New axiomatizations of the Shapley interaction index for bi-capacities

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Abstract

Bi-capacities are a natural generalization of capacities (or fuzzy measures) in a context of decision making where underlying scales are bipolar. They are able to capture a wide variety of decision behaviours. After a short presentation of the basis structure, we introduce the Shapley value and the interaction index for capacities. Afterwards, the case of bi-capacities is studied with new axiomatizations of the interaction index.

Keywords: bi-capacity, Shapley value, interaction index, partnership of criteria

1 Introduction

Real-valued set functions are widely used in operations research [11], while capacities [3] have become a fundamental tool in decision making. There

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have been some attempts to define more general concepts, among which can be cited *bi-cooperative games* [2], in game theory, which generalize the idea of *ternary voting games* [4]. In the field of multicriteria decision making, there has been a recent proposal of more general functions, motivated by multicriteria decision making, leading to *bi-capacities*, which have been introduced by Grabisch and Labreuche [7]. Specifically, let us consider a set N of criteria and a set X of alternatives in a multicriteria decision making problem, where each alternative x is described by a vector of real valued score $(x_1, \dots, x_n)$. A *decision maker* may provide a capacity ν defined over 2^N, where $\nu(A)$ for any $A \subseteq N$ is the score of every *binary alternative* $(1_A, 0_{A^c})$: all criteria of A have score 1 and others, 0. Then it is well known that the *Choquet integral* enables to compute an overall score of the alternative x by interpolation between binary alternatives. Motivated with perceptible limitations of such a model, the decision maker may score alternatives of X on a bipolar scale in this way: to each *bi-coalition* (A, B) of criteria — positive vs. negative ones — a *ternary alternative* $(1_A, -1_B, 0_{(A\cup B)^c})$ is associated: every criterion of A (the *positive part*) has a score equal to 1 (total satisfaction), every one in B (the *negative part*) has a score equal to -1 (total unsatisfaction) and the others have a score equal to 0 (neutrality). Scores are given to each ternary alternative, which defines a bi-capacity.

Different *values* for bi-cooperative games [1, 13] have already been proposed and characterized, based on the Shapley value [15]. The concept of interaction index can be seen as an extension of the notion of value or power index. It is fundamental for it enables to measure the interaction phenomena modeled by a capacity on a set of criteria; such phenomena can be for instance substitution or complementarity effects between some criteria [8]. Our aim is to provide axiomatizations of the *Shapley interaction index* of a bi-capacity. Two of them are proposed: at first a *recursive axiom* is used by extension of the one of Grabisch and Roubens [10], and subsequently we work out the *reduced-partnership-consistency axiom* using the concept of partnership [5].

2 Capacities and bi-capacities

Throughout the paper, $N := \{1, \dots, n\}$ denotes the finite referential set. Furthermore, cardinalities of subsets $S, T, \dots$ are denoted by the corresponding lower case letters $s, t, \dots$.

We begin by recalling basic notion about capacities for finite sets [3]. A *cooperative game* $\nu : 2^N \to \mathbb{R}^+$ is a set function such that $\nu(\emptyset) = 0$, and ν

is said to be a *capacity* if $A \subseteq B \subseteq N$ implies $\nu(A) \leq \nu(B)$ (monotonicity condition). If in addition $\nu(N) = 1$, the capacity is said to be *normalized.*

Let us denote $\mathcal{Q}(N) := \{(A, B) \in 2^N \times 2^N \mid A \cap B = \emptyset\}$.

Definition 1 *A function* $v : \mathcal{Q}(N) \to \mathbb{R}$ *is a* bi-capacity *if it satisfies:*
(i) $v(\emptyset, \emptyset) = 0$.
(ii) $A \subseteq B$ *implies* $v(A, \cdot) \leq v(B, \cdot)$ *and* $v(\cdot, A) \geq v(\cdot, B)$.
In addition, v *is* normalized *if* $v(N, \emptyset) = 1 = -v(\emptyset, N)$.

In a multicriteria decision making framework, $v(A, B)$ represents the score of the ternary alternative $(1_A, -1_B, 0_{(A\cup B)^c})$. Note that the definition implies that $v(\cdot, \emptyset) \geq 0$ and $v(\emptyset, \cdot) \leq 0$. Actually, bi-capacities are particular *bi-cooperative games* [2], that is, functions defined over $\mathcal{Q}(N)$ with only condition *(i)* holding.

From its definition, $\mathcal{Q}(N)$ is isomorphic to the set of mappings from N to $\{-1, 0, 1\}$, hence $|\mathcal{Q}(N)| = 3^n$. Also, it is easy to see that $\mathcal{Q}(N)$ is a lattice, when equipped with the order:

$$(A, B) \sqsubseteq (C, D) \text{ if } A \subseteq C \text{ and } B \supseteq D.$$

Supremum and infimum are respectively

$$\begin{aligned}(A, B) \sqcup (C, D) &= (A \cup C, B \cap D)\\ (A, B) \sqcap (C, D) &= (A \cap C, B \cup D),\end{aligned}$$

and top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$. We give in Fig. 1 the Hasse diagram of $(\mathcal{Q}(N), \sqsubseteq)$ for $n = 3$ (where top, bottom and the central point $(\emptyset, \emptyset)$ are represented by black circles).

Derivatives of bi-capacities play a central role in the definition of interaction [7] and are defined in this way: if v is a bi-capacity, and $i \in N$,

$$\begin{aligned}\Delta_{i,\emptyset} v(K, L) &:= v(K \cup i, L) - v(K, L), \quad \text{for any } (K, L) \in \mathcal{Q}(N \setminus i);\\ \Delta_{\emptyset,i} v(K, L) &:= v(K, L \setminus i) - v(K, L), \quad \text{for any } (K, L) \in \mathcal{Q}(N) \text{ with } i \in L.\end{aligned}$$

Recursively, we define $\Delta_{S,T} v$ for any $(K, L) \in \mathcal{Q}(N \setminus S)$ with $T \subseteq L$, for any $i \in S$ and any $j \in T$, by

$$\begin{aligned}\Delta_{S,T} v(K, L) &:= \Delta_{i,\emptyset}(\Delta_{S\setminus i,T} v(K, L))\\ &= \Delta_{\emptyset,j}(\Delta_{S,T\setminus j} v(K, L)),\end{aligned}$$

so that these values are always non-negative. This generalizes the notion of derivative for a capacity ν, that is $\Delta_i \nu(A) := \nu(A \cup i) - \nu(A)$ if $i \in N, A \subseteq$

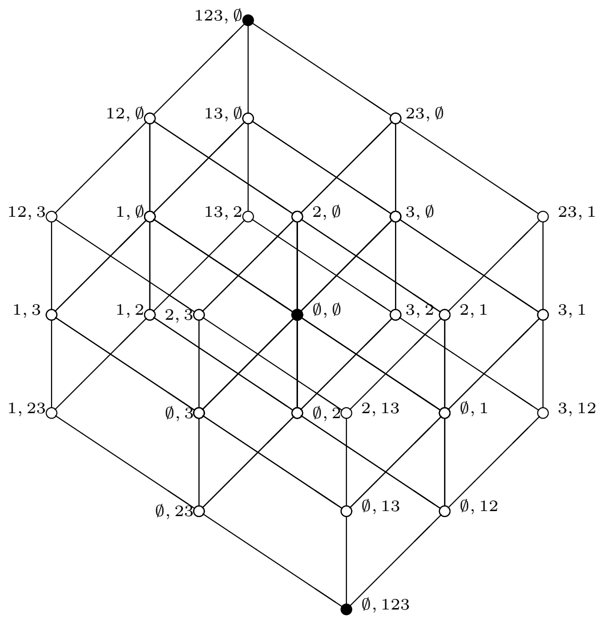

Figure 1: The lattice $\mathcal{Q}(N)$ for $n = 3$

$N \setminus i$ and $\Delta_S \nu(A) := \Delta_i(\Delta_{S\setminus i}\nu(A))$ if $A \subseteq N \setminus S$. The general expression for the (S,T)-derivative is given by, for any $(S,T) \in \mathcal{Q}(N)$, $(S,T) \neq (\emptyset, \emptyset)$ (cf. [9]):

$$\Delta_{S,T} v(K,L) = \sum_{\substack{S' \subseteq S \\ T' \subseteq T}} (-1)^{(s-s')+(t-t')} v(K \cup S', L \setminus T'),$$
$$\text{for all } (K,L) \in \mathcal{Q}(N \setminus S), L \supseteq T. \quad (1)$$

Although we develop our results for bi-capacities, we emphasize the fact that all subsequent results remain valid for bi-cooperative games.

3 Previous work on interaction index for capacities

We recall in this section two main ways which have been conducted to axiomatize the interaction index for capacities. Since the following axioms extend the ones of the Shapley value, we may adopt the terminology of *Shapley interaction index*.

A *value* $\boldsymbol{\varphi}$ maps every capacity ν on N to a real valued vector $\boldsymbol{\varphi}^\nu$, and an *interaction index* $\mathbf{I}$ maps every capacity ν to an allocation $\mathbf{I}^\nu$ over $2^N \setminus \{\emptyset\}$. In this section, ν denotes an arbirary capacity on N.

Let us recall the *Shapley value* of ν: for any element $i \in N$,

$$\phi^\nu(i) := \sum_{S \subseteq N \setminus i} p_s^1(n)\,(\nu(S \cup i) - \nu(S)),$$

where the coefficients $p_s^1(n) := \dfrac{(n-s-1)!\,s!}{n!}$ define a probability distribution over $\{S \subseteq N \setminus i\}$.

The classical axioms introduced by Shapley [15] (see also Weber [16]) are the following

- Linearity: for any $i \in N$, $\phi(i)$ is linear on the set of capacities on N.
- $i \in N$ is said to be *dummy* for ν if $\forall S \subseteq N \setminus i$, $\nu(S \cup i) = \nu(S) + \nu(i)$.
- Dummy axiom: For any capacity ν and any $i \in N$ dummy for ν, $\phi^\nu(i) = \nu(i)$.
- Symmetry axiom: for any permutation σ on N, any capacity ν and any $i \in N$, $\phi^{\nu \circ \sigma^{-1}}(\sigma(i)) = \phi^\nu(i)$. This means that ϕ^ν must not depend on the labelling of the criteria.
- Efficiency axiom (E^c): for any capacity ν, $\sum_{i \in N} \phi^\nu(i) = \nu(N)$; that is to say the values of the criteria must be divided in proportion of the overall score $\nu(N)$.

By generalizing Murofushi and Soneda [14], Grabisch has defined the *interaction index* of capacities [6]. A first axiomatization have been proposed by Grabisch and Roubens and rests on a recursivity axiom [10]. For this, they introduce the following definitions:

Let K a non-empty subset of N and $B \subseteq N \setminus K$. The *restricted capacity* ν^K is the capacity ν restricted to 2^K. The *restriction of ν to K in the presence of B* is the capacity defined by

$$\nu^K_{\cup B}(S) := \nu(S \cup B) - \nu(B)$$

for any $S \subseteq K$. Lastly, the *reduced capacity* $\nu^{[K]}$ is the capacity defined on $N_{[K]} := (N \setminus K) \cup \{[K]\}$ by

$$\nu^{[K]}(A) := \nu(A^\star)$$

where $A^\star := \begin{cases} A \text{ if } [K] \notin A \\ (A \setminus [K]) \cup K \text{ otherwise} \end{cases}$; $[K]$ actually indicates a single hypothetical player, which is the representative of the players in K.

Recursivity axiom 1 (R1^c): For any capacity ν, $\forall S \subseteq N$, $s > 1$, $\forall i \in S$,

$$I^\nu(S) = I^{\nu^{N\setminus i}_{\cup i}}(S \setminus i) - I^{\nu^{N\setminus i}}(S \setminus i).$$

Recursivity axiom 2 (R2^c): For any capacity ν, $\forall S \subseteq N$, $s > 1$,

$$I^\nu(S) = I^{\nu^{[S]}}([S]) - \sum_{\substack{K \subsetneq S \\ K \neq \emptyset}} I^{\nu^{N\setminus K}}(S \setminus K).$$

These axioms are well explained in [10]: to link value and interaction, the authors first consider the reduced capacity $\nu^{[ij]}$, and claim that the value of $[ij]$ should depend on the values of i when j is absent, and j when i is absent, as well as their interaction in ν. As a positive interaction (profitable cooperation) implies that the value of $[ij]$ should be greater than the sum of the above individual values, and on the contrary, a negative interaction (harmful cooperation) implies that the value of $[ij]$ should be less than the sum, the following formula is natural

$$\phi^{\nu^{[ij]}}([ij]) = \phi^{\nu^{N\setminus i}}(j) + \phi^{\nu^{N\setminus j}}(i) + I^\nu(ij),$$

that can be put also into this form

$$I^\nu(ij) = I^{\nu^{[ij]}}([ij]) - I^{\nu^{N\setminus i}}(j) - I^{\nu^{N\setminus j}}(i).$$

Axiom (R2^c) is a straightforward generalization, expressing interaction of S in terms of all successive interactions of subsets, whereas (R1^c) says that the interaction of the criteria in S is equal to the interaction between the criteria in $S \setminus i$ in the presence of i, minus the interaction between the criteria of $S \setminus i$ (in the absence of i).

Theorem 1 (Grabisch, Roubens [10]) *Under linear axiom, dummy axiom, symmetry axiom, efficiency axiom (E^c) and ((R1^c) or (R2^c)), for any capacity ν, $\forall S \subseteq N$, $S \neq \emptyset$,*

$$I^\nu(S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\, \Delta_S \nu(T),$$

where $p_t^s(n) := p_t^1(n-s+1) = \dfrac{(n-s-t)!\, t!}{(n-s+1)!}$.

Actually, the authors have shown that (R1^c) and (R2^c) are equivalent under the first axioms [10].

Now we present an axiomatization of Fujimoto, Kojadinovic and Marichal based on the concept of *partnership coalition* [5]; we use for this the following generalized axioms:

> **Linear axiom** (L^c): For any $S \subseteq N$, $I(S)$ is linear on the set of capacities on N.
>
> **Dummy axiom** (D^c): For any capacity ν and any $i \in N$ dummy for ν,
>
> $$\begin{cases} I^\nu(i) = \nu(i), \\ I^\nu(S \cup i) = 0, \quad \forall S \subseteq N \setminus i, S \neq \emptyset. \end{cases}$$
>
> This means that whenever i's contribution to any coalition is a constant worth, while i's value must be this worth, the interaction of any non-singleton coalition containing it must vanish.
>
> **Symmetry axiom** (S^c): For any permutation σ on N, any capacity ν and any $S \subseteq N$,
>
> $$I^{\nu \circ \sigma^{-1}}(\sigma(S)) = I^\nu(S).$$

For any $P \subseteq N$, P is said to be a *partnership* for ν if

$$\forall S \subsetneq P,\ \forall T \subseteq N \setminus P,\ \nu(S \cup T) = \nu(T).$$

In other words, as long as the elements of P are not present, the worth of any coalition outside P is left unchanged.

> **Reduced-partnership-consistency axiom** (RPC^c): For any capacity ν and $P \subseteq N$ partnership for ν,
>
> $$I^\nu(P) = I^{\nu^{[P]}}([P]).$$
>
> In words, the interaction of a partnership P in a game equals the value of its "representative player" $[P]$ in the associated reduced game.

Theorem 2 (Fujimoto, Kojadinovic, Marichal, [5]) *Under (L^c), (D^c), (S^c), (E^c) and (RPC^c), for any capacity ν, $\forall S \subseteq N$, $S \neq \emptyset$,*

$$I^\nu(S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\, \Delta_S \nu(T),$$

As in Theorem 1, I^ν is again the Shapley interaction index of ν.

Let us point out that I is *cardinal-probabilistic*, that is to say, $(p_t^s(n))_{T \subseteq N \setminus S}$ is a probability distribution, for any $S \subseteq N$, $S \neq \emptyset$ (see [5]).

4 Axiomatizations of the interaction for bi-capacities

In the sequel, v is a bi-capacity. Since criterion i has two possible situations (either being in the positive part or in the negative part of the bi-coalition), the effects of which being not necessarily symmetric on v, we should define a value $\Phi_{i,\emptyset}$ representing the contribution of i "joining the positive part" and a value $\Phi_{\emptyset,i}$ representing the contribution of i "leaving the negative part".

Therefore, a *value* maps every bi-capacity to a couple of real valued vectors. And as an interaction index for capacities is defined for non-empty coalitions, here an *interaction index* maps every bi-capacity to an allocation over $\mathcal{Q}(N) \setminus \{(\emptyset, \emptyset)\}$.

Labreuche and Grabisch have already axiomatized a Shapley value for bi-capacities [13], which is done by introducing axioms similar to the original ones of Shapley that we recalled above:

> **Linearity** (L): For any $i \in N$, $\Phi_{i,\emptyset}$ and $\Phi_{\emptyset,i}$ are linear on the set of bi-capacities on N.

$i \in N$ is said to be *left-null* (resp. *right-null*) for v if $\forall (K, L) \in \mathcal{Q}(N \setminus i)$,

$$v(K \cup i, L) \ \big(\text{resp. } v(K, L \cup i)\big) = v(K, L).$$

> **Left-null axiom** (LN): For any bi-capacity v and any $i \in N$ left-null for v,
> $$\Phi^v_{i,\emptyset} = 0.$$

> **Right-null axiom** (RN): For any bi-capacity v and any $i \in N$ right-null for v,
> $$\Phi^v_{\emptyset,i} = 0.$$

The interpretation of (LN) and (RN) is clear: if joining i to the positive (resp. negative) part of every bi-coalition of $\mathcal{Q}(N \setminus i)$ has no effect, the "left-value" $\Phi^v_{i,\emptyset}$ (resp. the "right-value" $\Phi^v_{\emptyset,i}$) must be null.

> **Invariance axiom** (I): For any two bi-capacities v, w, and any $i \in N$ such that $\forall (K, L) \in \mathcal{Q}(N \setminus i)$
> $$\begin{cases} v(K \cup i, L) = w(K, L), \\ v(K, L) = w(K, L \cup i), \end{cases}$$
> then $\Phi^v_{i,\emptyset} = \Phi^w_{\emptyset,i}$.

This axiom which, has no equivalent in the case of capacities, says that when a game w behaves symmetrically with v, then the Shapley values are the same.

> **Symmetry axiom** (S): For any permutation σ on N, any bi-capacity v and any $i \in N$,
>
> $$\Phi^{v\circ\sigma^{-1}}_{\sigma(i),\emptyset} = \Phi^v_{i,\emptyset} \text{ and } \Phi^{v\circ\sigma^{-1}}_{\emptyset,\sigma(i)} = \Phi^v_{\emptyset,i}.$$
>
> **Efficiency axiom** (E): For any bi-capacity v,
>
> $$\sum_{i\in N}(\phi^v_{i,\emptyset} + \phi^v_{\emptyset,i}) = v(N,\emptyset) - v(\emptyset,N).$$
>
> Sticking to the interpretation of the classical case, compared to the situation where all criteria would have been in the negative part, the gain is $v(N,\emptyset) - v(\emptyset,N)$, and this amount is to be shared among criteria ("left" and "right" contributions).

Theorem 3 (Labreuche, Grabisch [13]) *Under (L), (LN), (RN), (I), (S) and (E), for any bi-capacity v, $\forall i \in N$,*

$$\Phi^v_{i,\emptyset} = \sum_{S\subseteq N\setminus i} p^1_s(n)\,[v(S\cup i, N\setminus(S\cup i)) - v(S, N\setminus(S\cup i))],$$
$$\Phi^v_{\emptyset,i} = \sum_{S\subseteq N\setminus i} p^1_s(n)\,[v(S, N\setminus(S\cup i)) - v(S, N\setminus S)].$$

Now, since Grabisch and Labreuche have also defined an interaction index I^v over $\mathcal{Q}(N)$ for bi-capacities [9], it is necessary to give satisfactory properties to characterize it.

In the first place, as the interaction index for capacities can be obtained from the Shapley value by a recursion formula, we give here a similar approach to build $I^v_{S,T}$ from $\Phi^v_{i,\emptyset} =: I^v_{i,\emptyset}$ and $\Phi^v_{\emptyset,i} =: I^v_{\emptyset,i}$. Practically, $I^v_{S,T}$ denotes the interaction index when S is added to the positive part, and T is withdrawn from the negative part (i.e., the elements of T become neutral).

For any non-empty subset K, the *restricted* bi-capacity v^K is the restriction of v to $\mathcal{Q}(K)$. Besides, $v^{N\setminus i}_+$ and $v^{N\setminus i}_-$ are particular restricted bi-capacities defined by

$$v^{N\setminus i}_+(A,B) := v(A\cup i, B) - v(i,\emptyset)$$
$$v^{N\setminus i}_-(A,B) := v(A, B\cup i) - v(\emptyset,i),$$

for any $(A,B)\in\mathcal{Q}(N\setminus i)$. We respectively call $v_+^{N\setminus i}$ and $v_-^{N\setminus i}$ the *restrictions* of v in *positive* and *negative presence* of i. Note that the subtractions of $v(i,\emptyset)$ and $v(\emptyset,i)$ are necessary to constraint the nullity in $(\emptyset,\emptyset)$. The following axiom generalizes (R1^c).

> **Recursivity axiom** (R): For any bi-capacity v, $\forall(S,T)\in\mathcal{Q}(N)$, $s+t\geq 2$;
>
> $$\begin{aligned} \forall i\in S,\quad I^v_{S,T} &= I^{v_+^{N\setminus i}}_{S\setminus i,T} - I^{v^{N\setminus i}}_{S\setminus i,T},\ \text{if } s\geq 1,\\ \forall i\in T,\quad I^v_{S,T} &= I^{v^{N\setminus i}}_{S,T\setminus i} - I^{v_-^{N\setminus i}}_{S,T\setminus i},\ \text{if } t\geq 1. \end{aligned}$$
>
> (R) can be explained like (R1^c) in Section 3. The first equality says that the interaction of (S,T) is equal to the interaction of $(S\setminus i,T)$ with the restriction of v in the positive presence of i, minus the interaction of $(S\setminus i,T)$ in v restricted to $N\setminus i$. Similarly, the interaction $I^v(S,T)$ is given in terms of the interactions of $(S,T\setminus i)$ of the restriction of v in negative presence of i.

Theorem 4 *Under (L), (LN), (RN), (I), (S), (E) and (R), for any bi-capacity v, for any bi-coalition (S,T), $(S,T)\neq(\emptyset,\emptyset)$,*

$$I^v_{S,T} = \sum_{K\subseteq N\setminus(S\cup T)} p_k^{s+t}(n)\,\Delta_{S,T}v(K,N\setminus(K\cup S)). \tag{2}$$

Proof: By Theorem 3, $I^v_{i,\emptyset}$ and $I^v_{\emptyset,i}$ write for all $i\in N$:

$$\begin{aligned} I^v_{i,\emptyset} &= \sum_{S\subseteq N\setminus i}\frac{(n-s-1)!s!}{n!}\left[v(S\cup i,N\setminus(S\cup i)) - v(S,N\setminus(S\cup i))\right],\\ I^v_{\emptyset,i} &= \sum_{S\subseteq N\setminus i}\frac{(n-s-1)!s!}{n!}\left[v(S,N\setminus(S\cup i)) - v(S,N\setminus S)\right]. \end{aligned}$$

We show the result by induction on $m=s+t$.

- For $m=1$, it is immediate.
- Assume that (2) is shown for $m\in\{1,\dots,n-1\}$. Let $(S,T)\in\mathcal{Q}(N)$

and $s+t=m+1$. If $s \geq 1$ and $i \in S$ then

$$\begin{aligned}
I^v_{S,T} &= I^{v_+^{N\setminus i}}_{S\setminus i,T} - I^{v^{N\setminus i}}_{S\setminus i,T} \\
&= \sum_{K\subseteq (N\setminus i)\setminus((S\setminus i)\cup T)} \frac{((n-1)-(s-1)-t-k)!\,k!}{((n-1)-(s-1)-t+1)!} \\
&\qquad \Delta_{S\setminus i,T}[v_+^{N\setminus i} - v^{N\setminus i}](K,(N\setminus i)\setminus(K\cup(S\setminus i))) \\
&= \sum_{K\subseteq N\setminus(S\cup T)} \frac{(n-s-t-k)!\,k!}{(n-s-t+1)!} \\
&\qquad \Delta_{S\setminus i,T}[v(\cdot\cup i,\cdot) - v(i,\emptyset) - v + v(i,\emptyset)](K,N\setminus(K\cup S)) \\
&= \sum_{K\subseteq N\setminus(S\cup T)} \frac{(n-s-t-k)!\,k!}{(n-s-t+1)!} \\
&\qquad \Delta_{S\setminus i,T}\Delta_{i,\emptyset}v(K,N\setminus(K\cup S)).
\end{aligned}$$

If $t \geq 1$ and $i \in T$ then

$$\begin{aligned}
I^v_{S,T} &= I^{v^{N\setminus i}}_{S,T\setminus i} - I^{v_-^{N\setminus i}}_{S,T\setminus i} \\
&= \sum_{K\subseteq (N\setminus i)\setminus((S\cup(T\setminus i))} \frac{((n-1)-s-(t-1)-k)!\,k!}{((n-1)-s-(t-1)+1)!} \\
&\qquad \Delta_{S,T\setminus i}[v^{N\setminus i} - v_-^{N\setminus i}](K,(N\setminus i)\setminus(K\cup S)) \\
&= \sum_{K\subseteq N\setminus(S\cup T)} \frac{(n-s-t-k)!\,k!}{(n-s-t+1)!} \\
&\qquad \Delta_{S,T\setminus i}[v - v(\emptyset,i) - v(\cdot,\cdot\cup i) + v(\emptyset,i)](K,N\setminus(K\cup S\cup i)) \\
&= \sum_{K\subseteq N\setminus(S\cup T)} \frac{(n-s-t-k)!\,k!}{(n-s-t+1)!} \\
&\qquad \Delta_{S,T\setminus i}\Delta_{\emptyset,i}v(K,N\setminus(K\cup S)).
\end{aligned}$$

Since operators $\Delta_{S\setminus i,T}\Delta_{i,\emptyset}$ and $\Delta_{S,T\setminus i}\Delta_{\emptyset,i}$ are by definition $\Delta_{S,T}$, the result is shown for $s+t=m+1$.

■

Let us remark that a such result has also been derived from a generalization of (R2^c) (see [9]).

In the second place, one can take inspiration from the Fujimoto, Kojadinovic and Marichal's work [5] in working out an equivalent axiom of the above (RPC) axiom for capacities. Let us start by defining the concepts of partnership and reduced bi-capacity.

For any $P \subseteq N$, P is said a *partnership* for v if

$$\forall (S,T) \in \mathcal{Q}(N \setminus P), \quad \forall P_+, P_- \subsetneq P \text{ such that } P_+ \cap P_- = \emptyset,$$
$$v(S \cup P_+, T \cup P_-) = v(S,T).$$

The meaning is the same as for capacities, that is to say, if all elements of P are not joined together then they have a null effect on the worth of v.

For any non-empty subset K, the *reduced* bi-capacity $v^{[K]}$ is the bi-capacity defined on $N_{[K]} := (N \setminus K) \cup \{[K]\}$ by

$$v^{[K]}(S,T) := v(S^\star, T^\star),$$

where $A^\star := \begin{cases} A \text{ if } [K] \notin A \\ (A \setminus [K]) \cup K \text{ else} \end{cases}$, and $[K]$ is still comparable to a single macro player.

Reduced-partnership-consistency axiom (RPC): For any bi-capacity v and any partnership $P \subseteq N$ for v,

$$I^v_{P,\emptyset} = I^{v^{[P]}}_{[P],\emptyset}.$$

Like (RPC^c) for capacities (Section 3), the interpretation is quite clear: when we measure the interaction among the criteria of a partnership, it is as if we were measuring the value of a hypothetical player. Axiom (RPC) then simply states that the interaction of a bi-coalition splitted into a partnership P as the positive part and an empty negative part should be regarded as the left-value of the reduced partnership $[P]$ in the corresponding reduced game.

A first remark is that one could replace this axiom with its symmetric, that is, $I^v_{\emptyset,P} = I^{v^{[P]}}_{\emptyset,[P]}$, when P is still a partnership for v, one or the other being sufficient. On the other hand, from this axiom and the above ones (N), (LN), (RN), (I), (S) and (E), it is impossible to compute every $I^v_{S,T}$ whenever $T \neq \emptyset$. Consequently, we do it by generalizing these axioms:

Generalized linearity (GL): For any $(S,T) \in \mathcal{Q}(N)$, $I_{S,T}$ is linear on the set of bi-capacities on N.

Generalized left-null axiom (GLN): For any bi-capacity v and any $i \in N$ left-null for v,

$$I^v_{S\cup i,T} = 0, \quad \forall (S,T) \in \mathcal{Q}(N \setminus i).$$

Generalized right-null axiom (GRN): For any bi-capacity v and any $i \in N$ right-null for v,

$$I^v_{S,T\cup i} = 0, \quad \forall (S,T) \in \mathcal{Q}(N \setminus i).$$

Generalized invariance axiom (GI): For any two bi-capacities v, w and any $i \in N$ such that $\forall (K,L) \in \mathcal{Q}(N\setminus i)$, $\begin{cases} v(K \cup i, L) = w(K,L), \\ v(K,L) = w(K, L\cup i), \end{cases}$ we have

$$I^v_{S\cup i,T} = I^w_{S,T\cup i}, \quad \forall (S,T) \in \mathcal{Q}(N \setminus i).$$

Generalized symmetry axiom (GS): For any permutation σ on N, any bi-capacity v and any $(S,T) \in \mathcal{Q}(N)$,

$$I^{v\circ\sigma^{-1}}_{\sigma(S),\sigma(T)} = I^v_{S,T}.$$

Proposition 1 *Under (GL), (GLN), (GRN), (GI) and (GS), for any bi-capacity v, and any $(S,T) \in \mathcal{Q}(N) \setminus \{(\emptyset,\emptyset)\}$, $I^v_{S,T}$ is given by*

$$I^v_{S,T} = \sum_{(K,L)\in\mathcal{Q}(N\setminus(S\cup T))} p^{s+t}_{k,l}(n)\, \Delta_{S,T} v(K, L\cup T), \tag{3}$$

where $(p^u_{k,l}(n))_{(K,L)\in\mathcal{Q}(N\setminus U)}$, $U := S\cup T$, is a probability distribution.

Proof: We straightforwardly derive from (GL) that for any bi-capacity v

$$I^v_{S,T} = \sum_{(K,L)\in\mathcal{Q}(N)} p^{(S,T)}_{(K,L)}\, v(K,L) \quad \forall (S,T) \in \mathcal{Q}(N),$$

where the $p^{(S,T)}_{(K,L)}$'s are real numbers.

1. For all $i \in N$ and all $(S,T) \in \mathcal{Q}(N \setminus i)$,

$$I^v_{S\cup i,T} = \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} \Big[p^{(S\cup i,T)}_{(K\cup i,L)}\, v(K\cup i, L) + p^{(S\cup i,T)}_{(K,L)}\, v(K,L) + p^{(S\cup i,T)}_{(K,L\cup i)}\, v(K, L\cup i) \Big].$$

Then if i is left-null:

$$I^v_{S\cup i,T} = \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} \Big[\Big(p^{(S\cup i,T)}_{(K\cup i,L)} + p^{(S\cup i,T)}_{(K,L)}\Big)\, v(K,L) + p^{(S\cup i,T)}_{(K,L\cup i)}\, v(K,L\cup i)\Big].$$

From (GLN), $I^v_{S\cup i,T}$ vanishes for all v, which leads to equalities $\begin{cases} p^{(S\cup i,T)}_{(K,L)} = -p^{(S\cup i,T)}_{(K\cup i,L)} \\ p^{(S\cup i,T)}_{(K,L\cup i)} = 0 \end{cases}$.

Let $p^{S\cup i,T}_{K,L} := p^{(S\cup i,T)}_{(K\cup i,L)}$.

2. Similarly, if i is right-null, we have

$$I^v_{S,T\cup i} = \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} \Big[p^{(S,T\cup i)}_{(K\cup i,L)}\, v(K\cup i,L) + \Big(p^{(S,T\cup i)}_{(K,L)} + p^{(S,T\cup i)}_{(K,L\cup i)}\Big)\, v(K,L)\Big],$$

which implies from (GRN): $\begin{cases} p^{(S,T\cup i)}_{(K\cup i,L)} = 0 \\ -p^{(S,T\cup i)}_{(K,L\cup i)} = p^{(S,T\cup i)}_{(K,L)} =: p^{S,T\cup i}_{K,L} \end{cases}$,
$\forall i \in N, \forall (S,T) \in \mathcal{Q}(N\setminus i), \forall (K,L) \in \mathcal{Q}(N\setminus i)$.

3. Let$v$ w two bi-capacities, and $i \in N$; thus

$$\begin{aligned} I^v_{S\cup i,T} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} p^{S\cup i,T}_{K,L}\, (v(K\cup i,L) - v(K,L)), \\ I^w_{S,T\cup i} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} p^{S,T\cup i}_{K,L}\, (w(K,L) - w(K,L\cup i)). \end{aligned}$$

If we assume that $\begin{cases} v(K\cup i,L) = w(K,L) \\ v(K,L) = w(K,L\cup i), \end{cases}$ $\forall (K,L) \in \mathcal{Q}(N\setminus i)$ then the second equality above writes

$$I^w_{S,T\cup i} = \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} p^{S,T\cup i}_{K,L}\, (v(K\cup i,L) - v(K,L)).$$

since $I^v_{S\cup i,T} = I^w_{S,T\cup i}$ for all v, by (GI), we have $p^{S\cup i,T}_{K,L} = p^{S,T\cup i}_{K,L}, \forall (K,L) \in \mathcal{Q}(N\setminus i)$. Note that we get

$$\begin{aligned} I^v_{S\cup i,T} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} p^{S\cup i,T}_{K,L}\, \Delta_{i,\emptyset} v(K,L), \\ I^w_{S,T\cup i} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} p^{S,T\cup i}_{K,L}\, \Delta_{\emptyset,i} w(K,L\cup i) \text{ with } p^{S,T\cup i}_{K,L} = p^{S\cup i,T}_{K,L}. \end{aligned}$$

By applying (GI) for another criterion $j \neq i$ de N, for all $(S,T) \in \mathcal{Q}(N \setminus ij)$, we have

$$I^v_{S\cup ij,T} = \sum_{(K,L)\in\mathcal{Q}(N\setminus ij)} p^{S\cup ij,T}_{K,L}\, \Delta_{i,\emptyset}\Delta_{j,\emptyset} v(K,L),$$
$$I^v_{S\cup i,T\cup j} = \sum_{(K,L)\in\mathcal{Q}(N\setminus ij)} p^{S\cup i,T\cup j}_{K,L}\, \Delta_{i,\emptyset}\Delta_{\emptyset,j} v(K,L\cup j),$$
$$I^v_{S,T\cup ij} = \sum_{(K,L)\in\mathcal{Q}(N\setminus ij)} p^{S,T\cup ij}_{K,L}\, \Delta_{\emptyset,i}\Delta_{\emptyset,j} v(K,L\cup ij),$$

where $p^{S,T\cup ij}_{K,L} = p^{S\cup i,T\cup j}_{K,L} = p^{S\cup ij,T}_{K,L}$, $\forall (K,L) \in \mathcal{Q}(N \setminus ij)$. Thus, by successively applying (GI), we deduce that $(S,T) \in \mathcal{Q}(N) \setminus \{(\emptyset,\emptyset)\}$ and $(K,L) \in \mathcal{Q}(N \setminus (S \cup T))$, $p^{S,T}_{K,L}$ only depend on $S \cup T$ and (K,L). Let $U := S \cup T$ and $p^U_{K,L} := p^{U,\emptyset}_{K,L}$. We have

$$I^v_{S,T} = \sum_{(K,L)\in\mathcal{Q}(N\setminus(S\cup T))} p^{S\cup T}_{K,L}\, \Delta_{S,T} v(K,L\cup T).$$

4. Finally, let σ be any permutation of N. From (GS), we get $I^{v\circ\sigma^{-1}}_{\sigma(S),\sigma(T)} = I^v_{S,T}$ for all (S,T) of $\mathcal{Q}(N) \setminus \{(\emptyset,\emptyset)\}$. Besides,

$$I^{v\circ\sigma^{-1}}_{\sigma(S),\sigma(T)} = \sum_{(K,L)\in\mathcal{Q}(N\setminus\sigma(S\cup T))} p^{\sigma(S\cup T)}_{K,L}\, \underbrace{\Delta_{\sigma(S),\sigma(T)} v\circ\sigma^{-1}\,(K,L\cup T)}_{\Delta_{S,T} v(\sigma^{-1}(K),\sigma^{-1}(L,\cup T))}$$
$$= \sum_{(K,L)\in\mathcal{Q}(N\setminus(S\cup T)} p^{\sigma(S\cup T)}_{\sigma(K),\sigma(L)}\, \Delta_{S,T} v(K,L\cup T).$$

Thus $p^{\sigma(S\cup T)}_{\sigma(K),\sigma(L)} = p^{S\cup T}_{K,L}$, $\forall (K,L) \in \mathcal{Q}(N \setminus (S \cup T))$, that is to say, $p^U_{K,L}$ depend only on the cardinals of U, K, L. Let $p^u_{k,l} := p^U_{K,L}$, then (3) is shown.

■

Under this form, the mapping I is said to be *cardinal-probabilistic*, as a generalization of cardinal-probabilistic indices defined for capacities.

Finally, we have the following result:

Theorem 5 *Under (GL), (GLN), (GRN), (GI), (GS) and (E), axioms (R) and (RPC) are equivalent, thus for any bi-capacity v, for any bi-coalition (S,T), $(S,T) \neq (\emptyset,\emptyset)$,*

$$I^v_{S,T} = \sum_{K \subseteq N \setminus (S \cup T)} p^{s+t}_k(n)\, \Delta_{S,T} v(K, N \setminus (K \cup S)).$$

Proof: Note that (GL), (GLN), (GRN), (GI) and (GS) respectively imply (L), (LN), (RN), (I) and (S). Thus by Theorem 2, it is sufficient to prove that the formula holds with the first axioms and (RPC).

1. Clearly, for all $i \in N$,

$$\begin{aligned} I^v_{i,\emptyset} &= \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} [v(S \cup i, N \setminus (S \cup i)) - v(S, N \setminus (S \cup i))], \\ I^v_{\emptyset,i} &= \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} [v(S, N \setminus (S \cup i)) - v(S, N \setminus S)]. \end{aligned} \tag{4}$$

2. Let us compute $I^v_{S,\emptyset}$, $s \geq 2$.
By proposition 1, there are some real numbers $p^s_{k,l}(n)$, $k + l \leq n - s$ such that

$$\begin{aligned} I^v_{S,\emptyset} &= \sum_{(K,L) \in \mathcal{Q}(N \setminus S)} p^s_{k,l}(n)\, \Delta_{S,\emptyset} v(K,L) \\ &= \sum_{(K,L) \in \mathcal{Q}(N \setminus S)} p^s_{k,l}(n) \big(v(K \cup S, L) + \sum_{S' \subsetneq S} (-1)^{s-s'} v(K \cup S', L) \big), \end{aligned}$$

from the explicit expression (1) of $\Delta_{S,T} v$. Now, let S be a partnership, then for all $S' \subsetneq S$, $v(K \cup S', L) = v(K, L)$. Also, since

$$\begin{aligned} \sum_{S' \subsetneq S} (-1)^{s-s'} &= \sum_{S' \subseteq S} (-1)^{s-s'} - 1 \\ &= \sum_{s'=0}^{s} \binom{s}{s'} (-1)^{s-s'} - 1 \\ &= (1-1)^s - 1 \\ &= -1, \end{aligned}$$

$$\text{then} \quad I^v_{S,\emptyset} = \sum_{(K,L) \in \mathcal{Q}(N \setminus S)} p^s_{k,l}(n) \big(v(K \cup S, L) - v(K, L) \big). \tag{5}$$

Moreover, by (RPC) and (4) with $[S]=i$, we have also

$$\begin{aligned}
I^v_{S,\emptyset} &= I^{v^{[S]}}_{[S],\emptyset} \\
&= \sum_{K\subseteq(N\setminus S\cup[S])\setminus[S]} \frac{((n-s+1)-k-1)!\,k!}{(n-s+1)!} \\
&\qquad\qquad \Delta_{[S],\emptyset} v^{[S]}(K,(N\setminus S\cup[S])\setminus(K\cup[S])) \\
&= \sum_{K\subseteq N\setminus S} \frac{(n-s-k)!\,k!}{(n-s+1)!} \\
&\qquad\qquad \big(v(K\cup S, N\setminus(K\cup S)) - v(K, N\setminus(K\cup S))\big). \qquad (6)
\end{aligned}$$

Let $U := S$. By identifying coefficients of (5) in (6) (formulae are true for all v), we get $\forall u \in \{1,\ldots,n\}$, $\forall k \in \{0,\ldots,n-u\}$, $\forall l \in \{0\ldots,n-u-k\}$:

- For the terms of (5) that arise in (6): let $K\subseteq N\setminus U$ and $L = N\setminus(U\cup K)$. Note that $k+l=n-u$.

$$\begin{aligned}
p^u_{k,l}(n) &= p^u_{k,n-u-k}(n) \\
&= \frac{(n-u-k)!\,k!}{(n-u+1)!}.
\end{aligned}$$

  Note that these coefficients are identical to those given in Theorem 1, i.e., $p^u_{k,l}(n) = p^u_k(n)$.
- For all other coefficients, i.e., if $k+l\le n-u$, then

$$p^u_{k,l}(n) = 0.$$

This ends the proof in this case.

3. The computation of the $I^v_{S,T}$'s with $s+t\ge 2,\ t\ge 1$ is already given above. Indeed, all the $p^{s+t}_{k,l}(n)$'s of (3) are given with $s+t=u$.

4. Finally, for all $u\in\{1,\ldots,n\}$,

$$\begin{aligned}
\sum_{(K,L)\in\mathcal{Q}(N\setminus U)} p^u_{k,l}(n) &= \sum_{K\subseteq N\setminus U} p^u_k(n) \\
&= 1,
\end{aligned}$$

since the Shapley interaction index for capacities is cardinal-probabilistic (see Section 3, p. 7). Thus I for bi-capacities is also cardinal-probabilistic.

■

It is noteworthy that Kojadinovic has also proposed an alternative interaction index [12] for bi-capacities in the context of aggregation by the bipolar Choquet integral, however his solution is not completely axiomatized.

Conclusion

Axiomatic characterizations of the interaction index of bi-capacities have been proposed. The presented description is based on generalizations of the recursivity axiom and the reduced-partnership-consinstency axiom. According to the choice of one or the other, more or less powerful linearity, invariance and symmetry are required.

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