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Copulas: an Open Field for Risk Management

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Copulas: an open field for risk management

1 Introduction

One of the main issues of risk management is the aggregation of individual risks. A powerful concept to aggregate the risks — the copula function — has been introduced in finance by Embrechts, McNeil, and Straumann [1999,2000]. In their papers, the authors clarify the essential concepts of dependence and correlation and certainly will greatly influence the risk management industry. The goal of this paper is to provide simple applications for the practical use of copulas for risk management from an industrial point of view. First, we remind some basics about copulas. Then, some applications of copulas for market risk, credit risk and operational risk are given. We will not provide a full mathematical treatment of the subject and we refer interested readers to Joe [1997] or Nelsen [1999].

2 Copulas definition

A copula is a function that links univariate margins to the full multivariate distribution. Then, this function is the joint distribution function of N standard uniform random variables. Mathematically speaking, a function $\mathbf{C}$ is a copula function if it fulfills the following properties (Nelsen [1999]):

1. $\operatorname{Dom} \mathbf{C} = [0,1]^N$;
2. $\mathbf{C}$ is grounded and N-increasing[1];
3. The margins $\mathbf{C}_n$ of $\mathbf{C}$ satisfy $\mathbf{C}_n(u) = \mathbf{C}(1,\ldots,1,u,1,\ldots,1) = u$ for all u in $[0,1]$.

This class of function is very important because it permits to define the dependence structure between the margins of a multivariate distribution. Indeed, let think about N random variables $(X_1,\ldots,X_N)$ with multivariate distribution $\mathbf{F}$ and univariate margins $(\mathbf{F}_1,\ldots,\mathbf{F}_N)$. Then we have the canonical decomposition

$$\mathbf{F}(x_1,\ldots,x_N) = \mathbf{C}(\mathbf{F}_1(x_1),\ldots,\mathbf{F}_N(x_N))$$

Moreover, Abe Sklar proved in 1959 that the copula $\mathbf{C}$ is unique for a given distribution $\mathbf{F}$ if the margins are continuous. To illustrate the idea behind the copula function, we can think about the multivariate gaussian that is a 'standard' assumption in risk management. To postulate that a vector $(X_1,\ldots,X_N)$ is gaussian is equivalent to assume that:

1. the univariate margins $\mathbf{F}_1,\ldots,\mathbf{F}_N$ are gaussians;
2. these margins are linked by a unique copula function $\mathbf{C}$ (called Normal copula) such that:

$$\mathbf{C}_\rho(u_1,\ldots,u_N) = \Phi_\rho\left(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_N)\right) \tag{1}$$

   with Φ_ρ the multivariate normal cdf with correlation matrix ρ and Φ^{-1} the inverse of the standard univariate gaussian distribution.

It appears that the risk can be splitted into two parts: the individual risks and the dependence structure between them. Indeed, the assumption of normality for the margins can be removed and $\mathbf{F}_1,\ldots,\mathbf{F}_N$ may be fat-tailed distributions (e.g. Student, Weibull, Pareto) and the dependence may still be characterized by a Normal copula. This leads to a new multivariate distribution that takes into account, for example, the leptokurtic property of asset returns. This is illustrated by Figure 1.

From standard textbooks, we know that the density f of the distribution $\mathbf{F}$ is its N-derivative, if it exists:

$$f(x_1,\ldots,x_N) = \frac{\partial\, \mathbf{F}(x_1,\ldots,x_N)}{\partial x_1 \cdots \partial x_N}$$

[1] These properties mean that $\mathbf{C}$ is a positive probability measure.

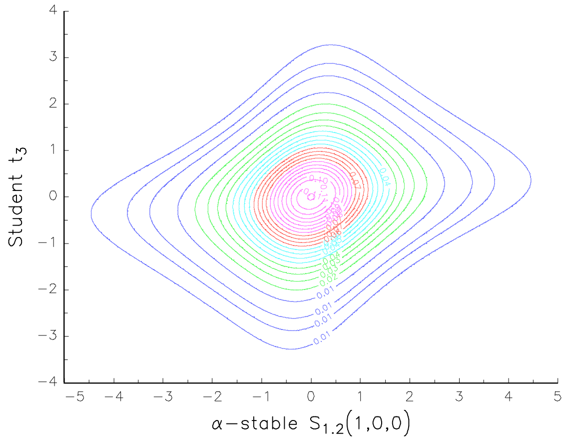


Figure 1: Contour of the bivariate density of two random variables with the first margin α-stable, the second margin Student and a Normal copula with $\rho = 0.3$.

Let f_n be the density function that corresponds to the n-th margin. The expression of the copula density c is

$$c(u_1, \ldots, u_N) = \frac{\partial\, C(u_1, \ldots, u_N)}{\partial\, u_1 \cdots \partial\, u_N}$$

It comes that the canonical decomposition of the density of F is

$$f(x_1, \ldots, x_N) = c(F_1(x_1), \ldots, F_N(x_N)) \times \prod_{n=1}^{N} f_n(x_n)$$

The formula of the Normal copula density is obtained by derivating equation (1):

$$c(u_1, \ldots, u_N; \rho) = \frac{1}{|\rho|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\varsigma^{\top}\left(\rho^{-1} - I\right)\varsigma\right)$$

with $\varsigma = (\varsigma_1, \ldots, \varsigma_N)$ where $\varsigma_n = \Phi^{-1}(u_n)$ for $n = 1, \ldots, N$. Other type of dependence could be postulated, for example the Student copula which has the following density[2]

$$|\rho|^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\nu+N}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^N}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^N \Gamma\left(\frac{\nu}{2}\right)} \frac{\left(1 + \frac{1}{\nu}\varsigma^{\top}\rho^{-1}\varsigma\right)^{-\frac{\nu+N}{2}}}{\prod_{n=1}^{N}\left(1 + \frac{\varsigma_n^2}{\nu}\right)^{-\frac{\nu+1}{2}}}$$

with $\varsigma_n = t_\nu^{-1}(u_n)$, Γ the gamma function and ν the degrees of freedom.

[2] We refer to Bouyé, Durrleman, Nickeghbali, Riboulet and Roncalli [2000] for the proof.

To illustrate the difference between the Normal copula and the Student copula, we have plotted bivariate simulations (i) with gaussian margins (ii) with normalized Student margins (such that the variances are the same).

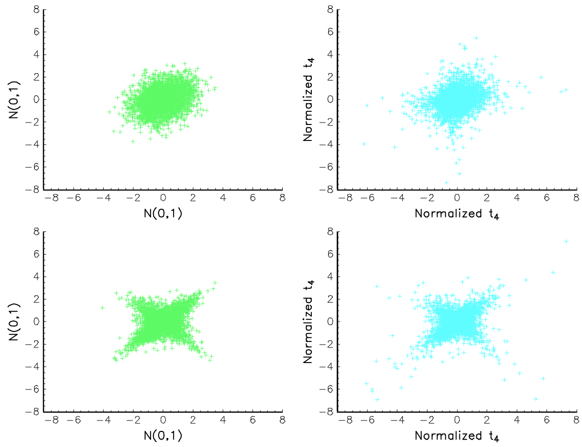


Figure 2: Monte Carlo simulation (5000 simulations) with respectively a Normal copula ($\rho = 0.3$) — upper plots — and a Student copula ($\rho = 0.3$, $\nu = 1$) — lower plots. The left plots assume gaussian margins whereas the right plots assume normalized Student margins.

3 Market risk management

The copula methodology can be applied both to compute Value at Risk (VaR) and to perform stress testing. The two approaches are explained.

3.1 VaR for portfolios

As noted by Embrechts, McNeil and Straumann [2000], the correlation is a special case through all measures that are available to understand the relationships between all the risks. If we assume a Normal copula, the empirical correlation is a good measure of the dependence only if the margins are gaussians. To illustrate this point, we can construct two estimators:

1. The empirical correlation $\hat{\rho}$;
2. The canonical correlation $\hat{\rho}_{\mathrm{CML}}$ obtained as follows: the data are mapped to empirical uniforms and transformed with the inverse function of the gaussian distribution. The correlation is then computed for the transformed data[3].

[3] $\hat{\rho}_{\mathrm{CML}}$ is also called the 'omnibus estimator'. It is consistent and asymptotically normally distributed (Genest, Ghoudi and Rivest [1995]).

The advantage of the canonical measure is that no distribution is assumed for individual risks. Indeed, it can be shown that a misspeci…cation about the marginal distributions (for example to assume gaussian margins if they are not) leads to a biased estimator of the correlation matrix. This is illustrated by the following example for asset returns. The database of the London Metal Exchange[4] is used and the spot prices of the commodities Aluminium Alloy (AL), Copper (CU), Nickel (NI), Lead (PB) and the 15 months forward prices of Aluminium Alloy (AL-15), dating back to January 1988, are considered. The two correlation measures of asset returns are reported below[5].

	AL	AL-15	CU	NI	PB
AL	1.00	0.82	0.44	0.36	0.33
AL-15		1.00	0.39	0.34	0.30
CU			1.00	0.37	0.31
NI				1.00	0.31
PB					1.00

Table 1: Correlation matrix ½ of the LME data

	AL	AL-15	CU	NI	PB
AL	1.00	0.85	0.49	0.39	0.35
AL-15		1.00	0.43	0.35	0.32
CU			1.00	0.41	0.36
NI				1.00	0.33
PB					1.00

Table 2: Correlation matrix $½_{CML}$ of the LME data

° = 2	AL	AL-15	CU	NI	PB
AL	1.00	0.82	0.33	0.25	0.19
AL-15		1.00	0.27	0.22	0.16
CU			1.00	0.27	0.22
NI				1.00	0.20
PB					1.00

Table 3: Correlation matrix $½_{ML}$ with Student copula (° = 1) of the LME data

Even if we assume that the margins are gaussians, we will show that the choice of the dependence structure has a great impact on the VaR computation of a portfolio. If we consider that the dependence of the LME data is a Student copula with 1 degree of freedom[6], the obtained parameter matrix (see Table 3) di¤ers from the Normal one[7] of Table 1. Then, let consider a portfolio a with P (t) the price vector of the assets at time t. The one period value-at-risk with ® con…dence level is de…ned by V aR = F i 1 (1 ¡ ®) with F the distribution of the random variate a> (P (t + 1) ¡ P (t)). Let assume we have three di¤erent portfolios (a negative number corresponds to a short position):

	AL	AL-15	CU	NI	PB
P_1	1	1	1	1	1
P_2	-1	-1	-1	1	1
P_3	2	1	-3	4	5

For these three portfolios, we assume that the margins are gaussians and compare[8] the VaRs under the assumption of Normal copula and Student copula with ° = 1. The higher the quantile[9], the more the Student dependence leads to a higher VaR (see Tables 4 and 5). An interesting point is that for the three portfolios and for a low level quantile (for example 90%), the Student copula leads to lower VaRs. In Table 6, we have reported the VaR when the copula is Normal and the margins are Student. If we compare this table with Table 4, we remark the impact of the choice of fat-tailed distributions on the VaR computation[10]. Note that if no analytical formula is available for the VaR computation, the results are obtained by simulation.

	90%	95%	99%	99.5%	99.9%
P_1	7.26	9.33	13.14	14.55	17.45
P_2	4.04	5.17	7.32	8.09	9.81
P_3	13.90	17.82	25.14	27.83	33.43

Table 4: VaR with Normal copula

	90%	95%	99%	99.5%	99.9%
P_1	5.69	7.95	13.19	15.38	20.06
P_2	3.82	5.55	9.75	11.65	16.41
P_3	13.41	19.36	34.16	40.55	54.48

Table 5: VaR with Student copula (° = 1)

[4] The database is available on the web site of the LME http://www.lme.co.uk.

[5] The standard errors are not reported here. However, the correlations of Table 2 and Table 3 are in italics if they are signi…cantly di¤erent from Table 1 at 5% con…dence level.

[6] We use the iterative algorithm described in [2] and [5] to estimate the parameters matrix ½.

[7] But if ° is equal 2, only three parameters among eleven are signi…cantly di¤erent at 5% con…dence level.

[8] All the parameters are estimated using maximum likelihood method.

[9] We have reported the VaR for the 99.9% quantile, which approximately corresponds to the rating target A.

[10] For low level quantiles (90% and 95%), we have lower VaRs whereas higher quantiles produce bigger VaRs.

	90%	95%	99%	99.5%	99.9%
P_1	6.51	8.82	14.26	16.94	24.09
P_2	3.77	5.00	7.90	9.31	13.56
P_3	12.76	17.05	27.51	32.84	49.15

Table 6: VaR with Normal copula and Student margins ($\nu = 4$)

Finally, we give in Table 7 an idea about the computational time needed to estimate the value-at-risk based on a Normal copula and Student margins. The number of simulations is 100000 and the computation has been done with the GAUSS software and a Pentium III 550 Mhz. These times are given for indication since the number of simulations is constant and does not depend on the number of assets (the problem of dimensionality is not treated).

Number of assets	Computational time
2	0.1 sc
10	24.5 sc
100	4 mn 7 sc
500	33 mn 22 sc
1000	1 hr 44 mn 45 sc

Table 7: Computational time for computing VaR

3.2 Stress testing

The extreme value theory is now familiar to practitioners. It allows, for example, to apply stress scenarios to a portfolio. However, the extension to the multivariate case is a difficult issue. There exists a special class of copula function that avoids the problem. Indeed, any copula function $C_\star$ such that

$$C_\star\left(u_1^t,\ldots,u_N^t\right) = C_\star^t(u_1,\ldots,u_N) \qquad \forall t > 0 \qquad (2)$$

can be used to construct a multivariate extreme value distribution (**Deheuvels** [1978]). We just write the equations for maxima as the problem is identical for minima. The maxima are defined componentwise

$$\chi_m^+ = \left(\chi_{1;m}^+,\ldots,\chi_{N;m}^+\right) := \left(\bigvee_{k=1}^{m} X_{1;k},\ldots,\bigvee_{k=1}^{m} X_{N;k}\right)$$

For each maxima $\chi_{n;m}^+$, its univariate generalized extreme value (GEV) distribution G_n with

$$G_n\left(\chi_{n;m}^+\right) = \exp\left\{-\left[1 + \xi_n\left(\frac{\chi_{n;m}^+ - \mu_n}{\sigma_n}\right)\right]^{-\frac{1}{\xi_n}}\right\}$$

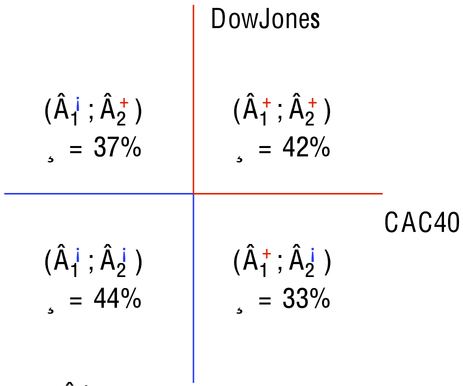
can be estimated for $n = 1,\ldots,N$. Then, the multivariate extreme value distribution for maxima G is

$$G\left(\chi_1^+,\ldots,\chi_N^+\right) = C_\star\left(G_1\left(\chi_1^+\right),\ldots,G_N\left(\chi_N^+\right)\right)$$

To illustrate how this result can be used for risk management, we consider an example which focuses on the extremes of the pair (CAC40,DowJones). First, the GEV univariate distributions are estimated for maxima and minima of CAC40 and DowJones respectively (that makes four estimations). Then, let assume a copula that fulfills the condition (2), for example the Gumbel copula:

$$C_\star(u_1,u_2) = \exp\left(-\left[(-\ln u_1)^\delta + (-\ln u_2)^\delta\right]^{\frac{1}{\delta}}\right)$$

with δ the dependence parameter ($\delta = 1$ for independence and $\delta = \infty$ for fully dependent extrema). It is then possible to construct a failure area that corresponds to the set of values $\left(\chi_1^+,\chi_2^+\right)$ such that $\Pr\left\{\chi_1^+ > \chi_1, \chi_2^+ > \chi_2\right\} = 1 - G_1(\chi_1) - G_2(\chi_2) + C_\star(G_1(\chi_1), G_2(\chi_2))$ equals a given level of probability. By applying the same methodology to the three other pairs (min/max, max/min and min/min), one can construct the failure area from the estimation of the dependence for the four quadrants of (CAC40,DowJones)



with χ_1^- and χ_2^- the minima. To characterize the dependence of extremal risks, the upper tail dependence coefficient λ (see **Joe** [1997]) is used:

$$\lambda = \lim_{\alpha \to 1} \Pr\left\{X_1 > F_1^{-1}(\alpha) \mid X_2 > F_2^{-1}(\alpha)\right\}$$

We can interpret λ as the probability that one random variable is extreme given that the other is extreme. In our example, the dependence of minima is not significantly different from the dependence of maxima, which means that bear markets are quite similar to bull markets from an economic point of view. Figure 3 provides an example for a probability that is equivalent to a 5 years waiting time. We remark that some past extremal events have a waiting time bigger than 5 years.

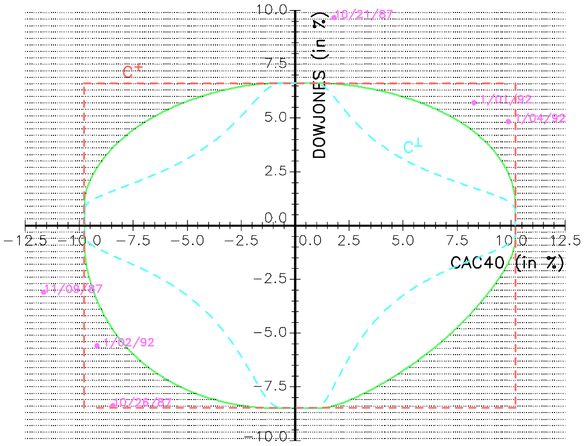


Figure 3: Failure areas for the pair (CAC40,DowJones) and a 5 years waiting time. Note that $C^{\perp}$ and C^{+} correspond to the cases of independence and perfect positive dependence between the two asset returns.

4 Credit risk management

One of the main issue concerning credit risk is without doubt the modelling of joint default distribution. Li [2000] and Maccarinelli and Maggiolini [2000] suggest that copulas could be a suitable tool for such a problem. Indeed, a default is generally described by a survival function $S(t) = \Pr\{T > t\}$, which indicates the probability that a security will attain age t. T is a random variable called the survival time, which is denoted time-until-default in Li [2000]. Let $\breve{\mathbf{C}}$ be a survival copula. A multivariate survival distribution S can be defined as follows

$$S(t_1,\ldots,t_N) = \breve{\mathbf{C}}(S_1(t_1),\ldots,S_N(t_N)) \qquad (3)$$

where $(S_1,\ldots,S_N)$ are the marginal survival functions. Nelsen [1999] notices that "C couples the joint survival function to its univariate margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins". Introducing correlation between defaultable securities can then be done using the copula framework.

4.1 Computing the risk of a credit portfolio

Using the previous framework, it is then possible to compute risk measure (or economic capital) of any

	Final rating							
Initial rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.54	6.48	0.86	0.06	0.06	0.00	0.00	0.0
AA	0.63	91.87	6.64	0.65	0.06	0.11	0.04	0.0
A	0.08	2.26	91.66	5.11	0.61	0.23	0.01	0.0
BBB	0.05	0.27	5.84	87.74	4.74	0.98	0.16	0.2
BB	0.04	0.11	0.64	7.85	81.14	8.27	0.89	1.0
B	0.00	0.11	0.30	0.42	6.75	83.07	3.86	5.4
CCC	0.19	0.00	0.38	0.75	2.44	12.03	60.71	23.5

Table 8: S&P one-year transition matrix (in %)

portfolio of risky securities. Thus, one could remark for instance that the CreditMetrics methodology implicitly uses the Normal Copula in (3) for their credit risk measure (Li [2000]). Indeed, in this (structural) approach the distribution of the joint default is obtained from the Asset Value Model of Merton where underlyings are assumed to be gaussian. To show that the dependence function has a great impact on the computation of the risk of a credit portfolio, we consider the example of joint default probability in the CreditMetrics framework with the one-year transition matrix of Table 8. In Figure 4, we remark that even if the copulas has the same Kendall's tau[11], we can obtain very different joint default probabilities, and of course very different credit risk VaRs.

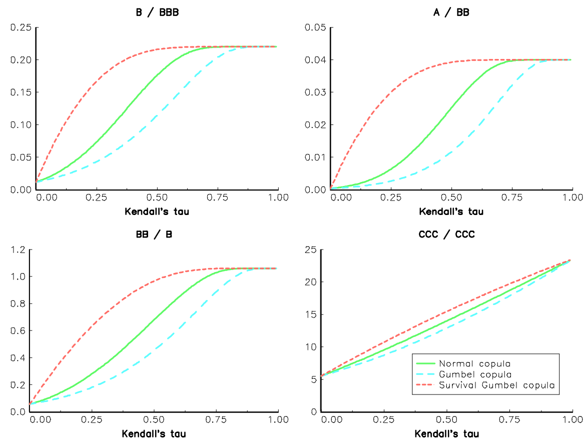


Figure 4: One-year joint default probabilities (in %). In order to compare them, we use Kendall's tau.

In the case of CreditRisk+, Coutant, Martineu, Messines, Riboulet and Roncalli [2001] show that the dependence function between defaults is related to a special class of Archimedean copulas called frailty models[12]. For alternative approach, we refer to the

[11] It is one of the most known measure to compare the concordance between copulas.

[12] For example, if the factors are Gamma distributed, the dependence function between defaults is the Cook-Johnson copula.

works of survival analysis and multivariate exponential distributions which provide a starting point for many extensions (see the survey [9]).

4.2 Pricing credit derivatives

Copulas may also apply to the pricing of credit derivatives. One may for instance consider the case of a contingent claim that depends on the …rst default among a list of N credit events (such an option is called a …rst-to-default). For simplicity, we assume here that the default of each credit event is given by the same Weibull survival function. In Figure 5, we have represented the hazard rate, the survival function, the mean residual time-until-default and the density[13].

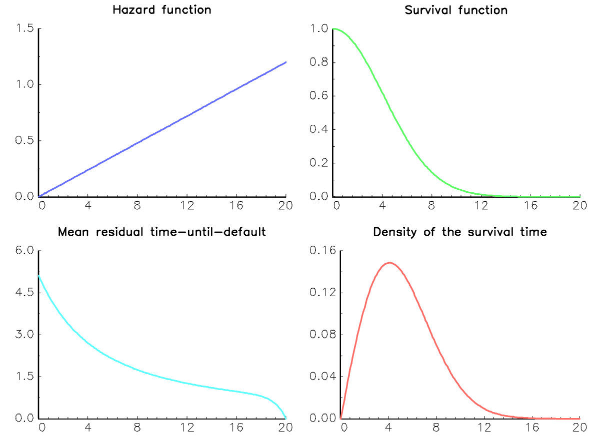


Figure 5: Weibull survival time.

Let us de…ne the …rst-to-default time ¿ as follows

$$¿ = \min(T_1, \ldots, T_N)$$

Nelsen [1999] shows that the survival function of ¿ is given by the diagonal section of the survival copula[14]:

$$S(¿) = \breve{C}(S_1(¿), \ldots, S_N(¿))$$

Figure 6 shows the in‡uence of the correlation parameter ½ of the Normal copula and the in‡uence of the number of securities N on the density of ¿. In Figure 7, we have reported the premium of the …rst-to-default option $1_{[¿ \le T]}$ in the case of deterministic interest rates. In the left plot, the maturity of the option T is two years. In the right plot, we take two securities. As noted by Coutant, Martineu, Messines, Riboulet and Roncalli [2001], we can …nd an analytical formula for the density of ¿ in the case of the Normal copula and compute easily the option prices even if the interest rates are stochastic thanks to numerical quadrature integration.

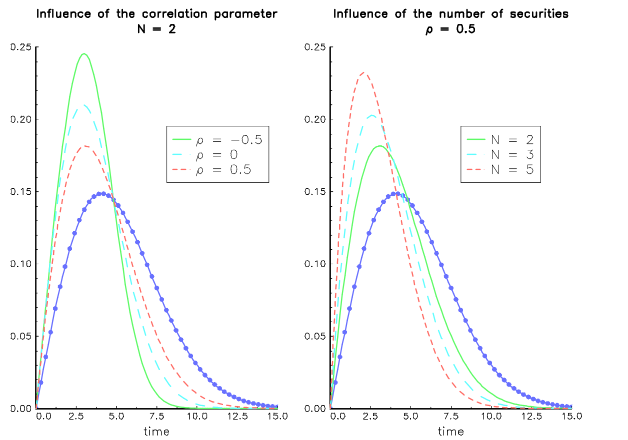


Figure 6: Density of the …rst-to-default. The solid line with circles corresponds to the density of one survival time.

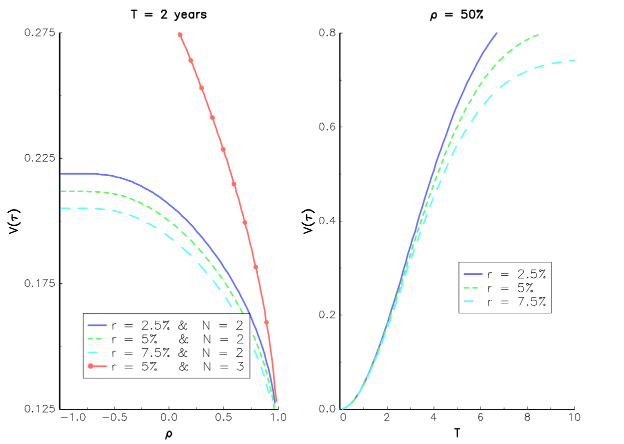


Figure 7: Premium of the …rst-to-default option.

[13] We assume that the baseline hazard is constant and equal to 3% per year and that the Weibull parameter is 2.

[14] Note that density of ¿ is then given by

$$f(¿) = \sum_{n=1}^{N} \partial_n \breve{C}(S_1(¿), \ldots, S_N(¿)) \times f_n(¿)$$

where f_n is the density of the survival time T_n.

5 Operational risk management

One of the standard measurement methodology for operational risk with internal data is the following[15]:

- Let ξ be the random variable that describes the severity of loss. We define also $\xi^k(t)$ as the random process of ξ for each operational risk k ($k = 1, \ldots, K$).
- For each risk, we assume that the number of events at time t is a random variable $N_k(t)$.
- The loss process $\vartheta(t)$ is also defined as

$$\vartheta(t) = \sum_{k=1}^{K} \sum_{j=1}^{N_k(t)} \xi_j^k(t) \tag{4}$$

- The Economic Capital with an α confidence level is usually defined as

$$\mathrm{EC} = \mathbf{F}^{-1}(\alpha) \tag{5}$$

with $\mathbf{F}^{-1}$ the inverse function of the loss distribution $\vartheta(t)$. This methodology can be viewed as the Loss Distribution Approach proposed by the Basel Committee on Banking Supervision (see document [1]). In the New Basel Capital Accord, dependence effects in operational risk are not considered:

> The capital charge is based on the simple sum of the operational risk VaR for each business line/risk type cell. Correlation effects across the cells are not considered in this approach (annex 6 of [1]).

But, from the point of view of economic capital allocation, "correlation effects" are a keypoint of the operational risk measure. One possibility is then to introduce dependence by using correlations between frequencies of different types of risk. Each individual frequency $N_k(t)$ is generally assumed to be a Poisson variable $\mathcal{P}$ with mean λ_k. However, multivariate Poisson distributions are relatively complicated for dimensions higher than two. Song [2000] suggests then an interesting alternative by using copulas. Assuming a Normal copula, we note $\mathcal{P}(\lambda, \rho)$ the multivariate Poisson distribution generated by the Normal copula with parameter ρ and univariate Poisson distribution $\mathcal{P}(\lambda_k)$. The next table contains the probability mass function $p_{i,j} = \Pr\{N_1 = i, N_2 = j\}$ of the bivariate Poisson distribution $\mathcal{P}(\lambda_1 = 1, \lambda_2 = 1, \rho = 0.5)$.

$p_{i,j}$	0	1	2	$\cdots$	$p_{i,\cdot}$
0	0.095	0.133	0.089		0.368
1	0.034	0.100	0.113		0.368
2	0.006	0.031	0.052		0.184
$\vdots$					$\vdots$
$p_{\cdot,j}$	0.135	0.271	0.271	$\cdots$	1

If $\rho = -0.5$, we obtain the following values for $p_{i,j}$.

$p_{i,j}$	0	1	2	$\cdots$	$p_{i,\cdot}$
0	0.014	0.062	0.101		0.368
1	0.044	0.112	0.111		0.368
2	0.044	0.068	0.046		0.184
$\vdots$					$\vdots$
$p_{\cdot,j}$	0.135	0.271	0.271	$\cdots$	1

The Economic Capital $\mathrm{EC} = \mathbf{F}^{-1}(\alpha)$ with an α confidence level for operational risk could then be calculated by assuming that $N = \{N_1, \ldots, N_K\}$ follows a multivariate Poisson distribution $\mathcal{P}(\lambda, \rho)$. Moreover, there are no computational difficulties, because the estimation of the parameters λ and ρ is straightforward and the quantile can be easily obtained with Monte Carlo methods. Figure 8 illustrates the simulation of a bivariate Poisson distribution.

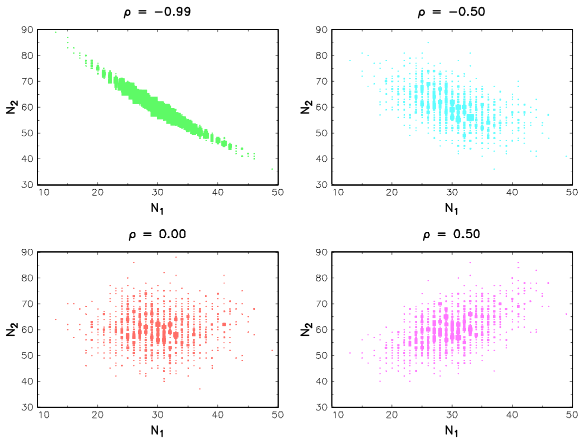


Figure 8: Random generation of bivariate Poisson variates $\mathcal{P}(30)$ and $\mathcal{P}(60)$.

6 Conclusion

In this paper, we show that copula is a very powerful tool for risk management since it fulfills one of its main goal: the modelling of dependence between the individual risks. That is why this approach is an open field for

[15] We assume that time is discrete — $t \in \mathbb{N}$ — and that the period of reference is one.

risk. Indeed, there is a need to …nd other 'industrial' copula functions such as Normal and Student. Before going further, copulas have to become more familiar to practitioners and we believe they will.

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References

[1] Basel Committee on Banking Supervision, Operational Risk — Consultative Document, Supporting document to the New Basel Capital Accord, January 2001

[2] Bouyé, E., V. Durrleman, A. Nikeghbali, G. Riboulet and T. Roncalli [2000], Copulas for …nance — a reading guide and some applications, Groupe de Recherche Opérationnelle, Crédit Lyonnais, Working Paper

[3] Coutant, S., P. Martineu, J. Messines, G. Riboulet and T. Roncalli [2001], Revisiting the dependence in credit risk models, Groupe de Recherche Opérationnelle, Crédit Lyonnais, Working Paper

[4] Deheuvels, P. [1978], Caractérisation complète des lois extrêmes multivariées et de la convergence des types extrêmes, Publications de l'Institut de Statistique de l'Université de Paris, 23, 1-36

[5] Durrleman, V., A. Nikeghbali and T. Roncalli [2000], Which copula is the right one?, Groupe de Recherche Opérationnelle, Crédit Lyonnais, Working Paper

[6] Embrechts, P., McNeil, A.J. and D. Straumann [1999], Correlation: pitfalls and alternatives, Risk magazine, 12, May, 69-71

[7] Embrechts, P., McNeil, A.J. and D. Straumann [2000], Correlation and dependency in risk management: properties and pitfalls, in M. Dempster and H.K. Mo¤att (Eds.), Risk Management: Value at Risk and Beyond, Cambridge University Press (yet to appear)

[8] Genest, C., K. Ghoudi and L-P Rivest [1995], A semiparametric estimation procedure for dependence parameters in multivariate families of distributions, Biometrika, 82, 543-552

[9] Georges, P., A-G. Lamy, E. Nicolas, G. Quibel and T. Roncalli [2001], Multivariate survival modelling: a uni…ed approach with copulas, Groupe de Recherche Opérationnelle, Crédit Lyonnais, Working Paper, in preparation

[10] Joe, H. [1997], Multivariate Models and Dependence Concepts, Monographs on Statistics and Applied Probability, 73, Chapmann & Hall, London

[11] Li, D.X. [2000], On default correlation: a copula function approach, Journal of Fixed Income, March, 43-54

[12] Maccarinelli, M. and V. Maggiolini [2000], The envolving practice of credit risk management in global …nancial institutions, Risk Conference, 26/27 September, Paris

[13] Nelsen, R.B. [1999], An Introduction to Copulas, Lectures Notes in Statistics, 139, Springer Verlag, New York

[14] Song, P. X-K. [2000], Multivariate dispersion models generated from Gaussian copula, Scandinavian Journal of Statistics, 27-2, 305-320

The following working papers on copulas are available upon request (thierry.roncalli@creditlyonnais.fr):

1. Bouyé, E., Durrleman, V., A. Nikeghbali, G. Riboulet and T. Roncalli [2000], **Copulas for finance: a reading guide and some applications**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*, 70 p., March 2000

   Abstract: Copulas are a general tool to construct multivariate distributions and to investigate dependence structure between random variables. However, the concept of copula is not popular in finance. In this paper, we show that copulas can be extensively used to solve many financial problems.

   Keywords: Copulas, dependence, markov processes, brownian copula, value-at-risk, stress-testing, operational risk.

   Available on the web page http://www.gloriamundi.org/var/wps.htm.

2. Durrleman, V., A. Nikeghbali and T. Roncalli [2000], **A simple transformation of copulas**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*, 15 p., July 2000

   Abstract: We study how copulas properties are modified after some suitable transformations. In particular, we show that using appropriate transformations permits to fit the dependence structure in a better way.

   Keywords: γ-transformation, Kendall's tau, Spearman's rho, upper tail dependence.

3. Durrleman, V., A. Nikeghbali and T. Roncalli [2000], **Which copula is the right one?**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*, 20 p., August 2000

   Abstract: In this paper, we give a few methods for the choice of copulas in financial modelling.

   Keywords: Maximum likelihood method, inference for margins, CML method, point estimator, non parametric estimation, Deheuvels copula, copula approximation, discrete L^p norm.

4. Durrleman, V., A. Nikeghbali and T. Roncalli [2000], **Copulas approximation and new families**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*, 25 p., August 2000

   Abstract: In this paper, we study the approximation procedures introduced by Li, Mikusinski, Sherwood and Taylor [1997]. We show that there exists a bijection between the set of the *discretized* copulas and the set of the *doubly stochastic* matrices. For the Bernstein and checkerboard approximations, we then provide analytical formulas for the Kendall's tau and Spearman's rho concordance measures. Moreover, we demonstrate that these approximations does not exhibit tail dependence. Finally, we consider the general case of approximations induced by *partitions of unity*. Moreover, we show that the set of copulas induced by *partition of unity* is a Markov sub-algebra with respect to the $*$-product of Darsow, Nguyen and Olsen [1992].

   Keywords: Doubly stochastic matrices, Bernstein polynomials approximation, checkerboard copula, partitions of unity, Markov algebras, product of copulas.

5. Durrleman, V., A. Nikeghbali, and T. Roncalli [2000], **How to get bounds for distribution convolutions? A simulation study and an application to risk management**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, 14 p., September 2000

   Abstract: In this paper, we consider the problem of bounds for distribution convolutions and we present some applications to risk management. We show that the upper Fréchet bound is not always the more risky dependence structure. It is in contradiction with the belief in finance that maximal risk correspond to the case where the random variables are comonotonic.

Keywords: Triangle functions, dependency bounds, infimal, supremal and σ-convolutions, Makarov inequalities, Value-at-Risk, 'square root' rule, Williamson's uniform quantisation method, Dall'Aglio problem, Kantorovich distance.

6. Costinot, A., T. Roncalli and J. Teiletche [2000], **Revisiting the dependence between financial markets with copulas**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*, 45 p., October 2000

   Abstract: We consider the problem of modelling the dependence between financial markets. In financial economics, the classical tool is the Pearson (or linear) correlation to compare the dependence structure. We show that this coefficient does not give a precise information on the dependence structure. Instead, we propose a conceptual framework based on copulas. Two applications are proposed. The first one concerns the study of extreme dependence between international equity markets. The second one concerns the analysis of the East Asian crisis.

   Keywords: Linear correlation, extreme value theory, quantile regression, concordance order, Deheuvels copula, contagion, Asian crisis.

7. Durrleman, V., A. Nikeghbali and T. Roncalli [2000], **A note about the conjecture on Spearman's rho and Kendall's tau**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*, 10 p., October 2000

   Abstract: In this paper, we consider the open question on Spearman's rho and Kendall's tau of Nelsen [1991]. Using a technical hypothesis, we can answer in the positive. However, one question remain open: how can we understand the technical hypothesis? Because this hypothesis is not right in general, we can find some pathological cases which contradicts the Nelsen's conjecture.

   Keywords: Spearman's rho, Kendall's tau, cubic copula.

8. Roncalli, T. [2000], **Financial Applications of Copulas**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Slides*, 106 p., November 2000

   Abstract: Slides for the seminar Financial Applications of Copulas, CREREG, Rennes, 11/16/2000.

   Keywords: Copulas, financial applications, risk management, statistical modelling, probabilistic metric spaces, markov operators, quasi-copulas.

9. Roncalli, T. [2001], **Copulas: A Tool For Modelling Dependence In Finance**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Slides*, 80 p., January 2001

   Abstract: Slides for the conference Statistical Methods in Integrated Risk Management, Frontières en Finance, Paris, 01/26/2001.

   Keywords: Copulas, two-assets option, Markov processes, Value-at-Risk, credit risk, credit derivatives.

   Available on the web page http://www.gloriamundi.org/var/links.htm.

10. Durrleman, V., A. Nikeghbali and T. Roncalli [2001], **What are the most important copulas in finance?**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Slides*, 40 p., March 2001

    Abstract: Slides for the International Finance Conference, Hammam-Sousse, Tunisia, 03/17/2001.

    Keywords: Copulas, risky dependence function, singular copulas, extreme points, quantile aggregation, spread option.

11. Bouyé, E., Durrleman, V., A. Nikeghbali, G. Riboulet and T. Roncalli [2001], **Copulas: an open field for risk management**, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*, 8 p., March 2001 (first version: October 2000)

    Abstract: In this paper, we show that copula is a very powerful tool for risk management since it fulfills one of its main goal: the modelling of dependence between the individual risks. That is why this approach is an open field for risk.

    Keywords: Copulas, value-at-risk, stress-testing, credit risk, operational risk.

List of other working papers:

2001

1. Soosung Hwang and Steve Satchell , GARCH Model with Cross-sectional Volatility; GARCHX Models, WP01-16
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15. Eric Bouyé, Multivariate Extremes at Work for Portfolio Risk Measurement, WP01-02
16. Erick Bouyé, Vado Durrleman, Ashkan Nikeghbali, Gael Riboulet and Thierry Roncalli, Copulas: an Open Field for Risk Management, WP01-01

2000

1. Soosung Hwang and Steve Satchell , Valuing Information Using Utility Functions, WP00-06
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1999

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1998

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