^{1-k}(\underline{E}))$.   □

Lemma A.3.6. Let $A \to M$ be a dull algebroid and d_∇ a degree-preserving A-connection on a graded vector bundle $\underline{E}$. The A-connections $\mathrm{d}_{\nabla^{\underline{\mathrm{End}}}}$ on $\underline{\mathrm{End}}(\underline{E})$ and d_A on $M \times \mathbb{R}$ give rise to the A-connection $\mathrm{d}_{\nabla^{\underline{\mathrm{Hom}}}}$ on the graded vector bundle $\underline{\mathrm{Hom}}(\underline{\mathrm{End}}(\underline{E}), \mathbb{R})$. Then

$$\mathrm{d}_{\nabla^{\underline{\mathrm{Hom}}}} \mathrm{str} = 0.$$

Proof. Consider a local frame $\{e_1^k, \ldots, e_{r_k}^k\}$ of E_k for each k and denote its dual frame on E_k^* by $\{\varepsilon_1^k, \ldots, \varepsilon_{r_k}^k\}$. Then, for all $a \in \Gamma(A_1)$ one computes

$$
\begin{aligned}
(\nabla_a^{\underline{\mathrm{Hom}}} \mathrm{str})(e_i^k \otimes \varepsilon_j^k) &= -\,\mathrm{str}(\nabla_a^{\underline{\mathrm{End}}}(e_i^k \otimes \varepsilon_j^k)) \\
&= -(-1)^k \sum_{p=1}^{r_k} \langle \varepsilon_p^k, \nabla_a^{\underline{\mathrm{End}}}(e_i^k \otimes \varepsilon_j^k)(e_p^k) \rangle \\
&= (-1)^{k+1} \sum_{p=1}^{r_k} \langle \varepsilon_p^k, \nabla_a(\delta_{jp} e_i^k) - e_i^k \langle \varepsilon_j^k, \nabla_a e_p^k \rangle \rangle \\
&= 0.
\end{aligned}
$$
□

Lemma A.3.7. Let $\mathcal{D}$ be a connection up to homotopy of $(\underline{A}, \mathcal{Q})$ on $\underline{E}$. Then

$$\mathrm{str} \circ \mathcal{D}_{\underline{\mathrm{End}}} = \mathcal{Q} \circ \mathrm{str}. \tag{A.1}$$

Proof. Due to the derivation rule of $\mathcal{Q}$ and $\mathcal{D}_{\underline{\mathrm{End}}}$, and the $\Gamma(\underline{S}(\underline{A}^*))$-linearity of str, it suffices to check the identity on graded endomorphism $\Phi \in \Gamma(\underline{\mathrm{End}}^i(\underline{E}))$ for all $i \in \mathbb{Z}$. Suppose first that $\Phi \in \Gamma(\underline{\mathrm{End}}^0(\underline{E}))$. This means that $\mathrm{str}(\Phi) \in \Gamma(\underline{S}^0(\underline{A}^*)) = C^\infty(M)$ and thus $\mathcal{Q}(\mathrm{str}(\Phi)) = \mathrm{d}_{A_1}(\mathrm{str}(\Phi)) \in \Gamma(A_1^*)$. Write $\mathcal{D} = \mathrm{d}_\nabla + \Theta$ with ∇ and Θ as in Lemma A.3.5 and compute

$$\mathcal{D}_{\underline{\mathrm{End}}}(\Phi) = \mathcal{D} \circ \Phi - (-1)^{|\Phi|} \Phi \circ \mathcal{D} = \mathrm{d}_{\nabla^{\underline{\mathrm{End}}}} \Phi + [\Theta, \Phi],$$

where $\mathrm{d}_{\nabla^{\underline{\mathrm{End}}}}$ is the (degree-preserving) A_1-connection on $\underline{\mathrm{End}}(\underline{E})$ induced by d_∇ on $\underline{E}$. Hence, by Proposition A.3.2:

$$\mathrm{str}(\mathcal{D}_{\underline{\mathrm{End}}}(\Phi)) = \mathrm{str}(\mathrm{d}_{\nabla^{\underline{\mathrm{End}}}} \Phi).$$

Thus, Lemma A.3.6 and the above give

$$
\begin{aligned}
\mathcal{Q}(\mathrm{str}\,\Phi) - \mathrm{str}(\mathcal{D}_{\underline{\mathrm{End}}} \Phi) &= \mathrm{d}_{A_1}(\mathrm{str}\,\Phi) - \mathrm{str}(\mathcal{D}_{\underline{\mathrm{End}}} \Phi) \\
&= \mathrm{d}_{A_1}(\mathrm{str}\,\Phi) - \mathrm{str}(\mathrm{d}_{\nabla^{\underline{\mathrm{End}}}} \Phi) \\
&= (\mathrm{d}_{\nabla^{\underline{\mathrm{Hom}}}} \mathrm{str})(\Phi) = 0.
\end{aligned}
$$

Suppose now that $\Phi \in \Gamma(\underline{\mathrm{End}}^i(\underline{E}))$ with $i \neq 0$. By definition, $\mathcal{Q}(\mathrm{str}(\Phi)) = \mathcal{Q}(0) = 0$. On the

other hand, the decomposition

$$\mathcal{D}_{\underline{\text{End}}}\Phi = \mathrm{d}_{\nabla^{\underline{\text{End}}}}\Phi + [\Theta, \Phi] \in \left(\Gamma(A_1^*) \otimes \Gamma(\underline{\text{End}}^i(\underline{E}))\right) \oplus \Gamma(\underline{\text{End}}^{i+1}(\underline{E}))$$

yields

$$\mathrm{str}(\mathcal{D}_{\underline{\text{End}}}(\Phi)) = \mathrm{str}(\mathrm{d}_{\nabla^{\underline{\text{End}}}}\Phi) = 0$$

since $i \neq 0$ and the graded trace of $[\Theta, \Phi]$ vanishes by Proposition A.3.2. $\qquad\square$

Let $(\mathcal{M}, \mathcal{Q})$ be a Lie n-algebroid with a connection up to homotopy $\mathcal{D}$ on $\underline{E}$. The product rule of $\mathcal{D}$ yields the identity

$$\mathcal{D}^2(\xi \wedge \omega) = \mathcal{D}(\mathcal{Q}(\xi) \wedge \omega + (-1)^{|\xi|}\xi \wedge \mathcal{D}(\omega)) = (-1)^{2|\xi|}\xi \wedge \mathcal{D}^2(\omega) = \xi \wedge \mathcal{D}^2(\omega)$$

for all homogeneous $\xi \in \mathcal{C}^\infty(\mathcal{M})$ and all $\omega \in \mathcal{C}^\infty(\mathcal{M}) \otimes (\Gamma(\underline{E}))$. In other words, $\mathcal{D}^2$ is graded $\mathcal{C}^\infty(\mathcal{M})$-linear and thus $\mathcal{D}^2 = R_\mathcal{D}$, where $R_\mathcal{D} \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\text{End}}(\underline{E}))$ of total degree 2.

Define now for each $k \geq 1$ the elements $R_\mathcal{D}^k := R_\mathcal{D} \circ \ldots \circ R_\mathcal{D} = \mathcal{D}^{2k} \in \mathcal{C}^\infty(\mathcal{M}) \otimes \Gamma(\underline{\text{End}}(\underline{E}))$ (k-times) of total degree $2k$. The **Bianchi identity**

$$\mathcal{D}_{\underline{\text{End}}}R_\mathcal{D}^k = [\mathcal{D}, \mathcal{D}^{2k}] = 0 \tag{A.2}$$

is immediate for all $k \geq 1$.

Proposition A.3.8. Let $(\mathcal{M}, \mathcal{Q})$ be a Lie n-algebroid and $\underline{E}$ a graded vector bundle over M.

1. For any connection up to homotopy $\mathcal{D}$ of $(\mathcal{M}, \mathcal{Q})$ on $\underline{E}$,

$$\mathcal{Q}(\mathrm{str}(R_\mathcal{D}^k)) = 0$$

   for all $k \geq 1$.

2. The cohomology classes $[\mathrm{str}(R_\mathcal{D}^k)] \in H^{2k}(\mathcal{M}, \mathcal{Q})$ do not depend on the connection $\mathcal{D}$.

Proof. The first part follows from the Bianchi identity and Lemma A.3.7. The second part is proved with the standard technique from classical TM-Pontryagin classes of vector bundles [81]. $\qquad\square$

Definition A.3.9. Let $(\mathcal{M}, \mathcal{Q})$ be a Lie n-algebroid over the smooth manifold M and let $\underline{E} \to M$ be a graded vector bundle. The $\mathcal{M}$-**Pontryagin algebra** of $\underline{E}$

$$\mathrm{Pont}_\mathcal{M}^\bullet(\underline{E}) \subset H^\bullet(\mathcal{M})$$

is defined as the subalgebra generated by the cohomology classes

$$\sigma_\mathcal{M}^i(\underline{E}) := [\mathrm{str}(R_\mathcal{D}^i)] \in H^{2i}(A)$$

for all $i \geq 1$.

Remark A.3.10. Note that after a choice of splitting for the Lie n-algebroid $\mathcal{M} \simeq A_1[1] \oplus \ldots \oplus A_n[n]$, one can represent the generators $\sigma^i_{\mathcal{M}}(\underline{E}) = [\mathrm{str}(R^i_{\mathcal{D}})]$ using a degree-preserving A_1-connection on $\underline{E} = \bigoplus_{i \in \mathbb{Z}} E_i$, i.e. $\mathcal{D} = (\nabla^i)_{i \in \mathbb{Z}}$ with ∇^i an A_1-connection on E_i. A choice of TM-connections ∇^i on the vector bundles E_i yields A_1-connections on E_i, denoted again by ∇^i, by the formula $\nabla^i_a e = \nabla^i_{\rho(a)} e$ for $a \in \Gamma(A_1)$ and $e_i \in \Gamma(E_i)$. Hence, one has that

$$\mathrm{Pont}^\bullet_{\mathcal{M}}(\underline{E}) = \rho^\star \mathrm{Pont}^\bullet(\underline{E}),$$

where $\mathrm{Pont}^\bullet(\underline{E})$ denotes the usual TM-Pontryagin algebra of $\underline{E}$, which implies that the $\mathcal{M}$-Pontryagin algebra of $\underline{E} \to M$ is completely characterised by those of TM. However, as [51] shows the Pontryagin classes with respect to a Lie algebroid $A \to M$ can be used in order to give obstructions for the existence of A-representations on a graded vector bundle $\underline{E} \to M$. In particular, [51, Thm. 4.13] proves that there is a 2-representation of $A \to M$ on the graded vector bundle $E_0[0] \oplus E_1[1] \to M$ only if

$$\mathrm{Pont}^\bullet_A(E_0) = \mathrm{Pont}^\bullet_A(E_1) \subset H^\bullet(A),$$

or alternatively, only if the pull-backs under $\rho : A \to TM$ of the TM-Pontryagin classes of E_0 and E_1 coincide.

Remark A.3.11. Characteristic classes of 2nd order in the context of Lie algebroids were defined by Mehta in [75]. These classes are associated to graded vector bundles $\underline{E} \to M$ equipped with the differential $\mathcal{D}$ of an A-representation up to homotopy, in contrast to classes of 1st order which vanish if $\mathcal{D}^2 = 0$. In particular, [75, Thm. 7.7] shows that characteristic classes of 2nd order are invariant with respect to **gauge transformations**; that is, DG A-module automorphisms μ between the representations up to homotopy $(\underline{E}, \mathcal{D})$ and $(\underline{E}, \mathcal{D}')$ on the same $\underline{E}$ that make the following diagram commute

$$
\begin{array}{ccc}
\Omega(A, \underline{E}) & \xrightarrow{\ \mu\ } & \Omega(A, \underline{E}) \\
{\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\
\Gamma(\underline{E}) & \xrightarrow[\ \mathrm{id}\ ]{} & \Gamma(\underline{E})
\end{array}
$$

Here, the map $\mathrm{pr} : \Omega(A, \underline{E}) \to \Gamma(\underline{E})$ is the natural projection whose kernel is $\bigoplus_{p \geq 1} \Omega^p(A) \otimes \Gamma(\underline{E})$.

Appendix B

Computations

B.1 Split Lie 2-algebroids in the geometric setting

Our goal in this appendix is to prove that Definition 3.3.4 and Definition 3.3.6 for split Lie 2-algebroids are equivalent. The original source of the following proof is [50]. For convenience, we repeat the two definitions here.

Definition B.1.1. A split Lie 2-algebroid is a split [2]-manifold $Q[1] \oplus B^*[2] \to M$ equipped with a homological vector field $\mathcal{Q}$.

Definition B.1.2. A split Lie 2-algebroid is given by a pair of an anchored vector bundle $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with a vector bundle map $\ell \colon B^* \to Q$, a skew-symmetric dull bracket $[\cdot, \cdot] \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$, a linear Q-connection ∇ on B, and a vector valued 3-form $\omega \in \Omega^3(Q, B^*)$ such that

  (i) $\nabla^*_{\ell(\beta_1)}\beta_2 + \nabla^*_{\ell(\beta_2)}\beta_1 = 0$, for all $\beta_1, \beta_2 \in \Gamma(B^*)$,

 (ii) $[q, \ell(\beta)] = \ell(\nabla^*_q \beta)$ for all $q \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,

(iii) $\mathrm{Jac}_{[\cdot, \cdot]} = \ell \circ \omega \in \Omega^3(Q, Q)$,

 (iv) $R_{\nabla^*}(q_1, q_2)\beta = -\omega(q_1, q_2, \ell(\beta))$ for $q_1, q_2 \in \Gamma(Q)$ and $\beta \in \Gamma(B^*)$,

  (v) $\mathrm{d}_{\nabla^*}\omega = 0$.

Note that the third condition in the last definition implies $\rho_Q \circ \ell = 0$. To prove this correspondence, we will need the following elements: the bundle map $\partial_B = \ell^* \colon Q^* \to B$, the **Dorfman connection** $\Delta \colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ that is dual to the (skew-symmetric) dull bracket $[\cdot, \cdot]$

$$\langle \Delta_q \tau, q' \rangle = \rho(q)\langle \tau, q' \rangle - \langle \tau, [q, q'] \rangle$$

for $q, q' \in \Gamma(Q), \tau \in \Gamma(Q^*)$, and the element $R_\omega \in \Omega^2(Q, \mathrm{Hom}(B, Q^*))$ defined by

$$R_\omega(q_1, q_2)b = \langle i_{q_2} i_{q_1} \omega, b \rangle.$$

One then easily checks that (ii) is equivalent to $\partial_B \circ \Delta_q = \nabla_q \circ \partial_B$ and (iii) is equivalent to $R_\omega(q_1, q_2) \circ \partial_B = R_\Delta(q_1, q_2)$ for $q, q_1, q_2 \in \Gamma(Q)$.

Starting with the data given in Definition B.1.2, the homological vector field $\mathcal{Q}$ is defined by the following equations:

$$\mathcal{Q}(f) = \rho_Q^* \mathrm{d}f \in \Gamma(Q^*), \tag{B.1}$$

$$\mathcal{Q}(\tau) = \mathrm{d}_Q \tau + \partial_B \tau \in \Omega^2(Q) \oplus \Gamma(B), \tag{B.2}$$

$$\mathcal{Q}(b) = \mathrm{d}_\nabla b - \langle \omega, b \rangle \in \Omega^1(Q, B) \oplus \Omega^3(Q), \tag{B.3}$$

for $f \in C^\infty(M), \tau \in \Gamma(Q^*), b \in \Gamma(B)$. Conversely, using the Leibniz identity for an arbitrary vector field of degree 1 on the [2]-manifold $\mathcal{M} = Q[1] \oplus B^*[2]$, one can show that it must be given by the equations above, defining an anchor map $\rho_Q : Q \to TM$, a dull bracket $[\cdot, \cdot]$ on Q, a Q-connection ∇ on B, a vector valued 3-form $\omega \in \Omega^3(Q, B^*)$, and a bundle morphism $\partial_B : Q^* \to B$. It remains to show that $\mathcal{Q}^2 = 0$ corresponds to the various relations between these objects. For $f \in C^\infty(M)$ we have

$$\mathcal{Q}^2(f) = \mathrm{d}_Q(\rho_Q^* \mathrm{d}f) + \partial_B(\rho_Q^* \mathrm{d}f) \in \Omega^2(Q) \oplus \Gamma(B).$$

Since for $q_1, q_2 \in \Gamma(Q)$

$$\begin{aligned}
\mathrm{d}_Q(\rho_Q^* \mathrm{d}f)(q_1, q_2) = & \rho_Q(q_1) \left(\langle \rho_Q^* \mathrm{d}f, q_2 \rangle \right) - \rho_Q(q_2) \left(\langle \rho_Q^* \mathrm{d}f, q_1 \rangle \right) - \langle \rho_Q^* \mathrm{d}f, [q_1, q_2] \rangle \\
= & \left(\rho_Q(q_1) \rho_Q(q_2) - \rho_Q(q_2) \rho_Q(q_1) - \rho_Q[q_1, q_2] \right) f,
\end{aligned}$$

we have that $\mathcal{Q}^2(f) = 0$ for all $f \in C^\infty(M)$ if and only if $\partial_B \circ \rho_Q^* = 0$ and $\rho_Q[q_1, q_2] = [\rho_Q(q_1), \rho_Q(q_2)]$ for all $q_1, q_2 \in \Gamma(Q)$.

Definition B.1.3. Given $\partial_B : Q^* \to B$ as above, we define the vector bundle morphism $\partial_B : \Omega^k(Q) \to \Omega^{k-1}(Q, B)$ by

$$\partial_B (\tau_1 \wedge \ldots \wedge \tau_k) = \sum_{i=1}^{k} (-1)^{i+1} \tau_1 \wedge \ldots \wedge \widehat{\tau_i} \wedge \ldots \wedge \tau_k \wedge \partial_B \tau_i.$$

Lemma B.1.4. For the homological vector field $\mathcal{Q}$ defined on $\Gamma(Q^*)$ by equation B.2, the following two identities hold:

(i) $\mathcal{Q} = \mathrm{d}_Q + \partial_B$ on $\Omega^\bullet(Q)$.

(ii) $\mathcal{Q} = \mathrm{d}_\nabla - \langle \omega, \cdot \rangle$ on $\Gamma(S^\bullet B)$, where we define

$$\langle \omega, b_1 b_2 \ldots b_k \rangle = \sum_{i=1}^{k} b_1 b_2 \ldots \langle \omega, b_i \rangle \ldots b_k,$$

for $b_1, b_2, \ldots, b_k \in \Gamma(B)$.

Proof.   (i) Consider an element $\theta = \tau_1 \wedge \ldots \wedge \tau_k \in \Omega^k(Q)$ with $\tau_1, \ldots, \tau_k \in \Gamma(Q^*)$. Using the

Leibniz identity, one computes

$$\mathcal{Q}(\theta) = \mathcal{Q}\left(\tau_1 \wedge \ldots \wedge \tau_k\right) = \sum_{i=1}^{k}(-1)^{i+1}\tau_1 \wedge \ldots \wedge \mathcal{Q}(\tau_i) \wedge \ldots \wedge \tau_k$$

$$= \sum_{i=1}^{k}(-1)^{i+1}\tau_1 \wedge \ldots \wedge \mathrm{d}_Q(\tau_i) \wedge \ldots \wedge \tau_k + \sum_{i=1}^{k}(-1)^{i+1}\tau_1 \wedge \ldots \wedge \partial_B(\tau_i) \wedge \ldots \wedge \tau_k$$

$$= \mathrm{d}_Q\theta + \partial_B\theta.$$

(ii) Similarly, considering an element $b = b_1 b_2 \ldots b_k \in \Gamma(S^k B)$, we compute

$$\mathcal{Q}(b) = \mathcal{Q}(b_1 b_2 \ldots b_k) = \sum_{i=1}^{k} b_1 b_2 \ldots \mathcal{Q}(b_i) \ldots b_k$$

$$= \sum_{i=1}^{k} b_1 b_2 \ldots \mathrm{d}_\nabla b_i \ldots b_k - \sum_{i=1}^{k} b_1 b_2 \ldots \langle \omega, b_i \rangle \ldots b_k$$

$$= \mathrm{d}_\nabla b - \langle \omega, b \rangle. \qquad \square$$

Let now $\tau \in \Gamma(Q^*)$. Then we have

$$\mathcal{Q}^2(\tau) = \mathcal{Q}(\mathrm{d}_Q\tau) + \mathrm{d}_\nabla\partial_B\tau - \langle \omega, \partial_B\tau \rangle,$$

and using Lemma B.1.4 it becomes

$$\mathcal{Q}^2(\tau) = \left(\mathrm{d}_Q^2\tau - \langle \omega, \partial_B\tau \rangle\right) + \left(\partial_B\mathrm{d}_Q\tau + \mathrm{d}_\nabla\partial_B\tau\right) \in \Omega^3(Q) \oplus \Omega^1(Q, B).$$

Since for all $q, q_1, q_2, q_3 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, we have $\mathrm{d}_Q^2\tau(q_1, q_2, q_3) = \langle \mathrm{Jac}_{[\cdot,\cdot]}(q_1, q_2, q_3), \tau \rangle$ and $\langle \beta, (\partial_B\mathrm{d}_Q\tau)(q) \rangle = -\langle \beta, \partial_B\Delta_q\tau \rangle$, it follows that $\mathcal{Q}^2(\tau) = 0$ if and only if $\mathrm{Jac}_{[\cdot,\cdot]}(q_1, q_2, q_3) = \ell(\omega(q_1, q_2, q_3))$ and $\partial_B\Delta_q\tau = \nabla_q(\partial_B\tau)$.

Lastly, for $b \in \Gamma(B)$ we have

$$\mathcal{Q}^2(b) = \mathcal{Q}(\mathrm{d}_\nabla b) - \mathrm{d}_Q\langle \omega, b \rangle - \partial_B\langle \omega, b \rangle.$$

Definition B.1.5. For $b \in \Gamma(B)$, we define the element $\langle \omega, \mathrm{d}_\nabla b \rangle \in \Omega^4(Q)$ by

$$\langle \omega, \mathrm{d}_\nabla b \rangle(q_1, q_2, q_3, q_4) = -\sum_{\sigma \in C_4} \mathrm{sgn}(\sigma) \left\langle \omega\left(q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)}\right), \nabla_{q_{\sigma(4)}} b \right\rangle$$

for all $q_i \in \Gamma(Q), i = 1, 2, 3, 4$, where C_4 is the group of 4-cyclic permutations of four elements.

Definition B.1.6. For $b \in \Gamma(B)$, we define the element $\nabla_{\partial_B^*} b \in \Gamma(S^2 B)$ given by

$$(\nabla_{\partial_B^*} b)(\beta_1, \beta_2) = \left\langle \nabla_{\partial_B^*\beta_1} b, \beta_2 \right\rangle + \left\langle \nabla_{\partial_B^*\beta_2} b, \beta_1 \right\rangle$$

for all $\beta_1, \beta_2 \in \Gamma(B^*)$.

Lemma B.1.7. Let $b \in \Gamma(B)$, then we have

$$\mathcal{Q}(\mathrm{d}_\nabla b) = \mathrm{d}_\nabla^2 b + \nabla_{\partial_B^*} b - \langle \omega, \mathrm{d}_\nabla b \rangle \in \Omega^2(Q, B) \oplus \Gamma(S^2 B) \oplus \Omega^4(Q).$$

Proof. Suppose that $\mathrm{d}_\nabla b = \sum \tau_i \wedge c_i$, where $\tau_i \in \Omega^1(Q), c_i \in \Gamma(B)$. Using the definition of $\mathcal{Q}$ together with the Leibniz identity, we compute

$$\mathcal{Q}(\mathrm{d}_\nabla b) = \mathcal{Q}\left(\sum_i \tau_i \wedge c_i\right) = \sum_i \mathrm{d}_Q \tau_i \wedge c_i - \tau_i \wedge \mathrm{d}_\nabla c_i + \partial_B \tau_i \wedge c_i + \tau_i \wedge \langle \omega, c_i \rangle$$

$$= \mathrm{d}_\nabla^2 b + \sum_i \partial_B \tau_i \wedge c_i + \sum_i \tau_i \wedge \langle \omega, c_i \rangle.$$

For the second term, let $\beta_1, \beta_2 \in \Gamma(B^*)$. Then we have

$$\left(\sum_i \partial_B \tau_i \wedge c_i\right)(\beta_1, \beta_2) = \sum_i \partial_B \tau_i(\beta_1) c_i(\beta_2) + \partial_B \tau_i(\beta_2) c_i(\beta_1)$$

$$= \sum_i \tau_i(\partial_B^* \beta_1) c_i(\beta_2) + \tau_i(\partial_B^* \beta_2) c_i(\beta_1)$$

$$= \nabla_{\partial_B^*} b(\beta_1, \beta_2).$$

To find the last term, consider $q_1, q_2, q_3, q_4 \in \Gamma(Q)$. Then we have

$$\left(\sum_i \tau_i \wedge \langle \omega, c_i \rangle\right)(q_1, q_2, q_3, q_4) = \sum_i \Big(\langle \omega(q_2, q_3, q_4), \tau_i(q_1) c_i \rangle - \langle \omega(q_1, q_3, q_4), \tau_i(q_2) c_i \rangle$$

$$+ \langle \omega(q_1, q_2, q_4), \tau_i(q_3) c_i \rangle - \langle \omega(q_1, q_2, q_3), \tau_i(q_4) c_i \rangle \Big)$$

$$= \langle \omega(q_2, q_3, q_4), \sum_i \tau_i(q_1) c_i \rangle - \langle \omega(q_1, q_3, q_4), \sum_i \tau_i(q_2) c_i \rangle$$

$$+ \langle \omega(q_1, q_2, q_4), \sum_i \tau_i(q_3) c_i \rangle - \langle \omega(q_1, q_2, q_3), \sum_i \tau_i(q_4) c_i \rangle$$

$$= \langle \omega, \mathrm{d}_\nabla b \rangle(q_1, q_2, q_3, q_4). \qquad \square$$

The remaining equations follow now by grouping together the different summands of $\mathcal{Q}^2(b)$ and equalising them to zero.

B.2   Signs for the linearity of $\pi^\sharp : \Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$

In this appendix, we explain the choice of our signs for the map of DG-modules $\pi^\sharp : \Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, where $(\mathcal{M}, \mathcal{Q})$ is a $\mathcal{Q}$-manifold equipped with a Poisson bracket $\{\cdot, \cdot\}_k$.

Recall that on exact 1-forms $\mathrm{d}\xi_1 \in \Omega^1(\mathcal{M})$, the map $\pi^\sharp$ is defined as $\pi^\sharp(\mathrm{d}\xi_1)(\xi_2) = \{\xi_1, \xi_2\}_k$, for all $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$. In order for $\pi^\sharp$ to be consistent with the properties of the de Rham differential

d on $\mathcal{C}^\infty(\mathcal{M})$, we need an appropriate choice of signs for the formulae

$$\pi^\sharp(\xi_1 d\xi_2) = \pm\, \xi_1 \pi^\sharp(d\xi_2) \qquad \text{and} \qquad \pi^\sharp((d\xi_1)\xi_2) = \pm\, \pi^\sharp(d\xi_1)\, \xi_2.$$

More precisely, given two homogeneous functions $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$, the following must hold:

$$\{\xi_1\xi_2, \cdot\}_k = \pi^\sharp(d(\xi_1\xi_2)) = \pi^\sharp((d\xi_1)\xi_2) + (-1)^{|\xi_1|}\pi^\sharp(\xi_1 d\xi_2). \tag{B.4}$$

This is true for the following linearity rules:

$$\pi^\sharp(\xi_1 d\xi_2) = (-1)^{|\xi_1|}\xi_1 \pi^\sharp(d\xi_2) \qquad \text{and} \qquad \pi^\sharp((d\xi_1)\xi_2) = (-1)^{|\xi_2|}\pi^\sharp(d\xi_1)\,\xi_2.$$

In order to see this, first observe that for homogeneous functions $\xi_1, \xi_2 \in \mathcal{C}^\infty(\mathcal{M})$ and a homogeneous vector field $\mathcal{X} \in \mathfrak{X}^{|\mathcal{X}|}(\mathcal{M}) = (\mathfrak{X}(\mathcal{M})[k])^{|\mathcal{X}|-k}$ we have

$$(\mathcal{X}\cdot\xi_1)(\xi_2) = (-1)^{|\xi_1|(|\mathcal{X}|+1-k)}\xi_1\mathcal{X}(\xi_2) = (-1)^{|\xi_1|(|\xi_2|+1-k)}\mathcal{X}(\xi_2)\xi_1.$$

Now we compute for a homogeneous $\xi_3 \in \mathcal{C}^\infty(\mathcal{M})$

$$\begin{aligned}
\{\xi_1\xi_2, \xi_3\}_k &= -\,(-1)^{(|\xi_1|+|\xi_2|+k)(|\xi_3|+k)}\{\xi_3, \xi_1\xi_2\}_k \\
&= -\,(-1)^{(|\xi_1|+|\xi_2|+k)(|\xi_3|+k)}\{\xi_3, \xi_1\}_k\xi_2 - (-1)^{(|\xi_3|+k)(|\xi_2|+k)}\xi_1\{\xi_3, \xi_2\}_k \\
&= (-1)^{(|\xi_3|+k)|\xi_2|}\{\xi_1, \xi_3\}_k\xi_2 + \xi_1\{\xi_2, \xi_3\}_k.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\pi^\sharp((d\xi_1)\xi_2)(\xi_3) &= (-1)^{|\xi_2|}(\pi^\sharp(d\xi_1)\cdot\xi_2)(\xi_3) \\
&= (-1)^{|\xi_2|}(-1)^{(|\xi_3|+1+k)|\xi_2|}\pi^\sharp(d\xi_1)(\xi_3)\cdot\xi_2 \\
&= (-1)^{(|\xi_3|+k)|\xi_2|}\{\xi_1, \xi_3\}_k\xi_2
\end{aligned}$$

and

$$\pi^\sharp(\xi_1 d\xi_2)(\xi_3) = (-1)^{|\xi_1|}\xi_1\pi^\sharp(d\xi_2)(\xi_3) = (-1)^{|\xi_1|}\xi_1\{\xi_2, \xi_3\}_k.$$

Therefore, equation (B.4) holds.

B.3   Adjoint representation of a split Lie 2-algebroid

In order to make the computations for the equations of the adjoint representation that appear in Proposition 6.3.1, we will need the following identity.

Lemma B.3.1. For all $q_1, q_2, q_3 \in \Gamma(Q)$ and $X \in \mathfrak{X}(M)$ we have

$$\left(d_{\nabla^{\mathrm{bas}, \mathrm{Hom}}} R_\nabla^{\mathrm{bas}}\right)(q_1, q_2, q_3)X = \left(\nabla_X^{\mathrm{Hom}} \mathrm{Jac}_{[\cdot, \cdot]}\right)(q_1, q_2, q_3).$$

Proof. Let $q_1, q_2, q_3 \in \Gamma(Q)$ and $X \in \mathfrak{X}(M)$. We compute:

$$
\begin{aligned}
\left(\mathrm{d}_{\nabla^{\mathrm{bas,Hom}}} R_{\nabla}^{\mathrm{bas}}\right)(q_1, q_2, q_3)X &= \nabla_{q_1}^{\mathrm{bas}}(R_{\nabla}^{\mathrm{bas}}(q_2, q_3)X) - R_{\nabla}^{\mathrm{bas}}(q_2, q_3)(\nabla_{q_1}^{\mathrm{bas}}X) \\
&\quad - R_{\nabla}^{\mathrm{bas}}([q_1, q_2], q_3)X + \mathrm{c.p.} \\
&= [q_1, R_{\nabla}^{\mathrm{bas}}(q_2, q_3)X] + \nabla_{\rho(R_{\nabla}^{\mathrm{bas}}(q_2, q_3)X)}q_1 - R_{\nabla}^{\mathrm{bas}}(q_2, q_3)(\nabla_{q_1}^{\mathrm{bas}}X) \\
&\quad + \nabla_X[[q_1, q_2], q_3] - [\nabla_X[q_1, q_2], q_3] - [[q_1, q_2], \nabla_X q_3] \\
&\quad - \nabla_{\nabla_{q_3}^{\mathrm{bas}}X}[q_1, q_2] + \nabla_{\nabla_{[q_1, q_2]}^{\mathrm{bas}}X}q_3 + \mathrm{c.p.} \\
&= \nabla_X(\mathrm{Jac}_{[\cdot, \cdot]}(q_1, q_2, q_3)) - \mathrm{Jac}_{[\cdot, \cdot]}(\nabla_X q_1, q_2, q_3) \\
&\quad - \mathrm{Jac}_{[\cdot, \cdot]}(q_1, \nabla_X q_2, q_3) - \mathrm{Jac}_{[\cdot, \cdot]}(q_1, q_2, \nabla_X q_3) \\
&= \left(\nabla_X^{\mathrm{Hom}} \mathrm{Jac}_{[\cdot, \cdot]}\right)(q_1, q_2, q_3). \qquad \square
\end{aligned}
$$

Proof of Proposition 6.3.1. A straightforward computation shows that the elements in the statement are $C^\infty(M)$-linear in all their arguments and thus well defined. Moreover, the equation $\rho \circ \ell = 0$, the commutativity of ρ with the dull bracket $[\cdot, \cdot]$, and equation (ii) of Definition 3.3.6 imply that the individual Q-connections on the vector bundles B^*, Q and TM commute with ℓ and ρ, and thus define a Q-connection on the complex $B^* \to Q \to TM$. Hence, we only need to check that the equations in the statement of Proposition 6.2.5, or equivalently the seven equations that are stated in the proof, hold. In the following, let $\beta, \beta_i \in \Gamma(B^*)$, $b \in \Gamma(B)$, $q, q_i \in \Gamma(Q)$ and $X \in \mathfrak{X}(M)$, for $i = 1, 2, \ldots$

(i) For the first equation we have the following three computations

$$
\left([\partial, \omega_2](X) + \mathrm{d}_{\nabla^{\mathrm{bas}}}^2 X\right)(q_1, q_2) = \rho\left(\omega_2(q_1, q_2)X\right) + R_{\nabla^{\mathrm{bas}}}(q_1, q_2)X = 0
$$

$$
\left([\partial, \omega_2](q) + \mathrm{d}_{\nabla^{\mathrm{bas}}}^2 q\right)(q_1, q_2) = -\ell\left(\omega_2(q_1, q_2)q\right) + \omega_2(q_1, q_2)\rho(q) + R_{\nabla^{\mathrm{bas}}}(q_1, q_2)q = 0
$$

$$
\left([\partial, \omega_2](\beta) + \mathrm{d}_{\nabla^*}^2 \beta\right)(q_1, q_2) = R_{\nabla^*}(q_1, q_2)\beta - \omega_2(q_1, q_2)\ell(\beta) = 0
$$

where in the third equation we use (iv) from Definition 3.3.6.

(ii) For the second equation we compute

$$
\left([\partial, \phi_0](X) + \partial_B(\mathrm{d}_{\nabla^{\mathrm{bas}}}X)\right)(\beta) = \rho(\nabla_X \ell(\beta)) - \rho(\ell(\nabla_X \beta)) + \nabla_{\ell(\beta)}^{\mathrm{bas}}X = 0
$$

$$
\left([\partial, \phi_0](\beta_1) + \partial_B(\mathrm{d}_{\nabla^*}\beta_1)\right)(\beta_2) = -\nabla_{\ell(\beta_2)}^*\beta_1 - \phi_0(\beta_2)\ell(\beta_1) = 0
$$

$$
\left([\partial, \phi_0](q) + \partial_B(\mathrm{d}_{\nabla^{\mathrm{bas}}}q)\right)(\beta) = \nabla_{\ell(\beta)}^{\mathrm{bas}}q + \phi_0(\beta)\rho(q) - \ell(\phi_0(\beta)q) = 0
$$

where for the last two equations we used (i) and (ii) from Definition 3.3.6.

(iii) The third equation splits into the following two cases

$$
\Big([\partial, \omega_3](X) + [d_{\nabla^{\mathrm{bas}}}, \omega_2](X)\Big)(q_1, q_2, q_3)
$$

$$
= \ell(\nabla_X(\omega(q_1, q_2, q_3))) + \left(\nabla_X^{\mathrm{Hom}} \mathrm{Jac}_{[\cdot,\cdot]}\right)(q_1, q_2, q_3) - \nabla_X\left(\mathrm{Jac}_{[\cdot,\cdot]}(q_1, q_2, q_3)\right)
$$

$$
+ \sum_{i<j}(-1)^{i+j} R_{\nabla}^{\mathrm{bas}}([q_i, q_j], q_{k\neq i,j})X + \sum_i (-1)^{i+1}\nabla_{q_i}^{\mathrm{bas}}\left(R_{\nabla}^{\mathrm{bas}}(q_{s\neq i}, q_{t\neq i})\right)X
$$

$$
- R_{\nabla}^{\mathrm{bas}}(q_1, q_2)(\nabla_{q_3}^{\mathrm{bas}}X) + R_{\nabla}^{\mathrm{bas}}(q_1, q_3)(\nabla_{q_2}^{\mathrm{bas}}X) - R_{\nabla}^{\mathrm{bas}}(q_2, q_3)(\nabla_{q_1}^{\mathrm{bas}}X)
$$

$$
= \ell(\nabla_X(\omega(q_1, q_2, q_3))) + \left(\nabla_X^{\mathrm{Hom}} \mathrm{Jac}_{[\cdot,\cdot]}\right)(q_1, q_2, q_3) - \nabla_X\left(\mathrm{Jac}_{[\cdot,\cdot]}(q_1, q_2, q_3)\right)
$$

$$
- \left(d_{\nabla^{\mathrm{bas}},\mathrm{Hom}} R_{\nabla}^{\mathrm{bas}}\right)(q_1, q_2, q_3)X
$$

$$
= \langle \omega(q_1, q_2, q_3), \phi_0(X)\rangle,
$$

$$
\Big(d_{\nabla^*}(\omega_2(q_4)) + \omega_2(d_{\nabla^{\mathrm{bas}}}q_4) + \omega_3(\partial(q_4))\Big)(q_1, q_2, q_3)
$$

$$
= \sum_{i<j}^{3}(-1)^{i+j}\omega([q_i, q_j], \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots) + \sum_{i=1}^{3}(-1)^{i+1}\nabla_{q_i}^*\left(\omega(\ldots, \hat{q}_i, \ldots)\right)
$$

$$
- \omega(q_1, q_2, \nabla_{q_3}^{\mathrm{bas}}q_4) + \omega(q_1, q_3, \nabla_{q_2}^{\mathrm{bas}}q_4) - \omega(q_2, q_3, \nabla_{q_1}^{\mathrm{bas}}q_4)
$$

$$
- \nabla_{\rho(q_4)}(\omega(q_1, q_2, q_3)) + \omega(\nabla_{\rho(q_4)}q_1, q_2, q_3)
$$

$$
+ \omega(q_1, \nabla_{\rho(q_4)}q_2, q_3) + \omega(q_1, q_2, \nabla_{\rho(q_4)}q_3)
$$

$$
= \nabla_{q_4}^*(\omega(q_1, q_2, q_3)) - \nabla_{\rho(q_4)}(\omega(q_1, q_2, q_3))
$$

$$
= \langle \omega(q_1, q_2, q_3), \phi_0(q_4)\rangle,
$$

where for the first equation we used Lemma B.3.1 together with (iii) from Definition 3.3.6 and for the second equation we used (v) from Definition 3.3.6.

(iv) For the fourth equation we compute

$$
\Big(d_{\overline{\nabla}}\phi_0(X) + [\partial, \phi_1](X) + \partial_B(\omega_2(X))\Big)(\beta, q)
$$

$$
= \nabla_q^{\mathrm{bas}}(\phi_0(\beta)X) - \phi_0(\nabla_q^*\beta)X - \phi_0(\beta)(\nabla_q^{\mathrm{bas}}X)
$$

$$
- \ell(\phi_1(\beta, q)X) + R_{\nabla}^{\mathrm{bas}}(q, \ell(\beta))X = 0,
$$

$$
\Big(d_{\overline{\nabla}}\phi_0(q_1) + [\partial, \phi_1](q_1) + \partial_B(\omega_2(q_1))\Big)(\beta, q_2)
$$

$$
= \nabla_{q_2}^{\mathrm{bas}}(\phi_0(\beta)q_1) - \phi_0(\nabla_{q_2}^*\beta)q_1 - \phi_0(\beta)(\nabla_{q_2}^{\mathrm{bas}}q_1)
$$

$$
+ \omega_2(\ell(\beta), q_2)q_1 + \phi_1(\beta, q_2)(\rho(q_1)) = 0
$$

where in the first calculation we used (ii) from Definition 3.3.6 and $\rho \circ \ell = 0$, and in the second calculation we used (iv) from Definition 3.3.6.

(v) For the fifth equation we have

$$\left(\mathrm{d}_{\nabla^*}(\omega_3(X)) + \omega_2(\omega_2(X)) + \omega_3(\mathrm{d}_{\nabla^{\mathrm{bas}}}X)\right)(q_1, q_2, q_3, q_4)$$

$$= \sum_{i<j}^{4}(-1)^{i+j}\omega_3([q_i, q_j], \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots) + \sum_{i=1}^{4}(-1)^{i+1}\nabla_{q_i}^*(\omega_3(\ldots, \hat{q}_i, \ldots))$$

$$+ \omega_2(q_1, q_2)(\omega_2(q_3, q_4)X) - \omega_2(q_1, q_3)(\omega_2(q_2, q_4)X)$$

$$+ \omega_2(q_1, q_4)(\omega_2(q_2, q_3)X) + \omega_2(q_2, q_3)(\omega_2(q_1, q_4)X)$$

$$- \omega_2(q_2, q_4)(\omega_2(q_1, q_3)X) + \omega_2(q_3, q_4)(\omega_2(q_1, q_2)X)$$

$$- \omega_3(q_2, q_3, q_4)(\nabla_{q_1}^{\mathrm{bas}}X) + \omega_3(q_1, q_3, q_4)(\nabla_{q_2}^{\mathrm{bas}}X)$$

$$- \omega_3(q_1, q_2, q_4)(\nabla_{q_3}^{\mathrm{bas}}X) + \omega_3(q_1, q_2, q_3)(\nabla_{q_4}^{\mathrm{bas}}X)$$

$$- \phi_1(\omega(q_2, q_3, q_4), q_1)X + \phi_1(\omega(q_1, q_3, q_4), q_2)X$$

$$- \phi_1(\omega(q_1, q_2, q_4), q_3)X + \phi_1(\omega(q_1, q_2, q_3), q_4)X$$

$$= \langle\omega, \phi(X)\rangle(q_1, q_2, q_3, q_4),$$

where we used (v) from Definition 3.3.6.

(vi) The sixth equation becomes

$$\left(\mathrm{d}_{\overline{\nabla}}\phi_1(X) + \omega_2(\phi_0(X)) + \phi_0(\omega_2(X)) + \partial_B(\omega_3(X))\right)(\beta, q_1, q_2)$$

$$= -\phi_1(\beta, [q_1, q_2])X + \phi_1(\nabla_{q_2}^*\beta, q_1)X - \phi_1(\nabla_{q_1}^*\beta, q_2)X$$

$$- \phi_1(\beta, q_2)(\nabla_{q_1}^*X) + \phi_1(\beta, q_1)(\nabla_{q_2}^*X) + \nabla_{q_1}^*(\phi_1(\beta, q_2)X)$$

$$- \nabla_{q_2}^*(\phi_1(\beta, q_1)X) + \omega(q_1, q_2, \ell(\nabla_X\beta)) - \omega(q_1, q_2, \nabla_X\ell(\beta))$$

$$- \phi_0(\beta)(R_\nabla^{\mathrm{bas}}(q_1, q_2)X) + \omega_3(q_1, q_2, \ell(\beta))X$$

$$= 0,$$

where we used (iv) from Definition 3.3.6 and that $\rho(R_\nabla^{\mathrm{bas}}(q_1, q_2)X) = R_{\nabla^{\mathrm{bas}}}(q_1, q_2)X$.

(vii) For the last equation we compute

$$\left(\phi_0(\phi_0(X)) + \partial_B(\phi_1(X))\right)(\beta_1, \beta_2) = \phi_0(\phi_0(\beta_1)X, \beta_2) + \phi_0(\phi_0(\beta_2)X, \beta_1)$$

$$+ \phi_1(\beta_1, \ell(\beta_2))X + \phi_1(\beta_2, \ell(\beta_1))X$$

$$= \nabla_{\ell(\nabla_X\beta_1)}^*\beta_2 + \nabla_{\ell(\nabla_X\beta_2)}^*\beta_1 + \nabla_{\ell(\beta_2)}^*\nabla_X\beta_1$$

$$- \nabla_X\nabla_{\ell(\beta_2)}^*\beta_1 + \nabla_{\ell(\beta_1)}^*\nabla_X\beta_2 - \nabla_X\nabla_{\ell(\beta_1)}^*\beta_2$$

$$= 0,$$

where in the last line we used (i) from Definition 3.3.6. $\square$

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Alphabetical Index