On differential systems related to generalized Meixner and deformed Laguerre orthogonal polynomials

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Abstract

In this paper we present a connection between systems of differential equations for the recurrence coefficients of polynomials orthogonal with respect to the generalized Meixner and the deformed Laguerre weights. It is well-known that the recurrence coefficients of both generalized Meixner and deformed Laguerre orthogonal polynomials can be expressed in terms of solutions of the fifth Painlevé equation but no explicit relation between systems of differential equations for the recurrence coefficients was known. We also present certain limits in which the recurrence coefficients can be expressed in terms of solutions of the Painlevé XXXIV equation, which in the deformed Laguerre case extends previous studies and in the generalized Meixner case is a new result.

1 Introduction

Orthogonal polynomials appear in a wide range of applications [Chi78, Ism05, Sze67]. One of the most important properties of a sequence of orthogonal polynomials is the so-called three-term recurrence relation. For orthonormal polynomials $p_n(x)$, this relation takes the following form:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x).$$

For the corresponding monic polynomials $P_n(x)$ the recurrence relation is given by

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x).$$

The recurrence coefficients of orthogonal polynomials for semi-classical weights are often related to Painlevé type equations (see, for instance, [VA18] and the numerous references therein). In [FVA11] it was shown that the coefficients of the three-term recurrence relation for the orthogonal polynomials with the generalized Meixner weight are related to the solutions of the fifth Painlevé equation. This weight is given by

$$w_M(x) = \frac{\Gamma(\beta)\Gamma(\gamma+x)c^x}{\Gamma(\gamma)\Gamma(\beta+x)\Gamma(x+1)},$$

with c, β, $\gamma > 0$, for which the orthogonal polynomials are considered on the lattice $\mathbb{N}$. In [SVA12] it was shown that the quantities u_n, v_n, defined in terms of the recurrence coefficients a_n, b_n by

$$a_n^2 = nc - (\gamma-1)u_n, \quad b_n = n + \gamma - \beta + c - (\gamma-1)v_n/c,$$

satisfy the following system of difference equations:

$$\begin{aligned}(u_n+v_n)(u_{n+1}+v_n) &= \frac{\gamma-1}{c^2}v_n(v_n-c)\left(v_n - c\frac{\gamma-\beta}{\gamma-1}\right), \\ (u_n+v_n)(u_n+v_{n-1}) &= \frac{u_n}{u_n - cn/(\gamma-1)}(u_n+c)\left(u_n + c\frac{\gamma-\beta}{\gamma-1}\right),\end{aligned} \tag{1.1}$$

with initial conditions at $n = 0$ expressed in terms of the confluent hypergeometric function. In the case of orthogonal polynomials on a shifted lattice (or a bi-lattice), the parameters in the weight must be modified, but the discrete system for the recurrence coefficients (1.1) is the same, though with different initial conditions.

To derive a differential system for the recurrence coefficients, see [FVA11], one needs to combine the discrete system (1.1) with a Toda-type differential-difference system. The Toda system, where differentiation is with respect to parameter c, is given by

$$\begin{aligned}(a_n^2)' &= \tfrac{a_n^2}{c}(b_n - b_{n-1}), \\ b_n' &= \tfrac{1}{c}(a_{n+1}^2 - a_n^2).\end{aligned} \tag{1.2}$$

Solving the first equation from the discrete system (1.1), we may express u_{n+1} in terms of u_n and v_n. Similarly, we can find an expression for v_{n-1} in terms of u_n and v_n from the second equation of (1.1). Substituting these expressions into (1.2), we obtain the following system of first order differential equations:

$$\begin{aligned} c^2(u_n+v_n)\,u_n' &= cnv_n^2 + u_n^2((n-\beta+2\gamma)c - 2(\gamma-1)v_n) + \\ &\quad + u_n((\gamma-\beta)c^2 + (2n+1)cv_n - (\gamma-1)v_n^2), \\ c^2(u_n+v_n)\,v_n' &= -c^2u_n^2 + (1-2c)cu_nv_n + v_n\left((\gamma-\beta)c^2 + \right. \\ &\quad \left. + c(2-c+\beta-2\gamma)v_n + (\gamma-1)v_n^2\right). \end{aligned} \tag{1.3}$$

It was shown in [FVA11] that this system can be further reduced to the fifth Painlevé equation.

A deformed Laguerre weight was considered in [MC19]. For this weight, the recurrence coefficients, as functions of the variable t, are also related to solutions of the fifth Painlevé equation. For this case, the weight is given by

$$w_L(x) = x^\alpha e^{-x}|x-t|^\delta (A + B\theta(x-t)), \tag{1.4}$$

where A, $A+B$, x, $t \geq 0$, α, $\delta > 0$ and θ is the Heaviside step function. The auxiliary functions $R_n(t)$ and $r_n(t)$, defined by

$$R_n(t) = \frac{\delta}{h_n}\int_0^\infty \frac{P_n^2(y)}{y-t} w(y)\, dy,$$
$$r_n(t) = \frac{\delta}{h_{n-1}}\int_0^\infty \frac{P_n(y)P_{n-1}(y)}{y-t} w(y)\, dy,$$

where $h_n(t) = \int_0^\infty P_n^2(x)w(x)\, dx$, are related to the recurrence coefficients as follows:

$$b_n = 2n + 1 + \alpha + \delta + tR_n,$$
$$a_n^2 = \frac{r_n^2 - \delta r_n}{R_n R_{n-1}} = \sum_{j=0}^{n-1} b_j + tr_n.$$

The functions $R_n(t)$ and $r_n(t)$ satisfy the following system of differential equations (see [MC19, Eqs. (4.6), (4.7)]):

$$\begin{aligned} tR_n' &= tR_n^2 + (2n+\alpha+\delta-t)R_n + 2r_n - \delta, \\ tr_n' &= \frac{r_n^2 - \delta r_n}{R_n} - \frac{r_n^2 - \delta r_n + (n(n+\alpha) + (2n+\alpha+\delta)r_n)R_n}{1-R_n}. \end{aligned} \tag{1.5}$$

A connection to the Painlevé XXXIV equation was also identified in [MC19]. The authors introduced a change of independent variables from t to s according to $t = 4n + 2^{4/3}n^{1/3}s = 4n + 4^{2/3}n^{1/3}s$ and considered the scaled functions $\tilde{R}_n(s) = R_n(4n + 2^{4/3}n^{1/3}s)$ and $\tilde{r}_n(s) = r_n(4n+2^{4/3}n^{1/3}s)$. It was shown that in the large n limit with t and s fixed, if these functions admit expansions of the form

$$\tilde{R}_n(s) = u(s)n^{-2/3} + w(s)n^{-1} + O(n^{-4/3}),$$
$$\tilde{r}_n(s) = u(s)n^{1/3} + 2^{-1/3}u'(s) + w(s) + \delta/2 + O(n^{-1/3}),$$

then $u(s)$ and $w(s)$ satisfy certain differential equations (see [MC19, Eqs. (5.12), (5.13)]). We shall present only the equation for the function u:

$$uu'' - 2^{-1}(u')^2 + 2^{8/3}u^3 - 2su^2 + 2^{-7/3}\delta^2 = 0.$$

By taking $\tilde{u} = -2^{2/3}u$, the equation above is reduced to

$$\tilde{u}'' = \frac{(\tilde{u}')^2}{2\tilde{u}} + 4\tilde{u}^2 + 2s\tilde{u} - \frac{\delta^2}{2\tilde{u}},$$

which is the Painlevé XXXIV equation. Moreover, a similar approach was used in [LC17] for the weight with $\delta = 0$, where the authors also distinguish the cases when the parameters of the differential system are of order $O(n)$, and the second Painlevé equation was obtained in the limit.

In what follows, we first recall and extend results of [MC19] on double scaling limits of the differential equations (1.5) from the deformed Laguerre case. We then present a new relation between this system (1.5) and the equations (1.3) from the case of the generalized Meixner weight. This leads us to ask whether similar results hold for this system, so we carry out a systematic study of the same kind of double scaling here. The relation between the differential systems was obtained using tools from the Okamoto-Sakai theory of Painlevé equations in terms of the geometry of certain rational surfaces, and will be explained in detail in a forthcoming paper [DFS]. The present paper serves to illustrate the kind of results of that can be obtained using this approach, without introducing the full machinery. We emphasize that in what follows we work with the differential systems only, without reference to the orthogonal polynomials themselves.

2 Main results

2.1 Double scaling limits in the deformed Laguerre case

Let us study the system (1.5) in detail. Our aim, building on the results of [MC19], is to make the change of variables $t = 4n + 4^{2/3}n^{1/3}s$ and identify the cases such that in the expansion

$$\tilde{R}_n(s) = a_0(s) + \frac{a_{-1}(s)}{n^{1/3}} + \frac{a_{-2}(s)}{n^{2/3}} + \frac{a_{-3}(s)}{n} + \dots, \tag{2.1}$$

one of the first non-zero coefficients $a_j(s)$, $j = 0, -1, -2, -3$, satisfies a second order differential equation and determine whether it can be reduced to one of the Painlevé equations. As we see below, in all cases we see reductions to the Painlevé XXXIV equation. We keep $\delta \neq 0$, since the system in the case $\delta = 0$ is known to be fundamentally different [LC17]. We shall also allow scalings of parameters to be of order $O(n)$. Clearly from the connection to orthogonal polynomials parameters should be real.

So that our approach is as systematic as possible, we proceed according to the following procedure for each scaling: We first consider a_0, and isolate conditions for the asymptotic expansions to balance, which may be algebraic constraints that determine a_0 as a function of s, or that a_0 is some constant, or that a_0 satisfies some differential equation. We discard cases where the leading behaviour is governed by a complicated algebraic expression, and for other cases when differential equations do not appear, we proceed to the next coefficient. This will then either be forced to take some constant value, be subject to an algebraic constraint, or satisfy a differential equation. Again discarding behaviours governed by algebraic functions, if we have no differential equation we proceed to the next coefficient, repeating this process. If a case does not yield a differential equation after considering a_{-3} we discard it, but this does not rule out the possibility that some differential equation might govern some higher order behaviour, but we limit ourselves to these first four coefficients for conciseness.

Case 1. Let us assume that we scale both parameters $\alpha = p_1 n$ and $\delta = p_2 n$. Then the first coefficient $a_0(s)$ in (2.1) satisfies a complicated algebraic equation (the procedure to obtain such an expansion will be described in detail in Case 2 below). So according to our approach outlined above, we discard this case.

Case 2. Let us scale the parameter $\alpha = p_1 n$ while keeping δ fixed with respect to n, and change variables $t = 4n + 2^{4/3}n^{1/3}s$. System (1.5) then becomes the following system for functions $\tilde{R}_n = \tilde{R}_n(s) = R_n(4n + 2^{4/3}n^{1/3}s)$ and $\tilde{r}_n = \tilde{r}_n(s) = r_n(4n + 2^{4/3}n^{1/3}s)$:

$$
\begin{aligned}
2^{-1/3}(2n^{2/3} + 2^{1/3}s)\tilde{R}_n' &= ((p_1 - 2)n - 2\cdot 2^{1/3}n^{1/3}s + \delta)\tilde{R}_n - \delta + 2\tilde{r}_n \\
&\quad + 2(2n + 2^{1/3}n^{1/3}s)\tilde{R}_n^2, \\
2^{-1/3}(2n^{2/3} + 2^{1/3}s)(\tilde{R}_n - 1)\tilde{R}_n\tilde{r}_n' &= \tilde{r}_n(\delta - 2\delta\tilde{R}_n + ((p_1 + 2)n + \delta)\tilde{R}_n^2) + (p_1 + 1)n^2\tilde{R}_n^2 \\
&\quad + \tilde{r}_n^2(2\tilde{R}_n - 1).
\end{aligned}
\tag{2.2}
$$

The first equation from the above system can be solved for $\tilde{r}_n(s)$. Substituting the resulting expression into the second equation, we find a second order differential equation for $\tilde{R}_n(s)$, which we omit due to its cumbersome expression. Assuming that the expansion (2.1) holds, we find several further subcases.

When $a_0(s) = 0$, we have the following subcases:

Subcase 2.1. Here $a_{-1}(s) = 0$ and further we have $(p_1 - 8)p_1 a_{-2}(s) = 0$. When $p_1 = 0$ or $p_1 = 8$, this leads to the subcases below. When $a_{-2}(s) = 0$ and $p_1 \neq 0; 8$, $a_{-3}(s) = \pm\delta/\sqrt{p_1(p_1 - 8)}$, and there is no differential equation for the first few coefficients in expansion (2.1). If we assume that $a_{-2}(s) = 0$ and $p_1 = 0; 8$, then necessarily we have $\delta = 0$, so this subcase is discarded.

Subcase 2.2. When $p_1 = 8$, we have

$$
\begin{aligned}
\tilde{R}_n(s) &= \frac{u(s)}{n^{2/3}} + \frac{a_{-3}(s)}{n} + O(n^{-4/3}), \\
\tilde{r}_n(s) &= -3u(s)n^{1/3} + v(s) + O(n^{-1/3}).
\end{aligned}
$$

Here the function v satisfies

$$v = \delta/2 - 3a_{-3} + 2^{-1/3}u',$$

and the function $a_{-2}(s) = u(s)$ satisfies

$$4\cdot 2^{1/3}uu'' - 2\cdot 2^{1/3}u'^2 - 96u^3 + 24\cdot 2^{1/3}su^2 + \delta^2 = 0.$$

Scaling $u(s) = \tilde{u}(-3^{1/3}s)/(2^{2/3}\cdot 3^{1/3})$, we obtain the Painlevé XXXIV equation for the function $\tilde{u}$. When we substitute expansions for $\tilde{R}_n$ and $\tilde{r}_n$ back into the system, use expression of u' in terms of v and a_{-3} from above, and let $n \to \infty$, we see that the first equation is satisfied identically, and from the second we have

$$2^{1/3}(\delta - v - 3a_{-3})(v + 2a_{-3}) + 2u(3\cdot 2^{1/3}(2^{1/3}s - 4u)u + v' + 2a_{-3}') = 0.$$

Making the change of variables $y = 2v + 6a_{-3}$, we find a system of first order differential equations in s:

$$2^{2/3}u' = y - \delta,$$
$$2 \cdot 2^{2/3}uy' = y^2 + 24u^2(4u - 2^{1/3}) - 2\delta y.$$

This system is equivalent to the second order differential equation for u above.

Subcase 2.3. When $p_1 = 0$, we have similar expressions:

$$\tilde{R}_n(s) = \frac{u(s)}{n^{2/3}} + \frac{a_{-3}(s)}{n} + O(n^{-4/3}),$$
$$\tilde{r}_n(s) = u(s)n^{1/3} + v(s) + O(n^{-1/3}),$$

where in this case the differential equation is

$$4 \cdot 2^{1/3}uu'' - 2 \cdot 2^{1/3}u'^2 + 32u^3 - 8 \cdot 2^{1/3}su^2 + \delta^2 = 0,$$

and $v = a_{-3} + (\delta + 2^{2/3}u')/2$. If we define $y = 2(v - a_{-3})$, we find

$$2^{2/3}u' = y - \delta,$$
$$2 \cdot 2^{2/3}uy' = y^2 + 8(2^{1/3}s - 4u)u^2 - 2\delta y,$$

which is equivalent to the above second order differential equation. Scaling the variable $u(s) = -2^{-2/3}\tilde{u}(s)$ we obtain the Painlevé XXXIV equation.

Case 3. Scaling $\delta = p_2 n$ while keeping α fixed with respect to n, we obtain subcases very similar to those from Case 2. The only difference is that this time p_2 satisfies $p_2^2 + 8p_2 + 32 = 0$, with complex roots. This case might be interesting from the point of view of differential equations (see the last section, where we discuss open problems), but clearly it is not relevant for the applied problem at hand.

Case 4. We do not scale α or δ with respect to n, in which case the procedure described in Case 2 gives the following result. The subcase $a_0(s) = 1$ is not interesting since we do not obtain a differential equation for any of the first few coefficients as outlined at the beginning of the section, namely, up to $a_{-3}(s)$. When $a_0(s) = 0$, we obtain $a_{-1}(s) = 0$, $a_{-2}(s) = u(s)$, where the function u satisfies the second order nonlinear differential equation

$$4 \cdot 2^{1/3}uu'' - 2 \cdot 2^{1/3}(u')^2 + 32u^3 - 8 \cdot 2^{1/3}su^2 + \delta^2 = 0,$$

which can be transformed to the Painlevé XXXIV equation by $u(s) = -2^{2/3}\tilde{u}(s)$. The functions $\tilde{R}_n(s)$ and $\tilde{r}_n(s)$ are given by

$$\tilde{R}_n(s) = \frac{u(s)}{n^{2/3}} + \frac{a_{-3}(s)}{n} + O(n^{-4/3}),$$
$$\tilde{r}_n(s) = u(s)n^{1/3} + v(s) + O(n^{-1/3}).$$

Taking $v = a_{-3} + (\delta + 2^2/3u')/2$, and denoting $y = 2(a_{-3} - v)$, we obtain the following system of first order differential equations, which is equivalent to the second order differential equation for u above:

$$2^{2/3}u' = -(y + \delta),$$
$$2 \cdot 2^{2/3}uy' = -(2\delta y + y^2 + 8(2^{1/3}s - 4u)u^2).$$

Hence, we recover the result in [MC19], which we described in the Introduction.

2.2 Relation between differential systems

Another important result of this paper is the following theorem. As remarked in the Introduction, we plan to give a geometric explanation of this transformation in terms of the Okamoto-Sakai theory of Painlevé equations in a forthcoming paper [DFS]. Let us make the following change of variables (this change is used in [LC17] for the case $\delta = 0$, and in [DFS]):

$$x_n = 1 - \frac{1}{R_{n-1}}, \quad y_n = -r_n,$$

or, equivalently,

$$r_n = -y_n, \quad R_n = \frac{x_n y_n(\delta + y_n)}{y_n(2n + \alpha - y_n + x_n(\delta + y_n)) - n(n + \alpha)}.$$

With this change of variables the differential system (1.5) becomes

$$\begin{aligned} tx_n' &= -(2n + \alpha - 2y_n - x_n^2(\delta + 2y_n) + x_n(t - 2n - \alpha + \delta + 4y_n)), \\ tx_n y_n' &= n(n + \alpha) - (2n + \alpha + \delta x_n^2)y_n - (x_n^2 - 1)y_n^2. \end{aligned} \tag{2.3}$$

Theorem 1. *Let $c = t$. Then the differential systems (1.3) and (2.3) are related by the following birational transformations:*

$$u_n = \frac{ty_n}{\delta}, \quad v_n = -\frac{ty_n(n - y_n + x_n(\delta + y_n))}{\delta(n - y_n)}, \tag{2.4}$$

$$x_n = -\frac{nt + u_n - \gamma u_n(u_n + v_n)}{(\gamma - 1)u_n(t + u_n)}, \quad y_n = \frac{(\gamma - 1)u_n}{t}, \tag{2.5}$$

and the parameters are related by

$$\begin{aligned} \alpha = -n + \beta - \gamma, \quad \delta = \gamma - 1, \\ \beta = 1 + n + \alpha + \delta, \quad \gamma = 1 + \delta. \end{aligned} \tag{2.6}$$

The change of variables in Theorem 1 follows from the Okamoto-Sakai geometric theory of Painlevé equations and it will be explained in detail in the forthcoming paper [DFS]. For the purposes of the present paper we only remark that this change of variables can also be verified by direct computation.

2.3 Double scaling limits in the generalized Meixner case

Here we exploit the transformation above to derive new double scaling limits for the differential system (1.3) in which the Painlevé XXXIV equation appears. Since $t = c$, and

$$x_n(t) = \frac{(r_n(t)+n)(\alpha+n+r_n(t))R_n(t)}{r_n(t)(r_n(t)-\delta)(R_n(t)-1)}, \quad y_n(t) = -r_n(t),$$

we may obtain expressions of u_n and v_n in terms of r_n and R_n directly:

$$u_n = -\frac{-tr_n}{\delta}, \quad v_n = -\frac{t(r_n+(n+\alpha)R_n)}{\delta(R_n-1)}.$$

Therefore, for the Subcases 2.2, 2.3 and Case 4 above we can immediately recalculate the corresponding expansions for u_n and v_n. Taking $t = 4n+2^{4/3}n^{1/3}s$ and defining new functions $\tilde{u}_n = \tilde{u}_n(s) = u_n(4n+2^{4/3}n^{1/3}s)$, and $\tilde{v}_n = \tilde{v}_n(s) = v_n(4n+2^{4/3}n^{1/3}s)$, we obtain the following results.

Subcase 2.2. Here $\alpha = p_1 n$ and $p_1 = 8$. Hence, the corresponding generalized Meixner parameters are $\beta = 1+9n+\delta$ and $\gamma = 1+\delta$. The corresponding expansions are

$$\tilde{u}_n(s) = 12\delta^{-1}u(s)n^{4/3} - 4\delta^{-1}v(s)n + O(n^{2/3}),$$
$$\tilde{v}_n(s) = 24\delta^{-1}u(s)n^{4/3} + 4\delta^{-1}(v(s)+a_{-3}(s))n + O(n^{2/3}).$$

Subcase 2.3. Here $\alpha = p_1 n$ and $p_1 = 0$. Hence, the corresponding generalized Meixner parameters are $\beta = 1+n+\delta$ and $\gamma = 1+\delta$. The corresponding expansions are

$$\tilde{u}_n(s) = -4\delta^{-1}u(s)n^{4/3} - 4\delta^{-1}v(s)n + O(n^{2/3}),$$
$$\tilde{v}_n(s) = 8\delta^{-1}u(s)n^{4/3} + 4\delta^{-1}(v(s)+a_{-3}(s))n + O(n^{2/3}).$$

Case 4. Here parameters α and δ are not scaled. Hence, the corresponding generalized Meixner parameters are as in (2.6). The corresponding expansions are

$$\tilde{u}_n(s) = -4\delta^{-1}u(s)n^{4/3} - 4\delta^{-1}v(s)n + O(n^{2/3}),$$
$$\tilde{v}_n(s) = 8\delta^{-1}u(s)n^{4/3} + 4\delta^{-1}(v(s)+a_{-3}(s))n + O(n^{2/3}).$$

We may also study system (1.3) without any reference to system (1.5), as in the Case 2 above for the deformed Laguerre system, but we will arrive at the same expansions as outlined in the cases here.

3 Conclusions and open problems

In this paper we have identified the cases when in the large n limit, solutions of the system of differential equations (1.5) are approximated by solutions of differential equations, and identified these as cases of the Painlevé XXXIV equation (Subcases 2.2, 2.3 and Case 4).

This extends the results in [MC19]. Moreover, finding a connection between systems (1.5) and (1.3), we obtained similar results for the system (1.3).
This leads to several important open problems. There exists a well-known degeneration scheme for the Painlevé equations, encoded in the so-called coalescence diagram. This consists of changes of variables, where in limits as certain parameters go to zero, the Painlevé equations degenerate from the sixth to the first one (see, for instance, [IKSY]). The geometric interpretation of this standard degeneration is well-known [KNY]. The degeneration presented in this paper is not straightforward and in particular one needs to introduce auxiliary functions (u, v, y) to obtain the Painlevé XXXIV equation from the fifth Painlevé equation. Therefore it is not immediately clear whether this degeneration is equivalent to the standard one or not. Further, the degeneration presented in this paper might hold only for special values of the parameters from the fifth Painlevé equation due to the connection to orthogonal polynomials and the presence of n in the parameters and, hence, might not be seen for arbitrary parameters. Case 3 suggests that there are more cases when the degeneration to the Painlevé XXXIV holds. This is an interesting question and warrants further investigation. We also note that a similar limit to the Painlevé XXXIV equation was recently obtained from the fourth Painlevé equation [CH20].

As remarked earlier, in the generalized Meixner case we have used the same scaling of the independent variable and ansatz for the expansions as in [MC19], motivated by the relation given in Theorem 1. Another interesting question is that of which scalings of independent variables with which assumed forms of asymptotic expansions will lead to other Painlevé equations appearing. Through this, we may identify cases when double scaling limits will lead to degenerations between Painlevé equations for other systems appearing in similar ways in the theory of orthogonal polynomials. We hope that the geometric approach taken in [DFS] might shed some light on these questions through a connection to the geometric picture of degenerations between Painlevé equations.

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