The twofold role of diagrams in Euclid's plane geometry

Marco Panza

The Twofold Role of Diagrams in Euclid's Plane Geometry

Marco Panza*
CNRS, IHPST
(UMR 8590 of CNRS, University of Paris 1, and ENS Paris)

August 12, 2011

Proposition I.1 of Euclid's *Elements* requires to "construct" an equilateral triangle on a "given finite straight line", or on a given segment, in modern parlance[1]. To achieve this, Euclid takes this segment to be AB (fig. 1), then describes two circles with its two extremities A and B as centres, and takes for granted that these circles intersect each other in a point C distinct from A and B. This last step is not warranted by his explicit stipulations (definitions, postulates, common notions). Hence, either his argument is flawed, or it is warranted on other grounds.

According to a classical view, "the Principle of Continuity" provides such another ground, insofar as it ensures "the actual existence of points of intersection" of lines ([7], I,

*Some views expounded in the present paper have been previously presented in [30], whose first version was written in 1996, during a visiting professorship at the Universidad Nacional Autónoma de México. I thank all the people who supported me during my stay there. Several preliminary versions of the present paper have circulated in different forms and one of them is available online at http://hal.archives-ouvertes.fr/hal-00192165. This has allowed me to benefit from many comments, suggestions and criticisms and to change some of my views. I thank in particular, for their comments, suggestions and criticisms: Carlos Alvarez, Andrew Arana, Jeremy Avigad, Jessica Carter, Karine Chemla, Annalisa Coliva, Davide Crippa, Paolo d'Allessandro, Enzo Fano, Michael Friedman, Massimo Galuzzi, Giovanna Giardina, Bruce Glymour, Pierluigi Graziani, Jan Lacki, Danielle Macbeth, Paolo Mancosu, Sébastien Maronne, John Mumma, Michael Hallett, Ken Manders, Michael Otte, Mircea Radu, Ferruccio Repellini, Giuseppina Ronziti, and Ken Saito.

[1]The text of the *Elements* I refer to is that established by Heiberg ([6]). Quotations from it are drawn from Heath's translation ([7]), possibly with some local changes (often inspired by Vitrac's French translation: [8]). The term 'straight line' is used here to translate the Greek 'εὐθεῖα' used in place of the larger 'εὐθεῖα γραμμή'. This translation is mandatory, though, in most cases, Euclid uses these last terms to refer to segments (of straight lines).

235 and 242). M. Friedman ([9], 60) has rightly remarked, however, that in the *Elements* "the notion of 'continuity' [...] is not logically analysed" and thus there is no room for a "valid syllogistic inference of the form: $\mathcal{C}_1$ is continuous[,] $\mathcal{C}_2$ is continuous[, then] C exists" (where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two circles, of course).

A possible solution of the difficulty is to admit that Euclid's argument is diagram-based and that continuity provides a ground for it insofar as it is understood as a property of diagrams.

Proposition I.1 is, by far, the most popular example used to justify the thesis that many of Euclid's geometrical arguments are diagram-based. Many scholars have recently articulated such a thesis in different ways and argued for it[2]. The purpose of my paper is to reformulate this thesis in a general way, by accounting for what I take to be the twofold role that diagrams play in Euclid's plane geometry[3] (EPG, from now on)[4]. This goes together with accounting for some crucial aspects of EPG, *i. e.* with offering a partial but basic account of EPG as a whole[5].

[2]For a survey of recent literature about diagram-based arguments in Euclid's geometry, cf. [19]. Some other works not mentioned by Manders will be in what follows.

[3]Diagrams have to be carefully distinguished from "figures", in the sense established in definition I.14 of the *Elements*: cf. footnote (19), below.

[4]With 'Euclid's plane geometry' I mean plane geometry as it is expounded by Euclid in the first six book of the *Elements* and in the *Data* (and was largely practised up to early-modern age: cf. [33]). This should be confounded neither with plane Euclidean geometry in general, nor with elementary synthetic plane geometry ([42]). I take it to be a theory. Nevertheless, this term should not be understood in modern logical terms. EPG is a theory merely insofar as it is a closed framework characterised by a precise system of (informal) rules for obtaining objects and drawing conclusions about them: on this matter, cf. [33], 43-58 (sections 1, 2.1 and 2.2).

[5]One could wonder about the status of this account. Among the works recently devoted to argue for the essential role of diagrams in EPG, many (like [24], [14], [22], [25], and Mumma's contribution to the present issue) have suggested and discussed appropriate formal systems intended to capture some features of Euclid's arguments. More generally, these and other works (like [20]) have aimed to provide a modern philosophical account of Euclid's geometry that, though trying to reveal some important aspects of it, is not primarily intended to be faithful to the relevant sources. My ambition is different. I would like to offer an understanding of EPG which depends on a faithful interpretation of these sources (but, on this matter, cf. the remarks advanced at the beginning of section 2). Another different exercise would be that of reading EPG in the light of some coeval philosophical views or discussions. Though my account contrasts with a more customary understanding of Euclid's geometry, often taken to be Platonic in spirit (and supposed to have been suggested by Proclus: [36] and [37]), according to which it would deal with purely ideal objects, and could rather be taken to be close to the Aristotelian view that geometric objects result by abstraction from physical ones, it is quite far from my purpose to argue that Euclid was actually guided by an Aristotelian, rather than a Platonic insight. One the one hand, the *Elements* and the *Data* offer no unquestionable evidence for supporting such a claim, that should, in any case, be defended or rejected by relying on a discussion of a number of sources that I cannot offer here. On the other hand, the Aristotelian notion of abstraction does not fit well with the relation between geometric objects and

I take arguments in EPG to be about geometrical objects: points, segments of straight lines (segments *tout court*, from now on), circles, plane angles (angles *tout court*, from now on), and polygons. Hence, in my view, they cannot be diagram-based unless diagrams are supposed to have an appropriate relation with these objects. I take this relation to be a quite peculiar sort of representation[6].

Its peculiarity depends on the two following claims that I shall argue for:

C.*i*) The identity conditions of EPG objects are provided by the identity conditions of the diagrams that represent them;

C.*ii*) EPG objects inherit some properties and relations from these diagrams.

For short, I say that diagrams play a global and a local role in EPG to mean, respectively, that they are such that claims (**C**.*i*) and (**C**.*ii*) hold[7].

diagrams that I describe (while I consider that the understanding of the notion of continuity that I suggest to be at work within EPG is close to Aristotle's: cf. footnote 40), and the considerations of Plato's views advanced in section 1.2 suggest that also the Platonic nature of my account could be plausibly questioned.

[6]For short, I use the term 'diagram' in a restricted sense, so as to refer only the particular sort of diagrams that occur in EPG. If the same term is used in its usual larger sense, one should distinguish between what Norman calls "intrinsically depictive" and "intrinsically non-depictive" diagrams ([28]: 78). According to him, the former are those that "can represent in virtue of a similarity of visual appearance with its object(s)", the latter those that cannot. This does not seem to me a good way to make the distinction, however: if the relevant objects are abstract, nothing can represent them in virtue of a similarity of visual appearance since, if taken as such, abstract objects have no visual appearance (at most, they have it insofar they are associated with something else which has such an appearance, like a diagram). I'd rather distinguish between diagrams that are taken to display some properties and relations of some other objects (possibly abstract ones) which are associated to them, and diagrams that do not. It is natural to call 'representation' the (quite complex) relation that the former have with the objects associated to them. I'm interested here with a particular case of representation, in this sense. The term 'representation' is also used by C. Parsons, in a second sense, which, though close to this first one, is more general in one respect, and more particular in another. According to him ([34], § 7 and ch. 5), some mathematical objects are "quasi-concrete". These are abstract objects "distinguished by the fact that they have an intrinsic relation to the concrete", to the effect that they are "determined" by some concrete objects (*ibid.* 33-34). For Parsons, the particular nature of the relation between quasi-concrete objects and the corresponding concrete ones differs according to the kind of the quasi-concrete objects considered. Still, he generally calls this relation 'representation'. In my view, the objects that EPG is about are quasi-concrete, and this just depends on the relation they have with the relevant diagrams (which I take to be concrete objects). Hence, I consider the case of representation, in the first sense, I'm interested in to be also a case of representation in Parsons's sense. The main purpose of my paper can thus be understood as that of accounting for the particular nature that the relation of representation, both in the first and in Parsons's sense, acquires in the case of EPG objects.

[7]This use of the adjectives 'local' and 'global' should thus not be taken to evoke some more general perspective on the role of diagrams in EPG. These adjectives are merely used to call forth claims (**C**.*i*) and (**C**.*ii*).

As a matter of fact, I have no direct argument in favour of these claims, and I have no clear idea about how a direct argument in favour of these or similar claims (regarding the way EPG works) could be shaped. All that I shall do in order to argue for these claims is to explain them and, based on such an explanation, offer my partial account of EPG[8]. If such an account is taken to be plausible (and/or favoured over other accounts), then it provides an indirect argument in favour of (**C**.*i*) and (**C**.*ii*).

My plan is as follows. In section 1, I present this account in general terms. While sub-sections 1.2 and 1.3 are respectively devoted to the global and the local roles of diagrams, sub-section 1.1 is concerned with a crucial related question: that of the generality of EPG results (namely, theorems and solutions of problems). In section 2 I illustrate, then, this account through examples, by applying it to a relevant fragment of the first book of the *Elements*. Finally, section 3 provides some concluding remarks.

1 The Global and Local Roles of Diagrams

J. Klein ([16], 119-123) has argued for a distinction that characterises the essential difference between Greek science, especially Euclid's mathematics, and early modern mathematics. This is the distinction between "the *generality of the method* and the *generality of the objects* of investigation". According to Klein, whereas early modern (and modern) mathematics is characterised by the generality of the method, Euclid's mathematics can reach generality only by dealing with general objects: whereas the former "determine its objects *by reflecting on the way in which these objects become accessible through a general method*", the latter "represents the whole complex of those '*natural*' *cognitions* which are implied in a prescientific activity", and "does *not* identify the object represented with the means of its representation", insofar as its concepts "are formed in continual dependence on 'natural' prescientific experience". Klein sums up this difference by saying that early modern mathematics is symbolic, while Greek science, and then Euclid's mathematics, in particular, is not. He seems then to suggest that in EPG, the acquaintance with the relevant objects is through an "immediate insight" of them, which makes it quite difficult to reach generality, since, in this setting, generality can only results from having such an immediate insight of general objects.

My account of EPG is basically different, even diametrically opposed. I concede that EPG objects can be conceived of as forms of concrete objects, that is, of objects that we have an immediate insight of. But I do not take EPG objects to be general, and, despite

[8] For different, but (at least partially) complementary, insights about the role of diagrams in Euclid's and, more generally, Greek geometry, cf., among others: [27], ch. 1; [41], part 1; [1]; [28]; [20]; [17]. For a complementary account to the one offered in the present paper, I refer the reader to [33], sect. 2.

this, I deny one can have an immediate insight of them. In my view, the acquaintance with EPG objects passes though diagrams conceived of as concrete objects, but EPG objects are neither the diagrams themselves, nor types of which diagrams are tokens[9]. Doing EPG rather relies on a number of abilities for operating with diagrams to draw conclusions about abstract objects which these diagrams represent. Insofar as diagrams can be taken to be a symbols of the objects they represent, EPG can then be taken to be symbolic, though the sense in which it is so is a quite particular one, namely a sense that claims ($\mathbf{C}.i$) and ($\mathbf{C}.ii$) are intended to specify, insofar as they are purported to account for the relation of representation that links diagrams with EPG objects. The way EPG is general depends, in turn, on this very relation. In short it is general neither because its objects are general, nor because its theorems and solutions of problems say something about determinate totalities of objects, but rather because they assert that certain repeatable procedures cannot but have certain outcomes.

Though my paper is not mainly concerned with the generality of EPG, it is appropriate to begin my account by saying something more about this matter. This will possibly avoid misunderstanding, by making clear a crucial difference between this account and a very widespread view on EPG.

1.1 The Problem of Generality and the Schematic View

Geometrical objects are abstract. By contrast, I take diagrams to be concrete objects, though I admit of course they are tokens belonging to appropriate types[10]. So understood, a diagram is a configuration of concrete lines[11] drawn on an appropriate flat material support[12].

[9]Taken as concrete objects, any diagram is, of course, a token of a certain type. But this type is not, in my view, a geometric object that EPG is about, but merely a type of diagram. It follows that, in my view, EPG is neither an empirical theory, nor a contentual one, in Hilbert's sense, that is, a theory of "extra-logical discrete objects, which exist intuitively as immediate experience before all thought" ([13], 202).

[10]Cf. footnote (9), above.

[11]Anytime the context is not clear enough to avoid confusion between terms denoting some EPG objects and terms denoting the configurations of lines representing them, I add to these terms the adjectives 'geometric' and 'concrete', according whether they refer to the former or the latter. The clarification of the distinction between concrete and abstract objects is a crucial philosophical task, that, of course, I cannot undertake here. I hope however that what I mean by saying that diagrams are to be concrete objects and EPG objects are to be abstract is clear enough, or at least is clarified by my very account.

[12]Through many forms of expositions of EPG arguments rely on diagrams that are not drawn throughout the exposition itself (but completely drawn beforehand, or printed), in order to follow these arguments one has to conceive these diagrams are drawn while the argument progresses. This is the how EPG diagrams are conceived here.

These are compositional objects: a diagram can be either elementary or composed of other diagrams. A composed diagram includes other distinct diagrams (either elementary or composed, in turn). But diagrams can also be distinct without being included in a single composed one. For example, a diagram drawn on December 1st, 2010 at 15h23 by a colleague of mine on the blackboard of a classroom of Paris 1 University, and another drawn by Euclid himself on a vax tablet during one if his courses at the Alexandria Museum are certainly distinct, and certainly not included in a single composed diagram. I say that diagrams like these are mutually independent, and that each of them is self-sufficient.

I suppose that the (cognitive) abilities necessary to understand and practice EPG include: recognising types that a diagram belongs to as token; identifying the distinct elementary or composed diagrams that are part of a single composed one; distinguishing two mutually independent diagrams; identifying a self-sufficient one.

Consider two mutually independent diagrams, and suppose that both of them represent an equilateral triangle. Take for example two diagrams like those just mentioned representing an equilateral triangle, or those that are printed on my copy of Heath's translation of the *Elements* on a side of the text of propositions I.1 and I.10, respectively, or, again, those that are printed respectively on my copy of this same book and on that of my friend Ken, on a side of the text of the same proposition I.1. It is necessary, in order to understand and practice EPG, to be able to establish whether they represent the same equilateral triangle? I take for granted that the answer is negative. More than that, I take for granted that the question makes little sense, or better that there is no need to be able to ascribe a clear sense to it in order to understand and practice EPG[13]. Hence, I take claim $(\mathbf{C}.i)$ to mean that EPG objects are distinct insofar as they are represented by distinct diagrams or distinct elements of diagrams entering into a single composed self-sufficient diagram[14], and that no other identity condition for EPG objects is available (except in some particular cases, where appropriate stipulations are made)[15].

[13]Cf. [21], 354: "[...] it is senseless to ask [...][whether] the vertex A of a triangle ABC [...] is equal to or distinct from vertex A of a square ABCD in another diagram. [...] If the points were distinct, then by postulate a unique [straight] line would join them; but a line between two diagrams is senseless in Euclid's practice."

[14]To be more precise, one should consider the case of angles separately. This is made clear in sections 1 and 2.1. Also the case of points is particular, since, according to my account, points are, properly speaking, not represented by diagrams but by elements of diagrams, namely by extremities or intersections of concrete lines (cf. footnote (38) and section 2.1, below). For short, I say, in general, that diagrams represent EPG objects to mean that these objects are individually represented either by diagrams or by elements of diagrams. But I use a more precise parlance when I rely on the relation of representation between diagrams and EPG objects in order to specify the identity conditions of points.

[15]These cases are very easy to conceive. Throughout the course of a single argument it can be convenient, for example, to reproduce a certain diagram to the side of another, or under it, or even on a fresh page

One could retort that in order to understand and practice EPG it is no more necessary to ascribe a clear sense to the question whether two mutually independent proofs are concerned with the same objects. Accordingly, one could argue that the reason why this is not necessary (and it is even impossible, in fact), as well as it is irrelevant whether two mutually independent diagrams represent the same EPG objects, rests on the fact that EPG arguments are not about singular objects, but rather about something like general schemas, or, better, only about concepts. N. Tennant has articulated this view by arguing that in geometrical proofs like Euclid's, a singular term like 'the triangle ABC' is "no more than a placeholder in schematic reasoning", and the corresponding diagram "stands for no particular triangle" ([44], 303-304). For future reference, label a singular term like this 'diagrammatic'.

This view can be specified in different ways, but, *mutatis mutandis*, it is quite common. In his paper included in the first tome of the present issue, J. Mumma argues, for example, that in the theorem "that the three angle bisectors of a triangle ABC intersect in a point", "the triangle ABC is not one individual triangle" since "nothing about its position or orientation is specified, nor is anything specified about the relative magnitude of its sides and angles[, but rather][...] all these can vary continuously, and the theorem still applies" ([26], ???). He concludes that the default way to account for this is to render the as an universally quantified statement of the form '$\forall x\,[(x \text{ is a triangle}) \Rightarrow \ldots]$'.

There is no doubt that this theorem is general, that is, it does not concern a single triangle, whatever this triangle might be. But if one wanted to state it in the language used in the *Elements*, one should not rely on any diagrammatic singular term. Propositions I.16-20 suggests to state it as follows: 'in any triangle, the three angle bisectors intersect in a point'. But other formulations are possible, for example the following ones, respectively suggested by propositions I.5 and I.6: 'in the triangles, the three angle bisectors intersect in a point', or 'in a triangle, the three angle bisectors intersect in a point'.

Diagrammatic singular terms never enter into the statement of a geometrical proposition (whether might it be a theorem or a problem) of the *Elements* and the *Data*. They enter rather into their proofs or solutions. It is just because these are proofs or solutions of propositions that are rightly taken to be general, that it is often denied that the diagram-

or blackboard, though admitting that the new diagram represents the same objects as the former. In cases like these, two mutually independent diagrams are explicitly taken to represent the same objects (the same situation could also be described by saying that there is only a diagram-type with two or more diagram-tokens belonging to it, and that the relevant objects are represented by the former). These cases are governed be explicit local stipulations and they can be easily accounted for within the framework of the simpler, more common case where no identity condition is available for objects represented by mutually independent diagrams (and diagrams are nothing but tokens). For simplicity, in what follows I limit myself to consider this simpler case.

matic singular terms that enter into them refer to particular objects. Insofar as diagrams are related with these terms in such a way that there is no doubt that one such term refers to a certain particular object if and only if the related diagram (or element of a diagram) represents this very object, this is also the reason why it is also often denied that diagrams represent particular objects in Euclid's arguments.

This leaves the problem open of understanding the role that diagrammatic singular terms and the related diagrams play in these arguments.

One possibility is to deny that diagrams have any effective role and to maintain that diagrammatic singular terms work as dummy letters. An obvious difficulty with this solution is just the one I have begun with: if such a solution is adopted, the possibility of arguing that some of these arguments are diagram-based has to be discarded; another explanation for these arguments has then to be offered, or they have to be frankly taken as flawed.

But the idea that diagrammatic singular terms work in Euclid's arguments as dummy letters is also compatible with the admission that diagrams enter indispensably into these arguments. This is the case, for example, if it is maintained that a diagram represents a variety (or a multitude, a family, a class, etc.) of geometric objects or configurations of geometric objects which include all those that are in the scope of the relevant proposition. This is just Mumma's option[16].

The main problem that I see with this option is that it requires that the abstract objects that EPG is about form a fixed domain of individuals within which the scope of any proposition is somehow selected. This is also required if Euclid theorems are understood as universally quantified statements where the range of quantifiers includes geometric objects, as in statements of the form '$\forall x\,[(x \text{ is a triangle}) \Rightarrow \ldots]$' mentioned above. For this view to be appropriate, one should then provide some fixed and global identity conditions for these objects. In other words, one should explain what makes it that any one of these objects is definitively distinct from any other, that is, it is just one particular geometric object, for example one particular triangle. This requirement can be quite easily satisfied: it is enough to admit that these objects eternally exist as such, independently from EPG and from our practice of EPG, in agreement with the usual Platonic view; or, at least, that they exist within the space of EPG (this space being appropriately identified), in such a

[16] If I understand his point well, according to Caveing, this is rather what happens in pre-Euclidean geometry (especially Thales's). For him ([4], 73-75 and 148-149) this geometry was concerned with "schemas" understood as diagrams "given in visual intuition", whose "mode of being" was "the same as that of the decorative drawing", but whose "sense" whose not "aesthetic", being rather that "of representing a problematic situation", so as to open "a field of possibilities". Caveing holds, however, that things change radically with Euclid's geometry ([3], 155 and 164), since in it "empirical intuition is out of the question" and the continuum is not "a simple intuitive determination", to the effect that diagrams lose their essential role.

way that each of them has a distinct location in this space, as it is the case in the picture Mumma finds himself led to. The problem is that both these views hardly fit with the constructive language Euclid adopts and the role he assigns to constructions and solutions of problems[17].

The account that I'd like to suggest is quite different, and fits quite well, instead, with this language and this role. According to it, in EPG, general propositions are proved or solved by working on particular individuals. Of course, this goes with the problem of explaining how this is possible. This problem is not very different, however, from that of explaining how general conclusions can be soundly reached by working with schemas as those that EPG would be about, according to the schematic view. In both cases, what requires explanation is how it is that Euclid's arguments, which are *prima facie* particular, can support general conclusions. But, while in the latter case, it is taken for granted that these last conclusions are general insofar as they can be rendered through quantified statements where the range of quantifiers includes geometric objects, my account suggests another way of understanding the generality of Euclid's propositions. According to it, they are general insofar as they assert that some admitted rules to be followed in constructing geometric objects are such that these objects cannot but be constructed so as to have certain properties or relations, to the effect that any time one of them is constructed what is obtained is an object having these properties or relations. According to the understanding of the notion of being given that I shall describe in the following subsection, this could be easily rendered by saying that any given object of a certain sort is so and so or has this or that relation with whatever other given objects of the same or other sorts. Indeed, according to this understanding, a given object is not an object selected within a fixed totality of geometric objects, but rather an object constructed in a certain way that has neither existence nor determination independently of the act of constructing it[18].

Suggesting this way of understanding the generality of Euclid's propositions is certainly not enough to explain how they can be soundly proved or solved through arguments like Euclid's. Still, this provides, I think, an appropriate ground for explaining it. This is not the purpose of my present paper, however (I leave it possibly for other occasions). This is rather devoted to a particular aspect of a more basic question: a question that any alleged solution of the problem of generality of Euclid's propositions has to deal with as a preliminary. This is the question of understanding how Euclid's arguments work, and the

[17]For my account of the role of problems in EPG, cf. [33], sect. 2.2.

[18]Hence, to come back to Mumma's example, though I admit that the theorem he considers can be rendered in the language of EPG through the statement 'in any triangle, the three angle bisectors intersect in a point', according to the example of propositions, I.16-20, I suggest to understand the universal quantifier occurring in this statement and in these proposition as ranging not on the totality of triangles, but on possible acts of construction.

particular aspect I tackle here is, of course, that of the role that diagrams play in these arguments.

1.2 The Global Role of Diagrams

Like any other mathematical theory, EPG relies on stipulations. In the case of EPG, these can be understood as prescriptions addressed to the members of a relevant community that are supposed to have appropriate abilities for understanding, applying and following them.

Some of these prescriptions are intended to provide appropriate conditions for a geometrical object to be of a certain sort, that is, for it to fall under a certain (sortal) concept: these are the application conditions of this concept. Others are intended to provide appropriate conditions for an object that fall under a certain concept to be distinct from any other object that also falls under this concept: these are the identity conditions of the objects falling under this concept.

As said, EPG objects are points, segments, circles, angles, and polygons. All of them can be understood as configurations of points and lines, or as (that which is common to) equivalence classes of such configurations, this latter case being that of angles[19]. To provide the application conditions of a concept under which some EPG objects are supposed to fall is thus the same as providing the conditions that a configuration of geometrical points and lines has to satisfy in order to be—or, in the case of angles, in order to determine—an object falling under this concept. These conditions typically include two different sorts of requirements. For example, according to definitions I.19 and I.22, squares are systems of four equal segments sharing an extremity two by two so as to form four equal angles (or figures contained by four such segments)[20]. The requirement that they be systems of four segments sharing an extremity two by two so as to form four angles is different in nature

[19] For Euclid, circles and polygons are "figures" and, according to definition I.14, "a figure [σχῆμά] is that which is contained by any boundary or boundaries". This suggests making a distinction between a line or configuration of lines and that which this line or configuration of lines contains, if it is contour-closed (a portion of the plane, perhaps?). Strictly speaking, this distinction is not necessary, however, for EPG to run. What is necessary, rather, is a distinction between two equivalence relations among contour-closed lines or configurations of lines, namely congruence and surface-equality, in modern language The same Euclid suggests that the former distinction is not essential as such by using quite often the term 'circle [κύκλος]' to refer to what in definitions I.17-18 he calls 'circumference [περιφέρεια]': this is for example the case when he refers to the intersection point of two circles, like in propositions I.1. Still, if the reader attaches some importance to this distinction, (s)he can take circles and polygons to be that which appropriate lines or configurations of lines contain (and oppose them to these very lines or configurations of lines, namely to circumferences and contour-closed configurations of segments). The necessity of some terminological adjustments apart, this will have no influence on what I have to say in the present paper about EPG.

[20] Cf. footnote 19, above.

from the requirement that these segments and angles be equal. The former fixes a clause relative to the intrinsic morphological nature of the relevant configuration, namely to the nature and number of the elements that form such a configuration, and of their mutual spatial disposition. Broadly speaking, it is topological. The latter fixes a further clause relative to a condition that does not merely depend on the morphological nature of this configuration. Broadly speaking, it is metric. I suggest that requirements of the former kind are in fact relative to the conditions that certain diagrams have to meet in order to be suitable for representing the relevant objects[21].

Hence, one might say: geometrical points are geometrical objects represented by extremities and intersections of concrete lines (which can be taken to be concrete points)[22]; segments are geometrical objects represented by appropriate[23] concrete contour-open lines; circles are geometrical objects represented by concrete contour-closed lines, which are supposed to meet a further equality condition[24]; angles are the geometrical objects represented by pairs of concrete lines that represent segments or circles; polygons are the geometrical objects represented by systems of concrete contour-open lines representing segments, which share an extremity two by two and do not intersect each other (so as to form a contour-closed configuration).

To practice EPG, it is not enough to provide the application conditions of appropriate

[21]The distinction between these two kinds of requirements is close to that advanced by Avigad, Dean and Mumma ([25], 703) between topological and metric components of the meaning of certain Euclid's assertions. This last distinction is inspired, in turn, by Manders's distinction between exact and co-exact attributions and attributes ([20], 91-94). I shall come back to this last distinction in section 1.3, but, because of its influence and pervasiveness in the recent discussion about the role of diagrams in Euclid's geometry, it is important to make clear from now that my own distinction between topological and metric requirements cannot be accounted for by saying the former are co-exact and the latter exact, in Manders's sense. There are at least two reasons for that. The first is that Manders take straightness or circularity of lines to be exact attributes (*ibid.*, 92), while I consider, for example, that the requirement that a certain EPG object be composed by segments to be topologic. To understand the second reason, consider the requirement that the segments and angles forming a system of four segments sharing an extremity two by two be unequal to each other. According to Manders, such a requirement would concern co-exact attributes, whereas it would be, in my view, of the same kind as the requirement that these four segments and angles be equal. The differences between mine and Manders's distinctions are not so important, however. What is much more relevant for my account is that the application conditions of the concepts under which EPG objects are supposed to fall include two sorts of requirements, of which the former are relative to the conditions that certain diagrams have to meet in order to be appropriate for representing the relevant objects.

[22]Cf. footnotes (38) and (60), above.

[23]What 'appropriate' means in this case is clarified in section 2.1.

[24]Note that in EPG there are only two sorts of lines: straight lines or segments, and circles. Hence, the only relevant distinction among lines is between these two sorts, the former being contour-open, the latter being contour-closed.

concepts, however. It is also necessary to provide identity conditions for the objects that are supposed to fall under these concepts. Typically, an identity condition for the objects of a certain sort is stated through an instance of the schema '$x = y$ IFF $C(x, y)$', where 'x' and 'y' refer, or are purported to refer (as names, descriptions, or schematic constants) to single (though possibly undetermined) objects of the sort, and '$C(x, y)$' designates an equivalence condition relative to these objects or to other associated entities. Hence, in order to provide identity conditions for the objects that are supposed to fall under a certain concept, it is necessary to have a way to refer, or to purport reference, to such single (though possibly undetermined) objects. But in EPG, only given or supposedly given objects are liable to individual reference. Hence, EPG only includes identity conditions for given or supposedly given objects[25]. To understand the nature of these conditions is thus necessary to understand what 'given' means in EPG.

Though in the *Elements*, geometrical objects are often said to be given, the conditions under which an object is given are never explicitly stated. And this is no more done in the *Data*, whose definitions 1, 3 and 4 establish, rather, under which conditions appropriate geometrical objects are given-in-magnitude, given-in-form, and given-in-position, respectively[26].

C. M. Taisbak has discussed these definitions in detail in the comments of his translation of the *Data* ([43]). He has argued that the term 'given [δεδομένος]' means there the same as it usually means: "that an object is given to us means that it is, in some relevant sense and scope, put at our disposal" (*ibid.*, 18). In other words, the term 'given' occurs in these definitions as "a primitive needing no definition", and "the very concept of given remains undefined" (*ibid.*, 25 and 22). In his view, definitions 1, 3 and 4 of the *Data* merely establish the conditions under which "some objects are *also* given (in the said respect), besides [...] those that are already given" (*ibid.*, 25). Take the example of definition 1: "Given in magnitude is said of figures and lines and angles for which we can provide equals " (*ibid.*, 17). According to Taisbak, this definition establishes that an appropriate object x is given-in-magnitude if and only if "we can provide" something equal to it, and this is equivalent to stating that an appropriate object x is given-in-magnitude if and only if it is equal to an already given object a (*ibid.*, 29).

This interpretation leaves the crucial questions open: what does it mean to say that a

[25]To be strictly faithful to the language that Euclid uses in the *Elements*, one should say that in EPG only given or constructed, or supposedly given or constructed objects are liable to individual reference. This is because, as I shall emphasise later, Euclid use of the verb 'to give [δίδωμι]' is quite restricted. However, in what follows, I shall suggest a larger understating of the past participle 'given', which justifies my previous claim.

[26]Definition 2 of the *Data* establishes under which condition a ratio is given. The status of ratios in EPG is controversial, but for my present purpose it is not useful to consider this matter.

geometrical object is put at our disposal in some sense and scope, and is thus given (to us)? And what does it mean to say that we can provide something? This is not because Taisbak's interpretation is deficient. It is rather because Euclid's definitions do not aim to establish what 'given' means, in general.

Taisbak seems to suggest that we can provide an object a to which another geometrical object x is equal if and only if a is already given and x is provably equal to a. Hence, in definition 1 of the *Data*, Euclid would not employ the verb 'to provide [πορίζω]' as a synonym of the verb 'to give [δίδωμι]' actively understood, that is, so used as to indicate the action of putting a at our disposal. This verb would rather be used to mean the same as the verb 'to give [δίδωμι]', passively understood—that is, so used as to indicate that a is already given, *i. e.* not put at our disposal, but already at our disposal—provided that the objects to which this verb applies be so given that it be possible to prove that something else is equal to it.

But even if this were correct, it would not be enough to clarify what 'to give' means, either actively or passively understood. For this purpose, Taisbak relies on Plato's account of *Republic* VII, 527*a*-*b* (to which I shall come back pretty soon) and, on the base of it, he argues that "when mathematicians are doing geometry, describing circles, constructing triangles, producing straight lines, they are not really *creating* these items, but only *drawing pictures* of them"(*ibid.*, 27). Hence, for him, giving a geometrical object concerns the "Realm of Intelligence", where "The Helping Hand [. . .] takes care that lines are drawn, points are taken, circles described, perpendiculars dropped, etc." and keeps these operations "free from contamination of our mortal fingers"(*ibid.*, 28-29). Taisbak offers the example of postulate I.1 of the *Elements*, that licenses one "to draw [a] straight line from any point to any point". According to him, such a postulate should be understood as follows: "whenever there are two points, there is also one (and only one) straight line joining them", and the geometer is "permitted to behave accordingly, that is to conceive a picture of this line" (*ibid.*, 28).

This view is ambiguous. I see at least two ways to understand it. According to the first understanding, a geometric object can be actively given only if it is already passively given, and this last condition merely consists in its existing in the Realm of Intelligence, whatever this Realm might be. Its being actively given would then merely consist in its being selected among other objects that are passively given insofar as they are the inhabitants of such a Realm, and diagrams would thus be nothing but pictures that geometers use for their convenience, in order to denote the objects they successively select. According to the second understanding, it is not required, for a geometric object to be actively given, that it be passively given, or that it exist in some sense, but what makes it comes to be actively given is not an act fulfilled by a human geometer. It is instead an act of The Helping Hand (or even the mere willing of such a transcendent subject), any act fulfilled by a human geometer, possibly using diagrams, being rather a sort of material echoing of this superhuman act (or willing). The former understanding fits with the usual Platonic view I mentioned at the end of section 1.1. But the latter also can be taken to be Platonic (or better neo-Platonic), in some sense. According to the former understanding, constructions are nothing but means for identifying objects that already exist as such and are already distinguished from each other by their intrinsic nature. According to the latter, there are two sorts of constructions: those fulfilled by the Helping Hand, and their human echoing, which are the only ones that diagrams possibly enter into.

I have already said that the former understanding hardly fits with Euclid's constructive language and with the role he assigns to constructions and solutions of problems. Let me offer here additional comments which also applies to the latter understanding.

In order to do it, let us come back to the passage from Plato mentioned by Taisbak[27]:

> This at least [...] will not be disputed by those who have even a slight acquaintance with geometry, that this science is in direct contradiction with the language employed in it by its adepts [...] Their language is most ludicrous, though they cannot help it, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and applying and adding and the like, whereas in fact the real object of the entire study is pure knowledge [...][,] the knowledge of that which always is, and not of a something which at some time comes into being and passes away.

The phrase rendered in this translation with 'their language is most ludicrous, though they cannot help it' is as follows, in Greek: 'λέγουσι μέν του μάλα γελοίως καὶ ἀναγκαίως'. The adjective 'ἀναγκαίως' clearly means inevitability. As remarked by P. Shorey in a footnote to his quoted translation, what Plato is saying here is that "geometers are compelled to use the language of sense perception". This has been emphasised by M. F. Burnyeat ([2], 219), according to which Plato would not advance here a criticism of the language of geometry, in the name of an idealistic conception of geometry; he would rather argue that the use of a practical language is indispensable, since human beings can speak of the eternal, unchangeable, and purely intelligible objects of geometry only by referring (at least apparently) to other objects, temporary, changeable and sensible.

Here is the point, which Plato himself (if this interpretation is correct), was making. Even if it were admitted that the objects EPG is about exist (eternally) as such, and are distinguished from each other by their intrinsic nature, and/or that the constructions fulfilled by the geometers merely echo some other transcendent constructions, these objects

[27] I quote P. Shorey's translation: [35], vol. 6, which is also that quoted by Taisbak.

would enter into EPG, understood as a human geometry[28], only insofar the geometer is able to identify them and to distinguish them from each other through some appropriate, human way. This is because any account of EPG cannot avoid explaining how these objects can be identified and distinguished by us. And if one wants to do this without appealing to their being actively given by or to us, the only other explanation I know of (and that is surreptitiously admitted in many accounts of EPG), is by appealing to their disposition in space[29]. But, its is very hard to appreciate this disposition in absence of any external system of reference. So, it is enough to remark that in EPG no such system is available[30] to bring to light how both understandings of Taisbak's views on giveness result in an account of EPG that is at least incomplete.

This being said, I can pass to explain my understanding of the notion of being given in EPG, which is quite different from Taisbak's.

I begin by observing that definition I of the *Data* could be understood in a way opposite to that suggested by Taisbak, namely as stating that an appropriate object a is given-in-magnitude if and only if it is given (either passively or actively) and we can provide another object x which is provably equal to it. I also suggest that the verb 'to provide' be understood here as a synonym of 'to give' actively understood, or more precisely, that 'we can provide x' should be understood as meaning that x can be actively given, and that 'can' be taken at face value, so as to indicate that a modal operator is implicitly involved in the definition.

I shall come back soon to the way how such an operator should be understood. Before doing that, it is necessary to say that I take geometric objects to be given in EPG if and only if diagrams appropriate to represent them are canonically drawn, or imagined to be canonically drawn. Hence, they are actively given, or provided, by canonically drawing these diagrams, or by imagining to draw them canonically, and they are passively given if and only if these diagrams have been canonically drawn, or imagined to have been canonically drawn. I shall say in a moment what 'canonically' means. At present, it is enough to notice that, whatever it means, from this condition—and from the fact that, as I have argued below, EPG includes only identity conditions for given or supposedly given objects—it follows that these conditions apply only to objects represented by appropriate

[28]EPG understood as a human geometry is perfectly conceivable in Plato's framework as a sort of knowledge, namely as that sort of knowledge that results by connecting true opinions to each other: cf. *Meno*, 97*e*-98*a*.

[29]This is just Taisbak's option ([43], 19): 'The Plane is supposed to be full of points, and one is free to choose among them. The same holds to a certain extent for lines and line segments."

[30]Of course, the adjective 'external' is crucial here. In EPG, any given object supplies, indeed, a system of reference with respect to which the spatial disposition of any other given object can be appreciated. The point is that the availability of such an obvious system of reference requires that some objects be given.

diagrams, actual or imagined.

To avoid any misunderstanding, a clarification concerning my appeal to imagination is needed. To argue that diagrams play an indispensable role in EPG is not the same as arguing that, in order to practice EPG, it is necessary to actually draw diagrams. In many cases it can be enough (or even necessary) to imagine them (or to imagine drawing them). More than that: diagrams play an indispensable role in EPG just insofar as EPG includes prescriptions on how they are to be drawn; hence the understanding of these prescriptions (or at least some of them) cannot require drawing diagrams. Understanding them is rather a condition for gaining the ability to draw them[31]. What is crucial is thus not that diagrams be actually drawn, bur rather that, in the case they or their drawing are imagined, imagination is just imagination of diagrams (understood as concrete tokens) and their material drawing, and not imagination of some abstract objects (whatever imagination of abstracta might be)[32].

Take now 'x' and 'y' to refer to some given or supposedly given EPG objects. Under what conditions is x the same object as y? I suggest that the right answer is the following: if x and y are segments, circles or polygons, then x is the same as y if and only if they are represented by the same concrete line or configuration of concrete lines; if x and y are points, then x is the same as y if and only if they are represented by the same extremity of one or more concrete lines, or by the same intersection of concrete lines; if x and y are angles, then x is the same as y if and only if they are respectively represented by appropriate configurations of concrete lines (namely configuration formed by two such lines sharing an extremity) belonging to the same appropriate equivalence class[33]. This is how I understand claims ($\mathbf{C}.i$).

With this in mind, we can go back to the modal nature that I suggest to be assigned to definition 1 of the *Data*. This will also allow me to clarify what I mean with 'canonically drawn'.

For Taisbak, this definition should be schematically understood as follows:

$$x \text{ is given-in-magitude} \quad \text{IFF} \quad \text{an } a \text{ such that } a = x \text{ is passively given.}$$

[31]Whether the appropriate conveyance of these prescriptions requires that the apprentice look at some diagrams or the teacher draw them in front of the apprentice is a separate issue which I shall not discuss here.

[32]Also the notion of being supposedly given deserves a clarification. I shall come back to it in footnote (34). A negative remark is appropriate at once, however: that an object is only supposedly given is not be confused with its being represented with an imagined diagram, a diagram that is not (or has not been) actually drawn. The supposition here is not relative to the drawing of the relevant diagrams but to the giveness of the corresponding objects.

[33]The nature of these equivalence classes will be made clear in section 2.1, p. 35

I rather suggest to understand it as follows:

$$a \text{ is given-in-magitude} \quad \text{IFF} \quad \begin{array}{l} a \text{ is passively given,} \\ \text{and an } x \text{ such that } a = x \text{ can be actively given.} \end{array}$$

According to my interpretation, that an x can be actively given means that a diagram appropriate to represent x can be canonically drawn (that is, we can canonically draw such a diagram). Therefore, I take definition 1 of the *Data* to establish, for example, that a certain segment is given-in-magnitude if and only if a concrete line appropriate to represent it has been canonically drawn, or imagined to have been drawn, and a new concrete line appropriate to represent another segment equal to it can be canonically drawn, in turn.

To better illustrate this interpretation, let us consider proposition 4 of the *Data*: "if a given magnitude be subtracted from a given magnitude, the remainder will be given" (*ibid.*, 43). A given magnitude is a geometrical object given-in-magnitude, and this is also the case of the remainder. Euclid's proof begins as follows: "For, since AB is given, it is possible to provide a [magnitude] equal to it. Let it have been provided, and let it be DZ" (*ibid.*, 44). Then Euclid continues by repeating the same argument for a second pair of magnitudes—AC and DE—and concludes that as AB = DZ and AC = DE, the remainder of AB and AC is equal to that of DZ and DE and thus the former is a given magnitude, that is, it is given-in-magnitude. Euclid does not say that, since AB is given, it is equal to another magnitude DZ. He separates the claim 'it is possible to provide a magnitude equal to AB', from the claim 'let it have been provided'. Under my interpretation, this is the same as distinguishing the admission that a geometrical objects x could be actively given from the assumption that x be actively given.

His argument is general, but it is illustrated by a diagram where AB and DZ are depicted as segments (fig. 2). Suppose they are segments. I suggest Euclid's argument be interpreted as follows: AB is a (passively) given segment represented by an appropriate concrete line (that is taken to have been canonically drawn); it is then possible to canonically draw another concrete line that represents another segment equal to it; let this line be drawn and let DZ be the segment that it represents; DZ is thus (actively) given (to the effect that AB is then given-in-magnitude).

A crucial question still remains open: what does it mean that a diagram is canonically drawn, and thus a geometric object is actively given?

The example of proposition 4 of the *Data* cannot help us in responding this question, since, in this proposition, Euclid is reasoning in general, and thus he can only suppose that certain geometric objects be given or can be given. Let us rather consider proposition I.3 of the *Elements*, which corresponds, clearly, to a particular case of proposition 4 of the *Data*, insofar as it is a problem whose solution shows how to give the remainder of two given magnitudes in the case where these magnitudes are two segments. Here it is: "Given two unequal straight lines, to cut off from the greater [a] straight line equal to the less".

To solve this problem, Euclid refers to a diagram (fig. 3) including two separate dashes representing two given segments that he calls 'AB' and 'C'. The diagram also includes a third dash representing a segment AD equal to C that is taken to have been placed, according to the solution of proposition I.2, so that it shares with the segment AB one of its extremities, namely the point A. Finally, the diagram includes a contour-closed line drawn around A which is taken to pass through the point D, representing the circle with centre A and radius AD described according to postulate I.3. Euclid tacitly admits that this circle intersects AB in a point E, and concludes that this point cuts AB as required.

This suggests that a diagram—call it $\mathcal{D}$—is canonically drawn in EPG if a certain procedure for drawing diagrams, starting from some other diagrams representing some given objects and resulting in $\mathcal{D}$, is licensed by the stipulations of EPG, or if these same stipulations licence that a diagram such as $\mathcal{D}$ be taken as a starting point of licensed procedures for drawing diagrams. Thus, that a geometrical object can be actively given in EPG means that a procedure for drawing diagrams resulting in a diagram representing such an object is licensed by these stipulations, or that these same stipulations license that a diagram representing such an object be taken as a starting point of a licensed procedure for drawing diagrams.

The plausibility of this interpretation depends on the nature of the relevant licensed procedures. I shall consider this matter in section 2.2[34]. Here, I only need to say that these procedures result in what is usually called a 'construction [κατασκευή]'. Accordingly, I suggest that in EPG a construction is a licensed procedure for drawing diagrams ([11], 19; [41], 137, footnote 8), and that a diagram is canonically drawn in EPG if and only if it results from an appropriate construction or is a licensed starting point of such a construction ([28], 21 and 33).

[34]There is no need to specify the nature of these procedures to understand what it means in EPG that a certain object is supposedly given. This means that it is supposedly represented by an appropriate diagram canonically drawn. According to my understanding of the notion of being given, in order for a geometrical object to be given, an appropriate diagrams has to be, or to have been canonically drawn or imagined to be, or to have been canonically drawn. Hence, supposing that such an object is given is the same as supposing that such a diagram has been so drawn, or imagined to have been so drawn, possibly while this same diagram is freely drawn, instead, or imagined to have been freely drawn. In the most interesting cases this occurs in situations in which the suitable procedure to be followed for drawing this diagram canonically has not been yet established. The connoisseurs should have no difficulty in understanding that this latter case is typical of analytical arguments occurring in solutions of problems (though in Greek and early-modern geometry, the beginning of such an argument is usually indicated with the phrase 'let it be done [γεγονέτω] [*iam factus sit*]' which does not include the verb 'to give [δίδωμι]'). The literature on geometrical analysis is quite large. For my views on this matter, cf. [29] and [32].

In the *Elements*, Euclid uses the verb 'to give [δίδωμι]' to refer to geometrical objects that are taken as being passively given, as in expressions of the form 'a given x' or 'given x, to do such and such'. He instead uses different verbs when he requires that some objects be actively given, or claims that they have been actively given. Five of these verbs occur, for example, in postulates I.1-3 and propositions I.1-2: in Heath's translation, they are the verbs 'to draw [ἄγω]', 'to produce [ἐκβάλλω]', 'to describe [γράφω]', 'to construct [συνίστημι]', and 'to place [τίθημι]'. In my view, these verbs are used to require that particular appropriate procedures be applied in order to give certain geometrical objects. They would thus be particular specifications of the verb 'to give', actively understood[35].

A last remark. As a matter of fact, Euclid's explicit use of modal operators is limited. From Euclid's practice it is clear however that not every construction that could be applied in a particular situation is actually applied. If the notion of being given is understood as I have suggested, the appeal to a modal operator is useful to account for this practice[36]. For simplicity, in section 2, I shall use the phrase 'to be susceptible of being given' and its cognates to appeal to such an operator. Hence I shall say of a geometrical object that it is susceptible of being given to mean that it can be actively given, in the sense that I have just explained.

1.3 The Local Role of Diagrams

As mentioned, a diagram is a compositional object. Thanks to the global role of diagrams, this is also the case for EPG objects. But compositionality requires distinctions. Up to now, I have only explained how the identity conditions of diagrams transfer to EPG objects. Another problem is that of understanding how diagrams—especially sub-diagrams entering into a single self-sufficient composed diagram—are distinct, and how they combine with each other so as to give rise to composed diagrams representing single EPG objects.

[35] Note that in my account the verbs 'to draw', 'to produce' and 'to describe', as well as other ones with a close meaning, like the verb 'to join', are susceptible of being used in two distinct senses: either as applied to EPG objects (as particular specifications of the verb 'to give'), or as applied to diagrams. This double use could be avoided by appropriate conventions, but these conventions would hide an essential feature of EPG that I want rather to emphasise: the fact that EPG objects can only be given by drawing diagrams. Hence, phrases like 'to produce a segment' or 'to join two points' necessarily indicate both the performing of a geometric construction and the material act of drawing a diagram (or at least the imagination of this act). This is not confusion: it is rather the symptom of a peculiar characteristic of EPG.

[36] The modal nature of EPG has been emphasised in [5], 10.

1.3.1 Continuity

Many such distinctions and modes of composition depend on obvious (often implicit) stipulations. For example, the fact that a configuration of three appropriate concrete lines sharing an extremity two by two is taken as a diagram representing a single object, namely a triangle, whereas a configuration of two such configurations external to each other but sharing a vertex is not taken as a diagram representing a single EPG object depends on the presence and lack of appropriate definitions. But stipulations like these apply only if elementary diagrams have been detected, that is, only if it is specified what counts as an elementary diagram: a diagram that is not composed of other diagrams.

EPG diagrams are drawn via constructions, and these proceed by steps, in each of which a line is drawn. Postulates I.1-I.3 provide the basic clauses according to which this is done. They respectively license one to "draw" and "produce" segments, and to "describe" circles. On my understanding, this means that they license the drawing of concrete lines representing segments and circles. In section 2.2, I shall argue that constructions comply in EPG with some other implicit constructive clauses, one of which allows taking some segments as starting points of a construction. I suggest taking as elementary diagrams both the lines representing segments and circles whose drawing amount to an elementary (*i. e.* single) construction step, according to postulates I.1-I.3, and those representing segments which are taken as a starting points of a construction according to this latter supplementary clause[37]. This is because their being so drawn confers to each of them—so to say, by stipulation—the property of being a single diagram in a way that is more fundamental than that in which any possible configuration of lines whose drawing results from several construction steps can be taken to be a single diagram. Let us say, for short, that, in being a single diagram in this fundamental way, an elementary diagram is intrinsically one.

From this criterion, it follows that geometrical points are not represented by elementary diagrams. They are rather represented by extremities or intersections of lines, some of which

[37]According to this criterion, any line representing a circle is an elementary diagram, since there is no other way do draw such a line in EPG than by applying the constructive clause stated in postulate I.3. But for lines representing segments this is not so. A clear reason for this depends on postulate I.2 (another will become clear in few lines). This postulate licenses one "to produce [a] limited straight [line] continuously in [a] straight [line]". The adverb 'continuously [κατὰ τὸ συνεχὲς]' is open to different interpretations. Under that which seems to me the most plausible, it is intended to mean that the given segment (or limited straight line) is so produced as to form a new segment of which this given segment is a part. Let AB such a given segment. Producing it "continuously in a straight line" is then the same as drawing a line representing a new segment BC, sharing an extremity with the line representing the segment AB, and so placed with respect to it that that the two lines taken together also represent a segment, namely AC (rather than two segments forming a non-flat angle). It follows that the application of this postulate allows constructing segments (like AC) represented by concrete lines whose drawing does not amount to an elementary construction step, and which are not, then elementary diagrams, according to my criterion.

are elementary diagrams, whereas others are parts of such elementary diagrams resulting from dividing them through intersection. I shall come back to this matter in section 2.1[38]. Here it is rather important to ask what underlies the representation of points by means of intersection of lines (representing either segments or circles).

The thought seems to be that intersection of lines yields a division of these lines into actual parts which share en extremity. Let me explain this. In EPG diagrams, an intersection results when a line is drawn so as to cross another line already drawn anywhere but in one of its extremities, if it has any. In this way, a division is obtained on the latter line, whereas the former, so to say, is divided at the same time it being drawn as a single line (which is even intrinsically one). Such a division yields actual parts of these lines. The extremities of these parts represent points, which are then constructed by intersection. Passing from diagrams to the geometric objects they represent, this means that intersection of geometric lines (segments or circles) results in the construction of points insofar as it prompts the division of these lines along with the construction of some actual parts of them having these points as extremities.

This is just the case of proposition I.1, discussed at the beginning of the paper. During the proof of this proposition, point C (fig. 1) is constructed by intersection of the circles with centres A and B. Hence, these circles mutually separate each other into two actual parts having a common extremity which is just this point.

A relevant question is then the following: does this construction, so explained, depend on the property of continuity, intended in some appropriate sense, and attributed to some appropriate objects?

A natural response is that it depends on the continuity of the concrete lines representing the relevant circles. But, what does it mean that these lines are continuous, exactly? One could argue that this means that they do not suffer of any spatial interruption. But then, what would ensure us that this is so? Certainly not the inspection of the lines themselves. There are two obvious reasons for this. The first is that no such inspection can be fine enough to ensure that some small gaps have not been missed. The second is

[38]In many of his arguments, Euclid takes some geometric points to be represented by isolated dots, rather than by extremities or intersections of lines. This could be accounted for, however, by supposing that these dots are used as shortcuts for extremities or intersections of lines that play no other role in these arguments but that of displaying these points. The obvious disadvantage of this interpretation is that it is not mandatory and is based on no textual positive evidence (a negative piece of textual evidence will be mentioned in section 2.1). The advantage is that it allows one to take only segments (rather than segments and points) as starting points of EPG constructions (this will become clear in sections 2.1 and 2.2). This is the reason I adopt it. This is all the more advantageous in that it has no substantial consequences with respect to the plausibility of my account. One who prefers to be more faithful to the (positive) textual evidence and take isolated dots as elementary diagrams providing possible starting points of a construction, has nothing else to do but complicate my account a little bit, without changing anything substantial in it.

that nothing forces one, in conducting the proof, actually to draw these lines, rather than merely imagining to draw them.

One might think, then, that what matters here is not how the concrete lines representing the relevant circles are (supposing that they are actually drawn), but rather how they are required to be, namely the fact that they are required to suffer no spatial interruption. This also seems open to obvious objections, however. If it were so, nothing could assure us that this requirement is met. Moreover, drawing two concrete lines that—despite being intended to represent the relevant circles—display some evident interruptions does not prevent one from rightly conducting the proof. For this proof to hold, it is enough, indeed, that one admits that these line intersect, and this intersection yields a division of them into two actual parts having the point C as an extremity, which is an admission that does not depend on the properties of the concrete lines that one could actually draw or imagine.

This suggests another option. What matters here, is neither how the concrete lines representing the relevant circles are, nor how they are required to be, but rather how they are taken to be[39]. The point is then that Euclid's proof of proposition I.1 holds (among other things) because these lines are taken to be continuous. In my view, this means, in turn, that each of them is taken to be both intrinsically one and liable to be divided into actual parts having extremities[40].

But, one could retort, if this is so, why does one need to take diagrams into account? After all, one could directly take each segment and circle constructed throughout an elementary construction step to be intrinsically one—that is, an elementary component of a configuration of geometric objects—and to be distinct from one other, without considering diagrams at all. My answer is that, in this case, both their intrinsic unity, and their identity would not have a spatial character: they would not be distinct elementary spatial objects having distinct spatial positions, but merely distinct but indeterminate contents of intentional thought. Hence, in the absence of diagrams conferring to these objects their

[39] Note that taking a concrete object to be so and so is not the same as imagining it as being so and so. Suppose that the relevant concrete lines are not drawn but merely imagined. What matters is not how they are imagined to be, but rather how, in the imagination, they are taken to be. Indeed, one can imagine a concrete object as being in a certain way and being taken to be in some other way. Also note, however, that in the parlance I adopt here, the requirement that diagrams be taken to be in a certain way is compatible with the fact that these diagrams are, appear, or are imagined to be just in this way, as it is generally the case when they are accurately drawn or imagined. Hence, saying that a certain diagram is taken to be P is in no way intended to imply that it is not, does not appear to be, or is not imagined to be P.

[40] In my view, this explanation fits quite well with Aristotle's notion of continuity, as it is expounded in books V, VI and VIII of the *Physics*. However, I do not have the space here to expound my understanding Aristotle's conception of continuity. I can only refer the reader to [31]. This is a quite old paper, however, and I hope to have soon the opportunity to come back on this matter to refresh my analysis.

spatial nature, it would be hard to understand what could it mean that they are liable to be divided into actual parts having extremities. This is essentially a spatial property that geometrical lines have only insofar as they inherit it from the diagrams that represent them.

As a matter of fact, this is only a particular aspect of a much more general fact that the present section is intended to stress: insofar as doing EPG is not a formal deductive activity, it requires that any step of its arguments be supplied with a clear meaning; taking diagrams into account, at least in imagination, is a necessary condition for suppling this meaning.

It is here that claim (**C**.*ii*) enters into the account. It is thus urgent to clarify it. For reasons of linguistic simplicity, let us use the term 'attribute' to refer either to properties or to relations. And so, accordingly, let us stipulate that to say that some objects have a certain attribute is the same as saying either that one or more objects have a certain property or that some objects stay in a certain relation.

This being admitted, I say that EPG objects inherit an attribute P from diagrams if and only if P is an attribute of diagrams—that is, an attribute that diagrams can be taken to have—, and it is admitted that:

i) some EPG objects have a certain attribute and there is no other way to explain, within the setting of EPG[41], what it means that they have this attribute besides saying that they have P (and possibly explaining what having P means for some diagrams);

ii) EPG objects have this attribute (that is, according to (*i*), they have P) if and only if the diagrams that represent them are taken to have P (to the effect that the fact that some EPG objects have P is the same as the fact of having taken the diagrams that represent them to have P);

iii) if some EPG objects have P, and Q is an attribute of diagrams that complies with the conditions (*i*) and (*ii*), and is such that diagrams that represent these objects

[41]With 'the setting of EPG' I do not want to refer merely to the space of possibilities conceded by Euclid's definitions, common notions or postulates, taken as such, but, more generally, to the intellectual resources that are required in order to appropriately do EPG (which excludes, of course, those resources that are merely required in order to provide some account of EPG, like the present one; in other terms, what matters here are resources to be used within EPG, and not in order to reason about EPG) This is, of course, a vague notion, but this is a sort of vagueness that seems to me unavoidable in an interpretative enterprise like mine. To clarify it a little bit, I could say, perhaps, that in my view the specification 'within the setting of EPG' is, strictly speaking, redundant, since the sort of explanation that I'm referring to, here, is an explanation of an attribute of EPG objects, rather then of geometrical objects in general. For example, I'm not here concerned with triangles in general, but with triangles as they are conceived and treated within EPG.

(have to be taken to) have Q if they (are taken to) have P, then these same objects have Q.

Suppose that EPG objects inherit a certain attribute P from diagrams. I shall say, for short, that P is a diagrammatic attribute of these objects. I shall also for short take the liberty of speaking of diagrammatic attributes of diagrams when I mean the attributes of diagrams inherited by the corresponding geometric objects.

This being said, let us come back to the concrete lines entering into Euclid's proof of proposition I.1 and the corresponding circles. I have no difficulty in admitting that these last circles are continuous and even that this proof holds (among other things) because these circles are so. Still, I maintain that continuity is a diagrammatic property of them, to the effect that they are continuous just because the concrete lines that represent them are taken to be so. It is just because of this (and the fact that other relevant properties of the objects entering into this proof are diagrammatic) that this proof is diagram-based.

So understood, continuity is an essential property of some of the geometric objects that EPG arguments are about. But it is never explicitly ascribed to them by Euclid.

The adjective 'continuous [συνεχής]' and its cognates occur in the *Elements* in three distinct senses. They are mostly used to identify a sort of proportion, *i. e.* continuous proportion. Much more seldom, they are used either to specify that a segment is produced continuously (like in postulate I.2[42] and in the proof of proposition XI.1), or that several equal chords of a circle are placed continuously with each other so as to form an inscribed regular polygon (like in the solution of propositions IV.16 and XII.16)[43]. In none of these cases, are this adjective and its cognates used to indicate a monadic property of some EPG objects (that is, in order to say that some EPG objects are continuous). Still, one could maintain that the fact that some such objects are continuous is crucial for many (or even all) Euclid's arguments to work.

If this is so, I argue, continuity enters into these arguments as a diagrammatic property of EPG objects. My reconstruction of the crucial step of the proof of proposition I.1 is intended to illustrate this through an example. Furthermore, if it is admitted that continuity consists in intrinsic unity plus divisibility in homogenous parts, as I have just suggested, it follows that in EPG the role of continuity, understood as a diagrammatic property of EPG objects, is pervasive, since the very global role of diagrams depends on it.

[42]Cf. footnote (37), above.

[43]I thank an anonymous referee for suggesting this classification to me.

1.3.2 More on Diagrammatic Attributes

Continuity is certainly not the only diagrammatic attribute of EPG objects playing an essential role in EPG. Among others, one can mention the properties of having extremities, and of being contour-open or contour-closed, and the relations of intersecting each other, being formed by, being part of, lying inside, being included in, lying on or on opposite sides of, passing through, having an extremity on, sharing an extremity, and in general all the relations that depend on the respective position in space of the relevant objects, since EPG objects have (respective) positions in space only insofar as this is the case of the diagrams that represent them[44].

Thus, in order to better clarify claim (**C**.*ii*), I have to say more on diagrammatic attributes and their role in EPG. This is the purpose of the present subsection.

Let us begin by coming back to my definition of diagrammatic attributes.

Take, as an example, the relation ⌜x and y intersect⌝. Clearly, this relation can be taken to hold for two appropriate elements of a diagram. On the other hand, it is clear that two (geometric) circles stand in this relation if and only if each lies partially inside the other. This second condition is clearly stated within the setting of EPG. Hence, one could say that, within this setting, there is a way to explain the meaning of circle intersection besides saying that the relevant circles have the same relation as two concrete lines intersecting each other. Accordingly, my definition would be open to an obvious objection[45], namely it would leave the following alternatives open: *i*) intersection of circles is not a diagrammatic relation of EPG objects since it can be explained in terms of the lying of circles partially inside each other; *ii*) intersection of circles is a diagrammatic relation of EPG objects, but the laying of circles partially inside each other is not, since the former can be explained in terms of the latter; *iii*) neither of these relations is diagrammatic, since each of them can be explained in terms of the other.

My reply is that a logic equivalence like the previous one does not provide any explanation within the setting of EPG. Within this setting, it is merely a consequence we can draw from our mutually independent understanding of the two relations involved in the equivalence, both of which are diagrammatic, since this understanding depends inescapably on our understanding of the corresponding relation of diagrams[46].

Another important clarification that my definition of diagrammatic attributes of EPG

[44] This fits quite well with Reed's claim that the function of a diagram in EPG is "to exhibit the relationship of figures and their parts" ([38], 42). But I do not see why one should also maintain, as Reed does, that "to ask other things of the diagrams is to misunderstand the nature of Euclid's demonstrations" (*ibid.*).

[45] I thank J. Mumma for having attracted my attention to this conundrum.

[46] It is only in a logical reconstruction of EPG that one of these relations is possibly reduced to the other.

objects calls for concerns the requirement that diagrams be taken to have certain attributes. I have discussed this matter with respect to continuity. More generally, my idea is that no argument in EPG is based (or could be based) on the inspection (visual or otherwise) of diagrams, whereby one is able to judge that these diagrams have an attribute that EPG objects inherit from them. What matters are not the real, possibly microscopic, features of actual or imagined diagrams, but rather the features that are attributed to actual or imagined diagrams in the attribution of certain diagrammatic attributes to the corresponding EPG objects. This is just because we attribute some features to diagrams by attributing diagrammatic attributes to the corresponding EPG objects that we are able, in conducting EPG arguments, to draw diagrams that macroscopically manifest diagrammatic attributes of the corresponding EPG objects. And it is only because of this that, if diagrams are drawn accurately, a superficial inspection of them can reveal that the corresponding geometric objects have some diagrammatic attributes ([1], 25).

A last thing I can do to clarify my notion of diagrammatic attribute is to consider some examples of non-diagrammatic attributes. The most obvious are the relation of equality and order. Consider the simplest case: equality of segments. To say that two segments are equal in EPG does not mean, certainly, that the concrete lines that represent them are so. For an EPG argument involving two equal segments to work it is certainly not necessary that the concrete (possibly imaginary) lines that represent these objects be taken to be equal. The reason is clear: equality of segments is (implicitly) defined within EPG in terms of other relations among EPG objects, to the effect that, within the setting of EPG, there is a way to explain what it means for two segments to be equal besides saying that they have to each other the same relation of equality as the two concrete lines that represent them have. Hence, these segments can be equal even if these concrete lines are not taken to be so.

The complete (implicit) definition of the equality of segments offered by Euclid in the book I of the *Elements* is complex enough[47]. But for my present purpose, it is enough to consider the simple case involved in the proof of proposition I.1, that is, the case of two segments sharing an extremity. What makes them equal is the fact that their other extremities are on the circle having their common extremity as centre. This last condition certainly involves a diagrammatic attribute (that of having an extremity on), but this is an attribute of the relevant segments (not merely one of the concrete line representing them), and it is, *a fortiori*, distinct from the relation of equality among these concrete lines[48].

[47]I shall come back to this matter in section 2.4, p. 44

[48]To clarify this point, suppose that the two circles entering into Euclid's solution of proposition I.1 are represented by the two concrete lines BCD and ACE in figure 4 (in order to better clarify my view I consider here a deliberately inaccurate diagram). For the argument to work there is no need that the concrete lines AC and BC be taken to be equal. It is enough that these lines be taken to represent two

All this being said, we are now ready to consider two distinctions introduced by K. Manders. As these distinctions have played a pivotal role in the recent discussion of the role of diagrams in EPG, it is important to clarify the relation I take them to have with my notion of diagrammatic attribute.

The first is the distinction between two components of a "demonstration" in EPG: the "discursive text"—that "consists of a reason-giving ordered progression of assertions, each with the surface form of an ascription of a feature to a diagram"—and the diagram itself ([20], 86). According to Manders, a step in the discursive text "is licensed by attributions either already in force in the discursive text or made directly based on the diagram as part of the step, or both" and "consists in an attribution in the discursive text, or a construction in the diagram or both" (*ibid.*, 86-87).

The second distinction is that between "exact" and "co-exact" attributes or attributions (*ibid*, sect.4.2.2)[49]. Exact attributes "are those which, for at least some continuous variation of the diagram, obtain only in isolated cases". The latter "are those [...] which are unaffected by some range of every continuous variation of a specified diagram" (*ibid.* 92).

Manders's crucial claim is that an "exact attribution is licensed only by prior entries in the discursive text; and may never be 'read off' from the diagram", whereas "co-exact attributions either arise by suitable entries in the discursive text [...] or *are licensed directly by the diagram*" (*ibid.* 93-94).

In Manders's framework, geometric objects are absent. Euclid's geometry is described as an activity involving diagrams and discursive text in such a way as to bring to attributions whose semantic content and aboutness are never specified by appealing to any kind of abstract objects. For his exact vs. co-exact distinction to be able to play a role within my own framework, it has thus to be modified a little bit, and namely applied to attributes of geometric objects. In the case of diagrammatic attributes, their being exact or co-exact can be made to depend on Manders's very definition, under the condition that it be made clear that the condition of obtaining only in isolated cases for some continuous variation is not concerned with the real, possibly microscopic, features of actual diagrams, but with those that actual or imagined diagrams are taken to have. But in the case of non diagrammatic attributes, like equality or order, a different definition is needed.

Suppose to limit ourselves to relations or monadic properties resulting from saturating

segments, the concrete lines BCD and ACE be taken to pass respectively through the two extremities of the concrete line AB and to represent two circles of centres A and B, and the two concrete lines AC and BC be taken to share one of their extremities respectively with the two distinct extremities of the concrete line AB, and to have their other extremities at the intersection of the two concrete lines BCD and ACE.

[49]Of course, Manders takes attributions to be exact or co-exact according to whether they are attributions of exact or co-exact attributes. I shall adopt a similar convention when I speak of diagrammatic and non-diagrammatic attributions.

n-1 places in a n-place relation. One could then take exact attributes to be those that EPG objects have insofar as the configuration they belong to has certain properties that, for at least some continuous variations of it[50], obtain only in isolated cases. But for many monadic properties (that do not result from saturation of relations) like circularity, contour-openness, or contour-closure, it is difficult to see which variation should be relevant, supposing that such a variation is supposed to conserve the property of being an EPG object. To take only a simple example, it is hard to imagine how a circle could vary so as to cease to be circle, without ceasing to be an EPG object. Fortunately, for my present purpose, one can limit the exact vs. co-exact distinction to relations or monadic properties resulting from them by saturation.

What is relevant, indeed, is that co-exact and diagrammatic attributes of EPG objects do not coincide with each other: there are both co-exact attributes of EPG objects that are not diagrammatic, and, vice versa, diagrammatic attributes of EPG objects that are exact[51]. Examples of the former are the relations of being greater or of being smaller than. Examples of the latter are the relations of passing through, of lying on, of having an extremity on, and of sharing an extremity with.

According to Manders, diagrams can directly license co-exact attributions, but not exact ones. My account is compatible with both claims, if, of course, the relevant attributions are understood as made of geometric objects. To grasp why, it is necessary to understand how, on my framework, an attribution to a geometric object could be licensed by a diagram. This happens when such an attribution is licensed by the fact that the relevant diagram cannot but have (or be taken to have) some attributes, if it has (or is taken to have) some other attributes. If these attributes are diagrammatic, the inferences that depend on them transmit to the relevant geometric objects, and their conclusions consist, then, in attributions licensed by diagrams. An example is provided by the case of two contour-closed concrete lines each of which (is taken to) lay partially inside the other, which is involved in the solution of proposition I.1: these lines (have) necessarily (to be taken to) intersect each other.

It follows that Euclid's solution of proposition I.1 provides evidence for the occurrence in EPG of co-exact diagrammatic attributions to geometric objects directly licensed by diagrams.

Attributions to geometric objects directly licensed by diagrams do not exhaust, however,

[50]Presumably, Manders takes the term 'continuous variation' in an informal modern sense, the variations being relative to diagrams In my adapted definition, the same informal modern sense has to be conserved, but the variations should be taken to be relative to the relevant configuration of geometric objects.

[51]This makes clear that the sense I ascribe to the adjective 'diagrammatic' in the expression 'diagrammatic attribute' is different from that ascribed to it in the expression 'diagrammatic assertion' by Avigad, Dean, and Mumma, which use this expression just to refer to Manders's co-exact attributions ([25], 701).

the contribution of diagrams to the justification of the attributions to geometric objects in EPG. I have observed above that the condition that makes two segments sharing an extremity equal involves a diagrammatic attribute. This is only an example of a more general and crucial fact that is now time to account for.

To this purpose, let us come back to the example of proposition I.3 of the *Elements* already considered above.

The diagrammatic attribution to the circle and the segment AB (fig. 3) of the co-exact relation of intersecting each other is not directly licensed by diagrams. It is rather because the segment AB is greater then the segment C and the segment AD is so constructed to be equal to C, to the effect that AB is also greater than AD, that the circle intersects AB. Hence, insofar as this relation is diagrammatic, the diagram has to be taken to be so as to display it. Hence, a diagram where the concrete line representing the circle patently does not intersect the one representing the segment AB (fig. 3.1) would be considered to be inapt for manifesting the relevant diagrammatic attributes. Still, Euclid's argument includes no explicit justification for the inference from the premise 'the segment AB is greater than the segment AD and shares the extremity A with it' to the conclusion 'the circle of radius AD and centre A intersects AB'. This argument seems rather to depend on the reduction of the co-exact non-diagrammatic relation of being greater than, applied to two segments, to two diagrammatic relations: the exact relation of sharing an extremity, applied to these same segments, and the co-exact relation of intersecting each other, applied to one of these segments and the circle having the other segment as a radius. This reduction results from taking for granted the following condition: if two segments a and b share an extremity, then a is greater than b if and only if the circle of radius b and centre in the shared extremity intersects a.

One could retort that the fact that Euclid's argument does not include the mentioned explicit justification does not mean that such a justification cannot be provided. And, in fact, this can be done as follows. Suppose that the circle of centre A and radius AD does not intersect AB. Hence, it either passes through B or it is possible to apply postulate I.1 and produce AB up to meet the circle in a point F so as to get the new segment AF (fig. 3.2). In the former case, it is enough to appeal to definition I.15 to conclude that AD and AB are equal, in contradiction to what is ensured by the very construction of AD. In the latter case, from definition I.15 it follows that AD and AF are equal; but AB is part of AF and thus, according to common notion I.5, the latter is greater than the former and then also greater than AD (admitting that the relation of being greater than is transitive). Hence, it is not possible that the circle of centre A and radius AD does not intersect AB.

Diagrams enter into this argument in at least three respects. Firstly, to argue that either the circle passes through B or it is possible to produce AB up to meet it in F, one relies on diagrammatic evidences. Secondly, definition I.15 is used twice to reduce the exact non-diagrammatic relation of being equal, applied to two segments (AD and AB, in the first occurrence of this definition, and AD and AF, in its second occurrence), to two exact diagrammatic relations: the relation of sharing an extremity, applied to these same segments, and the relation of passing through an extremity of, applied to the circle having this same extremity as centre and one of these segments as radius and the other segment. Thirdly, common notion I.5 is used to reduce the co-exact non-diagrammatic relation of being greater than, applied to two segments collinear by construction (AF and AB), to the co-exact diagrammatic relation of being part of, applied to these same segments.

This justification of the inference that Euclid's argument leaves implicit shows that the reduction of a non-diagrammatic relation to two diagrammatic ones that enters into this argument can be based on two similar reductions: one involving exact relations, justified by definition I.15; and the other involving (like the original one) co-exact relations, justified by common notion I.5. The former of these reductions results from appealing to definition I.15 to get the following condition: if two segments a and b share an extremity, then a is equal to b if and only if the circle of radius b and centre in the sharing extremity passes through a. The latter of these reductions results, instead, from appealing to common notion I.5 to get the other condition: if two segments a and b are collinear, then a is greater than b if and only if b is part of a.

Other justifications of this inference could be suggested, but it seems to me that no such justification can avoid some similar reductions. Indeed, reductions like these seem to me pervasive in EPG. They manifest an aspect of the local role of diagrams. This aspect can be generally described as follows: the reduction of some non-diagrammatic attributes (both exact and co-exact) to some diagrammatic ones (both exact and co-exact, too) is necessary for some EPG arguments to hold. But these reductions are not only necessary for this. They also provide an explanation for non-diagrammatic attributes. To take only a single example, what does it mean, ultimately, in EPG, that two segments are equal to each other? The answer leaves no doubt: it means that two appropriate circles respectively pass though an extremity of each of them. I have already considered above a particular case, I shall come to the general case, and justify my claim in section 2.4.

A last remark is appropriate before finishing with the local role of diagrams. My purpose, here, is to account for the positive roles that diagrams have in EPG arguments. Hence, I have not insisted on the obvious fact that there are limitations to the practice of appealing to them while conducting EPG arguments. The example of the well known all-triangle-are-isosceles fallacy, also discussed by Manders ([20], 94-96), is often mentioned to bring out the necessity of being aware of these limitations. Another example is the following[52]. Let ABC (fig. 5) be a given triangle, D and E the midpoints of its sides AB and

[52] I thank J. Mumma for having suggested me this nice example.

AC, respectively, and F the intersection point of the perpendicular to these sides through D and E. Suppose A is joined to F and the circle of centre F and radius BF is drawn. Taking the diagram to include, besides point A, also a distinct point G, resulting from the intersection of the segment AF or its prolongation and this circle, is a mistake, whatever the given triangle ABC might be. This is because this circle provably passes through A, as Euclid proves in the solution of proposition IV.5. In my view, this case, as well as that of the all-triangle-are-isosceles fallacy, do not show that taking diagrams to be so and so requires a special discipline (as suggested by Manders: ([20], 96-104); but merely that diagrams have to be taken to be in agreement with the diagrammatic attributes that the geometric objects they represent provably have (or are supposed to have). This is the very limitation that the practice of appealing to them while conducting a EPG argument is submitted to[53].

[53]The case of *reductio ad absurdum* is often taken to be different and problematic for an account of Euclid's geometry that assigns an essential role to diagrams. In the framework of my account, I do not see, however, why one should concede that this is so. Take two usually mentioned examples (also discussed by Manders: [20], 109-115). One is that of the proof of proposition I.6. What is proved here is that a triangle with two equal angles also has two equals sides. Let ABC (fig. 6.1) be a triangle with $\widehat{\mathsf{ABC}} = \widehat{\mathsf{BCA}}$ Suppose that $\mathsf{AB} > \mathsf{AC}$. One can construct on AB a point D such that $\mathsf{DB} = \mathsf{AC}$. Trace the segment DC, so as to construct a new triangle DBC. As the side BC is common to the two triangles, $\mathsf{DB} = \mathsf{AC}$, and $\widehat{\mathsf{DBC}} = \widehat{\mathsf{BCA}}$, the two triangles are equal for side-angle-side, which contradicts the fact that one of them is included in the other. Clearly, this argument does not show any impossibility relative to the diagram involved in it. What is showed to be impossible is rather that the sides DB and AC of the two triangles DBC and ABC stand to each other in the non-diagrammatic relation of equality, provided that these triangles stand to each other in the diagrammatic relation of being included in. One could argue that this same argument could be associated with a diagram different from that provided in figure 6.1 (which is like that occurring in Heath's translation: [7], I, 255). Still, supposing that the relevant diagram be drawn so as to appear as it has to be taken to be for the argument to work, it could differ from the diagram provided in figure 6.1 only for its metric features. It could be, for example, like that provided in figure 6.2. Clearly this would make no difference for my point: in any case, the diagram plays here the roles I ascribe to it in my account, without any difficulty. The situation is a little bit more complex in the case of the second example, which is that of proposition I.27. What is proved here is that if two segments form with a third one two equal alternate angles, then they are parallel (the understanding of this theorem as being about segments rather than straight lines, in modern sense of this term, is suggested, it seems to me, by definition I.23). In all the six manuscripts of the *Elements*, considered by K. Saito in his "preliminary study" of the diagrams of the *Elements* ([40], 123), this proof goes together with diagrams analogous to that provided in figure 7.1, which is also like that included in Heath's translation ([7], I, 307). But in one of these same manuscripts (ms. **B**: Bodleianus Dorvillianus 301), two other diagrams also occur, like those provided in figures 7.2 and 7.3. The argument is as follows. Let AB and CD be two segments cut by the third segment EF so as that $\widehat{\mathsf{AEF}} = \widehat{\mathsf{EFD}}$. If these segments, when produced, meet in a point G, a triangle GEF is constructed, and this is such that its exterior angle $\widehat{\mathsf{AEF}}$ is equal to its internal opposite angle $\widehat{\mathsf{EFG}}$. But this is impossible for proposition I.16. While it is clear that the diagram provided in figure 7.2 plays the roles I ascribe to diagrams in my account without any difficulty, one can doubt, at first glance, that this is also so for the

2 The Construction of a Right Angle

So far, I have spoken of EPG in quite general terms. It is time now to put my account to the test of some examples. I shall do that by reconstructing, in the light of this account, a fragment of book I of the *Elements*: that which leads up to the solution of the problems stated in propositions I.11-12. Both these problems require constructing a right angle: the former requires constructing a perpendicular to another given segment through a given point on it; the latter requires constructing a perpendicular to a given straight line through a given point outside it.

In EPG, a rectilinear angle is given when two segments that share an extremity are given; and it is said to be right if and only if it is equal to another one, adjacent to it. To account for Euclid's solutions of these two problems, it is thus necessary to account for the way in which two such segments are given and for the conditions of equality of angles. This involves a number of fundamental ingredients of EPG, which I shall consider one after the other.

As said in footnote (5), my account of EPG depends on what I take to be a faithful interpretation of Euclid's texts. Interpretation is, however, something different from mere description and/or exegesis. According to my understanding, these texts provide an exposition of a mathematical theory, and my aim is to account for some basic aspects of this theory as such, not merely to gloss these very texts. It is thus not surprising that, in what follows, some slight discrepancies appear between my reconstruction and what the *Elements* say, literally. This is not a symptom of unfaithfulness. Rather, it depends both on the fact that, in offering an exposition his theory, Euclid has recourse to convenient artifices or shortcuts (for example, by considering a whole straight line, rather than a segment, in proposition I.12, so as to avoid a case distinction), and on my wish of avoiding useless assumptions or detours in my reconstruction (as in the case of my limitation to segments as starting point of EPG constructions already mentioned in footnote (38), above, or in the case of my choice to avoid appealing to proposition I.8 in the solution of propositions I.11-12). In my view, what licenses the changes I suggest with respect to Euclid's arguments is thus not the simple fact that these changes are slight, but rather the fact that they involve no alteration of Euclid's theory as such, but merely result in a simplification of it.

---

diagrams provided in figures 7.1 and 7.3. But this doubt is unfounded. Nothing forbids, indeed, taking the concrete lines AG and CG in figure 7.1 to represent two segments constructed by producing the given segments AB and CD, which meet each other at G, for the purpose of proving that this cannot happen if $\widehat{\mathsf{AEF}} = \widehat{\mathsf{EFD}}$. In the same way, nothing forbids taking, in figure 7.2, the prolongation of the concrete lines AB and CD to meet in some point G, again for the purpose of proving that this cannot happen if $\widehat{\mathsf{AEF}} = \widehat{\mathsf{EFD}}$.

2.1 Some Definitions from Book I of the *Elements*

My reconstruction begins with definitions I.1-4. They respectively state that: "[a] point is that of which [there is] no part"; "[a] line [is] breathless length"; "[the] extremities of [a] line [are] points"; and "[a] straight line is that [line] which lies evenly with [the] points on itself".

Each of them appeals to some notions that never occur later in the proof of theorems and in the solution of problems presented in the *Elements*, to the effect that no inference occurring in these proofs and solutions is openly licensed by these definitions. This is why many commentators argued that these definitions play no effective role in EPG (and someone even suggested that they are interpolated)[54]. I disagree. I suggest that the function of these definitions is that of fixing the way in which geometrical points and lines, especially straight ones, are represented in EPG, and of specifying which properties of the concrete objects that provide their representation are relevant for this purpose[55].

Definitions I.1 and I.2 are independent of each other and present an important disanalogy. Definition I.2 seems to presuppose a primitive cognitive capacity of distinguishing and isolating lengths as such. If this capacity is granted, this definition is sufficient to prescribe that geometrical lines be represented by concrete lines regarded only for their having a length. Definition I.1 is unable, in contrast, to prescribe that points be represented by some sort of concrete objects. It is merely a premise of definition I.3, which, in the light of it and of definition I.2, prescribes, in turn, that geometrical points be represented by extremities of concrete (and then ended) contour-open lines, regarded only as boundaries of a length[56].

Taken as such, definition I.3 does not prescribes that geometric points be represented only in this way: it provides a sufficient, but not necessary condition for representation of

[54]This is Russo's view ([39]), according to which these definitions (as well as definitions I.5-1.8, which are similar to them in this respect, but concern surfaces) are due to Heron, in fact.

[55]A similar view is advanced by Azzouni ([1], 126), and is also suggested by Shabel which argues that Euclid's definitions (I suppose she means some Euclid's definitions, including I.1-4) "enable the geometer to understand the implications of diagrams" ([41], 12). One could object against this view by arguing that it conflicts with the fact that in the *Elements*, no diagram is associated to the definitions. But, it seems to me that there is no such conflict. The function of diagrams in EPG is that of representing single given geometric objects, and they comply with this function only insofar they are canonically drawn, or supposed to be so drawn. Hence, it is perfectly natural that definitions go without any diagrams, since they define sorts of geometric objects and are advanced before establishing the rules for canonically drawing diagrams.

[56]Whatever the function ascribed to definitions I.1-I.3 might be, definition I.1 seems to be incomplete if taken alone and seems then to require completion by definition I.3. Denying this, Proclus (*Commentary*, 93.6-94.7; [37], 76) advances that the subject matter of geometry is established in advance, to the effect that this definition states, in fact, that a point is that which has no part "in geometric matter". This is clearly unsatisfactory, however.

geometric points. It does not exclude, for example, that geometric points be represented by concrete dots, and does no more preclude the possibility that geometric lines be composed of geometric points throughout all their length. *Prima facie*, this last possibility is even suggested by definition I.4, insofar as this seems to characterise straight lines (or segments) on the basis of a relation they have with the points on them (and not just with the extremities of them). This contrasts, however, with the absence of any other characterisation of points, namely of any positive characterisation, other than that provided by definition I.3, apt to supplement the purely negative condition stated by definition I.1. Hence, if admitted, both the supposition that geometric points could be represented in EPG other than by extremities of concrete lines, and the supposition that geometric lines be composed by geometric points would remain without any explanation. I prefer then, in my account, to take geometric lines as not being composed of geometric points, and geometric points to be represented only by extremities of concrete lines (by admitting that, when Euclid takes into account isolated geometric points represented by dots he is implicitly supposing that an appropriate line is given, though it is not considered as such)[57]. This has a relevant consequence: that points are given (*i. e.* constructed), only insofar as contour-open lines are given (*i. e.* constructed).

This leaves open the problem of explaining definition I.4, however. A natural way to do it is to maintain that this definition does not refer to the points composing a line, but rather to those that one could take on it. Supposing that a concrete line is drawn, Euclid seems to appeal, then, to two primitive cognitive capacities: that of taking concrete points anywhere on this line and that of appreciating evenness of this line with respect to these points[58]. Accordingly, definition I.4 seems to advance the prescription that geometrical straight lines be represented by concrete lines that, when considered among any two points that one can take on them, always appear as even (*i. e.* present no apparent curvature), and are regarded only with respect to their having a length and to their evenness. For short, call these concrete lines 'straight', in turn. Any of them is bounded (and thus contour-open). In general (that is, except for particular cases explicitly signalled), this is also the case of geometrical straight lines in EPG: they are segments, in fact.

Once the way in which geometrical points and lines are represented in EPG is fixed through definitions I.1-4, definitions I.8-9 do the same for angles. They also state what sort of objects angles are. They respectively state that: "[a] plane angle is the inclination to one another of two lines in [a] plane which meet one another and [which] do not lie in [a] straight line"; and an angle is rectilineal if "the lines containing [...][it] are straight".

[57]On this matter, cf. also footnotes (38) and (60).

[58]This seems to fit with Heath's conjecture that definition I.4 results from Euclid's "attempt [...] to express [...] the same thing as the Platonic definition", according to which a straight line is "that of which the middle covers the ends" (*Parmenides*, 137*e*; [7], I, 165 and 168).

Hence, angles are not merely pairs of lines that share an extremity: such pairs determine angles, but are not angles. Though Euclid generically mentions lines that meet one another, it seems clear that when one or both of these lines continue besides the point where they meet, what counts for determining an angle are their portions that end in this point, that is, the lines which have this point as their common extremity. For my purpose, only rectilineal angles are relevant. Let us focus on them. To fix the way they are represented in EPG, Euclid seems to rely, at least, on two primitive cognitive capacities: that of realising what it means that a pair of concrete straight lines share an extremity, without forming, if taken together, another such line; and that of grasping what such a pair of lines would have in common with any other pair of such lines sharing the same extremity and having the same (mutual) "inclination". Each of these pairs of concrete straight lines represents a rectilinear angle (or angle *tout court*, from now on), and all of them represent the same, to the effect that only the properties of such a pair of lines that depend on their mutual inclination are relevant to their representing this angle. It is thus enough to admit that such an inclination does not depend on the length of these lines, to conclude that such a length is not relevant. But, as concrete straight lines represent geometric ones only insofar they are regarded as having a length, it follows that their mutual inclination only depends on their respective position. In other words, two pairs of concrete lines representing segments and sharing an extremity represent the same angle if and only if the common extremities of these lines coincide and these same lines are respectively collinear.

Right angles are angles of a particular sort. Definition I.10 establishes that "when [a] straight [line] having been set up on [a] straight [line] yields adjacent angles [which are] mutually equal, [then] each of the equal angles is right, and the straight [line] standing upon [the other] is called 'perpendicular' to that upon which it stands". If straight lines and angles are understood as I have just proposed, this definition is clear enough, but it leaves open the problem of specifying the conditions under which two (distinct) angles are equal. I shall consider it in section 2.4. Before coming to it, I need to say something on constructions and common notions. I shall do it in the next two sections.

2.2 Constructive Clauses and Constructive Rules

In section 1.2, I suggested that EPG constructions be understood as procedures for drawing diagrams. These procedures obey a number of rules. My purpose requires that some of them be clarified.

In EPG, any construction starts out from some given objects. Hence, among these rules there should be one specifying which objects can be taken as given without resulting from a previous construction, and thus represented by appropriate freely drawn diagrams.

At first glance, postulates I.1 and I.3[59] suggest that constructions begin with points (as argued, for example, by Shabel: [41], 17-18). But, as I have just argued, according to definition I.3 geometric points are given if bounded geometric lines are, and are represented by extremities of concrete lines. Hence, any construction step that applies to some given point can only apply if appropriate bounded lines are given. This suggests the following basic rule for EPG constructions (which I have already shortly mentioned in section 1.3.1):

R.0 Any (finite) number of unrelated segments can be taken as given as starting points of a construction and then respectively represented with appropriate concrete lines freely drawn[60].

Once this rule is stated, the question becomes the following: supposing that a suitable number of unrelated segments are given (*i. e.* that a suitable number of appropriate concrete lines are freely drawn in any mutual position), what rules does a construction of other EPG objects starting from these segments have to follow?

[59] I have quoted the former of these postulates in section 1.2, p. 13. The latter licenses to "describe a circle with any centre and interval".

[60] It is obvious why the relevant segments are required to be unrelated: in order for two segments so related as to comply with a certain condition (for example being perpendicular to one other) to be given, a construction is needed (I refer the reader to [33], section 2.2, for a more detailed elaboration of this point). As observed in footnote (38), rule **R**.0 could be coupled with an analogous rule—call it '$\mathbf{R}.0_p$', for short—where unrelated segments are replaced by unrelated points. More than that: because of postulate I.1, rule $\mathbf{R}.0_p$ would make rule **R**.0 strictly useless, since, according to this postulate, a segment can always be constructed if two points are given. But also rule **R**.0 makes rule $\mathbf{R}.0_p$ strictly useless, since, as observed in footnote (38), one can always consider a given isolated point as an extremity of a given segment that play no other role in the relevant argument than that of having this point as an extremity. Take the example of proposition I.2, which is a problem requiring "to place at a given point [as an extremity] a straight [line] equal to a given straight [line]". Nothing forbids to consider that the given point is an extremity of a given segment that, as such, plays no role in the solution. Hence, though a more faithful rendering of Euclid's practice might suggest adopting both **R**.0 and $\mathbf{R}.0_p$, logical economy suggests adopting only one them. In [33] (55, footnote 27), I maintained that definition I.3 provides a reason for preferring **R**.0 over $\mathbf{R}.0_p$: it suggests that segments have priority over points by implying that two points are *ipso facto* given if a segment is so, whereas a segment is not *ipso facto* given if two points are so. Two other reasons are implicit in what I said in sections 2.1 and 2.1, respectively: *i*) adopting **R**.0 results in supplementing the three sorts of elementary construction steps fixed by postulates I.1-3 with another sort of elementary construction step which, in pertaining to the constructions of segments, is similar to those fixed by postulates I.1-2. Adopting $\mathbf{R}.0_p$, on the other hand, would result in admitting a sort of elementary construction step that, in pertaining to the construction of points, would differ from any one of those fixed by postulates I.1-3 (hence adopting **R**.0 is compatible with my criterion for elementary diagrams, while adopting $\mathbf{R}.0_p$ would require entangling this criterion without there being a logical reason to do so); *ii*) adopting $\mathbf{R}.0_p$ would presuppose a positive characterization of isolated points, whereas the only positive characterization of points offered by Euclid is that provided by definition I.3 that describes points as extremities of lines.

Each such rule has to apply to diagrams representing already given objects, and establishes which other diagrams can be drawn in the presence of them, and which objects these diagrams represent. In doing that, it specifies which objects are susceptible of being given, provided that some other objects are so. It has thus to be twofold. It has to include both a clause licensing the drawing of certain diagrams provided that others are already drawn, and an inference rule stating that some geometrical objects are susceptible of being given, if some other geometrical objects, meeting certain conditions, are given.

Postulates I.1-3 provide three rules like these[61]. Those provided by postulates I.1 and I.3 are easy to grasp:

R.1) If two points are given, then one and only one concrete line representing a segment joining these points can be drawn; hence, if two points are given, one and only one segment joining these points is susceptible of being given.

R.3) If two points are given[62], then two and only two concrete lines, each of which represent a circle having its centre in one of the given points and passing through the other, can be drawn; hence, if two points are given, two and only two circles, each of which has its centre in one of two given points and passes through the other, are susceptible of being given.

The rule provided by postulate I.2 is less easy to grasp. I have already discussed an aspect of this postulate in footnote (37). Another relevant question concerns the conditions

[61] The idea that postulates I.1-3 provide rules for drawing diagrams is also advanced by [1], 123-124. Mäenpää and von Plato ([18]) capture the twofold nature of the rules provided by postulates I.1-3, by rendering them as rules of introduction. For example postulate I.1 is rendered as follows (I write 's' and 'Segment' instead of 'l' and 'Line' to adapt Mäenpää and von Plato's rule to my language):

$$\frac{a : Point \quad b : Point}{s(a, b) : Segment}$$

Any segment introduced through this rule is a value of a two-variable function defined on points: if the given points are a and b, this segment is a value of this function for a and b as arguments. This supposes that the totality of points is taken as given, and any construction step depending on an application of this rule is understood as a procedure for fixing the value of this function for two specified arguments. Moreover, since Mäenpää and von Plato's system does not include diagrams, geometric objects are identified only as that which appropriate terms refer to. Both these circumstances make this system unable to account for some essential features of EPG. This is openly admitted by Mäenpää and von Plato, who state that their system is suited for describing constructions, but not for accounting for their grounds (cf. [18], 288-289).

[62] The interval mentioned in postulate I.3 (cf. footnote 59, above) can be understand either as a segment having an extremity in the point taken as centre, or as the distance between two given points. According to postulate I.1 and definition I.3 these understandings do not contradict to each other.

under which it licenses producing the relevant given segment. Does Euclid admit the possibility of producing it at will, that is, by continuing it through to an arbitrarily long new segment? In many cases (like in the constructions relative to propositions I.2 and I.5), it seems so. Scrutiny of his applications of this postulate shows however that any argument where Euclid avails himself of this possibility can be recast so as to become independent of it. Another interpretation of postulate I.2 is thus possible: a given segment is produced at will only insofar as this allows one to abridge a more complex construction in which postulate I.2 is only applied to license producing this segment only to meet another given line. Hence, shortcuts apart, the application of this postulate seems to be subject to the implicit condition that the relevant given segment could be produced so that the result of the corresponding construction step is not just the construction of two new segments, but also of a new point on this other given line, which is thus divided into two portions (either two segments or two arcs of a circle)[63]. It remains, however, that Euclid does not provide a general criterion for deciding whether a given segment can be produced so as to meet a given line. He simply relies on diagrams to decide whether this is so.

Together with what I have said in footnote (37), this suggests that postulate I.3 be understood so as to provide the following rule:

R.2) If a segment is given and the concrete line representing it is such that it can be continued so as to meet a concrete line representing another given segment or a given circle, then the former segment can be produced up to meet this other segment or this circle; hence, if a segment a and another appropriate line b (either a segment, in turn, or a circle) are given, then the following other objects are susceptible of being given: two other segments, one of which, let us say c, extends a up to b, while the other, let's say d, is formed by a and c taken together; a point on b at which both c and d meet it; two portions of b having this last point as a common extremity (either two segments or two arcs of circle).

Not all the inference rules entering into constructions in EPG are associated with a constructive clause and have a modal nature. The construction of a right angle requires the application of two constructive rules of inference that are not so. One of them is implicit in definition I.3. The other depends on the admission that the intersection of two

[63] An analogous disclaimer as that advanced in footnote (38) also applies here. One who prefers a reconstruction more faithful to the textual evidence, according to which segments are allowed to be produced at will, has nothing else to do but liberalise my rule **R**.2 so as to allow applications of postulate I.2 which fits better with this evidence. As it was the case earlier, no substantial change of my account would be required.

lines results in a point dividing each of these lines into two portions having this point as a common extremity. These rules are the following[64]:

R.4) If a segment is given, two points, consisting in its extremities, are also given.

R.5) If two lines intersect each other, then, any time they meet, a point is given where they meet, and this point divides each of them into two portions (either two segments or two arcs of circle) having it as a common extremity, which are also given.

2.3 Common Notions

Not all the rules of inference occurring in EPG are constructive (*i. e.* concerned with being given). Among those that are not, the most relevant concern the relations of equality, being a part of, and being greater (or smaller) than, and the procedures[65] of taking together and cutting off. In EPG, these relations and procedures applies to segments, polygons and angles, and thus they cannot be fixed once for all, since their nature depends on the nature of the objects they are applied to. But, whatever sort of objects they apply to, each of these relations and procedures are taken to satisfy some appropriate general conditions. In my view, the task of the common notions in the *Elements* is that of fixing some of these conditions[66]. Insofar as these notions are not only common to any sort of EPG objects, but extend also to tridimensional geometrical objects and to numbers, the conditions they fix are not merely restricted to EPG. Here, I limit myself to the case of EPG, however.

To put it in modern terms, common notions I.1-1.3 state that any assumed relation of equality among EPG objects has to be transitive (provided that its being symmetrical be taken for granted), and preserved under taking these objects together or cutting off one from the other.

Common notions I.4-6 (in Heiberg's numeration) are probably interpolated ([7], I, 223-224), since they follow from the previous ones: they respectively establish that any assumed relation of equality among EPG objects has to be such that if equal objects are respectively taken together with unequal ones, the resulting objects are unequal, and has also to be preserved under the passage to the double and the half.

Common notions I.7-8 (always in Heiberg's numeration) are quite different. They state respectively that two geometric objects that coincide with each other are equal, and that

[64]These rules correspond to the second and third ways "in which points enter into arguments in the *Elements*", according to Mäenpää and von Plato ([18], 286).

[65]I say 'procedures' rather than 'operations', since I do not take operations, at least in the modern (functional) sense of this term, to be present in EPG.

[66]A similar interpretation of Euclid's common notions is also suggested by Stekeler-Weithofer ([42], 136).

the whole is greater than the part. I take them as referring to diagrams and as having several functions. Taken alone, common notion I.7 seems both to state that geometrical objects that are represented by the same diagram have to be equal to each other—which, according to the global role of diagrams, is the same as stating that any assumed relation of equality has to be reflexive[67]—and to license the (non-constructive) argument that Euclid relies on for proving proposition I.4 (on which I shall come back in section 2.4). Taken alone, too, common notions I.8 seems to establish that a geometric object represented by a diagram that includes another diagram that represents another geometrical object of the same sort has to be greater than this latter object. Taken together, these common notions state, moreover, that when defined for the same sort of geometrical objects, the relations of being equal to, and being greater (or smaller) than have to be exclusive and to respect trichotomy.

The following considerations will show how some of these conditions apply in certain cases. More generally, one could note that they are crucial for allowing reduction of non-diagrammatic attributes to diagrammatic ones. The simple examples of such a reduction advanced in section 1.3.2 should be enough for illustrating this point. Other examples will be offered below.

2.4 Constructing Perpendiculars

Solutions of problems in EPG include two steps: the construction of some objects, and the proof that these objects comply with the conditions of the problem. In the case of propositions I.11-12, the latter stage consists in proving that two angles constructed in the former stage are equal to each other. This requires invoking some sufficient conditions for equality of angles.

In Euclid's solution of both propositions, this condition is provided by proposition I.8. This is a theorem: "if two triangles have the two sides equal to [the] two sides, respectively, and have the base equal to the base, [they] will also have [any] angle equal to [any] angle, [namely] that contained by equal segments". It follows that two angles are equal if they are included, or are susceptible of being included, into two triangles whose sides are respectively equal.

Through this condition, equality of angles is reduced to equality of segments. But this reduction is not merely a matter of stipulation; it is rather a matter of proof. The proof

[67] This is relevant, of course, when a single geometric object is conceived as complying with two distinct functions, for example when the same segment is conceived as being a side of two distinct triangles. It is cases like this that, in her paper included in the first tome of present issue, A. Coliva accounts for with the notion of seeing as.

relies on a previous condition provided by proposition I.4[68]. This is also a theorem, and it states, in modern parlance, the side-angle-side condition for congruence of triangles. In Euclid's parlance, this results from three distinct conditions, namely equality of triangles, of their sides, and of their angles: "if two triangles have two sides equal to [the] two sides, respectively, and have [an] angle equal to [an] angle, [namely] that contained by the two equal segments, [they] will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which equal sides subtend". Euclid's proof of proposition I.8 depends on the following condition extracted by this theorem: if an internal angle of a certain triangle is equal to an internal angle of another triangle, and the sides of these triangles including these angles are also respectively equal, then the remaining internal angles of these same triangles are respectively equal. This is also a sufficient condition for equality of angles, but, contrary to the condition provided by proposition I.8, it does not result in a reduction of equality of angles to a circumstance independent of equality of angles: according to it, two appropriate angles are equal to each other, if (among other things) two other appropriate angles are so. This is, so to say, an inferentially conservative condition.

Despite this difference, also this last condition, like that provided by proposition I.8, is a matter of proof, rather than a matter of stipulation. This proof is very peculiar and often questioned[69], for it is based on the possibility of rigidly displacing a triangle, as if it were a rigid configuration of rigid bars[70]. It is not only diagram-based, but also mechanical, so to speak: the constructive clauses and rules of inference that are applied in other EPG proofs and constructions are not enough to make it work[71]. Still, there is no alternative argument, complying with these constructive constraints to replace it[72]. The reason for this is directly

[68]To be more precise, Euclid's proof of proposition I.8 depends on proposition I.7, whose proof depends on proposition I.5, which is proved in turn by relying on the condition provided by proposition I.4.

[69]For two opposed views about this proof, cf. [7], I, 225-231 and 249-250, and [23], 21-26.

[70]At first glance, this is also the case of the proof of proposition I.8. But it is easy to see that, unlike that of proposition I.8, this proof admits a rephrasing according to which it is independent of rigid displacement.

[71]Let ABC and DEF (fig. 8) two given triangles such that $\mathsf{AB} = \mathsf{DF}$, $\mathsf{AC} = \mathsf{DE}$, and $\widehat{\mathsf{CAB}} = \widehat{\mathsf{EDF}}$. Euclid claims that, if the triangle ABC is rigidly displaced so that the respective members of these equalities coincide with each other, then the points B and C coincide with the points F and E respectively, and thus also the side BC coincides with the side FE, then, by common notion I.7, these last sides, the whole triangles, and their other internal angles are equal to each other. The problems with this argument do not arise just from its appeal to rigid displacements of triangles. Another problem is that no stipulation explicitly made by Euclid ensures that when a triangle is so displaced, its sides and angles coincide with other segments and angles that are supposed to be equal to them, respectively. For it to be so, the converse of the common notion I.7 has to hold for segments and angles, which Euclid seems to take for granted ([11], 34).

[72]Shabel ([41], 31-34) interprets Euclid's argument in such a way that it does not rely on any displacement. The basic idea is to appeal to the constructions involved in the solution of proposition I.2 (which

connected with the fact that the sufficient condition for equality of angles that proposition I.4 provides is inferentially conservative: its proof is based on the hypothesis that two given angle are equal, but no previous statement is made in order to explain what it mean for two angles to be equal[73].

Still, once proposition I.4 is accepted and proposition I.5 is proved thank to it, propositions I.11 and I.12 can be easily solved without using either proposition I.8 or on any other sufficient condition of equality for angles. Euclid's choice to appeal to proposition I.8 possibly depends on his desire to replace as soon as possible the inferentially conservative condition for equality of angles provided by proposition I.4 with another condition reducing equality of angles to a circumstance independent of equality of angles. This is reasonable enough from the perspective of composing a comprehensive treatise as the *Elements*, but from the more limited perspective of solving propositions I.11-12, a simpler argument based on propositions I.4 and I.5 alone could be preferred. To shorten my reconstruction, I confine myself to this simpler argument.

It begins with the solution of proposition I.1, which requires, as mentioned, the construction of an equilateral triangle on a given segment. Provided that segments are defined, and diagrams play their twofold role (so as to make the definition intelligible), the definitions of triangles present no difficulty: "a boundary is that which is [the] extremity of something" (def. I.13); "a figure is that which is contained by some boundary or boundaries" (def. I.14)[74]; "rectilinear figures are those which are contained by straight lines", or better segments; and, among them, "three-sided [ones are] those [contained] by three" (def. I.19). Also the definitions of equilateral, isosceles, and scalene triangles are simple, at first glance (def. I.20), but, for it to be meaningful, equality of segments has to be explained.

Definition I.15 supplies a ground for doing this. It establishes that "[a] circle is a plane figure contained by one line such that all the straight [lines] falling upon it from one point of those lying inside the figure are equal to one another". Despite this definition, Euclid often considers a circle to be a line rather than a figure[75]. Needless to say that the intelligibility of this definition depends on the twofold role of diagrams, which also allows one to infer

I shall consider later) to construct two segments having point D as their common extremity and being respectively equal to AB and AC. But this does not save Euclid's argument from being flawed, since, as Shabel correctly remarks, the construction involved in the solution of proposition I.2 does not warrant that the two other extremities of the new segments constructed through it coincide respectively with F and E, under the supposition that $\widehat{\mathsf{CAB}} = \widehat{\mathsf{EDF}}$.

[73]This is the reason Hilbert included a weaker version of this proposition among the postulates of his own version of Euclidean geometry ([12], post. IV.6, or III.6 or III.5, in other editions of Hilbert's treatise).

[74]Cf. footnote 19, above.

[75]Cf. footnote (19), above.

from it that any given circle encloses a given point such that all the given segments that have it as an extremity and whose other extremity is on this same line are equal to each other[76]. Far from requiring a previous explanation of equality of segments, this provides, as already observed in section 1.3.2, a sufficient condition for equality of segments that share an extremity, a condition which depends on a reduction of a non-diagrammatic relation to a diagrammatic one: two such segments are equal if a circle passes trough their other extremities.

Relying on this condition, proposition I.1 is easily solved through an argument which I have already discussed thoroughly. This provides a new constructive rule to be added to **R**.0-**R**.5:

R.6) If a segment is given, then four concrete lines representing as many other segments forming with it two equilateral triangles can be drawn; hence, if a segment is given, two equilateral triangles having this segment as a common side are susceptible of being given.

Suppose now that a new segment AB be given (fig. 9). Apply rule **R**.6 so as to obtain[77] the two equilateral triangles ABC and DBA, and then rule **R**.1 so as to obtain the segment DC. According to rule **R**.5, the point E is thus given, and, for the twofold role of diagrams, also the four angles $\widehat{\mathsf{AEC}}$, $\widehat{\mathsf{CEB}}$, $\widehat{\mathsf{BED}}$ and $\widehat{\mathsf{DEA}}$ are given. These angles are right, but, without further resources, there is no way to prove it. The appropriate resources are provided by propositions I.4-5.

Hence, were our problem that of constructing a right angle, without any supplementary condition, the previous quite simple construction, together with these resources, would be enough to solve it. But propositions I.11 and I.12 require more; they require the construction of two right angles whose sides meet some conditions that segments AB and CD are not required to meet. Thus, in order to solve these propositions, other constructions are needed. Still, once these constructions are performed, the same resources provided by propositions I.4-5 are enough also to prove that they solve propositions I.11 and I.12. Let us consider, then, propositions I.4-5.

[76]This contrasts with proposition III.1. This is a problem requiring the construction of the centre of a given circle. It implies, then, that a circle can be given without its centre being given. EPG provides, however, no possibility of constructing a circle without having previously constructing its centre, unless a rule for circles analogous to **R**.0 is admitted. There is then a tension between proposition III.1 and the constructive clauses admitted in EPG.

[77]In agreement with the terminology I have used in [33], I use the verb 'to obtain', as synonymous with the verb 'to provide' understood as I have said, in section 1.2, p. 15. The phrase 'to obtain a', where 'a' refers to a geometric object, is then intended to indicate the action of giving a (actively), *i. e.* of putting a at one's disposal, which, in the context of EPG, means that a is constructed in an appropriate way.

They are theorems. I have already discussed the former above. Here, I need only to add that this proof is based on the supposition that two distinct triangles having two sides and the angle included by them respectively equal are given. These triangles are not supposed to be equilateral, and the generality of the proposition requires that they are not be necessarily so. Rule **R**.6 is thus not enough for licensing this supposition. This can be done by relying on the solution of proposition I.2. This is a problem and requires the construction of a segment equal to another given segment having one of its extremities in a given point[78]. There is no need to go into the details of Euclid's solution. It is enough to say that it applies rules **R**.1-3 and **R**.6 explicitly and rule **R**.4 implicitly, and it provides, in fact, the following new constructive rule:

R.7) If a segment and a point distinct from both its extremities are given, then a concrete line representing another segment equal to the given one and having one of its extremities in the given point can be drawn; hence, if a segment and a point distinct from the extremities of this segment are given another such segment is susceptible of being given.

It is easy to see how this new rule can be applied, together with a construction inspired by the one that solves proposition I.1, in order to construct a triangle whose sides are equal to three given segments[79]. But the solution of proposition I.2 implicitly provides also a sufficient condition of equality of two given segments that do not share an extremity: they are equal if one of them is given insofar as it is supposed to have been constructed starting from the other one by performing the construction that solves proposition I.2[80]. Hence, the solution of proposition I.2 provides all the necessary resources for understanding the supposition on which the proof of proposition I.4 is based, except for the condition of equality of the two relevant angles, which Euclid cannot but leave unclarified.

Also proposition I.5 is a theorem and states that "the angles at the base of isosceles triangles are equal to one another, and, if the equal straight [lines] are produced further,

[78]Cf. footnote (60), above.

[79]Of course, for the construction to be possible, these segments have to be such that any two of them are, if taken together, greater than the remaining one. This is made clear by proposition I.20, which is a theorem asserting that any two sides of a triangle are, if taken together, greater than the remaining one. This explains why Euclid postpones the exposition of this construction until proposition I.22, which is a problem requiring the construction of a triangle whose sides are equal to three given segments, under the condition that they satisfy this condition: clearly, he wants to avoid stating a problem that is possibly unsolvable. On the construction of generic triangles, cf. also Proclus' *Commentary*, 218.12-220.6 ([37], 171-172).

[80]By coupling this condition with the solution of proposition I.3, already discussed in sections 1.2 and 1.3.2, one then gets a sufficient condition for a segment to be greater or smaller than another one.

the angles under the base will [also] be equal to one another". Once proposition I.4 is admitted, its proof presents no difficulty.

Let ABC (fig. 10) be a given triangle whose sides BA and CA are equal to each other. Euclid applies postulate I.2, so as to produce these sides at will on the side of B and C, then takes a point F at random on the prolongation of BA and a point G on the prolongation of CA such that $\mathsf{GA} = \mathsf{FA}$. Since what is essential in the proof is merely this last equality, one could avoid both extending BA and CA at will and taking a point at random on the prolongation of the former[81], by rather proceeding as follows: apply rule $\mathbf{R}.3$ so as to obtain a circle passing through A with centre in B; apply $\mathbf{R}.2$ so as to extend BA up to meet this circle in F; apply $\mathbf{R}.3$ again so as to obtain a circle passing through F and centre A; and finally, apply $\mathbf{R}.2$ again so as to extend CA to meet this last circle in G.

Howsoever the points F and G are constructed, the rest of Euclid's argument goes as follows. Apply $\mathbf{R}.1$ so as to obtain two segments joining the points F and G to the points C and B, respectively. According to the twofold role of diagrams, the two triangles ABG and AFC are thus given, and include the same angle at vertex $\widehat{\mathsf{BAC}}$. Hence, since $\mathsf{BA} = \mathsf{CA}$, $\mathsf{FA} = \mathsf{GA}$ and equality of angles is required to be reflexive (by common notion I.7), these triangles satisfy the condition of proposition I.4, to the effect that: $\widehat{\mathsf{ABG}} = \widehat{\mathsf{FCA}}$, $\widehat{\mathsf{BGA}} = \widehat{\mathsf{AFC}}$, $\mathsf{FC} = \mathsf{GB}$. Moreover, according to rule $\mathbf{R}.2$ and, again, by the twofold role of diagrams, segments FB and GC, and triangles BFC and BGC are also given, and segment FA is formed by the two segments FB and BA, just as the segment GA is formed by the two segments GC and CA. Hence, it is enough to reduce the non-diagrammatic relation among three collinear segments a, b and c that these segments have to each other when c results from cutting off b from a to the diagrammatic relation that these segments have when a is formed by b and c, in order to conclude, according to the common notion I.3, that $\mathsf{FB} = \mathsf{GC}$.

The triangles BFC and BGC satisfy, then, the condition of proposition I.4, to the effect that $\widehat{\mathsf{CBF}} = \widehat{\mathsf{GCB}}$ and $\widehat{\mathsf{FCB}} = \widehat{\mathsf{CBG}}$. The former equality corresponds to the second part of the theorem to be proved. The latter provides a lemma to prove the first part. For this purpose, it is enough to apply to angles an argument analogous to that just applied to

[81] I have already discussed Euclid's practice of extending segments at will in section 2.2, above (cf. footnote (63), in particular). The practice of taking points at random, both on given lines or not is different and, strictly speaking, not assimilable to the practice of supposing that isolated points are given as starting points of constructions (on which, cf. footnotes (38) and (60), above). Still, like the latter practice, the former is not allowed by the constructive rules admitted in my reconstruction, and can be avoided though appropriate constructions agreeing with these rules. Hence, I understand Euclid's appeal to this practice as a shortcut that replace these latter constructions. But, as before, nothing forbids one who prefers a reconstruction more faithful to the textual evidence to admit another appropriate constructive rule. This would require no substantial change in my account.

segments: angle $\widehat{\mathsf{FCA}}$ is formed by the two angles $\widehat{\mathsf{FCB}}$ and $\widehat{\mathsf{BCA}}$, and angle $\widehat{\mathsf{ABG}}$ is formed by the two angles $\widehat{\mathsf{CBG}}$ and $\widehat{\mathsf{CBA}}$, hence, it is enough to reduce the non-diagrammatic relation among three angles with the same vertex α, β, ζ that these angles have to each other when γ results from cutting off β from α to the diagrammatic relation that these angles have when α is formed by β and γ, in order to conclude, according to the common notion I.3 again, that $\widehat{\mathsf{BCA}} = \widehat{\mathsf{CBA}}$, as it was to be proved.

Once propositions I.4 and I.5 are proved, the solution of propositions I.11-12 presents no further difficulties.

Consider first proposition I.11. Let AB be a given segment and C a given point on it (fig. 11). The problem consists in constructing a perpendicular to AB through C. The first step is the construction of a segment collinear to AB having C as is its middle point. To do it, Euclid suggests taking another point at random on AB in such a way that the circle of centre C that passes through this point, constructed according to rule $\mathbf{R}.3$, cuts AB. The same can be achieved by avoiding to take points at random[82], by directly applying rule $\mathbf{R}.3$ so as to obtain the circle of centre C passing through one of the extremities of AB, let's say A. If this circle cuts AB in another point D, then the segment AD having C as is its middle point is given by the twofold role of diagrams. If this circle does not cut AB in another point, apply rule $\mathbf{R}.2$ so as to extend AB on the side of B to meet this circle in D and to obtain the segment AD having, again, C as is its middle point. To construct the required perpendicular it is then enough to apply rule $\mathbf{R}.6$, so as to obtain an equilateral triangle AED, and then rule $\mathbf{R}.1$, so as to obtain the segment CE. By the twofold role of diagrams, triangles ACE and CDE are thus given, together with the adjacent angles $\widehat{\mathsf{ACE}}$ and $\widehat{\mathsf{ECD}}$. These angles are equal to each other and, consequently, right. To prove it, Euclid relies on proposition I.8, by remarking that the sides of the triangles ACE and CDE are respectively equal. It is however obvious that propositions I.4 and I.5 are also appropriate for this purpose, since triangle AED is isosceles, to the effect that $\widehat{\mathsf{EAC}} = \widehat{\mathsf{CDE}}$.

The solution of proposition I.12 requires a bit more work. Let AB be a given segment and C a given point off it, for example an extremity of another segment CD. If these objects are arbitrary, there is no warrant that a perpendicular to AB through C be susceptible of being given. To avoid distinguishing the case where it is (fig. 12.1.1) from that where it isn't (fig. 12.2.1), Euclid supposes to be given an "unlimited straight line" (that is, a straight line in the modern sense of this term)—which is a quite infrequent supposition in the *Elements*—and a point off it, and requires the construction of a perpendicular to the line from the point. There is however a quite simple way to proceed in order to construct a perpendicular through C either to AB or to a prolongation of AB, if C does not lay on this prolongation.

[82] Cf. footnote (81), above.

It is enough to apply **R**.3 so as to obtain the two circles with centre C passing through A and B, respectively, and distinguish two cases: *i*) one of these circles, for example the one passing through B, meets AB twice (fig. 12.1.2): *ii*) neither circle does. If case (*ii*) obtains, apply rule **R**.2, so as to produce AB up to meet one of these circles, for example the one passing through B, in G. Two sub-cases are possible: *ii.i*) point C does not lay on the prolongation of AB (fig. 12.2.2); *ii.ii*) point C does lay on the prolongation of AB (fig. 12.2.3). In this last case, no perpendicular through C either to AB or to a prolongation of AB is susceptible of being given. In the two other cases, this perpendicular can be constructed through analogous constructions.

Here is how it works in the case (*ii.i*): apply rule **R**.2 so as to obtain the segments BC and GC; then apply rule **R**.6 so as to obtain the equilateral triangle BKG; finally, apply rule **R**.1, so as to obtain the segment KC. By the twofold role of diagrams the triangles BGC, BKC, KGC, BHC and HGC are also given. By reasoning on these triangles and on the triangle BKG according to propositions I.4 and I.5, it is easy to prove that $\widehat{\mathsf{BHC}} = \widehat{\mathsf{CHG}}$, so that HC is the perpendicular that was to be constructed.

The argument goes as follows: as the triangle BKG is isosceles, by proposition I.5 it follows that $\widehat{\mathsf{GBK}} = \widehat{\mathsf{KGB}}$; as BC and GC are radii of the same circle, by definition I.5 they are equal, to the effect that triangle BG, is isosceles, in turn, and, by proposition I.5, $\widehat{\mathsf{CBG}} = \widehat{\mathsf{BGC}}$; it is then enough to reduce the non-diagrammatic relation among three angles with the same vertex α, β, ζ that these angles have to each other when α results from taking β and γ together to the diagrammatic relation that these angles have when α is formed by β and γ, in order to conclude, according to common notion I.2, that $\widehat{\mathsf{CBK}} = \widehat{\mathsf{KGC}}$; hence, the triangles BKC, KGC meet the conditions of proposition I.4, to the effect that $\widehat{\mathsf{KCB}} = \widehat{\mathsf{GCK}}$; but then also the triangles BHC and HGC meet these same conditions, to the effect that $\widehat{\mathsf{BHC}} = \widehat{\mathsf{CHG}}$, as it was to be proved.

In Euclid's own formulation, the problem does not split into different cases. Its solution requires, however, that a point D be taken at random on the other side of the given straight line from C (fig. 13)[83]. Once this is done, Euclid prescribes one to apply rule **R**.3 so as to obtain the circle with centre C passing through D. For the twofold role of diagrams, this circle cuts the given straight line twice, in G and E. It is then enough to apply rule **R**.6 so as to obtain the equilateral triangle GKE, and then rule **R**.1 so as to obtain the segment KC. According to the solution of proposition I.9, this segment bisects the angle $\widehat{\mathsf{ECG}}$, and, according to the solution of proposition I.10, it also bisects GE in H. Hence, the triangles GHC and HEC have sides respectively equal, to the effect that, according to proposition I.8, $\widehat{\mathsf{GHC}} = \widehat{\mathsf{CHE}}$, as it was to be proved.

It is easy to see that this argument differs from that I have offered only by the suppo-

[83]Cf. footnote (81), above.

sition that an "unlimited straight line" be given, the admission that a point be taken at random, and the appeal to the proposition I.8 instead of the proposition I.4.

3 Conclusions

The solutions of propositions I.11-12—that I have just reconstructed—should provide an example of the role that diagrams play in EPG. This role appears both in constructions and in proofs, and depends on the relation that diagrams have with EPG geometrical objects.

My account of this relation focuses on two crucial aspects, that respectively pertain to claims (**C**.*i*) and (**C**.*ii*), and that I have tried to clarify in section 1. More generally, I have also suggested that diagrams are able to play their role since they are compositional objects, and each of the elementary objects that compose them (which are concrete lines representing segments and circles that are drawn through a single construction step) is both intrinsically one and divisible: these elementary objects can both compose complex diagrams and be decomposed in parts.

Intrinsic unity and divisibility are, in my reading, the two components of the notion of continuity that is at work in EPG. This notion, I argue[84], is Aristotelian, in nature. Still, this is not a sufficient reason for taking my account of EPG to be Aristotelian, though it is certainly opposed to a conception of EPG according to which this is merely a matter of contemplation of ideal, eternal truths, as it is often said to be, according to a commonly accepted Platonic orthodoxy. But it is not my aim to take a side in the dispute over the Platonic or Aristotelian nature of EPG. In conclusion, I would like, instead, to say something about the relation between my account of EPG and Kant's understanding of Euclid's geometry.

As observed by Friedman, in his paper included in the present issue, the view that a diagram-based account of such a geometry can be used to motivate an interpretation of this understanding has been advanced by Shabel ([41]). In his paper, Friedman argues against this idea. His basic reason is that "Kant begins with general concepts [...] and then shows how to 'schematise' them sensibly by mean of an intellectual act or function of the pure productive imagination", and that this depends on Kant's basic thesis that "pure intuition [...] lies in wait prior to the reception of all sensations [...] as an *a priori* condition of the possibility of all sensory perceptions and their objects" ([10], ??? [ms. 11-12]), which is motivated, in turn, by his aim "to explain how [...] both space itself and physical nature in space necessarily acquire their objective mathematical nature" ([10], ??? [ms. 27]). This is fully convincing to me. But this is not all: for accounts like mine, which focus on the

[84]Cf. footnote (40), above.

relation between diagrams and abstract objects, the situation is even worst, since, in Kant's view, there is no precise role for something like abstract objects in my sense.

Still, as Friedman himself extensively argues, sensible intuition is for Kant a crucial (and not purely contingent) ingredient of geometry. After all, for Kant, "we cannot think a line without *drawing* it in thought, or a circle without *describing* it" (*KrV*, B 154; [15], 167; also quoted by Friedman: [10], ??? [ms. 17]), and "although [...][the] principles [of mathematics], and the representation of the object with which this science occupies itself, are generated in the mind completely *a priori*, they would mean nothing, were we not always able to present their meaning in appearances, that is, in empirical objects" (*KrV*, A 239-244, B 299; [15], 259-260). Moreover, geometric construction is, for Kant, "ostensive", and it is so insofar as it is "construction [...] of the objects themselves" (*KrV*, A 717, B 745; [15], 579). Hence, it seems to me that in order to account for Kant's view on geometry, it is necessary to account, among other things, for the relation between Kant's pure productive imagination and some concrete objects that, according to this views, necessarily enter into geometry. More than that, it is the very role of these objects that have to be explained.

Arguing that these objects result from general concepts through a schematisation operated by productive imagination cannot be but only a part of this account. It is also necessary to say what features of the relevant concepts are displayed by the corresponding empirical objects, and how these objects enter into geometrical arguments concerned with these features. I suggest that my notion of diagrammatic attribute can help achieve this purpose. But, of course, working on this matter is not a task to be pursued here.

References

[1] J. Azzouni. Proof and ontology in euclidean mathematics. In T. H. Kjeldsen, S. A. Pederson, and L. M. Sonne-Hansen, editors, *New Trends in he History and Philosophy of Mathematics*, pages 117–133. University Press of Southern Denmark, Odense, 2004.

[2] M. F. Burnyeat. Platonism and mathematics: A prelude to discussion. In A. Graeser, editor, *Mathematics and Metaphysics in Aristotle*, pages 213–240. Haupt, Bern and Stuttgart, 1987.

[3] M. Cavaing. Quelques remarques sur le traitement du continu dans les *éléments* d'euclide et la *Physique* d'aristote. In R. Apéry et alii, editor, *Penser les mathématiques*, pages 145–166. Éditions du Seui, Paris, 1982.

[4] M. Cavaing. *La figure et le nombre. Recherches sur les premires mathématiques des Grecs.* Presses Universitaires du Septentrion, Villeneuve d'Ascq, 1997.

[5] C. Chihara. *A structural Account of Mathematics.* Clarendon Press, Oxford, 2004.

[6] Euclid. *Elementa, vols. I-IV of Euclidi Opera Omnia.* B. G. Teubneri, Lipsiæ, 1883-1888. Edited by I. L. Heiberg and H. Menge. 8 vols. + 1 suppl. New edition by E. S. Stamatis, Teubner , Leipzig, 1969-1977 (5 volumes in 6 tomes).

[7] Euclid. *The Thirteen Books of the Elements.* Cambridge Univ. Press, Cambridge, 2nd edition, 1926. Translated with introduction and commentary by Sir Thomas L. Heath; 3 vols.

[8] Euclid. *Les Éléments.* PUF, Paris, 1990-2001. 4 vols. French translation and comments by B. Vitrac.

[9] M. Friedman. Kant's theory of geometry. *Philosophical Review*, 94:455–506, 1985. Also in: M. Friedman, *Kant and the Exact Sciences*, Harvard U. P., Cambridge (Mass), 1992, 55-95. I refer to this last edition.

[10] M. Friedman. Kant on geometry and spatial intuition. *Synthese*, ????:????, ????

[11] R. Hartshorne. *Geometry: Euclid and Beyond.* Springer, New York, Berlin, Heidelberg, 2000.

[12] D. Hilbert. *Grundlagen der Geometrie.* Teubner, Leipzig, 1899. 2nd edition: 1903; 3th edition: 1909.

[13] D. Hilbert. Neubegründung der mathematik: Erste mitteilung. *Abhandlungen aus dem Seminar der Hamburgischen Universität*, 3(1):157–177, 1998. English translation in P. Mancosu (ed.), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, Oxford, 198-214. I quote from this translation.

[14] J. Mumma J. Avigad, E. Dean. A formal system for euclid's *Elements*. *The Review of Symbolic Logic*, 2:700–768, 2009.

[15] I. Kant. *Critique of Pure Reason.* Macmillan, Basingstoke, London, 1929. Translated by N. Kemp Smith.

[16] J. Klein. Die griechische Logistik und die Entstehung der Algebra. *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Abteilung B (Studien)*, 3(1 and 3):18–105 (n. 1: 1934) and 122–235 (n. 3: 1936), 1934-1936. English translation (of a slightly amended version) by E. Brann: *Greek Mathematical Thought and the Origin of Algebra*, MIT Press, Cambridge (Mass), 1968.

[17] D. Macbeth. Diagrammatic reasoning in Euclid's *Elements*. In B. Van Kerkhove, J. De Vuyst, and J. P. Van Bendegem, editors, *Philosophical Perspectives on Mathematical Practice*, volume 12 of *Texts in Philosophy*. College Pubblications, London, 2010.

[18] P. Mäenpää and L. von Plato. The logic of euclidean construction procedures. *Acta Philosophica Fennica*, 39:275–293, 1990.

[19] K. Manders. Diagram-based geometric practice. In P. Mancosu, editor, *The Philosophy of Mathematical Practice*, pages 65–79. Clarendon Press, Oxford, 2008.

[20] K. Manders. The Euclidean diagram (1995). In P. Mancosu, editor, *The Philosophy of Mathematical Practice*, pages 80–133. Clarendon Press, Oxford, 2008.

[21] C. McLarty. What structuralism achives. In P. Mancosu, editor, *The Philosophy of Mathematical Practice*, pages 354–369. Oxford U.P., Oxford, New York, etc., 2008.

[22] N. Miller. *Euclid and his Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry*. CSLI, Stanford, 2008.

[23] I. Mueller. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. MIT Press, Cambridge (Mass.), London, 1981.

[24] J. Mumma. *Intuition Formalized: Ancient and Modern Methods of Proof in Elementary Geometry*. PhD thesis, Carnegie Mellon University, 2006.

[25] J. Mumma. Proofs, pictures, and Euclid. *The Review of Symbolic Logic*, Forthcoming.

[26] J. Mumma. Constructive geometrical reasoning and diagrams. *Synthese*, ???:???, ???

[27] R. Netz. *The Shaping of Deduction in Greek Mathematics. A Study in Cognitive History*. Cambridge U. P., Cambridge, New York, Melbourne, 1999.

[28] J. Norman. *After Euclid. Visual Reasoning and the Epistemology of Diagrams*. CSLI Publications, Stanford, 2006.

[29] M. Panza. Classical sources for the concepts of analysis and synthesis. In M. Panza and M.Otte, editors, *Analysis and synthesis in Mathematics. History and Philosophy*, pages 365–414. Kluwer A. P., Dordrecht, Boston London, 1982.

[30] M. Panza. Continuidad local aristotélica y geometria euclidiana. In C. Alvarez and A. Barahona, editors, *La Continuidad en las Ciencias*, pages 37–120. Fondo de Cultura Económica, México D. F., 1992.

[31] M. Panza. Le labyrinthe du continu. In J.-M. Salanskis and H. Sinaceur, editors, *De la continuité comme concept au continu comme objet mathématique*, pages 16–30. Springer-France, Paris, 1992.

[32] M. Panza. What is new and what is old in viète's analysis restituita and algebra nova, and where do they come from? some reflections on the relations between algebra and analysis before viète. *Revue d'Histoire des mathématiques*, 13:83–153, 2007.

[33] M. Panza. Rethinking geometrical exactness. *Historia Mathematica*, 38:42–95, 2011.

[34] C. Parsons. *Mathematical Thought and Its Objects*. Cambridge Univ. press, Cambridge, New York, etc., 2008.

[35] Plato. *Plato, in twelve volumes with an English translation*. Harvard U. P., Cambridge (Mass), 1967-1986. 12 vols.

[36] Proclus. *In primum Euclidis Elementorum librum commentarii*. B. G. Teubner, Lipsiæ, 1873. Ex recognitione G. Friedlein.

[37] Proclus. *A Commentary on the First Book of Euclid's Elements*. Princeton University Press, Princeton, 1970. Translated with Introduction and Notes by G. R. Morrow.

[38] V. Reed. *Figures of Thought. Mathematics and Mathematical Texts*. Routledge, London and New York, 1995.

[39] L. Russo. The definitions of fundamental geometric entities contained in book i of euclid's *Elements*. *Archive for History of Exact Sciences*, 52:195–219, 1988.

[40] K. Saito. A preliminary study in the critical assessment of diagrams in greek mathematical works. *Sciamus*, 7:81–144, 2006.

[41] L. Shabel. *Mathematics in Kant's Critical Philosophy*. Routledge, New York and London, 2003.

[42] P. Stekeler-Weithofer. On the concept of proof in elementary geometry. In M. Detlefsen, editor, *Proof and Knowledge in Mathematics*, pages 135–157. Routledge, London and New York, 1992.

[43] C. M. Taisbak. *ΔΕΔΟΜΕΝΑ. Euclid's Data or The Importance of Being Given*. Museum Tusculanum Press, Copenhagen, 2003.

[44] N. Tennant. The withering away of formal semantics? *Mind and Language*, 1:302–318, 1986.

Figures

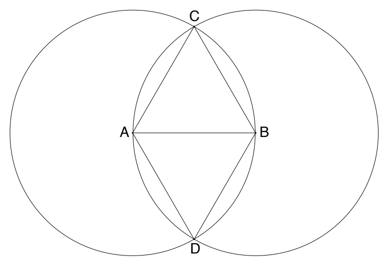

Figure 1

Figure 2

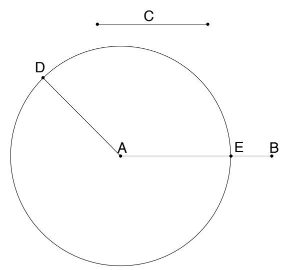

Figure 3

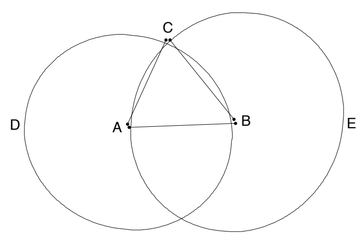

Figure 4

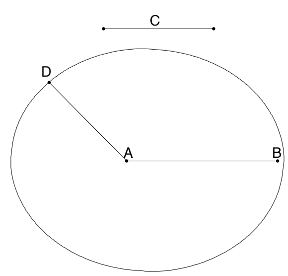

Figure 3.1

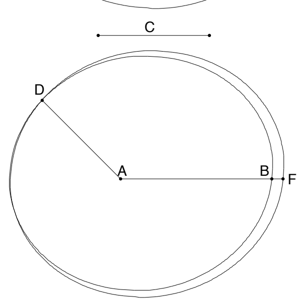

Figure 3.2

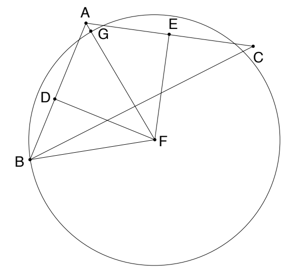

Figure 5

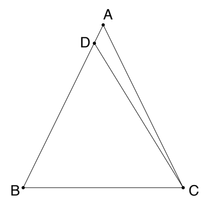

Figure 6.1

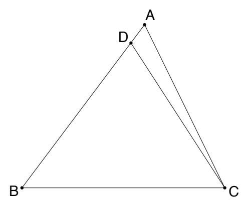

Figure 6.2

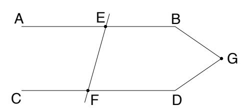

Figure 7.1

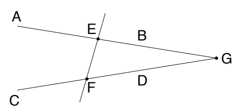

Figure 7.2

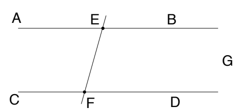

Figure 7.3

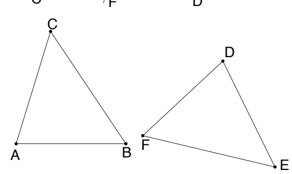

Figure 8

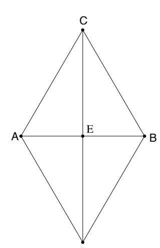

Figure 9

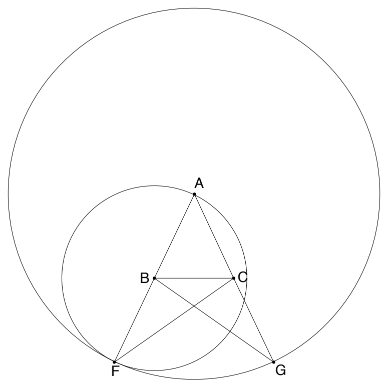

Figure 10

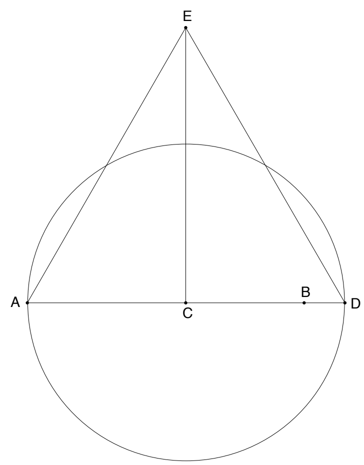

Figure 11

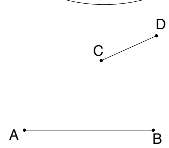

Figure 12.1.1

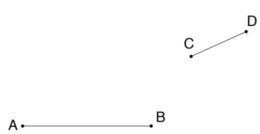

Figure 12.2.1

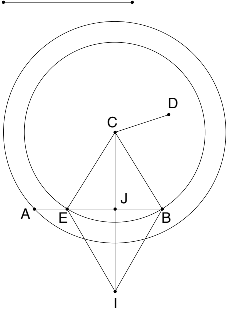

Figure 12.1.2

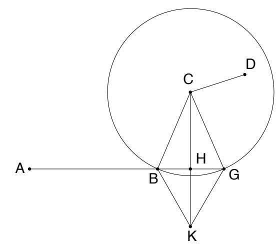

Figure 12.2.2

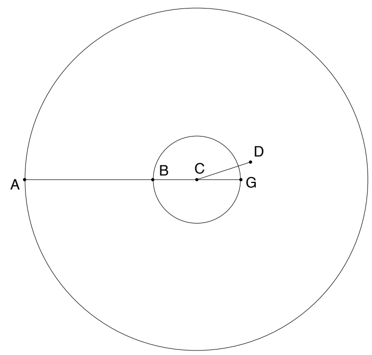

Figure 12.2.3

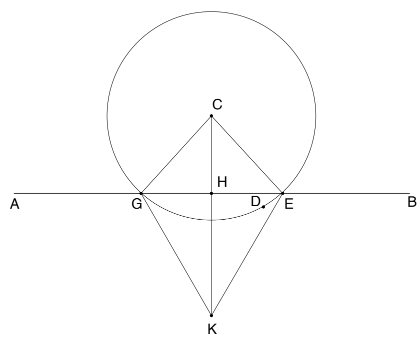

Figure 13