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# Towards the Homotopy Type of the Morse Complex

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# TOWARDS THE HOMOTOPY TYPE OF THE MORSE COMPLEX

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Abstract. Let $K$ be a simplicial complex. In 1998, Robin Forman developed gradient vector fields as a tool to study these complexes. Having gradient vector fields to study these simplicial complexes, in 2005, Chari and Joswig discovered the Morse complex. Although the Morse complex has been studied since 2005, there is little information regarding its homotopy type for different simplicial complexes. Pursuing our curiosity of the topic, we extend a result by Ayala et. al., stating that for a tree, $T$, $\mathscr{M}_p(T)$ is strongly collapsible. We also extend a result by Kozlov to show that a path, $P_{3n}$, is strongly collapsible. Additionally, we provide alternate proofs for the results by Ayala et. al. as well as Kozlov. Furthermore, we realize cocktail-party graphs as the $1-$skeleton of the core of $\mathscr{M}(P_t)$, compute the homotopy type for centipede graphs, cycles with a single leaf, and some paths with a single leaf. By using multiple partitioning and matching strategies, we provide a framework to pursue homotopy types of more involved Morse complexes.

## 1. Introduction

In 1998, Robin Forman [7, 8] developed a simple, yet powerful, tool of gradient vector fields on a simplicial complex $K$ to represent a series of collapses. This has become a very helpful way to estimate the number of critical simplices on a complex [7, Section 8]. A few years later, in 2005, Chari and Joswig [5] introduced a new complex, known as the Morse complex. The Morse complex of $K$, denoted $\mathscr{M}(K)$, was designed to show compatibility of different gradient vector fields on $K$, and thus has its own unique construction. Similarly, $\mathscr{M}(K)$ has the capability to, in a way, guide the reconstruction of $K$, up to isomorphism, because of the unique set of gradient vector fields that it represents [4]. However, interestingly enough, the same authors showed that the homotopy type of $\mathscr{M}(K)$ does not determine the simple homotopy type of $K$.

The first goal of this paper will be to explore the realm of determining homotopy types of a $\mathscr{M}(K)$ without needing to construct it. Independently, both Hersh and Jonsson [10, 11] discovered a result known as the "Cluster Lemma." The idea is that a complex can be split into smaller collections, gradient vector fields can be found on each collection, and then the complex can be reconstructed. In this paper, we will use the facets of $\mathscr{M}(K)$ to guide the partitioning of $\mathscr{M}(K)$, as well as the idea of a "star" and "link" of a vertex. These give simplicity to our collections, as well as increased manageability of the elements of $\mathscr{M}(K)$. Using this idea, we

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provide an alternate proof of the computation of the homotopy type of the Morse complex of a path [12] and an alternate proof that the pure Morse complex of a tree is collapsible [1].

However, the primary goal of this paper will be to examine the property of "strong collapsibility" of a Morse complex. This idea has been studied [3, 6] and has some interesting relationships to both the construction of a complex [6, Section 3.2] and the elementary collapsibility of a complex. Pursuing these ideas, we extend a result by Ayala et. al. [1], showing that the pure Morse complex of a tree is strongly collapsible, as well as extend Kozlov's proposition by showing that a path graph of length $3n$ has a strongly collapsible Morse complex. By studying the Morse complex of paths closely, we find that cocktail party graphs can be realized as the $1-$skeleton of the core of the Morse complex of certain paths. Additionally, we use strong collapses and certain results from [6] and [2] to compute the homotopy type of centipede graphs, finding that the Morse complex of a centipede graph, $\mathscr{M}(C_v)$, is homotopy equivalent to $\mathbb{S}^{n-1}$. We find that the Morse complex of the Morse complex of a cycle with one leaf, $\mathscr{M}(C_n \vee l)$, is either collapsible or homotopy equivalent to some $i-$dimensional sphere. Lastly, we find that the Morse complex of a path with one leaf, $\mathscr{M}(P_t \vee_{v_k} l)$, is homotopy equivalent to the join of the Morse complexes of two smaller paths and a leaf.

## 2. Background

Here we provide some basic notation, terminology, and important results that will be used throughout the rest of the paper. Our reference for notation and terminology is [13] unless stated otherwise.

### 2.1. Simplicial Complexes and the Morse Complex.

In this section, we provide some basic principles of simplicial complexes and the Morse complex.

We start with definitions regarding simplicial complexes, which allows us to define the Morse complex.

**Definition 1.** A set $\sigma$ of cardinality $i+1$ is called an ***$i$-dimensional simplex*** or ***$i$-simplex***.

Here we draw a definition from [14, Definition 2.1].

**Definition 2.** The ***$n$-simplex***, denoted $\Delta^n$, is a closed convex polyhedron of dimension $n$ created by joining $(n+1)$ vertices.

**Definition 3.** The ***(simplicial) $n-$sphere*** is defined by

$$\mathbb{S}^n := \Delta^{n+1} - \{v_{n+1}\}.$$

**Definition 4.** Let $n \geq 0$ be an integer and $[v_n] := \{v_0, v_1, \ldots, v_n\}$ a collection of $n+1$ symbols. A ***simplicial complex*** $K$ on $[v_n]$ or a ***complex*** is a collection of subsets of $[v_n]$, excluding $\emptyset$, such that

(1) if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$;
(2) $\{v_i\} \in K$ for every $v_i \in [v_n]$.

The set $[v_n]$ is called the ***vertex set*** of $K$ and the elements $\{v_i\}$ are called ***vertices*** or $0-$***simplices***. We sometimes write $V(K)$ for the vertex set of $K$.

The ***dimension*** of a simplicial complex, $K$, denoted by $\dim(K)$, is the maximum of all the dimensions of all its simplices. Additionally, any simplicial complex $K$ such that $\dim(K) = 1$ is called a ***graph***.

Any simplex of $K$ that is not properly contained in any other simplex of $K$ is called a ***facet*** of $K$.

It is important to distinguish the $n$-simplex, $\Delta^n$, which is a simplicial complex in itself, from an $i$-simplex, $\sigma$, which is an element of a simplicial complex $K$.

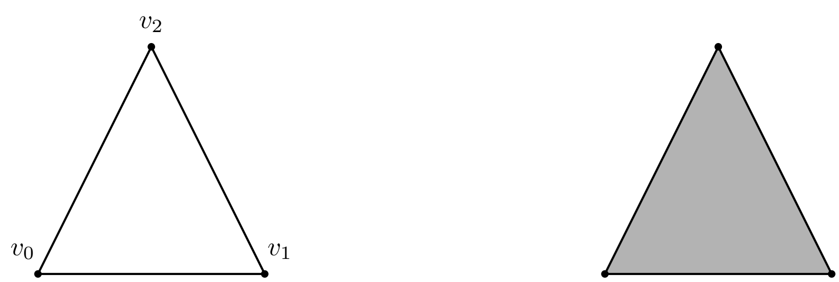


FIGURE 1. On the left, we have a simplicial complex $K := \{v_0, v_1, v_2, v_0v_1, v_1v_2, v_0v_2\}$. On the right, we have the 2-simplex, $\Delta^2$.

We would also like to introduce a special type of simplicial complex that will be of great interest to us come time for results. We study them directly, and we also build off of them to study slightly more complex graphs.



FIGURE 2. Here we have a ***path graph***, $P_v$, where $v$ is the number of vertices. Path graphs are one of the main simplicial complexes that we will study in this paper.

**Definition 5.** A ***discrete Morse function*** $f$ on $K$ is a function $f : K \to \mathbb{R}$ such that for every $p$-simplex $\sigma \in K$, we have

$$|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| \leq 1$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| \leq 1$$

It is worth noting that we do not use the definition of a discrete Morse function at all in this paper. However, it is needed to define a gradient vector field on simplicial complex, $K$.

**Definition 6.** Let $f$ be a discrete Morse function on $K$. The ***induced gradient vector field*** $V_f$, or $V$ when the context is clear, is defined by

$$V_f := \{(\sigma^{(p)}, \tau^{(p+1)} : \sigma < \tau, f(\sigma) \geq f(\tau)\}$$

If $(\sigma, \tau) \in V_f$, $(\sigma, \tau)$ is called a ***vector***, ***arrow***, or a ***matching***. The element $\sigma$ is a ***tail*** while $\tau$ is a ***head***.

If a induced gradient vector field has only one matching, we call it ***primitive***.

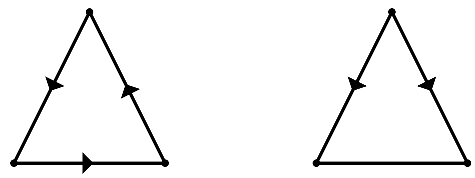

FIGURE 3. The gradient vector field on the left forms a cycle, and therefore is not allowed. The cycle on the right has two matchings using the top vertex, and therefore is not allowed.

We illustrate gradient vector fields that are *not* discrete Morse matchings (acyclic) in Figure 3.

After determining the induced gradient vector field, $f$, on a simplicial complex, any $i-$simplex that is not matched is ***critical***. We denote ***critical*** $i-$***simplices*** by $m_i$, and we can denote all critical simplices under $f$ by $\vec{f} := (m_0, m_1, \ldots, m_n)$.

**Definition 7.** Let $K$ be a simplicial complex. A gradient vector field $f = \{f_0, ..., f_n\}$ on $K$ is said to be ***maximum*** if $\dim(f) \geq \dim(g)$ for any gradient vector field $g$. In other words, as many simplices as possible are matched.

Additionally, $f$ is a ***maximum*** gradient vector field if and only if $f$ is a facet in $\mathscr{M}_p(G)$.

This idea is separate from that of a ***maximal*** vector field, which simply means a gradient vector field that is not contained in any other gradient vector field. If another matching is added to it, it is not longer a gradient vector field.

Here we illustrate the difference between a maximum gradient vector field and a gradient vector field that is only maximal.

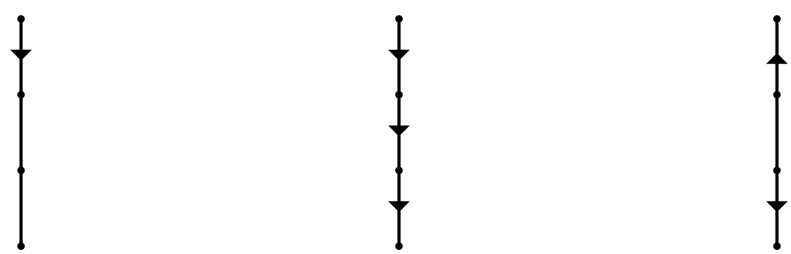

FIGURE 4. On the left is an example of a primitive gradient vector field on the path graph, $P_4$. In the center is an example of a maximum (and therefore maximal) gradient vector field, while the gradient vector field on the right is only maximal, but not maximum.

Using the concepts of simplices, simplicial complexes, and the gradient vector field, we can define the Morse complex. This is our main topic of study so it is important to understand what the Morse complex is, as well as what it represents. It holds the information of all the induced gradient vector fields on a simplicial complex, $K$.

**Definition 8.** The ***Morse Complex*** of $K$, denoted by $\mathscr{M}(K)$, is the simplicial complex whose vertices are given by primitive gradient vector fields and whose $n$-simplices are given by gradient vector fields with $n+1$ regular pairs.

A gradient vector field $f$ is then associated with all primitive gradient vector fields $f := \{f_0, ..., f_n\}$ with $f_i \leq f$ for all $0 \leq i \leq n$.

We provide another definition that allows us to give an alternate definition for the Morse complex that comes in handy. We define the ***Hasse Diagram***.

**Definition 9.** Let $K$ be a simplicial complex. The ***Hasse Diagram*** of $K$, denoted $\mathcal{H}_K$ or $\mathcal{H}$, is defined as the partially ordered set of simplices of $K$ ordered by the face relations. We organize the diagram by placing nodes in rows such that every node in the same row corresponds to a simplex of the same dimension.

Example

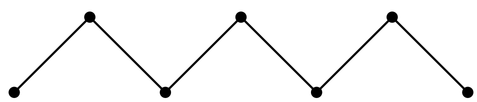

Figure 5. Here we illustrate the Hasse Diagram for $P_4$. The bottom row of nodes represent vertices in $P_4$, and the top row represents edges in $P_4$. The edges in the Hasse diagram show which vertices exist in which edge of $P_4$

Now we can provide an alternate definition for the Morse complex:

**Definition 10.** Let $K$ be a simplicial complex. The ***Morse Complex*** of $K$, denoted by $\mathscr{M}(K)$, is the simplicial complex on the set of edges of $\mathcal{H}(K)$ defined as the set of subsets of edges of $\mathcal{H}(K)$ which form discrete Morse matchings (acyclic matchings), excluding the empty matching.

Example

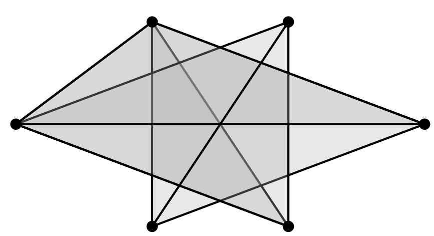

Figure 6. Here we have the Morse complex of $P_4$; the path graph with 4 vertices. It is much more involved!

2.2. **Elementary and Strong Collapsibility.** Here we draw out important ideas necessary for the comprehension of our results, as they are requisites for the understanding of homotopy and homotopy type.

**Definition 11.** Let $K$ be a simplicial complex and suppose that there is a pair of simplices $\{\sigma^{(p-1)}, \tau^{(p)}\}$ in $K$ such that $\sigma$ is a face of $\tau$ and $\sigma$ has no other cofaces. Then $K - \{\sigma, \tau\}$ is a simplicial complex called an ***elementary collapse*** of $K$. On the other hand, suppose $\{\sigma^{(p-1)}, \tau^{(p)}\}$ is a pair of simplices not in $K$ where $\sigma$ is a face of $\tau$ and all other faces of $\tau$ are in $K$. Then $K \cup \{\sigma^{(p-1)}, \tau^{(p)}\}$ is a simplicial complex called an ***elementary expansion*** of $K$.

We say that $K$ and $L$ are of the same ***simple homotopy type***, denoted by $K \sim L$, if there is a series of elementary collapses and expansions from $K$ to $L$.

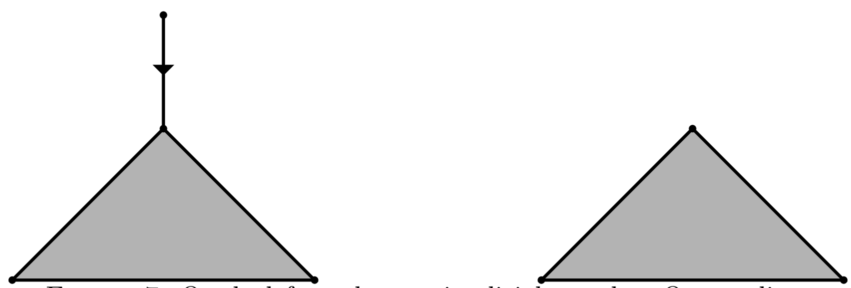

FIGURE 7. On the left, we have a simplicial complex. Our gradient vector field represents an elementary collapse that we can make. On the right, we see the simplicial complex after the elementary collapse.

**Definition 12.** A simplicial complex $K$ is ***collapsible*** if $K \simeq \{v\}$ for some vertex $v \in K$.

**Definition 13.** Let $K$ be a simplicial complex. A vertex $v$ is said to ***dominate*** $v'$ (or $v'$ is ***dominated*** by $v$) if every maximal simplex (facet) of $v'$ also contains $v$.

**Definition 14.** Let $K$ be a simplicial complex. If $v$ dominates $v'$, then the removal of $v'$ from $K$ is called an ***elementary strong collapse***, denoted $K \searrow\searrow K - \{v'\}$. The addition of a dominated vertex is an ***elementary strong expansion***, denoted by $\nearrow\nearrow$. A sequence of elementary strong collapses or elementary strong expansions is also called a strong collapse or strong expansion, respectively.

If there is a sequence of strong collapses or strong expansions from $K$ to $L$, they are said to have the same ***strong homotopy type***. A special case of this is if $K \approx *$, then $K$ is said to have the ***strong homotopy type of a point***.

If there is a sequence of elementary strong collapses from $K$ to a point, $K$ is called ***strongly collapsible***.

We illustrate a strong collapse in Figure 8. Notice how many elementary collapses seem to be happening simultaneously, as strong collapses are a sequence of elementary collapses.

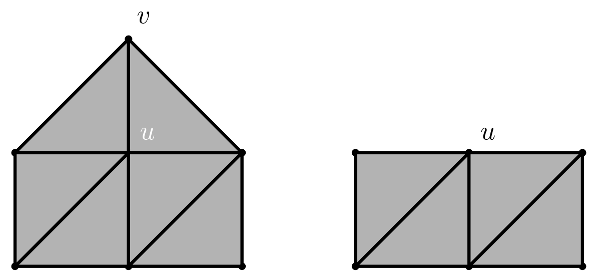


FIGURE 8. On the left we have a simplicial complex. It is evident that $u$ dominates $v$, so we can make a strong collapse, leaving the reduced simplicial complex on the right.

2.3. **Previous Results.** Now that we have provided the relevant terminology and notation to discuss our results, we will state the main previous results that aided in our research, as for convenience.

Throughout our results, we utilize a result by Forman:

**Theorem 15.** [7, Corollary 3.5] Let $K$ be a simplicial complex and $M$ an acyclic matching on $K$ with $m_i$ critical simplices of dimension $i$. Then $K$ has the homotopy type of a CW complex with exactly $m_i$ cells of dimension $i$.

In addition, we utilize the following result, discovered independently by J. Jonsson [11, Lemma 4.2]and P. Hersh [10, Lemma 4.1]. We will utilize a version for simplicial complexes due to R. Zax [14, Lemma 5.5].

**Lemma 16.** [Cluster Lemma] Let $\Delta$ be a simplicial complex which decomposes into collections $\Delta_\sigma$ of simplices, indexed by the elements $\sigma$ in a partial order $P$ which has a unique minimal element $\sigma_0 = \Delta_0$, Furthermore, assume that this decomposition is as follows:

(1) Each simplex belongs to exactly one $\Delta_\sigma$.
(2) For each $\sigma \in P$, $\bigcup_{\tau \leq \sigma} \Delta_\tau$ is a subsimplicial complex of $\Delta$.

For each $\sigma \in P$, let $M_\sigma$ be an acyclic matching in $\Delta_\sigma$. Then $\bigcup_{\sigma \in P} M_\sigma$ is an acyclic matching on $F(\Delta)$

## 3. RESULTS

We will start in Section 3.1 by examining simple homotopy types of both the pure Morse complex and the Morse complex through the usage of partitioning and matching, aided by the Cluster Lemma (Lemma 16). This is when we provide alternate proofs for results by both Ayala et. al [1] and Kozlov [12].

From there, we transition into working with strong collapses of a simplicial complex (Section 3.2). Here, we extend both the results by Ayala et. al. and Kozlov, while also providing a useful lemma (Lemma 28) that allows us to always perform certain strong collapses.

Afterward, we experiment with realizing $n-$cocktail party graphs in the Morse complex of a path, as well as compute the homotopy type of centipede (and similar) graphs (Sections 3.3 and 3.4).

We end our results by computing the homotopy type of a cycle with one leaf (Section 3.5), as well as providing a useful formula for computing the homotopy type of many paths with one leaf (Section 3.6).

3.1. **Utilizing Matchings to compute the Homotopy Type of a Morse Complex.** Let $K$ be any simplicial complex. If $a$ and $b$ are vertices of $K$, to represent a primitive gradient vector field, we sometimes write $(a)b := (a, ab)$.

In our results, we will refer to the ***pure Morse complex of*** $K$, denoted $\mathscr{M}_p(K)$. Let $n := \dim(\mathscr{M}(K))$. Then, $\mathscr{M}_p(K)$ is the subcomplex of $\mathscr{M}(K)$ generated by the facets of dimension $n$.

**Definition 17.** A ***tree***, $T$, is a connected graph that contains no closed loops (cycles).

**Remark 18.** We will utilize Lemma 16 for the alternate proof of Lemma 19 as follows:

We will create collections of the Morse complex, $\mathscr{M}(T)$, of a tree $T$, by separating it into its highest-dimensional facets $\sigma_i$. We order the facets such that each $\Delta_i$ collection is equal to $\sigma_i - \cup_{k=0}^{i-1}\sigma_k$. So, each gradient vector field is included in exactly one collection. Following this, we utilize Lemma 16 in order to put matchings on each collection, and glue them back together to put a matching on the whole $\mathscr{M}(T)$.

The following result was proved in [1, Corollary 4]. Here we offer an alternative proof.

**Lemma 19.** Let $T$ be a tree with at least 3 vertices. Then $\mathscr{M}_p(T)$ is collapsible.

*Proof.* Let $T$ be a tree with $v$ vertices. We define $V(T) := \{v_0, v_1, ..., v_n\}$. There is a bijection between the vertices of $T$ and the facets of $\mathscr{M}_p(T)$ [1, Proposition 2] so there are $v$ maximum gradient vector fields.

We can decompose $\mathscr{M}_p(T)$ into a collection of gradient vector fields, $\Delta_\sigma$, indexed by elements $\sigma$. We order $\sigma_0, ..., \sigma_{v-1}$ by the following:

(1) Let $\sigma_0, ..., \sigma_{v-1}$ be the elements of each maximum gradient vector field, respectively.
(2) Choose $\sigma_0$ to be the maximum gradient vector field with an inward-oriented vector on a ***leaf*** (vertex with a single edge), which we will call $(v_0)v_1$, and a critical vertex, $v_n$ on a leaf such that $d(v_n, v_0)$ (distance from $v_n$ to $v_0$) is maximized.

    We order $\sigma_1, ..., \sigma_{v-1}$ in relation to the distance of the critical vertex from $v_0$. So, let $\sigma_1$ be the maximum gradient vector field with critical vertex, $v_{n-1}$ such that $d(v_{n-1}, v_0) = d(v_n, v_0) - 1$. Let $\sigma_2$ be the maximum gradient vector field with critical vertex, $v_{n-2}$, such that $d(v_{n-2}, v_0) = d(v_{n-1}, v_0) - 1 = d(v_n, v_0) - 2$. Continue inductively until all elements, $\sigma$, are ordered.

    For at any point where there are two maximum gradient vector fields, $\sigma_i, \sigma_k$, with critical vertices, $v_i, v_k$, respectively, with equal distances from $v_0$, order them without loss of generality.
(3) Choose $\Delta_0 = \sigma_0$ to be the first subcollection.
(4) Let $\Delta_n = \sigma_n - \cup_{k=0}^{n-1}\Delta_k$ for $n > 0$.

Thus, each gradient vector field exists in exactly one $\Delta_\sigma$, and for each $\sigma$, $\cup_{\tau\leq\sigma}\Delta_\tau$ is a subcomplex of $\mathscr{M}_p(T)$.

Construct matchings, $M_\sigma$, on each $\Delta_\sigma$ by the following:

Choose a 0-simplex on $\Delta_0$ to be critical, which we will call $(v_0)v_1$. Match the remaining 0-simplices, $(v_i)v_j$ with the 1-simplices pairing them with $(v_0)v_1$, in the form $((v_0)v_1)((v_i)v_j)$. Match the remaining 1-simplices of the form $((v_i)v_j)((v_k)v_l)$ to the 2-simplices pairing them with $(v_0)v_1$ in the form $((v_0)v_1)((v_i)v_j)((v_k)v_l)$. Continue inductively until all $(k-1)$-simplices of the form $((v_{i_1})v_{i_2})...((v_{i_{k-1}})v_{i_k})$, not including $(v_0)v_1$, are matched with the corresponding $k$-simplex of the form $((v_0)v_1)((v_{i_1})v_{i_2})...((v_{i_{k-1}})v_{i_k})$.

Proceed similarly for $\Delta_2, ..., \Delta_{v-1}$ where $(v_1)v_0$ does not exist.

For $\Delta_{v-1}$ where $(v_1)v_0$ exists, match $(v_1)v_0$ with the 1-simplex of the form $((v_1)v_0)((v_i)v_j)$. Match the remaining 1-simplices of the form $((v_1)v_0)((v_k)v_l)$ to the corresponding 2-simplices, $((v_i)v_j)((v_1)v_0)((v_k)v_l)$. Continue inductively until all $(k-1)$-simplices, $((v_1)v_0)((v_{i_1})v_{i_2}), ..., ((v_{k-1})v_k)$ have been paired with the corresponding k-simplices, $((v_i)v_j)((v_1)v_0)((v_{i_1})v_{i_2}), ..., ((v_{k-1})v_k)$.

Because we have satisfied Lemma 16 and each of our $M_\sigma$ are acyclic, we know that $\cup_{\tau \leq \sigma} M_\sigma$ is an acyclic matching on $\mathscr{M}_p(T)$.

So, our only critical simplex is the original 0-simplex, $(v_0)v_1$, and thus $\mathscr{M}_p(T)$ is collapsible.

□

We can utilize a very similar ordering and matching scheme to the one used in the previous proof to find the homotopy type of many Morse complexes of trees. We know that the pure Morse complex is collapsible, so when we create our ordering, we simply put the simplicial complexes induced by maximum gradient vector fields first. Here we provide two examples.

Example

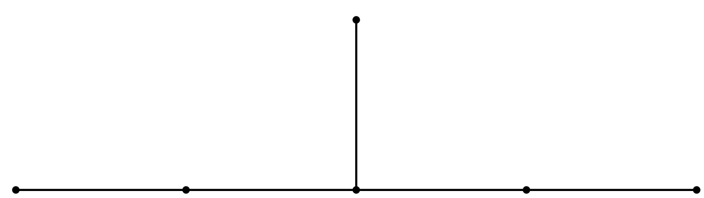

Figure 9. We have a simplicial complex, $K$, very similar to a typical path. It can be shown there is a gradient vector field on $\mathscr{M}(K)$ with $m_0 = 1, m_2 = 1$. Hence, $\mathscr{M}(K)$ has a homotopy type of $\mathbb{S}^2$ by Theorem 15.

Using a method of partitioning the gradient vector fields, as shown in the alternate proof for Lemma 19, it can be shown that after creating collections and applying matchings to each collection, a gradient vector field can be put on the Morse complex of $K$ (as seen in Figure 9) that has the homotopy type of $m_0 = 1, m_2 = 1$. We order the collections such that the first six form $\mathscr{M}_p(K)$. Thus, it is shown to be collapsible, only contributing one critical 0-simplex.

From there, we form collections using the simplicial complexes induced by the maximal gradient vector fields that are not maximum, and after applying matchings to these 7 collections, we find that there is a critical 2-simplex. Therefore, because the gradient vector field on the Morse complex has been maximized, by Theorem 15 $\mathscr{M}(K)$ has the homotopy type of $\mathbb{S}^2$.

EXAMPLE

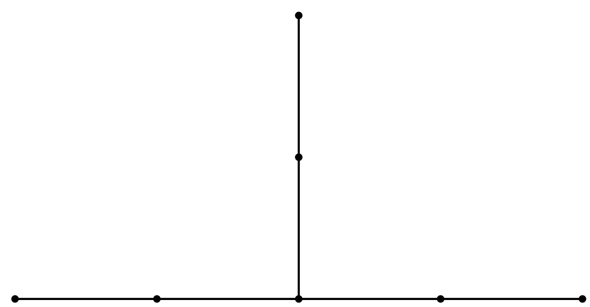

FIGURE 10. We have a simplicial complex, $K$, very similar to our last example. It can be shown there is a gradient vector field on $\mathscr{M}(K)$ with $m_0 = 1, m_3 = 2$. Hence, $\mathscr{M}(K)$ has a homotopy type of $\mathbb{S}^3 \vee \mathbb{S}^3$ by Theorem 15

Again, using a method of partitioning the gradient vector fields, as shown in the alternate proof for Lemma 19, it can be shown that after creating collections and applying matchings to each collection, a gradient vector field can be put on the Morse complex of $K$ ($K$ is as seen in Figure 10) with $m_0 = 1, m_3 = 2$. We order the collections such that the first seven form $\mathscr{M}_p(K)$, as there are only seven maximum gradient vector fields on $K$. Thus, it is shown to be collapsible, only contributing one critical 0-simplex.

After, we form collections using the simplicial complexes induced by the maximal gradient vector fields that are not maximum. Applying matchings to these 17 collections, we find that there are two critical 3-simplices. Therefore, because the gradient vector field on the Morse complex has been maximized, by Theorem 15 it is shown that the homotopy type is $\mathbb{S}^3 \vee \mathbb{S}^3$.

Now that we have introduced a matching strategy for putting a maximum gradient vector field on a Morse complex, we will examine the path graph. We can apply a different matching strategy for path graphs in order to compute the homotopy type, providing further insight into ways to pursue homotopy graphs of more involved complexes. Before continuing to our next result, we introduce new terminology.

**Definition 20.** Let $K$ be a simplicial complex and $v \in K$ be a vertex. The ***star*** of $v$ in $K$, denoted by $\text{st}(v)$, is the simplicial complex induced by the set of all simplices of $K$ containing $v$.

**Definition 21.** Let $K$ be a simplicial complex and $v \in K$ be a vertex. The ***link*** of $v$ in $K$, denoted by $\text{lk}(v)$, is the set $\text{lk}(v) := \text{st}(v) - \{\sigma \in K : v \in \sigma\}$.

**Definition 22.** Let $K$ be a simplicial complex with edges $\{a, b\}$ and $\{c, d\}$. Let $V_{\max}[(a)b, (c)d]$ be any maximal gradient vector field containing $(a)b$ and $(c)d$. We denote $\overline{\mathcal{V}_{\max}}[(a)b, (c)d]$ to be the set of all sub-simplices of all $V_{\max}[(a)b, (c)d]$.

The following is due to Kozlov [12], for which we provide an alternate proof.

**Proposition 23.** Let $P_t$ be the path on $t$ vertices, $t \geq 2$. Then

$$\mathscr{M}(P_t) \simeq \begin{cases} * & \text{if } t = 3n \\ \mathbb{S}^{2n-1} & \text{if } t = 3n+1 \\ \mathbb{S}^{2n} & \text{if } t = 3n+2 \end{cases}.$$

**Remark 24.** We will utilize Lemma 16 for the alternate proof of Proposition 23 as follows:

We will create collections of the Morse complex, $\mathscr{M}(P_t)$, of a path graph, $P_t$, by separating it into the star and link of certain vertices in $\mathscr{M}(P_t)$. We order these collections according to placement in the path of the primitive gradient vector fields which determine our collections. We start with the collection of a star and link of the inward-facing primitive gradient vector field on one of the leaves. We then utilize Lemma 16 in order to put matchings on each collection and glue them back together to put a matching on the whole $\mathscr{M}(P_t)$.

First, we will need a Lemma showing that our collections contain all possible gradient vector fields.

Let $v_0, v_2, \ldots, v_{t-1}$ denote the vertices of the path $P_t$ with $v_i$ adjacent to $v_{i+1}$ for every $0 \leq i \leq t-2$.

**Lemma 25.** Let $\sigma \in \mathscr{M}(P_t)$ be a maximal simplex. Then either

1. $(v_0)v_1 \in \sigma$
2. $(v_1)v_0$ and $(v_{3k-1})v_{3k-2}$ are in $\sigma$ for some $1 \leq k \leq \left\lfloor \frac{t}{3} \right\rfloor$
3. $(v_1)v_0$ and $(v_{3k-3})v_{3k-2}$ are in $\sigma$ for some $1 \leq k \leq \left\lfloor \frac{t+1}{3} \right\rfloor$
4. $(v_{3k-1})v_{3k}$ if $t = 3k+1$
5. $(v_{3k+1})v_{3k}$ if $t = 3k+2$

*Proof.* It is easy to see that the result holds for $t \leq 6$, so suppose that $t \geq 7$. Let $\sigma \in \mathscr{M}(P_t)$ be maximal. If $(v_0)v_1 \in \sigma$, we are done.

Observe that if $(v_0)v_1 \notin \sigma$, then $(v_1)v_0 \in \sigma$ since $\sigma$ is maximal by hypothesis. To show that the second condition of either (2) or (3) holds, suppose there is a $k$ such that neither $(v_{3k-1})v_{3k}$ nor $(v_{3k})v_{3k-1}$ is in $\sigma$. If $(v_{3k-2})v_{3k-1} \in \sigma$, then $(v_{3k-1})v_{3k} \in \sigma$ since $\sigma$ is maximal, a contradiction.

If there is no arrow on $v_{3k-2}v_{3k-1}$, then $(v_{3k-1})v_{3k} \in \sigma$ in order to make $\sigma$ maximal, again a contradiction. Hence, if there is no arrow on $v_{3k-1}v_{3k}$, then $(v_{3k-1})v_{3k-2} \in \sigma$ so condition (2) holds. Furthermore, a similar argument shows that $(v_{3k})v_{3k+1} \in \sigma$ so condition (3) also holds.

Otherwise, there is an arrow on edge $v_{3k-1}v_{3k}$ for every $k$. It is easy to see that if there are two arrows of the form $(v_{3k-1})v_{3k}$ and $(v_{3k+3})v_{3k+2}$, then at least one of $(v_{3k+2})v_{3k+1}$ or $(v_{3k})v_{3k+1}$ must be in $\sigma$ in order for $\sigma$ to be maximal. This corresponds to satisfying condition (2) and (3), respectively.

It is possible to organize these arrows on the edges $v_{3k-1}v_{3k}$ in such a way so that both conditions (2) and (3) are avoided. In this case, one of (4) or (5) must be satisfied depending on the value of $t$. Hence all maximal simplices are of the desired form.

□

We provide an illustration for Lemma 25 with Figure 11.

Example

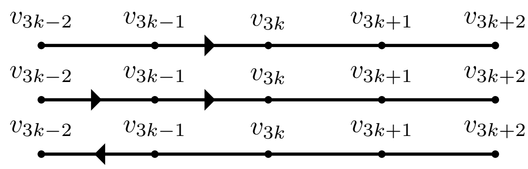


Figure 11. On our first path, it is evident that if there is no matching on $\{v_{3k-2}, v_{3k-1}\}$, we must have $(v_{3k-1})v_{3k}$, as this still allows for a maximal matching. Our second path, one can see that if $(v_{3k-2})v_{3k-1}$ is paired, there must be a matching on $\{v_{3k-1}, v_{3k}\}$ for there to exist a maximal gradient vector field. Our third path illustrates condition (2) in Lemma 25. By applying the matching $(v_{3k-1})v_{3k-2}$, this allows for no matching on $\{v_{3k-1}, v_{3k}\}$ while remaining maximal. A similar argument holds for condition (3) of Lemma 25.

We can now provide our alternate proof:

**Theorem 26.** Let $P_t$ be the path on $t$ vertices, $t \geq 3$. Then

$$\mathscr{M}(P_t) \simeq \begin{cases} * & \text{if } t = 3n \\ \mathbb{S}^{2n-1} & \text{if } t = 3n+1 \\ \mathbb{S}^{2n} & \text{if } t = 3n+2 \end{cases}$$

*Proof.* Define

$$\omega_t = \omega(t) = \begin{cases} 2n & \text{if } t = 3n \\ 2n & \text{if } t = 3n+1 \\ 2n+1 & \text{if } t = 3n+2 \end{cases}$$

We apply the Cluster Lemma (Lemma 16). In order to do so, we decompose $\mathscr{M}(P_t)$ into collections $\Delta_k$. As in Remark 24, we first construct collections of sub-simplices $\sigma_i$ as follows:

(1) Let $\sigma_0 := \text{st}((v_0)v_1) \cup \text{lk}((v_0)v_1)$.
(2) Let $\sigma_1 := \overline{\{V_{\max}[(v_1)v_0, (v_2)v_1]\}}$.
(3) Let $\sigma_2 := \overline{\{V_{\max}[(v_1)v_0, (v_3)v_4]}$.
(4) For $3 \leq j \leq \omega_t$, define the following.
   (a) If $j = 2k-1$, let
   $$\overline{\mathcal{V}}^{2k-1}_{\max} := \{V_{\max}[(v_1)v_0, (v_{3k-1})v_{3k-2}]\}.$$
   (b) If $j = 2k$, let
   $$\overline{\mathcal{V}}^{2k}_{\max} := \{V_{\max}[(v_1)v_0, (v_{3k})v_{3k+1}]\}.$$
   Let $\sigma_j := \overline{\mathcal{V}}^{j}_{\max}$.
(5) We have three cases:
   (a) If $t = 3k$, $\sigma_{\omega_t+1}$ is empty.
   (b) If $t = 3k+1, \sigma_{\omega_t+1} = \overline{\{V_{\max}[(v_1)v_0, (v_{3k-1})v_{3k}]\}}$

(c) If $t = 3k+2, \sigma_{\omega_t+1} = \overline{\{V_{\max}[(v_1)v_0, (v_{3k+1})v_{3k}]\}}$.

Now define $\Delta_0 := \sigma_0$ and $\Delta_j = \sigma_j - \cup_{k=0}^{j-1}\Delta_k$ for $1 \leq j \leq \omega_t + 1$. By Lemma 25, $\cup\sigma_j = \mathscr{M}(P_t)$ so that we may apply the Cluster Lemma (Lemma 16). We define an acyclic matching on each $\Delta_j$ as follows:

For $\Delta_0$, pair each element in $\sigma \in \mathrm{lk}((v_0)v_1)$ with $((v_0)v_1)\sigma \in \mathrm{st}((v_0)v_1)$. This is clearly an acyclic matching on all of $\Delta_0$ other than $(v_0)v_1$. Hence, $(v_0)v_1$ is a critical 0-simplex.

For $\Delta_1$, match $(v_1)v_0$ with $((v_1)v_0)((v_2)v_1)$ and match all $(i-1)$-simplices, $\sigma$, not containing $(v_2)v_1$, to the corresponding $i$-simplex, $\sigma((v_2)v_1)$. This produces an acyclic matching on $\Delta_1$ with no critical simplices.

For $\Delta_2$, note that because $(v_2)v_1$ is not a vertex of any simplex in $\sigma_2$, we have that $(v_2)v_3$ must be in every maximal simplex of $\Delta_2$. Hence, match the 1-simplex $((v_1)v_0)((v_2)v_3)$ with the 2-simplex $((v_1)v_0)((v_2)v_3)((v_3)v_4)$. Then we match all $(i-1)$-simplices $(\sigma)$ to the $i$-simplex, $\sigma((v_3)v_4)$. Therefore every simplex in $\Delta_2$ is matched, and we have no critical simplices.

Now let $3 \leq j \leq \omega_t$.

- Suppose $j = 2k-1$. To put an acyclic matching on $\Delta_{2k-1}$, notice that the maximal simplices of $\Delta_{2k-1}$ contain vertices $(v_1)v_0$ and $(v_{3k-1})v_{3k-2}$. The matching is given by pairing each $(i-1)$-simplex $\sigma$ to the corresponding $i$-simplex $\sigma((v_{3k-1})v_{3k-2})$ for all $\sigma \in \Delta_{2k-1}$ not containing $(v_{3k-1})v_{3k-2}$.
- Now suppose that $j = 2k$. Notice that the maximal simplices of $\Delta_{2k}$ contain $(v_1)v_0$ and $(v_{3k})v_{3k+1}$. The matching is given by pairing each $(i-1)$-simplex $\sigma$ to the corresponding $i$-simplex $\sigma((v_{3k})v_{3k+1})$ for all $\sigma \in \Delta_{2k}$ not containing $(v_{3k})v_{3k+1}$.

In either case, all simplices of $\Delta_j$ are matched, $3 \leq j \leq \omega_t$, and we have no new critical simplices.

It remains to put an acyclic matching on the simplices of $\Delta_{\omega_t+1}$. We consider cases.

**Case 1:** Suppose $t = 3n$ for some integer $n \geq 1$. Then $\sigma_{\omega_t+1} = \emptyset$ so that all simplices in $\mathscr{M}(P_t)$ are matched above. Since the only critical simplex is the 0-simplex found in $\Delta_0$ only contributing one critical $0-$simplex, and thus, $\mathscr{M}(P_{3n}) \simeq *$ by Theorem 15.

**Case 2:** : Suppose $t = 3n+1$ for some integer $n \geq 1$. Then $\sigma_{\omega_t+1}$ is generated by all maximal gradient vector fields containing $(v_1)v_0$ and $(v_{3n-1})v_{3n}$. Passing to $\Delta_{\omega_t+1}$, we see that the only maximal simplex in $\Delta_{\omega_t+1}$ is

$$\{((v_1)v_0)((v_2)v_3)((v_4)v_3)\ldots((v_{3k-1})v_{3k})((v_{3k+1})v_{3k})\ldots((v_{3n-2})v_{3n-3})((v_{3n-1})v_{3n})\}.$$

Furthermore, all proper sub-simplices of this $2n-1$-simplex have been matched in some previous $\Delta_i$. It follows that $\Delta_{\omega_t+1}$ consists of a single $(2n-1)$-simplex which is critical, and thus by Theorem 15, $\mathscr{M}(P_{3n+1}) \simeq \mathbb{S}^{2n-1}$.

**Case 3:** : Finally, suppose $t = 3n+2$ for some integer $n \geq 1$. Let

$$V := \{((v_1)v_0)((v_2)v_3)((v_4)v_3)\ldots((v_{3k-1})v_{3k})((v_{3k+1})v_{3k})\ldots((v_{3n-2})v_{3n-3})((v_{3n+1})v_{3n})\}.$$

Then,

$$\Delta_{\omega_t+1} = \{V, V \cup \{(v_{3n-1})v_{3n}\}, V \cup \{(v_{3n})v_{3n-1}\}.$$

We match $V$ with $V \cup \{(v_{3n-1})v_{3n}\}$ leaving $V \cup \{(v_{3n})v_{3n-1}\}$ as a critical $2n$-simplex. Thus $\mathscr{M}(P_{3n+2}) \simeq \mathbb{S}^{2n}$ by Theorem 15.

□

It is easy to show that this holds for $P_t$, $t = 2$, and can be used an exercise for the reader, as it follows directly. Our $\omega_t$ will be equal to 2, as our $\sigma_0$ is the star and link of $(v_0)v_1$ and $\sigma_1$ is simply flipping the one arrow. After applying matchings (which we find cannot be done), we see that this fits our results, as $\mathscr{M}(P_2) \simeq \mathbb{S}^0$.

3.2. **Extending Results by Ayala et. al. and Kozlov.** Previously, [6, Proposition 17] gave rise to motivation for further investigation of strong collapsibility of the Morse complex. Here we act on that curiosity, and are able to extend the previous results.

First, we extend the result from Lemma 19.

As previously defined, we once again let $v_0, v_2, \ldots, v_{t-1}$ denote the vertices of the path $P_t$ with $v_i$ adjacent to $v_{i+1}$ for every $0 \leq i \leq t-2$.

**Theorem 27.** The pure Morse complex of a tree is strongly collapsible.

*Proof.* Let $T$ be a tree. By definition, $T$ has at least one leaf, which we will call $\{v_0, v_1\}$, with inward-oriented primitive gradient vector field $(v_0)v_1$ and outward-oriented primitive gradient vector field $(v_1)v_0$. We claim that if $(v_1)v_0$ is dominated by another primitive gradient vector field, then $(v_0)v_1$ dominates all remaining primitive gradient vector fields.

Let $\{v_1, v_2\}$ be the adjacent edge to $\{v_0, v_1\}$, with $(v_2)v_1$ being the primitive gradient vector field oriented towards $\{v_0, v_1\}$. Then $(v_1)v_0$ is dominated by $(v_2)v_1$, as whenever $(v_1)v_0$ exists in a maximum gradient vector field, so does $(v_2)v_1$. Thus, $(v_1)v_0$ cannot exist in a facet of $\mathscr{M}_p(T)$ without $(v_2)v_1$. Therefore, we can make a strong collapse by removing $(v_1)v_0$.

Now, we claim $(v_0)v_1$ dominates all remaining primitive gradient vector fields. Let $(v_i)v_j$ be a remaining primitive gradient vector field. Consider any facet that $(v_i)v_j$ exists in. Because we are only considering $\mathscr{M}_p(T)$, any maximum gradient vector field in which $(v_i)v_j$ exists also contains $(v_0)v_1$, as $(v_0)v_1$ is compatible with all primitive gradient vector fields besides $(v_1)v_0$ (which has already been removed). Thus, $(v_0)v_1$ dominates $(v_i)v_j$. Therefore, we can make a strong collapse by removing $(v_i)v_j$ from $\mathscr{M}_p(T)$.

We can repeat this process, applying a strong collapse for all vertices $(v_i)v_j$, until we are left with only $(v_0)v_1$. □

We provide an example of the strong collapses to illustrate this result in Figure 12.

Example

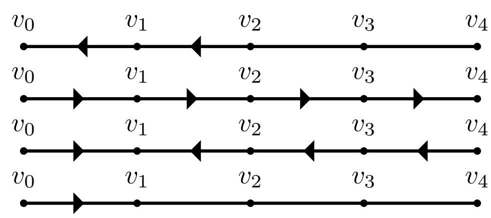


Figure 12. In our first path, $P_5$, we see that $(v_2)v_1$ dominates $(v_1)v_0$, as $(v_1)v_0$ cannot exist in a maximum gradient vector field without $(v_2)v_1$. Thus, we can apply a strong collapse on $(v_1)v_0$. In the second and third paths, we see that $(v_0)v_1$ dominates all remaining primitive gradient vector fields, and thus, we can apply strong collapses. So, we are left with $(v_0)v_1$ as our only remaining primitive gradient vector field. This corresponds to one vertex in $\mathscr{M}(P_5)$, and thus it is strongly collapsible.

Secondly, we extend the result from Proposition 23 and Theorem 26 by proving that a path $P_t$, $t = 3n$, has a strongly collapsible Morse complex. We provide a lemma that will aid in our proof.

**Lemma 28.** For any simplicial complex $K$ with leaf $\{a, b\}$ and $c$ a neighbor of $b$ not equal to $a$, then $(b)c$ is dominated in $\mathscr{M}(K)$.

*Proof.* Let $K$ be a simplicial complex with leaf $\{a, b\}$ and a neighbor of $b$, $c \neq a$. Consider any facet of $(b)c$ in $\mathscr{M}(K)$. A facet is a maximal gradient vector field on $K$, and thus, for any maximal gradient vector field in which $(b)c$ exists, so must $(a)b$. Thus, $(a)b$ dominates $(b)c$ in $\mathscr{M}(K)$. □

Example

Figure 13. On the left, we start with $C_3$. After performing the strong collapses that Lemma 28 allows, we then only have to take the Morse complex of the subcomplex on the right, which resembles a $3-$***ladder rung graph***. We will show in Corollary 39 and Corollary 40 that this yields $\mathscr{M}(C_3) \simeq \mathbb{S}^2$.

**Proposition 29.** Let $P_{3n}$ be the path on $3n$ vertices, $n \geq 1$. Then $\mathscr{M}(P_t) \searrow\searrow *$.

*Proof.* Let $P_t$ be the path with $t = 3n$ vertices, $n \geq 1$, and label the vertices as consistent with our previous proofs. By Lemma 28, $(v_0)v_1$ dominates $(v_1)v_2$. After removing $(v_1)v_2$, we see that $(v_2)v_1$ dominates $(v_3)v_2$, and so we remove $(v_3)v_2$. Continuing in this manner, we see that $(v_{3k-3})v_{3k-2}$ dominates $(v_{3k-2})v_{3k-1}$ for all $1 \leq k \leq n$, and $(v_{3k-1})v_{3k-2}$ dominates $(v_{3k})v_{3k-1}$ for all $1 \leq k < n$. Hence we may remove each of these primitive gradient vector fields.

Now the last primitive gradient vector field removed is $(v_{3n-2})v_{3n-1}$ since it was dominated by $(v_{3n-3})v_{3n-2}$. We now claim that $(v_{3n-1})v_{3n-2}$ dominates every remaining vertex. To see this, observe that because $(v_{3n-2})v_{3n-1}$ has been removed, $(v_{3n-1})v_{3n-2}$ is compatible with all remaining vertices $(v_i)v_j$, and no $(v_i)v_j$ can exist in a facet of the remaining Morse complex without $(v_{3n-1})v_{3n-2}$. We remove all $(v_i)v_j$ until we are only left with $(v_{3n-1})v_{3n-2}$. Thus $\mathscr{M}(P_t)$ is strongly collapsible. □

3.3. **Finding Cocktails in The Morse complex of a Path.** Before moving on, we must provide a few definitions.

We call a simplicial complex, $K$, ***minimal*** if it contains no dominating vertices. The ***core*** of $K$ is the minimal sub-complex $K_0 \subseteq K$ such that $K \searrow\searrow K_0$. Barmak [2, Theorem 5.1.10] proved that no matter the order of strong collapses, the core will be the same (the core is unique up to isomorphism).

**Definition 30.** The $n-$***cocktail party graph***, denoted $K_{n\times 2}$, is the complete $n$-partite graph where each partite set has size 2.

We illustrate two $n-$cocktail party graphs in Figure 14.

Example

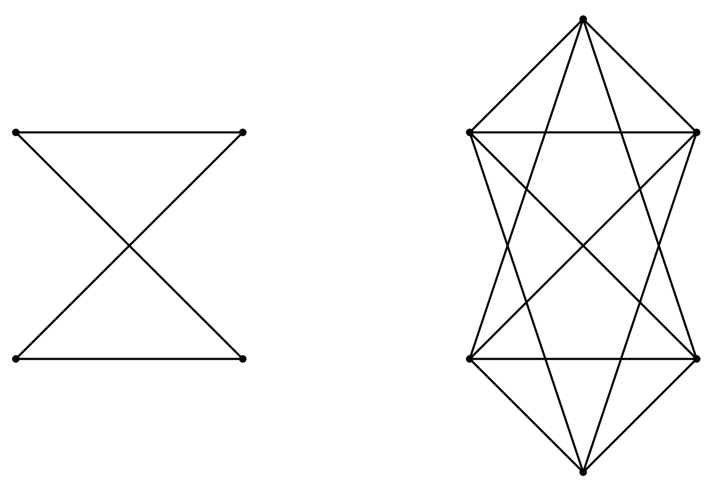

Figure 14. On the left, we have a $K_{2\times 2}$ cocktail party graph. On the right, we have a $K_{3\times 2}$ cocktail party graph.

The $n$***-skeleton of*** $K$, denoted by $K^{(n)}$, is the sub-complex of $K$ containing all simplices up to $n-$dimensional simplices.

For the following, let $\mathscr{M}_0(K)$ denote the core of the Morse complex of $K$.

**Proposition 31.** Let $P_t$ be the path on $t \geq 4$ vertices. Then

$$\mathscr{M}_0(P_t))^{(1)} = \begin{cases} K_{2k\times 2} & \text{if } t = 3k+1 \\ K_{(2k+1)\times 2} & \text{if } t = 3k+2 \end{cases}$$

*Proof.* Proceed as in the proof of Proposition 29 by removing dominated vertices from $\mathscr{M}(P_t)$ starting with $(v_1)v_2$. The last vertex removed along the path differs depending on $t$.

Suppose $t = 3k+1$. Then the last vertex removed from the Morse complex is $(v_{3n})v_{3n-1}$ as it is dominated by $(v_{3n-1})v_{3n-2}$. We claim there are no more dominated vertices. We show that for any remaining primitive gradient vector field $\sigma$, there exists a unique primitive gradient vector field $\tau$ such that $\sigma$ and $\tau$ are not compatible. If so, then $\sigma$ is compatible with every other primitive gradient vector field, thus creating a maximal simplex of $\sigma$ not containing $\tau$. Observe that $(v_0)v_1$ and $(v_1)v_0$ are not compatible with each other, and that $(v_2)v_1$ is not compatible with $(v_2)v_3$. Additionally, each remaining primitive gradient vector field, $(v_{3n})v_{1+3n}$ is not compatible with the corresponding $(v_{1+3n})v_{3n}$, as well as each remaining $(v_{2+3n})v_{1+3n}$ is not compatible with the corresponding $(v_{2+3n})v_{3+3n}$. Other than these incompatibilities, every primitive gradient vector field is compatible with any other. Thus no primitive gradient vector field can dominate another and we have arrived at $\mathscr{M}_0(P_t)$.

We now determine the structure of the 1-skeleton of $\mathscr{M}_0(P_t)$. There were $6k$ primitive gradient vector fields on $P_t$, and we removed $2k$ of these above, yielding $4k$ vertices in $\mathscr{M}_0(P_t)$. As determined above, every vertex of $\mathscr{M}_0(P_t)$ is compatible every other vertex of $\mathscr{M}_0(P_t)$ other than a unique vertex. In other words, there is an edge between vertex $v$ and every other vertex except a unique vertex $v'$. This is precisely the complete $2k$-partite graph with partite sets of size 2. Thus $\mathscr{M}_0(P_t))^{(1)} = K_{2k\times 2}$.

The $t = 3k+2$ case is similar, and so we omit it. □

3.4. **Classifying the Homotopy Type of Centipede Graphs.** We will continue to look closer at cases in which the homotopy type of the Morse complex is a sphere, as in the case of a path graph. The next results were inspired by considering centipede graphs. First, we will define a few key terms for these results.

**Definition 32.** Let $K$ and $L$ be two simplicial complexes with no vertices in common. Define the ***join*** of $K$ and $L$, written by $K * L$, by

$$K * L := \{\sigma, \tau, \sigma \cup \tau : \sigma \in K, \tau \in L\}.$$

**Definition 33.** Let $K$ be a simplicial complex with $v, w \notin K$, and let $w \neq v$. Define the **suspension** of $K$ by

$$\Sigma K := K * \{v, w\}$$

**Definition 34.** The $n-$***ladder rung graph***, denoted $nP_2$, is the disjoint union of $n$ copies of the $P_2$ graph.

**Definition 35.** A ***centipede graph***, $C_v$ is a graph obtained by adding a leaf to each vertex on a path graph $P_v$.

**Proposition 36.** [6, Proposition 23] Let $K$ and $L$ be disjointed connected simplicial complexes, each with at least one edge. Then $\mathscr{M}(K \sqcup L) = \mathscr{M}(K) * \mathscr{M}(L)$.

**Corollary 37.** [6, Corollary 24] Let $K$ be a simplicial complex. Then $\mathscr{M}(K \sqcup P_2) = \Sigma\mathscr{M}(K)$.

**Proposition 38.** For any simplicial complex, $K$, with $v$ vertices, in which we add a leaf to every vertex (we call the leaves $l_1, l_2, ..., l_v$) then, $\mathscr{M}(K \sqcup l_1 \sqcup l_2 \sqcup ... \sqcup l_v) \simeq \mathbb{S}^{v-1}$.

*Proof.* Let $K$ be a simplicial complex with $v$ vertices. Add a leaf, $\{a, b\}$ to each vertex. We will call these leaves, $l_1, l_2, ..., l_v$ We will call this expanded complex $K_l$, and its corresponding Morse complex $\mathscr{M}(K_l)$ Then for each neighbor of $b$, which we will call $c \neq a$, $(b)c$ is dominated in $\mathscr{M}(K_l)$ (Lemma 28). We added a leaf to each vertex, and so every primitive gradient vector field on $K$ has been dominated, and thus can be strongly collapsed in $\mathscr{M}(K_l)$.

By Proposition 36, this leaves us with

$$\mathscr{M}(l_1 \sqcup l_2 \sqcup ... \sqcup l_v) = \mathscr{M}(l_1) * \mathscr{M}(l_2) * ... * \mathscr{M}(l_v).$$

It is well-known [9] that $\Sigma\mathbb{S}^n = \mathbb{S}^{n+1}$. It follows that

$$\begin{aligned} \mathscr{M}(l_1) * \mathscr{M}(l_2) &= \mathbb{S}^1 \\ \mathbb{S}^1 * \mathscr{M}(l_3) &= \Sigma\mathbb{S}^1 \\ &= \mathbb{S}^2. \\ &\vdots \\ \mathbb{S}^{v-2} * \mathscr{M}(l_v) &= \Sigma\mathbb{S}^{v-2} \\ &= \mathbb{S}^{v-1} \end{aligned}$$

We continue inductively for all $l$, and we compute that $\mathscr{M}(K_l) \simeq \mathbb{S}^{v-1}$. □

The following corollaries are then immediate.

**Corollary 39.** Let $nP_2$ be an $n-$ladder rung graph. Then $\mathscr{M}(nP_2) \simeq \mathbb{S}^{n-1}$

**Corollary 40.** Let $C_v$ be a centipede graph. Then $\mathscr{M}(C_v) \simeq \mathbb{S}^{v-1}$.

**Corollary 41.** An $n-$sphere can be realized as the homotopy type of $\mathscr{M}(C_{n+1})$.

3.5. **Homotopy Type of a Cycle with a Leaf.** Using this idea of the join and suspension, and the intuition gained from the alternate proof in Lemma 25, we can prove the homotopy type of a cycle with a leaf. First, we provide a couple propositions.

**Proposition 42.** [6, Proposition 31] Let $v$ be a vertex of $C_n$. Then $\mathscr{M}(C_n \vee_v \ell) \searrow\searrow \mathscr{M}(P_{n-1} \sqcup \ell)$.

**Proposition 43.** [2, Proposition 5.1.16] Let $K$ and $L$ be disjoint simplicial complexes. Then $K * L$ is strongly collapsible if any only if $K$ or $L$ is strongly collapsible.

**Theorem 44.** Let $C_n$ be a cycle of length $n \geq 3$. Then

$$\mathscr{M}(C_n \vee l) \simeq \begin{cases} * & \text{if } n = 3k \\ S^{2k} & \text{if } n = 3k+1 \\ S^{2k+1} & \text{if } n = 3k+2. \end{cases}$$

*Proof.* Using prior results from [6] and [2], we have the following string of homotopy equivalences:

$$\begin{aligned}\mathscr{M}(C_n \vee l) &\simeq \mathscr{M}(P_{n-1} \sqcup l)\\ &\simeq \mathscr{M}(P_{n-1}) * \mathscr{M}(l)\\ &\simeq \Sigma\mathscr{M}(P_{n-1})\\ &\simeq \begin{cases} * & \text{if } n = 3k\\ S^{2k} & \text{if } n = 3k+1\\ S^{2k+1} & \text{if } n = 3k+2.\end{cases}\end{aligned}$$

The first equivalence is justified by Proposition 42. The second is justified by Proposition 36. The third is justified by Corollary 37. The last equivalences are justified by Proposition 43 and Proposition 23. □

Example

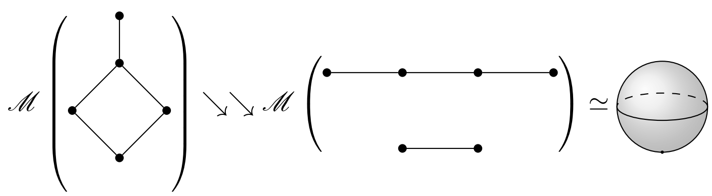

Figure 15.

### 3.6. **Homotopy Type of a Path with a Leaf.**

**Definition 45.** [6, Definition 26] Let $\mathbb{P}$ be the set of all (finite) posets, and let $\mathbb{K}$ be the set of all simplicial complexes. Define a function $f : \mathbb{P} \to \mathbb{K}$ as follows: for each $P \in \mathbb{P}$, construct a simplicial complex $f(P)$ whose vertex set is the edge set of $P$. Then let $\sigma = e_1e_2 \ldots e_k$ be a simplex of $f(P)$ if and only if the edges $e_1, e_2, \ldots, e_k$ oriented upward and all other edges oriented downward form an acyclic matching on $P$.

**Remark 46.** [6, Remark 27] Observe that $\mathscr{M}(K) \simeq f(\mathcal{H}(K))$. This generalizes our notion of taking the Morse complex of a degenerate Hasse Diagram, as will become evident in the proof of Lemma 47.

**Lemma 47.** Let $v_k$ be a vertex of $P_t$, $1 \le k \le t-2$ and $t \ge 3$. Then $\mathscr{M}(P_t \vee_{v_k} l) \searrow\searrow \mathscr{M}(P_{k+2} \sqcup P_{t-(k+2)} \sqcup l)$.

*Proof.* Write $l = v_k u$. By Lemma 28, $(u)v_k$ dominates $(v_k)v_{k+1}$ in $\mathscr{M}(P_t \vee_{v_k} l)$. In the corresponding Hasse diagram $\mathcal{H}(P_t \vee_{v_k} l)$, this corresponds to the removal of the edge between $v_k$ and $v_kv_{k+1}$.

Furthermore, by Lemma 28, $(v_t)v_{t-1}$ dominates $(v_{t-1})v_{t-2}$. This corresponds to the removal of the edge between $v_{t-1}$ and $v_{t-2}v_{t-1}$ on the Hasse diagram. Now, our Hasse diagram resembles

$$\mathcal{H}(P_{k+2}) \sqcup \mathcal{H}(P_{t-(k+2)}) \sqcup \mathcal{H}(l).$$

Therefore,

$$\mathscr{M}(P_t \vee_{v_k} l) \searrow\searrow f(\mathcal{H}(P_{k+2}) \sqcup \mathcal{H}(P_{t-(k+2)}) \sqcup \mathcal{H}(l)).$$

By Proposition 36 and Remark 46, we have that

$$f(\mathcal{H}(P_{k+2}) \sqcup \mathcal{H}(P_{t-(k+2)}) \sqcup \mathcal{H}(l)) \simeq \mathscr{M}(P_{k+2} \sqcup P_{t-(k+2)} \sqcup l).$$

□

Combining Lemma 47 and Proposition 36, we have

**Proposition 48.** Let $v_k$ be a vertex of $P_t$, $1 \leq k \leq t-2$. Then $\mathscr{M}(P_t \vee_{v_k} l) \simeq \mathscr{M}(P_{k+2}) * \mathscr{M}(P_{t-(k+2)}) * \mathscr{M}(l)$.

Considering Proposition 48, Proposition 43, and Proposition 29, we can conclude the following:

**Corollary 49.** Let $v_k$ be a vertex of $P_t$, $1 \leq k \leq t-2$. If $k+2 = 3j$ or $t-(k+2) = 3j$, then $\mathscr{M}(P_t \vee_{v_k} l) \searrow\searrow *$.

Example

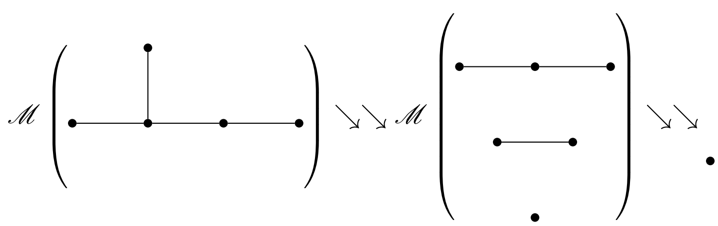

Figure 16.

## 4. Future Directions and Potential Pursuits

We want to devote the final section of this paper to raising some interesting questions about similar topics to those studied in this paper.

**Open Question 1.** Most immediately, we would like to raise the question of if there exists other ways to determine strong collapsibility in the Morse complex of a simplicial complex. Previously, it was shown that two leaves sharing a vertex on a simplicial complex, $K$, means that $\mathscr{M}(K)$ is strongly collapsible [6, Proposition 17]. Throughout our studies, we found multiple examples of graphs that had strongly collapsible Morse complexes but did not fit this description. Does there exist a way/ways to determine other structural causes of a strongly collapsible Morse complex?

We provide an example to demonstrate this question.

Example


Figure 17. $P_6$ is an example of a graph that has a collapsible Morse complex (proved in Theorem 26) that is also strongly collapsible (proved in Proposition 29).

By Lemma 28, we can see that $(v_1)v_2$ and $(v_4)v_3$ are both dominated in $\mathscr{M}(P_6)$. So, we can make strong collapses $\mathscr{M}(P_6) - (v_1)v_2 - (v_4)v_3 = \mathscr{M}_0(P_6)$. It follows immediately that $(v_3)v_2$ is dominated by $(v_2)v_1$ in $\mathscr{M}_0(P_6)$, and thus can be removed by a strong collapse.

It can then be noticed that $(v_3)v_4$ dominates the remaining primitive gradient vector fields. Therefore, a sequence of strong collapses can be made until only $(v_3)v_4$ is left, showing that $\mathscr{M}(P_6)$ is strongly collapsible.

Example

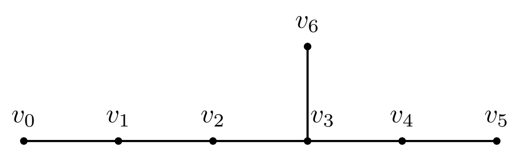


Figure 18. Here is a graph, $G$, very similar to that of $P_6$, and it still has a collapsible Morse complex. However, it can be shown that it is not strongly collapsible.

Using a matching scheme similar to that used in the proof for the homotopy type of a path, it can be shown that $G$ is collapsible. Our first collection is the star and link of $(v_0)v_1$, the next is $\overline{\mathcal{V}_{\max}}[(v_1)v_0, (v_2)v_1]$, the third is $\overline{\mathcal{V}_{\max}}[(v_1)v_0, (v_3)v_4]$, and the last is $\overline{\mathcal{V}_{\max}}[(v_1)v_0, (v_5)v_4]$. It follows that after putting matchings on each collection, $\mathscr{M}(G)$ is collapsible.

However, if we try to show that $G$ is strongly collapsible, we fail. We can start with strongly collapsing $(v_1)v_2, (v_4)v_3, (v_3)v_4$, and $(v_3)v_2$ by Lemma 28. However, then we are stuck. The two inner primitive gradient vector fields are not compatible, so they cannot dominate each other. They also cannot dominate any primitive gradient vector fields on the leaves as all primitive gradient vector fields on the leaves are compatible with both inner primitive gradient vector fields. Additionally, any primitive gradient vector fields on the leaves cannot dominate because all other primitive gradient vector fields are compatible with both the orientations on each leaf. So, we are stuck.

Here are two simplicial complexes which do not meet the condition in [6, Proposition 17]. The question exists of what determines that the first example has a Morse complex that is strongly collapsible, but the second does not? What is the structural cause, and can we generalize it for any other graphs or simplicial complexes?

**Open Question 2.** Secondly, we would like to further pursue using methods of matching, as illustrated in Lemma 19 and Theorem 26, to apply matchings to the Morse complexes of low-dimensional $n-$simplices. Most immediately, we would like to pursue the $3-$simplex. Chari and Joswig [5] showed that the $3-$simplex has $\{b_0 = 1, b_5 = 99\}$ using a software. However, this does not determine homotopy type. Can using these matching strategies, or one similar, provide further insight on how to apply a matching to the Morse complex of the $3-$simplex?

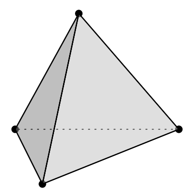

FIGURE 19. The $3-$simplex is still a relatively small complex, yet how may we even begin to visualize the Morse complex of the $3-$simplex?

**Open Question 3.** Lastly, and more abstractly, we are interested in how Reidemeister moves may affect the homotopy types of, both the Tait graph of a knot, as well as the Morse complex of the Tait graph of a knot. We have studied the effect that Reidemeister moves have on the Tait graph, and we will provide proofs for both type I and type II Reidemeister moves. First, let us informally define the Tait graph of a knot as well as the Reidemeister moves.

**Definition 50.** Let $D$ be a knot diagram. Its corresponding ***Tait graph***, denoted $\Gamma(D)$, is a graph corresponding to a checkerboard coloring of the knot diagram, where each region, as well as each crossing, corresponds to a vertex in $\Gamma(D)$. Then, each crossing has an edge connecting it to the vertices of its adjacent regions.

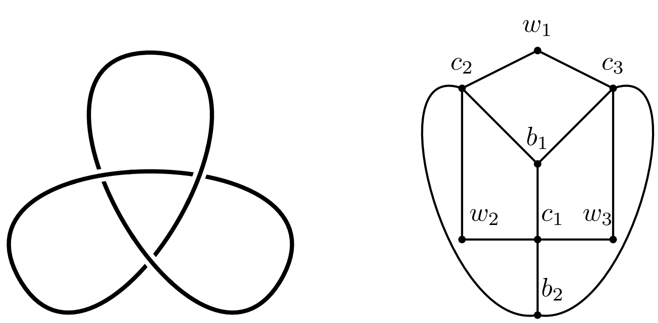


FIGURE 20. Here we have the Trefoil knot, $3_1$, and its corresponding Tait Graph, with crossings, and black and white regions, labeled.

**Definition 51.** Let $D$ be a knot diagram, with strand $s$ separating two regions. A ***Type I Reidemeister Move (RI)*** is the twisting of $s$, introducing a new crossing.

We illustrate a Type I Reidemeister move in Figure 21.

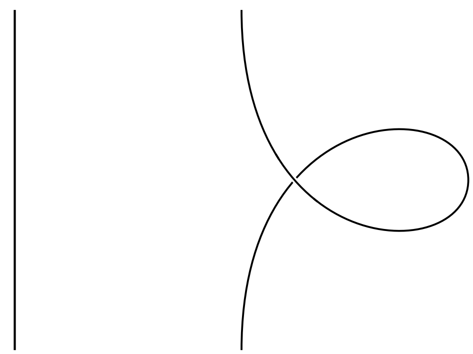

FIGURE 21. We illustrate a Type I Reidemeister move. It is a "twist" in the strand, introducing a new crossing and a new region.

**Definition 52.** Let $D$ be a knot diagram, with strands $s_1$ and $s_2$ such that they are not crossing. A ***Type II Reidemeister Move (RII)*** "pokes" $s_1$ either over or under $s_2$, introducing two new crossings.

We illustrate a Type II Reidemeister move in Figure 22.

FIGURE 22. We illustrate a Type II Reidemeister move. It is a "poke" move, meaning we slide one strand under or over the other, introducing two new crossings.

So, we want to investigate the effect these moves have on homotopy types, of both the Tait graph, and the Morse complex of the Tait graph. We have pursued the effect that these Reidemeister moves have on the Tait graph. We state those results, however, they are construction-based and so we define our result in the proofs.

For any region, $r_x$, in knot diagram $D$, we denote its corresponding vertex in the $\Gamma(D)$ by $v_{r_x}$. Also, we denote an edge between two vertices, $v_x, v_y$ in $\Gamma(D)$ by $v_x v_y$.

**Lemma 53.** Let $D$ be a knot diagram with Tait Graph $\Gamma(D)$. Then performing a Type I Reidemeister move on $D$ produces a new Tait Graph, $\Gamma(D_{RI}) = \Gamma(G_1)$.

*Proof.* Let $D$ be a knot diagram with Tait Graph $\Gamma(D)$. Let $s \in D$ be a strand which separates regions $r_1, r_2 \in D$.

We can perform a RI on $s$ by twisting the strand, protruding towards $r_1$, creating a new crossing, $c$, and a new region $r_3$. We can call knot projection $D$ with a RI, $D_{RI} = G_1$, with Tait graph $\Gamma(D_{RI}) = \Gamma(G_1)$.

Performing the RI, crossing $c$ and region $r_3$ are created.

Thus, $\Gamma(G_1)$ contains a new crossing vertex, $v_c$. From $v_c$, there will exist an edge connecting $v_c$ to $v_{r_3}$, $v_c v_{r_3}$; an edge connecting $v_c$ to $v_{r_2}$, $v_c v_{r_2}$; and two edges connecting $v_c$ to $v_{r_1}$, $v_c v_{r_1}$ and $v_c v_{r_1}{}'$.

Tait Graphs are planar so these new edges exist such that they do not overlap with any other edges.

All other elements of $\Gamma(D)$ are unchanged in $\Gamma(G_1)$ □

**Lemma 54.** Let $D$ be a knot diagram with Tait Graph $\Gamma(D)$. Then performing a Type II Reidemeister move on $D$ produces a new Tait Graph, $\Gamma(D_{RII}) = \Gamma(G_2)$.

*Proof.* Let $D$ be a knot diagram with Tait graph $\Gamma(D)$. Let there exist a strand $s_1 \in D$ between regions $r_1, r_2 \in D$ and a strand $s_2 \in D$ between regions $r_2, r_3 \in D$.

We can perform a RII by pushing $s_1$ across $r_2$, under/over (without loss of generality) $s_2$, and protruding into $r_3$. We can call this knot projection $D$ with a RII, $D_{RII} = G_2$, with Tait graph $\Gamma(D_{RII}) = \Gamma(G_2)$.

Performing the RII, two new crossings, $c_1, c_2 \in D$, are created, and two new regions, $r_4, r_5 \in D$, are created.

Thus, $\Gamma(G_2)$ contains a new crossing vertex $v_{c_1}$. From $v_{c_1}$, there exists edge $v_{c_1} v_{r_1}$, $v_{c_1} v_{r_2}$, $v_{c_1} v_{r_3}$, and $v_{c_1} v_{r_4}$. $\Gamma(G_2)$ also contains a new crossing vertex $v_{c_2}$. From $v_{c_2}$, there exists edge $v_{c_2} v_{r_1}$, $v_{c_2} v_{r_2}$, $v_{c_2} v_{r_3}$, and $v_{c_1} v_{r_5}$.

Tait graphs are planar, these new edges exist such that they do not overlap with any other edges.

All other elements of $\Gamma(D)$ are unchanged in $\Gamma(G_2)$. □

So, we restate our interest:

What effect, if any, do these moves have on the homotopy types of the Tait graph and the Morse complex of the Tait graph. We are specifically interested in these questions for the prime knots, as the prime knots are most heavily studied. Can the matching strategies utilizing Lemma 16 simplify this problem? If not, how might one need to approach this question?

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