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A study of the dynamic of influence through differential equations*

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Abstract. The paper concerns a model of influence in which agents make their decisions on a certain issue. It is assumed that each agent is inclined to make a particular decision, but due to a possible influence of the others, his final decision may be different from his initial inclination. Since in reality the influence does not necessarily stop after one step, but may iterate, we present a model which allows us to study the dynamic of influence. The use of continuous variables permits the application of differential equations systems to the analysis of the convergence of agents' decisions in long-time. In particular, by applying the approach based on differential equations to the influence model, we recover the results of the discrete model on classical influence functions, and the results on the boss and approval sets for the command games equivalent to some influence functions.

JEL Classification: C7, C6, D7

Keywords: social network, inclination, decision, influence function, differential equations

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1 Introduction

The phenomenon of influence between individuals or, more broadly, influence of a group of agents on an individual, is studied carefully in numerous works, e.g., in psychology, sociology, economics, and mathematics. The economics literature offers several surveys of different models of influence and of different approaches to this phenomenon; see, e.g., Jackson (2008), Grabisch and Rusinowska (2010d), Rusinowska (2010).

Some of the works on influence and interaction, in particular in the game theoretical literature, are one-step models. In reality, however, the influence does not necessarily stop after one step, but may iterate. Consequently, in order to study the iteration of influence, different dynamic models of influence are proposed by sociologists and economists. One of the leading works of opinion formation is introduced in DeGroot (1974). In that model, individuals start with initial opinions on a subject and place some weights on the current beliefs of other agents in forming their own beliefs for the next period. These beliefs

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are updated over time. Several other authors study the DeGroot model and propose its generalizations, in particular, models in which the updating of beliefs can vary in time and circumstances; see e.g. Berger (1981), DeMarzo et al. (2003), Krause (2000), Lorenz (2005), Friedkin and Johnsen (1990, 1997), Jackson (2008), Golub and Jackson (2010). A related dynamic framework of influence is also presented in Asavathiratham (2000), Asavathiratham et al. (2001) and Koster et al. (2010).

A cooperative approach to influence is presented in Hu and Shapley (2003b,a), where the command structure is applied to model players' interaction relations by simple games. Boss sets and approval sets for a player are defined, and based on these sets a simple game called the command game for the player is constructed. Also the concept of command function is defined. The authors introduce an authority distribution over an organization and define the authority equilibrium equation. In particular, they consider multi-step commands, where commands can be implemented through command channels.

The present paper is related to another framework of influence originally introduced in Hoede and Bakker (1982) and later refined in our several works. In the original one-step model, agents have to make their acceptance-rejection decision on a certain issue. Each agent has an inclination to say either YES or NO, but due to a possible influence of the other agents, his final decision (YES or NO) may be different from his initial inclination. Our first results concerning this model are presented in Grabisch and Rusinowska (2010a) where we investigate several tools to analyze the influence. In particular, we define the influence indices to measure the influence of a coalition on an agent, introduce several influence functions and study their properties, investigate the concept of a follower of a coalition. In Grabisch and Rusinowska (2010b) we generalize the YES-NO model of influence to a framework in which each agent has an ordered set of possible actions, and in Grabisch and Rusinowska (2010c) we assume a continuum of actions. Our results presented in Grabisch and Rusinowska (2009) concern a comparison of the influence model with the framework of command games (Hu and Shapley (2003b,a)). We show that our framework of influence is more general than the framework of the command games. In particular, we define several influence functions which capture the command structure. For some influence functions we define the equivalent command games. In Grabisch and Rusinowska (Forthcoming 2011) we establish exact relations between the key concepts of the influence model and the framework of command games. More precisely, we study the relations between: influence functions and follower functions, command games and command functions, and between command games and influence functions.

There are also some studies of the dynamic of influence in the model mentioned above, i.e., the model of initial inclinations and final decisions. In Grabisch and Rusinowska (2010e) the YES-NO model with a single step of mutual influence is generalized to a framework with iterated influence. We analyze the decision process in which the mutual influence does not stop after one step but iterates, and we study the convergence of an influence function. In particular, we investigate stochastic influence functions and apply Markov chains theory to the analysis of such functions. Also in the framework of influence with a continuum of actions (Grabisch and Rusinowska (2010c)), where we study in particular linear influence functions, the convergence of the linear influence functions in the iterated process is discussed.

The *aim of the present paper* is to apply another approach proposed in Maruani (2010) to study the dynamic of influence. We propose a dynamic model in which an agent gives a certain importance, reflected by a weight, to other agents in making his final decision. Such a weight might be positive, negative, or equal to zero, which corresponds to the stimulation, the inhibition, and the absence of relation, respectively. Furthermore, we define the exhortation of an agent which is the weighted sum of the opinions that the agent receives from the others. The opinions of the agents are updated over time and the updating rule is based on the sign of the exhortation. The use of continuous variables permits the application of differential equations systems to the analysis of the convergence of agents' decisions in long-time. By applying the approach based on differential equations to the influence model, we recover the results of the discrete model on classical influence functions, and the results on the boss and approval sets for command games equivalent to some selected influence functions.

The paper is structured as follows. In Section 2 the main concepts of the influence model in question and the framework of command games are presented. In Section 3 we introduce the dynamic model of influence. The dynamic of selected influence functions is studied in Section 4. Section 5 concern followers in the dynamic model and under selected influence functions. The analysis of the dynamic model in terms of command games, in particular, the determination of the boss and approval sets in the presented framework is delivered in Section 6. Concluding remarks are given in Section 7.

2 The model of influence and command games

2.1 Main concepts of the influence model

In this section we recapitulate main concepts of the YES-NO model of influence investigated in Grabisch and Rusinowska (2010a).

We consider a social network with the set of all players (agents, voters) denoted by $N := \{1, ..., n\}$. Each player has to make an acceptance-rejection decision concerning a certain issue, and he has an *inclination* to say either YES (denoted by $+1$) or NO (denoted by -1). An *inclination vector* is denoted by $I = (I_1, ..., I_n)$, where $I_k \in \{-1, +1\}$ indicates the inclination of agent k, for each $k \in N$. For convenience, $(1, 1, \ldots, 1) \in \{-1, +1\}^n$ is denoted by 1_N, $(-1, -1, \ldots, -1) \in \{-1, +1\}^n$ by -1_N, and mixed cases by $(-1_{N\setminus S}, 1_S)$.

It is assumed that agents may influence each other in the network, and due to the influences the final decision of an agent may be different from his original inclination. Formally, each inclination vector $I \in \{-1, +1\}^n$ is transformed into a *decision vector* $B(I) = (B_1(I), ..., B_n(I))$, where $B : \{-1, +1\}^n \to \{-1, +1\}^n, I \mapsto B(I)$ is the *influence function*[1], and $B_k(I)$ indicates the decision made by agent k, for each $k \in N$. The set of all influence functions is denoted by $\mathcal{B}$.

One of the key concepts of the influence model is the concept of *follower*. An agent is said to be *follower* of a coalition if he always decides according to the inclination of that coalition, assuming that the coalition in question is unanimously inclined. For any $S \subseteq N$, we denote by UI_S the set of all inclination vectors under which all members of S have the same inclination

$$UI_S := \{I \in \{-1, +1\}^n : \forall k, j \in S \; [I_k = I_j]\}.$$

[1] We can also speak of the influence function of agent k, $B_k : \{-1, +1\}^n \to \{-1, +1\}, I \mapsto B_k(I)$, for each $k \in N$.

In particular, $UI_k = \{-1, +1\}^n$ for any $k \in N$. We denote by I_S the value I_k for some $k \in S$, $I \in UI_S$. Formally, for $\emptyset \neq S \subseteq N$ and $B \in \mathcal{B}$, the *set of followers* of S under B is defined as

$$F_B(S) := \{j \in N : \forall I \in UI_S \; [B_j(I) = I_S]\}.$$

The cardinality of S will be denoted by s.

We recapitulate three particular influence functions that have been introduced and investigated in Grabisch and Rusinowska (2010a):

- Let $n \geq t > \lfloor \frac{n}{2} \rfloor$ and $I^+ := \{k \in N : I_k = +1\}$ for any $I \in \{-1, +1\}^n$. The *majority influence function* $\mathsf{Maj}^{[t]} \in \mathcal{B}$ is defined by

$$\mathsf{Maj}^{[t]}(I) := \begin{cases} 1_N, & \text{if } |I^+| \geq t \\ -1_N, & \text{if } |I^+| < t \end{cases}, \quad \forall I \in \{-1, +1\}^n.$$

  According to the majority influence function if a majority of players has the positive inclination, then all agents decide $+1$, otherwise all decide -1. For each $S \subseteq N$, the set of followers under the majority function is equal to

$$F_{\mathsf{Maj}^{[t]}}(S) = \begin{cases} N, & \text{if } \; s \geq t \\ \emptyset, & \text{if } \; s < t, \end{cases} \tag{1}$$

  i.e., everybody follows a coalition with a cardinality of at least t, and nobody follows a coalition with less than t members.
- Let $\widetilde{k} \in N$ be a particular player called the guru. The *guru influence function* $\mathsf{Gur}^{[\widetilde{k}]} \in \mathcal{B}$ is defined by

$$\mathsf{Gur}_j^{[\widetilde{k}]}(I) := I_{\widetilde{k}}, \quad \forall I \in \{-1, +1\}^n, \quad \forall j \in N.$$

  Hence, according to this function, when a guru exists, every agent follows the guru. For each $S \subseteq N$, the set of followers under the guru function is given by

$$F_{\mathsf{Gur}^{[\widetilde{k}]}}(S) = \begin{cases} N, & \text{if } \widetilde{k} \in S \\ \emptyset, & \text{if } \widetilde{k} \notin S. \end{cases} \tag{2}$$

  In other words, all agents follow a coalition containing the guru, and nobody follows a coalition without the guru.
- The *identity function* $\mathsf{Id} \in \mathcal{B}$ depicts the absence of any influence and is defined by

$$\mathsf{Id}(I) := I, \quad \forall I \in \{-1, +1\}^n.$$

  Moreover, we have for each $S \subseteq N$,

$$F_{\mathsf{Id}}(S) = S, \tag{3}$$

  which means that all members of a coalition and only them follow that coalition.

2.2 Command games and equivalence with influence functions

Next, we present some of the main concepts concerning command games that have originally been introduced by Hu and Shapley (2003b,a).

Let $N = \{1, ..., n\}$ be the set of agents (players, voters). For $k \in N$ and $S \subseteq N \setminus k$:

- S is a *boss set* for k if S determines the choice of k;
- S is an *approval set* for k if k can act with the approval of S.

It is assumed that any superset (in $N \setminus k$) of a boss set is a boss set. For each $k \in N$, a simple game $(N, \mathcal{W}_k)$ called the *command game for* k is created, with the set of winning coalitions given by

$$\mathcal{W}_k := \{S : S \text{ is a boss set for } k\} \cup \{S \cup k : S \text{ is a boss or approval set for } k\}.$$

We recover the boss sets for agent k

$$Boss_k = \{S \subseteq N \setminus k : S \in \mathcal{W}_k\} = \mathcal{W}_k \cap 2^{N \setminus k}$$

and the approval sets for k

$$App_k = \{S \subseteq N \setminus k : S \cup k \in \mathcal{W}_k \text{ but } S \notin \mathcal{W}_k\}.$$

Obviously, $Boss_k \cap App_k = \emptyset$. Given the set of command games $\Omega = \{(N, \mathcal{W}_k) : k \in N\}$, for any coalition $S \subseteq N$, the *command function* $\omega(S)$ is defined as the set of all members that are *commandable* by S:

$$\omega(S) := \{k \in N : S \in \mathcal{W}_k\}.$$

In Grabisch and Rusinowska (2009) the model of influence is applied to the framework of command games and the relations between these two frameworks are shown; see also Grabisch and Rusinowska (Forthcoming 2011). We present one of these relations, i.e., the equivalence between command games and (command) influence functions.

Let $\Omega = \{(N, \mathcal{W}_k) : k \in N\}$ be a set of command games, $\omega(S)$ be a set of agents commandable by S, and $F_B(S)$ denote the set of followers of S under an influence function B. The influence function B and the set of command games Ω are said to be *equivalent* if $F_B \equiv \omega$, i.e., if for each coalition $S \subseteq N$, the set of followers of S under the influence function B and the set of agents commandable by S under Ω coincide.

In Grabisch and Rusinowska (2009) we construct command games equivalent to the influence functions recapitulated in Section 2.1 and determine boss and approval sets for these command games:

(i) Let $n \geq t > \lfloor \frac{n}{2} \rfloor$ and $\mathsf{Maj}^{[t]} \in \mathcal{B}$ be the majority influence function and let $\{(N, \mathcal{W}_k^{\mathsf{Maj}^{[t]}}) : k \in N\}$ be a set of command games given by

$$\mathcal{W}_k^{\mathsf{Maj}^{[t]}} = \{S \subseteq N : s \geq t\}, \ \forall k \in N.$$

The majority influence function $\mathsf{Maj}^{[t]}$ *and the set of command games* $\{(N, \mathcal{W}_k^{\mathsf{Maj}^{[t]}}) : k \in N\}$ *are equivalent.* In other words, the command games in which winning coalitions for each player are the ones with the cardinality at least t, $n \geq t > \lfloor \frac{n}{2} \rfloor$, are equivalent to $\mathsf{Maj}^{[t]}$. Moreover, we have for $n > 2$, $n \geq t > \lfloor \frac{n}{2} \rfloor$, and $k \in N$

$$Boss_k^{\mathsf{Maj}^{[t]}} = \{S \subseteq N : s \geq t \ \wedge \ k \notin S\} \tag{4}$$

$$App_k^{\mathsf{Maj}^{[t]}} = \{S \subseteq N : s = t-1 \ \wedge \ k \notin S\}. \tag{5}$$

In particular, for $t = n$, $k \in N$,

$$Boss_k^{\mathsf{Maj}^{[t]}} = \emptyset, \quad App_k^{\mathsf{Maj}^{[t]}} = N \setminus k.$$

(ii) Let $\mathsf{Gur}^{[\widetilde{k}]} \in \mathcal{B}$ be the guru function with the guru $\widetilde{k} \in N$ and let $\{(N, \mathcal{W}_k^{\mathsf{Gur}^{[\widetilde{k}]}}) : k \in N\}$ be a set of command games given by

$$\mathcal{W}_k^{\mathsf{Gur}^{[\widetilde{k}]}} = \{S \subseteq N : \widetilde{k} \in S\}, \ \forall k \in N.$$

The guru function $\mathsf{Gur}^{[\widetilde{k}]}$ *and the set of command games* $\{(N, \mathcal{W}_k^{\mathsf{Gur}^{[\widetilde{k}]}}) : k \in N\}$ *are equivalent.* Hence, the command games in which winning coalitions for each player are the coalitions containing a certain player $\widetilde{k}$, are equivalent to the guru function $\mathsf{Gur}^{[\widetilde{k}]}$ with the guru $\widetilde{k}$. Moreover,

$$Boss_{\widetilde{k}}^{\mathsf{Gur}^{[\widetilde{k}]}} = \emptyset, \quad App_{\widetilde{k}}^{\mathsf{Gur}^{[\widetilde{k}]}} = 2^{N \setminus \widetilde{k}} \tag{6}$$

$$Boss_k^{\mathsf{Gur}^{[\widetilde{k}]}} = \{S \subseteq N : \widetilde{k} \in S \ \wedge \ k \notin S\}, \quad App_k^{\mathsf{Gur}^{[\widetilde{k}]}} = \emptyset, \quad for\ k \neq \widetilde{k}. \tag{7}$$

(iii) Let $\mathsf{Id} \in \mathcal{B}$ be the identity function and let $\{(N, \mathcal{W}_k^{\mathsf{Id}}) : k \in N\}$ be a set of command games given by

$$\mathcal{W}_k^{\mathsf{Id}} = \{S \subseteq N : k \in S\}, \ \forall k \in N.$$

The identity function Id *and the set of command games* $\{(N, \mathcal{W}_k^{\mathsf{Id}}) : k \in N\}$ *are equivalent.* This means that the command games, in which for each player k winning coalitions for k are the coalitions containing k, are equivalent to the identity function. We have also for each $k \in N$

$$Boss_k^{\mathsf{Id}} = \emptyset, \quad App_k^{\mathsf{Id}} = 2^{N \setminus k}. \tag{8}$$

3 The dynamic model of influence

3.1 Description of the model and stable states

In order to analyze the dynamic aspects of influence, we consider the following model introduced originally in Maruani (2010).

Let $N = \{1, 2, ..., n\}$ denote the set of agents. We are interested in the influence of all n agents on a player $i \in N$. Let e_j denote the inclination of agent $j \in N$, where $e_j = \pm 1$. Agent i gives to agent j a certain importance which is reflected by a weight p_{ij}. This means that the state (inclination) e_j will contribute to the decision of agent i with the weight $c_{ij} = p_{ij} e_j$. The *exhortation* E_i obtained by agent $i \in N$ is defined by the weighted sum of the inclinations of all agents, or saying differently, by the weighted sum of the opinions that i receives from the agents (including his own opinion):

$$E_i = \sum_{j \in N} c_{ij} = \sum_{j \in N} p_{ij} e_j. \tag{9}$$

Figure 1 presents the idea of the exhortation.

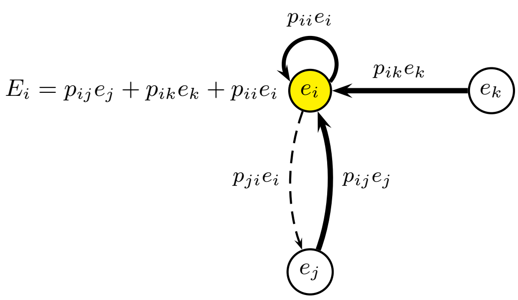

Fig. 1. The exhortation

The updating rule is the following. If $E_i > 0$, then agent i goes to the state $+1$. If $E_i < 0$, then he goes to the state -1, and if $E_i = 0$, then i stays in his present state. Hence, $p_{ij} > 0$ corresponds to the stimulation of i by j, $p_{ij} < 0$ corresponds to the inhibition, and $p_{ij} = 0$ means the absence of relation.

The *influence function* B_i of agent $i \in N$ is defined by

$$B_i(e) = \mathrm{sgn}(E_i), \text{ where } e = (e_1, \ldots, e_n) \text{ and } \mathrm{sign}(E_i) = \begin{cases} 1, & \text{if } E_i > 0 \\ -1, & \text{if } E_i < 0 \\ e_i, & \text{if } E_i = 0 \end{cases} \tag{10}$$

Remark 1 *In this dynamic model, we can express the influence functions recapitulated in Section 2.1 as follows:*

(i) The majority influence function with $t = \lfloor \frac{n}{2} \rfloor + 1$ in which every agent plays a role can be represented by $p_{ij} = \frac{1}{n}$.

(ii) The guru function with the guru $\widetilde{k}$ in which every agent is influenced only by the guru can be represented by $p_{ij} = 0$ for each $j \neq \widetilde{k}$ and $p_{i\widetilde{k}} = 1$.

(iii) The identity function in which every agent influences only himself can be represented by $p_{ij} = \delta_{ij}$.

Stable states of the system satisfy $e_i^{(k+1)} = e_i^{(k)}$ for each $i \in N$, starting from a certain k, where k is the number of iteration and $e_i^{(k)}$ denotes the inclination (state) of i at time k. This means that the state e_i and the exhortation E_i have the same sign, and therefore $e_i E_i > 0$. In other words, the stable states satisfy the following inequality:

$$\sum_{j \in N} p_{ij} e_i e_j > 0 \text{ for each } i \in N.$$

The evolution of the population can be expressed by

$$e_i^{(k+1)} = B_i(e_1^{(k)}, e_2^{(k)}, \ldots, e_j^{(k)}, \ldots, e_n^{(k)}) \tag{11}$$

where B_i is the influence function of voter $i \in N$. A stable state satisfies therefore

$$e_i^{(k)} = B_i(e_1^{(k)}, e_2^{(k)}, \ldots, e_j^{(k)}, \ldots, e_n^{(k)}).$$

An obvious way to get stability is to forbid any relation between agents, i.e., to assign the weights as $p_{ij} = \delta_{ij}$.

3.2 Dynamic of the model in the continuous framework

Let us consider an extension of the dynamic influence model to the continuous framework; see also Maruani (2010). The influence function remains

$$e_i^{(k+1)} = \operatorname{sgn}\left(E_i^{(k)}\right)$$

which together with $E_i^{(k)} = \sum_{j\in N} p_{ij} e_j^{(k)}$ can be written as

$$\frac{e_i^{(k+1)} - e_i^{(k)}}{(k+1)-k} = \operatorname{sgn}\left(\sum_j p_{ij} e_j^{(k)}\right) - e_i^{(k)}. \tag{12}$$

For a small time step, we can make the approximation

$$\frac{e_i^{(k+1)} - e_i^{(k)}}{(k+1)-k} \approx \frac{de_i}{dt} \tag{13}$$

and also the approximation

$$\operatorname{sgn}\left(\sum_j p_{ij} e_j\right) \approx \tanh\left(a \sum_j p_{ij} e_j\right) \tag{14}$$

where the parameter a controls the tendency towards the function sgn. This approximation keeps the properties of the function sgn that are essential for our analysis: it is increasing and bounded between -1 and 1 and nullifies in 0.

¿From (12), (13), and (14), in order to study the dynamic of the model, we need to solve the following system:

$$\frac{de_i}{dt} + e_i = \tanh\left(a \sum_j p_{ij} e_j\right), \quad i = 1, \ldots, n \tag{15}$$

and then put $e_i = \operatorname{sgn}[e_i(t)]$. In Section 4 we apply this dynamic approach of differential equations to classical influence functions. The results that we present have been originally shown in Maruani (2010).

4 The dynamic of selected influence functions

4.1 The majority influence function with $t = \lfloor \frac{n}{2} \rfloor + 1$

As mentioned in Remark 1(i), the majority influence function with $t = \lfloor \frac{n}{2} \rfloor + 1$ can be represented by the weights $p_{ij} = \frac{1}{n}$. The system (15) of differential equations that we need to solve is therefore

$$\frac{de_i}{dt} + e_i = \tanh\left(\frac{a}{n}\sum_j e_j\right), \quad i = 1, \dots, n. \tag{16}$$

The dynamic approach allows for the analysis of the influence model introduced in Grabisch and Rusinowska (2010c) in which agents have a continuum of actions, i.e., the set of all inclinations is equal to an interval $[x, y]$, where $x, y \in \mathbb{R}$.

Lemma 1 *For $a > 1$, the differential equation $\frac{dx}{dt} + x = \tanh(ax)$ has an unstable fixed point in 0 and another stable fixed point.*

Proof: In the equation

$$\frac{dx}{dt} + x = \tanh(ax)$$

we put $ax = y$. Hence,

$$\frac{dy}{dt} + y = a\tanh(y) = \frac{1}{p}\tanh(y)$$

with $p = \frac{1}{a}$. The fixed points satisfy $\tanh(y) = py$. There is always $y = 0$ and another fixed point y_0 iff $p < 1$ ($a > 1$). We are interested in the stability of fixed points. In the neighborhood of $y = 0$

$$\frac{dy}{dt} + y = \frac{1}{p}\tanh(y) \approx \frac{y}{p}$$

$$\frac{dy}{dt} + \frac{p-1}{p}y = 0$$

$$y(t) = y(0)\exp\frac{1-p}{p}t$$

For $0 < p < 1$ the origin is unstable.
In the neighborhood of y_0, we put $y(t) = y_0 + \varepsilon(t)$

$$\frac{dy}{dt} + y = \frac{1}{p}\tanh(y)$$

$$\begin{aligned}\frac{d\varepsilon}{dt} + y_0 + \varepsilon(t) &= \frac{1}{p}\tanh\left(y_0 + \varepsilon(t)\right)\\ &= \frac{1}{p}\frac{\tanh(y_0) + \tanh\left(\varepsilon(t)\right)}{1 + \tanh(y_0)\tanh\left(\varepsilon(t)\right)}\\ &= \frac{1}{p}\frac{py_0 + \tanh\left(\varepsilon(t)\right)}{1 + py_0\tanh\left(\varepsilon(t)\right)}\end{aligned}$$

where we have used $py_0 = \tanh(y_0)$. A first order approximation yields

$$\frac{d\varepsilon}{dt} + y_0 + \varepsilon(t) \approx \frac{1}{p}\frac{py_0 + \varepsilon(t)}{1 + py_0\varepsilon(t)} \approx \frac{1}{p}\left[py_0 + \varepsilon(t)\right]\left[1 - py_0\varepsilon(t)\right]$$
$$\approx y_0 - py_0^2\varepsilon(t) + \frac{\varepsilon(t)}{p}$$

and ultimately

$$\frac{d\varepsilon}{dt} + \left(1 + py_0^2 - \frac{1}{p}\right)\varepsilon(t) = 0$$

If $\left(1 + py_0^2 - \frac{1}{p}\right) > 0$, then the solution is the decreasing exponential and point y_0 is stable. Let us show that $\left(1 + py_0^2 - \frac{1}{p}\right) > 0$. Let

$$A = 1 + py_0^2 - \frac{1}{p}$$

$$py_0 = \tanh(y_0)$$

$$\begin{aligned}
A &= 1 + y_0\tanh(y_0) - \frac{y_0}{\tanh(y_0)} = 1 + y_0\left[\tanh(y_0) - \frac{1}{\tanh(y_0)}\right] \\
&= 1 + y_0\left[\frac{\tanh^2(y_0) - 1}{\tanh(y_0)}\right] = 1 - y_0\left[\frac{1}{\cosh^2(y_0)\tanh(y_0)}\right] \\
&= 1 - y_0\left[\frac{1}{\cosh(y_0)\sinh(y_0)}\right] = 1 - \frac{2y_0}{\sinh(2y_0)} \\
&= 1 - \frac{u}{\sinh(u)}
\end{aligned}$$

We have

$$\frac{\sinh(u)}{u} = 1 + \frac{u^2}{3!} + \frac{u^4}{5!} + ...$$
$$\frac{\sinh(u)}{u} > 1 \ \Rightarrow\ 1 - \frac{u}{\sinh(u)} > 0.$$

■

Proposition 1 *If agents make their decisions according to the majority influence function with $t = \lfloor \frac{n}{2} \rfloor + 1$, then the decision of each agent converges to the sign of the sum of the agents' inclinations.*

Proof: Let us consider the system

$$\frac{de_i}{dt} + e_i = \tanh\left[\frac{a}{n}\sum_j e_j(t)\right], \quad i = 1, \ldots, n.$$

With $\varepsilon_i = \frac{a}{n} e_i$, we have, for each i,

$$\frac{d\varepsilon_i}{dt} + \varepsilon_i = \frac{a}{n} \tanh \left[\sum_j \varepsilon_j(t) \right]$$

Adding term by term the equations of the system, the sum $S(t) = \sum_j \varepsilon_j(t)$ satisfies

$$\frac{dS}{dt} + S = a \tanh [S(t)]$$

According to Lemma 1, S converges to a stable fixed point S_∞, which satisfies for $a > 1$

$$S_\infty = a \tanh(S_\infty)$$

The system that we consider becomes

$$\frac{d\varepsilon_i}{dt} + \varepsilon_i = \frac{a}{n} \tanh [S(t)] \, .$$

By changing the function $\varepsilon_i(t) = E_i(t) \exp(-t)$ $(\varepsilon_i(0) = E_i(0))$, we get the equation

$$\frac{dE_i}{dt} = \frac{a}{n} \exp(t) \tanh [S(t)]$$

whose general solution is

$$E_i(t) = E_i(0) + \frac{a}{n} \int_0^t \exp(u) \tanh [S(u)] \, du$$

The solution for ε_i is therefore

$$\varepsilon_i(t) = \varepsilon_i(0) \exp(-t) + \frac{a}{n} \exp(-t) \int_0^t \exp(u) \tanh [S(u)] \, du \tag{17}$$

We are interested in the asymptotic form of the solution. The first term of the right hand side of (17) vanishes. Let us denote by η_i the second term, i.e.,

$$\eta_i(t) = \frac{a}{n} \exp(-t) \int_0^t \exp(u) \tanh [S(u)] \, du$$

Let us fix for the moment some T. Then, for $t > T$ we decompose the integral in two terms

$$\begin{aligned} \eta_i(t) &= \frac{a}{n} \exp(-t) \left[\int_0^T \exp(u) \tanh [S(u)] \, du + \int_T^t \exp(u) \tanh [S(u)] \, du \right] \\ &= \eta_i^{(1)}(t) + \eta_i^{(2)}(t) \end{aligned}$$

and consider successively the two components. We analyze the first term

$$\eta_i^{(1)}(t) = \frac{a}{n} \exp(-t) \int_0^T \exp(u) \tanh [S(u)] \, du$$

The tanh is bounded by 1, so we have

$$\left|\int_0^T \exp(u)\tanh\left[S(u)\right]du\right| \leq \int_0^T \exp(u)du = \exp(T) - 1$$

Given T, $\exp(T) - 1$ is a fixed number and we denote it by B. Then

$$\left|\eta_i^{(1)}(t)\right| \leq \frac{aB}{n}\exp(-t)$$

and therefore

$$\lim_{t\to\infty}\eta_i^{(1)}(t) = 0$$

The second term is

$$\eta_i^{(2)}(t) = \frac{a}{n}\exp(-t)\int_T^t \exp(u)\tanh\left[S(u)\right]du$$

For u large enough, which is the case, $S(u)$ is close to S_∞. By continuity $\tanh\left[S(u)\right]$ is close to $\tanh\left(S_\infty\right)$, which is equal to $\frac{S_\infty}{a}$. It is then appropriate to choose T as follows. Setting

$$\tanh\left[S(u)\right] = \frac{S_\infty}{a}\left[1 + \delta(u)\right]$$

introduces the function $\delta(u)$ which vanishes at infinity. The free parameter T is then chosen such that for any given positive ε and for $t > T$, $|\delta(t)| < \frac{n\varepsilon}{|S_\infty|}$. We have then

$$\begin{aligned}
\eta_i^{(2)}(t) &= \frac{a}{n}\exp(-t)\int_T^t \exp(u)\tanh\left[S(u)\right]du \\
&= \frac{a}{n}\exp(-t)\int_T^t \exp(u)\frac{S_\infty}{a}\left[1 + \delta(u)\right]du \\
&= \frac{S_\infty}{n}\exp(-t)\int_T^t \exp(u)du + \frac{S_\infty}{n}\exp(-t)\int_T^t \delta(u)\exp(u)du \\
&= \frac{S_\infty}{n}\left(1 - \exp(T - t)\right) + \varphi(t)
\end{aligned}$$

where the asymptotic value of the first term is $\frac{S_\infty}{n}$, and the second term is

$$\varphi(t) = \frac{S_\infty}{n}\exp(-t)\int_T^t \delta(u)\exp(u)du$$

$$\begin{aligned}
|\varphi(t)| &\leq \frac{S_\infty}{n}\exp(-t)\int_T^t \exp(u)\frac{n\varepsilon}{|S_\infty|}du \\
&\leq \varepsilon\exp(-t)\int_T^t \exp(u)du \\
&\leq \varepsilon
\end{aligned}$$

The asymptotic value of ε_i is then $\frac{S_\infty}{n}$, and the asymptotic value of $\sum_j \varepsilon_j$ is $\sum_{j=1}^n \frac{S_\infty}{n} = S_\infty$. ■

Remark 2 (The majority vote decreases costs)
The fixed points of the differential system $\frac{de_k}{dt} = g_k(e_1, e_2, ..., e_n)$, $k = 1, \ldots, n$ satisfy by definition $\frac{de_k}{dt} = 0$, $k = 1, \ldots, n$, which is equivalent to $\sum_k \left(\frac{de_k}{dt}\right)^2 = 0$.

Let us suppose the existence of a function G of n variables such that for each k

$$g_k(e_1, e_2, ..., e_n) = -\frac{\partial G}{\partial e_k}$$

and calculate

$$\frac{dG}{dt} = \sum_k \frac{\partial G}{\partial e_k}\frac{de_k}{dt} = -\sum_k g_k \frac{de_k}{dt} = -\sum_k \left(\frac{de_k}{dt}\right)^2 \leq 0$$

which means that G is decreasing in time and reaches its minimum when each $\frac{de_k}{dt} = 0$, i.e., for the fixed point of the system. We have

$$\frac{de_k}{dt} + e_k = f_k(e_1, e_2, ..., e_n) \;\Rightarrow\; \frac{de_k}{dt} = f_k(e_1, e_2, ..., e_n) - e_k \quad (g_k = f_k - e_k)$$

Let us apply this to our differential system (16), letting $\varepsilon_k = \frac{a}{n} e_k$:

$$\frac{d\varepsilon_k}{dt} = \frac{a}{n} \tanh\left[\sum_j \varepsilon_j(t)\right] - \varepsilon_k$$

(all f being identical). We multiply both sides of each differential equation by $\frac{d\varepsilon_k}{dt}$

$$\left(\frac{d\varepsilon_k}{dt}\right)^2 = \frac{a}{n}\frac{d\varepsilon_k}{dt} \tanh\left[\sum_j \varepsilon_j(t)\right] - \varepsilon_k \frac{d\varepsilon_k}{dt}$$

and obtain

$$\sum_k \left(\frac{d\varepsilon_k}{dt}\right)^2 = \left\{\frac{a}{n} \tanh\left[\sum_j \varepsilon_j(t)\right]\right\} \sum_k \frac{d\varepsilon_k}{dt} - \sum_k \varepsilon_k \frac{d\varepsilon_k}{dt}$$

With $S(t) = \sum_k \varepsilon_k(t)$ we have then

$$\sum_k \left(\frac{d\varepsilon_k}{dt}\right)^2 = \frac{a}{n}\frac{dS}{dt} \tanh\left[S(t)\right] - \frac{1}{2}\frac{d\sum_k (\varepsilon_k)^2}{dt}$$

Note that

$$\frac{dS}{dt} \tanh\left[S(t)\right] = \frac{1}{\cosh\left[S(t)\right]}\frac{d}{dt}\cosh\left[S(t)\right] = \frac{d}{dt}\ln\left\{\cosh\left[S(t)\right]\right\}$$

Hence, up to some unessential additive constant, we get the cost function

$$G(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) = \frac{1}{2}\sum_k \varepsilon_k^2(t) - \frac{a}{n}\ln\left\{\cosh\left[\sum_k \varepsilon_k(t)\right]\right\}.$$

4.2 The guru function and the identity function

As mentioned in Remark 1(ii), the guru function with the guru $\widetilde{k}$ can be represented by

$$\forall j \neq \widetilde{k},\ p_{ij} = 0,\ p_{i\widetilde{k}} = 1.$$

The system of differential equations given in (15) that we need to solve becomes

$$\frac{de_i}{dt} + e_i = \tanh(ae_{\widetilde{k}}), \quad i = 1, \ldots, n. \tag{18}$$

One can see that, on the one hand, the guru evolves only according to its own inclinations, and on the other hand, for all the remaining agents the guru evolution acts as a forcing term. The respective evolution equations of the agents differ only in their respective initial values. Hence, the following Proposition 2 is coherent with the intuition:

Proposition 2 *If agents make their decisions according to the guru function, then the decision of each agent converges to the inclination of the guru.*

Proof: Let $g_0 \neq 0$ be a stable fixed point of the equation $\frac{dy}{dt} + y = a\tanh(y)$ (it exists by virtue of Lemma 1). This means that asymptotically (in long term) we can write, letting $g = e_{\widetilde{k}}$ for simplicity:

$$g(t) = g_0 + \gamma(t)$$

where $pg_0 = \tanh(g_0)$, $p = \frac{1}{a}$, and $\gamma(t)$ converges to zero with t tending to infinity. There exists t_0 such that for $t > t_0$, $\gamma(t) < \frac{\epsilon}{pg_0}$. For each agent i, we have

$$\begin{aligned}
\frac{de_i}{dt} + e_i &= \frac{1}{p}\tanh[g(t)] = \frac{1}{p}\tanh\left[g_0 + \gamma(t)\right] \\
&= \frac{1}{p}\frac{\tanh(g_0) + \tanh\left[\gamma(t)\right]}{1 + \tanh(g_0)\tanh\left[\gamma(t)\right]} \approx \frac{1}{p}\frac{pg_0 + \gamma(t)}{1 + pg_0\gamma(t)} \\
&\approx \frac{1}{p}\left[pg_0 + \gamma(t)\right]\left[1 - pg_0\gamma(t)\right] \\
&\approx \frac{1}{p}\left(pg_0 - (pg_0)^2\gamma(t) + \gamma(t)\right) \\
&= g_0 + B\gamma(t)
\end{aligned}$$

where we used that $\frac{1}{1+\epsilon} = 1 - \epsilon$ ($|\epsilon| \ll 1$).

The value B is not important. The change of the function $e_i(t) = g_0 + z_i(t)$ (the aim is to show that $z_i(t)$ converges to 0) gives

$$\frac{dz_i}{dt} + z_i = B\gamma(t)$$

We will show that $\lim_{t\to\infty} z_i(t) = 0$. Let us consider the equation

$$\frac{dz}{dt} + z(t) = B\gamma(t)$$

The homogeneous solution is

$$z(t) = z_0\exp(-t)$$

For $t > t_0$ the particular solution is of the form

$$z(t) = u(t)\exp(-t)$$

and it must satisfy

$$\frac{dz}{dt} + z(t) = \frac{du}{dt}\exp(-t)$$

$$\frac{du}{dt} = B\gamma(t)\exp(t)$$

We have

$$u(t) = B\int_{t_0}^{t}\gamma(u)\exp(u)du$$

The general solution is therefore of the form

$$z(t) = z_0\exp(-t) + u(t)\exp(-t) = z_0\exp(-t) + B\exp(-t)\int_{t_0}^{t}\gamma(u)\exp(u)du$$

Asymptotically, the exponential disappears and

$$z(t) = B\exp(-t)\int_{t_0}^{t}\gamma(u)\exp(u)du$$

Hence,

$$\left|\int_{t_0}^{t}\gamma(u)\exp(u)du\right| \leq \left|\frac{\epsilon}{pg_0}\int_{t_0}^{t}\exp(u)du\right|$$

$$\left|\int_{t_0}^{t}\gamma(u)\exp(u)du\right| \leq \frac{\epsilon}{pg_0}\left[\exp(t) - \exp(t_0)\right]$$

and therefore

$$|z(t)| \leq \frac{B\epsilon}{pg_0}\exp(-t)\left[\exp(t) - \exp(t_0)\right] = \frac{B\epsilon}{pg_0}\left[1 - \exp(t_0 - t)\right]$$

and

$$|z(t)| \leq \frac{B\epsilon}{pg_0}.$$

Since this converges to zero, we have for each i,

$$e_i(t) = g_0 + z_i(t) \to g_0.$$

■

As mentioned in Remark 1(iii), the identity function can be represented by $p_{ij} = \delta_{ij}$. The system of differential equations that we need to solve for each i is

$$\frac{de_i}{dt} + e_i = \tanh(ae_i).$$

Proposition 3 *If agents make their decisions according to the identity function, then the decision of each agent converges to his own inclination.*

Proof: Evident from Lemma 1. ■

5 Followers in the dynamic model

As recapitulated in Section 2.1, *followers* of a coalition S are the agents who always follow the inclination of S, assuming that S is unanimously inclined. The set of followers of coalition S under the influence function B is therefore defined as

$$F_B(S) := \{i \in N : \forall e \in E_S \ [B_i(e) = e_S]\},$$

where in our model, $B_i(e) = \text{sgn}\left(\sum_j p_{ij} e_j\right)$.

In order to determine the followers for every influence function B, first we consider a particular case with $S = \{e_1\}$. Agent i is the follower of S if and only if

$$\forall e \in E, \ \text{sgn}\left(\sum_j p_{ij} e_j\right) = \text{sgn}(e_1)$$

$$\Leftrightarrow \ \forall e \in E, \ e_1 \sum_j p_{ij} e_j > 0$$

$$\Leftrightarrow \ \forall e \in E, \ p_{i1} + e_1 \sum_{j \geq 2} p_{ij} e_j > 0$$

$$\Leftrightarrow \ p_{i1} > \sum_{j \geq 2} |p_{ij}|$$

Proposition 4 *Agent i is the follower of coalition S if and only if*

$$\sum_{j \in S} p_{ij} > \sum_{j \notin S} |p_{ij}|.$$

Proof: In the general case, agent i is the follower of S if and only if

$$\forall e \in E_S, \ \text{sgn}\left(\sum_j p_{ij} e_j\right) = \text{sgn}(e_k) = \text{sgn}(e_l) = ... \text{ for each } e_k, e_l \in S$$

$$\Leftrightarrow \ \forall e \in E_S, \ e_k \sum_j p_{ij} e_j > 0$$

$$\Leftrightarrow \ \forall e \in E_S, \ \sum_{j \in S} p_{ij} + e_k \sum_{j \notin S} p_{ij} e_j > 0$$

$$\Leftrightarrow \ \sum_{j \in S} p_{ij} > \sum_{j \notin S} |p_{ij}|$$

■

Note that for $|S| = 1$, if $p_{ik} < 0$ for each i, then $F_B(e_k) = \emptyset$.

Let us apply Proposition 4 to the majority influence function with $t = \lfloor \frac{n}{2} \rfloor + 1$. We have $p_{ij} = \frac{1}{n}$ for each i, j. For each S, agent i is the follower of coalition S if and only if

$$\sum_{j \in S} p_{ij} > \sum_{j \notin S} |p_{ij}| \ \Leftrightarrow \ \frac{|S|}{n} > \frac{n - |S|}{n} \ \Leftrightarrow \ |S| > \frac{n}{2}.$$

This shows that in case of the majority influence function, the followers of S depend only on the cardinality of S and

$$F_{\mathsf{Maj}^{[t]}}(S) = \begin{cases} N, & \text{if } \ s > \frac{n}{2} \\ \emptyset, & \text{if } \ s \leq \frac{n}{2} \end{cases}$$

where $s = |S|$, which is coherent with (1).

Let player $\widetilde{k}$ be the guru. We have then for each i

$$p_{ij} = \begin{cases} 0 & \text{if } \ j \neq \widetilde{k} \\ 1 & \text{if } \ j = \widetilde{k}. \end{cases}$$

Hence,

$$\sum_{j \in S} p_{ij} = \begin{cases} 1 & \text{if } \ \widetilde{k} \in S \\ 0 & \text{if } \ \widetilde{k} \notin S \end{cases}, \qquad \sum_{j \notin S} |p_{ij}| = \begin{cases} 0 & \text{if } \ \widetilde{k} \in S \\ 1 & \text{if } \ \widetilde{k} \notin S \end{cases}$$

and therefore

$$\sum_{j \in S} p_{ij} > \sum_{j \notin S} |p_{ij}| \ \Leftrightarrow \ \widetilde{k} \in S. \tag{19}$$

This result shows that in case of the guru function, the followers of S depend only on the presence of the guru in coalition S. We get then

$$F_{\mathsf{Gur}^{[\widetilde{k}]}}(S) = \begin{cases} N, & \text{if } \widetilde{k} \in S \\ \emptyset, & \text{if } \widetilde{k} \notin S \end{cases}$$

which is equal to (2).

For the identity function $p_{ij} = \delta_{ij}$ for each i, j. We have

$$\sum_{j \in S} p_{ij} = \begin{cases} 1 & \text{if } \ i \in S \\ 0 & \text{if } \ i \notin S \end{cases}, \qquad \sum_{j \notin S} |p_{ij}| = \begin{cases} 0 & \text{if } \ i \in S \\ 1 & \text{if } \ i \notin S \end{cases}.$$

Hence,

$$\sum_{j \in S} p_{ij} > \sum_{j \notin S} |p_{ij}| \ \Leftrightarrow \ i \in S. \tag{20}$$

This means that the followers of S under the identity function are the players of S and only them, i.e.,

$$F_{\mathsf{Id}}(S) = S$$

which confirms (3).

6 Boss and approval sets in the dynamic model

Using our dynamic model, we can also determine the boss and approval sets of command games. We have the following:

Proposition 5 *(i) $S \subseteq N \setminus i$ is the boss set for agent i if and only if*

$$\forall e \in E, \ \sum_{j \in S} p_{ij} > \sum_{j \notin S} |p_{ij}|.$$

(ii) $S \subseteq N \setminus i$ is the approval set for agent i if and only if

$$\forall e \in E, \ \begin{cases} \sum_{j \in S} p_{ij} + p_{ii} > \sum_{i \neq j \notin S} |p_{ij}| \\ \sum_{j \in S} p_{ij} \leq \sum_{j \notin S} |p_{ij}|. \end{cases}$$

Proof: It is clear from the proof of Proposition 4. ∎

Let us apply Proposition 5 to the majority influence function with $t = \lfloor \frac{n}{2} \rfloor + 1$. We have $p_{ij} = \frac{1}{n}$ for each i, j. S is the boss set for agent i if and only if

$$\sum_{j \in S} p_{ij} > \sum_{j \notin S} |p_{ij}| \ \Leftrightarrow \ \frac{|S|}{n} > \frac{n - |S|}{n} \ \Leftrightarrow \ |S| > \frac{n}{2}.$$

We have therefore

$$Boss_i^{\mathsf{Maj}^{[t]}} = \left\{ S \subseteq N \setminus i : s > \frac{n}{2} \right\}$$

which is coherent with (4).

S is the approval set for agent i if and only if

$$\forall e \in E, \ \begin{cases} \sum_{j \in S} p_{ij} + p_{ii} > \sum_{i \neq j \notin S} |p_{ij}| \\ \sum_{j \in S} p_{ij} \leq \sum_{j \notin S} |p_{ij}| \end{cases} \quad \Leftrightarrow \quad \begin{cases} \frac{s}{n} + \frac{1}{n} > \frac{n-s-1}{n} \\ \frac{s}{n} \leq \frac{n-s}{n} \end{cases}$$

We have therefore

$$App_i^{\mathsf{Maj}^{[t]}} = \left\{ S \subseteq N \setminus i : s = \lfloor \frac{n}{2} \rfloor \right\}$$

which is coherent with (5).

Let $\widetilde{k}$ be the guru. By virtue of Proposition 5(i) and (19), we have

$$Boss_{\widetilde{k}}^{\mathsf{Gur}^{[\widetilde{k}]}} = \emptyset \quad \text{and} \quad Boss_k^{\mathsf{Gur}^{[\widetilde{k}]}} = \{S \subseteq N \setminus k : \widetilde{k} \in S\} \text{ for } k \neq \widetilde{k}.$$

Using Proposition 5(ii), we conclude that S is the approval set for agent i if and only if $p_{ii} = p_{\widetilde{k}\widetilde{k}} = 1$. Hence, every set not containing $\widetilde{k}$ is the approval set for $\widetilde{k}$, and every agent different from $\widetilde{k}$ has the empty approval set:

$$App_{\widetilde{k}}^{\mathsf{Gur}^{[\widetilde{k}]}} = 2^{N \setminus \widetilde{k}} \text{ and } App_k^{\mathsf{Gur}^{[\widetilde{k}]}} = \emptyset \text{ for } k \neq \widetilde{k}.$$

We get therefore the results given in (6) and (7).

For the identity function, we have $p_{ij} = \delta_{ij}$ for each i, j. Using Proposition 5(i) and (20), we get $Boss_i^{\mathsf{Id}} = \emptyset$. By virtue of Proposition 5(ii), we conclude that S is the approval set for agent i if and only if $p_{ii} = 1$, that is, each set not containing i is the approval set for i. Hence, $App_i^{\mathsf{Id}} = 2^{N \setminus i}$, which gives exactly (8).

7 Conclusions

The paper concerns the influence model originally introduced in Hoede and Bakker (1982) and later studied in several of our previous works. We have proposed a new approach to analyze the dynamic of the model. This approach is based on the use of differential equations. To the best of our knowledge, the differential equations approach has never been applied to this influence model before.

To be more precise, we have introduced a dynamic model of influence in which an agent may give a certain importance to other agents in making his final decision. This importance is reflected by a weight which, depending of the sign (positive, negative or zero), corresponds to the stimulation, the inhibition, and the absence of relation, respectively. We have defined the exhortation obtained by an agent as the weighted sum of the opinions that the agent receives from the others. The updating rule is based on the sign of the exhortation: its positive (negative) value means going to the positive (negative) state, and the exhortation equal to zero corresponds to staying in the present state.

The main ideas of the application of this approach to the model of influence is to switch to the continuum case and to apply some approximations allowing the use of differential equations systems. The solutions of these systems give the same results obtained earlier for the classical influence functions. Moreover, the approach allows the study of new concepts, like e.g. the weighted majority function. It also leads to the results on followers obtained for the discrete case, and to the results on the boss and approval sets of command games.

In the paper we have shown that if the majority function is used, then the decision of each agent converges to the sign of the sum of the agents' inclinations, and in case of the guru function, the decision of each agent converges to the inclination of the guru. Under the identity function, the agents' decisions converge obviously to their own inclinations. Furthermore, we have described the necessary and sufficient condition that an agent is the follower of a coalition in the dynamic framework. We have used that result to determine the sets of followers for the majority function, the guru function, and the identity function. We have also determined the necessary and sufficient conditions that a coalition is the boss set or the approval set for an agent in the dynamic model. We have applied these results to determine the boss and approval sets of command games equivalent to the three influence functions in question: the majority influence function, the guru function, and the identity function. Both in the case of followers and boss and approval sets, the conditions depend on the relation(s) between the weights that reflect the importance given by the agents to the inclinations of the others in making their own decisions.

There are several possible research issues related to this model that could still be raised. In the future research on this framework, it would be interesting to study, in particular, non-linear influence functions and to consider temporary effects in the behavior of a group of agents.

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