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Large Steklov eigenvalues via homogenisation on manifolds

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Abstract Using methods in the spirit of deterministic homogenisation theory we obtain convergence of the Steklov eigenvalues of a sequence of domains in a Riemannian manifold to weighted Laplace eigenvalues of that manifold. The domains are obtained by removing small geodesic balls that are asymptotically densely uniformly distributed as their radius tends to zero. We use this relationship to construct manifolds that have large Steklov eigenvalues. In dimension two, and with constant weight equal to 1, we prove that Kokarev's upper bound of 8π for the first nonzero normalised Steklov eigenvalue on orientable surfaces of genus 0 is saturated. For other topological types and eigenvalue indices, we also obtain lower bounds on the best upper bound for the eigenvalue in terms of Laplace maximisers. For the first two eigenvalues, these lower bounds become equalities. A surprising consequence is the existence of free boundary minimal surfaces immersed in the unit ball by first Steklov eigenfunctions and with area strictly larger than 2π. This was previously thought to be impossible. We provide numerical evidence that some of the already known examples of free boundary minimal surfaces have these

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properties and also exhibit simulations of new free boundary minimal surfaces of genus zero in the unit ball with even larger area. We prove that the first nonzero Steklov eigenvalue of all these examples is equal to 1, as a consequence of their symmetries and topology, so that they are consistent with a general conjecture by Fraser and Li. In dimension three and larger, we prove that the isoperimetric inequality of Colbois–El Soufi–Girouard is sharp and implies an upper bound for weighted Laplace eigenvalues. We also show that in any manifold with a fixed metric, one can construct by varying the weight a domain with connected boundary whose first nonzero normalised Steklov eigenvalue is arbitrarily large.

1 Introduction and main results

1.1 The Laplace and Steklov eigenvalue problems

Let (M, g) be a smooth, closed connected Riemannian manifold of dimension $d \geq 2$ and let $\Omega \subset M$ be a domain with smooth boundary $\partial\Omega$. Let $\beta \in C^{\infty}(M)$ be a smooth positive function. We study the weighted Laplace eigenvalue problem

$$-\Delta\varphi = \lambda\beta\varphi \qquad \text{in } M \tag{1}$$

and the Steklov eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ \partial_\nu u = \sigma u & \text{on } \partial\Omega; \end{cases} \tag{2}$$

where Δ is the Laplace operator and ν is the outwards unit normal. The spectra of the Laplace and Steklov problems are discrete and their eigenvalues form sequences

$$0 = \lambda_0 < \lambda_1(M, g, \beta) \leq \lambda_2(M, g, \beta) \leq \ldots \nearrow \infty$$

and

$$0 = \sigma_0 < \sigma_1(\Omega, g) \leq \sigma_2(\Omega, g) \leq \ldots \nearrow \infty$$

accumulating only at infinity. Problem (1) is a staple of geometric spectral theory, see e.g. [3,9]. The eigenvalues $\lambda_k(M, g, \beta)$ correspond to natural frequencies of a membrane that is non-homogeneous when β is not constant. It has recently been studied by Colbois–El Soufi [16] and Colbois–El Soufi–Savo [18] in the Riemannian setting. Problem (2) is a classic problem originating in mathematical physics [73] and which has received growing attention in the last few years. Its eigenvalues are those of the Dirichlet-to-

Neumann operator, which maps a function f on $\partial\Omega$ to the normal derivative on the boundary of its harmonic extension. See [37] for a survey.

Our main theorem states that for any positive $\beta \in C^\infty(M)$, Problem (1) may be realized as a limit of Problem (2) defined on carefully constructed domains $\Omega^\varepsilon \subset M$. Denote by $\mathrm{d}\mu_g$ the Lebesgue measure on M and for every domain $\Omega \subset M$ by $\mathrm{d}A_{\partial\Omega}$ the measure on M defined by integration against the Hausdorff measure on $\partial\Omega$.

Theorem 1.1 *Let (M, g) be a closed Riemannian manifold, and $\beta \in C^\infty(M)$ positive. There is a family of domains $\Omega^\varepsilon \subset M$ such that $\mathrm{d}A_{\partial\Omega^\varepsilon}$ converges weak-$*$ to $\beta\,\mathrm{d}\mu_g$ and*

$$\sigma_k(\Omega^\varepsilon, g) \xrightarrow{\varepsilon\to 0} \lambda_k(M, g, \beta).$$

The proof of Theorem 1.1 is in the spirit of Girouard–Henrot–Lagacé [35] where Neumann eigenvalues of a domain in Euclidean space are related to Steklov eigenvalues of subdomains through periodic homogenisation by obstacles.

Homogenisation theory is a branch of applied mathematics that is interested in the study of PDEs and variational problems in the presence of structures at many different scales; in the presence of two scales they are usually referred to as the *macrostructure* and *microstructure*. The methods are usually divided in two general categories: deterministic (or periodic), and stochastic. The effectiveness of homogenisation in shape optimisation, see for example the Allaire's influencial monograph [1] and the references therein, leads one to believe that it should also be useful elsewhere in geometric analysis.

The main obstacle to the application of deterministic homogenisation theory in the Riemannian setting is that most Riemannian manifolds do not exhibit any form of periodic structure. It is therefore not surprising that homogenisation theory in this setting has either been applied when an underlying manifold exhibits a periodic-like structure, see e.g. the work of Boutet de Monvel–Khruslov [4] and Contreras–Iturriaga–Siconolfi [23], or used the periodic structure of an ambient space in which a manifold is embedded, see Braides–Cancedda–Chiadò Piat [5], or relied on an imposed periodic structure in predetermined charts, see Dobberschütz–Böhm [25]. Our approach is distinct in that it is entirely intrinsic and does not require a periodic structure at any stage.

We note that stochastic homogenisation has been used in geometric contexts, see e.g. the recent paper by Li [62]. Chavel–Feldman [10,11] also studied the effect on the spectrum of the Laplacian of removing a large, but fixed, number of small geodesic balls on which Dirichlet boundary conditions are imposed. However, no consideration was given to the distribution of those geodesic balls, nor to asymptotic behaviour joint in the number of balls removed and their size. The construction that is the closest to our own can be found in the recent work of Anné–Post [2], where they consider perturbations of the Neumann spectrum by removing a large number of geodesic balls with Neumann boundary conditions, using the method of generalised resolvent convergence. We note that for technical reasons, such methods cannot be applied in our setting.

Remark 1.2 It is natural to expect that the Steklov eigenvalues of a domain Ω with smooth boundary would be related to the eigenvalues $\lambda_k(\partial\Omega)$ of the Laplace operator *of its boundary*, since the Dirichlet-to-Neumann map is an elliptic pseudo-differential operator that has the same principal symbol as the square root of the Laplace operator on $\partial\Omega$, see [74, Section 7.11]. Indeed, upper bounds for $\sigma_k(\Omega)$ in terms of $\lambda_k(\partial\Omega)$ have been obtained by Wang–Xia [77] for $k = 1$ and by Karpukhin [47] for higher eigenvalues. Quantitative estimate for $|\sigma_k(\Omega) - \sqrt{\lambda_k(\partial\Omega)}|$ have been obtained by Provenzano–Stubbe [69] for domains in Euclidean space and by Xiong [78] and Colbois–Girouard–Hassannezhad [20] in the Riemannian setting. The eigenvalues of various other spectral problems have also been compared with Steklov eigenvalues. See the work of Kuttler–Sigillito [57] and Hassannezhad–Siffert [40].

A different type of relationship was studied by Lamberti–Provenzano [59], who proved that the Steklov eigenvalues of a domain $\Omega \subset \mathbb{R}^d$ can be obtained as appropriate limits of non-homogeneous Neumann eigenvalues with the mass concentrated at the boundary of Ω.

1.2 Isoperimetric inequalities

Theorem 1.1 has several applications to the study of isoperimetric inequalities for Steklov eigenvalues. These are most naturally stated in terms of the scale invariant eigenvalues

$$\overline{\lambda}_k(M, g) := \mathrm{Vol}_g(M)^{\frac{2}{d}} \lambda_k(M, g, 1) \tag{3}$$

and

$$\overline{\sigma}_k(\Omega, g) := \mathscr{H}^{d-1}(\partial\Omega)^{\frac{1}{d-1}} \sigma_k(\Omega, g), \tag{4}$$

where $\mathrm{Vol}_g(M)$ is the volume of M and $\mathscr{H}^{d-1}(\partial\Omega)$ is the $(d-1)$-Hausdorff measure of the boundary $\partial\Omega$. It is natural to ask for upper bounds on the functionals (3) and (4), and as such to define

$$\Lambda_k(M) := \sup_{g \in \mathscr{G}(M)} \overline{\lambda}_k(M, g)$$

and

$$\Sigma_k(M) := \sup_{\Omega \subset M} \sup_{g \in \mathscr{G}(\overline{\Omega})} \overline{\sigma}_k(\Omega, g)$$

where for any manifold, with or without boundary, $\mathscr{G}(M)$ is the set of all Riemannian metrics on M. Spectral isoperimetric inequalities often have a wildly different behaviour in dimension two than in dimension at least three, as exhibited in the work of Colbois–Dodziuk [14] and Korevaar [56]. As such, we study these cases separately.

1.2.1 Isoperimetric inequalities in dimension two

From Colbois–El Soufi–Girouard [17] it is known that $\Sigma_k(M)$ is finite for each surface. The next result provides an effective lower bound.

Theorem 1.3 *For every $k \in \mathbb{N}$ and every smooth, closed, connected surface M,*

$$\Sigma_k(M) \geq \Lambda_k(M).$$

This should be compared with [35, Theorem 9] where a similar inequality was proved, relating Steklov and Neumann eigenvalues of a domain in Euclidean space. The storied study of Λ_k for various k and smooth surfaces M of different topologies yields explicit lower bounds for Σ_k, which we record in Sect. 6. Kokarev [55, Theorem A_1, Example 1.3] proved that $\Sigma_1(\mathbb{S}^2) \leq 8\pi$. Theorem 1.3 and the known value $\Lambda_1(\mathbb{S}^2) = 8\pi$ for the round sphere, shows that Kokarev's bound is sharp.

Corollary 1.4 *The following equality holds:*

$$\Sigma_1(\mathbb{S}^2) = 8\pi.$$

Remark 1.5 Both Corollary 1.4 and, further along, Theorem 1.13 along with (8) are in contradiction with parts of [32, Theorems 8.2], where the bound

$$\Sigma_1(\mathbb{S}^2) \leq 4\pi$$

is given. Further discussion and related results are delayed to "Appendix A".

Very recent work of Karpukhin–Stern [52, Theorem 5.2] in fact shows that for all surfaces M, and for $j \in \{1, 2\}$

$$\Sigma_j(M) \leq \Lambda_j(M),$$

using methods from the min-max theory of harmonic maps. In combination with Theorem 1.3, we obtain the following result, also presented as [52, Proposition 5.9], which extends Corollary 1.4.

Corollary 1.6 *For all closed surfaces M, the following equalities hold*

$$\Sigma_1(M) = \Lambda_1(M)$$

and

$$\Sigma_2(M) = \Lambda_2(M).$$

This leads naturally to the following conjecture.[1]

Conjecture *For all closed surfaces M and all $k \in \mathbb{N}$,*

$$\Sigma_k(M) = \Lambda_k(M).$$

1.2.2 Isoperimetric inequalities in dimension at least three

For $d \geq 3$, it follows from the work of Colbois–Dodziuk [14] that $\Lambda_1(M) = +\infty$. Together with Theorem 1.1 this gives $\Sigma_1(M) = +\infty$. Using the extra freedom provided by the weight β, we arrive at more precise statements, starting with the following corollary to Theorem 1.1.

Corollary 1.7 *Let (M, g) be a Riemannian manifold of dimension $d \geq 2$. For constant density $\beta > 0$, the domains $\Omega^\varepsilon \subset M$ obtained in* Theorem 1.1 *satisfy*

$$\overline{\sigma}_k(\Omega^\varepsilon, g) \xrightarrow{\varepsilon \to 0} \beta^{\frac{2-d}{d-1}} \operatorname{Vol}_g(M)^{\frac{2-d}{d(d-1)}} \overline{\lambda}_k(M, g). \tag{5}$$

Proof By Theorem 1.1 with constant density $\beta > 0$, one obtains a family of domains Ω_ε such that

$$\sigma_k(\Omega^\varepsilon, g) \to \lambda_k(M, g, \beta) = \frac{1}{\beta}\lambda_k(M, g)$$

and

$$\mathscr{H}^{d-1}(\partial\Omega^\varepsilon) \to \int_M \beta \, \mathrm{d}x = \beta \operatorname{Vol}_g(M).$$

Together with the definition (4) of $\overline{\sigma}_k$, we have

$$\overline{\sigma}_k(\Omega^\varepsilon, g) \to \beta^{\frac{2-d}{d-1}} \operatorname{Vol}_g(M)^{\frac{1}{d-1}} \lambda_k(M, g) = \beta^{\frac{2-d}{d-1}} \operatorname{Vol}_g(M)^{\frac{2-d}{d(d-1)}} \overline{\lambda}_k(M, g).$$

□

The previous corollary along with the results of Fraser and Schoen [33] lead to the following result.

[1] This conjecture was proved by the authors together with Karpukhin in the recent preprint [36].

Corollary 1.8 *Let (M, g) be a Riemannian manifold of dimension $d \geq 3$. Then there exists a family of domains $\Omega^\varepsilon \subset M$ with connected boundary such that*

$$\lim_{\varepsilon\to 0} \overline{\sigma}_1(\Omega^\varepsilon, g) = +\infty.$$

Proof Since $d \geq 3$, the righthand side in (5) diverges to $+\infty$ as $\beta \to 0$. Therefore, applying Corollary 1.7 with $\beta \to 0$ gives us a family $\Omega_\beta \subset M$ such that $\overline{\sigma}_1(\Omega_\beta, g) \to \infty$. By removing thin tubes joining boundary components of a domain $\Omega \subset M$, it is shown in [33] that in dimension $d \geq 3$, there is a family of domains $\Omega^\varepsilon_\beta \subset \Omega$, with connected boundary and such that $\left|\overline{\sigma}_1(\Omega_\beta) - \overline{\sigma}_1(\Omega^\varepsilon_\beta)\right| < \varepsilon$. The diagonal family $\Omega^\varepsilon := \Omega^\varepsilon_\varepsilon$ verifies our claim. □

In recent years several constructions of manifolds with large normalised Steklov eigenvalue $\overline{\sigma}_1$ have been proposed. Colbois–Girouard [19] and Colbois–Girouard–Binoy [21] constructed a sequence Ω_n of compact surfaces with connected boundary such that $\overline{\sigma}_1(\Omega_n) \to \infty$. Cianci–Girouard [12] proved that some manifolds M of dimension $d \geq 4$ carry Riemannian metrics that are prescribed on ∂M with uniformly bounded volume and arbitrarily large first Steklov eigenvalue σ_1. Corollary 1.8 provides a new outlook on this question.

1.2.3 Transferring bounds for Steklov eigenvalues to bounds for Laplace eigenvalues

If (M, g) is conformally equivalent to a Riemannian manifold with nonnegative Ricci curvature, it follows from Colbois–Girouard–El Soufi [17] that for each domain $\Omega \subset M$ with smooth boundary, and for each $k \geq 1$,

$$\sigma_k(\Omega) \leq C_d \frac{\mathrm{Vol}_g(\Omega)^{\frac{d-2}{d}}}{\mathscr{H}^{d-1}(\partial\Omega)} k^{\frac{2}{d}}. \tag{6}$$

Using the domains Ω^ε from Theorem 1.1 and taking the limit as $\varepsilon \to 0$ leads to the following.

Corollary 1.9 *Let (M, g) be a closed manifold with g conformally equivalent to a metric with nonnegative Ricci curvature. For each $\beta \in C^\infty(M)$ positive,*

$$\lambda_k(M, g, \beta) \int_M \beta \, d\mu_g \leq C_d \, \mathrm{Vol}_g(M)^{\frac{d-2}{d}} k^{\frac{2}{d}}. \tag{7}$$

This is a special case of an inequality that was proved in Grigor'yan–Netrusov–Yau [38, Theorem 5.9].

Corollary 1.10 *The exponent $2/d$ cannot be improved in* (6)*, and the exponents on* $\mathrm{Vol}_g(\Omega)$ *and* $\mathscr{H}^{d-1}(\partial\Omega)$ *cannot be replaced by any other exponents.*

Proof For $\beta > 0$ constant, inequality (7) becomes $\lambda_k(M,g)\,\mathrm{Vol}_g(M)^{\frac{2}{d}} \leq C_d k^{\frac{2}{d}}$, where the exponent on k carries over from (6). That it cannot be improved follows from the Weyl Law. Now, changing the exponent of $\mathscr{H}^{d-1}(\partial\Omega)$ in (6) would yield an inequality with a non-trivial exponent for β, while changing the exponent of $\mathrm{Vol}_g(\Omega)$ would lead to an inequality similar to (7), but not invariant under scaling of the Riemannian metric. □

Remark 1.11 Corollary 1.10 improves upon [17, Remark 1.4], where it was already observed that the exponent $2/d$ could not be replaced by $1/(d-1)$ in inequality (6). Note also that for an Euclidean domain $\Omega \subset \mathbb{R}^d$, it follows from the isoperimetric inequality and (6) that $\sigma_k(\Omega)|\partial\Omega|^{\frac{1}{d-1}} \leq Ck^{\frac{2}{d}}$. Deciding if the exponent $2/d$ can be improved in this inequality is still an open problem, which was proposed as [37, Open problem 5]. We also note that the very recent preprint of Karpukhin–Métras [50] discusses normalisation (6). It is shown that it is the most natural eigenvalue normalisation from the point of view of geometric analysis in dimension $d \geq 3$.

1.3 Free boundary minimal surfaces

In dimension $d = 2$, the striking connection between the Steklov eigenvalue problem and free boundary minimal submanifolds in the unit ball was revealed by Fraser and Schoen in [30–32].

Definition 1.12 (cf. [60, Theorem 2.2]) For $m \geq 3$, let $\mathbb{B}^m$ be the m-dimensional Euclidean unit ball and let $\Omega \subset \mathbb{B}^m$ be a k-dimensional submanifold with boundary $\partial\Omega = \overline{\Omega} \cap \partial\mathbb{B}^m$. We say that Ω is a *free boundary minimal submanifold* in $\mathbb{B}^m$ if one of the following equivalent conditions hold.

(1) Ω is a critical point for the area functional among all k-dimensional submanifolds of $\mathbb{B}^m$ with boundary on $\partial\mathbb{B}^m$.
(2) Ω has vanishing mean curvature and meets $\partial\mathbb{B}^m$ orthogonally.
(3) The coordinate functions $x^1, \dots, x^m$ restricted to Ω are solutions to the Steklov eigenvalue problem (2) with eigenvalue $\sigma = 1$.

Conditions (1) and (2) can be used to generalise Definition 1.12 to arbitrary background manifolds in place of $\mathbb{B}^m$, but the equivalence of condition (3) is a special property of the Euclidean unit ball. Conversely, Fraser and Schoen have shown in [32, Proposition 5.2] that maximal metrics for $\overline{\sigma}_1$ give rise to free boundary minimal surfaces in the unit ball. Indeed, given such a maximal metric g on Ω, the eigenfunctions associated with $\overline{\sigma}_1$ are the coordinates of an isometric immersion of Ω as a free boundary minimal surface inside $\mathbb{B}^m$ for some $m \geq 2$. The existence of those maximal metrics has recently been given for arbitrary genus and any number of boundary components by Matthiesen and Petrides in [64]. It is conjectured by Fraser and Li [29, Conjecture 3.3] that $\sigma = 1$ is actually equal to the *first* nonzero Steklov eigenvalue $\sigma_1(\Omega)$ for any given compact, properly embedded free boundary minimal hypersurface Ω in the unit ball.

Even in the case $m = 3$ it is a challenging problem to construct free boundary minimal surfaces with a given topology. The first nontrivial examples (apart from the equatorial disk and the critical catenoid) were found by Fraser and Schoen [32]. Their surfaces have genus 0 and an arbitrary number of boundary components. An independent construction of free boundary minimal surfaces with genus $\gamma \in \{0, 1\}$ and any sufficiently large number b of boundary components was given by Folha–Pacard–Zolotareva [28]. The sequence of surfaces converges as $b \to \infty$ to the equatorial disk with multiplicity two. McGrath [65, Corollary 4.3] proved that these surfaces indeed have the property that $\sigma_1 = 1$ as conjectured by Fraser and Li. We note that the existence of maximal metrics for the Steklov problem on surfaces of genus 0, the fact that the multiplicity of the first Steklov eigenvalue on surfaces of genus 0 is bounded above by 3 [49, Theorem 1.3.1], and Corollary 1.4 immediately give the existence of free boundary minimal surfaces of genus 0 immersed in $\mathbb{B}^3$ by first Steklov eigengunctions and with boundary length arbitrary close to 8π, and thus with area arbitrary close to 4π.

Let us now mention a few other constructions for which it is an open problem whether $\sigma_1 = 1$. Free boundary minimal surfaces with high genus were constructed by Kapouleas–Li [45] and Kapouleas–Wiygul [46] using desingularisation methods. The equivariant min-max theory developed by Ketover [53,54] allowed the construction of free boundary minimal surfaces of arbitrary genus with dihedral symmetry and of genus 0 with symmetry group associated to one of the platonic solids. If their genus is sufficiently high, Ketover's surfaces have three boundary components. More recently, Carlotto–Franz–Schulz [8] constructed free boundary minimal surfaces with dihedral symmetry, arbitrary genus and connected boundary.

For certain free boundary minimal surfaces which are invariant under the action of the symmetry group associated to one of the platonic solids (see [54, Theorem 6.1]) we confirm Fraser and Li's conjecture about the first Steklov eigenvalue in the following theorem based on the work of McGrath [65].

Theorem 1.13 *Let $\Omega \subset \mathbb{B}^3$ be an embedded free boundary minimal surface of genus* 0*. If Ω has tetrahedral symmetry and $b = 4$ boundary components or octahedral symmetry and $b \in \{6, 8\}$ boundary components or icosahedral symmetry and $b \in \{12, 20, 32\}$ boundary components, then $\sigma_1(\Omega) = 1$.*

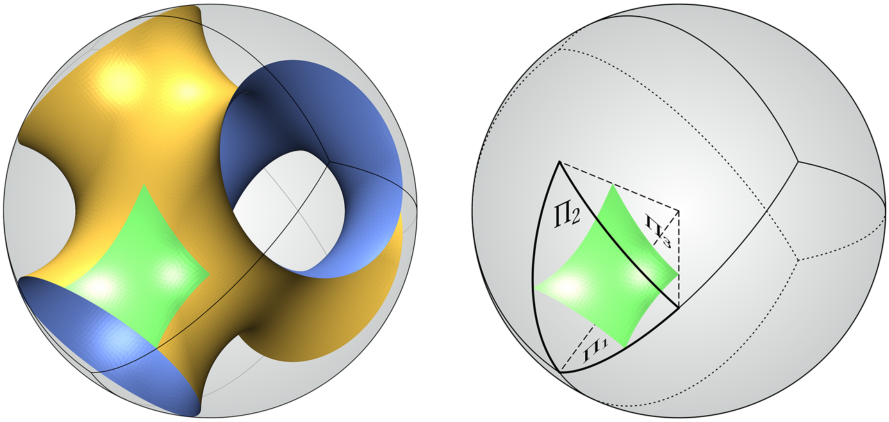

Fig. 1 Free boundary minimal surface of genus 0 with tetrahedral symmetry and 4 boundary components and its fundamental domain being a free boundary minimal disk inside a four-sided wedge

Remark 1.14 Ketover's result [54, Theorem 6.1] states the existence of free boundary minimal surfaces with tetrahedral symmetry and $b = 4$ boundary components, with octahedral symmetry and $b = 6$ boundary components and with icosahedral symmetry and $b = 12$ boundary components. We conjecture that free boundary minimal surfaces with $b \in \{8, 20, 32\}$ boundary components and corresponding symmetries as stated in Theorem 1.13 exist as well. In fact, we visualise all mentioned cases in Figs. 1, 2 and 3. The simulation is based on Brakke's surface evolver [6] which we use to approximate free boundary minimal disks D inside a four-sided wedge as shown on the right of Fig. 1. If the wedge is chosen suitably such that it forms a fundamental domain for the action of the symmetry group of one of the platonic solids (see Definition 7.1), then repeated reflection of D leads to an approximation of a free boundary minimal surface in the unit ball.

The simulations allow approximations for $\overline{\sigma}_1$. Indeed, in Table 1 we numerically compute the area of each surface shown in Figs. 1, 2 and 3 using the surface evolver. To increase accuracy, the area has been computed using a much finer triangulation than the one used to render the images. Since any free boundary minimal surface $\Omega \subset \mathbb{B}^3$ has boundary length equal to twice its area (see [60, Proposition 2.4]) and since symmetries and topology imply $\sigma_1(\Omega) = 1$ by Theorem 1.13, we observe in each case

$$\overline{\sigma}_1(\Omega) = \mathscr{H}^1(\partial\Omega)\,\sigma_1(\Omega) > 4\pi. \tag{8}$$

We emphasise that we do *not* answer the question whether or not any of the free boundary minimal surfaces discussed in Theorem 1.13 respectively Table 1 are maximisers for $\overline{\sigma}_1$ in the class of surfaces with the same topology. In

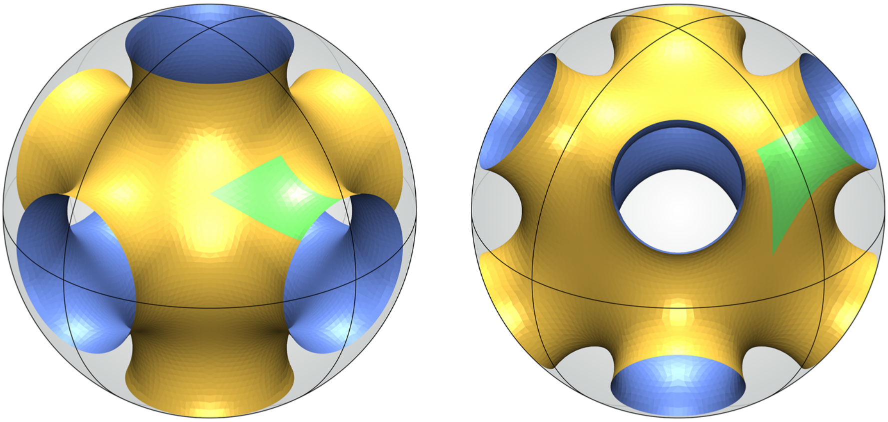

Fig. 2 Free boundary minimal surfaces of genus 0 with octahedral symmetry and 6 or 8 boundary components

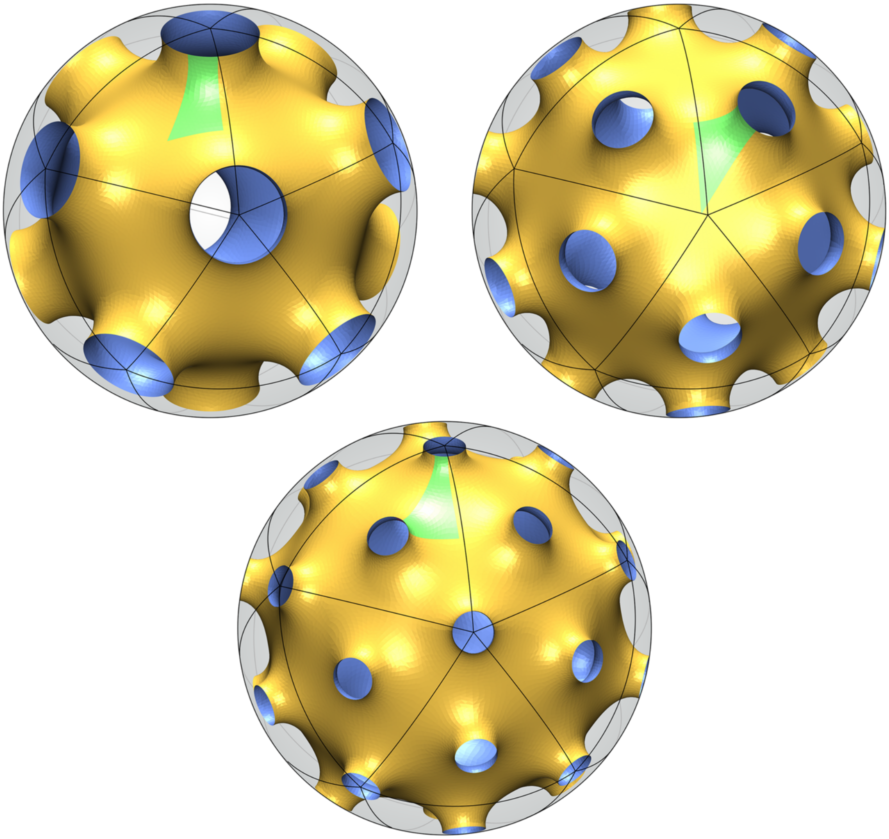

Fig. 3 Free boundary minimal surfaces of genus 0 with icosahedral symmetry and 12, 20 or 32 boundary components

Table 1 Areas and scale invariant eigenvalues of the surfaces shown in Figs. 1, 2 and 3

Symmetry	Boundary components	Area	$\overline{\sigma}_1(\Omega)$
Tetrahedral	4	$2.1752\,\pi$	$4.3505\,\pi$
Octahedral	6	$2.4549\,\pi$	$4.9099\,\pi$
Octahedral	8	$2.6141\,\pi$	$5.2282\,\pi$
Icosahedral	12	$2.8757\,\pi$	$5.7514\,\pi$
Icosahedral	20	$3.1149\,\pi$	$6.2299\,\pi$
Icosahedral	32	$3.3444\,\pi$	$6.6888\,\pi$

fact, after submission of this manuscript, Kao–Osting–Oudet [44] have found numerically for $b = 8$ and $b = 20$ free boundary minimal surfaces with a larger value of $\overline{\sigma}_1$. In that paper, the free boundary minimal surfaces are obtained via maximisation of the first Steklov eigenvalue.

Remark 1.15 For Laplace eigenvalues, the eigenfunctions of a critical metric g on M for $\overline{\lambda}_1$ realise an isometric immersion of M as a minimal surface in the sphere $\mathbb{S}^m$ for some m, see Nadirashvili [66].

Plan of the paper

In Sect. 2, we describe precisely the homogenisation construction in the Riemannian setting. Theorem 2.1 is a restatement of Theorem 1.1 in terms of the explicit sequence of domains for which the normalised Steklov eigenvalues converge to the weighted Laplace eigenvalues.

In Sect. 3 we prove various technical inequalities that will be used in the later stages. Some of these inequalities are known for domains in flat space and we extend their proofs to the Riemannian setting. We first need to control the norm of the traces $\gamma^\varepsilon : \mathrm{H}^1(\Omega^\varepsilon) \to \mathrm{L}^2(\partial\Omega^\varepsilon)$ and $\tau^\varepsilon : \mathrm{BV}(\Omega^\varepsilon) \to \mathrm{L}^1(\partial\Omega^\varepsilon)$ uniformly in the parameter ε. We also need to bound uniformly the norm of the harmonic extension operator from $\mathrm{H}^1(\Omega^\varepsilon)$ to $\mathrm{H}^1(M)$, and to have a uniform Poincaré–Wirtinger inequality for some topological perturbations of geodesically convex subsets of M. We point out that the usual sufficient conditions in term of conditions on tubular neighbourhoods of the boundary and inner cone conditions are not satisfied in our case, nevertheless we can use the structure of the problem to find the relevant bounds.

In Sect. 4, we prove boundedness properties for the Steklov eigenvalues and eigenfunctions of the domains Ω^ε. More precisely, we prove that for every fixed k, σ_k^ε is bounded in ε, and that the L^∞ norm of $u_k^{(\varepsilon)}$ is also bounded uniformly.

Section 5 is dedicated to the proof of Theorem 2.1. The proof proceeds in three main steps. The first one is to show that for every k, families of harmonic extensions $U_k^{(\varepsilon)}$ of $u_k^{(\varepsilon)}$ are bounded in $\mathrm{H}^1(M)$. This gives us the existence along a subsequence of a limit $\sigma_k^{\varepsilon} \to \lambda$ and of a $\mathrm{H}^1(M)$ weak limit $U_k^{(\varepsilon)} \to \varphi$. The second step consists in studying the weak formulations to show that the pair (φ, λ) is a solution to Problem (1). In the last step, we show that there is no mass lost in the process, and therefore that indeed $\lambda = \lambda_k(M, g, \beta)$.

In Sect. 6, we prove the isoperimetric inequality stated in Theorem 1.3 and give as a corollary explicit lower bounds on the maximiser for Steklov eigenvalues in terms of known bounds for Laplace eigenvalues.

Finally, in Sect. 7, we provide a proof of Theorem 1.13. This proof uses symmetries of the free boundary minimal surfaces, and properties of the nodal sets of first eigenfunctions.

2 The homogenisation construction

2.1 Notation

From this section on, we denote by c and C positive constants that may depend only on the closed connected Riemannian manifold (M, g), the dimension, and the positive smooth function $\beta \in C^{\infty}(M)$. Similarly, the homogenisation construction depends on a parameter $\varepsilon > 0$ which must be chosen smaller than $\varepsilon_0 > 0$, a value also depending only on (M, g), the dimension, and β. The precise values of c, C and ε_0 may change from line to line, but changes occur only a finite number of times so that at the end $0 < \varepsilon_0, c, C < \infty$.

We will reserve the letters φ, λ for general eigenfunctions and eigenvalues of Problem (1), and φ_k and λ_k representing specifically the kth ones. Similarly, we reserve $u^{(\varepsilon)}$ and $\sigma^{(\varepsilon)}$ for Steklov eigenvalues of the sequence of domains Ω^{ε}. We drop in this notation any specific reference to M, to the metric g and to the weight β as they are kept fixed. We assume that eigenfunctions $u_k^{(\varepsilon)}$ and φ_k are orthonormal, with respect to $\mathrm{L}^2(\partial\Omega^{\varepsilon})$ and $\mathrm{L}^2(M, \beta\, \mathrm{d}\mu_g)$ respectively. We make use of various asymptotic notation.

- Indiscriminately, writing $f = O(g)$ or $f \ll g$ means that there exists $C, \varepsilon_0 > 0$ such that $|f(x)| \leq Cg(x)$ for all $0 < x < \varepsilon_0$.
- Writing $f = o(g)$ means that $\frac{f}{g} \to 0$ as $\varepsilon \to 0$.
- Writing $f \asymp g$ means that both $f \ll g$ and $g \ll f$.
- Indices in the asymptotic notation (e.g. $f = O_M(g)$ or $f \ll_k g$) mean that the implicit constants, the range of validity or the rate of convergence to 0 for o depends only on those quantities. We use M as an index to represent dependence both on the manifold M and on the metric g.

2.2 Geodesic polar coordinates

Some of our proofs are formulated using geodesic polar coordinates, so let us recall their construction, see [9, Chapter XII.8]. For a point $p \in M$ and $\delta < \mathrm{inj}(M)$, the exponential map is a diffeomorphism from the ball of radius δ in T_pM to the geodesic ball $B_\delta(p) \subset M$. In $B_\delta(p)$, we use the polar coordinates (ρ, θ), where ρ is the geodesic distance from p and θ is a unit tangent vector in T_pM.

We recall that in those coordinates, the metric reads

$$g(\rho, \theta) = \mathrm{d}\rho^2 + \rho^2(1 + h(\rho, \theta))g_{\mathbb{S}^{d-1}},$$

where

$$\|h(\rho, \theta)\|_{\mathrm{C}^1(B_\delta(p))} = O_M(\delta). \tag{9}$$

We record as well that the volume element can be written in these coordinates as

$$\mathrm{d}V = \rho^{d-1}\left(1 + O_M\left(\delta^2\right)\right)\,\mathrm{d}\rho\,\mathrm{d}A_{\mathbb{S}^{d-1}} \tag{10}$$

and for any geodesic sphere of radius $r \leq \delta$, its area element is of the form

$$\mathrm{d}A = r^{d-1}\left(1 + O_M\left(r^2\right)\right)\,\mathrm{d}A_{\mathbb{S}^{d-1}}. \tag{11}$$

Compactness of M ensures that the implicit constants in (9), (10) and (11) can be chosen independently of p.

2.3 Homogenisation by obstacles

For every $\varepsilon > 0$, let $\mathbf{S}^\varepsilon$ be a maximal ε-separated subset of M, and let $\mathbf{V}^\varepsilon$ be the Voronoĭ tessellation associated with $\mathbf{S}^\varepsilon$, that is the set $\mathbf{V}^\varepsilon := \left\{V_p^\varepsilon : p \in \mathbf{S}^\varepsilon\right\}$, with

$$V_p^\varepsilon := \left\{x \in M : \mathrm{dist}(x, p) \leq \mathrm{dist}(x, q) \text{ for all } q \in \mathbf{S}^\varepsilon\right\}.$$

We note that for $\varepsilon < \varepsilon_0$ and every $p \in \mathbf{S}^\varepsilon$, V_p^ε is a domain with piecewise smooth boundary, and that

$$\mathrm{Vol}_g(V_p^\varepsilon) \asymp_M \varepsilon^d.$$

Indeed, by maximality of the ε-separated set $\mathbf{S}^\varepsilon$ we have that $B_{\varepsilon/2}(p) \subset V_p^\varepsilon \subset B_{3\varepsilon}(p)$. Let $\beta \in C^\infty(M)$ be a smooth positive function. For every $p \in \mathbf{S}^\varepsilon$, let $r_{\varepsilon,p} > 0$ be such that

$$\mathscr{H}^{d-1}(\partial B_{r_{\varepsilon,p}}(p)) = \beta(p)\,\mathrm{Vol}_g(V_p^\varepsilon). \tag{12}$$

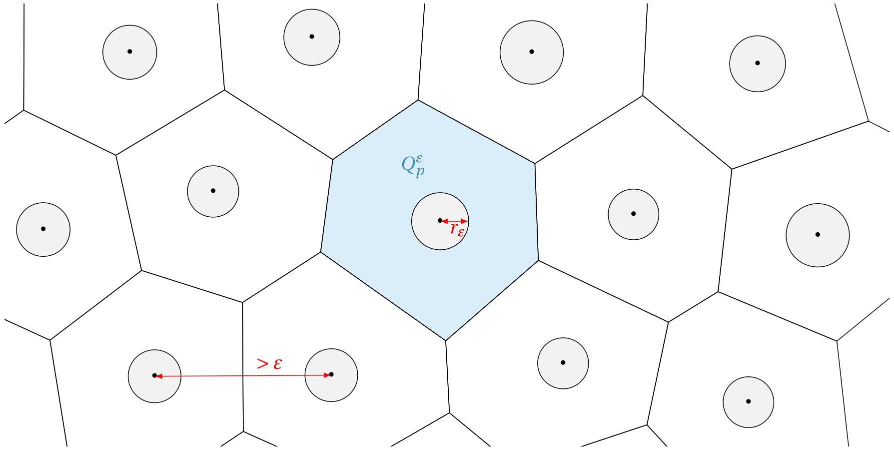

Fig. 4 Voronoĭ tessellation associated with a maximal ε-separated subset

It also follows from (10) and (11) that

$$r_{\varepsilon,p} \asymp_{M,\beta} \varepsilon^{\frac{d}{d-1}}. \tag{13}$$

Since the previous display holds uniformly for $p \in M$, we often abuse notation and write r_ε for $r_{\varepsilon,p}$. We set

$$\mathbf{T}^\varepsilon := \bigcup_{p \in \mathbf{S}^\varepsilon} B_{r_\varepsilon}(p),$$

$\Omega^\varepsilon = M \setminus \mathbf{T}^\varepsilon$, and $Q_p^\varepsilon = V_p^\varepsilon \setminus B_{r_\varepsilon}(p)$. We observe that $\mathrm{Vol}_g(\mathbf{T}^\varepsilon) = O_{M,\beta}\left(\varepsilon^{\frac{d}{d-1}}\right)$. See Fig. 4 for a depiction of this construction.

Furthermore, we have that $\mathrm{dist}(B_{r_{\varepsilon,p}}(p), B_{r_{\varepsilon,q}}(q)) \geq \varepsilon - O_{M,\beta}\left(\varepsilon^{\frac{d}{d-1}}\right)$ for all $p \neq q \in \mathbf{S}^\varepsilon$. We see that by construction, for every $0 < \varepsilon < \varepsilon_0$,

$$\mathscr{H}^{d-1}(\partial\Omega^\varepsilon) = \sum_{p \in \mathbf{S}^\varepsilon} \beta(p)\, \mathrm{Vol}_g(V_p^\varepsilon).$$

It also easy to see that the measure $\mathrm{d}A_g$ obtained on M by restriction of the Hausdorff measure $\mathscr{H}^{d-1}$ to $\partial\Omega^\varepsilon$ converges weak-$*$ to the weighted Lebesgue measure $\beta\, \mathrm{d}\mu_g$ on M. That is, for each continuous function f on M,

$$\int_{\partial\Omega^\varepsilon} f\, \mathrm{d}A_g \xrightarrow{\varepsilon \to 0} \int_M f\, \beta\, \mathrm{d}\mu_g. \tag{14}$$

This already addresses the first part of Theorem 1.1, and by considering $f \equiv 1$ in (14) we see that

$$\mathscr{H}^{d-1}(\partial\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} \int_M \beta \,\mathrm{d}\mu_g.$$

We study the sequence of eigenvalue problems on Ω^ε

$$\begin{cases} \Delta u^{(\varepsilon)} = 0 & \text{in } \Omega^\varepsilon; \\ \partial_\nu u^{(\varepsilon)} = \sigma^{(\varepsilon)} u^{(\varepsilon)} & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{15}$$

and for every eigenfunction $u_k^{(\varepsilon)}$, we define $U_k^{(\varepsilon)} : M \to \mathbb{R}$ as the unique function equal to $u_k^{(\varepsilon)}$ on $\overline{\Omega^\varepsilon}$ and harmonic in $\mathbf{T}^\varepsilon$. The next theorem is the central technical result of this paper, and is in the flavour of the main theorem of [35]. It is also readily seen to imply directly Theorem 1.1 by providing the appropriate sequence Ω^ε.

Theorem 2.1 *The eigenvalues $\sigma_k^{(\varepsilon)}$ of Problem (15) converge as $\varepsilon \to 0$ to the eigenvalue $\lambda_k(M, g, \beta)$ defined in Problem (1). Up to choosing a subsequence, the extensions $U_k^{(\varepsilon)}$ to M of the eigenfunctions $u_k^{(\varepsilon)}$ converge weakly in $\mathrm{H}^1(M)$ to the corresponding Laplace eigenfunction φ_k on M, where φ_k is normalised to $\mathrm{L}^2(M, \beta\,\mathrm{d}\mu_g)$ norm 1.*

The proof is split in two main steps and is the subject of Sect. 5.

The first step is to show that there is a subsequence $(\sigma_k^{(\varepsilon)}, U_k^{(\varepsilon)})$ converging to a weak solution (λ, φ) of the weighted Laplace eigenvalue problem. In other words, the pair (λ, φ) satisfies

$$\forall v \in \mathrm{H}^1(M), \qquad \int_M \nabla v \cdot \nabla \varphi \,\mathrm{d}\mu_g = \lambda \int_M v\,\varphi\,\beta\,\mathrm{d}\mu_g. \tag{16}$$

The second step consists in proving that (λ, φ) has to be the kth eigenpair of the weighted Laplace eigenvalue problem. This will be done by showing that in the limit the functions $U_k^{(\varepsilon)}$ do not lose any mass. Physically, this can be interpreted as an instance of the Fermi exclusion principle, see e.g. the work of Colin de Verdière [22] for an early application of such an idea to create manifolds whose first Laplace eigenvalue have large multiplicity. We mention also the work of Lohkamp [63], who has expanded on the result of Colin de Verdière to produce manifolds with a prescribed finite part of the spectrum and controlled volume, using a Besicovitch covering to create perforated (and otherwise wildly perturbed) manifolds.

3 Analytic properties of perforated domains

In this section we describe analytic properties of the perforated manifolds Ω^ε and of the Voronoĭ cells V_p^ε. More precisely, we show that trace and extension operators are well behaved in the homogenisation limiting process. We stress that many of the inequalities we show here would be obviously satisfied for a fixed domain Ω. However, the usual sufficient conditions under which those inequalities would hold uniformly for a family of domains Ω^ε either are not satisfied, or it is nontrivial to show that they are indeed satisfied. We start by proving three lemmata about norms of trace operators.

The statement of Lemma 3.1 below is a generalisation of [7, Proposition 5.1] for domains in a closed manifold. For any $E \subset M$ measurable and $\Omega \subset M$ open, we define the perimeter of E in Ω, or simply the perimeter of E if $\Omega = M$, as

$$\operatorname{Per}(E,\Omega) := \sup\left\{\int_E \operatorname{div}_g X \,\mathrm{d}\mu_g : X \in \Gamma(TM),\ \operatorname{supp}(X) \subset \Omega,\ \|X\|_\infty \le 1\right\},$$

where $\Gamma(TM)$ is the set of smooth vector fields on M. We note that E has finite perimeter if and only if its indicator function is in $\mathrm{BV}(M)$. The perimeter of E corresponds to the Hausdorff measure of its reduced boundary $\partial^* E$, which may in general be smaller than the topological boundary. See [27, Chapter 5] for further discussion on the perimeter and the reduced boundary.

Lemma 3.1 *Let $\{\Omega_n \subset M : n \in \mathbb{N}\}$ be a sequence of open, bounded domains such that $\mathscr{H}^{d-1}(\partial\Omega_n)$ is uniformly bounded. Assume that there exists $Q, \delta > 0$ such that for all $n \in \mathbb{N}$ and $x \in \partial\Omega_n$,*

$$\sup\left\{\frac{\mathscr{H}^{d-1}(\partial^* E \cap \partial^*\Omega_n)}{\mathscr{H}^{d-1}(\partial^* E \cap \Omega_n)} : E \subset \Omega_n \cap B_\delta(x), \operatorname{Per}(E,\Omega_n) < \infty\right\} < Q. \tag{17}$$

Then, the trace operators $\tau_n : \mathrm{BV}(\Omega_n) \to \mathrm{L}^1(\partial\Omega_n)$ are bounded uniformly in n.

Proof For any $\eta > 0$, since M is compact, we can choose δ small enough so that for every $x \in M$, the metric in geodesic polar coordinates in $B_{2\delta}(x)$ reads

$$g = \mathrm{d}\rho^2 + \rho^2(1 + h(\rho,\theta))\,\mathrm{d}\theta^2,$$

with $|h(\rho,\theta)| + |\nabla h(\rho,\theta)| \le \delta^{\frac{1}{2}} \le \eta$. In other word, the diffeomorphism provided by the inverse of the exponential map, from $B_{2\delta}(x)$ to the ball of radius 2δ in $\mathbb{R}^d$ is a C^1 η-perturbation of an isometry. For any n, the norms of $\mathrm{L}^1(\partial\Omega_n \cap B_\delta(x))$ and $\mathrm{BV}(\Omega_n \cap B_\delta(x))$ change uniformly continuously on bounded sets under C^1 diffeomorphisms, and the same is true of the Hausdorff measures in (17). By [7, Proposition 5.1], (17) implies that the trace operators are uniformly bounded on the pullbacks to the balls, and by the above discussion we can bring these estimates back to the manifold. □

Lemma 3.2 *The trace operators* $\tau^\varepsilon : \mathrm{BV}(\Omega^\varepsilon) \to \mathrm{L}^1(\partial\Omega^\varepsilon)$ *are bounded uniformly in* ε.

Proof In order to apply Lemma 3.1, we need to find $\delta, Q > 0$ such that for all $x \in \partial\Omega^\varepsilon$ and all $\varepsilon > 0$ small enough, (17) holds. A simple volume comparison yields that there is $c > 0$ such that for all $\delta > \varepsilon$ and $x \in \partial\Omega^\varepsilon$,

$$\#\left\{p \in \mathbf{S}^\varepsilon : Q_p^\varepsilon \cap B_\delta(x) \neq \varnothing\right\} \leq c\left(\frac{\delta}{\varepsilon}\right)^d. \tag{18}$$

Combining (18) with (12), for any $E \subset \Omega^\varepsilon \cap B_\delta(x)$ of finite perimeter,

$$\mathscr{H}^{d-1}(\partial^* E \cap \partial\Omega^\varepsilon) \leq \mathscr{H}^{d-1}(\partial\Omega^\varepsilon \cap B_\delta(x)) \leq C\delta^d,$$

where C depends on M, g and β. We may then assume that the supremum is taken over sets E such that

$$\mathscr{H}^{d-1}(\partial^* E \cap \Omega^\varepsilon) \leq C\delta^d,$$

otherwise the ratio in (17) is bounded by 1. Observe that

$$\mathscr{H}^{d-1}(\partial^* E \cap \partial\Omega^\varepsilon) = \sum_{p \in \mathbf{S}^\varepsilon} \mathscr{H}^{d-1}(\partial^* E \cap \partial B_{r_\varepsilon}(p)).$$

For $p \in \mathbf{S}^\varepsilon$ and $t \in (0, \varepsilon/4)$, define

$$F_{p,t} = E \cap \left\{x : \operatorname{dist}(x, \partial B_{r_\varepsilon}(p)) \leq t\right\}.$$

Assume that for some $t \in (0, \varepsilon/4)$ we have that

$$\mathscr{H}^{d-1}(\partial^* F_{p,t} \cap Q_p^\varepsilon) \leq 2\mathscr{H}^{d-1}(\partial^* E \cap Q_p^\varepsilon). \tag{19}$$

Without loss of generality, we have chosen δ small enough so that the retraction on a geodesic ball of radius $\delta' < \delta$ is a 2-Lipschitz map uniformly for $x \in M$. This means that

$$\begin{aligned}\mathscr{H}^{d-1}(\partial^* E \cap \partial B_{r_\varepsilon}(p)) &= \mathscr{H}^{d-1}(\partial^* F_{p,t} \cap \partial B_{r_\varepsilon}(p)) \\ &\leq 2\mathscr{H}^{d-1}(\partial^* F_{p,t} \cap Q_p^\varepsilon) \\ &\leq 4\mathscr{H}^{d-1}(\partial^* E \cap Q_p^\varepsilon).\end{aligned} \tag{20}$$

Let $\widetilde{\mathbf{S}}^\varepsilon = \{p \in \mathbf{S}^\varepsilon : (19) \text{ does not hold}\}$. If $\widetilde{\mathbf{S}}^\varepsilon$ is empty we are done, since in that case (20) implies that (17) holds with $Q = 4$. Let $p \in \widetilde{\mathbf{S}}^\varepsilon$. Setting

$$h_p(t) := \mathscr{H}^{d-1}\left(\partial^* F_{p,t} \cap \left\{x : \operatorname{dist}(x, \partial B_{r_\varepsilon}(p)) = t\right\}\right),$$

the coarea formula gives $\partial_t \operatorname{Vol}_g(F_{p,t}) = h_p(t)$. It follows from the relative isoperimetric inequality [27, Theorem 5.6.2] that there is a constant $c > 0$ depending on M such that

$$\begin{aligned} c \operatorname{Vol}_g(F_{p,t})^{\frac{d-1}{d}} &\le \mathscr{H}^{d-1}(\partial^* F_{p,t} \cap Q_p^\varepsilon) \\ &\le 2h_p(t), \end{aligned}$$

where the second inequality follows from (19) not holding at p. Integrating, we therefore have that

$$\begin{aligned} 2\operatorname{Vol}_g(F_{p,\varepsilon/4}) &= \left(\int_0^{\frac{\varepsilon}{4}} \frac{h_p(t)}{\operatorname{Vol}_g(F_{p,t})^{\frac{d-1}{d}}} \,\mathrm{d}t\right)^d \\ &\gg_M C\varepsilon^d \\ &\gg_{M,\beta} C\mathscr{H}^{d-1}(\partial B_{r_\varepsilon}(p)). \end{aligned} \tag{21}$$

On the other hand, it follows from the isoperimetric inequality and equation (20) that

$$\begin{aligned} \sum_{p \in \widetilde{\mathbf{S}}^\varepsilon} \operatorname{Vol}_g(E \cap Q_p^\varepsilon) &\ll_{M,\beta} \mathscr{H}^{d-1}(\partial^* E)^{\frac{d}{d-1}} \\ &\ll_{M,\beta} \left(\mathscr{H}^{d-1}(\partial^* E \cap \Omega^\varepsilon) + \mathscr{H}^{d-1}(\partial^* E \cap \partial\Omega^\varepsilon)\right)^{\frac{d}{d-1}} \\ &\ll_{M,\beta} \left(\mathscr{H}^{d-1}(\partial^* E \cap \Omega^\varepsilon) + \sum_{p \in \widetilde{\mathbf{S}}^\varepsilon} \mathscr{H}^{d-1}(B_{r_\varepsilon}(p))\right)^{\frac{d}{d-1}}. \end{aligned} \tag{22}$$

Summing over $p \in \widetilde{\mathbf{S}}^\varepsilon$ in (21) and inserting in (22), we obtain C depending only on M and β such that

$$1 \le C\left(\sum_{p \in \widetilde{\mathbf{S}}^\varepsilon} \mathscr{H}^{d-1}(B_{r_\varepsilon}(p))\right)^{\frac{1}{d-1}} \left(1 + \frac{\mathscr{H}^{d-1}(\partial^* E \cap \Omega^\varepsilon)}{\sum_{p \in \widetilde{\mathbf{S}}^\varepsilon} \mathscr{H}^{d-1}(\partial B_{r_\varepsilon}(p))}\right)^{\frac{d}{d-1}}.$$

It follows from the weak-$*$ convergence in (14) that for small enough ε,

$$\mathscr{H}^{d-1}(\partial\Omega^\varepsilon \cap B_\delta(x)) < 2\max_p \beta(p) \operatorname{Vol}_g(B_\delta(x)).$$

This means that we can choose δ small enough, depending on M and β but not on ε so that

$$C\left(\sum_{p\in\widetilde{\mathbf{S}}^{\varepsilon}} \mathscr{H}^{d-1}(\partial B_{r_\varepsilon}(p))\right)^{\frac{d}{d-1}} \le \frac{1}{4}, \tag{23}$$

which means that

$$1 \le \frac{\mathscr{H}^{d-1}(\partial^* E \cap \Omega^\varepsilon)}{\sum_{p\in\widetilde{\mathbf{S}}^{\varepsilon}} \mathscr{H}^{d-1}(\partial B_{r_\varepsilon}(p))}. \tag{24}$$

Combining estimates (20) with (24) gives us that for ε small enough,

$$\mathscr{H}^{d-1}(\partial^* E \cap \partial\Omega^\varepsilon) \le 4\mathscr{H}^{d-1}(\partial^* E \cap \Omega^\varepsilon),$$

establishing our claim for $Q = 4$ and δ small enough for (23) to hold. □

The following lemma describes the behaviour of the Sobolev trace operators on the cells Q_p^ε under rescaling of the metric. For every $\alpha > 0$, let $g_\alpha := \alpha^2 g$ be the rescaled metric. All sets involved defined using the distance (such as geodesic balls) are always defined using the reference metric g.

Lemma 3.3 *For any $\alpha > 0$, denote by g_α the metric $\alpha^2 g$, and by $\gamma_p^{\varepsilon,\alpha}$ the Sobolev trace operator*

$$\gamma_p^{\varepsilon,\alpha} : \mathrm{H}^1(Q_p^\varepsilon, g_\alpha) \to \mathrm{L}^2(\partial B_{r_\varepsilon}(p), g_\alpha).$$

There exists $c, \varepsilon_0 > 0$ depending only on M and β such that for all $\varepsilon < \varepsilon_0$,

$$\left\| \gamma_p^{\varepsilon,\alpha} \right\|^2 \le c \max\left\{ \frac{r_\varepsilon^{d-1}}{\alpha \varepsilon^d}, \alpha\varepsilon \right\}.$$

Proof Let $f \in \mathrm{H}^1(Q_p^\varepsilon, g_\alpha)$; by density we assume that f is smooth. We assume that ε is small enough that in normal coordinates around p,

$$\int_{B_{\varepsilon/3}\setminus B_{r_\varepsilon}} f^2 \,\mathrm{d}\mu_g = \left(1 + O_{M,\beta}(\varepsilon)\right) \int_{\mathbb{S}^{d-1}} \int_{r_\varepsilon}^{\varepsilon/3} f(\rho,\theta)^2 \rho^{d-1} \,\mathrm{d}\rho \,\mathrm{d}\theta;$$

and

$$\int_{\partial B_{r_\varepsilon}} f^2 \,\mathrm{d}A_g = \left(1 + O_{M,\beta}(\varepsilon)\right) \int_{\mathbb{S}^{d-1}} f(r_\varepsilon,\theta)^2 r_\varepsilon^{d-1} \,\mathrm{d}\theta.$$

Let $\widetilde{f}$ be the radially constant function given by $\widetilde{f}(\rho,\theta) = f(r_\varepsilon,\theta)$, and set $F = f - \widetilde{f}$. Note that F vanishes on $\partial B_{r_\varepsilon}(p)$ and $\partial_\rho F = \partial_\rho f$. It is a simple computation to see that

$$\begin{aligned}\|f\|_{\mathrm{L}^2(\partial B_{r_\varepsilon}(p),g_\alpha)} &= (\alpha r_\varepsilon)^{d-1}(1+O_{M,\beta}(\varepsilon))\int_{\mathbb{S}^{d-1}} f(r_\varepsilon,\theta)^2\,\mathrm{d}\theta \\ &\ll_{M,\beta} \frac{(\alpha r_\varepsilon)^{d-1}}{\varepsilon^d}\int_{r_\varepsilon}^{\varepsilon/3}\int_{\mathbb{S}^{d-1}} \widetilde{f}(\rho,\theta)^2\rho^{d-1}\,\mathrm{d}\theta\,\mathrm{d}\rho \\ &\ll_{M,\beta} \frac{(\alpha r_\varepsilon)^{d-1}}{\varepsilon^d}\int_{r_\varepsilon}^{\varepsilon/3}\int_{\mathbb{S}^{d-1}} \left(f(\rho,\theta)^2+F(\rho,\theta)^2\right)\rho^{d-1}\,\mathrm{d}\theta\,\mathrm{d}\rho.\end{aligned} \tag{25}$$

It follows from the fundamental theorem of calculus and the Cauchy–Schwarz inequality that for every $\theta\in\mathbb{S}^{d-1}$ and $\rho\in(r_\varepsilon,\varepsilon/3)$,

$$\begin{aligned}|F(\rho,\theta)|^2 &= \left|\int_{r_\varepsilon}^{\rho}\partial_t f(t,\theta)\,\mathrm{d}t\right|^2 \\ &\le \int_{r_\varepsilon}^{\rho} t^{1-d}\,\mathrm{d}t\int_{r_\varepsilon}^{\rho}(\partial_t f(t,\theta))^2 t^{d-1}\,\mathrm{d}t.\end{aligned}$$

Integrating this inequality we obtain that

$$\begin{aligned}\int_{r_\varepsilon}^{\varepsilon/3}\int_{\mathbb{S}^{d-1}} F(\rho,\theta)^2\rho^{d-1}\,\mathrm{d}\theta\,\mathrm{d}\rho &\le \int_{r_\varepsilon}^{\varepsilon/3}\rho^{d-1}\int_{r_\varepsilon}^{\rho} t^{1-d}\,\mathrm{d}t\,\|\partial_\rho f\|^2_{\mathrm{L}^2(B_\rho\setminus B_{r_\varepsilon})}\,\mathrm{d}\rho \\ &\ll_{M,\beta} \|\nabla_g f\|^2_{\mathrm{L}^2(Q_p^\varepsilon,g)}\int_{r_\varepsilon}^{\varepsilon/3}\rho^d r_\varepsilon^{1-d}\,\mathrm{d}\rho \\ &\ll \frac{\varepsilon^{d+1}}{r_\varepsilon^{d-1}}\|\nabla_g f\|^2_{\mathrm{L}^2(Q_p^\varepsilon,g)}.\end{aligned} \tag{26}$$

Finally, by scaling properties of the metric and the Dirichlet energy

$$\|f\|^2_{\mathrm{L}^2(Q_p^\varepsilon,g_\alpha)} = \alpha^d\,\|f\|^2_{\mathrm{L}^2(Q_p^\varepsilon,g)} \tag{27}$$

and

$$\left\|\nabla_{g_\alpha} f\right\|^2_{\mathrm{L}^2(Q_p^\varepsilon,g_\alpha)} = \left\|\alpha^{\frac{d-2}{2}}\nabla_g f\right\|^2_{\mathrm{L}^2(Q_p^\varepsilon,g)} \tag{28}$$

Inserting (26), (27), and (28) into (25) yields

$$\begin{aligned}\|f\|^2_{\mathrm{L}^2(\partial B_{r_\varepsilon},g_\alpha)} &\ll_{M,\beta} \frac{r_\varepsilon^{d-1}}{\alpha\varepsilon^d}\|f\|^2_{\mathrm{L}^2(Q_\varepsilon^p,g_\alpha)} + \varepsilon\alpha\left\|\nabla_{g_\alpha} f\right\|^2_{\mathrm{L}^2(Q_p^\varepsilon,g_\alpha)} \\ &\ll_{M,\beta} \max\left\{\frac{r_\varepsilon^{d-1}}{\alpha\varepsilon^d},\alpha\varepsilon\right\}\|f\|^2_{\mathrm{H}^1(Q_p^\varepsilon),g_\alpha},\end{aligned}$$

which is the desired estimate. □

The next Lemma describes the behaviour of the operator of harmonic extension inside the holes $\mathbf{T}^\varepsilon$. For $\delta>0$ and $p\in M$, we denote by $A_\delta(p)$ the geodesic annulus $B_{2\delta}(p)\setminus B_\delta(p)$.

Lemma 3.4 *There are $C, \delta_0 > 0$ depending on M and β such that the harmonic extension operator $h_p^\delta : \mathrm{H}^1(A_\delta(p)) \to \mathrm{H}^1(B_\delta(p))$ satisfies for all $f \in \mathrm{H}^1(B_\delta(p))$*

$$\left\| h_p^\delta f \right\|^2_{\mathrm{L}^2(B_\delta(p))} \leq C \left(\| f \|^2_{\mathrm{L}^2(A_\delta(p))} + \delta^2 \, \| \nabla f \|^2_{\mathrm{L}^2(A_\delta(p))} \right).$$

and

$$\left\| \nabla (h_p^\delta f) \right\|^2_{\mathrm{L}^2(B_\delta(p))} \leq C \, \| \nabla f \|^2_{\mathrm{L}^2(A_\delta(p))}$$

for all $\delta < \delta_0$ and $p \in M$. In particular, the harmonic extension operator $h^\varepsilon : \mathrm{H}^1(\Omega^\varepsilon) \to \mathrm{H}^1(\mathbf{T}^\varepsilon)$ has norm uniformly bounded in ε.

Proof For small enough δ, c.f. Eq. (10), geodesic balls and spherical shells are mapped to Euclidean balls and spherical shells by C^1-small perturbations of an isometry, and all quantities involved are uniformly continuous in such perturbations, therefore we only need to prove the bounds in the Euclidean setting. This has been done in [70, Example 1, p. 40], see also [2, Lemma 4.3] for more detailed computations.

The claim on the global harmonic extension operator then follows. Indeed, for all $f \in \mathrm{H}^1(\Omega^\varepsilon)$,

$$\| h^\varepsilon f \|^2_{\mathrm{H}^1(\mathbf{T}^\varepsilon)} = \sum_{p \in \mathbf{S}^\varepsilon} \left\| h_p^{r_\varepsilon} f \right\|^2_{\mathrm{H}^1(B_{r_\varepsilon}(p))} \leq C \sum_{p \in \mathbf{S}^\varepsilon} \| f \|^2_{\mathrm{H}^1(A_{r_\varepsilon}(p))} \leq C \, \| f \|^2_{\mathrm{H}^1(\Omega^\varepsilon)}.$$

□

Finally, we will require that the Poincaré–Wirtinger inequality of the perforated Voronoĭ cells Q_p^ε hold uniformly in both $p \in M$ and $\varepsilon > 0$. To this end, for any domain $\Omega \subset M$, denote by $\mu_1(\Omega)$ the first non-trivial Neumann eigenvalue of Ω, and for any $f : \Omega \subset M \to \mathbb{R}$,

$$m_f := \frac{1}{\mathrm{Vol}_g(\Omega)} \int_\Omega f \, \mathrm{d}\mu_g.$$

Lemma 3.5 *There is $c, \varepsilon_0 > 0$ depending only on M and β such that for $0 < \varepsilon < \varepsilon_0$, $p \in \mathbf{S}^\varepsilon$, and all $f \in \mathrm{H}^1(Q_p^\varepsilon)$*

$$\int_{Q_p^\varepsilon} \left| f - m_f \right|^2 \, \mathrm{d}\mu_g \leq c\varepsilon^2 \int_{Q_p^\varepsilon} |\nabla f|^2 \, \mathrm{d}\mu_g.$$

Equivalently, the first non-trivial Neumann eigenvalue satisfies

$$\mu_1(Q_p^\varepsilon) \geq c^{-1} \varepsilon^{-2}.$$

Proof The equivalent formulation follows from the variational characterisation for Neumann eigenvalues and the observation that $f - m_f$ is orthogonal to constants on Q_p^ε.

Since the Voronoĭ cells V_p^ε are geodesically convex and have diameter $\mathrm{diam}(V_p^\varepsilon) = O_{M,\beta}(\varepsilon)$, uniformly in $p \in \mathbf{S}^\varepsilon$, it follows from [39, Theorem 1.2] that there is a constant C depending only on the curvature and dimension of M such that

$$\mu_1(V_p^\varepsilon) \geq C\varepsilon^{-2}. \tag{29}$$

Let w be the first non-constant Neumann eigenfunction of Q_p^ε, normalised to $\|w\|_{\mathrm{L}^2(Q_p^\varepsilon)} = 1$, and let $\widehat{w}$ be the function defined on V_p^ε as the harmonic extension to $B_{r_\varepsilon}(p)$, i.e. as

$$\widehat{w}(x) = \begin{cases} w(x) & \text{if } x \in Q_p^\varepsilon \\ h^\varepsilon w(x) & \text{if } x \in B_{r_\varepsilon}(p), \end{cases}$$

where h^ε is defined in Lemma 3.4. It follows from the Cauchy–Schwarz inequality and the fact that $\int_{Q_p^\varepsilon} w \,\mathrm{d}\mu_g = 0$ that

$$\begin{aligned} m_{\widehat{w}} := \frac{1}{\mathrm{Vol}_g(V_p^\varepsilon)} \int_{V_p^\varepsilon} \widehat{w}(x)\,\mathrm{d}\mu_g &= \frac{1}{\mathrm{Vol}_g(V_p^\varepsilon)} \int_{B_{r_\varepsilon}(p)} h^\varepsilon w(x)\,\mathrm{d}A_g \\ &\leq \frac{\mathrm{Vol}_g(B_{r_\varepsilon}(p))^{\frac{1}{2}}}{\mathrm{Vol}_g\,(V_p^\varepsilon)} \left\|h^\varepsilon w\right\|_{\mathrm{L}^2(B_{r_\varepsilon}(p))} \\ &= O_{M,\beta}\left(\varepsilon^{\frac{d(2-d)}{2(d-1)}}\right), \end{aligned}$$

where the last line follows from Lemma 3.4 and estimate (13). This implies that

$$\|m_{\widehat{w}}\|_{\mathrm{L}^2(V_p^\varepsilon)} = m_{\widehat{w}}\,\mathrm{Vol}_g(V_p^\varepsilon)^{\frac{1}{2}} = O_{M,\beta}\left(\varepsilon^{\frac{d}{2(d-1)}}\right). \tag{30}$$

Using $\widehat{w} - m_{\widehat{w}}$ as a test function for the first Neumann eigenvalue in V_p^ε we have from Lemma 3.4 that there is a constant c such that

$$\begin{aligned} \mu_1(Q_p^\varepsilon) &= \int_{Q_p^\varepsilon} |\nabla(\widehat{w} - m_{\widehat{w}})|^2 \,\mathrm{d}\mu_g \\ &\geq c \int_{V_p^\varepsilon} |\nabla(\widehat{w} - m_{\widehat{w}})|^2 \,\mathrm{d}\mu_g \\ &\geq c\mu_1(V_p^\varepsilon)\,\|\widehat{w} - m_{\widehat{w}}\|^2_{\mathrm{L}^2(V_p^\varepsilon)} \\ &\gg_M \varepsilon^{-2}\left(\|\widehat{w}\|_{\mathrm{L}^2(V_p^\varepsilon)} - \|m_{\widehat{w}}\|_{\mathrm{L}^2(V_p^\varepsilon)}\right)^2, \end{aligned} \tag{31}$$

where the last inequality follows from (29). By (30) we have that

$$\|\widehat{w}\|_{\mathrm{L}^2(V_p^\varepsilon)} - \|m_{\widehat{w}}\|_{\mathrm{L}^2(V_p^\varepsilon)} \geq 1 + O_{M,\beta}\left(\varepsilon^{\frac{d}{2(d-1)}}\right).$$

Inserting this estimate in (31) concludes the proof. □

4 Analytic properties of Steklov eigenpairs

In this section, we obtain analytic properties of the Steklov eigenvalues $\sigma_k^{(\varepsilon)} := \sigma_k(\Omega^\varepsilon)$, and of Steklov eigenfunctions $u_k^{(\varepsilon)}$. We start by obtaining bounds on $\sigma_k^{(\varepsilon)}$ which are uniform in ε.

Lemma 4.1 *For all $k \in \mathbb{N}$ and $\beta \in C^\infty(M)$ positive, we have as $\varepsilon \to 0$*

$$\sigma_k^{(\varepsilon)} := \sigma_k(\Omega^\varepsilon) \leq \lambda_k(M, g, \beta) + o_{M,k,\beta}(1).$$

Proof It is clearly sufficient to prove this statement for ε small enough. It follows from the variational characterisation of Steklov eigenvalues that

$$\sigma_k^{(\varepsilon)} = \min_{E_{k+1}} \max_{u \in E_{k+1}} \frac{\int_{\Omega^\varepsilon} |\nabla u|^2 \, \mathrm{d}\mu_g}{\int_{\partial\Omega^\varepsilon} u^2 \, \mathrm{d}x},$$

where the minimum is taken over all $(k+1)$-dimensional subspaces $E_{k+1} \subset \mathrm{H}^1(\Omega^\varepsilon)$ whose trace remains $(k+1)$-dimensional in $\mathrm{L}^2(\partial\Omega^\varepsilon)$. Let $\varphi_0, \dots, \varphi_k$ be the first $k+1$ normalised eigenfunctions of the weighted Laplacian on M. They are pairwise $\mathrm{L}^2(M, \beta \, \mathrm{d}\mu_g)$ orthogonal, and since the $(d-1)$-dimensional Hausdorff measure restricted to $\partial\Omega^\varepsilon$ converges weak-$*$ to $\beta \, \mathrm{d}\mu_g$, for ε small enough they span a $k+1$ dimensional subspace of $\mathrm{L}^2(\partial\Omega^\varepsilon)$, and for $0 \leq j \leq k$,

$$\|\varphi_j\|^2_{\mathrm{L}^2(\partial\Omega^\varepsilon)} = \int_M \varphi_j^2(x)\beta(x) \, \mathrm{d}\mu_g + o_{M,k,\beta}(1).$$

Therefore, using $E = \operatorname{span}(\varphi_0, \dots, \varphi_k)$ as a trial subspace for σ_k^ε yields

$$\begin{aligned}\sigma_k^{(\varepsilon)} &\leq \max_{f \in E} \frac{\int_{\Omega^\varepsilon} |\nabla f|^2 \, \mathrm{d}\mu_g}{\int_{\partial\Omega^\varepsilon} f^2 \, \mathrm{d}\mu_g} \\ &\leq \lambda_k(M, g, \beta) + o_{M,k,\beta}(1),\end{aligned}$$

which is what we set out to prove. □

We turn to the boundedness of the sequence $\left\{u_k^{(\varepsilon)}\right\}$ in $\mathrm{L}^\infty(\Omega^\varepsilon)$.

Lemma 4.2 *There is $\varepsilon_0, C > 0$ depending only on k, β, M such that for all $0 < \varepsilon < \varepsilon_0$,*

$$\left\| u_k^{(\varepsilon)} \right\|_{\mathrm{L}^\infty(\Omega^\varepsilon)} \leq C$$

Proof It is shown in [7, Theorem 3.1] that for any Steklov eigenfunction u with eigenvalue σ on a domain Ω,

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \leq C \, \|u\|_{\mathrm{L}^2(\partial\Omega)} \,,$$

with C depending polynomially only on σ, $\mathrm{Vol}_g(\Omega)$ and the norm of the trace operator $\tau : \mathrm{BV}(\Omega) \to \mathrm{L}^1(\partial\Omega)$. Note that they only prove this statement for domains in $\mathbb{R}^d$, however a close inspection of their proof reveals that geometric dependence appears in only two places. The first one is on the norm of the extension operator from $\mathrm{BV}(\Omega) \to \mathrm{BV}(M)$, which depends only on the norm of τ (see [27][Theorem 5.4.1]), and therefore is already accounted for. The second one is on the norm of the Sobolev embedding $\mathrm{BV}(M) \to \mathrm{L}^{\frac{d}{d-1}}(M)$, whose norm depends only on the Gagliardo–Nirenberg–Sobolev inequality, which changes by at most a constant for M compact.

Lemma 3.2 gives a uniform bound for $\|\tau^\varepsilon\|$, Lemma 4.1 gives us a uniform bound for $\sigma_k^{(\varepsilon)}$ while $\mathrm{Vol}_g(\Omega)$ is obviously bounded by $\mathrm{Vol}_g(M)$ and $u_k^{(\varepsilon)}$ is normalised to $\|u_k^{(\varepsilon)}\|_{\mathrm{L}^2(\partial\Omega^\varepsilon)} = 1$. Thus $\|u_k^{(\varepsilon)}\|_{\mathrm{L}^\infty}$ is bounded, uniformly in ε. □

5 The homogenisation limit

In this section, we prove Theorem 2.1. While the general scheme of the proof follows the general idea in [35], we cannot use any periodic structure in order to define the auxiliary functions required to prove convergence. The major difference with general homogenisation methods will be the definition of those auxiliary functions on a cell by cell basis in such a way as to obtain the desired convergence.

Our first step is to show that there are converging subsequences. This is done in the following lemma. Recall that $u_k^{(\varepsilon)}$ are the Steklov eigenfunctions on Ω^ε and $U_k^{(\varepsilon)}$ their extension to M, harmonic in $\mathbf{T}^\varepsilon$.

Lemma 5.1 *There is a subsequence of $\left\{U_k^{(\varepsilon)}\right\}$, which we still label by ε, converging weakly in $\mathrm{H}^1(M)$.*

Proof It suffices to show that the sequence $\left\{U_k^{(\varepsilon)}\right\}$ is bounded in $\mathrm{H}^1(\Omega)$ as $\varepsilon \to 0$. By Lemma 3.4, we have that

$$\left\| U_k^{(\varepsilon)} \right\|_{\mathrm{H}^1(M)} \ll_{M,\beta} \left\| u_k^{(\varepsilon)} \right\|_{\mathrm{H}^1(\Omega^\varepsilon)}$$

On the other hand, we have that

$$\left\|\nabla u_k^{(\varepsilon)}\right\|_{\mathrm{L}^2(\Omega^\varepsilon)}^2 = \sigma_k^{(\varepsilon)} \leq \lambda_k(M, g, \beta) + o_{M,\beta,k}(1)\,,$$

where the last bound follows from Lemma 4.1. Furthermore, it follows from Lemma 4.2 that

$$\left\|u_k^{(\varepsilon)}\right\|_{\mathrm{L}^2(\Omega^\varepsilon)} \leq \mathrm{Vol}_g(\Omega^\varepsilon)^{\frac{1}{2}} \left\|u_k^{(\varepsilon)}\right\|_{\mathrm{L}^\infty(\Omega^\varepsilon)} = O_{M,\beta}(1)\,.$$

Combining all of this yields indeed that the sequence $\left\{U_k^{(\varepsilon)}\right\}$ is uniformly bounded in $\mathrm{H}^1(M)$, so that it has a subsequence weakly converging in $\mathrm{H}^1(M)$. $\square$

Proposition 5.2 *Let $k \in \mathbb{N}$. As $\varepsilon \to 0$, the pairs $(U_k^{(\varepsilon)}, \sigma_k^{(\varepsilon)})$ converge, up to a subsequence, to a pair (φ, λ), so that φ is an eigenfunction of the weighted Laplace problem on M with eigenvalue λ, the convergence of $U_k^{(\varepsilon)}$ being weak in* H^1.

Proof Denote by (φ, λ) the weak limit (up to a subsequence) of $(U_k^{(\varepsilon)}, \sigma_k^{(\varepsilon)})$, we now aim to show that they are weak solutions of the weighted Laplace eigenvalue problem on M, i.e. that they satisfy (16). For a real valued $v \in \mathrm{H}^1(M)$, we have, using the weak formulation of Problem (15) that

$$\int_M \nabla U_k^{(\varepsilon)} \cdot \nabla v \,\mathrm{d}\mu_g = \sigma_k^{(\varepsilon)} \int_{\partial\Omega^\varepsilon} U_k^{(\varepsilon)} v \,\mathrm{d}A_g + \int_{\mathbf{T}^\varepsilon} \nabla U_k^{(\varepsilon)} \cdot \nabla v \,\mathrm{d}\mu_g. \tag{32}$$

In order to be able to consider smooth test functions in this weak formulation, we need to ensure that the family of bounded linear functionals $\Phi^\varepsilon \in \mathrm{H}^1(M)^*$ given by

$$\Phi^\varepsilon(v) := \sigma_k^{(\varepsilon)} \int_{\partial\Omega^\varepsilon} U_k^{(\varepsilon)} v \,\mathrm{d}A_g.$$

is bounded uniformly in $\varepsilon < \varepsilon_0$. It indeed is, since we know from Lemma 4.1 that $\sigma_k^{(\varepsilon)}$ is bounded as $\varepsilon \to 0$, and we have

$$\begin{aligned}\left|\int_{\partial\Omega^\varepsilon} U_k^{(\varepsilon)} v \,\mathrm{d}A_g\right| &\leq \left\|\gamma^\varepsilon\right\|_{\mathrm{H}^1(\Omega^\varepsilon)\to\mathrm{L}^2(\partial\Omega^\varepsilon)} \left\|U_k^{(\varepsilon)}\right\|_{\mathrm{L}^2(\partial\Omega^\varepsilon)} \|v\|_{\mathrm{H}^1(M)}\\ &= \left\|\gamma^\varepsilon\right\|_{\mathrm{H}^1(\Omega^\varepsilon)\to\mathrm{L}^2(\partial\Omega^\varepsilon)} \|v\|_{\mathrm{H}^1(M)}\,,\end{aligned}$$

where γ^ε is the trace operator. Applying Lemma 3.3 with $\alpha = 1$, we see that for any $f \in \mathrm{H}^1(\Omega^\varepsilon)$,

$$\|f\|_{\mathrm{L}^2(\partial\Omega^\varepsilon)}^2 \leq \sum_{p\in\mathbf{S}^\varepsilon} \left\|\gamma_p^{\varepsilon,1}\right\|^2 \|f\|_{\mathrm{H}^1(Q_p^\varepsilon)}^2 \ll_M \|f\|_{\mathrm{H}^1(\Omega^\varepsilon)}^2$$

so that $\|\gamma^\varepsilon\|$ is bounded uniformly in ε for $0 < \varepsilon < \varepsilon_0$. By the Banach-Steinhaus theorem, the family $\{\Phi^\varepsilon\}$ is uniformly bounded. We may assume from now on that in the weak formulation of Problem (15), we consider only v in a dense subspace of $\mathrm{H}^1(M)$, in particular we assume $v \in \mathrm{C}^\infty(M)$.

By weak convergence of $U_k^{(\varepsilon)}$, the first term in (32) satisfies

$$\lim_{\varepsilon\to 0} \int_M \nabla U_k^{(\varepsilon)} \cdot \nabla v \,\mathrm{d}\mu_g = \int_M \nabla \varphi \cdot \nabla v \,\mathrm{d}\mu_g.$$

That the last term in (32) converges to 0 follows from the Cauchy–Schwarz inequality and the observation that since $v \in \mathrm{C}^\infty(M)$,

$$\int_{\mathbf{T}^\varepsilon} |\nabla v|^2 \,\mathrm{d}\mu_g \le \max_{x\in M} |\nabla v(x)|^2 \,\mathrm{Vol}_g(\mathbf{T}^\varepsilon) \xrightarrow{\varepsilon\to 0} 0.$$

We now study the boundary term in (32). For every $p \in \mathbf{S}^\varepsilon$, define a function $\Psi_p^\varepsilon : Q_p^\varepsilon \to \mathbb{R}$ satisfying the weak variational problem

$$\forall v \in \mathrm{H}^1(Q_p^\varepsilon), \qquad \int_{Q_p^\varepsilon} \nabla \Psi_p^\varepsilon \cdot \nabla v \,\mathrm{d}\mu_g = -c_{\varepsilon,p} \int_{Q_p^\varepsilon} v \,\mathrm{d}\mu_g + \int_{\partial B_{r_\varepsilon}(p)} v \,\mathrm{d}A_g.$$

It is an easy computation that under the metric rescaling $g \mapsto g_\alpha := \alpha^2 g$, Ψ_p^ε satisfies the variational problem

$$\forall v \in \mathrm{H}^1(Q_p^\varepsilon), \qquad \int_{Q_p^\varepsilon} \nabla_{g_\alpha} \Psi_p^\varepsilon \cdot \nabla_{g_\alpha} v \,\mathrm{d}\mu_{g_\alpha} = -\frac{c_{\varepsilon,p}}{\alpha^2} \int_{Q_p^\varepsilon} v \,\mathrm{d}\mu_{g_\alpha} + \frac{1}{\alpha} \int_{\partial B_{r_\varepsilon}(p)} v \,\mathrm{d}A_{g_\alpha}.$$

Choosing $v \equiv 1$, we see that a necessary and sufficient condition for the existence of a solution (see [75, Theorem 5.7.7]) is that

$$c_{\varepsilon,p} = \frac{\mathscr{H}^{d-1}(\partial B_{r_\varepsilon}(p))}{\mathrm{Vol}_g(Q_p^\varepsilon)} = \beta(p) + O_{M,\beta}\left(\varepsilon^{\frac{d}{d-1}}\right),$$

which holds uniformly in p. Uniqueness is guaranteed by requiring that $\int_{Q_p^\varepsilon} \Psi_p^\varepsilon \,\mathrm{d}A_g = 0$. The function Ψ_p^ε satisfies the differential equation

$$\begin{cases} \Delta \Psi_p^\varepsilon = c_{\varepsilon,p} & \text{in } Q_p^\varepsilon \\ \partial_\nu \Psi_p^\varepsilon = 1 & \text{on } \partial B_{r_\varepsilon}(p) \\ \partial_\nu \Psi_p^\varepsilon = 0 & \text{on } \partial V_p^\varepsilon. \end{cases}$$

We have that for all trial functions v,

$$\int_{\partial\Omega^\varepsilon} u_k^{(\varepsilon)} v \,\mathrm{d}A_g = \sum_{p\in\mathbf{S}^\varepsilon} \int_{Q_p^\varepsilon} \nabla\Psi_p^\varepsilon \cdot \nabla(u_k^{(\varepsilon)} v)\,\mathrm{d}\mu_g + \sum_{p\in\mathbf{S}^\varepsilon} c_{\varepsilon,p} \int_{Q_p^\varepsilon} u_k^{(\varepsilon)} v \,\mathrm{d}\mu_g. \tag{33}$$

The last term can be written as

$$\begin{aligned}
&\sum_{p\in\mathbf{S}^\varepsilon} c_{\varepsilon,p} \int_{Q_p^\varepsilon} u_k^{(\varepsilon)} v \,\mathrm{d}\mu_g \\
&\quad= \sum_{p\in\mathbf{S}^\varepsilon} \Bigg[\beta(p) \int_{V_p^\varepsilon} U_k^{(\varepsilon)} v \,\mathrm{d}\mu_g - \beta(p) \int_{B_{r_\varepsilon}(p)} U_k^{(\varepsilon)} v \,\mathrm{d}\mu_g \\
&\qquad + O_{M,\beta}\left(\varepsilon^{\frac{d}{d-1}}\right) \int_{Q_p^\varepsilon} u_k^{(\varepsilon)} v \,\mathrm{d}\mu_g \Bigg].
\end{aligned}$$

Now, by the generalised Hölder inequality

$$\begin{aligned}
&\left|\sum_{p\in\mathbf{S}^\varepsilon} \beta(p) \int_{B_{r_\varepsilon}(p)} U_k^{(\varepsilon)} v \,\mathrm{d}\mu_g\right| \\
&\quad\le \|\beta\|_{\mathrm{C}^0(M)} \|v\|_{\mathrm{C}^0(M)} \left\|U_k^{(\varepsilon)}\right\|_{\mathrm{L}^2(M)} \mathrm{Vol}_g(\mathbf{T}^\varepsilon)^{\frac{1}{2}} = O_{\beta,v,M}\left(\varepsilon^{\frac{d}{2(d-1)}}\right).
\end{aligned} \tag{34}$$

Furthermore,

$$\begin{aligned}
&\left|\sum_{p\in\mathbf{S}^\varepsilon} O_{M,\beta}\left(\varepsilon^{\frac{d}{d-1}}\right) \int_{Q_p^\varepsilon} u_k^{(\varepsilon)} v \,\mathrm{d}\mu_g\right| \ll_{M,\beta} \varepsilon^{\frac{d}{d-1}} \left\|u_k^{(\varepsilon)}\right\|_{\mathrm{L}^2(\Omega^\varepsilon)} \|v\|_{\mathrm{L}^2(M)} \\
&\quad= O_{M,\beta,v}\left(\varepsilon^{\frac{d}{d-1}}\right).
\end{aligned}$$

Finally,

$$\sum_{p\in\mathbf{S}^\varepsilon} \beta(p) \int_{V_p^\varepsilon} U_k^{(\varepsilon)} v \,\mathrm{d}\mu_g = \sum_{p\in\mathbf{S}^\varepsilon} \beta(p) \left(\int_{V_p^\varepsilon} \varphi v \,\mathrm{d}\mu_g + \int_{V_p^\varepsilon} (U_k^{(\varepsilon)} - \varphi) v \,\mathrm{d}\mu_g\right).$$

This time,

$$\sum_{p\in\mathbf{S}^\varepsilon} \beta(p) \int_{V_p^\varepsilon} \varphi v \,\mathrm{d}\mu_g \xrightarrow{\varepsilon\to 0} \int_M \varphi v \beta \,\mathrm{d}\mu_g,$$

this follows from the definition of a Riemann integral along with the mean value theorem. On the other hand

$$\left|\sum_{p\in\mathbf{S}^\varepsilon}\beta(p)\int_{V_p^\varepsilon}(U_k^{(\varepsilon)}-\varphi)v\,\mathrm{d}\mu_g\right|\le\|\beta\|_{\mathrm{C}^0(M)}\|v\|_{\mathrm{L}^2(M)}\left\|U_k^{(\varepsilon)}-\varphi\right\|_{\mathrm{L}^2(M)}\xrightarrow{\varepsilon\to 0}0$$

since $U_k^{(\varepsilon)}\to\varphi$ strongly in $\mathrm{L}^2(M)$. The upshot from those computations is that the last term in (33) satisfies

$$\sum_{p\in\mathbf{S}^\varepsilon}c_{\varepsilon,p}\int_{Q_p^\varepsilon}u_k^{(\varepsilon)}v\,\mathrm{d}\mu_g\to\int_M\varphi v\beta\,\mathrm{d}\mu_g.$$

We show that the other term in the righthand side of (33) converges to 0. Applying the generalised Hölder inequality, we obtain

$$\left|\int_{Q_p^\varepsilon}\nabla\Psi_p^\varepsilon\cdot\nabla(u_k^{(\varepsilon)}v)\,\mathrm{d}\mu_g\right|\le\|v\|_{\mathrm{C}^1(Q_p^\varepsilon)}\left\|\nabla\Psi_p^\varepsilon\right\|_{\mathrm{L}^2(Q_p^\varepsilon)}\left\|U_k^{(\varepsilon)}\right\|_{\mathrm{H}^1(Q_p^\varepsilon)}. \tag{35}$$

Since v is smooth, $\|v\|_{\mathrm{C}^1(M)}$ is bounded, and a fortiori the restriction to Q_p^ε is bounded as well. By applying the variational characterisation of Ψ_p^ε to itself, we obtain

$$\begin{aligned}\left\|\nabla\Psi_p^\varepsilon\right\|_{\mathrm{L}^2(Q_p^\varepsilon)}^2&=\alpha^{2-d}\int_{Q_p^\varepsilon}\left|\nabla_{g_\alpha}\Psi_p^\varepsilon\right|^2\,\mathrm{d}\mu_{g_\alpha}\\&=\alpha^{1-d}\int_{\partial B_{r_\varepsilon}(p)}\Psi_p^\varepsilon\,\mathrm{d}A_{g_\alpha}\\&\le\alpha^{1-d}\left\|\gamma_p^{\varepsilon,\alpha}\right\|\sqrt{\mathscr{H}_{g_\alpha}^{d-1}(\partial B_{r_\varepsilon}(p))}\left\|\Psi_p^\varepsilon\right\|_{\mathrm{H}^1(Q_p^\varepsilon,g_\alpha)}.\end{aligned}$$

Here, $\gamma_p^{\varepsilon,\alpha}$ is given in Lemma 3.3 and satisfies

$$\left\|\gamma_p^{\varepsilon,\alpha}\right\|^2\ll_M\max\left\{\frac{r_\varepsilon^{d-1}}{\alpha\varepsilon^d},\alpha\varepsilon\right\}.$$

Since Ψ_p^ε has average 0 on Q_p^ε, the Poincaré–Wirtinger inequality tells us that

$$\begin{aligned}\left\|\Psi_p^\varepsilon\right\|_{\mathrm{H}^1(Q_p^\varepsilon,g_\alpha)}&\le\left(1+\frac{1}{\mu_1(Q_p^\varepsilon,g_\alpha)}\right)^{\frac12}\left\|\nabla_{g_\alpha}\Psi_p^\varepsilon\right\|_{\mathrm{L}^2(Q_p^\varepsilon,g_\alpha)}\\&=\alpha^{\frac{d-2}{2}}\left(1+\frac{1}{\mu_1(Q_p^\varepsilon,g_\alpha)}\right)^{\frac12}\left\|\nabla_g\Psi_p^\varepsilon\right\|_{\mathrm{L}^2(Q_p^\varepsilon)}.\end{aligned}$$

By Lemma 3.5 and the usual scaling of Laplace eigenvalues, $\mu_1(Q_p^\varepsilon, g_\alpha) \gg_{M,\beta} (\alpha\varepsilon)^{-2}$ as long as $\alpha\varepsilon$ is small enough. We also have that that $\mathscr{H}_{g_\alpha}^{d-1}(\partial B_{r_\varepsilon}(p)) \asymp \alpha^{d-1}\varepsilon^d$. Combining those estimates yield, for $\alpha = \varepsilon^{-\frac{1}{2}}$ that

$$\left\| \nabla \Psi_p^\varepsilon \right\|_{\mathrm{L}^2(Q_p^\varepsilon)} = O_M\left(\varepsilon^{\frac{d+1}{2}}\right). \tag{36}$$

Putting this estimate and (35) into (33) yields from successive applications of the Hölder inequality

$$\begin{aligned} \sum_{p \in \mathbf{S}^\varepsilon} \int_{Q_p^\varepsilon} \nabla \Psi_p^\varepsilon \cdot \nabla (u_k^{(\varepsilon)} v) \, \mathrm{d}\mu_g &\leq \sum_{p \in \mathbf{S}^\varepsilon} \|v\|_{\mathrm{C}^1(M)} \left\| \nabla \Psi_p^\varepsilon \right\|_{\mathrm{L}^2(Q_p^\varepsilon)} \left\| U_k^{(\varepsilon)} \right\|_{\mathrm{H}^1(Q_p^\varepsilon)} \\ &\ll_{M,\beta,v} \varepsilon^{\frac{1}{2}} \left\| U_k^{(\varepsilon)} \right\|_{\mathrm{H}^1(M)}, \end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. Therefore, in view of (33) and (32), we have that if (φ, λ) are the limits of $(U_k^{(\varepsilon)}, \sigma_k^\varepsilon)$ they do indeed satisfy the weak variational problem

$$\forall v \in \mathrm{H}^1(M), \qquad \int_M \nabla \varphi \cdot \nabla v \, \mathrm{d}\mu_g = \lambda \int_M \varphi v \beta \, \mathrm{d}\mu_g,$$

in other word φ is a weak eigenfunction of the weighted Laplacian on M with eigenvalue λ. □

Now that we have established convergence to solutions of the limit problem, we need the following lemma to show that there is no mass lost in the interior, in other words the weak limits form an orthonormal family.

Lemma 5.3 *Let $j, k \in \mathbb{N}$, φ be the weak limit in H^1 of $U_k^{(\varepsilon)}$ and ψ be the weak limit of $U_j^{(\varepsilon)}$. Then,*

$$\delta_{j,k} = \lim_{\varepsilon \to 0} \int_{\partial\Omega^\varepsilon} U_k^{(\varepsilon)} U_j^{(\varepsilon)} \, \mathrm{d}A_g = \int_M \varphi \psi \beta \, \mathrm{d}\mu_g,$$

where $\delta_{j,k}$ is the Kronecker delta.

Proof By considering $v = u_j^{(\varepsilon)}$ in Eq. (33) for $u_k^{(\varepsilon)}$ we have that

$$\delta_{j,k} = \int_{\partial\Omega^\varepsilon} u_k^{(\varepsilon)} u_j^{(\varepsilon)} \, \mathrm{d}A_g = \sum_{p\in\mathbf{S}^\varepsilon} \int_{Q_p^\varepsilon} \nabla\Psi_p^\varepsilon \cdot \nabla(u_k^{(\varepsilon)} u_j^{(\varepsilon)}) \, \mathrm{d}\mu_g \\ + \underbrace{\sum_{p\in\mathbf{S}^\varepsilon} c_{\varepsilon,p} \int_{Q_p^\varepsilon} u_k^{(\varepsilon)} u_j^{(\varepsilon)} \, \mathrm{d}\mu_g,}_{\to \int_M \varphi\psi\beta \, \mathrm{d}\mu_g}$$

where the last term is shown to converge in essentially the same way as the last term in (33). The difference being that the C^0 norm of v in (34) is replaced by the L^∞ norm of $u_j^{(\varepsilon)}$, which is bounded by Lemma 4.2. Once again, we have to show that the other term converges to 0 as $\varepsilon \to 0$. From the generalised Hölder inequality, we see that

$$\int_{Q_p^\varepsilon} \nabla\Psi_p^\varepsilon \cdot \nabla(u_k^{(\varepsilon)} u_j^{(\varepsilon)}) \, \mathrm{d}\mu_g = \int_{Q_p^\varepsilon} \left(u_k^{(\varepsilon)} \nabla u_j^{(\varepsilon)} + u_j^{(\varepsilon)} \nabla u_k^{(\varepsilon)} \right) \cdot \nabla\Psi_p^\varepsilon \, \mathrm{d}\mu_g \\ \leq \left\| \nabla\Psi_p^\varepsilon \right\|_{\mathrm{L}^2(Q_p^\varepsilon)} \left(\left\| u_k^{(\varepsilon)} \right\|_{\mathrm{L}^\infty(Q_p^\varepsilon)} \left\| \nabla u_j^{(\varepsilon)} \right\|_{\mathrm{L}^2(Q_p^\varepsilon)} \right. \\ \left. + \left\| u_j^{(\varepsilon)} \right\|_{\mathrm{L}^\infty(Q_p^\varepsilon)} \left\| \nabla u_j^{(\varepsilon)} \right\|_{\mathrm{L}^2(Q_p^\varepsilon)} \right).$$

It follows from Lemmas 4.2 and 5.1 that the L^∞ norms are bounded, uniformly in ε. Furthermore, it follows from Eq. (36) that $\left\| \nabla\Psi_p^\varepsilon \right\|_{\mathrm{L}^2(Q_p^\varepsilon)} \ll \varepsilon^{\frac{d+1}{2}}$, so that, from the Cauchy–Schwarz inequality,

$$\sum_{p\in\mathbf{S}^\varepsilon} \int_{Q_p^\varepsilon} \nabla\Psi_p^\varepsilon \cdot \nabla(u_k^{(\varepsilon)} u_j^{(\varepsilon)}) \, \mathrm{d}\mu_g \\ \ll_{M,\beta} \sum_{p\in\mathbf{S}^\varepsilon} C\varepsilon^{\frac{d+1}{2}} \left(\left\| \nabla u_k^{(\varepsilon)} \right\|_{\mathrm{L}^2(Q_p^\varepsilon)} + \left\| \nabla u_j^{(\varepsilon)} \right\|_{\mathrm{L}^2(Q_p^\varepsilon)} \right) \\ \ll_{M,\beta} C\varepsilon^{\frac{1}{2}} \left(\left\| \nabla u_k^{(\varepsilon)} \right\|_{\mathrm{L}^2(\Omega^\varepsilon)} + \left\| \nabla u_j^{(\varepsilon)} \right\|_{\mathrm{L}^2(\Omega^\varepsilon)} \right),$$

which goes to 0 as $\varepsilon \to 0$, thereby finishing the proof. □

Proof of Theorem 2.1 We first prove that all the eigenvalues converge, proceeding by induction on the rank k. The base case $k = 0$ is trivial : indeed, the eigenvalue $\sigma_0^{(\varepsilon)}$ obviously converges to $\lambda_0 = 0$, and the normalised constant eigenfunctions of each problem satisfy by construction

$$
\begin{aligned}
U_0^{(\varepsilon)}(x) &= \mathscr{H}^{d-1}(\partial\Omega^\varepsilon)^{-\frac{1}{2}} \\
&\xrightarrow{\varepsilon\to 0} \left(\int_M \beta \,\mathrm{d}\mu_g\right)^{-\frac{1}{2}} \\
&= \varphi_0(x).
\end{aligned}
$$

Suppose now that for all $0 \le j \le k-1$, $U_j^{(\varepsilon)}$ converges to φ_j weakly in $\mathrm{H}^1(\Omega)$. We have already shown in Lemma 4.1 that for all k, $\sigma_k^{(\varepsilon)} \le \lambda_k(M, g, \beta) + o(1)$. We now show that the eigenvalues $\lambda_k(M, g, \beta)$ are bounded above by $\sigma_k^{(\varepsilon)} + o(1)$. Suppose that the limit eigenpair for $(\sigma_k^{(\varepsilon)}, u_k^{(\varepsilon)})$ is (λ_j, φ_j) for some $0 \le j \le k-1$. We have that

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \int_{\partial\Omega^\varepsilon} u_k^{(\varepsilon)} u_j^{(\varepsilon)} \,\mathrm{d}A_g \\
&= \lim_{\varepsilon\to 0} \int_{\partial\Omega^\varepsilon} u_k^{(\varepsilon)} \varphi_j \,\mathrm{d}A_g + \int_{\partial\Omega^\varepsilon} u_k^{(\varepsilon)} (u_j^{(\varepsilon)} - \varphi_j) \,\mathrm{d}A_g.
\end{aligned}
$$

The first term converges to 1 by the assumption that

$$
\int_M \varphi_j^2 \beta \,\mathrm{d}\mu_g = 1.
$$

For the second term, Cauchy-Schwarz inequality and the normalisation of $u_k^{(\varepsilon)}$ tells us that

$$
\int_{\partial\Omega^\varepsilon} u_k^{(\varepsilon)} (u_j^{(\varepsilon)} - \varphi_j) \,\mathrm{d}A_g \le \left\| u_j^{(\varepsilon)} - \varphi_j \right\|_{\mathrm{L}^2(\partial\Omega^\varepsilon)}.
$$

It follows from Lemma 5.3 that this limit converges to 0, resulting in a contradiction. This means that the eigenvalue λ_j to which $\sigma_k^{(\varepsilon)}$ converges has a rank higher than $k-1$. Combining this with the upper bound on λ_j implies that $\sigma_k^{(\varepsilon)}$ converges indeed to λ_k. Weak convergence of the eigenfunctions to the complete orthonormal set of weighted Laplace eigenfunctions $\{\varphi_k : k \in \mathbb{N}\}$ therefore follows from Lemma 5.3, up to taking a subsequence when the eigenvalues are multiple. $\square$

6 Isoperimetric inequalities

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3 Let $\delta > 0$ and g_δ be a metric on the surface M such that

$$\overline{\lambda}_k(M, g_\delta) \geq \Lambda_k(M) - \frac{\delta}{2}.$$

By taking $\beta = 1$ in Theorem 1.1, there is a family of domains $\Omega^\varepsilon \subset M$ such that for all $0 < \varepsilon < \varepsilon_0$, $\mathscr{H}^1(\partial\Omega^\varepsilon) \to \mathrm{Vol}_g(M)$ and such that $\sigma_k(\Omega^\varepsilon, g_\delta) \to \lambda_k(M, g_\delta)$ as $\varepsilon \to 0$. In other words,

$$\lim_{\varepsilon\to 0} \overline{\sigma}_k(\Omega^\varepsilon, g_\delta) = \overline{\lambda}_k(M, g_\delta),$$

so that there is $\varepsilon > 0$ such that $\overline{\sigma}_k(\Omega^\varepsilon, g_\delta) \geq \Lambda_k(M) - \delta$. Since δ is arbitrary, we have that

$$\Sigma_k(M) \geq \Lambda_k(M).$$

for all $k \in \mathbb{N}$ and surfaces M. $\square$

6.1 Lower bounds and exact values for Σ_k

For any closed surface M for which $\Lambda_k(M)$ is known, Theorem 1.3, along with Corollary 1.6 leads to an exact value for Σ_k when $k \in \{1, 2\}$, whereas it yields lower bounds when $k \geq 3$. We have already seen that $\Sigma_1(\mathbb{S}^2) = \Lambda_1(\mathbb{S}^2) = 8\pi$ in Corollary 1.4. More generally, it follows from Karpukhin–Nadirashvili–Penskoi–Polterovich [51] that

$$\Sigma_k(\mathbb{S}^2) \geq \Lambda_k(\mathbb{S}^2) = 8\pi k,$$

with equality when $k \leq 2$. The supremum is saturated by a sequence of Riemannian metrics degenerating to k kissing spheres of equal area. It follows from Nadirashvili [66] that

$$\Sigma_1(\mathbb{T}^2) = \Lambda_1(\mathbb{T}^2) = \frac{8\pi^2}{\sqrt{3}}.$$

The maximizer is the equilateral flat torus. For the orientable surface M of genus two, it follows from Nayatoni–Shoda [68] that

$$\Sigma_1(M) = \Lambda_1(M) = 16\pi.$$

Where the equality $\Lambda_1(M) = 16\pi$ was initially conjectured in the paper [41] by Jakobson–Levitin–Nadirashvili–Nigam–Polterovich. This time the maximizer is realized by a singular conformal metric on the Bolza surface. Some results are also known for non-orientable surfaces. For instance, it follows from the work of Li–Yau [61] that for the projective plane,

$$\Sigma_1(\mathbb{RP}^2) = \Lambda_1(\mathbb{RP}^2) = 12\pi,$$

where the maximal metric is the canonical Fubini–Study metric. It follows from Nadirashvili–Penskoi [67] that

$$\Sigma_2(\mathbb{RP}^2) = \Lambda_2(\mathbb{RP}^2) = 20\pi,$$

and from Karpukhin [48] that for all $k \geq 3$,

$$\Sigma_k(\mathbb{RP}^2) \geq \Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1).$$

This time the maximal metric is achieved by a sequence of surfaces degenerating to a union of a projective plane and $k-1$ spheres with their canonical metrics, the ratio of the area of the projective planes to the area of the union of the spheres being 3 : 2.

Finally, it follows from El Soufi–Giacomini–Jazar [26] and Cianci–Karpukhin–Medvedev [13] that for the Klein bottle $\mathbb{KL}$

$$\Sigma_1(\mathbb{KL}) = \Lambda_1(\mathbb{KL}) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right),$$

where E is the complete elliptic integral of the second type. The supremum for is realized by a bipolar Lawson surface corresponding to the $\tau_{3,1}$-torus. The equality for Λ_1 was first conjectured by Jakobson–Nadirashvili–Polterovich [42].

There are also situations where lower bounds for Λ_k can be transferred to Σ_k. For instance, restricting to flat metrics on $\mathbb{T}^2$, it follows from Kao–Lai–Osting [43] and Lagacé [58] that

$$\Lambda_k(\mathbb{T}^2)_{\text{flat}} := \sup_{g\in\mathscr{G}(M) g \text{ flat}} \overline{\lambda}_k(M,g) \geq \frac{4\pi^2 \left\lceil\frac{k}{2}\right\rceil^2}{\sqrt{\left\lceil\frac{k}{2}\right\rceil^2 - \frac{1}{4}}} \tag{37}$$

and that Λ_k is realised by a family of flat tori degenerating to a circle as $k \to \infty$. It follows from Theorem 1.3 that

$$\Sigma_k(\mathbb{T}^2)_{\text{flat}} := \sup_{\substack{g \text{ flat}\\ \Omega\subset M}} \overline{\sigma}_k(\Omega) \geq \frac{4\pi^2 \left\lceil\frac{k}{2}\right\rceil^2}{\sqrt{\left\lceil\frac{k}{2}\right\rceil^2 - \frac{1}{4}}}.$$

Note that it is also conjectured in [43] that (37) is an equality. We record one last general result following from the same strategy.

Corollary 6.1 *For every $k \geq 1$,*

$$\Sigma_k(M) \geq \Lambda_1(M) + 8\pi(k-1)$$

Proof This follows from the work of Colbois–El Soufi [15], see also [43,51] for further discussion, where it is shown that one can glue in appropriate ratios maximisers for the first eigenvalue in a topological class with spheres to obtain bounds on the kth normalised eigenvalue of the Laplacian. □

7 First Steklov eigenvalue of free boundary minimal surfaces

In view of the proof of Theorem 1.13, we recall a few definitions. For the definition of fundamental domains, we follow [76, Definition I.1.5].

Definition 7.1 A *fundamental domain* for the action of a group G on $\Omega \subset \mathbb{R}^3$ is a closed connected subset $W \subset \Omega$ such that

$$\Omega = \bigcup_{\psi \in G} \psi(W) \tag{38}$$

and for every $\psi \neq \eta \in G$,

$$\mathrm{int}(\psi(W)) \cap \mathrm{int}(\eta(W)) = \varnothing. \tag{39}$$

If W is a fundamental domain for the action of G on $\mathbb{R}^3$ and Ω is G-invariant, then $W \cap \Omega$ is a fundamental domain for the action of G on Ω.

Recall that when we say that Ω has tetrahedral, octahedral or icosahedral symmetry, we mean that it is invariant under the action of the full symmetry group G of the related platonic solid. We note that the symmetries of a cube are also octahedral, and that those of a dodecahedron are also icosahedral. Following [24, Chapter 5.4], we know that those groups are generated by reflections across three planes Π_1, Π_2, Π_3 through the origin, and that a fundamental domain for the action of G on $\mathbb{R}^3$ is given by a closed three-sided wedge W bounded by Π_1, Π_2, Π_3. Figure 1 on the right shows an example of such a fundamental domain restricted to the ball $\mathbb{B}^3$, which is G-invariant.

We mention a very explicit construction for the reflection planes. Given a platonic solid centred at the origin, let v_1 and v_2 be two of its adjacent vertices, let $c_1 = \frac{1}{2}(v_1 + v_2)$ and let c_2 be the centre of a face adjacent to the edge between v_1 and v_2. Then, we can choose Π_1 as the plane through v_1, v_2 and the origin, Π_2 as the plane through v_1, c_1 and the origin and Π_3 as the

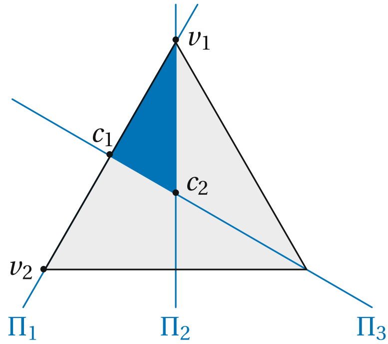

Fig. 5 Reflection symmetries

Table 2 Smallest angle between the reflection planes generating the full symmetry groups of the platonic solids and order of those groups

Symmetry	Order	$\angle(\Pi_1, \Pi_2)$	$\angle(\Pi_2, \Pi_3)$	$\angle(\Pi_1, \Pi_3)$
Tetrahedral	24	$\pi/3$	$\pi/3$	$\pi/2$
Octahedral	48	$\pi/3$	$\pi/4$	$\pi/2$
Icosahedral	120	$\pi/3$	$\pi/5$	$\pi/2$

plane through c_1, c_2 and the origin (see Fig. 5). In particular, Π_1 and Π_3 are orthogonal, and we give in Table 2 the angles between the reflection planes as well as the order of the groups. Given Ω invariant under the action of G, we refer to the closed subset $W \subset \Omega$ bounded by Π_1, Π_2, Π_3 as the standard fundamental domain for the action of G on Ω.

We prove that connected G-invariant sets in Ω have connected intersection with the standard fundamental domain. Note that this is a special property of the groups at hand. It would be false, for instance, for fundamental domains of rotation groups.

Lemma 7.2 *Let G be the symmetry group of a platonic solid and let W be the standard fundamental domain for the action of G on $\mathbb{B}^3$. Let $E \subset W$ be such that $G(E)$ is connected. Then, E is connected.*

In order to prove this Lemma we require the following two intermediate results. The covering property (38) means that fundamental domains intersect every orbit of G, while the non-intersection condition (39) means that the orbits intersect the interior of a fundamental domain at most once. The first intermediate result shows that for the symmetry groups of platonic solids this non-intersection property is also satisfied on the boundary of the fundamental domain.

Lemma 7.3 *Let G be the symmetry group of a platonic solid and W a standard fundamental domain for the action of G on $\mathbb{B}^3$. Then, every orbit of G intersects W exactly once.*

Proof It is sufficient to verify that for every $x \in W$,

$$\#\,\mathrm{Stab}(x) = \#\{\psi \in G : x \in \psi(W)\}\,. \tag{40}$$

For $x = 0$, the stabiliser is the whole group and $x \in \psi(W)$ for every $\psi \in W$. If $x \notin \Pi_j$ for any $j \in \{1, 2, 3\}$, then $x \notin \psi(W)$ for all non-trivial $\psi \in G$, and x has trivial stabiliser.

If x belongs to exactly one of the boundary plane Π_j, then it is stabilised only by reflection along that boundary plane, and belongs to W and to the reflection along Π_j of W.

Finally, if x belongs to the intersection of two boundary planes Π_j and Π_k, then its stabiliser is the group generated by reflections along those two planes. This is a group of order $2q$ where the angle between Π_j and Π_k is π/q, c.f. Table 2. Similarly, since the angles can sum to at most 2π,

$$\#\{\psi \in G : x \in \psi(W)\} \le 2q.$$

Since the right-hand side in (40) is always an upper bound for the left-hand side, this implies equality. The claim holds, having exhausted all possibilities. □

Lemma 7.4 *Let G be the symmetry group of a platonic solid, and let W be the standard fundamental domain for the action of G on $\mathbb{B}^3$. Then, the composition $W \hookrightarrow \mathbb{B}^3 \to \mathbb{B}^3/G$ is a homeomorphism.*

Proof By Lemma 7.3, W is homeomorphic to its own orbit space W/G since the quotient is trivial. It follows from [76, Proposition I.1.6] that since W is a fundamental domain, the natural embedding $W/G \hookrightarrow \mathbb{B}^3/G$ is a homeomorphism. The claim follows readily. □

Proof of Lemma 7.2 Since $G(E)$ is connected its image in the quotient $\mathbb{B}^3/G$ is also connected. By Lemma 7.4, the map $W \hookrightarrow \mathbb{B}^3 \to \mathbb{B}^3/G$ is a homeomorphism, so that E is also connected. □

The following Lemma states that the surfaces satisfying the hypotheses of Theorem 1.13 have fundamental domains with the same structure as those visualised in Figs. 1, 2 and 3.

Lemma 7.5 *Let $\Omega \subset \mathbb{B}^3$ be an embedded free boundary minimal surface of genus 0 which has tetrahedral symmetry and $b = 4$ boundary components or octahedral symmetry and $b \in \{6, 8\}$ boundary components or icosahedral* *symmetry and $b \in \{12, 20, 32\}$ boundary components. Then Ω has a simply connected fundamental domain D with piecewise smooth boundary ∂D. If $b = 32$ then ∂D consists of five edges and five right-angled corners. In the other cases, ∂D has four edges and four corners, three of which are right-angled.*

Proof Let G be the symmetry group of the platonic solid under which Ω is invariant and Π_1, Π_2, Π_3 be the three reflection planes generating G, see Table 2. Let W be the standard fundamental domain for the action of G in $\mathbb{B}^3$ and put $D := W \cap \Omega$. Since Ω is an embedded free boundary minimal surface in the unit ball, it is connected [29, Lemma 2.4]. By Lemma 7.2 this implies that D is connected. Moreover, D meets ∂W orthogonally. Along the planar faces of W, this follows from the assumption that Ω is embedded and invariant under reflection and along $\partial\mathbb{B}^3 \cap \partial W$ it is a direct consequence from the free boundary condition. Hence, the curve ∂D is piecewise smooth with corners where it meets the edges of W. Moreover, the exterior angles along ∂D are given by the angles between the faces of ∂W. Let $\alpha_1\pi$ be the larger angle between Π_1 and Π_2 and let $\alpha_2\pi$ be the larger angle between Π_2 and Π_3. All the other faces of ∂W are pairwise orthogonal. Let j, ℓ_1, ℓ_2 be the numbers of exterior angles along ∂D with values $\frac{\pi}{2}$, $\alpha_1\pi$, $\alpha_2\pi$ respectively. By the argument above, these are all possible cases. We first observe that

$$\int_\Omega K = |G| \int_D K, \qquad \int_{\partial\Omega} \kappa = |G| \int_{\partial D} \kappa,$$

where we denote the Gauß curvature of a surface (here Ω or D) by K, the geodesic curvature of its boundary by κ and the number of elements in the symmetry group G by $|G|$. By the Gauß–Bonnet theorem, we have the following formula for the Euler characteristic $\chi(\Omega)$ of Ω.

$$\begin{aligned} 2\pi\chi(\Omega) &= \int_\Omega K + \int_{\partial\Omega} \kappa = |G| \left(\int_D K + \int_{\partial D} \kappa \right) \\ &= |G| \big(2\pi\chi(D) - j\tfrac{\pi}{2} - \ell_1\alpha_1\pi - \ell_2\alpha_2\pi \big). \end{aligned} \tag{41}$$

Since Ω has genus 0 and b boundary components, $\chi(\Omega) = 2 - b$ and Eq. (41) yields

$$2\,|G|\,\chi(D) = |G|\,\tfrac{j}{2} + |G|\,\ell_1\alpha_1 + |G|\,\ell_2\alpha_2 + 2(2-b). \tag{42}$$

In the case of tetrahedral symmetry we have $|G| = 24$ and $b = 4$ as well as $\alpha_1 = \alpha_2 = \frac{2}{3}$. Simplifying Eq. (42), we obtain

$$12\chi(D) = 3j + 4(\ell_1 + \ell_2) - 1. \tag{43}$$

Any connected surface D with boundary has Euler characteristic $\chi(D) \leq 1$. Since j, ℓ_1, ℓ_2 must be nonnegative integers, the right hand side of Eq. (43) is bounded from below by -1 and does not vanish which implies $\chi(D) = 1$. Moreover, Eq. (43) implies $j, \ell_1, \ell_2 \leq 4$. By testing all combinations we obtain $j = 3$ and $\ell_1 + \ell_2 = 1$ as the only possibility. In particular, D has $j + \ell_1 + \ell_2 = 4$ corners and the topology of a disk as claimed.

In the octahedral case, we have $|G| = 48$ and $b \in \{6, 8\}$ as well as $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = \frac{3}{4}$. In this case, Eq. (42) implies

$$24\chi(D) = 6j + 8\ell_1 + 9\ell_2 - \begin{cases} 2 & \text{if } b = 6, \\ 3 & \text{if } b = 8. \end{cases}$$

As before, we conclude $\chi(D) = 1$ and obtain $(j, \ell_1, \ell_2) = (3, 1, 0)$ if $b = 6$ or $(j, \ell_1, \ell_2) = (3, 0, 1)$ if $b = 8$.

With icosahedral symmetry, we have $|G| = 120$ and $b \in \{12, 20, 32\}$ as well as $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = \frac{4}{5}$. Then, Eq. (42) implies

$$60\chi(D) = 15j + 20\ell_1 + 36\ell_2 - \begin{cases} 5 & \text{if } b = 12, \\ 9 & \text{if } b = 20, \\ 15 & \text{if } b = 32. \end{cases} \tag{44}$$

If $b \in \{12, 20\}$ we obtain $\chi(D) = 1$ and $(j, \ell_1, \ell_2) = (3, 1, 0)$ respectively $(j, \ell_1, \ell_2) = (3, 0, 1)$ as above. In the case $b = 32$, Eq. (44) has the solution $(j, \ell_1, \ell_2) = (1, 0, 0)$ with $\chi(D) = 0$ which we need to exclude. Since the group order $|G| = 120$ exceeds the number $b = 32$ of boundary components, there are no closed curves in $\partial D \cap \partial \mathbb{B}^3$. Consequently, and since Ω is embedded with boundary, ∂D must have at least two corners on $\partial \mathbb{B}^3$ which implies $j \geq 2$. In this case, the right hand side of (44) is positive which implies $\chi(D) = 1$. The equation simplifies to

$$45 = 15(j - 2) + 20\ell_1 + 36\ell_2$$

and the only solution with integers $(j - 2), \ell_1, \ell_2 \geq 0$ is $(j, \ell_1, \ell_2) = (5, 0, 0)$. □

We are now ready to prove our main result regarding free boundary minimal surfaces.

Proof of Theorem 1.13 A result by McGrath [65, Theorem 4.2] states $\sigma_1(\Omega) = 1$ provided that $\Omega \subset \mathbb{B}^3$ is an embedded free boundary minimal surface which is invariant under a finite group G of reflections satisfying the following two conditions.

(1) The fundamental domain for the action of G on $\mathbb{B}^3$ is a four-sided wedge W bounded by three planes and $\partial\mathbb{B}^3$.
(2) The fundamental domain $D = W \cap \Omega$ for Ω is simply connected with boundary ∂D which has at most five edges and intersects $\partial\Omega$ in a single connected curve.

Let D be the fundamental domain for Ω as given by Lemma 7.5. Interpreting D as free boundary minimal disk inside W, a result by Smyth [72, Lemma 1] states that

$$\int_{\partial D} \nu \, \mathrm{d}s = 0$$

where ν is the outward normal vector field of ∂W. In particular, for any $x \in \mathbb{R}^3$,

$$\int_{\partial D} \nu \cdot x \, \mathrm{d}s = 0.$$

For $j \in \{1, 2, 3\}$, let n_j be the outward normal to Π_j, and let $x_j = n_i \times n_k$ where $i \neq j \neq k$ and $i < k$. Then, $x_j \cdot \nu = 0$ on Π_k for $k \neq j$, and $x_j \cdot \nu$ are nonzero and have opposite sign on Π_j and $\partial\mathbb{B}_3$. Consequently, for all j we have that $\partial D \cap \Pi_j \neq \varnothing$ if and only if $\partial D \cap \partial\mathbb{B}_3 \neq \varnothing$, which implies that D meets all four faces of W at least once.

Hence, in the cases where ∂D has exactly four edges, $\partial D \cap \partial\Omega$ must be connected and [65, Theorem 4.2] applies.

In the case $b = 32$ where ∂D has five edges and right angles, $\partial D \cap \partial\Omega$ could be disconnected which would violate condition (2). We recall from the proof of Lemma 7.5 that the plane Π_2 intersects Π_1 and Π_3 at angles different from $\frac{\pi}{2}$.

Since ∂D has only right angles, it must avoid these two intersections while still meeting the adjacent faces of W (see Fig. 3 lower image). Hence, $\gamma = \partial D \cap \partial\Omega$ has indeed two connected components γ_1 and γ_2. Let e_i be the edge of ∂D on Π_i for $i \in \{1, 2, 3\}$ such that in consecutive order

$$\partial D = e_1 \cup \gamma_1 \cup e_2 \cup \gamma_2 \cup e_3.$$

In the following, we adapt McGrath's [65] approach to prove $\sigma_1(\Omega) = 1$ for the case at hand. Towards a contradiction, suppose that $\sigma_1(\Omega) < 1$ and let u be a first eigenfunction for the Steklov eigenvalue problem satisfying

$$\int_{\partial\Omega} u \, \mathrm{d}s = 0. \tag{45}$$

Let $\mathscr{N} = \{x \in \Omega \mid u(x) = 0\}$ denote the *nodal set* of u. As remarked in [65], $\mathscr{N}$ consists of finitely many arcs which intersect in a finite set of points.

By definition a *nodal domain* of u is a connected component of $\Omega \setminus \mathscr{N}$. By Courant's nodal domain theorem, u has exactly two nodal domains $\mathscr{N}^{\pm} := \{x \in \Omega \mid \pm u(x) > 0\}$, being a first non-trivial eigenfunction, see [65, Lemma 2.2] for a proof of this theorem for Steklov eigenfunctions.

We recall that the symmetry group G of Ω is generated by reflections. Let $R \in G$ be any such reflection. According to [65, Lemma 3.2] we have $u = \frac{1}{2}(u + u \circ R)$ since Ω is R-invariant with $\sigma_1(\Omega) < 1$.

This implies that $u = u \circ \psi$ for any $\psi \in G$ which means that the two nodal domains $\mathscr{N}^{\pm}$ are invariant under the group action, i.e. they must intersect every fundamental domain of Ω and both must still be connected. Below we show that this contradicts the fact that the order of an element of the icosahedral group is at most 10.

Assumption (45) implies that u restricted to $\gamma = \partial D \cap \partial\Omega$ changes sign because being a Steklov eigenfunction, u does not vanish on all of $\partial\Omega$. Consequently, an arc η in $\mathscr{N}$ either meets one connected component of γ or separates them by connecting two edges e_i and e_j. In the latter case, at most ten alternating reflections on Π_i and Π_j close up the curve η and the enclosed region of Ω intersects at most ten fundamental domains. However, Ω has $|G| = 120$ pairwise disjoint fundamental domains in total. This contradicts the fact that there are only two nodal domains which are invariant under the group action. If the nodal arc η meets γ_1, then it is closed by six alternating reflections along Π_1 and Π_2. Similarly, if it meets γ_2 then it is closed by ten alternating reflections along Π_2 and Π_3. In either cases, u restricted to the corresponding connected component Υ of $\partial\Omega$ changes sign after each reflection, hence at least six times. Let ρ be a path in $\mathscr{N} \cup (\partial\Omega \setminus \Upsilon)$ connecting two zeroes of u on Υ. Since Ω has genus 0, ρ disconnects Ω. Since u changes sign at least six times (four would be enough), at least one of the nodal domains $\mathscr{N}^{\pm}$ lies on both sides of ρ and is therefore disconnected which again contradicts Courant's nodal domain theorem. This completes the proof. $\square$

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Appendix A: On the monotonicity of Steklov eigenvalues

In this appendix, we elaborate on Remark 1.5, following communication with Fraser and Schoen [34]. Given a compact orientable surface Ω of genus γ with b boundary components, we recall the notation from (4) and set

$$\sigma_1^*(\gamma, b) := \sup_{g \in \mathscr{G}(\Omega)} \overline{\sigma}_1(\Omega, g)$$

as in [32]. The limit result [32, Theorem 8.2] states that $\sigma_1^*(0, b) \to 4\pi$ as $b \to \infty$ and that the associated free boundary minimal surfaces Ω_b converge to a double disk. In the proof, it is shown that the area of Ω_b cannot concentrate near its boundary. While this is true, a gap appears where this non-concentration phenomenon is used to deduce that all Ω_b must intersect a fixed smaller ball. In [32] this is used to show convergence of Ω_b to a non-trivial limit. There is another possibility: that the sequence of maximisers Ω_b converge to the boundary $\mathbb{S}^2$. It is this latter behaviour that is suggested by Theorem 1.1 and Corollary 1.4, which leads us to state the following conjecture.

Conjecture *There is a sequence $\{\Omega_b : b \in \mathbb{N}\} \subset \mathbb{B}^3$ of free boundary minimal surfaces of genus 0 with b boundary components which enjoys all the following properties.*

(1) *For every b, Ω_b maximises $\overline{\sigma}_1$ among surfaces of genus 0 with b boundary components.*
(2) *As $b \to \infty$, the measure on $\mathbb{R}^3$ obtained by restriction of the Hausdorff measure $\mathscr{H}^1$ to $\partial\Omega_b$ converges weak-$*$ to twice the measure obtained by restriction of $\mathscr{H}^2$ on $\mathbb{S}^2$.*
(3) *As $b \to \infty$, Ω_b converges in the sense of varifolds to $\mathbb{S}^2$.*

Furthermore, $\mathbb{S}^2$ is the unique limit point for $\{\Omega_b\}$ under the condition that they maximise $\overline{\sigma}_1$.

It follows from a recent preprint of Matthiesen and Petrides [64] that for any genus γ and any number b of boundary components there exists a maximal metric for the Steklov problem on surfaces of genus γ with b boundary components. In other words, sequences satisfying only (1) always exist and this is the case for any genus. It would therefore also make sense to extend the conjecture to free boundary minimal surfaces converging to minimal immersions of closed surfaces of genus γ into some sphere $\mathbb{S}^{m-1}$ given by maximal metrics for the Laplacian. However, in such a case we make no claim as to the uniqueness of the limit, the dimension of the sphere in which it is embedded, nor whether convergence is along a subsequence in b or along the whole sequence.

We remark that our conjecture is not in contradiction with the existence of free boundary minimal surfaces converging to the double disk as the number of boundary components goes to infinity, it simply means that they are not global maximisers for $\overline{\sigma}_1$. We also remark that a part of the gap in the proof of [32, Theorem 8.2] appears also in the monotonicity result [32, Proposition 4.3], stating that $\sigma_1^*(\gamma, b) < \sigma_1^*(\gamma, b+1)$. This was also mentioned to us in [34], along with a statement that the result still holds and that a corrigendum is in preparation.

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