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ABSTRACT

We prove the existence of ballistic transport for a Schrödinger operator with a generic quasi-periodic potential in any dimension $d > 1$.

I. INTRODUCTION

A. Prior results on ballistic transport

It is well known that the spectral and dynamical properties of Schrödinger operators $H = -\Delta + V$ acting in $\mathcal{H} = L^2(\mathbb{R}^d)$ are related. A general correspondence of this kind is given by the RAGE theorem, e.g., Ref. 28. Stated briefly, it says that solutions $\Psi(\cdot, t) = e^{-iHt}\Psi_0$ of the time-dependent Schrödinger equation are "bound states" if the spectral measure μ_{Ψ_0} of the initial state Ψ_0 is pure point, while $\Psi(\cdot, t)$ is a "scattering state" if μ_{Ψ_0} is (absolutely) continuous. However, knowing the spectral type is not sufficient to quantify transport properties more precisely, for example, in terms of diffusion exponents β. These exponents, if they exist, characterize how time-averaged m-moments,

$$\langle\langle X^m_{\Psi_0}\rangle\rangle_T := \frac{2}{T}\int_0^\infty \exp\left(-\frac{2t}{T}\right)\|X^{m/2}\Psi(\cdot,t)\|^2_{\mathcal{H}}\,dt, \qquad m > 0, \tag{1.1}$$

of the position operator X grow as a power $T^{m\beta}$ of time T, where $(Xu)(x) := |x|u(x)$ ($x \in \mathbb{R}^d$ and m is a positive real number). The special cases $\beta = 1$, $\beta = 1/2$, and $\beta = 0$ are interpreted as ballistic transport, diffusive transport, and dynamical localization, respectively.

In general, due to the possibility of fast traveling small tails, β may depend on m. In this paper, we will restrict our attention to the most frequently considered case of the second moment $m = 2$. The ballistic upper bound

$$\|X\Psi(\cdot,t)\|^2_{\mathcal{H}} \le C_1(\Psi_0)T^2 + C_2(\Psi_0), \tag{1.2}$$

and thus also its averaged version $\langle\langle X^2_{\Psi_0}\rangle\rangle_T \le C_1(\Psi_0)T^2 + C_2(\Psi_0)$, is known to hold for general potentials V with relative Δ-bound less than 1 (in particular, all bounded potentials) and initial states

$$\Psi_0 \in \mathcal{S}_1 := \{f \in L^2(\mathbb{R}^d) : |x|f \in L^2(\mathbb{R}^d),\ |\nabla f| \in L^2(\mathbb{R}^d)\} \tag{1.3}$$

(see Ref. 27). We will work with the Abel mean used in (1.1), but note that the existence of a ballistic upper bound can be used to show that Abel means and Cesaro means $T^{-1}\int_0^T \cdots\, dt$ lead to the same diffusion exponents (see, for example, Theorem 2.20 of Ref. 10).

In the late 1980s and 1990s, methods were developed that led to more concrete bounds on diffusion exponents by also taking fractal dimensions of the associated spectral measures into account and showing that this gives lower transport bounds. In particular, again for the special case of the second moment, the Guarneri–Combes theorem[5,15,16,23] says that

$$\langle\langle X^2_{\Psi_0}\rangle\rangle_T \geq C_{\Psi_0} T^{2\alpha/d} \tag{1.4}$$

for initial states Ψ_0 with a uniformly α-Hölder continuous spectral measure (and satisfying an additional energy bound in the continuum case[5]). In dimension $d = 1$, this says that states with an absolutely continuous spectral measure ($\alpha = 1$) also will have ballistic transport [as by (1.2), the transport cannot be faster than ballistic]. In particular, this means that in cases where the spectra of one-dimensional Schrödinger operators with limit or quasi-periodic potentials were found to have an a.c. component, e.g., Refs. 2, 4, 11, 12, and 24–26, one also gets ballistic transport.

The bound (1.4) does not suffice to conclude ballistic transport from the existence of the a.c. spectrum in dimension $d \geq 2$. In fact, examples of Schrödinger operators with absolutely continuous spectra but slower than ballistic transport have been found: A two-dimensional "jelly-roll" example with an a.c. spectrum and diffusive transport is discussed in Ref. 22, while Ref. 3 provides examples of separable potentials in dimension $d \geq 3$ with an a.c. spectrum and sub-diffusive transport.

In general, the growth properties of generalized eigenfunctions should be used in addition to spectral information for a more complete characterization of the dynamics. General relations between eigenfunction growth and spectral type as well as dynamics were found in Ref. 22. A series of works studied one-dimensional models with $\alpha < 1$ and related the dynamics to transfer matrix bounds, e.g., Refs. 6–9, 14, 18, and 29. In particular, these methods can establish lower transport bounds in models with sub-ballistic transport, such as the Fibonacci Hamiltonian and the random dimer model.

Until recently, much less has been done for $d \geq 2$. Ballistic lower bounds and thus the existence of waves propagating at non-zero velocity were known only for $V = 0$, where this is classical, e.g., Ref. 28, and for periodic potentials.[1] Scattering theoretic methods show that this extends to potentials of sufficiently rapid decay or sufficiently rapidly decaying perturbations of periodic potentials. In Ref. 19, two results on ballistic lower bounds in dimension $d = 2$ were obtained, one for limit-periodic and one for quasi-periodic potentials. Our goal here is to generalize these results to any dimension $d \geq 2$ and a generic quasi-periodic potential.

We have already proved, in Ref. 20, that generic quasi-periodic potentials have absolutely continuous spectra for high energies. Here, we will combine the results obtained in Ref. 20 (in particular, the properties of the generalized eigenfunctions constructed there) with the methods of Ref. 19 to prove the existence of ballistic transport.

B. The main result

We study the initial value problem

$$i\frac{\partial\Psi}{\partial t} = H\Psi, \quad \Psi(\mathbf{x}, 0) = \Psi_0(\mathbf{x}), \tag{1.5}$$

for the multidimensional Schrödinger operator H acting on $L^2(\mathbb{R}^d)$, $d \geq 2$, defined in the following way. Let $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_l \in \mathbb{R}^d$, $l > d$, be a collection of vectors that we will call *the basic frequencies*. It will be convenient to form a "vector" out of the basic frequencies: $\vec{\boldsymbol{\omega}} := (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_l)$. We consider the operator

$$H := -\Delta + V, \tag{1.6}$$

where

$$V := \sum_{|\boldsymbol{n}|\leq Q} V_{\boldsymbol{n}} \boldsymbol{e}_{\boldsymbol{n}\vec{\boldsymbol{\omega}}}. \tag{1.7}$$

The last sum is finite and taken over all vectors $\boldsymbol{n} = (n_1, \ldots, n_l) \in \mathbb{Z}^l$ with

$$|\boldsymbol{n}| := \max_{j=1,\ldots,l} |n_j| < Q, \quad Q \in \mathbb{N}. \tag{1.8}$$

We have also denoted

$$\boldsymbol{e}_{\boldsymbol{\theta}}(\mathbf{x}) := e^{i\langle\boldsymbol{\theta},\mathbf{x}\rangle}, \quad \boldsymbol{\theta}, \mathbf{x} \in \mathbb{R}^d, \tag{1.9}$$

and

$$\boldsymbol{n}\vec{\boldsymbol{\omega}} := \sum_{j=1}^{l} n_j \boldsymbol{\omega}_j \in \mathbb{R}^d, \tag{1.10}$$

with the vectors $\boldsymbol{n}\vec{\boldsymbol{\omega}}$ being called *the frequencies*. For convenience and without loss of generality, we assume that the basic frequencies $\boldsymbol{\omega}_j \in [-1/2, 1/2]^d$ and thus $\vec{\boldsymbol{\omega}} \in [-1/2, 1/2]^{dl}$ (so that the Lebesgue measure of this set is one; obviously, we can always achieve this by rescaling). We assume that the frequencies $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_l$ are linearly independent over rationals. We also assume that $V_{-\boldsymbol{n}} = \bar{V}_{\boldsymbol{n}}$. Clearly, V is real valued.

Consider the evolution equation (1.5) for operators H described above. Clearly, the ballistic upper bound of Ref. 27 can be applied, and we have (1.2) for initial conditions Ψ_0 satisfying (1.3). We prove that for these operators, there are corresponding *ballistic lower bounds* for a large class of initial conditions. To formulate our main result, we use the infinite-dimensional spectral projection E_∞ for H whose construction is described in Sec. II.

Theorem I.1. *For any given set of Fourier coefficients* $\{V_{\boldsymbol{n}}\}$, $V_{-\boldsymbol{n}} = \bar{V}_{\boldsymbol{n}}$, $|\boldsymbol{n}| \le Q$, $Q \in \mathbb{N}$, *there exists a subset* $\Omega_* = \Omega_*(\{V_{\boldsymbol{n}}\}) \subset [-1/2, 1/2]^{dl}$ *of basic frequencies with* $\mathrm{meas}(\Omega_*) = 1$ *such that for any* $\vec{\boldsymbol{\omega}} \in \Omega_*$, *there is an infinite-dimensional projection* $E_\infty = E_\infty(V)$ *in* $L^2(\mathbb{R}^d)$ *(described in Sec. II) with the following property: For any*

$$\Psi_0 \in \mathcal{C}_0^\infty \quad \text{with} \quad E_\infty \Psi_0 \neq 0, \tag{1.11}$$

there are constants $c_1 = c_1(\Psi_0) > 0$ *and* $T_0 = T_0(\Psi_0)$ *such that the solution* $\Psi(\mathbf{x}, t)$ *of* (1.5) *satisfies the estimate*

$$\frac{2}{T}\int_0^\infty e^{-2t/T} \|X\Psi(\cdot, t)\|^2_{L^2(\mathbb{R}^d)} dt > c_1 T^2 \tag{1.12}$$

for all $T > T_0$.

Remark I.2. The set Ω_* in the formulation of the theorem is implicit. More specifically, it is the very same set for which the results from Ref. 20 are valid. In particular, the frequencies in this set satisfy strong Diophantine condition (see Ref. 20 for more details). In what follows, we will assume that the potential V is fixed and corresponding frequencies belong to Ω_*. We also remark that the notation in this paper, while following in most symbols the notation of Refs. 19 and 20, sometimes differs slightly from it. For example, the projection E_∞ is denoted by $E^{(\infty)}$ in Ref. 20.

In Sec. II, we show that E_∞ is close in norm to $\mathcal{F}^*\chi(\mathcal{G}_\infty)\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform and $\chi(\mathcal{G}_\infty)$ is the characteristic function of a set $\mathcal{G}_\infty$, which has an asymptotically full measure in $\mathbb{R}^d$ [see (2.5) and (2.34)].

As already remarked in Sec. I A, due to the validity of the ballistic upper bound (1.2) for all initial conditions $\Psi_0 \in \mathcal{C}_0^\infty \subset \mathcal{S}_1$, Theorem I.1 remains true if the Abel means are replaced by Cesaro means.

Theorem I.1 will be proven in two steps. First, we will show the following proposition.

Proposition I.3. If $\Psi_0 \in E_\infty \mathcal{C}_0^\infty$, *with* $\Psi_0 \neq 0$ *and* E_∞ *being defined as in* Ref. 20, *then the solution* $\Psi(\mathbf{x}, t)$ *of* (1.5) *satisfies the ballistic lower bound (1.12).*

Note that Proposition I.3 differs from Theorem I.1 by the fact that the initial condition Ψ_0 for which the ballistic lower bound is concluded is in the image of $\mathcal{C}_0^\infty$ under the projection E_∞ (but that Ψ_0 itself is not in $\mathcal{C}_0^\infty$ here). This proposition takes the role of our core technical result, i.e., most of the technical work toward proving Theorem I.1 will go into the proof of the proposition. Theorem I.1 gives a more explicit description of the initial conditions for which ballistic transport can be established. In fact, one easily combines Theorem I.1 with the ballistic upper bound (1.2) to get ballistic transport in the form of a two-sided bound for many initial conditions.

Corollary I.4. There is an $L^2(\mathbb{R}^d)$*-dense and relatively open subset* $\mathcal{D}$ *of* $\mathcal{C}_0^\infty(\mathbb{R}^d)$ *such that for every* $\Psi_0 \in \mathcal{D}$, *there are constants* $0 < c_1 \le C_1 < \infty$ *such that the ballistic upper bound (1.2) and the ballistic lower bound (1.12) hold for* $T > T_0(\Psi_0)$.

This follows by an elementary argument using only that E_∞ is not the zero projection and $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ [and that $\mathcal{C}_0^\infty(\mathbb{R}^d)$ functions also satisfy (1.3)].

It is certainly desirable to go beyond this corollary and to more explicitly characterize classes of the initial conditions for which (1.11) holds. This requires to much better describe and exploit the nature of the projection E_∞. While we believe that $E_\infty\Psi_0 \neq 0$ for any non-zero $\Psi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, we do not have a proof of this. We will return to this question later (see Remark III.1), where we will more explicitly construct the initial conditions, which lead to both upper and lower ballistic transport bounds. These will have the form of suitably regularized generalized eigenfunction expansions.

As mentioned above, the Proof of Theorem I.1 is very similar to the two-dimensional proof.[19] We just need to use the recent results from Ref. 20 instead of those from Ref. 21. In what follows, we present the main steps in the proof and explain the changes we need to make in the proof due to the increase in dimension.

II. SPECTRAL PROPERTIES OF THE OPERATOR H

Our proofs of Proposition I.3 and Theorem I.1 are based on the results and properties of quasi-periodic Schrödinger operators derived in Ref. 20. While that work has derived, in particular, the existence of an absolutely continuous component of the spectrum, we will show now how the bounds obtained in Ref. 20 for the spectral projections can be used to prove the existence of ballistic transport. In this section, we give a thorough discussion of the results and some of the methods from Ref. 20. In particular, we give a detailed construction of the spectral projection E_∞ used in our main results. Unless stated otherwise, all statements in this section have been proved in Ref. 20.

A. Prior results

For any given set of Fourier coefficients $\{V_{\boldsymbol{n}}\}$, $V_{-\boldsymbol{n}} = \bar{V}_{\boldsymbol{n}}$, $|\boldsymbol{n}| \le Q$, $Q \in \mathbb{N}$, there exists a subset $\Omega_* = \Omega_*(\{V_{\boldsymbol{n}}\}) \subset [-1/2, 1/2]^{dl}$ of basic frequencies with meas$(\Omega_*) = 1$ such that for any $\vec{\boldsymbol{\omega}} \in \Omega_*$, the following statements hold, for sufficiently small positive number σ, depending on V, l, and d only.

1. The spectrum of operator (1.6) contains a semi-axis.
2. There are generalized eigenfunctions $U_\infty(\mathbf{k}, \mathbf{x})$, corresponding to the semi-axis, which are close to the unperturbed exponentials. More precisely, for every $\mathbf{k}$ in an extensive [in the sense of (2.5) below] subset $\mathcal{G}_\infty$ of $\mathbb{R}^d$, there is a solution $U_\infty(\mathbf{k}, \mathbf{x})$ of the equation

$$HU_\infty = \lambda_\infty U_\infty$$

that satisfies the following properties:

$$U_\infty(\mathbf{k}, \mathbf{x}) = e^{i\langle \mathbf{k}, \mathbf{x}\rangle}(1 + u_\infty(\mathbf{k}, \mathbf{x})), \tag{2.1}$$

$$\|u_\infty\|_{L^\infty(\mathbb{R}^d)} =_{|\mathbf{k}|\to\infty} O(|\mathbf{k}|^{-\gamma_1}), \quad \gamma_1 = 1 - \sigma > 0, \tag{2.2}$$

where $u_\infty(\mathbf{k}, \mathbf{x})$ is a quasi-periodic function,

$$u_\infty(\mathbf{k}, \mathbf{x}) := \sum_{\boldsymbol{r}\in\mathbb{Z}^l} c_{\boldsymbol{r}}(\mathbf{k}) \boldsymbol{e}_{\boldsymbol{r}\vec{\boldsymbol{\omega}}}(\mathbf{x}), \tag{2.3}$$

the series converging in $L_\infty(\mathbb{R}^d)$. The eigenvalue $\lambda_\infty(\mathbf{k})$ corresponding to $U_\infty(\mathbf{k}, \mathbf{x})$ is close to $|\mathbf{k}|^2$,

$$\lambda_\infty(\mathbf{k}) =_{|\mathbf{k}|\to\infty} |\mathbf{k}|^2 + O(|\mathbf{k}|^{-\gamma_2}), \quad \gamma_2 = 2 - \sigma > 0. \tag{2.4}$$

The "non-resonant" set $\mathcal{G}_\infty$ of the vectors $\mathbf{k}$, for which (2.1) to (2.4) hold, can be expressed as $\mathcal{G}_\infty = \cap_{n=1}^\infty \mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^\infty$ is a decreasing sequence of sets in $\mathbb{R}^d$. Each $\mathcal{G}_n$ has a finite number of holes in each bounded region. Typically, as n increases, more holes of smaller sizes appear in the intersection. As a result, the overall intersection $\mathcal{G}_\infty$ is, typically, a Cantor type set (i.e., it has empty interior). This set satisfies the estimate

$$\begin{aligned} &\frac{\text{meas}(\mathcal{G}_\infty \cap B_R)}{\text{meas}(B_R)} =_{R\to\infty} 1 + O(R^{-c\sigma}), \\ &\sigma > 0, \quad c = c(l, d, \vec{\boldsymbol{\omega}}), \end{aligned} \tag{2.5}$$

where B_R is the ball of radius R centered at the origin.
3. The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\mathbf{k})$,

$$\mathcal{D}_\infty(\lambda) = \{\mathbf{k} \in \mathcal{G}_\infty : \lambda_\infty(\mathbf{k}) = \lambda\},$$

is a slightly distorted sphere, typically with infinite number of holes. It can be described by the formula

$$\mathcal{D}_\infty(\lambda) = \{\mathbf{k} : \mathbf{k} = \varkappa_\infty(\lambda, \vec{\nu})\vec{\nu}, \ \vec{\nu} \in \mathcal{B}_\infty(\lambda)\}, \tag{2.6}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit sphere $\mathbb{S}^{d-1}$. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for the almost plane waves (2.1). The set $\mathcal{B}_\infty(\lambda)$ typically has a Cantor type structure and has an asymptotically full measure on $\mathbb{S}^{d-1}$ as $\lambda \to \infty$,

$$\text{meas}\,(\mathcal{B}_\infty(\lambda)) =_{\lambda\to\infty} \text{meas}\,(\mathbb{S}^{d-1}) + O(\lambda^{-c\sigma}). \tag{2.7}$$

The value $\varkappa_\infty(\lambda, \vec{\nu})$ in (2.6) is the "radius" of $\mathcal{D}_\infty(\lambda)$ in a direction $\vec{\nu}$. The function $\varkappa_\infty(\lambda, \vec{\nu}) - \lambda^{1/2}$ describes the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect sphere of radius $\lambda^{1/2}$. It is proven that the deviation is asymptotically small, uniformly in $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$,

$$\varkappa_\infty(\lambda, \vec{\nu}) =_{\lambda\to\infty} \lambda^{1/2} + O(\lambda^{-\gamma_3}), \ \gamma_3 = (3 - \sigma)/2 > 0. \tag{2.8}$$

4. The part of the spectrum corresponding to $\{U_\infty(\mathbf{k}, \mathbf{x})\}_{\mathbf{k}}$ is absolutely continuous.

Remark II.1. While parameter σ can be chosen arbitrarily small, all constants in $O(\cdot)$ depend on σ. For the purposes of this paper, we will not need to impose any additional assumptions on σ on top of those assumed in Ref. 20 [in particular, $\sigma < (100d)^{-1}$].

B. Description of the method

To prove the results formulated in Sec. II A, in Ref. 20, we have considered the sequence of operators $H_n = H_n(\mathbf{k})$, each being restriction of the operator H onto the linear subspace of $\mathbb{Z}^l$ spanned by the exponentials $\boldsymbol{e}_{\mathbf{k}+\boldsymbol{n}\vec{\omega}}$, $|\boldsymbol{n}| \le |\mathbf{k}|^{r_n}$. Here, r_n is a super exponentially growing sequence of numbers of the form $r_0 = \sigma_1$, $r_n = |\mathbf{k}|^{\sigma_2 r_{n-1}}$, $\sigma_j = \sigma_j(\{V_n\}) > 0$.

Each operator H_n, $n \ge 0$, is considered as a perturbation of the previous operator H_{n-1} ($H_{-1} = -\Delta$). For every operator H_n, there is one eigenvalue located sufficiently far (at least ~ $|\mathbf{k}|^{-r_n}$ away) from the rest of the spectrum of H_n. The corresponding eigenvector is close to the unperturbed exponential. More precisely, for every $\mathbf{k}$ in a certain subset $\mathcal{G}_n$ of $\mathbb{R}^d$, there is a solution $U_n(\mathbf{k},\mathbf{x})$ of the differential equation $H_n U_n = \lambda_n U_n$ that satisfies the following asymptotic formula:

$$U_n(\mathbf{k},\mathbf{x}) = e^{i\langle \mathbf{k},\mathbf{x}\rangle}(1 + u_n(\mathbf{k},\mathbf{x})), \quad \|u_n\|_{L^\infty(\mathbb{R}^d)} \underset{|\mathbf{k}|\to\infty}{=} O(|\mathbf{k}|^{-\gamma_1}), \tag{2.9}$$

where $u_n(\mathbf{k},\cdot)$ is quasi-periodic, a finite combination of $\boldsymbol{e}_{\boldsymbol{r}\vec{\omega}}(\mathbf{x})$,

$$u_n(\mathbf{k},\mathbf{x}) := \sum_{\boldsymbol{r}\in\mathbb{Z}^l, |\boldsymbol{r}|<M_n} c_{\boldsymbol{r}}^{(n)}(\mathbf{k})\boldsymbol{e}_{\boldsymbol{r}\vec{\omega}}(\mathbf{x}), \quad M_n \to \infty \text{ as } n \to \infty. \tag{2.10}$$

The corresponding eigenvalue $\lambda_n(\mathbf{k})$ is close to $|\mathbf{k}|^2$,

$$\lambda_n(\mathbf{k}) =_{|\mathbf{k}|\to\infty} |\mathbf{k}|^2 + O\big(|\mathbf{k}|^{-\gamma_2}\big). \tag{2.11}$$

The non-resonant set $\mathcal{G}_n$, for which (2.11) holds, is proven to be extensive in $\mathbb{R}^d$,

$$\frac{\text{meas}(\mathcal{G}_n \cap B_R)}{\text{meas}(B_R)} =_{R\to\infty} 1 + O(R^{-\sigma}). \tag{2.12}$$

The estimates (2.9)–(2.12) are uniform in n. The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for the non-resonant eigenvalue $\lambda_n(\vec{k})$,

$$\mathcal{D}_n(\lambda) := \{\mathbf{k} \in \mathcal{G}_n : \lambda_n(\mathbf{k}) = \lambda\}.$$

This set is a slightly distorted sphere with a finite number of holes; it can also be described by the following formula:

$$\mathcal{D}_n(\lambda) = \{\mathbf{k} : \mathbf{k} = \varkappa_n(\lambda,\vec{v})\vec{v},\ \vec{v} \in \mathcal{B}_n(\lambda)\}, \tag{2.13}$$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit sphere $\mathbb{S}^{d-1}$. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (2.9). The sequence of sets $\{\mathcal{B}_n(\lambda)\}_{n=0}^{\infty}$ is decreasing since on each step more and more directions are excluded. Each $\mathcal{B}_n(\lambda)$ has an asymptotically full measure on $\mathbb{S}^{d-1}$ as $\lambda \to \infty$,

$$\text{meas}\,(\mathcal{B}_n(\lambda)) =_{\lambda\to\infty} \text{meas}\big(\mathbb{S}^{d-1}\big) + O\big(\lambda^{-\sigma/2}\big), \tag{2.14}$$

with the estimate being uniform in n. The set $\mathcal{B}_n(\lambda)$ has only a finite number of holes; however, their number is growing with n. The value $\varkappa_n(\lambda,\vec{v}) - \lambda^{1/2}$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the perfect sphere of radius $\lambda^{1/2}$ in direction $\vec{v}$. This deviation is asymptotically small uniformly in n,

$$\varkappa_n(\lambda,\vec{v}) = \lambda^{1/2} + O\big(\lambda^{-\gamma_3}\big), \qquad \frac{\partial\varkappa_n(\lambda,\vec{v})}{\partial\vec{\varphi}} = O\big(\lambda^{-\gamma_3}\big), \tag{2.15}$$

with $\vec{\varphi}$ being an angle variable associated with natural spherical coordinates (see Ref. 20 for more details).

More and more points are excluded from the non-resonant sets $\mathcal{G}_n$ on each step. Thus, $\{\mathcal{G}_n\}_{n=0}^{\infty}$ is a decreasing sequence of sets. The set $\mathcal{G}_\infty$ is defined as the limit set $\mathcal{G}_\infty = \cap_{n=0}^{\infty}\mathcal{G}_n$. It has typically an infinite number of holes in each bounded region but nevertheless satisfies relation (2.5). For every $\mathbf{k} \in \mathcal{G}_\infty$ and every n, there is a generalized eigenfunction of H_n of the type (2.9). It is proven that the sequence of $U_n(\mathbf{k},\mathbf{x})$ has a limit in $L^\infty(\mathbb{R}^d)$ as $n \to \infty$ when $\mathbf{k} \in \mathcal{G}_\infty$. The function $U_\infty(\mathbf{k},\mathbf{x}) = \lim_{n\to\infty} U_n(\mathbf{k},\mathbf{x})$ is a generalized eigenfunction of H. It can be written in the form (2.1) and (2.2). Naturally, the corresponding eigenvalue $\lambda_\infty(\mathbf{k})$ is the limit of $\lambda_n(\mathbf{k})$ as $n \to \infty$. Expansion with respect to the generalized eigenfunctions $\Psi_\infty(\mathbf{k},\cdot)$, $\mathbf{k} \in \mathcal{G}_\infty$, will give a reducing subspace for H, with the corresponding spectral resolution arising as the limit of spectral resolutions for operators H_n.

To study them, one needs properties of the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$,

$$\mathcal{B}_\infty(\lambda) = \bigcap_{n=0}^{\infty} \mathcal{B}_n(\lambda), \quad \mathcal{B}_n(\lambda) \subset \mathcal{B}_{n-1}(\lambda).$$

This set has an asymptotically full measure, as (2.7) follows from (2.14). The sequence $\varkappa_n(\lambda,\vec{\nu})$, $n = 0, 1, 2, \ldots$, describing the isoenergetic sets $\mathcal{D}_n(\lambda)$, quickly converges as $n \to \infty$. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense of (2.6), where $\varkappa_\infty(\lambda,\vec{\nu}) = \lim_{n\to\infty}\varkappa_n(\lambda,\vec{\nu})$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. The derivatives of the functions $\varkappa_n(\lambda,\vec{\nu})$ (with respect to the angle variable $\vec{\varphi}$) have a limit as $n \to \infty$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. We denote this limit by $\frac{\partial \varkappa_\infty(\lambda,\vec{\nu})}{\partial \vec{\varphi}}$. We also have

$$\frac{\partial \varkappa_\infty(\lambda,\vec{\nu})}{\partial \vec{\varphi}} = O\big(\lambda^{-\gamma_3}\big). \tag{2.16}$$

Thus, the limit set $\mathcal{D}_\infty(\lambda)$ takes the form of a slightly distorted sphere with, possibly, infinite number of holes.

Let $\mathcal{G}_n'$ be a bounded Lebesgue measurable subset of $\mathcal{G}_n$. We consider the spectral projection $E_n\big(\mathcal{G}_n'\big)$ of H_n, corresponding to functions $U_n(\mathbf{k},\mathbf{x})$, $\mathbf{k} \in \mathcal{G}_n'$. By Ref. 13, $E_n\big(\mathcal{G}_n'\big) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ can be represented by the following formula:

$$E_n\big(\mathcal{G}_n'\big)F = \frac{1}{(2\pi)^d}\int_{\mathcal{G}_n'}\Big(F, U_n(\vec{k})\Big)U_n(\vec{k})\,d\mathbf{k} \tag{2.17}$$

for any $F \in \mathcal{C}_c(\mathbb{R}^d)$, the space of continuous, compactly supported functions on $\mathbb{R}^d$. Here and below, $(\cdot,\cdot)$ is the integral corresponding to the canonical scalar product in $L^2(\mathbb{R}^d)$, i.e.,

$$(F, U_n(\mathbf{k})) = \int_{\mathbb{R}^d} F(\mathbf{x})\overline{U_n(\mathbf{k},\mathbf{x})}\,d\mathbf{x}.$$

The above formula can be rewritten in the form

$$E_n\big(\mathcal{G}_n'\big) = S_n\big(\mathcal{G}_n'\big)T_n\big(\mathcal{G}_n'\big), \tag{2.18}$$

$$T_n : \mathcal{C}_c(\mathbb{R}^d) \to L^2\big(\mathcal{G}_n'\big), \qquad S_n : L^\infty\big(\mathcal{G}_n'\big) \to L^2(\mathbb{R}^d),$$

$$(T_nF)(\mathbf{k}) := \frac{1}{(2\pi)^{d/2}}\Big(F, U_n(\vec{k})\Big) \text{ for any } F \in \mathcal{C}_c(\mathbb{R}^d), \tag{2.19}$$

with T_nF being in $L^\infty\big(\mathcal{G}_n'\big)$, and

$$(S_nf)(\mathbf{x}) := \frac{1}{(2\pi)^{d/2}}\int_{\mathcal{G}_n'} f(\mathbf{k})U_n(\vec{k},\mathbf{x})\,d\mathbf{k} \text{ for any } f \in L^\infty\big(\mathcal{G}_n'\big). \tag{2.20}$$

By Ref. 13,

$$\|T_nF\|_{L^2(\mathcal{G}_n')} \le \|F\|_{L^2(\mathbb{R}^d)} \tag{2.21}$$

and

$$\|S_nf\|_{L^2(\mathbb{R}^d)} \le \|f\|_{L^2(\mathcal{G}_n')}. \tag{2.22}$$

Hence, T_n and S_n can be extended by continuity from $\mathcal{C}_c(\mathbb{R}^d)$ and $L^\infty\big(\mathcal{G}_n'\big)$ to $L^2(\mathbb{R}^d)$ and $L^2\big(\mathcal{G}_n'\big)$, respectively. Obviously, $T_n^* = S_n$. Thus, the operator $E_n\big(\mathcal{G}_n'\big)$ is described by (2.18) in the whole space $L^2(\mathbb{R}^d)$.

In what follows, we will use these operators for the case where $\mathcal{G}_n'$ is given by

$$\mathcal{G}_n' = \mathcal{G}_{n,\lambda} := \{\mathbf{k} \in \mathcal{G}_n : \lambda_n(\mathbf{k}) < \lambda\} \tag{2.23}$$

for finite sufficiently large λ. This set is Lebesgue measurable since $\mathcal{G}_n$ is open and $\lambda_n(\mathbf{k})$ is continuous on $\mathcal{G}_n$.

Let

$$\mathcal{G}_{\infty,\lambda} = \{\mathbf{k} \in \mathcal{G}_\infty : \lambda_\infty(\mathbf{k}) < \lambda\}. \tag{2.24}$$

The function $\lambda_\infty(\mathbf{k})$ is a Lebesgue measurable function since it is a pointwise limit of a sequence of measurable functions. Hence, the set $\mathcal{G}_{\infty,\lambda}$ is measurable. The sets $\mathcal{G}_{n,\lambda}$ and $\mathcal{G}_{\infty,\lambda}$ are also bounded. The measure of the symmetric difference of the two sets $\mathcal{G}_{\infty,\lambda}$ and $\mathcal{G}_{n,\lambda}$ converges to zero as $n \to \infty$, uniformly in λ in every bounded interval,

$$\lim_{n\to\infty}\operatorname{meas}(\mathcal{G}_{\infty,\lambda}\,\Delta\,\mathcal{G}_{n,\lambda}) = 0.$$

Next, we consider the sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ given by (2.20) and with $\mathcal{G}_n' = \mathcal{G}_{\infty,\lambda}$,

$$S_n(\mathcal{G}_{\infty,\lambda}) : \; L^2(\mathcal{G}_{\infty,\lambda}) \to L^2(\mathbb{R}^d). \tag{2.25}$$

This sequence has a limit $S_\infty(\mathcal{G}_{\infty,\lambda})$ in the operator norm sense as $n\to\infty$, uniform in λ. Moreover, the estimate

$$\|S_\infty(\mathcal{G}_{\infty,\lambda})-S_{-1}(\mathcal{G}_{\infty,\lambda})\|<c\lambda_*^{-\gamma_1} \tag{2.26}$$

holds for $\lambda>\lambda_*$, c not depending on λ,λ_*. Here, we put $U_{-1}=e^{i\langle\mathbf{k},\mathbf{x}\rangle}$ and define S_{-1} by (2.20). The operator $S_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|S_\infty\|=1$ and can be described by the following formula:

$$(S_\infty f)(\vec{x})=\frac{1}{(2\pi)^{d/2}}\int_{\mathcal{G}_{\infty,\lambda}}f(\mathbf{k})\Psi_\infty(\vec{k},\mathbf{x})\,d\mathbf{k} \tag{2.27}$$

for any $f\in L^\infty(\mathcal{G}_{\infty,\lambda})$.

Similarly, we consider the sequence of operators $T_n(\mathcal{G}_{\infty,\lambda})$, which are given by (2.19), and act from $L^2(\mathbb{R}^d)$ to $L^2(\mathcal{G}_{\infty,\lambda})$. Since $T_n=S_n^*$, the sequence $T_n(\mathcal{G}_{\infty,\lambda})$ has a limit $T_\infty(\mathcal{G}_{\infty,\lambda})=S_\infty^*(\mathcal{G}_{\infty,\lambda})$ in the operator norm sense. The operator $T_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|T_\infty\|\le 1$ and can be described by the formula $(T_\infty F)(\mathbf{k})=\frac{1}{(2\pi)^{d/2}}(F,\Psi_\infty(\mathbf{k}))$ for any $F\in\mathcal{C}_c(\mathbb{R}^d)$. The convergence is uniform in λ and

$$\|T_\infty(\mathcal{G}_{\infty,\lambda})-T_{-1}(\mathcal{G}_{\infty,\lambda})\|<c\lambda_*^{-\gamma_1}. \tag{2.28}$$

Spectral projections $E_n(\mathcal{G}_{\infty,\lambda})$ converge in norm to $E_\infty(\mathcal{G}_{\infty,\lambda})$ in $L^2(\mathbb{R}^d)$ as n tends to infinity since $T_n=S_n^*$. The operator $E_\infty(\mathcal{G}_{\infty,\lambda})$ is a spectral projection of H. It can be represented in the form $E_\infty(\mathcal{G}_{\infty,\lambda})=S_\infty(\mathcal{G}_{\infty,\lambda})T_\infty(\mathcal{G}_{\infty,\lambda})$. For any $F\in\mathcal{C}_c(\mathbb{R}^d)$, we have

$$E_\infty(\mathcal{G}_{\infty,\lambda})F=\frac{1}{(2\pi)^d}\int_{\mathcal{G}_{\infty,\lambda}}\Big(F,\Psi_\infty(\vec{k})\Big)\Psi_\infty(\vec{k})\,d\mathbf{k}, \tag{2.29}$$

$$HE_\infty(\mathcal{G}_{\infty,\lambda})F=\frac{1}{(2\pi)^d}\int_{\mathcal{G}_{\infty,\lambda}}\lambda_\infty(\mathbf{k})(F,\Psi_\infty(\mathbf{k}))\Psi_\infty(\mathbf{k})\,d\mathbf{k}. \tag{2.30}$$

Since E_∞ is a projection, one has the Parseval formula

$$\|E_\infty(\mathcal{G}_{\infty,\lambda})F\|^2=\frac{1}{(2\pi)^d}\int_{\mathcal{G}_{\infty,\lambda}}|(F,\Psi_\infty(\mathbf{k})|^2\,d\mathbf{k}. \tag{2.31}$$

It is easy to see that

$$\|E_\infty(\mathcal{G}_{\infty,\lambda})-S_{-1}T_{-1}(\mathcal{G}_{\infty,\lambda})\|<c\lambda_*^{-\gamma_1}, \tag{2.32}$$

$$S_{-1}T_{-1}(\mathcal{G}_{\infty,\lambda})=\mathcal{F}^*\chi(\mathcal{G}_{\infty,\lambda})\mathcal{F}. \tag{2.33}$$

Projections $E_\infty(\mathcal{G}_{\infty,\lambda})$ are increasing in λ and have a strong limit $E_\infty(\mathcal{G}_\infty)$ as λ goes to infinity. Hence, the operator $E_\infty(\mathcal{G}_\infty)$ is a projection. The projections $E_\infty(\mathcal{G}_{\infty,\lambda})$, $\lambda\ge\lambda_*$, and $E_\infty(\mathcal{G}_\infty)$ reduce the operator H. The family of projections $E_\infty(\mathcal{G}_{\infty,\lambda})$ is the resolution of the identity of the operator $HE_\infty(\mathcal{G}_\infty)$ acting in $E_\infty(\mathcal{G}_\infty)L^2(\mathbb{R}^d)$. Let us denote $E_\infty:=E_\infty(\mathcal{G}_\infty)$ and use

$$\|E_\infty-\mathcal{F}^*\chi(\mathcal{G}_\infty)\mathcal{F}\|<c\lambda_*^{-\gamma_1}. \tag{2.34}$$

Obviously, the rhs can be made arbitrarily small by an appropriate choice of $\mathcal{G}_\infty$.

The restriction of H to the range of E_∞ has purely absolutely continuous spectrum. In addition to the above mentioned convergence of the spectral projections of H_n to those of H, uniform in $\lambda\ge\lambda_*$ for sufficiently large $\lambda_*=\lambda_*(V)$, this requires an analysis of the continuity properties of the level sets $\mathcal{D}_\infty(\lambda)$ with respect to λ.

C. Extension of $\lambda_\infty(\mathbf{k})$ from $\mathcal{G}_\infty$ to $\mathbb{R}^d$

Let M be a large natural number; for the purposes of this paper, taking $M:=[3d/2+6]$ would do. We want to extend the function $\lambda_\infty(\mathbf{k})$ from $\mathcal{G}_\infty$ to $\mathbb{R}^d$, with the result being a $\mathcal{C}^M(\mathbb{R}^d)$ function. Note that the extended function is not going to be a generalized eigenvalue outside of $\mathcal{G}_\infty$.

The first step is representing $\lambda_\infty(\mathbf{k})-k^2$, $k:=|\mathbf{k}|$, $\mathbf{k}\in\mathcal{G}_\infty$, in the form

$$\lambda_\infty(\mathbf{k})-k^2=\lambda_0(\mathbf{k})-k^2+\sum_{n=1}^{\infty}(\lambda_n(\mathbf{k})-\lambda_{n-1}(\mathbf{k})).$$

Let $\mathbf{m}=(m_1,\ldots,m_d)$ be a multi-index and put $D_{\mathbf{k}}^{\mathbf{m}}:=\partial_1^{m_1}\cdots\partial_d^{m_d}$. We have (see Ref. 20, Lemma 11.3)

$$\left|D_{\mathbf{k}}^{\mathbf{m}}\big(\lambda_0(\mathbf{k})-k^2\big)\right|<Ck^{-\gamma_2+\sigma|m|},\quad \gamma_2=2-\sigma, \tag{2.35}$$

when $\mathbf{k}$ is in the $k^{-\sigma}$-neighborhood of $\mathcal{G}_0 \supset \mathcal{G}_\infty$ and

$$\left|D^{\mathbf{m}}_{\mathbf{k}}(\lambda_n(\mathbf{k}) - \lambda_{n-1}(\mathbf{k}))\right| < Ck^{-k^{r_{n-1}}+|m|k^{\sigma r_{n-1}}} \tag{2.36}$$

in the $k^{-k^{\sigma r_{n-1}}}$-neighborhood of $\mathcal{G}_n$ for all m. Here, the constants depend only on V and m.

Now, we introduce a function $\eta_0(\mathbf{k}) \in \mathcal{C}^\infty(\mathbb{R}^d)$ with support in the (real) $k^{-\sigma}$-neighborhood of $\mathcal{G}_0$, satisfying $\eta_0 = 1$ on $\mathcal{G}_0$ and

$$\left|D^m \eta_0(\mathbf{k})\right| < Ck^{\sigma|m|}. \tag{2.37}$$

This is possible since we can take a convolution of the characteristic function of the $\frac{1}{2}k^{-\sigma}$-neighborhood of $\mathcal{G}_0$ with $\omega(2k^\sigma \mathbf{k})$, where $\omega(\mathbf{k})$ is a non-negative $C_0^\infty(\mathbb{R}^d)$-function with a support in the unit ball centered at the origin and integral one. Similarly, let $\eta_n(\mathbf{k})$, $n \geq 1$, be a $\mathcal{C}^\infty$ function with support in the $k^{-k^{\sigma r_{n-1}}}$-neighborhood of $\mathcal{G}_n$, satisfying $\eta_n = 1$ on $\mathcal{G}_n$ and

$$\left|D^m_{\mathbf{k}} \eta_n(\mathbf{k})\right| \leq Ck^{|m|k^{\sigma r_{n-1}}}. \tag{2.38}$$

Next, we extend $\lambda_\infty(\mathbf{k}) - k^2$ from $\mathcal{G}_\infty$ to $\mathbb{R}^d$ using the formula

$$\lambda_\infty(\mathbf{k}) - k^2 = (\lambda_0(\mathbf{k}) - k^2)\eta_0(\mathbf{k}) + \sum_{n=1}^{\infty}(\lambda_n(\mathbf{k}) - \lambda_{n-1}(\mathbf{k}))\eta_n(\mathbf{k}). \tag{2.39}$$

It follows from (2.35)–(2.38) that the series converges in $\mathcal{C}^M(\mathbb{R}^d)$. Taking into account that $\sigma > 0$ could be chosen arbitrarily small (note that λ_* increases and $\mathcal{G}_\infty$ is getting smaller when σ decreases) gives the following lemma:

Lemma II.2. For every natural number M, there exists $\lambda_*(V, M) > 0$ such that the function $\lambda_\infty(\mathbf{k}) - k^2$ can be extended, as a $\mathcal{C}^M$ function, from $\mathcal{G}_\infty$ to $\mathbb{R}$, and it satisfies

$$\left|D^{\mathbf{m}}_{\mathbf{k}}(\lambda_\infty(\mathbf{k}) - k^2)\right| < C_M k^{-\gamma_2+\sigma|m|} \tag{2.40}$$

for any $m \in \mathbb{N}_0^d$ with $|m| \leq M < \sigma^{-1}$.

D. Extension of $U_\infty(\mathbf{k}, \mathbf{x})$ from $\mathcal{G}_\infty$ to $\mathbb{R}^d$

We now define $U_\infty(\mathbf{k}, \mathbf{x})$ for arbitrary $\mathbf{k} \in \mathbb{R}^d$ by a formula analogous to (2.39),

$$\begin{aligned} U_\infty(\mathbf{k}, \mathbf{x}) - e^{i\langle \mathbf{k}, \mathbf{x}\rangle} &= \left(\Psi_0(\mathbf{k}, \mathbf{x}) - e^{i\langle \mathbf{k}, \mathbf{x}\rangle}\right)\eta_0(\mathbf{k}) \\ &+ \sum_{n=1}^{\infty}(U_n(\mathbf{k}, \mathbf{x}) - U_{n-1}(\mathbf{k}, \mathbf{x}))\eta_n(\mathbf{k}). \end{aligned} \tag{2.41}$$

Here, U_n are described by (2.9), (2.10), and (see Ref. 20, Lemma 11.3)

$$\|D^m_{\mathbf{k}}\left(u^{(0)} - \boldsymbol{e}_{\mathbf{k}}\right)\|_{L_\infty(\mathbb{R}^d)} < Ck^{-\gamma_1+\sigma|m|}, \quad \gamma_1 = 1 - \sigma, \tag{2.42}$$

$$\|D^m_{\mathbf{k}}\left(u^{(n)} - u^{(n-1)}\right)\|_{L_\infty(\mathbb{R}^d)} < Ck^{-k^{r_{n-1}}+|m|k^{\sigma r_{n-1}}}. \tag{2.43}$$

Thus, series (2.41) is convergent in $L_\infty(\mathbb{R}^d)$. Using the last formula and (2.27), we define $S_\infty(\widetilde{\mathcal{G}}_\infty)$ for any $\widetilde{\mathcal{G}}_\infty \supset \mathcal{G}_\infty$,

$$\left(S_\infty(\widetilde{\mathcal{G}}_\infty)f\right)(\mathbf{x}) := \frac{1}{(2\pi)^{d/2}}\int_{\widetilde{\mathcal{G}}_\infty} f(\mathbf{k})U_\infty(\mathbf{k}, \mathbf{x})\, d\mathbf{k}. \tag{2.44}$$

It is easy to see that

$$S_\infty(\widetilde{\mathcal{G}}_\infty) = S_{-1}(\widetilde{\mathcal{G}}_\infty) + \sum_{n=0}^{\infty}\left(S_n(\widetilde{\mathcal{G}}_\infty) - S_{n-1}(\widetilde{\mathcal{G}}_\infty)\right)\eta_n, \tag{2.45}$$

where $S_{-1}(\widetilde{\mathcal{G}}_\infty)$ is defined by

$$S_{-1}(\widetilde{\mathcal{G}}_\infty)f = \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty} f(\mathbf{k})e^{-i\langle \mathbf{k}, \mathbf{x}\rangle}\, d\mathbf{k}.$$

η_n is multiplication by $\eta_n(\mathbf{k})$, and $S_n(\widetilde{\mathcal{G}}_\infty)$ is given by (2.20), with $\mathcal{G}'_n$ being the intersection of $\widetilde{\mathcal{G}}_\infty$ with the $k^{-k^{\sigma r_{n-1}}}$-neighborhood of $\mathcal{G}_n$ for $n \geq 1$ and the $k^{-\sigma}$-neighborhood of $\mathcal{G}_0$ for $n = 0$.

Similar to (2.26), we show that

$$\|S_\infty(\widetilde{\mathcal{G}}_\infty) - S_{-1}(\widetilde{\mathcal{G}}_\infty)\| < c(V)\lambda_*^{-\gamma_1}. \tag{2.46}$$

In what follows, we assume that λ_* is chosen so that, in particular, $c(V)\lambda_*^{-\gamma_1} \leq 1/2$ (in fact, this is already the case under conditions from Ref. 20). Clearly, $\|S_{-1}(\widetilde{\mathcal{G}}_\infty)\| = 1$. Thus, we have

$$\|S_\infty(\widetilde{\mathcal{G}}_\infty)\| \leq 2. \tag{2.47}$$

Similarly, with T_{-1} being the Fourier transform,

$$\begin{aligned}(T_\infty F)(\mathbf{k}) &:= \frac{1}{(2\pi)^{d/2}}(F(\cdot), U_\infty(\mathbf{k}, \cdot)) \\ &= (T_0 F)(\mathbf{k}) + \sum_{n=0}^{\infty}((T_n - T_{n-1})F)(\mathbf{k})\eta_n(\mathbf{k}).\end{aligned} \tag{2.48}$$

Lemma II.3. For any given $L \in \mathbb{N}$, there exists $\lambda_*(V, L)$ such that for any $F \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, the function $T_\infty F$ as defined above is in $\mathcal{C}^L(\mathbb{R}^d)$. Moreover, if $0 \leq j \leq L$ and $m \in \mathbb{N}_0^d$, $|m| \leq L$, then

$$\left||\mathbf{k}|^j D^{\mathbf{m}}(T_\infty F)(\mathbf{k})\right| < C(L, F) \tag{2.49}$$

for all $\mathbf{k} \in \mathbb{R}^d$.

We prove the lemma using (2.48) and then (2.19) for each T_n. Integrating by parts j times and considering (2.38), (2.42), and (2.43), we arrive at (2.49).

Remark II.4. For our needs $L = M = [3d/2 + 6]$ is sufficient, so we may assume that such L and M are fixed.

III. PROOFS OF PROPOSITION I.3 AND THEOREM I.1

Let $\mathcal{S} := T_\infty \mathcal{C}_0^\infty(\mathbb{R}^d)$ [see (2.48)]. Let $\widehat{\Psi}_0 \in \mathcal{S}$. As shown in Lemma II.3, then

$$||\mathbf{k}|^j D^m(\widehat{\Psi}_0)(\mathbf{k})| < C(j, m, \widehat{\Psi}_0) \tag{3.1}$$

for any $\mathbf{k} \in \mathbb{R}^d$.

Now, we define

$$\Psi(\mathbf{x}, t) := \frac{1}{(2\pi)^{d/2}} \int_{\mathcal{G}_\infty} U_\infty(\mathbf{k}, \mathbf{x}) e^{-it\lambda_\infty(\mathbf{k})} \widehat{\Psi}_0(\mathbf{k})\, d\mathbf{k}, \tag{3.2}$$

then this function solves initial value problem (1.5), where

$$\Psi_0(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathcal{G}_\infty} U_\infty(\mathbf{k}, \mathbf{x}) \widehat{\Psi}_0(\mathbf{k})\, d\mathbf{k} \tag{3.3}$$

and $\Psi_0(\mathbf{x}) \in S_\infty \mathcal{S} = E_\infty \mathcal{C}_0^\infty$. Obviously, $S_\infty \mathcal{S}$ is dense in $E_\infty L^2(\mathbb{R}^d)$.

The next step of the proof is to replace $\mathcal{G}_\infty$ by a small neighborhood $\tilde{\mathcal{G}}_\infty$ and to estimate the resulting errors in the integrals. This is an important step since $\mathcal{G}_\infty$ is a closed Cantor type set, while $\tilde{\mathcal{G}}_\infty$ is an open set. Then, we would like to integrate by parts in the integral over $\tilde{\mathcal{G}}_\infty$ with the purpose of obtaining (1.12); the fact that $\tilde{\mathcal{G}}_\infty$ is open being used for handling the boundary terms.

To get the lower bound (1.12), we first note that

$$\|X\Psi\|^2_{L^2(\mathbb{R}^d)} \geq \|X\Psi\|^2_{L^2(B_R)} \geq \frac{1}{2}\|Xw\|^2_{L^2(B_R)} - \|X(\Psi - w)\|^2_{L^2(B_R)},$$

where B_R is the open disc with radius R centered at the origin, $R = c_0 T$, c_0 to be chosen later, and $w(\mathbf{x}, t)$ is an approximation of Ψ when $\mathcal{G}_\infty$ is replaced by its small neighborhood $\tilde{\mathcal{G}}_\infty$. Namely,

$$w(\mathbf{x}, t) := \frac{1}{(2\pi)^{d/2}} \int_{\widetilde{\mathcal{G}}_\infty} U_\infty(\mathbf{k}, \mathbf{x}) e^{-it\lambda_\infty(\mathbf{k})} \widehat{\Psi}_0(\mathbf{k}) \tilde{\eta}_\delta(\mathbf{k})\, d\mathbf{k}, \tag{3.4}$$

with $\tilde{\eta}_\delta$ being a smooth cutoff function with support in a δ-neighborhood $\widetilde{\mathcal{G}}_\infty$ of $\mathcal{G}_\infty$ and $\tilde{\eta}_\delta = 1$ on $\mathcal{G}_\infty$. The parameter $\delta(0 < \delta < 1)$ will be chosen later to be sufficiently small and depend only on $\widehat{\Psi}_0$. We take $\tilde{\eta}_\delta$ to be a convolution of a function $\omega(\mathbf{k}/2\delta)$ with the characteristic function of the $\delta/2$-neighborhood of $\mathcal{G}_\infty$, where ω is a smooth cutoff function defined in Sec. II C. Then, $\tilde{\eta}_\delta \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}&0 \le \tilde{\eta}_\delta \le 1, \ \tilde{\eta}_\delta(\mathbf{k}) = 1 \text{ when } \mathrm{k} \in \mathcal{G}_\infty, \ \tilde{\eta}_\delta(\mathbf{k}) = 0 \text{ when } \mathrm{k} \notin \widetilde{\mathcal{G}}_\infty,\\ &\|D^m \tilde{\eta}_\delta\|_{L^\infty} < C_m \delta^{-|m|}.\end{aligned} \tag{3.5}$$

To prove (1.12), we will show that there exist a positive constant c_1 and constants c_2 and c_3 such that

$$\frac{2}{T}\int_0^\infty e^{-2t/T}\|Xw(\cdot,t)\|^2_{L^2(B_R)}dt \ge 6c_1T^2 - c_2T - c_3 \tag{3.6}$$

as long as c_0 in the definition of R exceeds a certain value depending only on $\widehat{\Psi}_0$. In formula (3.6), the constant $c_1 = c_1(\widehat{\Psi}_0)$ depends on $\widehat{\Psi}_0$, but not on δ or c_0, while the constants $c_2 = c_2(\widehat{\Psi}_0, \delta)$ and $c_3 = c_3(\widehat{\Psi}_0, \delta)$ depend on $\widehat{\Psi}_0$ and δ, but not on c_0.

We also prove that

$$\frac{2}{T}\int_0^\infty e^{-2t/T}\|X(\Psi - w)(\cdot,t)\|^2_{L^2(B_R)}dt \le \gamma(\delta, \widehat{\Psi}_0)c_0^2T^2, \tag{3.7}$$

$\gamma(\delta, \widehat{\Psi}_0) = o(1)$ as $\delta \to 0$ uniformly in c_0.

The proofs of (3.6) and (3.7) are completely analogous to those from Ref. 19. The only difference is in the estimate of the integral of the form

$$\tilde{\phi}_1(\vec{z}t,t) := \frac{1}{(2\pi)^{d/2}}\int_{\widetilde{\mathcal{G}}_\infty \cap \{\mathbf{k}:|\mathbf{k}-\mathbf{k}_0|<2\}} e^{it(\langle \mathbf{k},\vec{z}\rangle - \lambda_\infty(\mathbf{k}))} g_3(\mathbf{k})(1-\hat{\eta}(\mathbf{k}))\,d\mathbf{k}, \quad \vec{z} := \frac{\mathbf{x}}{t}, \tag{3.8}$$

where $g_3(\mathbf{k}) := \nabla\lambda_\infty(\mathbf{k})\widehat{\Psi}_0(\mathbf{k})\tilde{\eta}_\delta(\mathbf{k})$, $\hat{\eta}$ is a smooth cutoff function satisfying

$$\hat{\eta}(\mathbf{k}) = \begin{cases} 0, & |\mathbf{k} - \mathbf{k}_0| \le 1 \\ 1, & |\mathbf{k} - \mathbf{k}_0| \ge 2, \end{cases}$$

and

$$\mathbf{k}_0 = \mathbf{k}_0(\vec{z}) = \frac{1}{2}\vec{z} + O(|\vec{z}|^{-\gamma_4}), \quad \gamma_4 > 0,$$

is the unique solution (see (2.39) and Lemma II.2) of the equation for a stationary point

$$\vec{z} - \nabla\lambda_\infty(\mathbf{k}) = 0, \quad |\vec{z}|^2 > \lambda_*.$$

As in Ref. 19, we apply Theorem 7.7.5 of Ref. 17 but for arbitrary $d > 1$. The number of derivatives required depends on the dimension ($M := [3d/2 + 6]$ is enough). We have

$$\tilde{\phi}_1(\vec{z}t,t) = \frac{1}{(2i)^{d/2}} e^{it(\langle \mathbf{k}_0,\vec{z}\rangle - \lambda_\infty(\mathbf{k}_0))}\big(1 + O(|\vec{z}|^{-\gamma_4})\big) g_3(\mathbf{k}_0)t^{-d/2} + \epsilon(g_3)t^{-d/2-1} \tag{3.9}$$

for $|\vec{z}|^2 > \lambda_*$ and 0 otherwise. Here,

$$\begin{aligned}|\epsilon(g_3)| &\le c\sum_{|m|\le d+3}\sup_{|\mathbf{k}-\mathbf{k}_0|<2}|D^m g_3(\mathbf{k})|\\ &\le c\Big\||\mathbf{k}|^{d/2+2}\widehat{\Psi}_0(\mathbf{k})\Big\|_{C^{d+3}(\mathbb{R}^d)}\delta^{-d-3}|\vec{z}|^{-d/2-1}.\end{aligned}$$

Now, the end of the Proof of Proposition I.3 follows as in Sec. III of Ref. 19. The Proof of Theorem I.1 is identical to the proof in Sec. IV of Ref. 19.

Remark III.1. (a) The above proofs show that Theorem I.1 remains true if we replace C_0^∞ in (1.11) with

$$\mathcal{S}_d := \{f : |x|^s D^{\mathbf{m}} f(x) \in L^2(\mathbb{R}^d), 0 \le s, |m| \le C(d)\},$$

i.e., for initial conditions that are sufficiently smooth and of sufficiently rapid power decay.

(b) Using the constructions mentioned in the above proofs, we can now also describe more explicitly how to choose the initial conditions Ψ_0 for the solution of (1.5), which give simultaneous ballistic upper and lower bounds. Essentially, one has to regularize elements in the range of E_∞ in two different ways, first at the boundary of $\mathcal{G}_\infty$ using the cutoff function $\tilde{\eta}_\delta$ as in (3.5) and then at high momentum $\mathbf{k}$. For the latter, let $\varphi \in \mathcal{S}_d$ on $\mathbb{R}^d$ such that φ does not vanish identically on $\mathcal{G}_\infty$.

Choose

$$\Psi_0(\mathbf{x}) := \frac{1}{(2\pi)^{d/2}} \int_{\widetilde{\mathcal{G}}_\infty} \varphi(\mathbf{k})\, \tilde{\eta}_\delta(\mathbf{k})\, U_\infty(\mathbf{k}, \mathbf{x})\, d\mathbf{k}. \tag{3.10}$$

As $\delta \to 0$, this converges to $F_0(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathcal{G}_\infty} \varphi(\mathbf{x}) U_\infty(\mathbf{k}, \mathbf{x})\, d\mathbf{k}$ in the range of E_∞ with $\|F_0\|^2 = \int_{\mathcal{G}_\infty} |\varphi|^2\, d\mathbf{k}/(2\pi)^d \neq 0$. Thus, for $\delta > 0$ being sufficiently small, $E_\infty \Psi_0 \neq 0$.

Furthermore, our methods show that the choice of $\varphi \in \mathcal{S}_d$ gives $\Psi_0 \in \mathcal{S}_d$. Thus, the initial condition Ψ_0 leads to a ballistic lower bound on transport. At the same time, the condition of Ref. 27 for the ballistic upper bound (1.2) is satisfied.

DEDICATION

The authors would like to dedicate this paper to the memory of Jean Bourgain.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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