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LAGRANGEAN FUNCTIONS AND AFFINE MINORANTS

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We give hypotheses, valid in reflexive Banach spaces (such as L^p for $\infty > p > 1$ or Hilbert spaces), for a certain modification of the ordinary lagrangean to close the duality gap, in convex programs with (possibly) infinitely many constraint functions.

Our modification of the ordinary lagrangean is to perturb the criterion function by a linear term, and to take the limit of this perturbed lagrangean as the norm of this term goes to zero.

We also review the recent literature on this topic of the "limiting lagrangean".

Key words: Convexity, Lagrangean, Nonlinear Programming.

0. Introduction

In an earlier paper [6], Duffin proved this result, for convex functions F_h defined on all of R^n:

$$\lim_{\epsilon\to 0^+} \sup_{\substack{a\in R^n \\ \|a\|<\epsilon}} \sup_{\lambda} \inf_{x\in R^n} \left\{F_0(x) + ax + \sum_{h\in H} \lambda_h F_h(x)\right\} = v(P). \tag{1}$$

In (1), $v(P)$ is the value of the convex program

$$\text{(CP)} \qquad \inf \quad F_0(x) \qquad \text{subject to} \quad F_h(x) \le 0, \quad h \in H$$

where H is a finite, non-empty index set.

A purpose of this paper is to extend (1) to proper lower semi-continuous (l.s.c.) convex functions defined on a convex subset of certain infinite-dimensional spaces, specifically reflexive Banach spaces, and also to obtain information on "affine minorants" of the convex functions. The L^p spaces for $\infty > p > 1$ and Hilbert spaces are treated by our results. A goal of the paper will

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be to establish the following result in this setting, under suitable hypotheses:

$$\text{(LL)} \qquad \lim_{\epsilon\to 0^+} \sup_{\substack{g\in X^* \\ \|g\|<\epsilon}} \sup_{\lambda} \inf_{x\in K} \left\{ F_0(x) + g(x) + \sum_{h\in H} \lambda_h F_h(x) \right\} = v(P)$$

where the index set H may be infinite, and K will be a convex subset of the space in which the variable x is constrained to lie by an explicit set constraint "$x \in K$", in addition to functional constraints such as those in (CP) above (see (8) below).

We use the theory of infinite sets of linear inequalities to obtain our results. Our approach has its source in the literature of "semi-infinite programming" (see e.g. [1] and [9]), and is the basic idea for proofs of various strengthenings and refinements of our result in infinite-dimensional spaces.

Professor R.T. Rockafellar has informed us (private communication) that the result (LL) is implicit in his monograph [15], under suitable hypotheses, and it is indeed the case that [15, eq. (4.20)] can be applied to [15, example 4′, p. 26] to derive (LL) under the hypotheses used in [15]. We strengthen the result due to the additional information in Theorem 6, and, as we will point out in Section 3, our mode of analysis easily extends to set-valued maps in locally convex spaces, without the hypotheses of semi-continuity used in [15]. See [2] for a counter-example to (LL) when the semi-continuity hypothesis is dropped.

Our present paper contains an exposition of a part of the semi-infinite approach to convex optimization. For related work which utilizes the theory of conjugate functions see [13], [14], and [15].

1. Preliminary results, conventions, and general assumptions

Throughout the results, X will denote a reflexive Banach space. Thus $X^{**} = X$, where Y^* denotes the space of all continuous linear functionals on the linear topological space Y.

The following result is well-known (see e.g. [5]).

Theorem 1. *Let C be a closed cone in a locally convex linear topological space Y.*

Then the following two statements are equivalent:

(i) $y_0 \in C$;

(ii) *if $f \in Y^*$, and $f(y) \geq 0$ for all $y \in C$, then $f(y_0) \geq 0$.*

In what follows, we view functions as points, so that, e.g., $f = h$ abbreviates $f(x) = h(x)$ for all $x \in X$.

Corollary 2. *Let $\{f_i \mid i \in I\}$ be a family of continuous linear functionals on the* *reflexive Banach space X, and suppose that, for any $x \in X$,*

$$f_i(x) \geq 0 \quad \text{for all } i \in I \tag{2}$$

implies

$$f(x) \geq 0 \tag{3}$$

for the continuous linear functional f.

Then for any real scalar $\epsilon > 0$ there exists a finite subset $J \subseteq I$ and non-negative numbers $\lambda_j \geq 0$, $j \in J$, and a continuous linear functional g, satisfying both these conditions:

$$f = g + \sum_{j \in J} \lambda_j f_j, \tag{α}$$

$$\|g\| < \epsilon. \tag{β}$$

Proof. Let $C = \mathrm{cl}(\mathrm{cone}(\{f_i \mid i \in I\}))$, where cone $(\{f_i \mid i \in I\})$ is the cone (algebraically) generated by the set $\{f_i \mid i \in I\}$, and $\mathrm{cl}(S)$ denotes the closure, here in the norm topology, of the set $S \subseteq X^*$.

The conclusion of this corollary can be restated as "$f \in C$", for if $\|f - \sum_{j \in J} \lambda_j f_j\| < \epsilon$ then $g = f - \sum_{j \in J} \lambda_j f_j$ satisfies (α) and (β).

Since C is a closed cone in the locally convex linear topological space X^*, Theorem 1 applies. Thus if $f \notin C$, we reach a contradiction as follows, where we take $y_0 = f$ in Theorem 1.

There exists a continuous linear functional $\tilde{F}$ on X^* with $\tilde{F}(h) \geq 0$ for all $h \in C$ and $\tilde{F}(f) < 0$. In particular, $\tilde{F}(f_i) \geq 0$ for all $i \in I$, and $\tilde{F}(f) < 0$.

Since $\tilde{F} \in X^{**}$, there exists $\tilde{x} \in X$ with $\tilde{F}(h) = h(\tilde{x})$ for all $h \in X^*$. In particular, $f_i(\tilde{x}) \geq 0$ for all $i \in I$ and $f(\tilde{x}) < 0$, contradicting the hypothesis. This shows that $f \in C$. □

In what follows, we view (γ, h), where h is a function on X, and $\gamma \in R$, as the functional on $R \times X$ such that $(\gamma, h)(p, x) = h(x) + \gamma p$, for $(p, x) \in R \times X$.

For any linear topological space Y, the continuous dual $(R \times Y)^*$ of $R \times Y$ is $(R \times Y)^* = R \times Y^*$, with the evaluation $(r, f)(s, y) = f(y) + rs$, where $(r, f) \in R \times Y^*$, $f \in Y^*$, and $(s, y) \in R \times Y$, $y \in Y$. In particular, as X is reflexive, $(R \times X)^{**} = (R \times X^*)^{**} = R \times X^{**} = R \times X$, so $R \times X$ is reflexive. We need this latter observation in the next result.

Corollary 3. *Let $\{f_i \mid i \in I\}$ be a family of continuous linear functionals on the reflexive Banach space X and let $\{\alpha_i \mid i \in I\}$ be a correspondingly-indexed family of real scalars, such that there is a solution to*

$$f_i(x) \geq \alpha_i, \quad i \in I. \tag{4}$$

Suppose that every solution x to (4) *also satisfies*

$$f(x) \geq \alpha \tag{5}$$

for the continuous linear functional f and scalar $\alpha \in R$.

Then for any real scalar $\epsilon > 0$ *there exists a finite subset* $J \subseteq I$, *non-negative numbers* λ_j, $j \in J$, *a non-negative scalar* $\theta \geq 0$, *and a continuous linear functional g on X, and* $\beta \in R$, *satisfying*:

$$(-\alpha, f) = \theta(1,0) + (-\beta, g) + \sum_{j \in J} \lambda_j(-\alpha_j, f_j), \tag{α'}$$

$$\|(-\beta, g)\| < \epsilon. \tag{β'}$$

In particular,

$$f = g + \sum_{j \in J} \lambda_j f_j, \tag{6a}$$

$$\|g\| < \epsilon, \tag{6b}$$

$$\alpha \leq \epsilon + \sum_{j \in J} \lambda_j \alpha_j. \tag{6c}$$

Proof. The particular conclusions (6a)–(6c) follow from (α') and (β') by taking components in (α'), and noting that (β') implies $\|g\| < \epsilon$ and $|\beta| < \epsilon$. We prove only (α') and (β').

To do so, note that, in the space $R \times X$,

$$-\alpha_i r + f_i(x) \geq 0, \quad i \in I,\ r \geq 0 \tag{4'}$$

implies

$$-\alpha r + f(x) \geq 0. \tag{5'}$$

Indeed, if $r > 0$, (4′) implies (5′) by the fact that (4) implies (5) and through the linearity of the functionals $\{f_i \mid i \in I\}$ and f. If $r = 0$, again (4′) implies (5′), as we see by the following contradiction.

Let $\bar{x}$ be such that $f_i(\bar{x}) \geq 0$ for $i \in I$ yet $f(\bar{x}) < 0$. By hypothesis there exists x^* with $f_i(x^*) \geq \alpha_i$ for $i \in I$. Then for any scalar $\rho \geq 0$, $f_i(x^* + \rho\bar{x}) = f_i(x^*) + \rho f_i(\bar{x}) \geq f_i(x^*) + 0 \geq \alpha_i$ for all $i \in I$. However for large ρ, $f(x^* + \rho\bar{x}) = f(x^*) + \rho f(\bar{x}) < \alpha$ as $f(\bar{x}) < 0$. This contradicts that (4) implies (5), and proves that (4′) implies (5′).

We apply Corollary 2 to the system (4′), (5′) with (2) taken as (4′), and the functionals $\{f_i \mid i \in I\}$ of (2) taken as $\{(-\alpha_i, f_i) \mid i \in I\} \cup \{(1,0)\}$. Likewise the functional f of (3) is $(-\alpha, f)$ in (5′). The corollary applies since $R \times X$ is a reflexive Banach space.

Upon application of Corollary 2, we at once obtain (α') and (β'), since θ is simply the multiplier of the functional (1, 0), where here "0" is the identically zero linear functional on X. □

In what follows, we shall consider convex functions F on subsets of X, by which we mean a function $F:D\to R$ where D is a non-empty convex subset of X. (We do not use the extended reals $\bar{R}=R\cup\{-\infty\}\cup\{+\infty\}$ here.) As usual, the epigraph epi(F) of F is defined as:

$$\mathrm{epi}(F)=\{(z,x)\in R\times D \mid z\geq F(x)\}. \tag{7}$$

We say that F is *closed* if epi(F) is closed in $R\times X$, i.e., if F is a proper lower semi-continuous convex function.

This paper is concerned with the following convex program, where each function F_h for $h\in H\cup\{0\}$ (H an index set of arbitrary cardinality) is finite and lower semi-continuous on a domain D_h, K is a non-empty and closed convex set in X, and $D_h\supseteq K$ for $h\in\{0\}\cup H$:

$$\begin{aligned} \inf\quad & F_0(x), \\ \text{s.t.}\quad & F_h(x)\leq 0, \quad h\in H, \\ & x\in K. \end{aligned} \tag{8}$$

This program (8) is assumed to have a finite value $v(P)$; thus (8) is assumed consistent, but the infimum need not be attained.

We shall be concerned with this lagrangean, which we call the "limiting lagrangean":

$$L(x,\lambda,g)=F_0(x)+g(x)+\sum_{h\in H}\lambda_hF_h(x). \tag{9}$$

In (9), $x\in X$, $\lambda=(\lambda_h \mid h\in H)$ is a vector of non-negative components $\lambda_h\geq 0$ only finitely many of which are non-zero, and g is a continuous linear functional on X. The summation in (9) is understood as:

$$\sum_{h\in H}\lambda_hF_h(x)=\sum_{h\in H'}\lambda_hF_h(x), \tag{10}$$

where H' is the finite set $H'=\{h\in H \mid \lambda_h>0\}$ (and summation over an empty set is taken to be zero). All infinite sums of this paper have finite support and are construed analogously. Thus the sum $\sum_{i\in J}\lambda_if_i$ on the right-hand side of (6a) will also be written $\sum_{i\in I}\lambda_if_i$ with the understanding that we have set $\lambda_i=0$ for $i\in I\backslash J$.

With the notation (9), equation (LL) can be rewritten as:

$$\lim_{\epsilon\to 0^+}\sup_{\substack{g\in X^*\\ \|g\|<\epsilon}}\sup_{\lambda}\inf_{x\in K}L(x,\lambda,g)=v(P). \tag{LL'}$$

It is the limiting operation in (LL) from which we derived the term "limiting lagrangean". If the limiting operation is deleted and one sets $g=0$, one obtains an ordinary lagrangean.

It turns out that the limiting lagrangean result (LL) holds with our present assumptions, which are far weaker than the assumptions usually needed for a lagrangean result. For one thing, the index set H is not constrained in cardinality, yet the sums $\sum_{h\in H}\lambda_h F_h(x)$ always have finite support. Also, even for $|H|$ finite, the usual examples in R^n of duality gaps involve closed functions (in fact, everywhere-defined functions), and no duality gap is possible with the limiting lagrangean in this case (or even for $|H|$ infinite).

The next preliminary result is relevant to the "easy part" of (LL).

Lemma 4.

$$\limsup_{\epsilon\to 0^+}\sup_{\substack{g\in X^*\\ \|g\|<\epsilon}}\sup_{\lambda}\inf_{x\in K} L(x,\lambda,g)\le v(P). \tag{11}$$

Proof. For each integer $n\ge 1$, choose $x^{(n)}\in K$ such that

$$F_0(x^{(n)})\le v(P)+\frac{1}{n}\quad\text{and}\quad F_h(x^{(n)})\le 0\quad\text{for } h\in H. \tag{12}$$

Then for any g and λ, as $\lambda\ge 0$ we have

$$\begin{aligned}\inf_{x\in K} L(x,\lambda,g)&\le L(x^{(n)},\lambda,g)\\ &\le F_0(x^{(n)})+g(x^{(n)})\\ &\le v(P)+g(x^{(n)})+\frac{1}{n}.\end{aligned}\tag{13}$$

From (13) it follows at once that

$$\sup_{\lambda}\inf_{x\in K} L(x,\lambda,g)\le v(P)+g(x^{(n)})+\frac{1}{n} \tag{14}$$

and hence

$$\limsup_{\epsilon\to 0^+}\sup_{\substack{g\in X^*\\ \|g\|<\epsilon}}\sup_{\lambda}\inf_{x\in K} L(x,\lambda,g)\le v(P)+\frac{1}{n}. \tag{15}$$

Since (15) is valid for any n, so is (11). ☐

We remark that (11) can also be proven if $v(P)=-\infty$.

2. The main result

We shall use these notations, which exist by the fact that a closed, convex set in a locally convex space X (or $R\times X$) is the intersection of closed half spaces where the f_j written below are continuous linear functions on X, and the a^j are real scalars:

$$K = \{x \in X \mid f_j(x) \geq a^j,\ j \in I(-1)\} \tag{16a}$$

i.e.

$$x \in K \leftrightarrow f_j(x) \geq a^j, \quad \text{for all } j \in I(-1),$$

$$\text{epi}(F_h) = \{(z, x) \in R \times X \mid b^j z + f_j(x) \geq a^j,\ j \in I(h)\}, \text{ for } h \in \{0\} \cup H,$$

i.e.

$$(z, x) \in \text{epi}(F_h) \leftrightarrow b^j z + f_j(x) \geq a^j, \quad \text{for all } j \in I(h).$$

Since $(z, x) \in \text{epi}(F_h)$ and $z' \geq z$ implies $(z', x) \in \text{epi}(F_h)$, we see that all $b^j \geq 0$. In (16) all the index sets $I(h)$ for $h \in \{-1\} \cup \{0\} \cup H$ are (without loss of generality) disjoint.

We will use the fact that all $b_j \geq 0$ in the proof of Theorem 6, and we will also use the fact that if $b^j = 0$ then $f_j(x) \geq a^j$ for all $x \in K$. The latter is a consequence of our assumption that the domain of F_h contains K. Hence if $x \in K$ and $b^j = 0$, and $j \in I(h)$, we have $a^j \leq 0 \cdot F_h(x) + f_j(x) \leq f_j(x)$.

With this notation, one preliminary result remains before we can obtain our main result (Theorem 6).

Lemma 5. *Every solution to the inequalities*

$$\begin{aligned} b^j z + f_j(x) &\geq a^j, \quad j \in I(0) \\ f_j(x) &\geq a^j, \quad j \in I(h) \text{ and } h \in \{-1\} \cup H \end{aligned} \tag{17}$$

also satisfies

$$z \geq v(P). \tag{18}$$

Proof. It suffices to prove that if $(z, x) \in R \times X$ satisfies (17), then $(z, x) \in \text{epi}(F_0)$ and also x satisfies the constraints of (8). From the definitions of (16), this will be accomplished once we prove:

$$F_h(x) \leq 0 \text{ if and only if } f_j(x) \geq a^j \text{ for all } j \in I(h). \tag{19}$$

However, (19) is immediate:

$$\begin{aligned} F_h(x) \leq 0 &\leftrightarrow (0, x) \in \text{epi}(F_h) \\ &\leftrightarrow b^j \cdot 0 + f_j(x) \geq a^j \text{ for all } j \in I(h) \\ &\leftrightarrow f_j(x) \geq a^j \qquad \text{for all } j \in I(h). \quad \square \end{aligned} \tag{20}$$

Since Lemma 5 concerns an implication among linear functionals in the reflexive Banach space $R \times X$, and since the constraints (17) are consistent (and in fact satisfied by any feasible solution x to (8), with $z = F_0(x)$), it is natural to wish to apply Corollary 3 to the "fully-infinite" system (17). If one does so, our next and main result is obtained after some purely algebraic manipulation. (Recall that an affine linear functional is a linear functional plus a constant.)

Theorem 6. *Let X be a reflexive Banach space, assume that all functions F_h, $h \in \{0\} \cup H$ are finite on a set $D_h \supseteq K$, and are lower semi-continuous, that K is a non-empty, closed convex set in X, and that* (8) *is consistent.*

For any $\epsilon > 0$, there exists a finitely non-zero vector $\lambda = (\lambda_h \mid h \in H)$ of non-negative components, continuous affine linear functionals g_h for $h \in \{-1\} \cup \{0\} \cup H$, a continuous linear functional p, and a scalar $\lambda_0 > 0$ satisfying these five conditions:

Condition 1. *$g_{-1}(x) \le 0$ for $x \in K$;*

Condition 2. *$F_h(x) \ge g_h(x)$ for $h \in \{0\} \cup H$ and x in the domain of F_h;*

Condition 3. *$\|p\| < \epsilon$;*

Condition 4. *$|\lambda_0 - 1| < \epsilon$;*

Condition 5. *For all $x \in X$,*

$$g_{-1}(x) + \lambda_0 g_0(x) + p(x) + \sum_{h \in H} \lambda_h g_h(x) \ge v(P) - \epsilon. \tag{21}$$

Proof. Note that $z = z \cdot 1 + 0 \cdot x$ in (18), and $z \cdot 1 + 0 \cdot x = (1,0)(z,x)$, where $1 \in R$ and 0 is the zero functional on X. The left-hand side of the inequalities in (17) are $b_j z + f_j(x) = (b^j, f_j)(z,x)$ for $j \in I(0)$, and are $0 \cdot z + f_j(x) = (0, f_j)(z,x)$ for $j \in I(h)$, $h \in H \cup \{-1\}$. (We use the notation introduced above Corollary 3.)

We apply Corollary 3 to the implication from (17) to (18), with $\{f_i \mid i \in I\}$ taken as

$$\{(b^j, f_j) \mid j \in I(0)\} \cup \bigcup_{h \in \{-1\} \cup H} \{(0, f_j) \mid j \in I(h)\},$$

f taken as $(1,0)$, $\{\alpha_i \mid i \in I\}$ taken as $\bigcup_{h \in \{-1,0\} \cup H} \{\alpha^j \mid j \in I(h)\}$, and α taken as $v(P)$. The conclusions (6a), (6b) and (6c) of Corollary 3 become:

$$(1,0) = (\beta, -p) + \sum_{h \in \{-1\} \cup H} \sum_{j \in I(h)} \phi_{h,j}(0, f_j) + \sum_{j \in I(0)} \phi_{0,j}(b^j, f_j) \tag{6a'}$$

$$\|(\beta, -p)\| < \epsilon \tag{6b'}$$

$$v(P) \le \epsilon + \sum_{h \in \{-1,0\} \cup H} \sum_{j \in I(h)} \phi_{h,j}\, a^j. \tag{6c'}$$

In (6a'), (6b'), and (6c'), β is a real scalar, p is a continuous linear functional, and the quantities $\phi_{h,j} \ge 0$ are non-negative real scalars, only finitely many of which are actually different from zero (i.e., for only finitely many $h \in \{-1, 0\} \cup H$ there are only finitely many $\phi_{h,j} > 0$ for some $j \in I(h)$). Thus we have $\|p\| < \epsilon$ from (6b'), i.e., Condition 3.

From the first components of the vectors of (6a'), we obtain

$$1 = \beta + \sum_{j \in I(0)} \phi_{0,j}\, b^j \tag{22}$$

and from (6b′) we infer

$$|\beta| < \epsilon. \tag{23}$$

Therefore, upon setting, for $h \in \{0\} \cup H$,

$$\lambda_h = \sum_{\substack{j \in I(h) \\ b^j > 0}} \phi_{h,j}\, b^j \tag{24}$$

and (upon recalling our convention, that the empty summation is zero) we obtain by (22) and (23) the result $|\lambda_0 - 1| < \epsilon$, i.e. Condition 4. Without loss of generality, $\lambda_0 > 0$ also (by taking $\epsilon > 0$ smaller if necessary).

From the second components of the vectors in (6a′), recalling that $b^j = 0$ for all $j \in I(-1)$, and using an auspicious partitioning of the (actually finite) summation, we obtain:

$$\begin{aligned} 0 = -p + \Bigg(\sum_{j \in I(-1)} \phi_{-1,j}\, f_j + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j = 0}} \phi_{h,j}\, f_j\Bigg) \\ + \sum_{h \in \{0\} \cup H} \sum_{b^j > 0} (\phi_{h,j}\, b^j)(f_j / b^j). \end{aligned} \tag{25}$$

The same partitioning of the sum in (6c′) yields:

$$\begin{aligned} \Bigg(\sum_{j \in I(-1)} \phi_{-1,j}\, a^j + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j = 0}} \phi_{h,j}\, a^j\Bigg) \\ + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j > 0}} (\phi_{h,j}\, b^j)(a^j / b^j) \geq v(P) - \epsilon. \end{aligned} \tag{26}$$

We next evaluate the functionals of (25) at an arbitrary point $x \in X$, and add the negative of the resulting real numbers to those of (26), keeping the partitioning. We find

$$g_{-1}(x) + p(x) + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j > 0}} (\phi_{h,j}\, b^j)((a^j - f_j(x)) / b^j) \geq v(P) - \epsilon. \tag{27}$$

In (27), we have used this notation:

$$g_{-1}(x) = \sum_{j \in I(-1)} \phi_{-1,j}\, (-f_j(x) + a^j) + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j = 0}} \phi_{h,j}\, (-f_j(x) + a^j). \tag{28}$$

Clearly, $g_{-1}(x)$ is linear affine and continuous. From (16a), if $h = -1$, $(-f_j(x) + a^j) \leq 0$ for $x \in K$; and we recall from our previous discussion that, for $h \in \{0\} \cup H$ and $j \in I(h)$ with $b^j = 0$, we have $(-f_j(x) + a^j) \leq 0$ whenever $x \in K$. Using this information in (28), we obtain $g_{-1}(x) \leq 0$ for $x \in K$, i.e. Condition 1.

By the definition (16b), we have, for $h \in \{0\} \cup H, b^j F_h(x) + f_j(x) \geq a_0^j$

whenever x is in the domain of F_h; thus if $b^j > 0$ for $j \in I(h)$,

$$F_h(x) \geq (a^j - f_j(x))/b^j. \tag{29}$$

Now if $\lambda_h = 0$, we let $g_h(x)$ be $(a^j - f_j(x))/b^j$ for any $j \in I(h)$ with $b^j \neq 0$ (there is at least one such $j \in I(h)$, by our assumption that F_h is defined and finite on all of $K \neq \phi$). We at once have Condition 2, and the part $\lambda_h g_h(x)$ of the sum in (24) is zero, as is the corresponding part

$$\sum_{\substack{j \in I(h) \\ b^j > 0}} (\phi_{h,j}\, b^j)((a^j - f_j(x))/b^j)$$

of the sum in (27) (since $\lambda_h = 0$ implies $\phi_{h,j}\, b^j = 0$ for all $j \in I(h)$, using $\phi_{h,j} \geq 0$ and $b^j > 0$).

In the case that $\lambda_h > 0$, we use (29) to deduce this inequality (via the definition (24)):

$$\lambda_h F_h(x) = \Bigg(\sum_{\substack{j \in I(h) \\ b^j > 0}} \phi_{h,j}\, b^j\Bigg) F_h(x) \geq \sum_{\substack{j \in I(h) \\ b^j > 0}} \phi_{h,j}\, b^j (a^j - f_j(x))/b^j. \tag{30}$$

Upon setting

$$g_h(x) = \frac{1}{\lambda_h} \sum_{\substack{j \in I(h) \\ b^j > 0}} \phi_{h,j}(a^j - f_j(x)) \tag{31}$$

we at once obtain Condition 2 from (30) and (31) when $\lambda_h > 0$. Moreover, (27) becomes

$$g_{-1}(x) + p(x) + \sum_{\substack{h \in \{0\} \in H \\ \lambda_h > 0}} \lambda_h \Bigg(\frac{1}{\lambda_h} \sum_{\substack{j \in I(h) \\ b^j > 0}} \phi_{h,j}(a^j - f_j(x))\Bigg) \geq v(P) - \epsilon. \tag{27'}$$

which is identical to Condition 5.

All five conditions have been verified, and the proof is complete. □

Corollary 7. *Assume the hypotheses of Theorem* 6.

For any $\epsilon > 0$, there exists a finitely non-zero vector $\lambda' = (\lambda'_h \mid h \in H)$ of non-negative components, a continuous linear affine functional $a(x)$, and a continuous linear functional q, satisfying these stipulations:

Stipulation 1. $\|q\| < \epsilon$;

Stipulation 2. $a(x) \leq 0$ *for* $x \in K$;

Stipulation 3. *For all x in the common domain of F_h*,

$$a(x) + F_0(x) + q(x) + \sum_{h \in H} \lambda'_h F_h(x) \geq v(P)(1+\epsilon)/(1+2\epsilon) - \epsilon \tag{32}$$

with $h \in \{0\} \cup H$.

Proof. Put $\epsilon' = \epsilon/(1+\epsilon)$, so that $\epsilon = \epsilon'/(1-\epsilon')$, and note that $\epsilon' > 0$. We apply Theorem 6 for $\epsilon' > 0$.

After dividing through in (21) by $\lambda_0 > 0$, and using the facts that

$$\|p(x)/\lambda_0\| = \|p(x)\|/|\lambda_0| \le \|p(x)\|/(1-\epsilon') < \epsilon'/(1-\epsilon') = \epsilon, \tag{33}$$

$$\epsilon'/\lambda_0 \le \epsilon'/(1-\epsilon) = \epsilon,$$

$$v(P)/\lambda_0 \ge v(P)/(1+\epsilon') \ge v(P)(1+\epsilon)/(1+2\epsilon), \tag{34}$$

we obtain this corollary at once, with these settings:

$$a(x) = g_{-1}(x)/\lambda_0, \tag{35a}$$

$$q(x) = p(x)/\lambda_0, \tag{35b}$$

$$\lambda'_h = \lambda_h/\lambda_0 \text{ for } h \in H. \quad \square \tag{35c}$$

Note that, using Stipulation 2 of Corollary 7, the inequality (32) yields

$$\inf_{x\in K}\left\{F_0(x) + q(x) + \sum_{h\in H} \lambda'_h F_h(x)\right\} \ge v(P)(1+\epsilon)/(1+2\epsilon) - \epsilon. \tag{36}$$

Thus, for any $\epsilon > 0$, there is a linear continuous functional q with $\|q\| < \epsilon$ and

$$\sup_{\lambda} \inf_{x\in K}\left\{F_0(x) + q(x) + \sum_{h\in H} \lambda_h F_h(x)\right\} \ge v(P)(1+\epsilon)/(1+2\epsilon) - \epsilon. \tag{37}$$

It follows at once that

$$\liminf_{\|g\|\searrow 0^+} \sup_{\lambda} \inf_{x\in K}\left\{F_0(x) + g(x) + \sum_{h\in H} \lambda_h F_h(x)\right\} \ge v(P) \tag{38}$$

From (38), one has

$$\liminf_{\epsilon\to 0^+} \sup_{\substack{g\in X^* \\ \|g\|<\epsilon}} \sup_{\lambda} \inf_{x\in K} L(x, \lambda, g) \ge v(P) \tag{39}$$

with L as defined in (9). We now combine (39) with Lemma 4, and obtain the limiting lagrangean equation (LL).

By use of the norm of the Banach space X, a result about the ordinary lagrangean can also be obtained, in the case that K is norm-bounded (but not necessarily compact) in X. In fact, let $B = \sup\{\|x\| \mid x \in K\} < +\infty$; then if $\|q\| < \epsilon$, (36), becomes

$$\inf_{x\in K}\left\{F_0(x) + \sum_{h\in H} \lambda'_h F_h(x)\right\} \ge v(P)(1+\epsilon)/(1+2\epsilon) - \epsilon B - \epsilon. \tag{36'}$$

We at once obtain our next and final result, as $\epsilon > 0$ is arbitrary.

Corollary 8. *Assume the hypotheses of Theorem 6 and also assume that K is bounded.*

Then

$$\sup_{\lambda} \inf_{x\in K} \left\{ F_0(x) + \sum_{h\in H} \lambda_h F_h(x) \right\} = v(P). \tag{40}$$

3. Related literature, concluding remarks

The phenomenon of the "limiting lagrangean" (LL) was discovered by Duffin [6]. Jeroslow [11] showed that, for $X = R^n$, (LL) could be sharpened, in that the limit as $g \to 0$ could be taken to be one-dimensional. To be specific, for $X = R^n$ there exists one fixed $w \in R^n$ such that, with the hypotheses of Theorem 6,

$$\lim_{\theta\to 0} \sup_{\lambda} \inf_{x\in K} \left\{ F_0(x) + \theta wx + \sum_{h\in H} \lambda_h F_h(x) \right\} = v(P). \tag{41}$$

An alternative proof of (41) has been provided by Borwein [3], using Helly's theorem.

Extensions of the limiting lagrangean equation to infinite-dimensional spaces, in the form (LL), occur in [4] and [8]; the present paper presents a simpler result than [7], since only lower semi-continuous (convex) functions F_h are treated here.

In the paper [4], an infinite set of real-valued convex functions and a single cone-convex constraining function are used; moreover, only a general reflexivity property is used and the space X need not be normed.

In [8] the limiting lagrangean result is generalized to set-valued convex functions, and the need for a norm is dropped; and these results are further extended, in that a treatment is given of the case that the constraints are not lower semi-continuous. In addition, [8] has an extension of the result in [11], for $X = R^n$, to set-valued convex functions. It does not appear, at this writing, that the "most general" statement of limiting phenomena has been achieved; improvements will no doubt continue.

It is significant that Borwein in [4] uses the elegant theory of convex conjugate functions (as developed in [14, 15]) to shorten proofs regarding the limiting lagrangean, by citation of results from that theory. In contrast, we have preferred to cite separation principles in order to get representations of the convex program (8) as an infinite system of linear inequalities (17), and then to manipulate the resulting linear system by elementary algebra. All the refinements and extensions of the results of this paper, as mentioned above, are obtained by our method also; in fact, proofs in the set-valued case actually simplify, as one does not need to use an auspicious partitioning (as in (25)) when affine minorant results are not of concern. For further results on affine minorants, see [13].

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M-50-634

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Cover Page

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Systems Theory and Operations Research

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Nonlinear Programming and Cutting-plane Theory

REQUESTED AMOUNT	PROPOSED DURATION	DESIRED STARTING DATE
$86,396	36 months	July 1, 1980

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Robert G. Jeroslow	151-32-1110	PhD, 1969	X	
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NAME Robert G. Jeroslow	NAME Phyllis Oliver	NAME Milton Blood
SIGNATURE Robert G Jeroslow	SIGNATURE Phyllis Oliver	SIGNATURE Milton R Blood
TITLE Professor	TITLE Contracting Officer	TITLE Associate Dean
DATE September 27, 1979	DATE 10/3/79	DATE 10/1/79

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Robert G. Jeroslow; Operations Research

4. TITLE OF PROJECT

Nonlinear Programming and Cutting-plane Theory

5. SUMMARY OF PROPOSED WORK (LIMIT TO 22 PICA OR 18 ELITE TYPEWRITTEN LINES)

We shall continue our investigations of nonlinear programming, with emphasis on the nonconvex case. The limiting linear perturbation for the ordinary lagrangean in the convex case, which we have already investigated, will become a limiting norm perturbation for a penalty method in the nonconvex case. We will utilize the fact that the existence of exact global or local norm penalties is related to the existence of lagrange multipliers for "sections" of the nonconvex optimization.

We shall also continue our work on the theory of cutting-planes, with emphasis on generalizations of Balas' facial constraints theorem, a study of the non-facial case, and the relationships between the intersection operation of Balas and the construction of functionals of a generalized subadditive type.

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		CAL.	ACAD.	SUMR.		
	1. P.I./P.D.				$	$
	2. CO P.I./P.D.				$	$
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11115	6. ( ) ◄— SUBTOTALS A1 - A5 —►				$	$
	FACULTY AND OTHER SENIOR ASSOCIATES (ATTACH EXTRA SHEET IF NECESSARY)	▨	▨	▨	▨	▨
	7.				$	$
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	11.				$	$
11117	12. ( ) ◄— SUBTOTALS A7 - A11 —►				$	$
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	TOTAL SALARIES AND WAGES (A+B)				$	$
11200	C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS)				$	$
	TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A+B+C)				$	$
	D. EQUIPMENT (LIST ITEMS AND DOLLAR AMOUNTS FOR EACH ITEM)				▨	▨
23181	TOTAL EQUIPMENT				$	$
	E. MATERIALS AND SUPPLIES				▨	▨
32630					$	$
	F. DOMESTIC TRAVEL					
42111					$	$
	G. FOREIGN TRAVEL (LIST DESTINATION AND AMOUNT FOR EACH TRIP; GPM 731)				▨	▨
42112					$	$

Summary of Progress from September 1978 to Date and Its Relation to Proposed Work

In a recent paper ("The Limiting Lagrangean"), R. J. Duffin and the writer studied the program

$$\underline{P} \qquad \inf\ f(u) \quad \text{subject to} \quad 0 \in g(u)$$

where f is a multi-valued function from a locally convex space U into (subsets of) the reals R, and g is a multi-valued function from U into a locally convex space W. The problem $\underline{P}$ is taken to be consistent with value $v(P) < +\infty$.

Under suitable assumptions, we proved this "limiting lagrangean" statement:

$$\text{(1a)} \qquad \lim_{M\to 0}\ \sup_{u^*\in M}\ \sup_{\lambda^*\in W^*}\ \inf_{u\in U}\{f(u) + u^*(u) + \lambda^* g(u)\} = v(P)$$

In (1a), the notation "$M\to 0$" indicates a net of neighborhoods with limit $0 \in U^*$, where U^* is the continuous dual of U; also W^* is the continuous dual of W. In the case that the domain space $U = R^n$ is finite-dimensional real space R^n, (1a) can be strengthened, for then there is one fixed vector $u \in R^n$ such that

$$\text{(1b)} \qquad \lim_{\theta\to 0^+}\ \sup_{\lambda^*\in W^*}\ \inf_{x\in R^n}\{f(x) + \theta ux + \lambda^* g(x)\} = v(P)\ .$$

Our exact hypotheses were: (1) f and g are closed convex (i.e. the sets $\{(z,u)|z \in f(u)\}$ resp. $\{(w,u)|w \in g(u)\}$ are closed and convex in $R \times U$

resp. $W \times U$); (2) U^* is locally convex and U is semi-reflexive with dual U^* (i.e. the continuous linear functionals u^{**} on U^* arise exactly from point evaluations for a suitable point $u \in U^*$: we have the identity $u^{**}(u^*) = u^*(u)$ for all $u^* \in U^*$).

The program P is a very general formulation into which point-valued programs can be put. For example:

$$\begin{aligned} (2) \qquad & \inf \ f(x) \\ \text{subject to} \ & k(x) \le 0 \\ & h(x) = 0 \\ & x \in K \end{aligned}$$

is formulated as P by setting

$$(3) \qquad g(x) = \{r \in R | r \ge k(x)\} \times \{h(x)\} \times i(x)$$

where $i(x)$ is the set-valued function

$$(4) \qquad i(x) = \begin{cases} \{0\} , & x \in K ; \\ \phi , & x \notin K ; \end{cases}$$

(thus $g(x) = \phi$ if $x \notin K$), and in P the zero vector abbreviates $(0,0,0) \in R^3$.

Our result (1b) applies to the convex program

$$\begin{aligned} (5) \qquad & \inf \ -y \\ \text{subject to} \ & (x^2 + y^2) - x \le 0 \end{aligned}$$

of value $v(P) = 0$, with feasible solutions $(x,0)$ for all $x \ge 0$. Yet the perturbation function of this program is given by

$$(6) \qquad p(u) = \inf\{-y \mid (x^2 + y^2)^{1/2} - x \leq u\} = \begin{cases} +\infty \text{ (inconsistency)} , & \text{if } u < 0 ; \\ 0 , & \text{if } u = 0 ; \\ -\infty , & \text{if } u > 0 ; \end{cases}$$

so that no lagrangean vector, or augmented lagrangean function with finite values can close the duality gap. We also have shown ("A Limiting Lagrangean for Infinitely-Constrained Convex Optimization in R^n", [G32]) how the vector $u \in R^n$ of (1b) can be found.

Our results extended R. J. Duffin's result [G20] and our earlier result on the limiting lagrangean [G32] by consideration of the infinite-dimensional setting, the set-valued map, plus the determination of the limiting lagrangean value in all cases. Specifically, we defined a closure of the program P, if f and g are convex, and we showed that (1a) and (1b) always hold if v(P) is replaced by the value v(P´) of the closure (as one expects, if f and g are closed, P is its own closure). Thus (1a) and (1b) hold exactly if v(P) = v(P´), and jointly with Blair and Borwein, we proved [G12] that the existence of a Slater point for a convex program in point-valued functions in R^n implies v(P) = v(P´) (yet from (5) and (6) above, the converse is not true). Although (1a), (1b) are statements of very broad applicability, a simple example in [G12] shows they may fail (i.e. one can have v(P´) < v(P) if no hypotheses are made on the convex functions f and g in P). In addition, for $U = R^n$ we have established a lower bound on the convergence to v(P) as a function of $\theta > 0$.

Our interest in (1a) and (1b) is that they show how a small linear perturbation, which is sent to zero, can close duality gaps in convex programming, when other complex methods cannot.

Jointly with C. E. Blair and R. J. Duffin, we extended this kind of result to a general minimax problem, and proved the following, under suitable hypotheses:

$$\lim_{M\to 0} \sup_{x^*\in M} \sup_{y\in D} \inf_{x\in C}\{x^*(x) + F(x,y)\} = \inf_{x\in C} \sup_{y\in D} F(x,y) \tag{7a}$$

The finite dimensional version (i.e. $C \subseteq R^n$) asserts the existence of a vector $u \in R^n$ such that

$$\lim_{\theta\to 0^+} \sup_{y\in D} \inf_{x\in C}\{\theta ux + F(x,y)\} = \inf_{x\in C} \sup_{y\in D} F(x,y) \tag{7b}$$

From (7b) of course follows the usual result on minimax problems (as in [G48]) when C is compact, for then there is a uniform bound on $|ux|$, and (7b) becomes:

$$\sup_{y\in D} \inf_{x\in C} F(x,y) = \inf_{x\in C} \sup_{y\in D} F(x,y) \tag{8}$$

The exact hypotheses are given in our joint paper, "A Limiting Infisup Theorem," August 1979, and they are: (1) C is a non-empty, closed convex set in a semi-reflexive locally convex space X;

(2) $\inf_{x\in C} \sup_{y\in D} F(x,y) < +\infty$; (3) For each fixed $y \in Y$, $F(x,y)$ is a closed, convex function of $x \in X$; (4) $F(x,y)$ is concavelike in y on $C \times D$.

Our proof proceeds by a reduction of an arbitrary minimax problem to a programming problem in infinitely many constraints. We showed that the same device allowed us to generalize results of Sion [G50] from which Sion derived the Kneser-Fan minimax theorem directly.

The writer's interest has turned to strongly nonconvex problems--i.e. problems not reducible by transformations to convex programs, quasi- or pseudo-convex programs, etc. Among the simpler versions of such non-convex programs are those with an underlying linear structure, in which the non-convexity is introduced by integrality constraints and/or logical conditions. This simpler problem could, nevertheless, provide ideas and methods for further research. We studied this problem jointly with C. E. Blair:

$$\underline{\text{MIP}} \qquad \begin{aligned} \inf \quad & cx + dy \\ \text{subject to} \quad & Ax + By = b \\ & x,y \geq 0 \\ & x \text{ integer} \end{aligned}$$

where x and y are finite-dimensional vectors, as are b, c and d, while A and B are matrices, all conformally dimensioned.

In our paper, "An Exact Penalty Method for Mixed-integer Programs," we proved an exact penalty result, assuming that <u>MIP</u> has finite value v(MIP) and all data (c, d, b, A and B) are rational (without the rationality hypothesis, the penalty result can fail). Specifically, there is a finite value of the penalty parameter $\rho \geq 0$, depending on b, such that

$$\inf_{\substack{x,y\geq 0 \\ x \text{ integer}}} \{cx + dy + \rho\|b - Ax - By\|\} = v(\text{MIP}) \tag{9}$$

and such that the infimum on the left-hand-side in (9) is attained precisely for the optima to <u>MIP</u>. We showed that a result completely analogous to (9) still holds if complementarity constraints (e.g. $x_1x_2x_3 = 0$)

are appended to MIP.

While the techniques of proof we used to establish (9) in our paper are based on our earlier joint work on the value function for (MIP) (see [G11]), actually there is an alternative proof which derives from our work on the limiting lagrangean. It is this alternative proof which will extend to more general nonconvex frameworks than (MIP).

Some of the ideas of the alternative proof will be sketched in the Description of Proposed Research. The proposed research is simply a continuation of our research to date under the current NSF grant ENG-7900284.

Description of Proposed Research

We propose to continue our research under the previous grant ENG-7900284, in both the nonlinear programming and cutting-plane theory aspects of that grant.

1. Proposed Research in Nonlinear Programming

We shall study the program P in our "Summary of Progress to Date," from the point of view of extending our earlier results to highly nonconvex situations. It is expected that the lagrangean function will become a norm penalty in the nonconvex setting; the existence of an optimal dual vector in the lagrangean will correspond to an exact norm penalty; and the basic "limiting lagrangean" phenomenon discovered by R. J. Duffin in the convex case will go over to a "limiting double-norm penalty" for the nonconvex case, in different senses, to be described below.

The limiting double-norm penalty will allow reduction of constrained optimization to a sequence of unconstrained optimizations, even in cases when this is not possible by norm penalties (as for example in the case of the program (5) - (6) of the "Summary"). It is likely to yield an exceptionally broad treatment of the cases when such a reduction is possible.

To indicate some of the ideas involved, particularly the concept of "sectioning," assume in P that f is linear (this involves no loss of generality). Let a polyhedral norm be defined on R^m, the range space of g of P, as the maximum of finitely many linear forms:

$$\|w\| = \max_i f^i w \tag{10}$$

where the vectors $f^i \in R^m$ for $i = 1,\ldots,t$ are chosen solely to satisfy

$$\max_i f^i w \leq 0 \leftrightarrow w = 0 \ . \tag{11}$$

(The other axioms for a norm always will be satisfied; only (11) is dependent on the f^i). The choice of the vectors f^i are at one's convenience; as few as $(m + 1)$ can do, but the more convenient norm $\|w\| = |w_1| + |w_2| + \ldots + |w_m|$ requires $t = 2^m$. (The number t, however large, does not enter in the final results.) All such norms are of course equivalent, so that exact penalty results, etc., for one norm are true for any other.

When a program P is nonconvex, it is natural to "section" it into pieces which either are convex, or can be convexified without disturbing the optimal value. We use the norm $\|w\|$ to provide the sections (although substantially more general constructions are also available), by defining the polyhedral cones

$$C_k = \{w \,|\, f^k w \geq f^i w \quad \text{for all } i = 1,\ldots,t\} \tag{12$_k$}$$

and in turn using these cones to define t optimization problems from P, namely

$$\underline{P_k} \qquad \begin{array}{c} \inf \quad f(u) \\ \text{subject to} \quad 0 \in g_k(u) \end{array}$$

In $\underline{P_k}$, $g_k(u)$ is a set-valued function given by

$$g_k(u) = g(u) \cap C_k \ . \tag*{$(13)_k$}$$

It is then possible to show that each program $\underline{P_k}$ has an optimal lagrange multiplier $\lambda_k \in R^m$ satisfying

$$f(u) + \lambda_k \, g_k(u) \geq v(P) \quad \text{for all } u \in U \ , \tag*{$(14)_k$}$$

i.e. $r + \lambda_k w \geq v(P)$ for all $r \in f(u)$, $w \in g(u)$, and $u \in U$, if and only if there exists a global exact norm penalty, i.e. if and only if there exists $\rho \geq 0$, satisfying

$$f(u) + \rho \|g(u)\| \geq v(P) \quad \text{for all } u \in U \ . \tag*{(15)}$$

In fact, the size of ρ in (15) can easily be related to the norms $\|\lambda_k\|$ of $(14)_k$. The existence of an inexact norm penalty (i.e. equivalence of value as $\rho \nearrow +\infty$) is likewise equivalent to the fact that the supremum of the lagrangean dual in each program $\underline{P_k}$ is $v(P)$. In the framework of set-valued functions, local (rather than global) exact or inexact penalties can be discussed by modifying the set function $g(u)$ to be empty off some neighborhood of interest (compare with the function $i(x)$ of (4) in the "Summary"). The sectioning operation $(13)_k$ is natural precisely because we work with set-valued functions.

Under suitable conditions on g, which are quite broad, our earlier work on the limiting lagrangean can be applied to the programs $\underline{P_k}$, and the kind of result one obtains is of the form

$$\lim_{\theta \to 0^+} \ \sup_{\rho \geq 0} \ \inf_{x \in R^n} \{f(x) + \theta\|x\| + \rho\|g(x)\|\} = v(P) \tag*{(16)}$$

for the case $U = R^n$. In (16), two norms appear (this is the "double-norm penalty" we mentioned)--one for the domain space ($\|x\|$) and one for the value space ($\|w\|$)--but they are asymmetrical, in that $\theta \searrow 0^+$ with no necessary effect on $\rho \geq 0$, and it can occur that ρ continually increases as $\theta \to 0^+$ (always increasing to a finite value, but one depending on $\theta > 0$) in order to attain the limit of (16). Actually, substantially more information can be obtained than (16); but we are only attempting to illustrate the general direction of these kinds of limiting results.

To cite one example of the kind of hypotheses which give (16), this hypothesis is both necessary and sufficient if (16) is to hold for all linear functions $f(x)$ (the constraint function $g(x)$ being held fixed):

(17) For each $k = 1,\ldots,t$ the closure of the convex span
of $\{(w,x) | w \in g_k(x)\}$, when intersected with $w = 0$,
is the closure of the convex span of $\{(0,x) | 0 \in g_k(x)\}$.

In particular, if the convex span of $\{(w,x) | w \in g_k(x)\}$ is closed, (16) holds. The condition (17) often (but not always) holds because the flat $w = 0$ is always on the boundary of $\{(w,x) | w \in g_k(x)\}$, and each cone C_k of $(12)_k$ is pointed. Moreover, if one fixed function $f(x)$ is considered, (17) can be further weakened as a sufficient condition.

Generally the programs $\underline{P_k}$ are difficult to analyze by conventional means, even if they are convex, because the type of operation $(13)_k$ involved in the definition of g_k is a "truncation" which a priori rules out the existence of a Slater point. However, a joint unpublished result we obtained earlier with R. J. Duffin is capable of dealing with many cases of such truncation, and still allows one to conclude the existence of an optimal multiplier λ_k in $(14)_k$. Several results in the literature can also be adapted to this end. Thus we do expect to obtain exact penalty results (15) by this kind of "sectioning" analysis.

We also are interested in algorithms for the sectioned problems $(14)_k$, which, under mild hypotheses, can be applied to suitable convexifications of $(14)_k$. However, consideration of algorithms will be deferred until the basic phenomena have been explored.

2. Proposed Research on the Theory of Cutting-Planes

We shall be continuing our work on the theory of cutting planes, as applied to discrete structures (see e.g. [G30], [G33], [G34], [G35]). Cutting-planes are finding wider use in applications (see e.g. [G5], [G6]).

Under the current grant ENG 7900284, which runs to June 30, 1980, we are exploring a subadditive treatment of constraint sets of the type:

$$\begin{aligned}
(18) \qquad & Ax = v \\
& x \geq 0 \quad (x = (x_1,\ldots,x_r)) \\
& x_j \in T_j \quad \text{for } j = 1,\ldots,r \\
& x \text{ satisfies the logical constraints } \mathscr{L}
\end{aligned}$$

where $\mathscr{L}$ is of a generalized complementarity nature, e.g.
$\mathscr{L} = (x_1x_5 = 0 \wedge x_1x_3x_7 = 0)$.

We had previously explored similar constraint sets [G33], from the point of view of the disjunctive methods; the subadditive treatment is of course an extension of the algebraic methods [G35]. The interrelations between the two approaches is well-known for the mixed integer program (MIP) below (see [G35]), but has not been worked out for problems of the form (18), as a subadditive treatment of (18) was not available.

Recently Wolsey gave an economic interpretation [G51] of the earlier subadditive dual in [G30] for (MIP), as an analogue to Koopman's economic interpretation of linear programming duality, and such an interpretation may be possible in this case.

We also plan to do further research on a result of Balas [G2], on "facial constraints," which plays a central role in the disjunctive approach to cutting-planes.

To state Balas' construction in a broader setting, let S be a (not necessarily closed) convex set in R^n, and let $A_1,\ldots,A_t$ be sets which can be written in the form

$$(18)_i \qquad A_i = \bigcup_{j \in J(i)} E_j$$

for an index set $J(i) \neq \phi$, where the sets $\phi \neq E_j \subseteq S$ are convex sets which are extreme in S. Balas considers the iterated operation

$$(19a) \qquad S_0 = S$$

$$(19b) \qquad S_i = \operatorname{conv}(S_{i-1} \cap A_i) \ , \quad 1 \leq i \leq t \ ,$$

where conv(T) is the convex span of the set T. Then it is true that

$$(20) \qquad S_t = \operatorname{conv}(S \cap A_1 \cap \ldots \cap A_t) \ .$$

It is also known that (20) can fail when the extremality property of the E_j's in $(18)_i$ are not present.

Balas' original statement of (20) is for S a polytope, and for each E_j an exposed subset (i.e. a face) of S (and $|I(i)|$finite), but the result (20) is true in the generality above. For S compact, each set S_i is compact if each set E_j is compact and $|I(i)|$ is finite; and in this case the operation of convex closure is equivalent to (19b). In other words, if we set

$$(19a)' \qquad S_0' = S$$

$$(19b)' \qquad S_i' = \mathit{cl}\,\mathrm{conv}(S_{i-1}' \cap A_i) \ , \quad 1 \le i \le n \ ,$$

where $\mathit{cl}\,\mathrm{conv}(T)$ is the closure of the convex span of the set T, we have $S_i' = S_i$ $(1 \le i \le t)$ under the above compactness assumptions. In general $S_i' \supseteqq S_i$ and $\neq$ can hold; there are simple (unbounded) polyhedral examples of S for which $S_t' \supsetneqq \mathit{cl}\,\mathrm{conv}(S \cap A_1 \cap \ldots \cap A_t)$. Nevertheless, it also is possible to give a description of S_t', which we omit.

One example of the construction (19), (20) occurs for complementarity constraints, where an underlying linear structure with nonnegativities

$$(21) \qquad Cx = d \ , \quad x \ge 0$$

is subject to several complementarity conditions. For example, the complementarity constraints of $\mathcal{L} = (x_1x_5 = 0 \wedge x_1x_3x_7 = 0)$ can be represented with $S = \{x | Cx = d,\ x \ge 0\}$, $A_1 = E_1 \cup E_2$, $A_2 = E_1 \cup E_3 \cup E_7$, $t = 2$, where each $E_j = \{x \in S | x_j = 0\}$. Then $S_t = \mathrm{conv}(S \cap A_1 \cap A_2)$ is the convex hull of the points feasible for the complementarity problem. Also bivalent mixed-integer programming problems are examples of (19), (20). We are thus discussing a disjunctive programming approach to a problem for which we are now developing a subadditive treatment.

We have begun to discuss the issue of facial and nonfacial constraints with C. E. Blair, and plan collaboration with him on this topic. The nonfacial case is more complex; Blair has shown that any finite number of Balas' intersection operations (19b) may fail to produce the convex hull of feasible points (as in (20)); although for S compact convex, countably many intersection operations do "converge" to the hull. We plan to continue work in this area.

These discoveries raise several issues with regard to finiteness proofs for cutting-plane algorithms which do use the intersection operation (19b), even for nonfacial constraints, and which are nevertheless finitely convergent. Examples of such algorithms are Gomory's mixed-integer algorithm [G26] (without the cut-strengthening for integer variables) and an algorithm we described in [G34]. Essentially, while these algorithms do not necessarily reach the convex hull, they do compute "lexicographically maximal points" in a certain sense, and these points are sufficient to solve the programming problem. In contrast, our algorithm for facial constraints in [G37] can reach the hull.

We plan to study the issue of finite convergence, and the related issue of efficient use of cuts, in greater detail. While there are now many proofs of finiteness for various algorithms on different classes of problems, there is not much understanding of what unifies them.

Currently there are several different ideas about the choice of cuts. Blair has suggested that complexity measures be used, which are well-ordered, and such that a cut never increases and (at least every so often) strictly decreases complexity (see [G10]); the specific measure he suggests counts the intersection of faces of various dimensions with the extremal subsets. One programme which has never successfully been carried out is that of relating "deep cuts" (under some measure of "depth" or "strength") to faster finite convergence, or even to better relative progress after several cuts have been added. One rule frequently used now is to seek easily-computed facets of the convex hull of feasible points; this is often possible, at least in part, on problems with a very specific structure.

Further research is needed to develop an understanding of which of the above approaches to the choice of a cut are most appropriate under different circumstances.

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Experience

University of Minnesota
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Research Interests

Mathematical programming, with emphasis on cutting-plane theory and its uses in integer programming and linear complementarity; nonlinear programming; integer programming; programming aspects of computational complexity.

Teaching Interests

Research interests, plus production and applications of Operations Research techniques; management strategy.

Journals

Member of the Editorial Board (Associate Editor), the SIAM Journal on Control and Optimization and Discrete Applied Mathematics.

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Reviewer for Bulletin of the American Mathematical Society.

Grants and Fellowships

Ford A Fellowship 1964-66
NSF Graduate Fellowship 1966-69
NSF Research Grant GP 21067, Principal Investigator, 1971-72
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Organizational Responsibilities

Representative to the Faculty Senate from the business school, 1977-78
Organizer of the Operations Research seminar for the business school,
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Representative of the management college to the Seminar on Operations Research, 1978-80 (co-sponsored with the School of Industrial and Systems Engineering and the School of Mathematics)
Member of the Program Committee of the symposium in honor of R. J. Duffin, Constructive Approaches to Mathematical Models, July 10-15, 1978
Co-organizer (with Cedric Suzman) of the Colloquium on Strategic Planning, September 28, 1979

Hobbies and Personal Interests

Light hiking; swimming; jogging; reading in literature, archaeology, history, and politics.

Published Articles

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30. "Two Lectures on the Theory of Cutting-Planes," for Combinatorial Optimization, edited by N. Christofides et al., John Wiley and Sons, Ltd.

Book Review

M. R. Hestenes' Optimization Theory: the Finite-Dimensional Case, reviewed in Bulletin of the American Mathematical Society (83), May 1977, pp. 324-334.

To Be Published

1. "Representations of Unbounded Optimizations as Integer Programs," in Journal on Optimization Theory and Its Applications.

2. "Lagrange Dual Problems with Linear Constraints on the Multipliers," (with C. E. Blair) for the proceedings of the conference, Constructive Approaches to Mathematical Models, held at Pittsburgh, Pa., July 1978.

3. "An Introduction to the Theory of Cutting-Planes," for volume based on NATO Advanced Research Institute on Discrete Optimization and Systems Applications (May 1977).

4. "A Cutting-Plane Game and Its Algorithms," for SIAM Journal on Control and Optimization.

Submitted for Publication

1. "A Limiting Lagrangean for Infinitely-Constrained Convex Optimization in R^n," to Journal of Optimization Theory and Applications (revision requested).

2. "Strengthening Cuts for Mixed Integer Programs," (with E. Balas), to the European Journal of Operations Research.

3. "Lagrangean Functions and Affine Minorants," with R. J. Duffin, to Mathematical Programming.

Recent Papers

1. "The Limiting Lagrangean," with R. J. Duffin, June 1979.

2. "A Limiting Infisup Theorem," with C. E. Blair and R. J. Duffin, August 1979.

3. "An Exact Penalty Method for Mixed Integer Programs," with C. E. Blair, August 1979.

Invited Talks

1. "On Godel's Consistency Theorem," University of Texas at Austin, October 1971.

2. "K-descriptions," Lakehead University, Thunder Bay, Ontario, Canada, January 1972.

3. "Asymptotic Linear Programming," Carnegie-Mellon University, February 1972.

4. "Trial-and-error Logics," State University of New York at Buffalo, February 1973.

5. "On a Theorem of Chvátal and Gomory," SIGMAP-UW Symposium on Nonlinear Programming, University of Wisconsin, April 1974.

6. "Proof Theory and Hilbert's Finitism," a series of nine lectures, Universita di Siena, Instituto di Matematica, Siena, Italy, May 1974.

7. "Cutting-Planes for Relaxations of Integer Programs," ORSA/TIMS meeting in San Juan, P.R., October 1974.

8. "Some Results and Constructions of Cutting-Plane Theory," NSF Regional Conference on Convex Polytypes and Mathematical Programming, Tuscaloosa, Alabama, June 9-13, 1975.

9. "Algebraic Methods, Disjunctive Methods," for the Workshop in Integer Programming, Bonn, Germany, September 8-11, 1975.

10. "Completeness Theorems for Cutting Planes," seminar at the University of North Carolina, November 13, 1975.

11. "Minimal Inequalities," ORSA/TIMS meeting in Las Vegas, November 17-19, 1975.

12. "Completeness Results in Cutting-Plane Theory," Centre de Recherches Mathematiques, Montreal, January 1976.

13. "Cutting-Planes for Complementarity Constraints," IX International Symposium on Mathematical Programming, Budapest, August 1976.

14. "Bracketing Discrete Problems by Two Problems of Linear Optimization," Symposium on Operations Research, Heidelberg, September 1976.

15. "Treeless Searches," ORSA/TIMS Joint National Meeting (joint with C. E. Blair), Miami Beach, November 1976.

16. "The Complexity of Certain Linear and Integer Programming Algorithms," University of Bonn, February 1977.

17. "Subadditivity and Value Functions of Mixed-Integer Programs," University of Cologne, May 1977.

18. "Linear Programs Dependent on a Single Parameter," University of Aachen, May 1977.

19. "An Introduction to Cutting-Plane Theory," Summer School on Combinatorial Optimization at Sogesta, in Urbino, Italy, June 1977.

20. "An Introduction to the Theory of Cutting-Planes," NATO Advanced Research Institute on Discrete Optimization and Systems Applications, Vancouver, August 1977.

21. "Cutting-Planes and Cutting-Plane Algorithms for Complementarity Constraints," International Symposium on Extremal Methods and and Systems Analysis, Austin, September 1977.

22. "A Cutting-Plane Game and Its Algorithms," Georgia Institute of Technology, Atlanta, February 1978.

23. "A Limiting Lagrangean for Infinitely Constrained Convex Optimization in R^n," at Constructive Approaches to Mathematical Models, Pittsburgh, July 1978.

24. "Representations of Unbounded Optimizations as Integer Programs," ORSA/TIMS meeting in New Orleans, April 30-May 2, 1979.

25. "Recent Results in Nonlinear and Integer Programming," at the meeting on mathematical programming, at Mathematisches Forschungsinstitut in Oberwolfach, Germany, May 6-12, 1979.

26. "A Limiting Infisup Theorem," at the Tenth International Symposium on Mathematical Programming, Montreal, August 27-31, 1979.

27. "Nonlinear Optimization Treated by Linear Inequalities," ORSA/TIMS meeting in Milwaukee, October 15-17, 1979.

Papers Published, August 1978-August 1979

1. "A Converse for Disjunctive Constraints," (with C. E. Blair), Journal of Optimization Theory and Its Applications, June 1978.
2. "Some Relaxation Methods for Linear Inequalities," Cahiers du Centre d'Etudes de Recherche Operationelle 21 (1979), pp. 43-53.
3. "Cutting-plane Theory: Algebraic Methods," Discrete Mathematics 23 (1978), pp. 121-150.
4. "The Value Function of a Mixed Integer Program: II," (with C. E. Blair), Discrete Mathematics 24 (1979), pp. 7-19.
5. "Minimal Inequalities," Mathematical Programming 17 (1979), pp. 1-15.
6. "Two Lectures on the Theory of Cutting-planes," in Combinatorial Optimization, edited by N. Christofides et al., John Wiley and Sons, Ltd., 1979.

Papers Accepted for Publication, August 1978-August 1979

1. "A Cutting-Plane Game for Facial Disjunctive Programs," accepted for the SIAM Journal on Control and Optimization.
2. "An Introduction to the Theory of Cutting-Planes," for volume based on NATO Advanced Research Institute on Discrete Optimization and Systems Applications (held in August, 1977).
3. "Lagrange Dual Problems with Linear Constraints on the Multipliers," (with C. E. Blair), for proceedings of Constructive Approaches to Mathematical Models (held in July, 1978).

Combined Budget

July 1980-June 1983

SUMMARY PROPOSAL BUDGET

FOR NSF USE ONLY
PROPOSAL NO.
DURATION (MONTHS)
PROPOSED REVISED

ORGANIZATION AND ADDRESS
Georgia Tech Research Institute
North Avenue, Atlanta, Georgia 30332

PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR
Robert G. Jeroslow

NSF USE	A. SENIOR PERSONNEL (LIST BY NAME; SHOW NUMBERS OF PEOPLE IN BRACKETS; SALARY AMOUNTS MAY BE LISTED ON SEPARATE SCHEDULE) GPM 205.1b	NSF FUNDED MAN MONTHS			FUNDS REQUESTED BY PROPOSER	FUNDS GRANTED BY NSF (IF DIFFERENT)
		CAL.	ACAD.	SUMR.		
	1. P.I./P.D. Robert G. Jeroslow		3	6	$ 44,464	$
	2. CO P.I./P.D. none				$	$
	3. CO P.I./P.D.				$	$
	4. CO P.I./P.D.				$	$
	5. CO P.I./P.D.				$	$
11115	6. (1) ← SUBTOTALS A1 - A5 →		3	6	$ 44,464	$
	FACULTY AND OTHER SENIOR ASSOCIATES (ATTACH EXTRA SHEET IF NECESSARY)					
	7.				$	$
	8.				$	$
	9.				$	$
	10.				$	$
	11.				$	$
11117	12. ( ) ← SUBTOTALS A7 - A11 →				$ 0	$
	B. OTHER PERSONNEL (LIST NUMBERS IN BRACKETS)					
11141	1. ( ) POSTDOCTORAL ASSOCIATES				$	$
11149	2. ( ) OTHER PROFESSIONALS				$	$
11150	3. ( ) GRADUATE STUDENTS				$	$
11152	4. ( ) UNDERGRADUATE STUDENTS				$	$
11182	5. ( ) SECRETARIAL - CLERICAL				$	$
11183	6. ( ) TECHNICAL, SHOP, OTHER				$	$
	TOTAL SALARIES AND WAGES (A+B)				$ 44,464	$
11200	C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS) (10.51% of 6. + 12.)				$ 4,673	$
	TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A+B+C)				$ 49,137	$
	D. EQUIPMENT (LIST ITEMS AND DOLLAR AMOUNTS FOR EACH ITEM) none					
23181	TOTAL EQUIPMENT				$ 0	$
	E. MATERIALS AND SUPPLIES none					
32630					$ 0	$
	F. DOMESTIC TRAVEL					
42111					$ 2,300	$
	G. FOREIGN TRAVEL (LIST DESTINATION AND AMOUNT FOR EACH TRIP; GPM 731) Eleventh International Symposium on Mathematical Programming, Location to be in Europe					
42112					$ 1,000	$

NOTICE OF RESEARCH PROJECT
SCIENCE INFORMATION EXCHANGE

SMITHSONIAN INSTITUTION

NATIONAL SCIENCE FOUNDATION
PROJECT SUMMARY

PROJECT NO. (Do not use this space)

NSF AWARD NO.

1. NAME OF INSTITUTION (INCLUDE BRANCH/CAMPUS & SCHOOL OR DIVISION)

2. MAILING ADDRESS

3. PRINCIPAL INVESTIGATOR AND FIELD OF SCIENCE/SPECIALTY

4. TITLE OF PROJECT

5. SUMMARY OF PROPOSED WORK (LIMIT TO 22 PICA OR 18 ELITE TYPEWRITTEN LINES)

FOR NSF USE ONLY

DIVISION (OFFICE) AND DIRECTORATE	PROGRAM	
SECTION	PROPOSAL NO.	F.Y.

FOR DGC USE ONLY

TART AND END DATES	AMOUNT GRANTED

FORM 4 (7-78)
1. Proposal Folder
2. Program Suspense
3. Division of Grants & Contracts
4. Science Information Exchange
5. Principal Investigator
6. Off. of Govt. & Pub. Progs.
7. Assistant Director

Combined Budget

July 1980-June 1983

(continued)

SUMMARY PROPOSAL BUDGET

PROPOSAL NO.

52500	H. PUBLICATION COSTS/PAGE CHARGES	$ 900	$
62315	I. COMPUTER (ADPE) SERVICES	$ 600	$
	J. CONSULTANT SERVICES (IDENTIFY CONSULTANTS BY NAME AND AMOUNT; GPM 516) none		
		$ 0	$
	K. PARTICIPANT SUPPORT COSTS, IF ALLOWED BY PROGRAM GUIDE (ITEMIZE) GPM 518 1. STIPENDS $ 2. TRAVEL $ 3. SUBSISTENCE $ 4. OTHER - SPECIFY $ none		
	5. TOTAL PARTICIPANT COSTS (K1 + K2 + K3 + K4)	$ 0	$
	L. ALL OTHER DIRECT COSTS (List items and dollar amounts. Details of subcontracts, including work statements and budget, should be explained in full in proposal.) $ $ $ $ none		
65001	TOTAL OTHER DIRECT COSTS	$ 0	$
	M. TOTAL DIRECT COSTS (A THROUGH L)	$ 53,937	$
	N. INDIRECT COSTS (Specify rate(s) and base(s) for on/off campus activity. Where both are involved, identify itemized costs included in on/off campus bases in remarks.) on campus: 73% of (A + B)		
74100	TOTAL INDIRECT COSTS	$ 32,459	$
	O. TOTAL DIRECT AND INDIRECT COSTS (M + N)	$ 86,396	$
74500	P. LESS RESIDUAL FUNDS (If for further support of current project; GPM 252 and 253)	$ 0	$
75000	Q. AMOUNT OF THIS REQUEST (O MINUS P)	$ 86,396	$

REMARKS

Georgia Institute of Technology will cost share in accordance with current Foundation policy

NOTE: SIGNATURES REQUIRED ONLY FOR REVISED BUDGET (GPM 233). THIS IS REVISION NO.

SIGNATURE OF PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR	DATE OF SIGNATURE	TYPED OR PRINTED NAME AND TITLE
SIGNATURE OF AUTHORIZED ORGANIZATIONAL REPRESENTATIVE	DATE OF SIGNATURE	TYPED OR PRINTED NAME AND TITLE

FOR NSF USE ONLY

INDIRECT COST RATE VERIFICATION

Date Checked	Date of Rate Sheet	Signature

PROGRAM OFFICER APPROVAL

Grant Number	Amend No.	Institution	Organization	Fund Acct.	Program	Object	

	Proposal Number	Dur.	Chg.	Award Date	Proposed Amount	Propsd. Dur.	

Proposed Budget

July 1980-June 1981

SUMMARY PROPOSAL BUDGET

ORGANIZATION AND ADDRESS
Georgia Tech Research Institute
North Avenue, Atlanta, Georgia 30332

PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR
Robert G. Jeroslow

FOR NSF USE ONLY
PROPOSAL NO.
DURATION (MONTHS)
PROPOSED
REVISED

NSF USE	A. SENIOR PERSONNEL (LIST BY NAME; SHOW NUMBERS OF PEOPLE IN BRACKETS; SALARY AMOUNTS MAY BE LISTED ON SEPARATE SCHEDULE) GPM 205.1b	NSF FUNDED MAN MONTHS CAL.	ACAD.	SUMR.	FUNDS REQUESTED BY PROPOSER	FUNDS GRANTED BY NSF (IF DIFFERENT)
	1. P.I./P.D. Robert G. Jeroslow		1	2	$ 13,433	$
	2. CO P.I./P.D. none				$	$
	3. CO P.I./P.D.				$	$
	4. CO P.I./P.D.				$	$
	5. CO P.I./P.D.				$	$
11115	6. (1) SUBTOTALS A1 - A5		1	2	$ 13,433	$
	FACULTY AND OTHER SENIOR ASSOCIATES (ATTACH EXTRA SHEET IF NECESSARY)					
	7.				$	$
	8.				$	$
	9.				$	$
	10.				$	$
	11.				$	$
11117	12. ( ) SUBTOTALS A7 - A11				$ 0	$
	B. OTHER PERSONNEL (LIST NUMBERS IN BRACKETS)					
11141	1. ( ) POSTDOCTORAL ASSOCIATES				$	$
11149	2. ( ) OTHER PROFESSIONALS				$	$
11150	3. ( ) GRADUATE STUDENTS				$	$
11152	4. ( ) UNDERGRADUATE STUDENTS				$	$
11182	5. ( ) SECRETARIAL - CLERICAL				$	$
11183	6. ( ) TECHNICAL, SHOP, OTHER				$	$
	TOTAL SALARIES AND WAGES (A+B)				$ 13,433	$
11200	C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS) (10.51% of 6. + 12.)				$ 1,412	$
	TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A+B+C)				$ 14,845	$
	D. EQUIPMENT (LIST ITEMS AND DOLLAR AMOUNTS FOR EACH ITEM) none					
23181	TOTAL EQUIPMENT				$ 0	$
	E. MATERIALS AND SUPPLIES none					
32630					$ 0	$
	F. DOMESTIC TRAVEL					
42111					$ 700	$
	G. FOREIGN TRAVEL (LIST DESTINATION AND AMOUNT FOR EACH TRIP; GPM 731) none					
42117					$ 0	$

NOTICE OF RESEARCH PROJECT

SCIENCE INFORMATION EXCHANGE

SMITHSONIAN INSTITUTION

NATIONAL SCIENCE FOUNDATION

PROJECT SUMMARY

PROJECT NO. (Do not use this space)
NSF AWARD NO.

1. NAME OF INSTITUTION (INCLUDE BRANCH/CAMPUS & SCHOOL OR DIVISION)

2. MAILING ADDRESS

3. PRINCIPAL INVESTIGATOR AND FIELD OF SCIENCE/SPECIALTY

4. TITLE OF PROJECT

5. SUMMARY OF PROPOSED WORK (LIMIT TO 22 PICA OR 18 ELITE TYPEWRITTEN LINES)

FOR NSF USE ONLY

IVISION (OFFICE) AND DIRECTORATE	PROGRAM	
ECTION	PROPOSAL NO.	F.Y.

FOR DGC USE ONLY

TART AND END DATES	AMOUNT GRANTED

FORM 4 (7-78)

1. Proposal Folder
2. Program Suspense
3. Division of Grants & Contracts
4. Science Information Exchange
5. Principal Investigator
6. Off. of Govt. & Pub. Progs.
7. Assistant Director

Proposed Budget

July 1980-June 1981

(continued)

SUMMARY PROPOSAL BUDGET

PROPOSAL NO.

52500	H. PUBLICATION COSTS/PAGE CHARGES	$ 300	$
62315	I. COMPUTER (ADPE) SERVICES	$ 100	$
	J. CONSULTANT SERVICES (IDENTIFY CONSULTANTS BY NAME AND AMOUNT; GPM 516) none	$ 0	$
	K. PARTICIPANT SUPPORT COSTS, IF ALLOWED BY PROGRAM GUIDE (ITEMIZE) GPM 518 none		
	1. STIPENDS $		
	2. TRAVEL $		
	3. SUBSISTENCE $		
	4. OTHER - SPECIFY $		
	5. TOTAL PARTICIPANT COSTS (K1 + K2 + K3 + K4)	$ 0	$
	L. ALL OTHER DIRECT COSTS (List items and dollar amounts. Details of subcontracts, including work statements and budget, should be explained in full in proposal.) none $ $ $ $		
65001	TOTAL OTHER DIRECT COSTS	$ 0	$
	M. TOTAL DIRECT COSTS (A THROUGH L)	$ 15,945	$
	N. INDIRECT COSTS (Specify rate(s) and base(s) for on/off campus activity. Where both are involved identify itemized costs included in on/off campus bases in remarks.) on campus: 73% of (A + B)		
74100	TOTAL INDIRECT COSTS	$ 9,806	$
	O. TOTAL DIRECT AND INDIRECT COSTS (M + N)	$ 25,751	$
74500	P. LESS RESIDUAL FUNDS (If for further support of current project; GPM 252 and 253)	$ 0	$
75000	Q. AMOUNT OF THIS REQUEST (O MINUS P)	$ 25,751	$

REMARKS

Georgia Institute of Technology will provide cost sharing in accordance with current Foundation policy

NOTE: SIGNATURES PEQUIRED ONLY FOR REVISED BUDGET (GPM 233). THIS IS REVISION NO.

SIGNATURE OF PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR	DATE OF SIGNATURE	TYPED OR PRINTED NAME AND TITLE
SIGNATURE OF AUTHORIZED ORGANIZATIONAL REPRESENTATIVE	DATE OF SIGNATURE	TYPED OR PRINTED NAME AND TITLE

FOR NSF USE ONLY

INDIRECT COST RATE VERIFICATION

Date Checked	Date of Rate Sheet	Signature

PROGRAM OFFICER APPROVAL

Grant Number	Amend No.	Institution	Organization	Fund Acct.	Program	Object

Proposal Number	Dur.	Chg.	Award Date	Proposed Amount	Prpsd. Dur.

Proposed Budget

July 1981-June 1982

SUMMARY PROPOSAL BUDGET

FOR NSF USE ONLY
PROPOSAL NO.
DURATION (MONTHS)
PROPOSED REVISED

ORGANIZATION AND ADDRESS
Georgia Tech Research Institute
North Avenue, Atlanta, Georgia 30332

PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR
Robert G. Jeroslow

NSF USE		NSF FUNDED MAN MONTHS CAL.	ACAD.	SUMR.	FUNDS REQUESTED BY PROPOSER	FUNDS GRANTED BY NSF (IF DIFFERENT)
	A. SENIOR PERSONNEL (LIST BY NAME; SHOW NUMBERS OF PEOPLE IN BRACKETS; SALARY AMOUNTS MAY BE LISTED ON SEPARATE SCHEDULE) GPM 205.1b					
	1. P.I./P.D. Robert G. Jeroslow		1	2	$ 14,776	$
	2. CO P.I./P.D. none				$	$
	3. CO P.I./P.D.				$	$
	4. CO P.I./P.D.				$	$
	5. CO P.I./P.D.				$	$
11115	6. (1) ◄— SUBTOTALS A1 - A5 —►		1	2	$ 14,776	$
	FACULTY AND OTHER SENIOR ASSOCIATES (ATTACH EXTRA SHEET IF NECESSARY)					
	7.				$	$
	8.				$	$
	9.				$	$
	10.				$	$
	11.				$	$
11117	12. ( ) ◄— SUBTOTALS A7 - A11 —►				$ 0	$
	B. OTHER PERSONNEL (LIST NUMBERS IN BRACKETS)					
11141	1. ( ) POSTDOCTORAL ASSOCIATES				$	$
11149	2. ( ) OTHER PROFESSIONALS				$	$
11150	3. ( ) GRADUATE STUDENTS				$	$
11152	4. ( ) UNDERGRADUATE STUDENTS				$	$
11182	5. ( ) SECRETARIAL - CLERICAL				$	$
11183	6. ( ) TECHNICAL, SHOP, OTHER				$	$
	TOTAL SALARIES AND WAGES (A+B)				$ 14,776	$
11200	C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS) (10.51% of 6. + 12.)				$ 1,553	$
	TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A+B+C)				$ 16,329	$
	D. EQUIPMENT (LIST ITEMS AND DOLLAR AMOUNTS FOR EACH ITEM) none					
23181	TOTAL EQUIPMENT				$ 0	$
	E. MATERIALS AND SUPPLIES none					
32630					$ 0	$
	F. DOMESTIC TRAVEL					
42111					$ 800	$
	G. FOREIGN TRAVEL (LIST DESTINATION AND AMOUNT FOR EACH TRIP; GPM 731) none					
42112					$ 0	$

NOTICE OF RESEARCH PROJECT
SCIENCE INFORMATION EXCHANGE

SMITHSONIAN INSTITUTION

NATIONAL SCIENCE FOUNDATION
PROJECT SUMMARY

PROJECT NO. (Do not use this space)

NSF AWARD NO.

1. NAME OF INSTITUTION (INCLUDE BRANCH/CAMPUS & SCHOOL OR DIVISION)

2. MAILING ADDRESS

3. PRINCIPAL INVESTIGATOR AND FIELD OF SCIENCE/SPECIALTY

4. TITLE OF PROJECT

5. SUMMARY OF PROPOSED WORK (LIMIT TO 22 PICA OR 18 ELITE TYPEWRITTEN LINES)

FOR NSF USE ONLY

IVISION (OFFICE) AND DIRECTORATE	PROGRAM	
ECTION	PROPOSAL NO.	F.Y.

FOR DGC USE ONLY

TART AND END DATES	AMOUNT GRANTED

FORM 4 (7-78)

1. Proposal Folder
2. Program Suspense
3. Division of Grants & Contracts
4. Science Information Exchange
5. Principal Investigator
6. Off. of Govt. & Pub. Progs.
7. Assistant Director

Proposed Budget

July 1981-June 1982

(continued)

SUMMARY PROPOSAL BUDGET

PROPOSAL NO

52500	H. PUBLICATION COSTS/PAGE CHARGES	$ 300	$
62315	I. COMPUTER (ADPE) SERVICES	$ 200	$
	J. CONSULTANT SERVICES (IDENTIFY CONSULTANTS BY NAME AND AMOUNT; GPM 516) none	▨	▨
		$ 0	$
	K. PARTICIPANT SUPPORT COSTS, IF ALLOWED BY PROGRAM GUIDE (ITEMIZE) GPM 518	▨	▨
	1. STIPENDS $	▨	▨
	2. TRAVEL $	▨	▨
	3. SUBSISTENCE none $	▨	▨
	4. OTHER - SPECIFY $	▨	▨
	5. TOTAL PARTICIPANT COSTS (K1 + K2 + K3 + K4)	$ 0	$
	L. ALL OTHER DIRECT COSTS (List items and dollar amounts. Details of subcontracts, including work statements and budget, should be explained in full in proposal.) $ $ none $ $	▨	▨
65001	TOTAL OTHER DIRECT COSTS	$ 0	$
	M. TOTAL DIRECT COSTS (A THROUGH L)	$ 17,629	$
	N. INDIRECT COSTS (Specify rate(s) and base(s) for on/off campus activity. Where both are involved, identify itemized costs included in on/off campus bases in remarks.) on campus: 73% of (A + B)	▨	▨
74100	TOTAL INDIRECT COSTS	$ 10,787	$
	O. TOTAL DIRECT AND INDIRECT COSTS (M + N)	$ 28,416	$
74500	P. LESS RESIDUAL FUNDS (If for further support of current project; GPM 252 and 253)	$ 0	$
75000	Q. AMOUNT OF THIS REQUEST (O MINUS P)	$ 28,416	$

REMARKS

Georgia Institute of Technology will provide cost sharing in accordance with current Foundation policy

NOTE: SIGNATURES REQUIRED ONLY FOR REVISED BUDGET (GPM 233). THIS IS REVISION NO.

SIGNATURE OF PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR	DATE OF SIGNATURE	TYPED OR PRINTED NAME AND TITLE
SIGNATURE OF AUTHORIZED ORGANIZATIONAL REPRESENTATIVE	DATE OF SIGNATURE	TYPED OR PRINTED NAME AND TITLE

FOR NSF USE ONLY

INDIRECT COST RATE VERIFICATION			PROGRAM OFFICER APPROVAL
Date Checked	Date of Rate Sheet	Signature	

Grant Number	Amend No.	Institution	Organization	Fund Acct.	Program	Object

	Proposal Number	Dur.	Chg.	Award Date	Proposed Amount	Propsd. Dur.

Proposed Budget

July 1982-June 1983

SUMMARY PROPOSAL BUDGET

FOR NSF USE ONLY
PROPOSAL NO.
DURATION (MONTHS)
PROPOSED | REVISED

ORGANIZATION AND ADDRESS
Georgia Tech Research Institute
North Avenue, Atlanta, Georgia 30332

PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR
Robert G. Jeroslow

NSF USE	A. SENIOR PERSONNEL (LIST BY NAME; SHOW NUMBERS OF PEOPLE IN BRACKETS; SALARY AMOUNTS MAY BE LISTED ON SEPARATE SCHEDULE) GPM 205.1b	NSF FUNDED MAN MONTHS			FUNDS REQUESTED BY PROPOSER	FUNDS GRANTED BY NSF (IF DIFFERENT)
		CAL.	ACAD.	SUMR.		
	1. P.I./P.D. Robert G. Jeroslow		1	2	$ 16,255	$
	2. CO P.I./P.D.				$	$
	3. CO P.I./P.D.				$	$
	4. CO P.I./P.D.				$	$
	5. CO P.I./P.D.				$	$
11115	6. ( ) ◄—— SUBTOTALS A1 - A5 ——►		1	2	$ 16,255	$
	FACULTY AND OTHER SENIOR ASSOCIATES (ATTACH EXTRA SHEET IF NECESSARY)					
	7.				$	$
	8.				$	$
	9.				$	$
	10.				$	$
	11.				$	$
11117	12. ( ) ◄—— SUBTOTALS A7 - A11 ——►				$	$
	B. OTHER PERSONNEL (LIST NUMBERS IN BRACKETS)					
11141	1. ( ) POSTDOCTORAL ASSOCIATES				$	$
11149	2. ( ) OTHER PROFESSIONALS				$	$
11150	3. ( ) GRADUATE STUDENTS				$	$
11152	4. ( ) UNDERGRADUATE STUDENTS				$	$
11182	5. ( ) SECRETARIAL - CLERICAL				$	$
11183	6. ( ) TECHNICAL, SHOP, OTHER				$	$
	TOTAL SALARIES AND WAGES (A+B)				$ 16,255	$
11200	C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS) (10.51% of 6. + 12.)				$ 1,708	$
	TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A+B+C)				$ 17,963	$
	D. EQUIPMENT (LIST ITEMS AND DOLLAR AMOUNTS FOR EACH ITEM) none					
23181	TOTAL EQUIPMENT				$ 0	$
	E. MATERIALS AND SUPPLIES none					
32630					$ 0	$
	F. DOMESTIC TRAVEL					
42111					$ 800	$
	G. FOREIGN TRAVEL (LIST DESTINATION AND AMOUNT FOR EACH TRIP; GPM 731) Eleventh International Symposium on Mathematical Programming, location to be in Europe					
42112					$ 1,000	$

NOTICE OF RESEARCH PROJECT
SCIENCE INFORMATION EXCHANGE

SMITHSONIAN INSTITUTION
NATIONAL SCIENCE FOUNDATION
PROJECT SUMMARY

PROJECT NO. (Do not use this space)

NSF AWARD NO.

1. NAME OF INSTITUTION (INCLUDE BRANCH/CAMPUS & SCHOOL OR DIVISION)

2. MAILING ADDRESS

3. PRINCIPAL INVESTIGATOR AND FIELD OF SCIENCE/SPECIALTY

4. TITLE OF PROJECT

5. SUMMARY OF PROPOSED WORK (LIMIT TO 22 PICA OR 18 ELITE TYPEWRITTEN LINES)

FOR NSF USE ONLY

IVISION (OFFICE) AND DIRECTORATE	PROGRAM	
ECTION	PROPOSAL NO.	F.Y.

FOR OGC USE ONLY

TART AND END DATES	AMOUNT GRANTED

FORM 4 (7-78)

1 Proposal Folder
2 Program Suspense
3. Division of Grants & Contracts
4 Science Information Exchange
5 Principal Investigator
6. Off. of Govt. & Pub. Progs.
7 Assistant Director

Proposed Budget

July 1982-June 1983

(continued)

SUMMARY PROPOSAL BUDGET

PROPOSAL NO.

52500	**H. PUBLICATION COSTS/PAGE CHARGES**	$ 300	$
62315	**I. COMPUTER (ADPE) SERVICES**	$ 300	$
	J. CONSULTANT SERVICES (IDENTIFY CONSULTANTS BY NAME AND AMOUNT; GPM 516) none		
		$ 0	$
	K. PARTICIPANT SUPPORT COSTS, IF ALLOWED BY PROGRAM GUIDE (ITEMIZE) GPM 518 1. STIPENDS $ 2. TRAVEL $ 3. SUBSISTENCE $ 4. OTHER - SPECIFY $ none		
	5. TOTAL PARTICIPANT COSTS (K1 + K2 + K3 + K4)	$ 0	$
	L. ALL OTHER DIRECT COSTS (List items and dollar amounts. Details of subcontracts, including work statements and budget, should be explained in full in proposal.) $ $ $ $ none		
65001	TOTAL OTHER DIRECT COSTS	$ 0	$
	M. TOTAL DIRECT COSTS (A THROUGH L)	$ 20,363	$
	N. INDIRECT COSTS (Specify rate(s) and base(s) for on/off campus activity. Where both are involved, identify itemized costs included in on/off campus bases in remarks.) on campus: 73% of (A + B)		
74100	TOTAL INDIRECT COSTS	$ 11,866	$
	O. TOTAL DIRECT AND INDIRECT COSTS (M + N)	$ 32,229	$
74500	**P. LESS RESIDUAL FUNDS** (If for further support of current project; GPM 252 and 253)	$ 0	$
75000	**Q. AMOUNT OF THIS REQUEST** (O MINUS P)	$ 32,229	$

REMARKS

Georgia Institute of Technology will provide cost sharing in accordance with current Foundation policy

NOTE: SIGNATURES REQUIRED ONLY FOR REVISED BUDGET (GPM 233). THIS IS REVISION NO.

SIGNATURE OF PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR	DATE OF SIGNATURE	TYPED OR PRINTED NAME AND TITLE
SIGNATURE OF AUTHORIZED ORGANIZATIONAL REPRESENTATIVE	DATE OF SIGNATURE	TYPED OR PRINTED NAME AND TITLE

FOR NSF USE ONLY

INDIRECT COST RATE VERIFICATION

Date Checked	Date of Rate Sheet	Signature

PROGRAM OFFICER APPROVAL

Grant Number	Amend No.	Institution	Organization	Fund Acct.	Program	Object

Proposal Number	Dur.	Chg.	Award Date	Proposed Amount	Propsd. Dur.

Expenditures to Date
Under Existing NSF Contract
ENG 7900284

A.	Senior Personnel P.I. Robert G. Jeroslow (2 summer mos.)	$ 7,777
B.	Other Personnel Graduate Students	0
	Total Salaries and Wages (A + B)	7,777
C.	Fringe Benefits (9.83% of 6. + 12.)	764
	Total Salaries, Wages, and Fringe Benefits	8,541
F.	Domestic Travel	0
H.	Publication Costs	0
I.	Computer Services	0
M.	Total Direct Costs (A through L)	8,542
N.	Indirect Costs (on campus: 76% of (A + B))	5,910
O.	Total Direct and Indirect Costs, Total Expenditures to Date (9/11/79)	$14,452

Estimated Future Expenditures
Under Contract ENG 7900284, Ending June 30, 1980,
Including a Proposed Budget for Ensuing Period

A.	Senior Personnel Robert G. Jeroslow	$ 0
B.	Other Personnel One Graduate Student	3,600
	Total Salaries and Wages (A + B)	3,600
C.	Fringe Benefits	0
	Total Salaries, Wages, and Fringe Benefits	3,600
F.	Domestic Travel	600
H.	Publication Costs	200
I.	Computer Services	200
M.	Total Direct Costs (A through L)	4,600
N.	Indirect Costs (on campus: 76% of (A + B))	2,736
O.	Total Direct and Indirect Costs (Total Estimated Future Expenditures to June 30, 1979)	$7,336

Description of Travel and Its Relationship to Proposed Research

All travel is to professional conferences, where the Principal Investigator will give talks to disseminate research, listen to professional talks, and enter into discussions on developments and results in Operations Research.

Current and Pending Support Statement

The principal investigator does not have any other research support, and no other application is pending or contemplated.

M50-634

NATIONAL SCIENCE FOUNDATION
Washington, D.C. 20550

FINAL PROJECT REPORT

NSF FORM 98A

PLEASE READ INSTRUCTIONS ON REVERSE BEFORE COMPLETING

PART I–PROJECT IDENTIFICATION INFORMATION

1. Institution and Address Georgia Tech Research Institute North Avenue, Atlanta, GA 30332	2. NSF Program Engineering	3. NSF Award Number ENG-7900284
	4. Award Period From 7/1/79 To 12/31/80	5. Cumulative Award Amount $21,788

6. Project Title
"Cutting-Plane Theory and Nonlinear Programming"

PART II–SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

We have extended the limiting lagrangean equation to a wide variety of infinite dimensional settings in its broadest (i.e., set-valued) formulation, and obtained the most general conditions known for the equation to hold. As a consequence of this work, we have been able to provide a treatment of minimax problems from a limiting equation perspective, and have extended basic results in this area.

Our work has led to a penalty-method approach to mixed-integer programs and generalized complementarity problems. We have substantially sharpened earlier results on the valid form of cutting-planes for integer programs, by a study of the value function of these programs (i.e., the variation of the optimal value in terms of the right-hand-sides, which includes both variations in factors of production and variations in logical constraints). Specifically, we have found closed-form expressions for these value functions in terms of the operations of nonnegative combinations, maxima, and integer round-up, as applied initially to linear functions. The value function provides the sensitivity analysis for changes in the right-hand-sides, which is of particular interest in applications.

Our work has also provided "integer analogues" for concepts which occur in linear programming (eg., linear function, polyhedral function, dual program, etc.). By replacing each concept mentioned, in a true statement of linear programming, by its integer analogue, one often obtains a true statement of integer programming, perhaps with suitable regularity conditions.

PART III–TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM	
				Check (✓)	Approx. Date
a. Abstracts of Theses	X				
b. Publication Citations		X			
c. Data on Scientific Collaborators	X				
d. Information on Inventions	X				
e. Technical Description of Project and Results	▨	X			
f. Other *(specify)* See following page "Table of Contents"	▨	X			

2. Principal Investigator/Project Director Name *(Typed)* Robert G. Jeroslow	3. Principal Investigator/Project Director Signature Robert G. Jeroslow	4. Date March 11, '81

NSF Form 98A (5-78) Supersedes All Previous Editions

Form Approved OMB No. [illegible]

NSF Grant ENG-7900284

FINAL REPORT

Part III - Table of Contents

b. Publication Citations for Papers Written Under this Grant

(see also f1. for related information)

Published

1. "Lagrangean Functions and Affine Minorants," with R.J. Duffin, Mathematical Programming Study no. 14, 1981, pp. 48-60.

Accepted for Publication

2. "An Exact Penalty Method for Mixed Integer Programs," with C.E. Blair, to appear in Mathematics of Operations Research.

3. "The Value Function of an Integer Program," with C.E. Blair, to appear in Mathematical Programming.

Submitted for Publication

4. "The Limiting Lagrangean," with R.J. Duffin, June 1979.

5. "A Limiting Infisup Theorem," with C.E. Blair and R.J. Duffin, August 1979.

e. Technical Description of Project and Results

In citations here, we use the numbering of our papers as given on page three, for "b. Publication Citations for Papers Written Under this Grant." The five reports cited are given as appendices here (item f3. on page two, Table of Contents). We also refer to our paper, "Some Influences of Generalized and Ordinary Convexity in Disjunctive and Integer Programming," under the citation "[f4]," since it is given as item f4. on page two, Table of Contents, and is an Appendix here. (The latter paper was written after the funded part of the grant period and will be reported under a later grant.)

A set-valued mapping $h:X \to Y$ from a linear space X to a linear space Y is called <u>convex</u>, if $\{(x,y) \mid y \varepsilon h(x)\}$ is a convex set in $X \times Y$. This concept is evidently due to Blashke, and is more general than cone-convex functions.

In [4] we study the optimization problem

$$\text{(P)} \qquad \begin{array}{l} \inf f(u) \\ \text{subject to } 0 \varepsilon g(u) \end{array}$$

where $g:U \to W$ and $f:U \to R$ are convex set functions, U and W are locally convex linear topological spaces, and W is semi-reflexive. The usual convex programming problem, with function constraints and a set constraint, can be cast in the form (P), as we show in [4], and the case $U=R^n$ and $W=R^m$ is a special instance.

Suppose that (P) is consistent, and denote its value by $v(P)$. In [4] we give necessary and sufficient conditions for this "limiting lagrangean equation" to hold:

$$\text{(LL)} \qquad \lim_{M \to 0} \; \sup_{u^* \varepsilon M} \; \sup_{\lambda^* \varepsilon W^*} \; \inf_{u \varepsilon U} \{f(u)+u^*(u)+\lambda^* g(u)\} = v(P)$$

In (LL), U* respectively W* is the dual of U resp. W, M is an open set in U^*, and "M→0" denotes the filter consisting of a local base of open sets about 0εU*. When $U=R^n$, the case of particular interest to us, (LL) can be simplified to this form, for some fixed vector $u_0^* \varepsilon R^n$:

$$(LL)^n \quad \lim_{\theta \downarrow 0^+} \; \sup_{\lambda \varepsilon W^*} \; \inf_{u \varepsilon U} \{f(u)+\theta u_0^*.u+\lambda g(u)\} = v(P)$$

From our results, it easily follows that (LL) and $(LL)^n$ hold in instances in which the ordinary lagrangean has a duality gap, including many instances in which the convex functions involved are not closed. These results complement the usual Lagrangean duality results, for we show that, by adding a "limiting perturbation" (i.e., θu_0^* (u) in $(LL)^n$) to the criterion function, most duality gaps are closed. While results of this type can be inferred from earlier results in conjugate duality, we explicitly exhibit (LL) and $(LL)^n$ and have obtained the broadest hypotheses for which these equations, and similar ones, are valid.

We use the work in [4] to obtain conditions sufficient for this "limiting infisup" equation, which we state in the particular case that $X=R^n$:

$$(LIS)^n \quad \lim_{\theta \downarrow 0^+} \; \sup_{y \varepsilon D} \; \inf_{x \varepsilon C} \{\theta x_0^* \cdot x+F(x,y)\} = \inf_{x \varepsilon C} \; \sup_{y \varepsilon D} \; F(x,y)$$

In $(LIS)^n$, $x_0^* \varepsilon R^n$ is some fixed vector, C is a convex set in R^n, D is a convex set in the linear space Y, F:CxD→R satisfies some convexity/concavity assumptions, and some additional hypotheses are met, which are exactly specified in [5]. We provide the broadest hypotheses known for $(LIS)^n$, and show how the "limiting perturbation" term "$\theta x_o^*.x$" allows closure of duality gaps in situations where minimax or infisup results fail.

In [5], we also provide conditions under which the following "finite infisup" result holds:

(FIS) $$\sup_{y\in D}\ \inf_{x\in C}\ F(x,y) = \sup_{\substack{G\in D\\ G\text{ finite}}}\ \inf_{x\in C}\ \max_{y\in G}\ F(x,y)$$

In doing so, we generalize well-known results of Sion and Kneser and Fan (cited in [5]).

The paper [1] is a specialized account of our work in [4], which illustrates our methods of proof and our approach to the equations (LL), $(LL)^n$, and $(LIS)^n$.

The paper [2] concerns the mixed-integer program:

(MIP)
$$\begin{aligned} &\inf\ cx+dy\\ \text{subject to }&Ax+By=b\\ &x,y\geq 0\\ &x\text{ integer}\end{aligned}$$

We always assume that A,B,b,c and d are rational, and that (MIP) is consistent of finite value v(P). We establish in [2] that there is a finite value $\rho_0\geq 0$ for the "penalty parameter", such that the following "norm penalty" result holds:

(NP) $$\min_{\substack{x,y\geq 0\\ x\text{ integer}}}\ \{cx+dy+\rho_0||Ax+by-b||\}=v(P)$$

We also extend (NP) to more complex constraint sets, including complementarity constraints. The value of ρ_0 in (NP) varies (typically discontinuously) with the right-hand-side b.

Our research in paper [3] was done toward the end of the funded period of the grant, and represents a development which is surprising to us. Specifically, for the pure integer program

(IP)
$$\begin{aligned} &\min\ cx\\ \text{subject to }&Ax=b\\ &x\geq 0\text{ and integer}\qquad x=(x_1,\ldots,x_r);\ A=[a^j](\text{cols.}))\end{aligned}$$

in which no continuous variables occur, we were able to give, in principle, a closed form expression F(b) for the optimal value to (IP) as a function of its right-hand-side (rhs)b. These closed form expressions, called the "Gomory functions" in [3], are built up from the linear functions by inductive application of nonnegative combinations, maxima, and integer round-up operations.

The optimal value function F(b) is of obvious importance in applications since it embodies all sensitivity analysis for the rhs. In [3] we also obtain sensitivity analysis information as the criterion function cx varies, and we obtain closed-form expressions for the optimal solution vector x^o.

Moreover, a study of the optimal value function is essentially a study of all the valid cutting-planes for (IP). It is well-known, for example, that for any optimal value function F the inequality

$$\text{(CP)} \qquad \sum_{j=1}^{r} F(a^j)x_j \geq F(b)$$

is a valid cutting-plane for (IP); and moreover, cutting-planes of the form (CP) are all that are needed to obtain all valid cutting-planes (as the nonnegativity conditions $x \geq 0$ are enforced via the pivoting of the simplex Algorithm). A converse is also true for a pair of Gomory functions F and G, when G satisfies a condition specified in [3] ($G(b)>0$ for b non-integer is one example of that condition): these exist an integer program (IP), consistent exactly when $G(b) \leq 0$, having optimal value (when consistent) of F(b). In this manner, we have exactly identified the class of optimal value functions for (IP), in terms of the inductively-defined class of Gomory functions.

Those closed form expressions which are built up from the linear functions by inductive application of nonnegative combinations and maxima alone (i.e., no use of the integer round up) are the polyhedral convex functions. These can be shown to provide the class of optimal value functions of linear programs. Consequently, we cannot expect a characterization of (IP) value functions which is much simpler than the one we have obtained, as the use of integer round-up operations is a minimal concession to the occurence of integer variables in (IP).

An alternative perspective on our results in [3] is provided in the brief discussion in [f4, pages 6-10] on "integer analogues." Put briefly, we have found discrete analogues of the linear concepts of "linear function," "polyhedral convex function," "polyhedral cone," etc. which allow a nearly automatic way of producing valid theorems in integer programming from known theorems of linear programming. Basically, if one inserts in a linear programming theorem the integer analogue names for the linear objects named there, one obtains a statement which is true, perhaps with some additional "regularity conditions". However, the proof of the linear programming theorem typically does not go over routinely to produce a proof of the integer programming theorem. New methods of proof have been necessary up to the present time. A discussion of this "nearly automatic" procedure for producing theorems, together with some remarks on its limitations, is in [f4].

In research previous to the grant that is the subject of this report, we saw no chance of an inductive characterization of the value function for (MIP). Indeed, the value functions for (MIP) are not closed under addition. We were fortunate in reconsidering this issue in the context of the more specialized problem (IP), and since the end of this grant we have obtained a (noninductive) characterization of the value functions for (MIP), using the results and concepts which occured in our study for (IP).

Publication Activity, July 1979 to March 1981

Papers Published

1. "Representations of Unbounded Optimizations as Integer Programs", Journal on Optimization Theory and Its Applications (30), 1980, pp. 339-351.

2. "Lagrange Dual Problems with Linear Constraints on the Multipliers", with C.E. Blair, Constructive Approaches to Mathematical Models, C.V. Coffman and G. Fix (eds.), Academic Press, 1979, pp. 137-152.

3. "An Introduction to the Theory of Cutting-Planes", Annals of Discrete Mathematics (5), 1979, pp. 71-95.

4. "A Cutting-Plane Game for Facial Disjunctive Programs", SIAM Journal on Control and Optimization (18), 1980, pp. 264-281.

5. "Strengthening Cuts for Mixed-Integer Programs", with E. Balas, European Journal of Operations Research (4), 1980, pp. 224-234.

6. "Lagrangean Functions and Affine Minorants", with R.J. Duffin, Mathematical Programming Studies no. 14, 1981, pp. 48-60.

Papers Accepted for Publication

1. "A Limiting Lagrangean for Infinitely-Constrained Conves Optimization in R^n", Journal of Optimization and Theory Applications.

2. "An Exact Penalty Method for Mixed-Integer Programs", with C.E. Blair, Mathematics of Operations Research.

3. "The Value Function of an Integer Program", with C.E. Blair, Mathematical Programming.

Other Papers Submitted for Publication

1. "The Limiting Lagrangean", with R.J. Duffin, June 1979.

2. "A Limiting Infisup Theorem", with C.E. Blair and R.J. Duffin, August 1979.

3. "Some Influences of Generalized and Ordinary Convexity in Disjunctive and Integer Programming", August 1980.

DR. ROBERT G. JEROSLOW

February 1981

rsonal

Birthdate: September 13, 1942
Soc. Sec.: 151-32-1110
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ucation:

B. S. 1964	Columbia University School of Engineering Department of Industrial Engineering
1964 - 1966	Cornell University School of Engineering Department of Operations Research (completed Comprehensive Examination)
Ph. D. 1969	Cornell University Department of Mathematics Professor Anil Nerode, Advisor

perience:

University of Minnesota
School of Mathematics
September 1969 - August 1972
Assistant Professor

Carnegie-Mellon University
Graduate School of Industrial Administration and Department of Mathematics
Associate Professor
September 1972 - August 1976
Professor
September 1976 - June 1980

Georgia Institute of Technology
College of Management
Professor
September 1978 to date

search Interests

Mathematical programming, with emphasis on cutting-plane theory and its uses in integer programming and linear complementarity; nonlinear programming; integer programming; programming aspects of computational complexity; multicriteria optimization. Strategic planning systems, and the utilization of quantitative models to assist in nonquantitative decision making.

aching Interests

Research interests, plus production and applications of Operations Research techniques; management strategy.

urnals

Member of the Editorial Board (Associate Editor), *Discrete Applied Mathematics*, *Mathematical Programming* and *Mathematical Programming Studies*.

Referee for *Operations Research*, *SIAM Journal on Applied Mathematics*, *Management Science*, *Mathematical Programming*, and *Discrete Mathematics*.

Reviewer for *Bulletin of the American Mathematical Society*.

ınts and Fellowships

Ford A Fellowship 1964-1966
NSF Graduate Fellowship 1966-1969
NSF Research Grant GP 21067, Principle Investigator, 1971-1972
  (Grant Awarded 1970)
NSF Research Grant GP-37510X, Associate Investigator, 1973-1975
NSF Research Grant MCS76-12026, Co-Principle Investigator, 1976-1978
Research Fellowship, January - June 1977, from the Center for
  Operations Research and Econometrics, Belgium
NSF Research Grant ENG-79000284, Principle Investigator, 1979-1980
NSF Research Grant ECS-8001763, Principle Investigator, 1980-1982

anizational Responsibilities

Representative to the Faculty Senate from the business school, 1977-1978
Organizer of the Operations Research Seminar for the business school,
  1977-1978

Representative of the management college to the Seminar on Operations Research, 1978-1981 (co-sponsored with the School of Industrial and Systems Engineering and the School of Mathematics)

Member of the Program Committee of the symposium in honor of R. J. Duffin, Constructive Approaches to Mathematical Models, July 10-15, 1978

Co-organizer (with Cedric Suzman) of the Colloquiua on Strategic Planning, September 28, 1979, and October 10, 1980 (third Colloquium projected for Fall 1981)

Member of the Organizing Committee of the 1981 International Symposium on Semi-infinite Programming and Applications

Member of the Teaching Evaluation Committee in the College of Management

Member of the Personnel Committee in the College of Management

bies and Personal Interests

Light hiking; swimming; jogging; weight training; reading in literature, archaeology and history.

lished Articles

"Consistency Statements in Formal Theories", Fundamentae Mathematicae, LXXXII (1971), pp. 17-40.

"Non-effectiveness in S. Orey's Arithematical Compactness Theorem", Zeithschrift f. math. Logik and Grundlagen d. Math., Bd. 17, 1971, pp. 285-289.

"On Semi-infinite Systems of Linear Inequalities", with K. O. Kortanek, Israel Journal of Mathematics, vol. 10, no. 2, 1971, pp. 252-258.

"A Note on Some Classical Methods in Constrained Optimization and Positively Bounded Jacobians", with K. O. Kortanek, Operations Research, vol. 15, no. 5, 1967, pp. 964-969.

"Comments on Integer Hulls of Two Linear Constraints", Operations Research, vol. 19, no. 4, July-August 1971, pp. 1061-1069.

"On an Algorithm of Gomory", with K. O. Kortanek, SIAM Journal on Applied Mathematics, vol. 21, no. 1, July 1971, pp. 55-59.

"Canonical Cuts on the Unit Hypercube", with E. Balas, SIAM Journal on Applied Mathematics, vol. 23, no. 1, July 1972, pp. 61-69.

"There Cannot be any Algorithm for Integer Programming with Quadratic Constraints", Operations Research, Programming Volume, vol. 21, no. 1, January-February 1973, pp. 221-224.

"Redundancies in the Hilbert-Bernays Derivability Conditions for Godel's Second Incompleteness Theorem", Journal of Symbolic Logic, vol. 38, no. 3, September 1973, pp. 359-367.

"Linear Programs Dependent on a Single Parameter", Discrete Mathematics (6), 1973, pp. 119-140.

"The Simplex Algorithm with the Pivot Rule of Maximizing Criterion Improvement", Discrete Mathematics (4), 1973, pp. 367-378.

"An Exposition on the Constructive Decomposition of the Group of Gomory Cuts and Gomory's Round-off Algorithm", with K. O. Kortanek, Cahiers du Centre d'Etudes de Recherche Operationnelle, no. 2, 1971, pp. 63-84.

"Asymptotic Linear Programming", Operations Research (21), 1973, pp. 1128-1141.

"On Algorithms for Discrete Problems", Discrete Mathematics (7), 1974, pp. 273-280.

"Trivial Integer Programs Unsolvable by Branch and Bound", Mathematical Programming (6), 1974, pp. 105-109.

"On Defining Sets of Vertices of the Hypercube by Linear Inequalities", Discrete Mathematics (11), 1975, pp. 119-124.

"A Generalization of a Theorem of Chvatal and Gomory", pp. 313-332 in Nonlinear Programming·2, edited by O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, Academic Press, New York, 1975.

"Experimental Results on Hillier's Linear Search", with T. H. C. Smith, Mathematical Programming (9), 1975, pp. 371-376.

"Experimental Logics and 2^0-Theories", Journal of Philosophical Logic (4), 1975, pp. 253-267.

"Cutting-plane Theory: Disjunctive Methods", Annals of Discrete Mathematics, vol. 1, May 1977, pp. 293-330.

"Cutting-planes for Complementary Constraints", SIAM Journal on Control and Optimization, vol. 16, no. 1, January 1978, pp. 56-62.

"Bracketing Discrete Problems by Two Problems of Linear Optimization", in Operations Research Verfahren (Methods of Operations Research) XXV, 1977, pp. 205-216, Verlag Anton Hain, Meisenheim an Glan.

"The Value Function of a Mixed Integer Program: I", with C. E. Blair, Discrete Mathematics, vol. 19, 1977, pp. 121-138.

"Some Basis Theorems for Integral Monoids", Mathematics of Operations Research 3, 1978, pp. 145-154.

"Cutting-plane Theory: Algebraic Methods:, Discrete Mathematics 23, 1978, pp. 121-150.

"A Converse for Disjunctive Constraints", with C. E. Blair, Journal of Optimization Theory and Its Applications, June 1978.

"Some Relaxation Methods for Linear Inequalities", Cahiers du Centre d'Etudes de Recherche Operationnelle, vol. 21, no. 1, 1979, pp. 43-53.

"The Value Function of a Mixed-Integer Program: II", with C. E. Blair, Discrete Mathematics (25), 1979, pp. 7-19.

"Minimal Inequalities", Mathematical Programming (17), 1979, pp. 1-15.

"Two Lectures on the Theory of Cutting-planes", for Combinatorial Optimization, edited by N. Christofides et al., John Wiley and Sons, Ltd.

"Representations of Unbounded Optimizations as Integer Programs", Journal on Optimization Theory and Its Applications (30), 1980, pp. 339-351.

"Lagrange Dual Problems with Linear Constraints on the Multipliers", with C. E. Blair, Constructive Approaches to Mathematical Models, C. V. Coffman and G. Fix (eds.), Academic Press, 1979, pp. 137-152.

"An Introduction to the Theory of Cutting-planes", Annals of Discrete Mathematics (5), 1979, pp. 71-95.

"A Cutting-plane Game for Facial Disjunctive Programs", SIAM Journal on Control and Optimization (18), 1980, pp. 264-281.

"Strengthening Cuts for Mixed Integer Programs", with E. Balas, European Journal of Operations Research (4), 1980, pp. 224-234.

k Review

M. R. Hestenes' Optimization Theory: The Finite-dimensional Case, reviewer in Bulletin of the American Mathematical Society (83), May 1977, pp. 324-334.

epted for Publication

"A Limiting Lagrangean for Infinitely-constrained Convex Optimization in R^n", Journal of Optimization Theory and Applications.

"Lagrangean Functions and Affine Minorants", with R. J. Duffin, Mathematical Programming.

"An Exact Penalty Method for Mixed Integer Programs", with C. E. Blair, Mathematics of Operations Research.

"The Value Function of an Integer Program", with C. E. Blair, Mathematical Programming.

bmitted for Publication

"The Limiting Lagrangean", with R. J. Duffin, June 1979.

"A Limiting Infisup Theorem", with C. E. Blair and R. J. Duffin, August 1979.

"Some Influences of Generalized and Ordinary Convexity in Disjunctive and Integer Programming", August 1980.

rrent Research (Papers in Preparation)

"Duality in Semi-infinite Linear Programming", with R. J. Duffin and L. A. Karlovitz.

"Duality in Semi-infinite Convex Programming".

"Sensitivity Analysis for Mixed Integer Programs", with C. E. Blair.

"Proceedings of the Second Annual Georgia Tech Colloquium on Strategic Planning", C. Suzman, co-editor.

vited Talks

"On Godel's Consistency Theorem", University of Texas at Austin, October 1971.

"K-descriptions", Lakehead University, Thunder Bay, Ontario, Canada, January 1972.

"Asymptotic Linear Programming", Carnegie-Mellon University, February 1972.

"Trial-and-error Logics", State University of New York at Buffalo, February 1973.

"On a Theorem of Chvatal and Gomory", SIGMAP-UW Symposium on Nonlinear Programming, University of Wisconsin, April 1974.

"Proof Theory and Hilbert's Finitism", a series of nine lectures, Universita di Siena, Instituto di Matematica, Siena, Italy, May 1974.

"Cutting-planes for Relaxations of Integer Programs", ORSA/TIMS meeting in San Juan, P. R., October 1974.

"Some Results and Constructions of Cutting-plane Theory", NSF Regional Conference on Convex Polytypes and Mathematical Programming, Tuscaloosa, Alabama, June 9-13, 1975.

"Algebraic Methods, Disjunctive Methods", for the Workshop in Integer Programming, Bonn, Germany, September 8-11, 1975.

"Completeness Theorems for Cutting-planes", seminar at the University of North Carolina, November 13, 1975.

"Minimal Inequalities", ORSA/TIMS meeting in Las Vegas, November 17-19, 1975.

"Completeness Results in Cutting-plane Theory", Centre de Recherches Mathematiques, Montreal, January 1976.

"Cutting-planes for Complementary Constraints", IX International Symposium on Mathematical Programming, Budapest, August 1976.

"Bracketing Discrete Problems by Two Problems of Linear Optimization", Symposium on Operations Research, Heidelberg, September 1976.

"Treeless Searches", ORSA/TIMS Joint National Meeting (joint with C. E. Blair), Miami Beach, November 1976.

"The Complexity of Certain Linear and Integer Programming Algorithms", University of Bonn, February 1977.

"Subadditivity and Value Functions of Mixed-Integer Programs", University of Cologne, May 1977.

"Linear Programs Dependent on a Single Parameter", University of Aachen, May 1977.

"An Introduction to the Theory of Cutting-planes", Summer School on Combinatorial Optimization at Sogesta, in Urbino, Italy, June 1977.

"An Introduction to the Theory of Cutting-planes", NATO Advanced Research Institute on Discrete Optimization and Systems Applications, Vancouver, August 1977.

"Cutting-planes and Cutting-plane Algorithms for Complementary Constraints", International Symposium on Extremal Methods and Systems Analysis, Austin, September 1977.

"A Cutting-plane Game and Its Algorithms", Georgia Institute of Technology, Atlanta, February 1978.

"A Limiting Lagrangean for Infinitely Constrained Convex Optimization in R^n", at Constructive Approaches to Mathematical Models, Pittsburgh, July 1978.

"Representations of Unbounded Optimizations as Integer Programs", ORSA/TIMS meeting in New Orleans, April 30 - May 2, 1979.

"Recent Results in Nonlinear and Integer Programming", at the meeting on Mathematical Programming, at Mathematisches Forschungsinstitut in Obersolfach, Germany, May 6-12, 1979.

"A Limiting Infisup Theorem", at the Tenth International Symposium on Mathematical Programming, Montreal, August 27-31, 1979.

. "Nonlinear Optimization Treated by Linear Inequalities", ORSA/TIMS meeting in Milwaukee, October 15-17, 1979.

. "Sensitivity Analysis for Integer Programs", ORSA/TIMS National Meeting at Colorado Springs, November 10-12, 1980.

. "Integer Analogues", Mathematical Sciences Department, University of Delaware, November 1980.

. "Sensitivity Analysis for Mixed Integer Programs", CORS/ORSA/TIMS Joint National Meeting, Toronto, May 3-6, 1981.

M-50-634

ABSTRACT

We give hypotheses, valid in reflexive Banach spaces (such as L^p for $\infty > p > 1$ or Hilbert spaces), for a certain modification of the ordinary lagrangean to close the duality gap, in convex programs with (possibly) infinitely many constraint functions.

Our modification of the ordinary lagrangean is to perturb the criterion function by a linear term, and to take the limit of this perturbed lagrangean, as the norm of this term goes to zero.

We also review the recent literature on this topic of the "limiting lagrangean."

Key Words

1) Convexity
2) Lagrangean
3) Nonlinear programming

Final Report
NSF Grant ENG-7900284
Ref. 1

Math Prog Studies no. 14
pp. 48-60

Revised

LAGRANGEAN FUNCTIONS AND AFFINE MINORANTS

by R. J. Duffin[1] and R. G. Jeroslow[2]

In an earlier paper [6], the first author proved this result, for convex functions F_h defined on all of R^n:

$$1) \quad \lim_{\epsilon \to 0^+} \sup_{\substack{a \in R^n \\ \|a\| < \epsilon}} \sup_{\lambda} \inf_{x \in R^n} \{F_0(x) + ax + \sum_{h \in H} \lambda_h F_h(x)\} = v(P) \ .$$

In 1), v(P) is the value of the convex program

$$(CP) \qquad \inf F_0(x) \quad \text{subject to } F_h(x) \leq 0, \ h \in H$$

where H is a finite, non-empty index set.

A purpose of this paper is to extend 1) to proper lower-semi-continuous (l.s.c.) convex functions defined on a convex subset of certain infinite-dimensional spaces, specifically reflexive Banach spaces, and also to obtain information on "affine minorants" of the convex functions. The L^p spaces for $\infty > p > 1$ and Hilbert spaces are treated by our results. A goal of the paper will be to establish the following result in this setting, under suitable hypotheses:

$$(LL) \quad \lim_{\epsilon \to 0^+} \sup_{\substack{g \in X^* \\ \|g\| < \epsilon}} \sup_{\lambda} \inf_{x \in K} \{F_0(x) + g(x) + \sum_{h \in H} \lambda_h F_h(x)\} = v(P)$$

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where the index set H may be infinite, and K will be a convex subset of the space in which the variable x is constrained to lie by an explicit set constraint "$x \in K$," in addition to functional constraints such as those in (CP) above (see (8) below).

We use the theory of infinite sets of linear inequalities to obtain our results. Our approach has its source in the literature of "semi-infinite programming" (see e.g. [1] and [9]), and is the basic idea for proofs of various strengthenings and refinements of our result in infinite-dimensional spaces.

Professor R. T. Rockfellar has informed us (private communication) that the result (LL) is implicit in his monograph [15], under suitable hypotheses, and it is indeed the case that [15, equation (4.20)] can be applied to [15, Example 4, page 26] to derive (LL) under the hypotheses used in [15]. We strengthen the result due to the additional information in Theorem 6, and, as we will point out in Section III, our mode of analysis easily extends to set-valued maps in locally convex spaces, without the hypotheses of semi-continuity used in [15]. See [2] for a counter-example to (LL) when the semi-continuity hypothesis is dropped.

Our present paper contains an exposition of a part of the semi-infinite approach to convex optimization. For related work which utilizes the theory of conjugate functions see [13], [14], and [15].

SECTION I: PRELIMINARY RESULTS, CONVENTIONS, AND GENERAL ASSUMPTIONS

Throughout the results, X will denote a reflexive Banach space. Thus $X^{**} = X$, where Y^* denotes the space of all continuous linear functionals on the linear topological space Y.

The following result is well-known; see e.g. [5].

THEOREM 1: Let C be a closed cone in a locally convex linear topological space Y.

Then the following two statements are equivalent:

(i) $y_0 \in C$;

(ii) If $f \in Y^*$, and $f(y) \geq 0$ for all $y \in C$, then $f(y_0) \geq 0$.

In what follows, we view functions as points, so that, e.g., $f = h$ abbreviates $f(x) = h(x)$ for all $x \in X$.

COROLLARY 2: Let $\{f_i | i \in I\}$ be a family of continuous linear functionals on the reflexive Banach space X, and suppose that, for any $x \in X$,

2) $$f_i(x) \geq 0 \quad \text{for all } i \in I$$

implies

3) $$f(x) \geq 0$$

for the continuous linear functional f.

Then for any real scalar $\epsilon > 0$ there exists a finite subset $J \subseteq I$ and non-negative numbers $\lambda_j \geq 0$, $j \in J$, and a continuous linear functional g, satisfying both these conditions:

α) $$f = g + \sum_{j \in J} \lambda_j f_j$$

β) $$\|g\| < \epsilon .$$

PROOF: Let $C = \mathrm{cl}(\mathrm{cone}(\{f_i | i \in I\}))$, where cone $(\{f_i | i \in I\})$ is the cone (algebraically) generated by the set $\{f_i | i \in I\}$, and cl(S) denotes the closure, here in the norm topology, of the set $S \subseteq X^*$.

The conclusion of this corollary can be restated as "$f \in C$," for if $\|f - \sum_{j \in J} \lambda_j f_j\| < \epsilon$ then $g = f - \sum_{j \in J} \lambda_j f_j$ satisfies α) and β).

Since C is a closed cone in the locally convex linear topological space X^*, Theorem 1 applies. Thus if $f \notin C$, we reach a contradiction as follows, where we take $y_0 = f$ in Theorem 1.

There exists a continuous linear functional $\tilde{F}$ on X^* with $\tilde{F}(h) \geq 0$ for all $h \in C$ and $\tilde{F}(f) < 0$. In particular, $\tilde{F}(f_i) \geq 0$ for all $i \in I$, and $\tilde{F}(f) < 0$.

Since $\tilde{F} \in X^{**}$, there exists $\tilde{x} \in X$ with $\tilde{F}(h) = h(\tilde{x})$ for all $h \in X^*$. In particular, $f_i(\tilde{x}) \geq 0$ for all $i \in I$ and $f(\tilde{x}) < 0$, contradicting the hypothesis. This shows that $f \in C$.

Q.E.D.

In what follows, we view (γ,h), where h is a function on X, and $\gamma \in R$, as the functional on $X \times R$ such that $(\gamma,h)\ (p,x) = h(x) + \gamma p$, for $(p,x) \in R \times X$.

For any linear topological space Y, the continuous dual $(R \times X)^*$ of $R \times Y$ is $(R \times Y)^* = R \times Y^*$, with the evaluation $(r,f)(s,y) = f(y) + rs$, where $(r,f) \in R \times Y$, $f \in Y^*$, and $(s,y) \in R \times Y$, $y \in Y$. In particular, as

X is reflexive, $(R \times X)^{**} = (R\times X^*)^{**} = R \times X^{**} = R \times X$, so $R \times X$ is reflexive. We need this latter observation in the next result.

COROLLARY 3: Let $\{f_i | i \in I\}$ be a family of continuous linear functionals on the reflexive Banach space X and let $\{\alpha_i | i \in I\}$ be a correspondingly-indexed family of real scalars, such that there is a solution to

4) $$f_i(x) \geq \alpha_i, \quad i \in I \ .$$

Suppose that every solution x to 4) also satisfies

5) $$f(x) \geq \alpha$$

for the continuous linear functional f and scalar $\alpha \in R$.

Then for any real scalar $\epsilon > 0$ there exists a finite subset $J \subseteq I$, non-negative numbers λ_j, $j \in J$, a non-negative scalar $\theta \geq 0$, and a continuous linear functional g on X, and $\beta \in R$, satisfying:

α)′ $$(-\alpha, f) = \theta(1,0) + (-\beta, g) + \sum_{j\in J} \lambda_j(-\alpha_j, f_j) \ .$$

β)′ $$\|(-\beta, g)\| < \epsilon \ .$$

In particular,

6a) $$f = g + \sum_{j\in J} \lambda_j f_j$$

6b) $$\|g\| < \epsilon$$

6c) $$\alpha \leq \epsilon + \sum_{j\in J} \lambda_j \alpha_j$$

<u>PROOF</u>: The particular conclusions 6a)-6c) follow from $\alpha)'$ and $\beta)'$ by taking components in $\alpha)'$, and noting that $\beta)'$ implies $\|g\| < \epsilon$ and $|\beta| < \epsilon$. We prove only $\alpha)'$ and $\beta)'$.

To do so, note that, in the space $R \times X$,

$$4)' \qquad -\alpha_i r + f_i(x) \geq 0, \quad i \in I$$
$$r \geq 0$$

implies

$$5)' \qquad -\alpha r + f(x) \geq 0 \; .$$

Indeed, if $r > 0$, $4)'$ implies $5)'$ by the fact that 4) implies 5) and the linearity of the functionals $\{f_i | i \in I\}$ and f. If $r = 0$, again $4)'$ implies $5)'$, as we see by the following contradiction.

Let $\tilde{x}$ be such that $f_i(\tilde{x}) \geq 0$ for $i \in I$ yet $f(\tilde{x}) < 0$. By hypothesis there exists x^* with $f_i(x^*) \geq \alpha_i$ for $i \in I$. Then for any scalar $\rho \geq 0$, $f_i(x^* + \rho\tilde{x}) = f_i(x^*) + \rho f_i(\tilde{x}) \geq f_i(x^*) + 0 \geq \alpha_i$ for all $i \in I$. However for large ρ, $f(x^* + \rho\tilde{x}) = f(x^*) + \rho f(\tilde{x}) < \alpha$ as $f(\tilde{x}) < 0$. This contradicts that 4) implies 5), and proves that $4)'$ implies $5)'$.

We apply Corollary 2 to the system $4)'$, $5)'$ with 2) taken as $4)'$, and the functionals $\{f_i | i \in I\}$ of 2) taken as $\{(-\alpha_i, f_i) | i \in I\} \cup \{(1,0)\}$. Likewise the functional f of 3) is $(-\alpha, f)$ in $5)'$. The corollary applies since $R \times X$ is a reflexive Banach space.

Upon application of Corollary 2, we at once obtain $\alpha)'$ and $\beta)'$, since θ is simply the multiplier of the functional $(1,0)$, where here "0" is the identically zero linear functional on X.

Q.E.D.

In what follows, we shall consider convex functions F on subsets of X, by which we mean a function $F:D\to R$ where D is a non-empty convex subset of X. (We do not use the extended reals $\bar{R} = R \cup \{-\infty\} \cup \{+\infty\}$ here.) As usual, the epigraph epi(F) of F is defined as:

$$\text{7)} \qquad \operatorname{epi}(F) = \{(z,x) \in R \times D \mid z \geq F(x)\} \; .$$

We say that F is closed if epi(F) is closed in $R \times X$, i.e., if F is a proper lower-semi-continuous convex function.

This paper is concerned with the following convex program, where each function F_h for $h \in H \cup \{0\}$ (H an index set of arbitrary cardinality) is finite and lower-semi continuous on a domain D_h, K is a non-empty and closed convex set in X, and $D_h \supseteq K$ for $h \in \{0\} \cup H$:

$$\text{(8)} \qquad \begin{array}{c} \inf F_0(x) \\ \text{subject to } F_h(x) \leq 0, \quad h \in H \\ \text{and } x \in K \end{array}$$

This program (8) is assumed to have a finite value v(P); thus (8) is assumed consistent, but the infimum need not be attained.

We shall be concerned with this Lagrangean, which we call the "limiting Lagrangean":

$$\text{9)} \qquad L(x,\lambda,g) = F_0(x) + g(x) + \sum_{h\in H} \lambda_h F_h(x) \; .$$

In 9), $x \in X$, $\lambda = (\lambda_h \mid h \in H)$ is a vector of non-negative components $\lambda_h \geq 0$ only finitely many of which are non-zero, and g is a continuous linear functional on X. The summation in 9) is understood as:

10) $$\sum_{h\in H} \lambda_h F_h(x) = \sum_{h\in H'} \lambda_h F_h(x) ,$$

where H' is the finite set $H' = \{h \in H | \lambda_h > 0\}$ (and summation over an empty set is taken to be zero). All infinite sums of this paper have finite support and are construed analogously. Thus the sum $\sum_{j\in J} \lambda_j f_j$ on the r.h.s. of 6a) will also be written $\sum_{i\in I} \lambda_i f_i$ with the understanding that we have set $\lambda_i = 0$ for $i \in I\backslash J$.

With the notation (9), equation (LL) can be rewritten as:

(LL)′ $$\lim_{\epsilon\to 0^+} \sup_{\substack{g\in X^* \\ \|g\|<\epsilon}} \sup_{\lambda} \inf_{x\in K} L(x,\lambda,g) = v(P)$$

It is the limiting operation in (LL) from which we derived the term "limiting lagrangean." If the limiting operation is deleted and one sets $g = 0$, one obtains an ordinary lagrangean.

It turns out that the limiting lagrangean result (LL) holds with our present assumptions, which are far weaker than the assumptions usually needed for a lagrangean result. For one thing, the index set H is not constrained in cardinality, yet the sums $\sum_{h\in H} \lambda_h F_h(x)$ always have finite support. Also, even for $|H|$ finite, the usual examples in R^n of duality gaps involve closed functions (in fact, everywhere-defined functions), and no duality gap is possible with the limiting lagrangean in this case (or even for $|H|$ infinite).

The next preliminary result is relevant to the "easy part" of (LL).

LEMMA 4:

11) $$\limsup_{\epsilon\to 0^+} \sup_{\substack{g\in X^* \\ \|g\|<\epsilon}} \sup_{\lambda} \inf_{x\in K} L(x,\lambda,g) \le v(P)$$

PROOF: For each integer $n \ge 1$, choose $x^{(n)} \in K$ such that

12) $$F_0(x^{(n)}) \le v(P) + \frac{1}{n} \text{ and } F_h(x^{(n)}) \le 0 \text{ for } h \in H .$$

Then for any g and λ, as $\lambda \ge 0$ we have

13) $$\begin{aligned}\inf_{x\in K} L(x,\lambda,g) &\le L(x^{(n)},\lambda,g) \\ &\le F_0(x^{(n)}) + g(x^{(n)}) \\ &\le v(P) + g(x^{(n)}) + \frac{1}{n} .\end{aligned}$$

From 13) it follows at once that

14) $$\sup_{\lambda} \inf_{x\in K} L(x,\lambda,g) \le v(P) + g(x^{(n)}) + \frac{1}{n}$$

and hence

15) $$\limsup_{\epsilon\to 0^+} \sup_{\substack{g\in X^* \\ \|g\|<\epsilon}} \sup_{\lambda} \inf_{x\in K} L(x,\lambda,g) \le v(P) + \frac{1}{n} .$$

Since 15) is valid for any n, so is 11).

Q.E.D.

We remark that 11) can also be proven if $v(P) = -\infty$.

SECTION II: THE MAIN RESULT

We shall use these notations, which exist by the fact that a closed, convex set in a locally convex space X (or R × X) is the intersection of closed half spaces where the f_j written below are continuous linear functions on X, and the a^j are real scalars:

16a) $$K = \{x \in X | f_j(x) \geq a^j,\ j \in I(-1)\}$$

$$\text{i.e. } x \in K \leftrightarrow f_j(x) \geq a^j, \text{ for all } j \in I(-1)$$

16b) $$\mathrm{epi}(F_h) = \{(z,x) \in R \times X | b^j z + f_j(x) \geq a^j,\ j \in I(h)\},$$

$$\text{for } h \in \{0\} \cup H;$$

$$\text{i.e. } (z,x) \in \mathrm{epi}(F_h) \leftrightarrow b^j z + f_j(x) \geq a^j, \text{ for all } j \in I(h).$$

Since $(z,x) \in \mathrm{epi}(F_h)$ and $z' \geq z$ implies $(z',x) \in \mathrm{epi}(F_h)$, we see that all $b^j \geq 0$. In 16) all the index sets I(h) for $h \in \{-1\} \cup \{0\} \cup H$ are (without loss of generality) disjoint.

We will use the fact that all $b^j \geq 0$ in the proof of Theorem 6, and we will also use the fact that if $b^j = 0$ then $f_j(x) \geq a^j$ for all $x \in K$. The latter is a consequence of our assumption that the domain of F_h contains K. Hence if $x \in K$ and $b^j = 0$, and $j \in I(h)$, we have $a^j \leq 0 \cdot F_h(x) + f_j(x) \leq f_j(x)$.

With this notation, one preliminary result remains before we can obtain our main result (Theorem 6).

LEMMA 5: Every solution to the inequali_ies

17) $$b^j z + f_j(x) \geq a^j, \quad j \in I(0)$$
$$f_j(x) \geq a^j, \quad j \in I(h) \text{ and } h \in \{-1\} \cup H$$

also satisfies

18) $$z \geq v(P) .$$

PROOF: It suffices to prove that if $(z,x) \in R \times X$ satisfies 17), then $(z,x) \in \text{epi}(F_0)$ and also x satisfies the constraints of (8). From the definitions 16), this will be accomplished once we prove:

19) $$F_h(x) \leq 0 \text{ if and only if } f_j(x) \geq a^j \text{ for all } j \in I(h) .$$

However, 19) is immediate:

20) $$F_h(x) \leq 0 \leftrightarrow (0,x) \in \text{epi}(F_h)$$
$$\leftrightarrow b^j \cdot 0 + f_j(x) \geq a^j \text{ for all } j \in I(h)$$
$$\leftrightarrow f_j(x) \geq a^j \text{ for all } j \in I(h)$$

Q.E.D.

Since Lemma 5 concerns an implication among linear functionals in the reflexive Banach space $R \times X$, and since the constraints 17) are consistent (and in fact satisfied by any feasible solution x to (8), with $z = F_0(x)$), it is natural to wish to apply Corollary 3 to the "fully-infinite" system (17). If one does so, our next, and main, result is obtained after some purely algebraic manipulation. (Recall that an affine linear functional is a linear functional plus a constant.)

THEOREM 6: Let X be a reflexive Banach space, assume that all functions F_h, $h \in \{0\} \cup H$ are finite on a set $D_h \supseteq K$, and are lower semi-continuous, that K is a non-empty, closed convex set in X, and that 8) is consistent.

For any $\epsilon > 0$, there exists a finitely non-zero vector $\lambda = (\lambda_h | h \in H)$ of non-negative components, continuous affine linear functionals g_h for $h \in \{-1\} \cup \{0\} \cup H$, a continuous linear functional p, and a scalar $\lambda_0 > 0$ satisfying these five conditions:

CONDITION 1: $g_{-1}(x) \leq 0$ for $x \in K$;

CONDITION 2: $F_h(x) \geq g_h(x)$ for $h \in \{0\} \cup H$ and x in the domain of F_h;

CONDITION 3: $\|p\| < \epsilon$;

CONDITION 4: $|\lambda_0 - 1| < \epsilon$;

CONDITION 5: For all $x \in X$,

$$g_{-1}(x) + \lambda_0 g_0(x) + p(x) + \sum_{h \in H} \lambda_h g_h(x) \geq v(P\ ) - \epsilon . \tag{21)}$$

PROOF: Note that $z = z \cdot 1 + 0 \cdot x$ in 18), and $z \cdot 1 + 0 \cdot x = (1,0)(z,x)$, where $1 \in R$ and 0 is the zero functional on X. The left-hand-side of the inequalities in 17) are $b^j z + f_j(x) = (b^j,f_j)(z,x)$ for $j \in I(0)$, and are $0 \cdot z + f_j(x) = (0,f_j)(z,x)$ for $j \in I(h)$, $h \in H \cup \{-1\}$. (We use the notation introduced above Corollary 3.)

We apply Corollary 3 to the implication from 17) to 18), with $\{f_i | i \in I\}$ taken as $\{(b^j,f_j) | j \in I(0)\} \cup \bigcup_{h \in \{-1\} \cup H} \{(0,f_j) | j \in I(h)\}$, f taken as (1,0), $\{\alpha_i | i \in I\}$ taken as $\bigcup_{h\in\{-1,0\}\cup H} \{a^j | j \in I(h)\}$, and α taken as v(P). The conclusions 6a), 6b) and 6c) of Corollary 3 become:

6a)′ $$(1,0) = (\beta,-p) + \sum_{h\in\{-1\}\cup H} \sum_{j\in I(h)} \phi_{h,j}(0,f_j) + \sum_{j\in I(0)} \phi_{0,j}(b^j,f_j)$$

6b)′ $$\|(\beta,-p)\| < \epsilon$$

6c)′ $$v(P) \le \epsilon + \sum_{h\in\{-1,0\}\cup H} \sum_{j\in I(h)} \phi_{h,j}\, a^j$$

In 6a)′, 6b)′, and 6c)′, β is a real scalar, p is a continuous linear functional, and the quantities $\phi_{h,j} \ge 0$ are non-negative real scalars, only finitely many of which are actually different from zero (i.e., for only finitely many $h \in \{-1,0\} \cup H$ there are only finitely many $\phi_{h,j} > 0$ for some $j \in I(h)$). Thus we have $\|p\| < \epsilon$ from 6b)′, i.e., Condition 3.

From the first components of the vectors of 6a)′, we obtain

22) $$1 = \beta + \sum_{j\in I(0)} \phi_{0,j}\, b^j$$

and from 6b)′ we infer

23) $$|\beta| < \epsilon .$$

Therefore, upon setting, for $h \in \{0\} \cup H$,

24) $$\lambda_h = \sum_{\substack{j\in I(h)\\ b^j>0}} \phi_{h,j}\, b^j$$

(and upon recalling our convention, that the empty summation is zero), we obtain by 22) and 23) the result $|\lambda_0 - 1| < \epsilon$, i.e. Condition 4.

Without loss of generality, $\lambda_0 > 0$ also (by taking $\epsilon > 0$ smaller if necessary).

From the second components of the vectors in 6a)′, recalling that $b^j = 0$ for all $j \in I(-1)$, and using an auspicious partitioning of the (actually finite) summation, we obtain:

25)
$$0 = -p + \Big(\sum_{j \in I(-1)} \phi_{-1,j} f_j + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j = 0}} \phi_{h,j} f_j \Big) + \sum_{h \in \{0\} \cup H} \sum_{b^j > 0} (\phi_{h,j} b^j)(f_j / b^j)$$

The same partitioning of the sum in 6c)′ yields:

26)
$$\Big(\sum_{j \in I(-1)} \phi_{-1,j}\, a^j + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j = 0}} \phi_{h,j}\, a^j \Big) + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j > 0}} (\phi_{h,j}\, b^j)(a^j / b^j) \geq v(P) - \epsilon \,.$$

We next evaluate the functionals of 25) at an arbitrary point $x \in X$, and add the negative of the resulting real numbers to those of 26), keeping the partitioning. We find:

27)
$$g_{-1}(x) + p(x) + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j > 0}} (\phi_{h,j}\, b^j)((a^j - f_j(x))/b^j) \geq v(P) - \epsilon \,.$$

In 27), we have used this notation:

28) $$g_{-1}(x) = \sum_{j \in I(-1)} \phi_{-1,j}(-f_j(x) + a^j) + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b^j = 0}} \phi_{h,j}(-f_j(x) + a^j)$$

Clearly, $g_{-1}(x)$ is linear affine and continuous. From 16a), if $h = -1$, $(-f_j(x) + a^j) \le 0$ for $x \in K$; and we recall from our previous discussion that, for $h \in \{0\} \cup H$ and $j \in I(h)$ with $b^j = 0$, we have $(-f_j(x) + a^j) \le 0$ whenever $x \in K$. Using this information in 28), we obtain $g_{-1}(x) \le 0$ for $x \in K$, i.e. Condition 1.

By the definition 16b), we have, for $h \in \{0\} \cup H$, $b^j F_h(x) + f_j(x) \ge a_0^j$ whenever x is in the domain of F_h; thus if $b^j > 0$ for $j \in I(h)$,

29) $$F_h(x) \ge (a^j - f_j(x))/b^j \ .$$

Now if $\lambda_h = 0$, we let $g_h(x)$ be $(a^j - f_j(x))/b^j$ for any $j \in I(h)$ with $b^j \ne 0$ (there is at least one such $j \in I(h)$, by our assumption that F_h is defined and finite on all of $K \ne \phi$). We at once have Condition 2, and the part $\lambda_h g_h(x)$ of the sum in 24) is zero, as is the corresponding part

$$\sum_{\substack{j \in I(h) \\ b^j > 0}} (\phi_{h,j}\, b^j)(a^j - f_j(x))/b^j)$$

of the sum in 27) (since $\lambda_h = 0$ implies $\phi_{h,j}\, b^j = 0$ for all $j \in I(h)$, using $\phi_{h,j} \ge 0$ and $b^j > 0$).

In the case that $\lambda_h > 0$, we use 29) to deduce this inequality (via the definition 24)):

30)
$$\lambda_h F_h(x) = (\sum_{\substack{j\in I(h)\\ b^j>0}} \phi_{h,j}\, b^j)\, F_h(x)$$
$$\geq \sum_{\substack{j\in I(h)\\ b^j>0}} \phi_{h,j}\, b^j(a^j - f_j(x))/b^j$$

Upon setting

31)
$$g_h(x) = \frac{1}{\lambda_h} \sum_{\substack{j\in I(h)\\ b^j>0}} \phi_{h,j}(a^j - f_j(x))$$

we at once obtain Condition 2 from 30) and 31) when $\lambda_h > 0$. Moreover, 27) becomes

27)´
$$g_{-1}(x) + p(x) + \sum_{\substack{h\in\{0\}\cup H\\ \lambda_h>0}} \lambda_h(\frac{1}{\lambda_h} \sum_{\substack{j\in I(h)\\ b^j>0}} \phi_{h,j}(a^j - f_j(x))) \geq v(P\ ) - \epsilon'$$

which is identical to Condition 5.

All five conditions have been verified, and the proof is complete.

Q.E.D.

COROLLARY 7: Assume the hypotheses of Theorem 6.

For any $\epsilon > 0$, there exists a finitely non-zero vector $\lambda' = (\lambda'_h | h \in H)$ of non-negative components, a continuous linear affine functional a(x), and a continuous linear functional q, satisfying these stipulations:

STIPULATION 1: $\|q\| < \epsilon$;

STIPULATION 2: $a(x) \leq 0$ for $x \in K$;

STIPULATION 3:

32) $$a(x) + F_0(x) + q(x) + \sum_{h \in H} \lambda'_h F_h(x) \geq v(P)(1 + \epsilon)/(1 + 2\epsilon)$$

for all x in the common domain of F_h, $h \in \{0\} \cup H$.

PROOF: Put $\epsilon' = \epsilon/(1 + \epsilon)$, so that $\epsilon = \epsilon'/(1 - \epsilon')$, and note that $\epsilon' > 0$. We apply Theorem 6 for $\epsilon' > 0$.

After dividing through in 21) by $\lambda_0 > 0$, and using the facts that

33) $$\|p(x)/\lambda_0\| = \|p(x)\|/|\lambda_0| \leq \|p(x)\|/(1 - \epsilon') < \epsilon'/(1 - \epsilon') = \epsilon ,$$

34) $$\epsilon'/\lambda_0 \leq \epsilon'/(1 - \epsilon) = \epsilon, \quad v(P)/\lambda_0 \geq v(P)/(1 + \epsilon') \geq v(P)(1 + \epsilon)/(1 + 2\epsilon) ,$$

we obtain this corollary at once, with these settings:

35a) $$a(x) = g_{-1}(x)/\lambda_0$$

35b) $$q(x) = p(x)/\lambda_0$$

35c) $$\lambda'_h = \lambda_h/\lambda_0 \quad \text{for } h \in H .$$

Q.E.D.

Note that, using Stipulation 2 of Corollary 7, the inequality 32) yields

36) $$\inf_{x \in K}\{F_0(x) + q(x) + \sum_{h \in H} \lambda'_h F_h(x)\} \geq v(P)(1 + \epsilon)/(1 + 2\epsilon) - \epsilon .$$

Thus, for any $\epsilon > 0$, there is a linear continuous functional q with $\|q\| < \epsilon$ and

37) $$\sup_{\lambda} \inf_{x \in K} \{F_0(x) + q(x) + \sum_{h \in H} \lambda_h F_h(x)\} \geq v(P)(1 + \epsilon)/(1 + 2\epsilon) - \epsilon.$$

It follows at once that

38) $$\liminf_{\|g\| \to 0^+} \sup_{\lambda} \inf_{x \in K} \{F_0(x) + g(x) + \sum_{h \in H} \lambda_h F_h(x)\} \geq v(P)$$

From 38), one has

39) $$\liminf_{\epsilon \to 0^+} \sup_{\substack{g \in X^* \\ \|g\| < \epsilon}} \sup_{\lambda} \inf_{x \in K} L(x,\lambda,g) \geq v(P)$$

with L as defined in 9). We now combine 39) with Lemma 4, and obtain the limiting lagrangean equation LL).

By use of the norm of the Banach space X, a result about the ordinary lagrangean can also be obtained, in the case that K is norm-bounded (but not necessarily compact) in X. In fact, let $B = \sup\{\|x\| \mid x \in K\} < +\infty$; then if $\|q\| < \epsilon$, 36) becomes

36)´ $$\inf_{x \in K} \{F_0(x) + \sum_{h \in H} \lambda'_h F_h(x)\} \geq v(P_{-\epsilon B - \epsilon})(1 + \epsilon)/(1 + 2\epsilon)$$

We at once obtain our next and final result, as $\epsilon > 0$ is arbitrary.

<u>COROLLARY 8</u>: Assume the hypotheses of Theorem 6 and also assume that K is bounded.

Then

40) $$\sup_{\lambda} \inf_{x \in K} \{F_0(x) + \sum_{h \in H} \lambda_h F_h(x)\} = v(P) .$$

SECTION III: RELATED LITERATURE, CONCLUDING REMARKS

The phenomenon of the "limiting lagrangean" LL) was discovered by the first author [6]. The second author showed [11] that, for $X = R^n$, LL) could be sharpened, in that the limit as $g \rightarrow 0$ could be taken to be one-dimensional. To be specific, for $X = R^n$ there exists one fixed $w \in R^n$ such that, with the hypotheses of Theorem 6,

41) $$\lim_{\theta\to 0} \sup_{\lambda} \inf_{x\in K}\{F_0(x) + \theta wx + \sum_{h\in H} \lambda_h F_h(x)\} = v(P)$$

An alternate proof of 41) has been provided by Borwein [3], using Helly's theorem.

Extensions of the limiting lagrangean equation to infinite-dimensional spaces, in the form LL), occur in [4] and [8]; the present paper presents a simpler result than [7], since only lower semi-continuous (convex) functions F_h are treated here.

In the paper [4], an infinite set of real-valued convex functions and a single cone-convex constraining function are used; moreover, only a general reflexivity property is used and the space X need not be normed.

In [8] the limiting lagrangean result is generalized to set-valued convex functions, and the need for a norm is dropped; and these results are further extended, in that a treatment is given of the case that the constraints are not lower semi-continuous. In addition, [8] has an extension of the result in [11], for $X = R^n$, to set-valued convex functions. It does not appear, at this writing, that the "most general" statement of limiting phenomena has been achieved; improvements will no doubt continue.

It is significant that Borwein in [4] uses the elegant theory of convex conjugate functions (as developed in [14], [15]), to shorten proofs regarding the limiting lagrangean, by citation of results from that theory. In contrast, we have preferred to cite separation principles in order to get representations of the convex program 8) as an infinite system of linear inequalities 17), and then to manipulate the resulting linear system by elementary algebra. All the refinements and extensions of the results of this paper, as mentioned above, are obtained by our method also; in fact, proofs in the set-valued case actually simplify, as one does not need to use an auspicious partitioning (as in 25)) when affine minorant results are not of concern. For further results on affine minorants, see [13].

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ABSTRACT

It is shown that a norm penalty method is exact for mixed integer programs in rational data, in the sense that the minimization of the criterion plus penalty over the nonnegativities and integrality constraints has the same set of globally optimal solutions as does the mixed integer program with the equality constraints present. This result is then extended to mixed-integer programs with complementarity constraints.

An example shows that no differentiable penalty can be exact for mixed integer programs.

Key Words

1) Mixed integer programs
2) Penalty functions
3) Exact penalties

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AN EXACT PENALTY METHOD FOR MIXED-INTEGER PROGRAMS

by C. E. Blair and R. G. Jeroslow[1]

In [3] the authors studied the relationship between the integer program

$$\text{(IP)} \qquad \begin{aligned} &\min\ cx \\ &\text{subject to } Ax = b \\ &\qquad x \geq 0 \text{ and integer} \end{aligned}$$

and the quadratic dual problem

$$(\mathrm{QD})_\lambda \qquad \min_{\substack{x\geq 0 \\ x \text{ integer}}} \quad (c - \lambda A)x + \rho\|Ax - b\|^2 + \lambda b$$

We showed [3, Theorem 1.5] that, if (IP) is consistent and bounded in value, and A, b, and c are rational, then for any λ, the optimum solutions to $(\mathrm{QD})_\lambda$ are the optimal solutions to (IP) when ρ becomes sufficiently large. Hence, as ρ increases, the value of $(\mathrm{QD})_\lambda$ becomes equal to the value of (IP). $(\mathrm{QD})_\lambda$ may be interpreted as an exact penalty method for (IP).

The program $(\mathrm{QD})_\lambda$ does not work for (MIP), as shown by the following one-dimensional example:

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$$(1)\qquad \min\ y + z$$
$$\text{subject to } 2x + y - z = 1$$
$$x, y, z \geq 0,\ x \text{ integer}$$

which has an optimum solution $x = 0$, $y = 1$, $z = 0$ of value one.

If F is any differentiable function with $F(0) = 0$ then the dual program

$$(2)\qquad \min_{\substack{x,y,z\geq 0 \\ x \text{ integer}}} \quad y + z + F(1 - 2x - y + z)$$

fails to be exact and have the solution noted above for (1); in fact, it fails to have value one. If $F'(0) \leq 0$, then for small $\epsilon > 0$, $x = 0$, $y = 1 - \epsilon$, $z = 0$ gives $(1 - \epsilon) + F(\epsilon) < 1$. If $F'(0) > 0$ then $x = 1$, $y = 0$, $z = 1 - \epsilon$ gives $(1 - \epsilon) + F(-\epsilon) < 1$. In particular, the differentiable function $F(\alpha) = \lambda\alpha + \rho\alpha^2$ fails to provide the same value as (1), regardless of λ and ρ. For related results on exact penalties in the convex case, see Bertsekas [1].

For the general mixed-integer program

$$(\text{MIP})\qquad \min\ cx + dy$$
$$\text{subject to } Ax + By = b$$
$$x, y \geq 0,\ x \text{ integer}$$

with A, B, c, d, and b rational, we propose a "norm-penalty" method

$$(\text{NP})\qquad \inf\ cx + dy + \rho\|b - Ax - By\|$$
$$x, y \geq 0,\ x \text{ integer.}$$

Theorem: Suppose that A, B, c, d, and b are rational, and that (MIP) is consistent and has a finite value.

Then for ρ sufficiently large, the optimal solutions to (NP) are exactly the optimal solutions to (MIP). In particular, the value of (NP) is that of (MIP) for ρ large.

We begin by establishing a result from which our theorem follows easily.

Lemma: Let $G(z)$ denote the value of (MIP) with b replaced by z ($+\infty$ if the MIP is not feasible). There is a $\rho' > 0$ (depending on b) such that

$$(3) \qquad G(z) \geq G(b) - \rho'\|z - b\| \text{ for all } z.$$

Proof: In [2, Theorem 2.1(2)] we showed there are $E,F \geq 0$ such that

$$(4) \qquad |G(z) - G(b)| \leq E\|z - b\| + F \text{ for all } z.$$

Let $\rho_1 = E + F$. Then if $\|z - b\| \geq 1$

$$(5) \quad G(z) \geq G(b) - |G(z) - G(b)| \geq G(b) - E\|z - b\| - F \geq G(b) - \rho_1\|z - b\|.$$

Let (x^*,y^*) be an optimum solution to (MIP) with right-hand side b. (The existence of (x^*,y^*) is a result of Meyer [4]). In [2, Theorem 2.1(1)] we showed there are $C,D > 0$ such that if $G(b') < +\infty$ then (MIP) with r.h.s. b' has an optimal solution x' with $\|x^* - x'\| \leq C\|b - b'\| + D$. Let $x_1,\ldots x_N$ be those finitely-many integer vectors x' such that $\|x^* - x'\| \leq C + D$. Then, if $\|z - b\| \leq 1$

$$(6) \qquad G(z) = \inf_{1\leq i\leq N} cx_i + L(z - Ax_i)$$

where L(w) is the linear programming value function

$$\text{(7)}\qquad \begin{aligned} &\inf\ dy \\ &By = w \\ &y \geq 0\ . \end{aligned}$$

(L(w) is $+\infty$ if the LP is not feasible.) It is well known from the theory of parametric linear programming that there are polyhedral cones $Q_1,\ldots,Q_M$ such that L is a linear function on each Q_i and $L(w) = +\infty$ for $w \notin \bigcup_1^M Q_i$.

For $1 \leq i \leq N$ define $C_i = \{z|L(z - Ax_i) < +\infty\}$, and let $C_{i,1},\ldots,C_{i,M}$ denote polyhedra, with union C_i, in which some linear affine form equals $cx_i + L(z - Ax_i)$. The collection of all these sets $C_{i,j}$ for $1 \leq i \leq N$ and $1 \leq j \leq M$, intersected with $\{z|\,\|z - b\| \leq 1\}$, forms closed sets $S_1,\ldots,S_T$. Then there are $\alpha_i \in R^m$, where m is the dimension of z (i.e. the number of rows in A or B), and $\beta_i \in R$, such that, if $\|z - b\| \leq 1$,

$$\text{(8)}\qquad G(z) = \min_{i \in J(z)} \alpha_i z + \beta_i$$

where $J(z) = \{i|z \in S_i\} \subseteq \{1,2,\ldots,T\}$.

Let $\rho_2 = \max\limits_{i \in J(b)} \|\alpha_i\|$. Then for $i \in J(b)$

$$\text{(9)}\qquad \alpha_i z + \beta_i = \alpha_i b + \beta_i + \alpha_i(z - b) \geq G(b) - \rho_2\|z - b\|\ .$$

Since the S_i are closed, there is a $\delta > 0$ such that $\|z - b\| \geq \delta$ for all $z \in \bigcup\limits_{i \notin J(b)} S_i$. Let $\rho_3 = \frac{1}{\delta}\max\{G(b) - G(z)|\ \|z - b\| \leq 1\}$. If $i \notin J(b)$ and $z \in S_i$, then

$$\text{(10)}\qquad \alpha_i z + \beta_i \geq G(z) = G(b) - (G(b) - G(z)) \geq G(b) - \rho_3\|z - b\|\ .$$

From (8), (9), and (10) we conclude that if $\|z - b\| \le 1$, then $G(z) \ge G(b) - \max\{\rho_2,\rho_3\}\|z - b\|$. Therefore setting $\rho' = \max\{\rho_1,\rho_2,\rho_3\}$, we see that (3) holds.

Q.E.D.

We complete the proof of our theorem by noting that if $\rho > \rho'$ and $Ax + By \neq b$ then

$$(11) \quad cx + dy + \rho\|Ax + By - b\| \ge G(Ax + By) + \rho\|Ax + By - b\| > G(Ax + By) + \rho'\|Ax + By - b\| \ge G(b) = cx^* + dy^* .$$

Hence the only optimal solutions to (NP) are optimum solutions to (MIP).

Q.E.D.

Norm penalties may also be used for mixed-integer programs with complementarity constraints. In detail if $P_1,P_2,\ldots,P_J$ are J finite sets of variables, the program

(MIPCa) $\min cx + dy$

$Ax + By = b$

$x,y \ge 0$, x integer

(MIPCb) at least one variable from each set P_i is zero

has the corresponding dual

(NPC) $\min cx + dy + \rho\|Ax + By - b\|$

$x,y \ge 0$ x integer

at least one variable from each set P_i is zero

<u>Theorem</u>: Suppose that A, B, c, d, b are rational, and MIPC is consistent

with finite value. Also assume that the program (MIPCa) alone is bounded below in value.

Then if ρ becomes sufficiently large the optimum solutions to (NPC) are the optimal solutions to (MIPC).

Lemma: If (MIP) is inconsistent, but is bounded below in value for some r.h.s. for which it is consistent, then as $\rho\to\infty$ the value of (NP) approaches infinity.

Proof of Lemma: If (MIP) is inconsistent the program

$$\begin{gathered} \min z + z' \\ Ax + By + zI - z'I = b \\ x, y, z, z' \geq 0,\ x \text{ integer} \end{gathered} \tag{12}$$

has value >0, by theorem of Meyer [4]. Hence there is a $\delta > 0$ such that if $G(w) < +\infty$ then $\|w - b\| \geq \delta$.

Let z_0 be such that $G(z_0) < +\infty$. By (3) there is a ρ' such that $G(z) \geq G(z_0) - \rho'\|z - z_0\|$ for all z, since a mixed integer program in rationals which is bounded below in value for one r.h.s., is also bounded below for all r.h.s.

Let N be arbitrarily given. Also let $\|z_0 - b\| = \theta \geq \delta$ and let ρ be sufficiently large so that $(\rho - \rho')\delta + G(z_0) - \rho'\theta \geq N$. Then if $x, y \geq 0$, x integer, and $Ax + By = z$, we have by the triangle inequality $\|z - z_0\| \leq \|z - b\| + \|z_0 - b\|$, that $cx + dy + \rho\|z - b\| \geq G(z) + \rho\|z - b\| \geq G(z_0) + \rho\|z - b\| - \rho'\|z - z_0\| \geq (\rho - \rho')\|z - b\| + G(z_0) - \rho'\theta \geq (\rho - \rho')\delta + G(z_0) - \rho'\theta \geq N$.

As N is arbitrary, the Lemma is proven.

Q.E.D.

Proof of Theorem: An assignment for (MIPC) is defined to be a subset S of the variables of (MIPC) such that $S \cap P_i \neq \phi$ for all i. Corresponding to each assignment S is an (MIP_S) obtained by setting all the S-variables equal to zero. The optimum solution to (MIPC) is the optimum among the solutions to the (MIP_S) for all assignments S. Similarly the optimum solution to (NPC) is the optimum solution to (NP_S) for all assignments S. For those S such that (MIP_S) is consistent the theorem gives a ρ_S such that the value of (MIP_S) equals the value of (NP_S). For those S such that (MIP_S) is inconsistent the lemma gives a ρ_S such that the value of (NP_S) exceeds the value of (MIPC). Letting $\rho = \max_S \rho_S$ we obtain the desired result.

Q.E.D.

It is worth noting that the penalty parameter ρ, which makes (NP) an exact penalty for (MIP), can be unbounded in bounded regions of r.h.s. space b. Consider, for example, the mixed integer program

$$\begin{aligned} &\min y \\ &\text{subject to } x + y = b \\ &\qquad x, y \geq 0,\ x \text{ integer} \end{aligned} \tag{13}$$

which extracts the fractional part of the real number $b \geq 0$. For $b = 1 - \epsilon$, $\epsilon > 0$ small, exactness of (NP) requires that

$$\inf_{\substack{x,y \geq 0 \\ x \text{ integer}}} y + \rho \|(1 - \epsilon) - x - y\| \geq 1 - \epsilon \tag{14}$$

and so setting $x = 1$, $y = 0$, we have $\rho\|\epsilon\| \geq 1 - \epsilon$, i.e. $\rho \geq (1 - \epsilon)/\epsilon$. Thus $\rho \nearrow +\infty$ as $\epsilon \searrow 0^+$.

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The Value Function of an Integer Program

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Abstract

We consider integer programs in which the objective function and constraint matrix are fixed while the right-hand side varies. The value function gives, for each feasible right-hand side, the criterion value of the optimal solution. We provide a precise characterization of the closed-form expression for any value function.

The class of Gomory functions consists of those functions constructed from linear functions by taking maximums, sums, non-negative multiples, and ceiling (i.e. next highest integer) operations.

The class of Gomory functions is identified with the class of all possible value functions by the following results: (1) For any Gomory function g, there is an integer program which is feasible for all integer vectors v and has g as value function; (2) For any integer program, there is a Gomory function g which is the value function for that program (for all feasible right-hand sides); (3) For any integer program there is a Gomory function f such that $f(v) \leq 0$ if and only if v is a feasible right-hand side.

Applications of (1) - (3) are also given.

Key Words:

1) Integer programming

2) Cutting-planes

3) Subadditive duals

THE VALUE FUNCTION OF AN INTEGER PROGRAM

by

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1. Introduction

The value function of the pure integer program

$$\begin{aligned} &\min cx \\ &\text{subject to } Ax = b \\ &x \geq 0, \quad x \text{ integer} \end{aligned} \tag{1.1}$$

provides the sensitivity analysis of (1.1) to changes in the right-hand-side b. Specifically, it is the function G such that G(b) is the optimal value of (1.1). When (1.1) is inconsistent (i.e. when there is no $x \geq 0$, x integer, with $Ax = b$) we put $G(b) = +\infty$. We also allow values $G(b) = -\infty$ if no lower bound can be put on cx over the set of solutions to the constraints. We shall assume throughout the paper that

$$\text{A, b, and c are rational matrices and vectors, and } G(0) > -\infty \tag{1.2}$$

The hypothesis $G(0) > -\infty$ discards only the trivial case that $G(b) = -\infty$ for all b such that (1.1) is feasible.

This paper provides an exact description of the class of value functions, by showing how they are iteratively constructed by simple operations, and by showing also that all functions thus constructed are value functions. In order to give the intuitive content of our results, we provide this verbal sketch of the class of functions involved: they are exactly the functions (which we call

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"Gomory functions" in Section 2 below) which are obtained by starting with the linear functions λb, and finitely often repeating the operations of sums, maxima, and nonnegative multiples of functions already obtained, and rounding up to the nearest integer. Thus, for example the Gomory function $G(b_1, b_2) = \max\{-3b_1 + \frac{1}{2}b_2 \ , \ b_1 + b_2 + \lceil b_2/3 \rceil\}$ is the value function of some two-constraint pure integer program, where $\lceil r \rceil$ denotes the least integer which is greater than or equal to the real number r.

Perhaps the main deficiency of our intuitive summary is that it ignores the domain of definition of the value function, which, as it turns out, is defined by the vectors for which a second Gomory function is not positive (see Theorem 3.13 and Theorem 5.2 below). In Section 2 below we give precise definitions for the terms to be used later on, further motivation and discussion of related literature, and some preliminary results.

Our intuitive summary shows that, once the "technology matrix" A and "criterion function" c are fixed in the integer program (1.1), there is a simple (although perhaps lengthy) closed form expression for the value of the solution in terms of the right-hand-side (r.h.s.) b. This result is in exact analogy to the similar result for a linear program: in fact, the value functions of linear programs are built up precisely in the same way, except that the rounding-up operation is not used. The characterization of linear programming value functions does not require the rationality hypotheses in (1.2).

This paper is a continuation of our earlier investigations [1], [11]. We extend work of Gomory [4], particularly from the perspective of [2], and we have benefitted from Shrijver's note [13] and two papers of Wolsey [14], [15]. These are the most immediate influences on our results here, and recent related work has been done by Edmonds and Giles [3]. The literature on this topic, which is part of the theory of cutting-planes, is extensive and partially summarized in the references of the survey [9].

This completes our introductory remarks. The plan of the remainder of the paper is as follows. Section 2 defines the Gomory functions and establishes some of their important properties. In Section 3, we show that Gomory functions provide value functions, by means of the monoid basis results of [10]. Section 4 is devoted to the proof of some elementary principles which are used later, and seem to have some interest in their own right. In Section 5 we prove that value functions are Gomory functions. Section 6 is devoted to the proof of two results (Theorem 6.2 and Theorem 6.3) which are closely related to our study of the value function, the first of which (Theorem 6.2) is a result announced by Wolsey [15]. In Section 7 we work an example to illustrate our characterization of the value function.

We conclude this section with some notational issues. In (1.1) A is an m by n matrix with columns denoted by a_j: $A = [a_j]$ [cols]. Also b is an m by one vector, c is one by n, and x is n by one; for components we write $c = (c_j) = (c_1 \ldots, c_n)$, $b = (b_i) = (b_1, \ldots, b_m)$, and $x = (x_j) = (x_1 \ldots, x_n)$. With this notation, $Ax = b$ can also be written $\sum_{j=1}^{n} a_j x_j = b$, and we use the second form generally when some specific column of A has to be identified (as in Section 5 below).

All variables below, such as the x_j, are understood as continuous throughout, which here means rational; if a variable is to be restricted to be integer this will be explicity stated. In many contexts below, it does not actually matter whether our continuous variables are rationals or reals, but we shall not treat the latter distinction. We let Q denote the rationals. If v and w are vectors we will use vw for the inner product.

2. Chvâtal Functions and Gomory Functions; General Background

The class $\acute{\mathcal{C}}$ of Chvâtal functions consists of essentially the Gomory functions built up without taking maximums. The exact definition follows.

Definition 2.1: The class $\acute{\mathcal{C}}_m$ of m-dimensional Chvâtal functions is the smallest class $\mathcal{C}$ of functions with these properties:

i) $f \in \mathcal{C}$ if $f(v) = \lambda v$ and $\lambda \in Q^m$ (here $v = (v_1, \ldots, v_m)$);

ii) $f, g \in \mathcal{C}$ and $\alpha, \beta \geq 0$ with $\alpha, \beta \in Q$ implies $\alpha f + \beta g \in \mathcal{C}$

iii) $f \in \mathcal{C}$ implies $\lceil f \rceil \in \mathcal{C}$, where $\lceil f \rceil$ is the function defined by the condition

$$\lceil f \rceil (v) = \lceil f(v) \rceil \tag{2.1}$$

Definition 2.2: The class $\acute{\mathcal{C}}$ of Chvâtal functions is defined by

$$\acute{\mathcal{C}} = \cup\{\acute{\mathcal{C}}_m | m \geq 1, \quad m \text{ integer}\} \tag{2.2}$$

Note that, while non-negative multipliers $\alpha, \beta \geq 0$ occur in clause (ii) of Definition 2.1, the vector $\lambda \in Q^m$ of clause (i) is unrestricted in sign.

We similarly obtain an exact definition of the class   of Gomory functions.

Definition 2.3: The class $\mathcal{G}_m$ of m-dimensional Gomory functions is the smallest class $\mathcal{C}$ of functions with the properties (i) - (iii) of Definition 2.1, and also this fourth property:

iv) $f, g \in \mathcal{C}$ implies $\max\{f,g\} \in \mathcal{C}$

Definition 2.4: The class $\mathcal{G}$ of Gomory functions is defined by

$$\mathcal{G} = \cup\{\mathcal{G}_m | m \geq 1, \quad m \text{ integer}\} \tag{2.3}$$

In Definitions 2.1 and 2.3 the function notation is understood in the usual way. For example, the function $\alpha f + \beta g$ of Definition 2.1 (ii) is defined by the condition:

$$(\alpha f + \beta g)(v) = \alpha f(v) + \beta g(v) \quad \text{for all } v \in Q^m \tag{2.4}$$

Similarly, the defining condition for $\max\{f,g\}$ in Definition 2.3 (iv) is $\max\{f,g\}\ (v) = \max\{f(v),\ g(v)\}$. Note that functions $f \in \mathcal{G}_m$ or $f \in \mathcal{C}_m$ are defined for all $v \in Q^m$, although in several instances below, we shall have occasion to restrict their domains to smaller sets, as e.g. integer vectors $v \in Z^m$.

Of course, the device of phrasing $\mathcal{C}_m$ and $\mathcal{G}_m$ in terms of smallest classes of functions, which contain the linear function, and have certain closure properties, is equivalent to saying that these clases are built up from the linear functions by iterative finite application of the operations defined in the closure properties. Our next definition makes the concept of "iterative application" exact.

Definition 2.5: A function f has pre-rank zero if it is a linear function. It has pre-rank (r+1) exactly if there are functions g, h of pre-rank $\leq$ r which satisfy at least one of these conditions:

(i) $f = \alpha g + \beta h$ for some rational scalars α, $\beta \geq 0$;

or

(ii) $f = \max\{g,h\}$

or

(iii) $f = \lceil g \rceil$

In general, a function has several pre-ranks.

Definition 2.6: If f has at least one pre-rank, its rank is its least pre-rank.

We now can state and prove the equivalence of e.g. Definition 2.3 with one by iterative application.

Proposition 2.7: For an m-dimensional function f, $f \in \mathcal{G}_m$ if and only if f has a pre-rank.

Proof: Let $\mathcal{K}$ be the class of all m-dimensional functions f which have a pre-rank. If $f \in \mathcal{K}$, one proves $f \in \mathcal{G}_m$ by induction on the rank of f. Thus $\mathcal{K} \subseteq \mathcal{G}_m$. Conversely it is easy to prove that $\mathcal{K}$ satisfies (i) to (iv) of Definition 2.3. Therefore $\mathcal{G}_m \subseteq \mathcal{K}$, hence $\mathcal{G}_m = \mathcal{K}$. Q.E.D.

Many results about Chvátal and Gomory functions are most easily proven by induction on rank. We will sometimes use the phrase "induction on the formation of f" to mean induction on the rank of f.

We next define a class of functions which we shall need in Section 5, to discuss the components of an optimal solution to (1.1).

Definition 2.8: The class $\mathcal{G}_m^{\pm}$ of unrestricted m-dimensional Gomory functions is the smallest class $\mathcal{K}$ with properties (i) and (iii) of Definition 2.1, and (iv) of Definition 2.3 and also this property:

(ii)' $f, g \in \mathcal{K}$ and $\alpha, \beta \in Q$ implies $\alpha f + \beta g \in \mathcal{K}$

The class $G^{\pm}$ is defined by

(2.5) $$\mathcal{G}^{\pm} = \bigcup_{m} \{\mathcal{G}_m^{\pm} \mid m \geq 1, \text{ m integer}\}$$

We remark that the composition of unrestricted Gomory functions is an unrestricted Gomory function.

The rounding-up operation $\lceil r \rceil$ (actually, truncation $\lfloor r \rfloor$, but $\lceil r \rceil = -\lfloor -r \rfloor$) occurs in Gomory's Method of Integer Forms. It also occurs in the following "rule of deduction" which is due to Chvátal [2], which we here adapt to non-negative (rather than unconstrained) integer variables:

(2.6) If the inequality

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq a_o$$

is valid, and if the x_j are non-negative integers, then the inequality

$$\lceil a_1 \rceil x_1 + \lceil a_2 \rceil x_2 + \ldots + \lceil a_n \rceil x_n \geq \lceil a_o \rceil$$

is also valid.

For example, if $\frac{1}{3}x_1 \geq 1/6$ (i.e. $x_1 \geq \frac{1}{2}$) is valid, and if x_1 is a non-negative integer, then $x_1 \geq 1$ is valid.

Chvátal's rule can be justified in two steps. For if its hypothesis is valid, then by adding suitable multiples of the non-negativities $x_j \geq 0$, we see that the weaker statement

$$\lceil a_1 \rceil x_1 + \lceil a_2 \rceil x_2 + \ldots + \lceil a_n \rceil x_n \geq a_o \tag{2.7}$$

is valid. Since the left-hand-side of (2.7) is an integer for integral x_j, and is not less than a_o, it also is not less than $\lceil a_o \rceil$. This justifies Chvátal's rule.

Chvátal and Hoffman observed [2] that Gomory's algorithm proceeds by certain instances of the rule (2.6). The precise mode of its implementation of (2.6) is affected by the way it introduces variables for cuts, and in its given form Gomory's algorithm is not convenient for analysis. If the Chvátal operation is repeatedly applied, and is viewed as parametric in the right-hand-side, it constructs a Chvátal function [14].

The Chvátal functions are essentially the discrete analogue of linear functions. We will see below that their carrier is linear and that they are pointwise close to it (Definition 2.9 and Proposition 2.10). Now if this analogy holds true, just as the value functions of linear programs are the finite maximum of linear functions, the value function of an integer program should be a finite maximum of Chvatal functions. That is why one might conjecture that value functions are Gomory functions, at least on their domain of definition.

The technical difficulties toward establishing the equivalence of Gomory functions and integer value functions should be clear enough. For one thing further operations, beyond maxima, might be necessary. For another, it is conceivable that infinitely many different Chvâtal functions occur for the infinitely many possible right-hand-sides b. In fact, our result, that the value function G is a Gomory function, can be construed as a "hyper-finiteness" result concerning Gomory-type algorithms based on the Chvâtal operation (2.6).

We establish as a consequence of our work, that not only can such algorithms be designed to be finitely convergent, but one uniform finite upper bound on the number of cuts needed is valid for all r.h.s. (once A and c are fixed in (1.1)).

We associate with each Gomory function $f \in \mathcal{G}$ a set of homogeneous polyhedral functions called "carriers," in our next definition. The carrier will turn out to be unique.

Definition 2.9: To every $f \in \mathcal{G}_m$ we assign a set S(f) of functions inductively as follows:

(i) If $f \in \mathcal{G}_m$ is linear (i.e. $f(v) = \lambda v$ for some $\lambda \in Q^m$) then $f \in S(f)$.

(ii) If $f \in \mathcal{G}_m$ can be written as $f = \alpha g + \beta h$ with $\alpha, \beta \in Q$ non-negative and $g, h \in \mathcal{G}_m$, and if $g' \in S(g)$ and $h' \in S(h)$, then $\alpha g' + \beta h' \in S(f)$.

(iii) If $f \in \mathcal{G}_m$ can be written as $f = \lceil g \rceil$ with $g \in \mathcal{G}_m$, and if $g' \in S(g)$, then $g' \in S(f)$.

(iv) If $f \in \mathcal{G}_m$ can be written as $f = \max\{g,h\}$ with $g, h \in \mathcal{G}_m$ and if $g' \in S(g)$ and $h' \in S(h)$, then $\max\{g', h'\} \in S(f)$.

(v) The sets $S(f)$, $f \in \mathcal{G}_m$, are formed by inductive application of rules (i) - (iv) preceeding.

Because of clause (iii) in Definition 2.9 a carrier, i.e. an element of $S(f)$, of $f \in \mathcal{J}_m$ is trivially obtained by simply deleting the integer round-up operations. For example, if $f(v) = \max\{-b_1 + \frac{3}{4}b_2, 2b_1 + \lceil -b_1 \rceil\}$, then one carrier of f is $\max\{-b_1 + \frac{3}{4}b_2, b_1\}$.

<u>Proposition 2.10</u>: If $f' \in S(f)$, $f \in \mathcal{G}$, then f' is a homogeneous function iteratively constructed from linear functions by taking sums and maximums, and f' satisfies, for some constant $k \geq 0$ (depending on the formation of f'):

$$(2.8) \qquad 0 \leq f(v) - f'(v) \leq k \qquad \text{for all } v \in Q^m$$

Moreover, if $f \in \mathcal{C}'$ then f' is linear.

<u>Proof</u>: The nature of f' is evident as the clauses (i) - (iv) of Definition 2.9 do not involve the round-up operation, and such functions f' are easily proven to be homogenous by induction on their iterative formation.

Similarly, the inequality $f(v) \geq f'(v)$ is easily seen to be preserved in clauses (i) - (iv). For example, if $f = \alpha g + \beta h$, then since $g \geq g'$ and $h \geq h'$, and $\alpha, \beta \geq 0$, we have $f \geq \alpha g' + \beta h' = f'$. We now examine the bound $f(v) - f'(v) \leq k$ of (2.8)

If f' is a carrier of f due to clause (1), $k=0$ since $f=f'$.

If f' is a carrier of f due to clause (ii), let k_1 and k_2 be such that

$$(2.9) \qquad g(v) - g'(v) \leq k_1 \qquad \text{for all } v \in R^m$$

$$h(v) - h'(v) \leq k_2 \qquad \text{for all } v \in R^m$$

k_1 and k_2 exist by induction on the number of steps in the inductive formation of g' and h' under the clauses of Definition 2.9. Then we have, as $f' = \alpha g' + \beta h'$,

$$(2.10) \qquad f(v) - f'(v) \leq \alpha(g(v) - g'(v)) + \beta(h(v) - h'(v))$$

$$\leq \alpha k_1 + \beta k_2$$

so we make take $k = \alpha k_1 + \beta k_2$.

If f' is a carrier of f due to clause (iii), let k' be such that

(2.11) $$g(v) - g'(v) \leq k' \qquad \text{for all } v \in R^m$$

. Then as $f' = g'$, we have

(2.12) $$f(v) - f'(v) = \lceil g(v) \rceil - g'(v) < k' + 1$$

and we may take $k = k' + 1$.

Clause (iv) formation is handled in a manner similar to clause (ii). For $f \in \mathcal{E}'$, f' is linear, since no application of maximums (clause (iv)) occurs.

Q.E.D.

Corollary 2.11: For $f \in \mathcal{J}$, $S(f)$ contains exactly one function.

Proof: Clearly $S(f) \neq \emptyset$ by induction on the rank of f. Let f_1', $f_2' \in S(f)$. If $f_1' \neq f_2'$, let v_o be such that $f_1'(v_o) \neq f_2'(v_o)$. Let k_1, k_2 be such that, for all v,

(2.13) $$\begin{aligned} 0 &\leq f(v) - f_1'(v) \leq k_1. \\ 0 &\leq f(v) - f_2'(v) \leq k_2. \end{aligned}$$

For all $\lambda \geq 0$, (2.13) applied to $v = \lambda v_o$ gives

(2.14) $$\begin{aligned} \lambda |f_1'(v_o) - f_2'(v_o)| &= |f_1'(\lambda v_o) - f_2'(\lambda v_o)| \\ &\leq |f_1'(\lambda v_o) - f(\lambda v_0)| + |f(\lambda v_o) - f_2'(\lambda v_o)| \\ &\leq k_1 + k_2 \end{aligned}$$

But (2.14) is impossible for $\lambda > (k_1 + k_2)/(|f_1'(v_o) - f_2'(v_o)|)$, and this contradicts $f_1' \neq f_2'$.

Q.E.D.

Definition 2.12: A monoid is a set M of vectors of Q^m which forms a semi-group under addition in Q^m. To be precise: (i) $0 \in M$; and (ii) If $v, w \in M$ then $v + w \in M$. The monoid M is integral if it contains only integer vectors.

Any monoid $M \neq \{0\}$ contains infinitely many elements. Any set of vectors generates a monoid, by taking all non-negative integer combinations of vectors in the set.

A function f: $M \rightarrow R$, with M a monoid, is called subadditive if:

(2.15) $$f(v + w) \leq f(v) + f(w) \qquad \text{for all } v, w \in M.$$

The interest in subadditive functions is that they generate valid cutting-planes, as summarized in our next result.

Proposition 2.13: [5], [11]

If f is a subadditive function on the monoid generated by the columns of $A = [a_j]$, then the inequality

(2.16) $$\sum_{j=1}^{n} f(a_j)x_j \geq f(b)$$

is satisfied by all solutions to (1.1).

A converse to Proposition 2.13 is also true.

Proposition 2.14: [11]

Assume that (1.1) is consistent. If the inequality

(2.17) $$\sum_{j=1}^{n} \Pi_j x_j \geq \Pi_o$$

is satisfied by all solutions to (1.1), then there is a subadditive function f, defined on the monoid generated by the columns of $A = [a_j]$, which satisfies

(2.18) $$f(0) = 0,\ f(a_j) \leq \Pi_j \text{ for } j=1 \ldots,\ n \text{ and } f(b) \geq \Pi_o.$$

We remark that it is easy to derive (2.17) as a consequence of (2.18) and (2.16), if one simply notes that $x \geq 0$ for all solutions to (1.1).

An alternate form of Propositions 2.13 and 2.14 is the "subadditive dual" we referred to earlier.

Theorem 2.15: [11]

If (1.1) is consistent and has a finite value, then this program has the same finite value:

(2.19) $$\begin{aligned} &\max f(b) \\ \text{subject to } \quad & f(a_j) \leq c_j \qquad\qquad j=1, \ldots, n \\ & f \text{ subadditive on the monoid generated} \\ & \text{by the columns of } A = [a_j] \end{aligned}$$

Moreover, the value function G is always an optimal solution to (2.19).

We next relate subadditivity to Gomory functions (Proposition 2.17).

<u>Lemma 2.16</u>: Suppose that f and g are subadditive on M, and $\alpha, \beta \geq 0$. Then the following functions are subadditive on M:

(i) $\alpha f + \beta g$

(ii) $\lceil f \rceil$

(iii) $\max\{f,g\}$.

<u>Proof</u>: Let $v, w \in M$ be given. Then we have

$$\begin{aligned} (\alpha f + \beta g)(v + w) &= \alpha f(v + w) + \beta g(v + w) \\ &\leq \alpha f(v) + \alpha f(w) + \beta g(v) + \beta g(w) \\ &\leq (\alpha f(v) + \beta g(v)) + (\alpha f(w) + \beta g(w) \\ &= (\alpha f + \beta g)(v) + (\alpha f + \beta g)(w) \end{aligned} \tag{2.20}$$

which establishes (i). Also

$$\begin{aligned} \lceil f \rceil (v + w) &= \lceil f(v + w) \rceil \\ &\leq \lceil f(v) + f(w) \rceil \\ &\leq \lceil f(v) \rceil + \lceil f(w) \rceil = \lceil f \rceil (v) + \lceil f \rceil (w) \end{aligned} \tag{2.21}$$

which establishes (ii). The first inequality in (2.21) is due to the subadditivity of f (see (2.15)) and the fact that r is a non-decreasing function of r. The second inequality in (2.21) is due to the easily verified subadditivity of the function $\lceil r \rceil$.

Moreover, for f and g subadditive,

$$\begin{aligned} f(v + w) &\leq f(v) + f(w) \leq \max\{f(v), g(v)\} + \max\{f(w), g(w)\} \\ g(v + w) &\leq g(v) + g(w) \leq \max\{f(v), g(v)\} + \max\{f(w), g(w)\} \end{aligned} \tag{2.22}$$

By taking the maximum over both sides in (2.22), we prove (iii).

Q.E.D.

Proposition 2.17: All Gomory functions $f \in \mathcal{G}_m$ are subadditive on Q^m.

Proof: By induction on the rank of $f \in \mathcal{G}_m$. Q.E.D.

Thus, Gomory functions can be used to obtain valid cutting-planes (in Proposition 2.13).

The fact that Chvâtal functions are subadditive, and usually somewhere strictly subadditive (i.e. in (2.15) there is strict inequality for at least some choice of v, w), shows that the negative of a Chvâtal function is not usually subadditive. For example, $-\lceil v \rceil$ is not subadditive (although it is a typical element of $\mathcal{G}^{\pm}$, because $-1 = -\lceil 1 \rceil = -\lceil .5 + .5 \rceil > -2 = (-\lceil .5 \rceil) + (-\lceil .5 \rceil)$, which contradicts (2.15).

The following simple result is a "normal form" for Gomory functions.

Proposition 2.18: Every Gomory function $f \in \mathcal{G}_m$ is a maximum of finitely many Chvâtal functions:

$$f = \max\{g_1, \ldots, g_t\}, \qquad \text{all } g_i \in \mathcal{C}_m \tag{2.23}$$

Proof: By induction on the rank of f. If f is a linear function the result is immediate.

Suppose that $f = \alpha g + \beta h$ where $\alpha, \beta \geq 0$ are rational and g and h are of lower rank than f. We write

$$g = \max_{i \in I} \{g_i\}$$

$$h = \max_{j \in J} \{h_j\} \tag{2.24}$$

for finite non-empty index sets I and J, where g_i and h_j are Chvâtal functions. Then one easily verifies that

$$f = \max_{\substack{i \in I \\ j \in J}} \{\alpha g_i + \beta h_j\} \tag{2.25}$$

Suppose $f = \lceil g \rceil$, where g has lower rank than f. We may again assume (2.24) holds for g, and we can conclude

(2.26) $$f = \max_{i \in I} \{ \lceil g_i \rceil \}$$

Suppose that $f = \max\{g, h\}$, where g and h have lower rank than f. We again may assume (2.24), and we have

(2.27) $$f = \max\{\max_{i \in I}\{g_i\}, \max_{j \in J}\{h_j\}\}$$

so that again the inductive hypothesis is preserved. Q.E.D.

3. Gomory Functions are Value Functions

Just as we have been using small letters f, g, h, ... for Chvátal and Gomory functions we shall reserve capital letters F, G, H, ... for value functions.

In this section, we derive sufficient closure properties for value functions, to insure that Gomory functions are value functions, at least when their domains are suitably restricted. The issue regarding the domain of definition is, of course, that value functions are defined, i.e. are not $+\infty$, only for certain r.h.s. b in (1.1), while Gomory functions are defined in all Q^m.

In this section, we will confine ourselves to showing how Gomory functions arise in the setting of programs (1.1) with A, b, and c integral. The value functions associated with such programs we shall call integral value functions. The extension of our work to the rational case (i.e. hypothesis (1.2)) is straightforward; see e.g. Corollary 3.14 below.

We proceed by use of certain results in [10], particularly Theorem 3.2 below.

A set $S \subseteq Q^m$ is a slice [10] precisely if S has the form

(3.1) $$S = T + M$$

where $T \neq \phi$ is a finite set of integer vectors in Q^m, and M is an integer monoid in Q^m which has a finite set of generators.

A monoid is the discrete analogue of a convex cone with vertex at the origin; a slice is the discrete analogue of a polyhedron. It is trivial for polyhedra, that their intersection is a polyhedron. The analogous result is true for slices (but see also [1], [10] for a continuous result which has a false integer analogue).

Theorem 3.1: [10]

If T_1 and T_2 are slices and $T_1 \cap T_2 \neq \emptyset$, then $T_1 \cap T_2$ is a slice.

Corollary 3.2:

If M_1 and M_2 are integer monoids which are finitely generated, then $M_1 \cap M_2$ is also a finitely generated monoid.

Proof: It is trivial that $M_1 \cap M_2$ is a monoid.

Since $M_1 \cap M_2 \neq \{0\}$, $M_1 \cap M_2$ is a slice:

$$M_1 \cap M_2 = T + M \tag{3.2}$$

where T is a non-empty finite set of integer vectors, and M is a finitely generated integer monoid. As $M_1 \cap M_2$ is a monoid, so is T + M, hence

$$T + T + M = (T + M) + (T + M) = T + M. \tag{3.3}$$

Let $T = \{t_1, \ldots, t_a\}$ and let M be generated by $s_1 \ldots, s_b$. We claim that T + M is generated by $U = \{t_1, \ldots, t_a, s_1, \ldots, s_b\}$.

It is clear that any element $t + m \in T + M$ $(t \in T, m \in M)$ is generated by U. Conversely, let v be generated by U:

$$v = \sum_{i=1}^{a} n_i t_i + \sum_{j=1}^{b} m_j s_j. \tag{3.4}$$

One may easily prove, by induction on $\rho = \sum_{i=1}^{a} n_i'$, that any vector of the form $\sum_{i=1}^{a} n_i' t_i$ is an element of T + M, using (3.3) for the inductive step, and the fact that $0 \in T + M$ for $\rho=0$ (the latter by (3.2) and the fact that $0 \in M_1 \cap M_2$).

Thus in (3.4) $\sum_{i=1}^{a} n_i t_i \in T + M$, and as $\sum_{j=1}^{b} m_j s_j \in M$, we have $v \in T + M + M = T + M$.

This completes the proof of our claim. Q.E.D.

We recall our assumption at the start of the section that A is integer. G(b) will be defined (i.e. $G(b) < +\infty$) only for certain integer vectors b. In what follows, we may interchangeably write row vectors as column vectors, or vice-versa, simply to improve readability.

Lemma 3.3: If $M_1, \ldots, M_r$ are finitely generated integral monoids, so is their Cartesian product $M_1 \times \ldots \times M_r$.

Proof: Without loss of generality, r = 2. Let M_j be generated by $v^{j1}, \ldots, v^{jt}$ for j=1,2 (we may take t to be the same as $0 \in M_j$). Then $M_1 \times M_2$ is generated by

(3.5) $$\begin{pmatrix} v^{11} \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} v^{1t} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v^{21} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ v^{2t} \end{pmatrix}$$

Q.E.D.

Lemma 3.4: If M is a finitely generated integral monoid, then so is the projection

(3.6) $$M_1 = \{v^1 \mid \text{for some } v^2, \ (v^1, v^2) \in M\}$$

Proof: If M is generated by (v^{1j}, v^{2j}) for j=1, ..., t, then M_j is generated by v^{1j} for j=1, ..., t. Q.E.D.

Proposition 3.5: Let G be a function G: $Q^m \to R \cup \{+\infty\} \cup \{-\infty\}$.

Then G is the integral value function of some integer program (1.1) if and only if the set M defined by

$$(3.7) \qquad M = \{(z, b) \mid z \text{ is an integer and } z \geq G(b)\}$$

is a finitely generated integer monoid.

Proof: Suppose that M is a finitely generated integer monoid, and let its generators be (c_j, a_j). Then

$$(3.8) \qquad M = \left\{ \begin{pmatrix} z \\ b \end{pmatrix} \;\middle|\; \begin{array}{l} \text{there is an integer vector } x \text{ with} \\ z = cx, \quad Ax = b, \; x \geq 0 \end{array} \right\}$$

where $A = [a_j]$ (cols) and $c = (c_j)$. Then the value function of the integer program (1.1) for this A and c is

$$\min\{cx \mid Ax = b,\ x \geq 0 \text{ and } x \text{ integer}\}$$

$$(3.9) \qquad = \min\{z \mid (z, b) \in M\} = G(b)$$

Conversely, if G(b) is the integral value function of (1.1), we have

$$(3.10) \qquad \left\{ \begin{pmatrix} z \\ b \end{pmatrix} \;\middle|\; \begin{array}{l} z \text{ is integral} \\ \text{and } z \geq G(b) \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} z \\ b \end{pmatrix} \;\middle|\; \begin{array}{l} \text{for some non-negative integers } x_0, x_1, \ldots, x_m, \\ \begin{pmatrix} z \\ b \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{n} x_j \begin{pmatrix} c_j \\ a_j \end{pmatrix} \end{array} \right\}.$$

Q.E.D.

Via the same ideas as in the proof of Proposition 3.5, one easily establishes the following result.

Corollary 3.6: M is the domain of some integral value function G (i.e. $M = \{b \mid G(b) < +\infty\}$) if and only if M is a finitely generated integer monoid.

Throughout this paper, the infimum over an empty set is $+\infty$.

Theorem 3.7: Let H and $H_1, \ldots, H_r$ be integral value functions and let Q and $Q^1, \ldots, Q^r$ be matrices of rationals.

Then the function defined by

$$(3.11)\qquad G(b) = \inf \left\{ H(w_1, \ldots, w_r) \;\middle|\; \begin{array}{l} w_1, \ldots, w_r \text{ and } b \text{ are integral, and} \\ \text{there are integer vectors } b^1, \ldots, b^r \\ \text{such that} \\ \qquad Qb + \sum_{j=1}^{r} Q^j b^j \geq 0 \\ \qquad \text{and moreover all } w_j \geq H_j(b^j),\ j=1, \ldots, r \end{array} \right\}$$

is an integral value function

(In (3.11), $w_1, \ldots, w_r$ are integers, the vectors $b, b^1, \ldots, b^r$ may be of different dimensions, and the matrices $Q, Q^1, \ldots, Q^r$ are dimensioned to make all expressions displayed compatible).

Proof: The monoid

$$(3.12)\qquad M = \left\{ \begin{pmatrix} z \\ w_1' \\ \vdots \\ w_r' \end{pmatrix} \;\middle|\; \begin{array}{l} z \text{ is integer and} \\ \\ z \geq H(w_1', \ldots, w_r') \end{array} \right\}$$

and the monoids

$$(3.13)\qquad M_j = \left\{ \begin{pmatrix} w_j \\ b^j \end{pmatrix} \;\middle|\; \begin{array}{l} w_j \text{ is integer} \\ \text{and } w_j \geq H_j(b^j) \end{array} \right\}, \qquad j=1, \ldots, r$$

are all integer monoids with a finite set of generators, by Proposition 3.5. By Lemma 3.3 so is $M \times M_1 \times \ldots \times M_r$.

It is well known that (the result goes back to Hilbert [8]; for one proof, see [10]) any monoid, defined by imposing integrality conditions on the solutions to homogeneous linear inequalities in rationals, has a finite set of generators. In particular, this monoid is finitely generated:

$$(3.14) \qquad P = \left\{ \begin{pmatrix} z \\ w_1' \\ \vdots \\ w_r' \\ w_1 \\ b^1 \\ \vdots \\ w_r \\ b^r \\ b \end{pmatrix} \middle| \begin{array}{l} z,\ w_1',\ \ldots,\ w_r',\ w_1,\ b^1,\ \ldots,\ w_r,\ b^r, \text{ and} \\ b \text{ are integral, and} \\ \qquad w_j' = w_j \qquad \text{for } j=1, \ldots, r \\ \text{and} \\ \qquad Qb + \sum_{j=1}^{r} Q^j b^j \geq 0 \end{array} \right\}$$

By Corollary 3.2, the monoid

$$(3.15) \quad M' = (M \times M_1 \times \ldots \times M_r) \cap P$$

has a finite set of generators. Let M* denote the projection of M′ onto its co-ordinates (z, b). By Lemma 3.4, M* has a finite set of generators. One also checks from (3.11) that

$$(3.16) \quad (z, b) \in M^* \text{ if and only if } z \text{ is integral and } z \geq G(b).$$

By Propositon 3.5, G is an integral value function. Q.E.D.

In what follows, when we write a composition of functions such as

$$(3.17) \qquad G(b) = H(H_1(b), \ldots, H_r(b))$$

we shall understand that G(b) is defined (i.e. $G(b) < +\infty$) exactly if each quantity $w_j = H_j(b) < +\infty$ and also $H(w_1, \ldots, w_r) < +\infty$, in which case $G(b) = H(w_1, \ldots, w_r)$.

Corollary 3.8: If H is a monotone non-decreasing integral value function, and $H_1, \ldots, H_r$ are integral value functions which are nowhere $-\infty$, then the function G in (3.17) is an integral value function.

Proof: Note that, by the monotonicity of H,

$$(3.18)\quad G(b) = \inf\left\{ H(w_1, \ldots, w_r) \;\middle|\; \begin{array}{l} w_1, \ldots, w_r \text{ are integral and, there are} \\ \text{integral } b^j = b \text{ with } w_j \geq H_j(b^j) \text{ for } j=1, \ldots, r \end{array} \right\}$$

Theorem 3.7 applies.

Q.E.D.

Corollary 3.9: If H_1 and H_2 are integral value functions, n_1 and n_2 are non-negative integers, and D is an integer, then these three functions are integral value functions:

i) $G = n_1 H_1 + n_2 H_2$

ii) $G = \lceil H_1/D \rceil$

iii) $\max\{H_1, H_2\}$

Proof: In cases i) and iii), it suffices to show that $G(b) = H(H_1(b), H_2(b))$, where H is a monotone non-decreasing value function. In case ii), we show that $G(b) = H(H_1(b))$, where H is a monotone non-decreasing integral value function. Corollary 3.8 then yields the desired result.

For i), the value function H is that of this two row integer program:

$$(3.19)\qquad \begin{array}{lll} & \inf & n_1x_1 + n_2x_2 \\ & \text{subject to} & x_1 = b_1 \\ & & x_2 = b_2 \\ & & x_1, x_2 \text{ integral} \end{array}$$

where we can obtain a formulation in non-negative variables by setting $x_j = x_j' - x_j''$ where x_j' and x_j'' are integral and nonnegative. The value function is non-decreasing because $n_1, n_2 \geq 0$.

For ii), the value function is that of the integer program

$$(3.20)\qquad \begin{array}{ll} & \inf x_1 \\ \text{subject to} & Dx_1 - x_2 = b \\ & x_2 \geq 0 \\ & x_1, x_2 \text{ integral,} \end{array}$$

and again a formulation in nonnegative variables easily follows. The function $H(b) = \lceil b/D \rceil$ is clearly non-decreasing.

For iii), H is the value function of the integer program

$$\begin{array}{lllll} & \inf & x_1 & & \\ \text{subject to} & & x_1 - x_2 & = & b_1 \\ & & x_1 \quad - x_3 & = & b_2 \\ & & x_2, x_3 \geq 0 & & \\ & & x_1, x_2, x_3 \text{ integer} & & \end{array}$$

and the desired properties are easily verified. Q.E.D.

Proposition 3.10: If ρ is an integer vector, then the function $F(v) = \rho v$ is an integral value function.

Proof: F is the value function of this integer program:

$$\begin{array}{ll} & \min \rho x \\ \text{subject to} & Ix = b \\ & x \text{ integer} \end{array} \tag{3.22}$$

and by the usual device of setting $x = x' - x''$ with $x', x'' \geq 0$ we can put (3.22) in the form (1.1) Q.E.D.

The statement that "Gomory functions are value functions" has to be properly construed. The domain of a Gomory function g is all of Q^m, while that of a value function G is some subset of the integer vectors Z^m; hence a Gomory function g must first be restricted to Z^m for any such statement to hold. A second issue derives from the fact that a Gomory function g need not have an integer value g(v) even for an integer vector $v \in Z^m$, yet the value G(v) for a value function is always integral, since c is assumed integral in this section. A precise statement follows next.

Theorem 3.11: If g is a Gomory function, there is an integral value function and non-negative integer $D \geq 1$ such that

$$g(v) = G(v)/D \qquad \text{for all } v \in \mathbb{Z}^m \tag{3.23}$$

Proof: By induction on the rank of g. If $g(v) = \lambda v$ for some $\lambda \in Q^m$, write $\lambda = \rho/D$ for ρ integer and $D \geq 1$ integer. Then $g(v) = \rho v/D$ and the result follows by Proposition 3.10.

If $g = \alpha h_1 + \beta h_2$ where $\alpha = n_1/D_1$ and $\beta = n_2/D_2$ are non-negative rationals, D_1 and $D_2 \geq 1$, and h_1 and h_2 are Gomory functions, let D_3 and D_4 be non-negative integers such that

$$\begin{aligned} h_1(v) &= H_1(v)/D_3 \qquad \text{for all } v \in \mathbb{Z}^m \\ h_2(v) &= H_2(v)/D_4 \qquad \text{for all } v \in \mathbb{Z}^m \end{aligned} \tag{3.24}$$

for value functions H_1 and H_2. Then for $v \in \mathbb{Z}^m$,

$$g(v) = \frac{n_1H_1(v)}{D_1D_3} + \frac{n_2H_2(v)}{D_2D_4} = \frac{(D_2D_4n_1)H_1(v) + (D_1D_3n_2)\ H_2(v)}{D_1D_3D_2D_4} \tag{3.25}$$

Since $n_1' = D_2D_4n_1$ and $n_2' = D_1D_3n_2$ are non-negative integers, $n_1'H_1 + n_2'H_2$ is a value function by Corollary 3.9 (i).

If $g = \lceil h_1 \rceil$ let (3.24) hold. Then for v integer, $g(v) = \lceil H_1(v)/D_3 \rceil$ and $\lceil H_1/D_3 \rceil$ is a value function by Corollary 3.9 (ii).

If $g = \max\{h_1, h_2\}$ let (3.24) hold. For $v \in \mathbb{Z}^m$ we have

$$\begin{aligned} g(v) &= \max\{H_1(v)/D_3, H_2(v)/D_4\} \\ &= \frac{1}{D_3D_4} \max\{D_4H_1(v), D_3H_2(v)\} \end{aligned} \tag{3.26}$$

Now $D_4H_1(v)$ and $D_3H_2(v)$ are value functions by Corollary 3.9 (i), and so is $\max\{D_4H_1, D_3H_2\}$ by Corollary 3.9 (iii). Q.E.D.

We also wish to be able to restrict Gomory functions g by a non-negativity condition $h \leq 0$ on another Gomory function h, and still have a value function. In this context, the domain of g and h will be Q^m, not Z^m, hence some hypothesis on the Gomory function h will be needed. This hypothesis will take the form

(3.27) $$h(v) > 0 \quad \text{if} \quad v \notin Z^m$$

so that, in essence, the compositely-defined function is $< +\infty$ only for $v \in Z^m$.

We proceed toward our goal in the next two results.

Theorem 3.12: Let G and H be integral value functions. Then the function defined by

(3.28) $$F(v) = \begin{cases} G(v), & \text{if } H(v) \leq 0; \\ +\infty \quad , & \text{if } H(v) > 0; \end{cases}$$

is also an integral value function.

Proof: We have $F(v) = K(G(v), H(v))$, where, for $w_1, w_2 \in Z$

(3.29) $$K(w) = \begin{cases} w_1, & \text{if } w_2 \leq 0; \\ +\infty, & \text{if } w_2 > 0. \end{cases}$$

K is non-decreasing, and it is the value function of this two-row integer program:

(3.30) $$\begin{aligned} \inf \quad & x_1 \\ \text{subject to} \quad & x_1 = w_1 \\ & -x_2 = w_2 \\ & x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{aligned}$$

Then F is a value function by Corollary 3.8. Q.E.D.

Theorem 3.13: Suppose that g and h are Gomory functions and that h satisfies (3.27). Let the function f be defined by

$$(3.31) \qquad f(v) = \begin{cases} g(v), & \text{if} \quad h(v) \leq 0; \\ +\infty, & \text{if} \quad h(v) > 0. \end{cases}$$

Then there is an integral value function F and an integer $D \geq 1$ with

$$(3.32) \qquad f(v) = F(v)/D \qquad \text{for all } v \in Q^m$$

Proof: By Theorem 3.11 there are value functions G and H and integers D_1, $D_2 \geq 1$ with

$$(3.33) \qquad \begin{aligned} g(v) &= G(v)/D_1 && \text{for all } v \in Z^m \\ h(v) &= H(v)/D_2 && \text{for all } v \in Z^m \end{aligned}$$

Note that, by (3.27),

$$(3.34) \qquad h(v) \leq 0 \quad \text{if and only if } H(v) \leq 0 \text{ for all } v \in Q^m.$$

From (3.31), we have, using (3.33)

$$(3.35) \qquad D_1 f(v) = \begin{cases} G(v), & \text{if} \quad H(v) \leq 0; \\ +\infty, & \text{if} \quad H(v) > 0. \end{cases}$$

By Theorem 3.12, $D_1 f(v)$ is a value function. Q.E.D.

Corollary 3.14: Suppose that g and h are Gomory functions and that there is a rational non-singular m by m matrix B such that, for all $v \in Q^m$,

$$(3.36) \qquad h(v) > 0 \qquad \text{if } Bv \notin Z^m$$

Then there is a value function F arising from a program (1.1) with rational A, c such that

$$(3.37) \qquad F(v) = \begin{cases} g(v), & \text{if} \quad h(v) \leq 0; \\ +\infty, & \text{if} \quad h(v) > 0. \end{cases}$$

Proof: Define $h'(v) = h(B^{-1}v)$, $g'(v) = g(B^{-1}v)$, and apply Theorem 3.13 to g', h' to obtain an integer matrix A' and an integer vector c', and an integer D, such that it has value function

$$(3.38) \qquad F'(v) = \begin{cases} Dg(B^{-1}v), & \text{if} \quad h(B^{-1}v) \leq 0; \\ +\infty \quad , & \text{if} \quad h(B^{-1}v) > 0. \end{cases}$$

Let $c = c'/D$, $A = B^{-1}A'$. Then the value function G of (1.1) satisfies

$$\begin{aligned} (3.39) \qquad F(v) &= \min\{c'x/D \mid B^{-1}A'x = v,\ x \geq 0 \text{ and integer}\} \\ &= \frac{1}{D}\min\{c'x \mid A'x = Bv,\ x \geq 0 \text{ and integer}\} \\ &= \frac{1}{D}F'(Bv) = \begin{cases} g(v), & \text{if} \quad h(v) \leq 0; \\ +\infty\ , & \text{if} \quad h(v) > 0. \end{cases} \end{aligned}$$

Q.E.D.

4. Some Results on the relation between an integer program and its LP Relaxation

We begin with two results showing that if an integer program is inconsistent, then a perturbation of the linear programming relaxation is also inconsistent. Throughout this section $a_1, \dots a_n \in Q^m$ are fixed, e is a vector with all components equal to one.

Theorem 4.1: There exists a $k > 0$ such that, for all $v \in R^m$, if there are no integer x_j such that

$$(4.1) \qquad \sum_1^n a_j x_j \geq v$$

then there are no x_j such that

$$(4.2) \qquad \sum_1^n a_j x_j \geq v + ke.$$

Proof: Let k be n times larger than any non-negative component of any a_j. If $x = (x_1, \ldots x_n)$ satisfies (4.2) then replacing each x_i by the next lower integer provides an integer solution that satisfies (4.1). Hence if (4.1) has no integer solution (4.2) has no continous solution. Q.E.D.

Next we examine the analogous problem for integer programs whose constraints are given as equations rather than inequalities.

For $v \in R^m$ define

$$(4.3) \qquad I_v = \{(x_1, \ldots x_n) \mid \sum_1^n a_j x_j = v;\ x_j \geq 0;\ x_j \text{ integer}\}$$

Theorem 4.2: There exists a $K_1 \geq 0$ such that, for all v, either: (i) I_v is non-empty;or (ii) there are no integer x_j (positive or negative) such that $\Sigma a_j x_j = v$;or (iii) there is no $x \geq K_1 e$ such that $\sum_1^n a_j x_j = v$.

In other words, if an integer program with right-hand-side v is inconsistent, then either it remains inconsistent when the non-negativity constraints are dropped or else the LP relaxation is inconsistent if lower bounds of K_1 are imposed on all the variables.

Proof: Let $S = \{(\alpha_1, \ldots \alpha_n) \mid \sum_1^n \alpha_j a_j = \vec{0}\}$. Let $F \subset Z^n$ be a basis for S.

Let K_1 be larger than the dimension of S multiplied times the largest non-negative component of any member of F. If v is such that (ii) and (iii) are false then there is an integer $\underline{x} = (\underline{x}_1, \ldots \underline{x}_n)$ such that $\sum_1^n a_j \underline{x}_j = v$ and scalars α_f, $f \in F$ such that $\underline{x} + \sum_{f \in F} \alpha_f f \geq K_1 e$. If α_f' is the largest integer $\leq \alpha_f$ then $\underline{x} + \sum_{f \in F} \alpha_f' f$ is integer, because $F \subset Z^n$, and non-negative because $\sum_{f \in F} (\alpha_f' - \alpha_f) f \geq - K_1 e$. Hence $\underline{x} + \sum_{f \in F} \alpha_f' f \in I_v$. Q.E.D.

Remark: An alternate form of theorem 4.2 replaces (iii) by (iii)$'$: there is a $1 \leq J \leq n$ dependent on v such that $\{x \mid \sum_1^n a_j x_j = v;\ x \geq \vec{0},\ x_J \geq k\} = \emptyset$. The k constructed here is n times the K_1 constructed in the proof of theorem 4.2. We will not use this result later, and omit the detailed proof.

Next we present some results relating the optimal solution to an integer program to the optimal solution to linear programming problems. The results we require later are Theorem 4.6 and Corollary 4.7. These can be deduced from [1] but our presentation is self-contained. Also, we believe the value of the constant K_2 is new.

For $v \in R^m$ define

$$L_v = \{x \mid \Sigma a_j x_j = v,\ x_j \geq 0\} \tag{4.4}$$

$$R_c(v) = \inf\{cx \mid x \in L_v\} \tag{4.5}$$

$$G_c(v) = \inf\{cx \mid x \in I_v\} \tag{4.6}$$

Lemma 4.3: There exists $K_2 > 0$ and a finite $F \subset Z^n$ such that, for every c, if every component of x is either zero or $\geq K_2$ then either: (i) $R_c(\Sigma a_j x_j) = cx$; or (ii) there is $y \in F$ such that $\Sigma a_j(x_j + y_j) = \Sigma a_j x_j$, $c(x + y) < cx$, and $x + y \geq \vec{0}$.

Proof: For $S \subset \{1, 2, \ldots n\}$ let

$$U_S = \{x \mid \Sigma a_j x_j = \vec{0} \text{ and } x_j = 0 \text{ if } j \notin S\}. \tag{4.7}$$

For each S such that U_S is one-dimensional, let $x^S \in U_S$ be a non-zero integer vector. We take F to be x^S and $-x^S$ for all such S. K_2 is chosen to be as large as any component of any member of F. If x, c are such that (i) is false there is a $z \in R^n$ such that: (α) $\Sigma a_j z_j = \vec{0}$; (β) $cz < 0$; (γ) $z_j \geq 0$ if $x_j = 0$. Let z^* satisfying (α) - (γ) be such that $\{j | z_j^* = 0\}$ is maximal. By definition of F, there is a $w \in F$ such that $w_j = 0$ if $z_j^* = 0$.

We claim that $z^* = \Theta w$ for some scalar Θ. Let Θ be such that $z' = z^* - \Theta w$ satisfies

(4.8) $\qquad z'_j \geq 0$ if $z_j^* \geq 0$

(4.9) $\qquad z'_j \leq 0$ if $z_j^* \leq 0$

(4.10) $\qquad$ For at least one $j, z'_j = 0$ and $z_j^* \neq 0$.

z' satisfies (α) and (γ). (4.10) and the maximality property of z^* imply $cz' \geq 0$. If $z' \neq \vec{0}$, we could find a scalar Θ' such that $z^* - \Theta' z'$ satisfies (α) - (γ) and has more zero components that z^*. Since this would contradict the maximality we must have $z' = \vec{0}$, $z = \Theta w$, and our claim is established.

Hence there is a $y \in F$ satisfying (α) - (γ) [$y = w$ or $-w$]. If every component of x is zero or $\geq K_2$, then $x + y \geq \vec{0}$; hence (ii) holds. Q.E.D.

Corollary 4.4: Let the set F be as in Lemma 4.3. For $x \epsilon Z^n$, $x \geq \vec{0}$, define $\bar{x}$ by

$$(4.11) \qquad \bar{x}_j = \begin{cases} x_j & \text{if } x_j \geq K_2 \\ 0 & \text{otherwise} \end{cases}$$

Then, for every c, either $R_c(\Sigma a_j \bar{x}_j) = c\bar{x}$ or there exists $y \in F$ such that $x + y \geq \vec{0}$ and $c(x + y) < cx$.

Proof: Apply Lemma 4.3 to $\bar{x}$. Q.E.D.

Lemma 4.5: Let c, v be such that $I_v \neq \emptyset$ and $G_c(v) > -\infty$. Then, for every $x \in I_v$, there is an $x^* \in I_v$ such that

(4.12) $$cx^* \leq cx$$

(4.13) $$\{a_i \mid x_i^* \geq K_2\} \quad \text{is linearly independent}$$

(4.14) $$R_c(\Sigma a_j \bar{x}_j^*) \;=\; c\bar{x}^* \quad [\bar{x}^* \text{ defined by (4.11)}]$$

Proof: Apply Corollary 4.4 to x. Either $R_c(\Sigma a_i \bar{x}_i) = c\bar{x}$ or there is an $x' = x + y \in I_v$ with $cx' \leq cx - \min\{cy \mid y \in F,\ cy < 0\}$. Then we apply 4.4 to x' etc. Since $G_c(v) > -\infty$ we must eventually obtain an $x^{(n)}$ such that (4.12) and (4.14) hold. By (4.14) and the complementary slackness theorem there is a $w \in R^m$ such that $wa_j \leq c_j$ for all j and $wa_j = c_j$ if $\bar{x}_j^{(n)} > 0$. If (4.13) fails there is a $y \in F$ such that $cy \leq 0$, $y_j = 0$ if $\bar{x}_j^{(n)} = 0$, and at least one component of y is negative (recall $G_c(v) > -\infty$). For some integer $\Theta > 0$, $x + \Theta y \in I_v$ and $x + \Theta y$ has fewer components $\geq K_2$. This process is repeated until an x^* is obtained such that (4.12) and (4.13) hold, and $\bar{x}_j^{(n)} > 0$ if $\bar{x}_j^* > 0$, hence $wa_j = c_j$ if $\bar{x}_j^* > 0$. To verify (4.14) note that if $x \geq 0$ and $\Sigma a_j x_j = \Sigma a_j \bar{x}_j^*$ then $cx \geq \Sigma(wa_j)x_j = w(\Sigma a_j x_j) = w(\Sigma a_j \bar{x}_j^*) = c\bar{x}^*$. Q.E.D.

Theorem 4.6 : For any c, v such that $I_v \neq \emptyset$ and $G_c(v) > -\infty$ there is an $x^* \in I_v$ satisfying (4.13), (4.14) and

(4.15) $$G_c(v) = cx^*$$

Proof: Any $x \in I_v$ can be decomposed as $\bar{x}$ plus a vector x' all of whose components are between 0 and K_2. Since there are only finitely many x' and at most one $x \in I_v$ satisfying (4.13) for each choice of x' and linearly independent set, there are only finitely many $x \in I_v$ satisfying (4.13) and (4.14). Let x^* be an x with cx minimal. Q.E.D.

Corollary 4.7: For every $c \in R^n$ there is a K_3 such that if $I_v \neq \emptyset$ and $R_c(v) > -\infty$ then

$$R_c(v) \leq G_c(v) \leq R_c(v) + K_3 \tag{4.16}$$

Proof: $R_c(v) \leq G_c(v)$ is immediate. Parametric linear programming theory[*] implies that there is an M_1 such that $|R_c(v) - R_c(w)| \leq M_1 \| v - w \|$. Let $M_2 = \max\{|cx| \mid \vec{0} \leq x \leq K_2 e\}$ Let $M_3 = \max\{\| \sum_1^n a_j x_j \|, \vec{0} \leq x \leq K_2 e\}$. From theorem 4.6 we know there is an $x^* \in I_v$ such that $cx^* = G_c(v)$ and $c\bar{x}^* = R_c(\Sigma a_j \bar{x}_j^*)$. Since $0 \leq x^* - \bar{x}^* \leq K_2 e$ we have $G_c(v) = cx^* \leq c\bar{x}^* + M_2 = R_c(\Sigma a_j \bar{x}_j^*) + M_2 \leq R_c(v) + M_1 M_3 + M_2$, so we may take $K_3 = M_1 M_3 + M_2$ Q.E.D.

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[*] This follows from the fact that $R_c(v) = \max \lambda_i v$, where the λ_i are the extreme points of the polyhedron, $\{\lambda | \lambda a_j \leq c_j, 1 \leq j \leq n\}$. We take $M_1 = \max \| \lambda_j \|$.

5. Value Functions are Gomory Functions

We will use notation (especially (4.3) - (4.6)) and results from section 4.

Let $a_1, \dots a_n \in Q^m$ and $c \in Q^n$ be fixed. The two main results of this section are

Theorem 5.1: There is a Gomory function $f: R^m \to R$ such that, for every v, $f(v) \leq 0$ if and only if $I_v \neq \emptyset$.

Theorem 5.2: There is a Gomory function g such that, for every v such that $I_v \neq \emptyset$, $g(v) = G_c(v)$.

The function f is a "consistency tester" for the integer program. g is a function which is equal to the value function of the given integer program, whenever it is consistent. Our proof of 5.2 uses 5.1, which requires several preliminary results. The proofs are constructive.

Lemma 5.3: Let

(5.1) $S = \{v \mid v = \Sigma a_j x_j,\ x_j$ integer (positive or negative)$\}$

There is a linearly independent $U \subset Q^m$ such that $S = \{v \mid v = \sum_{u \in U} \alpha_u u,\ \alpha_u$ integer$\}$.

Proof: Let $H = \{j \mid$ some member of S has first j-1 components zero and j^{th} component positive$\}$. For each $j \in H, u_j \in U$ is a vector where first (j-1) components are zero and whose j^{th} component is the smallest possible positive number. Set $U = \{u_j \mid j \in H\}$. It is easy to show that if $v \in S$ and the first (j-1) components of v are zero, then $v - \alpha u_j$ will have first j components zero, for some integer α. This process can be continued to yield a representation of v as an integer linear combination of the u_j. Q.E.D.

Remark: The proof of lemma 5.3 consists essentially of taking the Smith normal form of A.

Corollary 5.4: Let S be as in lemma 5.3. There is a Gomory function f_1 such that $v \in S$ if and only if $f_1(v) \leq 0$.

Proof: Let d be the dimension of L(S), the linear span of S. There are $w_1, \dots w_{m-d} \in Q^m$ such that $v \in L(S)$ if and only if $w_i v = 0$, $1 \leq i \leq m-d$. There are $z_u \in Q^m$ such that if $v \in L(S)$ then $v = \sum_{u \in U} (z_u v)u$. Hence $v \in S$ if $w_i v = 0$ for all i and $z_u v$ is integer for all $u \in U$. Hence we may take

$$(5.2) \qquad f_1(v) = \max\{w_i v, -w_i v;\ 1 \leq i \leq m-d; \lceil z_u v \rceil - z_u v;\ u \in U\}$$

Q.E.D.

Theorem 4.2 says that if $I_v = \emptyset$ then either S above is empty or else inserting lower bounds of K_1 on the variables produces an inconsistent linear program. 5.4 shows that the first situation can be detected by a Gomory function. Our next lemma is a fact from parametric linear programming. It states that if lower bounds produce an inconsistent LP, this inconsistency can be detected in a uniform manner over all v.

Lemma 5.5: There exist $\lambda_1, \dots \lambda_M \in Q^m$ such that

$$(5.3) \qquad \lambda_i a_j \leq 0 \text{ for all } 1 \leq i \leq M,\ 1 \leq j \leq n;$$

(5.4) For every $v \in R^m$, $k \geq 0$, if there is no $x \in L_v$ with $x \geq ke$ then, for some i, $\lambda_i v + k \sum_{j=1}^{n} (-\lambda_i a_j) > 0$.

Proof: $\{w | a_j w \leq 0 \quad 1 \leq j \leq n\}$ is a cone. We apply the finite basis theorem to obtain $\lambda_1, \dots \lambda_M$ such that every member of the cone is a non-negative linear combination of the λ_i.

Standard results of linear programming show that if there is no $x \in L_v$ with $x \geq ke$ then there exists $w \in R^m$ and scalars $s_1, \dots s_n \geq 0$ such that $wa_j + s_j = 0$, $1 \leq j \leq n$ and $wv + k(\Sigma s_j) > 0$. There are non-negative α_i such that $w = \Sigma \alpha_i \lambda_i$. If $s_{ij} = -\lambda_i a_j$ then $s_j = \sum_{i=1}^{M} \alpha_i s_{ij}$. Since $\sum_{i=1}^{M} \alpha_i (\lambda_i v + k \sum_{j=1}^{n} s_{ij}) = wv + k \Sigma s_j > 0$ there must be at least one i such that $\lambda_i v + k \sum_{j=1}^{n} (-\lambda_i a_j) > 0$. Q.E.D.

Our next result is motivated by the fact that Gomory functions, being subadditive, generate valid inequalities. Suppose A and v are such that every $x \in I_v$ satisfies $x_1 \geq p$, for some integer p. Suppose we also know that every $x \in I_{v-pa_1}$ satisfies an inequality $\Sigma\beta_j x_j \geq \gamma$. Then we can conclude that every $x \in I_v$ satisfies $\Sigma\beta_j x_j \geq \gamma + \beta_1 p$. We will show that if there are Gomory functions generating the first two inequalities, then one can construct a Gomory function that generates the third one.

Lemma 5.6: Let p be a Gomory function such that $p(a_1) = 1$; $p(a_j) \leq 0$ $2 < j \leq n$ [i.e., p generates an inequality of the form $x_1 \geq$ something]. Let h be any Gomory function. Then there is a Gomory function s such that

(5.5) $$s(a_j) \leq h(a_j) \quad 2 \leq j \leq n$$

(5.6) $$s(a_1) = h(a_1)$$

(5.7) For any v, if p(v) is integer then

$$p(v)s(a_1) + h(v - p(v)a_1) = s(v)$$

Proof: Our argument proceeds by induction on the formation of h. If h is linear we take $s(v) = h(v)$. If $h(v) = \lceil h_1(v) \rceil$ where h_1 is a Gomory function, then by induction hypothesis there is an s_1 such that (5.5) - (5.7) hold for s_1, h_1. We define

(5.8) $$s(v) = \lceil s_1(v) + (\lceil s_1(a_1) \rceil - s_1(a_1))p(v) \rceil$$

For $2 \leq j \leq n$ $s(a_j) \leq \lceil s_1(a_j) \rceil \leq \lceil h_1(a_j) \rceil = h(a_j)$, hence (5.5) holds. $s(a_1) = \lceil s_1(a_1) \rceil = \lceil h_1(a_1) \rceil = h(a_1)$ so (5.6) holds. If p(v) is integer $s(v) = \lceil s_1(v) - s_1(a_1)p(v) \rceil + \lceil s_1(a_1) \rceil p(v) = \lceil h_1(v-p(v)a_1) \rceil + s(a_1)p(v) = h(v-p(v)a_1) + p(v)s(a_1)$ so (5.7) holds.

If $h(v) = \alpha h_1(v)$ where $\alpha \geq 0$ we take $s(v) = \alpha s_1(v)$

If $h(v) = h_1(v) + h_2(v)$, we take $s(v) = s_1(v) + s_2(v)$

If $h(v) = \max\{h_1(v), h_2(v)\}$ and $h_1(a_1) \geq h_2(a_1)$ we take

(5.9) $$s(v) = \max\{s_1(v), s_2(v) + (h_1(a_1) - h_2(a_1))p(v)\}.$$

For $2 \leq j \leq n, s(a_j) \leq \max\{s_1(a_j), s_2(a_j)\} \leq \max\{h_1(a_j), h_2(a_j)\} = h(a_j)$. Also $s(a_1) = s_1(a_1) = h(a_1)$. If $p(v)$ is integer, $s(v) = \max\{p(v)s_1(a_1) + h_1(v - p(v)a_1), h_2(v - p(v)a_1) + h_1(a_1)p(v)\} = p(v)s_1(a_1) + h(v - p(v)a_1)$ so (5.5) - (5.7) hold in this case and the induction is complete.

Q.E.D.

Remark: The construction of s is based on the idea that a Gomory function represents a method of obtaining valid inequalities, with each step in the formation of the function corresponding to the generation of a new valid inequality from those previously obtained. The function s represents the same sequence of operations on inequalities as the function h, except (see (5.8) and (5.9)) that whenever h uses the inequality $x_1 \geq 0$, s uses the inequality $x_1 \geq p(v)$ generated by p.

Our next task is to show how we can use information about the consistency of an integer program with n-1 columns to obtain valid inequalities for an integer program with n columns. Let

(5.10) $$LI_v = \{(x_2, \dots x_n) | \sum_2^n a_j x_j = v, x_j \geq 0 \text{ and integer}\}$$

Suppose we know that $x_1 \geq p$ (p non-negative integer) if $x \in I_v$ and that $LI_{v-pa_1} = \emptyset$. Then we may conclude that $x_1 \geq p + 1$ if $x \in I_v$. Our next result uses this idea in the context of Gomory functions.

Lemma 5.7: Suppose there is a Gomory function h such that $h(v) \leq 0$ if and only if $LI_v \neq \emptyset$. Then for any k there is a Gomory function p_k such that

(5.10) $\qquad p_k(a_1) \leq 1$

(5.11) $\qquad p_k(a_j) \leq 0 , \quad 2 \leq j \leq n$

(5.12) $\qquad$ For any v, $p_k(v) \geq k + 1$ if $I_v = \emptyset$

Proof: We argue by induction on k. For $k = 0$ we may take $p_o(v) = \lceil \alpha h(v) \rceil$ for some $\alpha > 0$. If $p_k(v)$ satisfies (5.10) - (5.12) and $p_k(a_1) < 1$ we may take $p_{k+1}(v) = \left\lceil \frac{1}{\alpha} p_k(v) \right\rceil$ where $\alpha = \max\{ p_k(a_1), \tfrac{1}{2}\}$*. If $h(a_1) \leq 0$ we may take $p_{k+1}(v) = (k + 2)\lceil h(v) \rceil$.

The interesting case is $p_k(a_1) = 1$, $h(a_1) > 0$. By scaling we may assume $h(a_1) = 1$. We apply lemma 5.6 with $p = p_k$ to obtain s such that $s(a_j) \leq h(a_j) \leq 0$ and $s(a_1) = h(a_1) = 1$. We define $p_{k+1}(v) = \lceil \max\{p_k(v), s(v)\} \rceil$. If $I_v = \emptyset$ then by (5.12) either $p_k(v) > k + 1$ or $p_k(v) = k + 1$. Since $p_{k+1}(v) \geq \lceil p_k(v) \rceil$ we are done in the first case. If $p_k(v) = k + 1$ then (5.7) implies $s(v) = p_k(v) + h(v - (k+1)a_1) > k + 1$, hence $p_{k+1}(v) \geq \lceil s(v) \rceil \geq k + 2$

We are ready to carry out

Proof of Theorem 5.1: Our proof proceeds by induction on n.

First we deal with the case n=1. There are $\lambda_1, \ldots \lambda_{m-1}$ such that v is a scalar multiple of a_1 if $\lambda_j v = 0$, $1 \leq j \leq m-1$. There is w such that if $v = \alpha a_1$ then $\alpha = wv$ (e.g. we may take $w = (1/\|a_1\| a_1)$. We may take

$f(v) = \max\{\lambda_i v, - \lambda_i v, - wv, \lceil wv \rceil - wv\}$.

Now we deal with the induction step. We are assuming that for every n-1 rational vectors there is a Gomory function f such that $f(v) > 0$ if and only if v is not a non-negative integer combination of the n-1 vectors. In

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* The original manuscript used an incorrect choice of α, as was remarked to us by P. Carstensen.

particular we are assuming there are Gomory functions h_j, $1 \leq j \leq n$ such that $h_j(v) > 0$ if and only if there is no $x \in I_v$ with $x_j = 0$. We apply lemma 5.7 with $k = K_1$ to obtain functions T_j such that

(5.13) $$T_j(a_j) \leq 1$$

(5.14) $$T_j(a_i) \leq 0, i \neq j$$

(5.15) $$T_j(v) \geq K_1 \quad \text{if there are no } x \in I_v \text{ with } x_j = 0$$

Let $\lambda_1, \ldots \lambda_M$ be as in lemma 5.5. Define

(5.16) $$f_2(v) = \max_{1 \leq i \leq M}\{\lambda_i v + \sum_{j=1}^{n}(-\lambda_i a_j)T_j(v)\}$$

(5.17) $$f(v) = \max\{f_1(v), f_2(v)\}$$

where $f_1(v)$ was constructed in lemma 5.4.

If $I_v = \emptyset$, then by lemma 4.2, either $S = \emptyset$ (hence $f_1(v) > 0$ by lemma 5.4) or there is no $x \in L_v$ with $x \geq K_1 e$. In this last case lemma 5.5 implies that, for some i, $\lambda_i v + K_1 \sum_{j=1}^{n} (-\lambda_i a_j) > 0$. Using (5.15), (5.3)

$f_2(v) \geq \lambda_i v + \sum_{j=1}^{n} (-\lambda_i a_j)T_j(v) > 0$, hence $f(v) > 0$ if $I_v = \emptyset$.

To show $f(v) \leq 0$ if $I_v \neq \emptyset$ it suffices to show $f(a_j) \leq 0$ and use the subadditivity of f. $f_1(a_j) \leq 0$ by corollary 5.4. By (5.16), (5.3), (5.13), (5.14) $f_2(a_j) \leq 0$ hence $f(a_j) \leq 0$. Q.E.D.

Our next task is the proof of Theorem 5.2.

Let $N > 0$ be such that

(5.18) Nc_j is integer, $1 \leq j \leq n$; N integer.

Our method of proof is to deduce valid inequalities by making use of information about I_v together with information about $I_v \cap \{x| \ cx = p\}$. Suppose we know that if $x \in I_v$ then $cx \geq p$, where Np is integer; and that if $x \in I_v$ and $cx = p$ then $\alpha x \geq \beta$. There should be some way of combining these two inequalities into a single inequality $(\alpha + Lc)x \geq \beta + Lp$ for some $L \geq 0$. The next result shows that this does happen when the inequalities are generated by Gomory functions.

Lemma 5.8: Let $p \colon R^m \to R$ be a Gomory function such that $p(a_j) \leq c_j$ for all j. Let $f \colon R^{m+1} \to R$ be any Gomory function. Let $p'(v) = \frac{1}{N}\lceil Np(v)\rceil$ ($p'(a_j) \leq c_j$ by (5.18)). There is a Gomory function $h \colon R^m \to R$ and an $L \geq 0$ such that

$$h(a_j) \leq f(c_j, a_j) + Lc_j \quad ; \quad 1 \leq j \leq n \tag{5.19}$$

$$\text{For every } v, \quad h(v) \geq f(p'(v), v) + Lp'(v) \tag{5.20}$$

Proof: We construct h by induction on the formation of f.

If f is a linear functional $f(r,v) = \alpha r + wv$ with $\alpha \leq 0$ we take $h(v) = wv$, $L = -\alpha$ and (5.19) and (5.20) hold as equations.

If $f(r,v) = \alpha r + wv$ with $\alpha > 0$ take $h(v) = \alpha p'(v) + wv$, $L = 0$. (5.20) is an equation, (5.19) follows because $p'(a_j) \leq c_j$.

If $f(r,v) = \lceil f_1(r,v)\rceil$ then by induction hypothesis there are h_1, L_1 such that (5.19), (5.20) hold. Take $L = N\lceil L_1\rceil$ and define

$$h(v) = \lceil h_1(v) + (L-L_1)p'(v)\rceil = \lceil h_1(v) - L_1 p'(v)\rceil + Lp'(v) \tag{5.21}$$

We have $h(a_j) \leq \lceil h_1(a_j) + (L-L_1)c_j\rceil = \lceil h_1(a_j) - L_1 c_j\rceil + Lc_j \leq f(c_j, a_j) + Lc_j$, so (5.19) holds. Also $h(v) \geq \lceil f_1(p'(v), v)\rceil + Lp'(v) = f(p'(v), v) + Lp'(v)$ so (5.20) holds.

If $f(r,v) = \alpha f_1(r,v)$, $\alpha \geq 0$ we take $h = \alpha h_1$, $L = \alpha L_1$

If $f(r,v) = f_1(r,v) + f_2(r,v)$ we take $h = h_1 + h_2$, $L = L_1 + L_2$.

If $f(r,v) = \max\{f_1(r,v); f_2(r,v)\}$ with $L_1 \geq L_2$ take $L = L_1$ and

(5.22) $$h(v) = \max\{h_1(v);\ h_2(v) + (L_1 - L_2)p'(v)\}$$

We have $h(a_j) \le \max\{f_1(c_j, a_j) + L_1c_j;\ f_2(c_j, a_j) + L_1c_j\} = f(c_j, a_j) + Lc_j$. Also $h(v) \ge \max f_1(p'(v), v) + L_1p'(v);\ f_2(p'(v), v) + L_1p'(v)\} =$

$f(p'(v), v) + Lp'(v)$. Thus (5.19), (5.20) hold in this case and the induction is complete.

Remark: The idea behind this construction is similar to that for lemma 5.6. The function h represents the same sequence of operations as the function f, except that at every step at which f uses the equation $cx = p$, h uses the inequality $cx \ge p'$ generated by the Gomory function p'.

Corollary 5.9: For every $k \ge 0$ there is a Gomory function $T_k: R^m \to R$ such that

(5.23) $$T_k(a_j) \le c_j; \qquad 1 \le j \le n$$

(5.24) For all v, $NT_k(v)$ is integer

(5.25) For all v, if $I_v \neq \emptyset$, $T_k(v) \ge \min\{G_c(v), \frac{1}{N}\lceil NR_c(v)\rceil + \frac{k}{N}\}$

Recall N defined by (5.18), R_c by (4.5). Note that (5.23) and subadditivity imply $T_k(v) \le G_c(v)$.

Proof: We argue by induction on k. We take $T_o(v) = \frac{1}{N}\lceil NR_c(v)\rceil$.* $R_c(a_j) \le c_j$ is immediate and (5.23) follows because Nc_j is integer. (5.24) and (5.25) are also easy.

Suppose we have constructed $T_k(v)$. By theorem 5.1 applied to $a_j' = (c_j, a_j)$ there is an $f: R^{m+1} \to R$ such that $f(r, v) \le 0$ if and only if there is an $x \in I_v$ such that $cx = r$. We apply lemma 5.8 with $p = T_k$, f as described. By (5.24) $p' = p = T_k$. Define

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*R_c is a Gomory function by the remark at the end of Corollary 4.7.

(5.26) $$T_{k+1}(v) = \max\{T_k(v), \frac{1}{N}\lceil Nh(v)/L\rceil\}, \text{ if } L > 0$$

(5.27) $$T_{k+1}(v) = \max\{T_k(v), T_k(v) + \frac{1}{N}\lceil Nh(v)\rceil\} \quad \text{if } L = 0$$

(5.23) holds for T_{k+1} because $T_k(a_j) \leq c_j$ and $f(c_j, a_j) \leq 0$, in (5.19). (5.24) is immediate. If $T_k(v) = G_c(v)$ we see that (5.25) holds for T_{k+1}, because $T_{k+1}(v) \geq T_k(v)$. If $T_k(v) < G_c(v)$, then $f(T_k(v), v) = f(p'(v), v) > 0$. (5.20) implies $T_{k+1}(v) > T_k(v)$, which implies that $T_{k+1}(v)$ is a rational with denominator N and hence $T_{k+1}(v) \geq \frac{1}{N}\lceil NR_c(v)\rceil + \frac{(k+1)}{N}$. Since $NG_c(v)$ is integer, $T_k(v) < G_c(v)$ implies $G_c(v) \geq \frac{1}{N}\lceil NR_c(v)\rceil + \frac{(k+1)}{N}$ hence (5.25) is established for T_{k+1}.

Q.E.D.

Now we can return to

<u>Proof of Theorem 5.2</u>: With K_3 constructed by corollary 4.7*, we let $f = T_{NK_3}$. Using (4.16) condition (5.25) becomes $T_{NK_3}(v) \geq G_c(v)$. As remarked above, (5.23) implies the opposite inequality, hence $f(v) = G_c(v)$.

Q.E.D.

Next we consider the dependence of the optimal solution to (IP) on the right-hand side v. Consider the one-row problem

$$\begin{aligned} \min\ & x + y \\ & 3x + y = v \\ & x,y \geq 0 \quad \text{and integer} \end{aligned}$$

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*The assumption $R_c(v) > -\infty$ needed to invoke corollary 4.7 is not restrictive. It is easy to show that if $R_c(v) = -\infty$ for any v, then for all v, either $I_v = \emptyset$ or $G_c(v) = -\infty$.

The optimal solutions for $v = 4, 5, 9$ have $x = 1, 1, 3$ respectively. The optimal solution value for x is not a subadditive function of v, hence cannot be a Gomory function. However, our next result shows that the optimal solution can be obtained by using unrestricted Gomory functions (defined in 2.7).

To deal with cases involving more than one optimal solution, we define the lexicographically smallest optimal solution to be that optimal solution which makes x_1^* as small as possible. If there is more than one such x we make x_2^* as small as possible, given the specified value of x_1^*, etc.

Corollary 5.10: Assume $R_c(v) > -\infty$ for all right-hand sides v. If $I_v \neq \emptyset$ let x_v^* be the lexicographically smallest member of I_v such that $cx_v^* = G_c(v)$ [i.e. x_v^* is an optimal solution]. Then there are unrestricted Gomory functions $f_j\colon R^m \to R$ such that if $I_v \neq \emptyset$ then the j^{th} component of x_v^* is $f_j(v)$.

Proof: The first component of x_v^* is the value of the optimal solution to the integer program

$$
(5.28)\qquad \begin{aligned} &\min \quad x_1 \\ &\text{subject to} \quad cx = \alpha_0 \\ &\qquad \Sigma a_j x_j = v \\ &\qquad x \geq \vec{0}, \ \ x \text{ integer} \end{aligned}
$$

when we set $\alpha_0 = G_c(v)$. By theorem 5.2, there are Gomory functions $g_0(v)$, $g_1(\alpha, v)$ such that $g_0(v) = G_c(v)$ and $g_1(\alpha_0, v)$ is the optimal value of (5.28). Then the first component of x_v^* is $g_1(g_0(v), v)$, which is an unrestricted Gomory function of v. Similarly, the second component of x_v^* is the value of the optimal solution to

$$\begin{aligned} & \min x_2 \\ (5.29) \quad & \text{subject to } cx = \alpha_0 \\ & x_1 = \alpha_1 \\ & \Sigma a_j x_j = v \\ & x_j \geq \vec{0}, \quad x \text{ integer} \end{aligned}$$

where $\alpha_0 = g_0(v), \alpha_1 = g_1(g_0(v), v)$. By theorem 5.2 there is a Gomory function $g_2(\alpha_0, \alpha_1, v)$ which is the optimal value of (5.29). Hence the second component of x_v^* is $g_2(g_0(v), g_1(g_0(v), v), v)$. The other components of x_v^* are developed similarly. Q.E.D.

We next present the analogues of theorems 5.1 and 5.2 for an integer program in inequality format:

$$\begin{aligned} & \min \quad c_1x_1 + \ldots c_nx_n \\ (5.30) \quad & \text{subject to} \quad a_1x_1 + \ldots a_nx_n \geq v \\ & x_1, \ldots x_n \geq 0 \text{ and integer.} \end{aligned}$$

We will assume that the vectors a_j have all components integer (the extension to the rational case is straightforward). Then (5.30) is equivalent to the integer program in equation form

$$\begin{aligned} & \min \quad c_1x_1 + \ldots c_nx_n \\ (5.31) \quad & \text{subject to} \quad a_1x_1 + \ldots a_nx_n - e_1y_1 - \ldots e_my_m = \lceil v \rceil \\ & x, y \geq 0 \text{ and integer} \end{aligned}$$

where $e_i \in R^m$ has one in i^{th} component, zero in other components, and v is taken componentwise. Application of theorems 5.1, 5.2 yields

Corollary 5.11: There is a Gomory function f such that (5.30) is consistent if and only if $f(\lceil v \rceil) \leq 0$.

Corollary 5.12: There is a Gomory function g such that $g(\lceil v \rceil)$ is the value of (5.30) for any v for which $f(\lceil v \rceil) \leq 0$.

We can extract further information about f, g. A Gomory function h is specified by a definition giving the precise order in which the various operations (sums, round-ups, etc.) are carried out. A Gomory function can have several different definitions, e.g. $\frac{3}{2}x$ defines the same function as $x + \lceil \frac{1}{2}x \rceil$. We will use $\hat{h}$ to denote a definition of h.

Definition 5.13: For a given $\hat{h}$ we associate $T(\hat{h}) \subseteq Q^m$, the set of all λ occuring in linear functionals used in $\hat{h}$. Formally $T(\hat{h})$ is defined by

i) if $\hat{h}(v) = \lambda v$ then $T(\hat{h}) = \{\lambda\}$;

ii) if $\hat{h} = \alpha \hat{h}_1$ or $\hat{h} = \lceil \hat{h}_1 \rceil$ then $T(\hat{h}) = T(\hat{h}_1)$;

iii) if $\hat{h} = \hat{h}_1 + \hat{h}_2$ or $\hat{h} = \max\{\hat{h}_1, \hat{h}_2\}$ then $T(\hat{h}) = T(\hat{h}_1) \cup T(\hat{h}_2)$

The class $\mathcal{MG}$ consists of those Gomory functions h for which there is $\hat{h}$ such that every $\lambda \in T(\hat{h})$ has non-negative components.

Every $h \in \mathcal{MG}$ is a monotone non-decreasing Gomory function (the converse is also true, but non-trivial).* $\mathcal{MG}$ is closed under composition in the sense that if $f\colon R^Q \to R$, and $g_i\colon R^m \to R$, $1 \leq i \leq Q$, are in $\mathcal{MG}$ then so is $h(v) = f(g_1(v), \ldots g_Q(v))$. In particular, if $f \in \mathcal{MG}$ then $h(v) = f(\lceil v \rceil) \in \mathcal{MG}$.

Lemma 5.14: Let $1 \leq j \leq n$. Let h_0 be a Chvátal function defined by $\hat{h}_0$ such that $h_0(-e_j) \leq 0$. Then there is a Chvátal function h_1 defined by $\hat{h}_1$ such that: (i) $h_1(v) = h_0(v)$ for all v with integer components; (ii) If $\lambda \in T(\hat{h}_1)$ then $\lambda e_j \geq 0$; (iii) If $\lambda \in T(\hat{h}_1)$, $\lambda = \lambda' + ke_j$, where λ' is a non-negative linear combination of members of $T(h_0)$.

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*The proof is by induction on the formation of h. The key step is that if h = f+g is a monotone Gomory function then for some linear function λ, $f + \lambda$ and $g - \lambda$ are monotone Gomory functions.

Proof: We construct $\hat{h}_1$ by moving integer quantities through the round-up operations $\lceil \; \rceil$ which occur in $\hat{h}_0$. For example, if $h_0(v) = 4\frac{1}{3}e_1 v + \lceil -2\frac{1}{2}e_1 v \rceil$ we could take $h_1(v) = 1\frac{1}{3}e_1 v + \lceil \frac{1}{2}e_1 v \rceil$.

Formally, we proceed by induction on the number of round-up operations used in $\hat{h}_0$. For any $\hat{h}$ define $n(\hat{h})$ by

i) if $\hat{h}(v) = \lambda v$, $n(\hat{h}) = 0$;

ii) if $\hat{h} = \alpha\hat{f}$, $n(\hat{h}) = n(\hat{f})$;

iii) if $\hat{h} = \hat{f} + \hat{g}$, $n(\hat{h}) = n(\hat{f}) + h(\hat{g})$;

iv) if $\hat{h} = \lceil \hat{f} \rceil$, $n(\hat{h}) = n(\hat{f}) + 1$.

If $n(\hat{h}_0) = 0$ then we may take $h_1(v) = \lambda v$, since h_0 is linear.

If $n(\hat{h}_0) > 0$ then there is $\lambda \epsilon Q^m$; $\alpha_1, \ldots \alpha_k \geq 0$; $f_1, \ldots f_k$ such that $\hat{h}_0(v) = \lambda v + \alpha_1 \lceil \hat{f}_1(v) \rceil + \alpha_2 \lceil \hat{f}_2(v) \rceil + \ldots \alpha_k \lceil \hat{f}_k(v) \rceil$ where $n(f_i) < n(\hat{h}_o)$. Since $h_0(-e_j) \leq 0$ there are integers $m_0, m_1, \ldots m_k$ such that: (i) $\lambda m_0 + \sum_1^k \alpha_i m_i = 0$ (ii) $\lambda e_j + m_0 \leq 0$; (iii) $\lceil f_i(-e_j) \rceil + m_i = 0$. Define $h_1(v) = (\lambda - m_0 e_j)v + \Sigma\alpha_i \lceil g_i(v) \rceil$ where $g_i(v) = f_i(v) - (m_i e_j)v$. $h_1(v) = h_0(v)$ for integer v by (i), and (ii), (iii) mean we may apply the induction hypothesis to produce suitable $\hat{g}_i$. Q.E.D.

Corollary 5.15: If h_0 is a Gomory function and $h_0(-e_j) \leq 0$ for all j, then there is an $h \epsilon \mathcal{G}$ such that $h(v) = h_0(v)$ for all integer v.

Proof: By proposition 2.16, h_0 is a maximum of Chvâtal functions. Use lemma 5.13 on each Chvâtal function for each $1 \leq j \leq n$ to get the desired representation. Q.E.D.

Now the strengthening of corollary 5.4 and 5.5 is immediate.

Theorem 5.16: There is an $f \in \mathcal{N}^{\mathcal{G}}$ such that (5.30) is consistent if and only if $f(v) \leq 0$.

Theorem 5.17: There is a $g \in \mathcal{N}^{\mathcal{G}}$ such that $g(v)$ = optimum value of (5.30) if $f(v) \leq 0$.

The next result was first proven by Wolsey [14] by an analysis of Gomory's Method of Integer Forms [4]. However, [4] assumes that the initial linear programming relaxation has a tableau of lexicographically positive columns [4, bottom page 286; also p. 287 and p. 289]. Hence the method of proof in [14] cannot be used for all integer programs.

Theorem 5.18: If (1.1) is consistent and has finite value, there is an optimal solution f to the subadditive dual problem (2.19) which is a Chvátal function.

Proof: By Theorem 2.15, the value function G of (1.1) optimally solves (2.19); hence by Theorem 5.2, there is a Gomory function g which is an optimum in (2.19). By Proposition 2.18, $g = \max\{f_1, \ldots, f_t\}$ for certain Chvátal functions f_i, $1 \leq i \leq t$. If f_j is such that $g(b) = f_j(b)$, then f_j is an optimum for (2.19). Q.E.D.

Remark: Several alternative proofs of Theorem 5.18 are possible. Schrivjer, building on work of Edmonds and Giles [3], has recently established [13] that finitely many applications of Chvátal's operation (as in (2.6)) yields the convex hull of integer points for any integer program (without the restriction in [4]). This can be used to construct the appropriate f in Theorem 5.18. Another proof is based on (non-trivial) modifications of the method of integer forms so that it will work for all integer programs.

An interesting "separation principle" follows from Theorem 5.18 which we give next.

Corollary 5.19:

If b is not an element of a finitely generated integer monoid M, there is a Chvátal function f such that: (i) $f(m) \leq 0$ for all $m \in M$; and (ii) $f(b) > 0$.

Proof: Let the generators of M be $a_1, \ldots, a_n$. Then the following integer program is consistent and has finite value one:

$$\text{minimize} \quad x_{n+1}$$

$$(5.32) \qquad \text{subject to} \quad \sum_{j=1}^{n} a_j x_j + b x_{n+1} = b$$
$$x_j \geq 0 \text{ and integer}$$

The subadditive dual of (5.32) is the program:

$$(5.33) \qquad \max F(b)$$
$$\text{subject to } F(a_j) \leq 0, \; j = 1, \ldots, n$$
$$\text{and } F(b) \leq 1$$

and by Theorem 5.18, the optimum value of this dual is achieved by a Chvátal function f; hence $f(b) = 1$. From $f(a_j) \leq 0$ for $j = 1, \ldots, n$ one easily derives $f(m) \leq 0$ for all $m = \sum_{j=1}^{n} a_j x_j$ ($x_j \geq 0$ and integer) by induction on $\alpha = \sum_{j=1}^{n} x_j$. Q.E.D.

We conclude this section with a result which relates the value function G_c of (1.1) to that of the linear relaxation.

Theorem 5.20: Let g be any Gomory function such that

$$(5.34) \qquad g(v) = G_c(v) \quad \text{whenever } I_v \neq \emptyset.$$

and let $\tilde{g}$ be the carrier of g. Then

$$(5.35) \qquad \tilde{g}(v) = R_c(v) \text{ whenever } R_c(v) < +\infty .$$

Proof: Suppose that there is a v_0 with $R_c(v_0) < +\infty$ and $\tilde{g}(v_0) \neq R_c(v_0)$. Then for suitably large integral $D \geq 1$, $I_{Dv_0} \neq \emptyset$, and as $\tilde{g}$ and R_c are homogeneous functions, $\tilde{g}(Dv_0) \neq R_c(Dv_0)$. Then without loss of generality $D = 1$ and $I_{v_0} \neq \emptyset$.

We established in Proposition 2.10 that there exists $k_1 \geq 0$ with

$$(5.36) \qquad 0 \leq g(v) - \tilde{g}(v) \leq k_1 \quad \text{for all } v \in Q_m$$

By Corollary 4.7, there exists $k_2 \geq 0$ such that

$$(5.37) \qquad 0 \leq G_c(v) - R_c(v) \leq k_2, \quad \text{whenever } I_v \neq \emptyset,$$

and hence

$$(5.38) \qquad 0 \leq g(v) - R_c(v) \leq k_2, \quad \text{whenever } I_v \neq \emptyset.$$

Starting from (5.36) and (5.38), we may apply the kind of reasoning as in the proof of Corollary 2.11 (particularly as in the display (2.14)) to the homogeneous functions $\tilde{g}$ and R_c, and we obtain a contradiction from our supposition that $\tilde{g}(v_0) \neq R_c(v_0)$. Q.E.D.

Theorem 5.20 has this interpretation: if we start with a closed-form Gomory expression g for the optimal value of (1.1), and simply go through the expression erasing all round-up symbols, we obtain a closed-form expression for the optiaml value of the linear relaxation of (1.1)

6. The Structure of $G_c(v)$ as c Varies

Throughout this section $a_1, \dots a_n \in Q^m$ will be fixed. In section 5 we determined the parametric form of the value of (1.1) in its right-hand-side; now we seek a simultaneous uniformity in the criterion vector c.

We begin with a result which says that there is a finite set F such that, if x is any feasible but not not optimal solution to an integer program, there is a better feasible solution obtained by adding some member of F to x. The set F is independent of the criterion vector c. This type of result was first established by Graver [7]; we give an alternate proof (and a somewhat different statement of the result) via monoid basis results.

Lemma 6.1: There is a finite $F \subset Z^n$ such that, for any $v \in Q^m$, $c \in R^n$, $x \in I_v$ either: (i) $cx = G_c(v)$; or (ii) for some $y \in F$, $x + y \in I_v$ and $c(x + y) < cx$.

Proof: Define $M \subseteq Z^{2n}$ by

(6.1) $$M = \{(\alpha_1, \dots \alpha_n, \beta_1, \dots \beta_n) | \Sigma a_j \alpha_j = \Sigma a_j \beta_j \ \alpha_j, \beta_j \geq 0 \text{ and integer}\}$$

M is a monoid defined by rational polyhedral constraints. Theorem 7 of [10] (indeed, Hilbert's result [8]) implies that there is a finite $W \subset M$ such that every member of M is a non-negative integer combination of members of W. Define $F \subseteq Z^n$ by

(6.2) $$F = \{y | y = \alpha - \beta \ \text{ where } (\alpha, \beta) \in W\}$$

If $v \in Q^m$, $x \in I_v$ and (i) fails there is a $z \in I_v$ with $cz < cx$. Since $(z, x) \in M$ there are non-negative integers n_w such that

(6.3) $$\sum_{w \in W} n_w w = (z, x)$$

Since $cz < cx$ there is at least one $\underline{w} = (\alpha, \beta) \in W$ such that $n_{\underline{w}} \geq 1$ and $c\alpha < c\beta$. As $(\Sigma n_w w) - \underline{w} + (\alpha, \alpha) \in M$, $x - \beta + \alpha \in I_v$. Then $(\alpha - \beta) \in F$ and $c(\alpha - \beta) < 0$. Q.E.D.

Theorem 6.2: There is a finite set $T = \{d_1, \ldots d_N\} \subset Q^n$ such that, for any $c \in R^n$, $v \in R^m$, if $I_v \neq \emptyset$ and $G_c(v) > -\infty$ there are $\alpha_d \geq 0$, $d \in T$ such that: (i) $\sum_{d \in T} \alpha_d d = c$ and (ii) $\sum_{d \in T} \alpha_d G_d(v) = G_c(v)$.

The algebraic content of Theorem 6.2 is that any inequality $cx \geq G_c(v)$ valid for I_v can be obtained by taking non-negative linear combinations of the inequalities $dx \geq G_d(v)$, $d \in T$. Geometrically, this means that for every v the finitely many inequalities $dx \geq G_d(v)$, $d \in T$ include the facets of I_v, uniformly in v.

Essentially the same result has been stated by Wolsey [15, Theorem 2].

Proof: Define

$$(6.4) \qquad C = \{c \mid \text{for some } w \in R^m, wa_j \leq c_j, 1 \leq j \leq n\} \subset R^n$$

C is a polyhedral cone. If $I_v \neq \emptyset$, $G_c(v) > -\infty$ if and only if $c \in C$, since $c \in C$ if and only if $R_c(v) > -\infty$. (The "if" part is easy. The "only if" follows from the remark at the end of Theorem 5.2).

Let F be as in Lemma 6.1. For each $H \subseteq F$ define the polyhedral cone

$$(6.5) \qquad B_H = \{c \mid c \in C \text{ and } cy \geq 0 \text{ for every } y \in H\} \subseteq R^n$$

By the Finite Basis Theorem there is a finite $A_H \subset B_H$ such that the cone generated by A_H is B_H. We define

$$(6.6) \qquad T = \bigcup_{H \subseteq F} A_H$$

We must establish that T has the desired properties. Let $v \in Q^m$, $c \in R^n$ satisfy our hypotheses. By theorem 4.6 (or [12]) there is an $\underline{x} \in I_v$ with $G_c(v) = c\underline{x}$. Let $\underline{H} = \{y \in F \mid \underline{x} + y \in I_v\}$. Clearly $c \in B_{\underline{H}}$. Hence (i) holds for some $\alpha_d \geq 0$ where we may further specify $\alpha_d = 0$ if $d \notin A_{\underline{H}}$. By Lemma 6.1, $G_d(v) = d\underline{x}$ for every $d \in B_{\underline{H}}$. Hence $\sum_{d \in A_H} \alpha_d G_d(v) = (\sum_{d \in A_H} \alpha_d d)\, \underline{x} = c\underline{x} = G_c(v)$, and (ii) holds. Q.E.D.

By Theorem 5.2, there is, for each $d \in T$, a Gomory function g_d such that $g_d(v) = G_d(v)$ if $I_v \neq \emptyset$. Also, it follows from the definition of $G_d(v)$ that $\Sigma\alpha_d G_d(v) \leq G_c(v)$ for all non-negative α_d such that $\Sigma\alpha_d d = c$.

Thus we can strengthen Theorem 6.2:

Theorem 6.3: There are finitely many Gomory functions g_d, $d \in T$ such that, if $I_v \neq \emptyset$ and $G_c(v) > -\infty$, then $G_c(v)$ is the value of the optimal solution to this programming problem with linear constraints:

$$(6.7) \qquad \text{maximize} \quad \sum_{d \in T} \alpha_d g_d(v)$$

$$\text{subject to} \quad \sum_{d \in T} \alpha_d d = c$$

$$\alpha_d \geq 0$$

Remark: If c is fixed and v varies, only finitely many optimal solutions $\underline{\alpha}$ to (6.7) arise, as each optimal $\underline{\alpha}$ is an extreme point to the linear constraints. Each of the optimal solutions gives a Gomory function $\Sigma\underline{\alpha}_d g_d(v) \leq G_c(v)$, where, for all v, $G_c(v)$ is the maximum of this finite family of Gomory functions. Thus we have extended Theorem 5.2 to $c \in R^n$.

7. Examples of Valid Inequalities Generated by Chvâtal Functions

In [5 , p. 524] Gomory tabulates the facets of the group problem

$$(7.1) \qquad t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 \equiv 0 \pmod 6$$

$$t_i \geq 0;\ t_i \text{ integer; not all } t_i = 0$$

This is equivalent to an integer programming problem with a single constraint

$$(7.2) \qquad x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 - 6x_6 = 6$$

$$x_i \geq 0;\ x_i \text{ integer}$$

One facet given in [4] is

(7.3) $$5t_1 + 4t_2 + 3t_3 + 2t_4 + t_5 \geq 6$$

This is generated by the Chvâtal function

(7.4) $$f(\alpha) = 6\lceil \alpha/5 \rceil - \alpha$$

More generally, the inequality

(7.5) $$kt_1 + (k-1)t_2 + \ldots t_k \geq k + 1$$

is valid for the group program constraint

(7.6) $$t_1 + 2t_2 + 3t_3 + \ldots kt_k \equiv 0 \pmod{k+1}$$

(7.5) is generated by the Chvâtal function $f(\alpha) = (k+1)\lceil \alpha/k \rceil - \alpha$.

Theorem 5.11 guarantees that for any valid inequality for an integer program with fixed right-hand side is generated by a Chvâtal functions. However, it seems too much to expect that the facets will be generated by particularly simple functions.

Another facet of (7.1) is [4]

(7.7) $$4x_1 + 2x_2 + 3x_3 + 4x_4 + 2x_5 \geq 6$$

One function that generates (7.7) is

(7.8) $$f(\alpha) = 3\left\lceil -\frac{2}{3}\alpha + \frac{2}{3}\left\lceil \left(\frac{3}{4}\left\lceil -\frac{2}{5}\alpha \right\rceil - \frac{1}{2}\alpha\right) \right\rceil \right\rceil + 4\alpha$$

(7.8) was obtained by using the method of integer forms [5] to solve (7.2) with objective function $4x_1 + 2x_2 + 3x_3 + 4x_4 + 2x_5$. (7.7) can probably be generated by a simpler function, but it can be shown that (7.7) cannot be generated by a function of the form $f(\alpha) = \lambda_1\alpha + \lambda_2\lceil \lambda_3\alpha \rceil$. There is room for further investigation.

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SOME INFLUENCES OF GENERALIZED AND ORDINARY CONVEXITY IN DISJUNCTIVE AND INTEGER PROGRAMMING

by

Robert Jeroslow[1]

Abstract

Three extensions of linear programming are convex, disjunctive, and integer programming. Each generalization represents a different direction, and is attuned to specific distinctive features of the phenomena studied. While e.g. in convexity, and certian of its generalizations, line segment containment provides the crucial property of polyhedra which is retained while a curvature of the feasible region is then permitted, in integer programming the linearity of the region is retained while the discrete nature of the variables departs entirely from the continuum and is the primary complicating factor. Nevertheless, developments in convex programming and its generalizations have influenced disjunctive and integer programming.

Conversely, parts of the infinitary disjunctive programming may be useful in nonconvex nonlinear programming. Similarly "integer analogues" recently discovered in integer programming represent developments somewhat parallel to the generalized duality schemes which extend Lagrangean duality for convex

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programs to more general duality results for nonconvex programs. These integer analogues are in fact the primary focus of our present report.

Most of the influence of convexity in disjunctive and integer programming is in the theory of cutting-planes, which we earlier surveyed in detail in Jeroslow (1977, 1978). Here we make some additional remarks from the somewhat different perspective of developments in nonlinear programming.

Key Words

Convexity, generalized concavity, disjunctive programming, integer programming, subadditive duality.

1. Introduction

Convexity is a generalization of linearity. If each linear constraint of a linear program is replaced by a convex constraint, a convex program arises.

However, there are other ways of producing important extensions of the linear program, and these include: adding (generally non-convex) logical conditions, thereby producing what is called a disjunctive program in Balas (1979); or adding the requirement that the variables be integer, producing an integer program.

These three extensions of the linear program are different in nature, but elementary results from convexity have been used in disjunctive programming and integer programming. We have surveyed these uses of convexity as part of the earlier papers Jeroslow (1977, 1978). Partially in order not to repeat ourselves, our emphasis here will be to reverse the perspective given earlier, and show how the ideas from disjunctive programming may be beneficial in nonconvex nonlinear programming.

In addition, recent results in integer programming (see eg. Blair and Jeroslow (1980), Shrijver (1979), and Wolsey (1978, 1979)) have led to the concept of an <u>integer analogue</u> to a statement of linear duality. The recent work has been strongly influenced by Gomory (1963). To some extent, this development corresponds to the development of generalized duality schemes for nonconvex programming. (see e.g. Tind and Wolsey (1978)).

Our focus throughout is on cutting-planes, i.e. valid linear inequalities implied by a given set of constraints. The primary influence of convexity in disjunctive and integer programming has been on the theory of cutting-planes. Also, cutting-planes relate to nonlinear programming in a more direct fashion than one might first intuit.

For example, an approach to the study of the general nonlinear program

$$\begin{array}{c} \inf f(x) \\ \text{subject to } g(x) \leq 0,\ x \in K, \end{array} \tag{1}$$

where $f:R^n \to R$ and $g:R^n \to R^m$, is to develop all the valid linear inequalities for the set $S = \{(z,w) \mid \text{for some } x \in K,\ f(x) \leq z \text{ and } g(x) \leq w\}$.

as was done in Duffin and Jeroslow (1979) in the convex case. A valid linear inequality $z + \sum_i \lambda_i w_i \geq u$ is equivalent to the Lagrangean statement: $\inf_x \{f(x) + \sum_i \lambda_i g_i(x)\} \geq u$.

For a second example, let $\{a^1, \ldots, a^t\}$ be any finite set of vectors in R^m, $S^k = \{(z,w) \mid (z,w) \in S \text{ and } a^k w = \max_i a^i w\}$. If, for each k, $z + \sum_i \lambda_i^k w_i \geq u$ is a valid linear inequality for S^k it is not difficult to show that, for some $p \geq 0$, $z + p\|w\| \geq u$ is valid for S, where $\|w\|$ denotes the norm of w. The latter fact is, in turn, equivalent to the norm-penalty statement: $\inf_x \{f(x) + p\|g(x)\|\} \geq u$.

2. Disjunctive Programming: Co-propositions

We consider a propositional logic, built up from finitary or infinitary uses of the logical connectives 'Λ' (for: 'and') and 'V' (for: 'or'), starting from linear inequality statements of the form $ax \geq b$ (see Tait (1968) for such propositional logic). The symbol 'V' for 'or' is also called the 'disjunction'.

Any system of convex constraints can be stated in this logic, as e.g. the requirement of being in the level set of a quasi-convex function. In fact, the constraint $z \geq x^2$ is equivalent to

$$z \geq 2x_o\,(x - x_o) + x_o^2, \text{ for all } x_o \varepsilon R. \tag{2}$$

To state it another way, $z \geq x^2$ is equivalent to this infinite 'and' statement of the propositional logic: $\Lambda_{x_o \varepsilon R}(z - 2x_o x \geq - x_o^2)$. The infinite 'and' is simply a notational variant of (2). Note that convex constraints constitute that part of the logic in which only the 'and' connective 'Λ' is used. The logic contains propositions asserting many nonconvex statements, via the disjunction 'V', as for example $(x \leq 1) \vee (x \geq 2)$.

We now describe an inductive assignment of closed convex cones to propositions of this logic, which is called the 'co-proposition assignment'. We denote propositions by Greek letters α, β, γ, etc., and the 'co-proposition' assigned to a proposition α is denoted $CT(\alpha)$. On occasion, we write $\alpha(x)$ to emphasize the dependence of the proposition α on $x \varepsilon R^n$.

If α is a linear inequality statement $ax \geq b$, let

$$CT(\alpha) = \{\, \lambda(a, -b) + \theta(0,1) \mid \lambda, \theta \geq 0;\ \lambda, \theta \varepsilon R \} \tag{3}$$

If $\alpha = \Lambda_{h \varepsilon H}\ \alpha_h$, for H a (possibly infinite) index set, we define

$$CT(\alpha) = \text{clconv}\left(\bigcup_F \{\textstyle\sum_{h \varepsilon F} CT(\alpha_h) \mid F \subseteq H, \text{ F finite}\}\right) \tag{4}$$

where $\sum_{h \varepsilon F} CT(\alpha_h) = \{ \sum_{h \varepsilon F} a_h \mid a_h \varepsilon CT(h) \text{ for each } h \varepsilon F\}$. If $\alpha = V_{h \varepsilon H}\ \alpha_h$, we define

$$CT(\alpha) = \bigcap_{h \varepsilon H} CT(\alpha_h). \tag{5}$$

Quite possibly $CT(\alpha) = \{(0,b) \mid b \leq 0\}$, which indicates that no nontrivial linear inequalities are obtained from α. If H is finite and $CT(\alpha_h)$ is a polyhedral cone for $h \in H$, then (4) simplifies to:

$$CT(\alpha) = \sum_{h \in H} CT(\alpha_h) \qquad (4)'$$

The co-proposition assignment has the property that:

If $\alpha = \alpha(x)$ is true for $x \in R^n$, and $(\pi, -\pi_o) \in CT(\alpha)$, then $\pi x \geq \pi_o$ is true. (6)

Indeed, (6) is correct for the ground step (3) of our inductive construction, and it is a property preserved by the inductive steps (4) and (5). Indeed, (4) in essence provides that the sum of valid linear inequalities, and their closure, yield valid inequalities. Similarly (5) provides that those inequalities common to all propositions α_h, $h \in H$, must be valid, provided only that at least one of these propositions holds.

As one application of the co-propositions, we obtain cutting-planes from the nonconvex condition:

$$x \notin C_1 \cup C_2 \cup \dots \cup C_t \text{ and } x \geq 0 \qquad (7)$$

where $C_k = \{x \in R^n \mid ax \leq b, (a,b) \in H_k\}$, $1 \leq k \leq t$, is a closed convex set, and where H_k is an arbitrary non-empty index set.

The co-propositions provide this family of inequalities:

$$\sum_{j=1}^{n} x_j \max \{\sum_{k=1}^{t} \lambda_k(a^k, b^k)\, a_j^k \mid (a^k, b^k) \in H_k \text{ for } k = 1, \dots, t\} \qquad (8)$$

$$\geq \min \{ \sum_{k=1}^{t} \lambda_k(a^k, b^k)\, b^k \mid (a^k, b^k) \in H_k \text{ for } k = 1, \dots, t\}.$$

In (8), $x = (x_1, \dots, x_n)$; a_j^k is the j-th component of a^k; and we are permitted to arbitrarily select $\lambda_k(a^k, b^k) \geq 0$ as $(a^k, b^k) \in H_k$ varies.

An interesting special case occurs when $t = 1$, all $b^1 > 0$ and the choice $\lambda_1(a^1, b^1) = b^1$ is made for each $(a^k, b^k) = (a,b) \varepsilon H = H_1$. This gives the cutting-planes:

$$\sum_{j=1}^{n} x_j \max \{a_j/b \mid (a,b) \varepsilon H\} \geq 1 \qquad (8)'$$

For $|H_1|$ finite, these are some of the cuts obtained in Glover (1973) and Balas (1975); see Jeroslow (1977) for a discussion of relationships between disjunctive constructions and the intersection cut constructions. The co-propositions were introduced in Jeroslow (1974) as a generalization of disjunctive constructions.

For $t=1$, the cut is as drawn in Figure 1. Specifically, a plane is passed through the intersection points of the convex set $C=C_1$ with the co-ordinate axes. Since $x \notin C$, we can restrict x to be in the half space that lies to the side of the hyperplane which does not contain the origin. For $t=2$, two cuts from the family of cuts are drawn in Figure 2. Both figures assume that the intercepts exists and that the origin lies in the interior of the convex regions, as depicted.

Our limited space has required us to sketch only a few fundamental points about the disjunctive methods, which were first introduced in Balas (1975, 1979); in particular, Balas (1979) contains an important result on "facial constraints", which we have not touched on here and which has a number of consequences.

3. Integer Programming: Analogues

A Chvatal function is one which is built up from linear functions λb, with λ rational, by repeatedly rounding-up to the nearest integer, and then taking

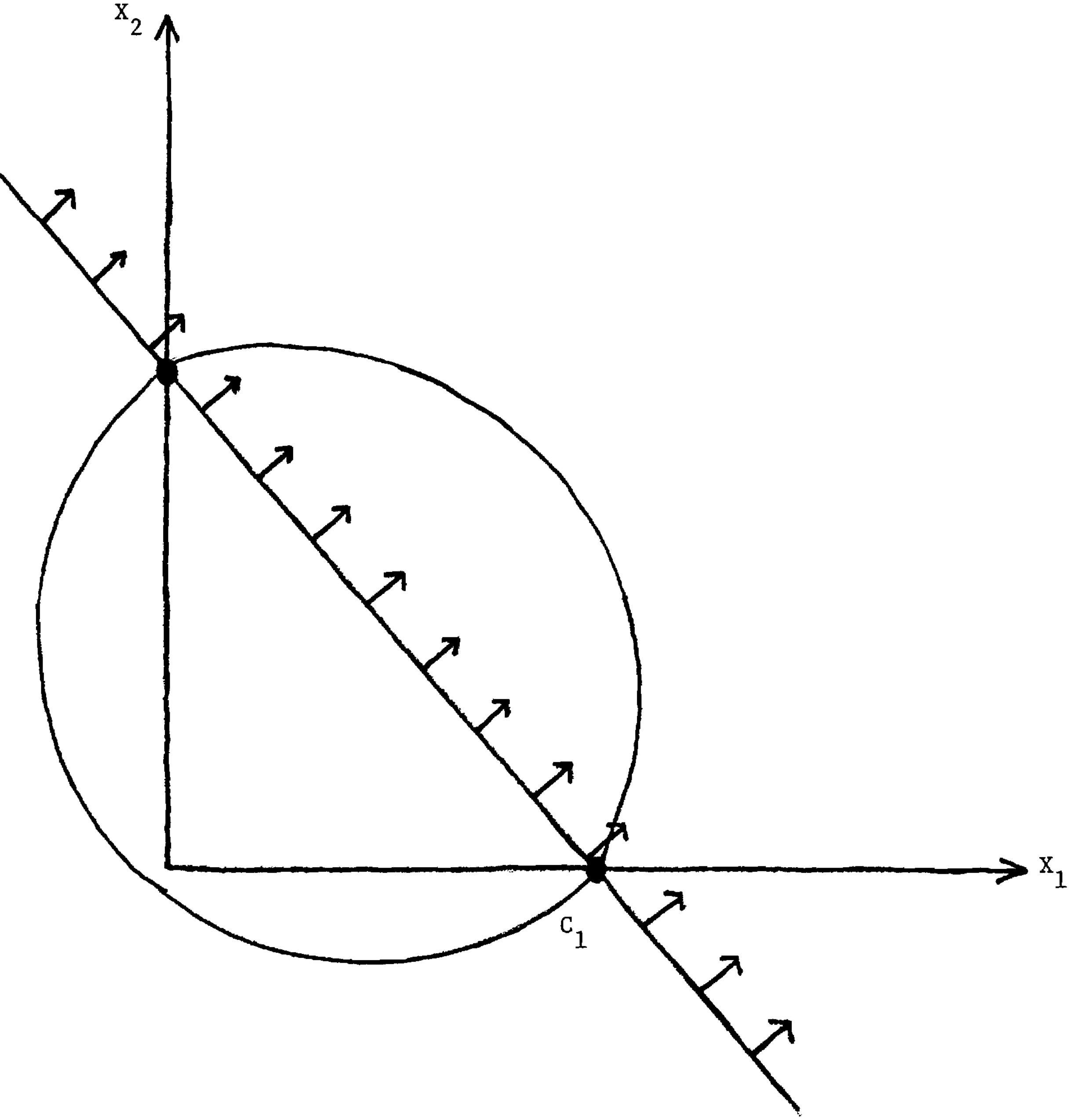

Figure 1

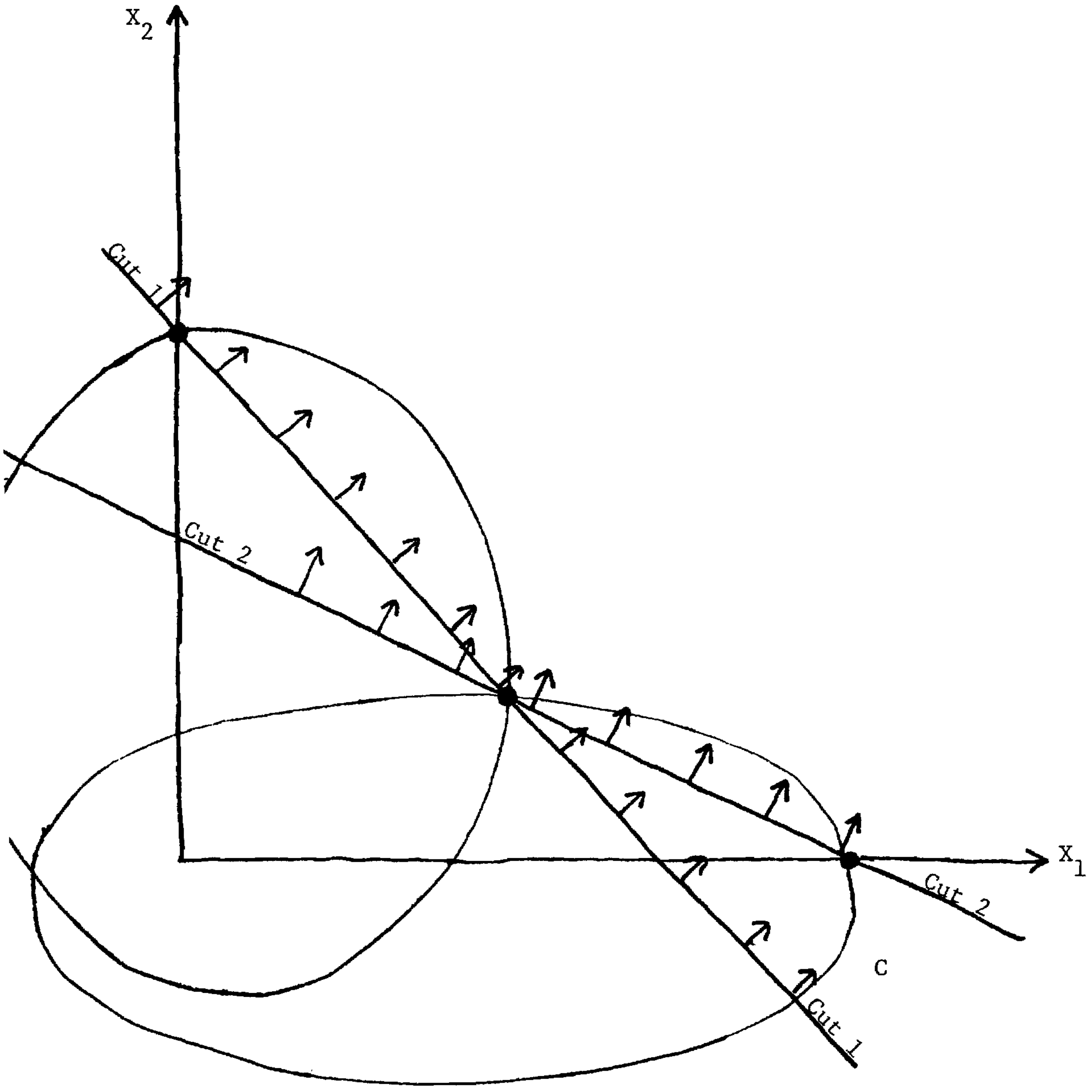

Figure 2

rational non-negative combinations of the functions thus obtained. In the original linear functions λb we can have λ of arbitrary signs, but thereafter the non-negativity of multipliers must be observed. For example, a Chvatal function of two variables b_1, b_2 is given by $f(b_1, b_2) = 3[-b_1 + 2b_2] + 2(3b_1 + [-b_2])$, where $\lfloor u \rfloor$ denotes the round-up of the real number u (i.e., the smallest integer not smaller than u).

Chvatal functions appear to play the role in integer programming, that linear functions play in the dual form of two equivalent linear statements. In other words, among the various equivalency theorems regarding linear inequalities, one tends to obtain true statements when the variables of the "primal" are required to be integer, and the linear functions of the "dual" are allowed to become Chvatal functions. We assume that all quantities of an integer program are rational, and that the proper choice of "primal" and "dual" statements has been made.

For example, it is well known that the linear program

$$\begin{aligned} &\min cx \\ \text{subject } &Ax = b \\ &x \geq 0 \end{aligned} \tag{9}$$

has as its dual the program

$$\begin{aligned} &\max \theta b \\ \text{subject } &\theta A \leq c. \end{aligned} \tag{10}$$

According to the heuristic principle annunciated in the last paragraph, the dual of the integer program in rationals

$$\begin{aligned} &\min cx \\ \text{subject } &Ax = b \\ &x \geq 0 \text{ and } x \text{ integer} \end{aligned} \tag{9)'$$

ought to be

$$\begin{aligned} &\max f(b) \\ \text{subject to } &f(a^j) \leq c_j \\ &f \text{ Chvatal} \end{aligned} \tag{10)'$$

where $A = [a^j]$ (cols) and $c = (c_j)$. Indeed, it is the case, that if (9)' is consistent, its value is that of (10)' (see Blair and Jeroslow (1980)).

As a second illustration of the heuristic principle, recall the linear theorem that a finitely generated polyhedral cone, i.e. a set of the form $\{b \mid$ there is an $x \geq 0$ with $Ax = b\}$, has a definition in terms of homogeneous linear inequalities. If the later definition is termed "dual", then we would conclude that a set of the form

$$\{b \mid \text{there is an integer } x \geq 0 \text{ with } Ax = b\} \tag{11}$$

will have a definition of the form

$$\{ b \mid f_i (b) \leq 0, i = 1,\ldots, p\} \tag{12}$$

for certain Chvatal functions $f_1,\ldots, f_p$ (at least when A is rational). This is indeed the case, though we remark that in (12) the direction of the homogeneous inequalities cannot be reversed.

A third linear theorem is that the optimal value of a linear program, as a function of its right-hand-side (r.h.s.) b, i.e. the function given by

$$L(b) = \inf \{ cx \mid Ax = b, x \geq 0 \} \tag{13}$$

is the maximum of a finite number of linear functions, where it is defined. (L(b) is defined precisely if there is $x \geq 0$ with $Ax = b$). And it is indeed the case that the value function of an integer program, given by

$$G(b) = \inf \{ cx \mid Ax = b, x \geq 0 \text{ and integer}\} \tag{14}$$

is the maximum of finitely many Chvatal functions, where it is defined. A fourth linear theorem states that L(b) is defined exactly where a certain finite set of linear functions are all nonpositive; and, as one would expect, G(b) is defined where a certain finite set of Chvatal functions are all nonpositive.

The role of Chvatal functions as an <u>integer analogue</u> of linear functions is even more pronounced. To each Chvatal function f is associated a linear function $\bar{f}$ called its <u>carrier</u>. The carrier is, intuitively, obtained by erasing all round-up operations and collecting terms. The carrier of the Chvatal function in the first paragraph of this section is therefore $\bar{f}(b_1, b_2)=3(-b_1 + 2b_2) + 2(3b_1 - b_2) = 3b_1 + 4b_2$. Now the role that the Chvatal function f plays in the discrete version of a linear theorem, appears to be the role its carrier $\bar{f}$ plays in the original theorem. For example, the carrier of G(b) in (14) is the linear optimal value L(b) of (13).

A fifth linear theorem states that, if $Ax = b$, $x \geq 0$ is inconsistent, there is a linear form θw such that $\theta Ax \leq 0$ for all $x \geq 0$ and $\theta b > 0$. This linear theorem is a version of the Farkas Lemma; for other linear theorems of the alternative, see Mangasarian (1969, table 2.4.1). As one would expect, if there is no solution to $Ax = b$, $x \geq 0$ and integer, then there is a Chvatal function f with $f(Ax) \leq 0$ for all integer $x \geq 0$, and $f(b) > 0$.

A sixth linear theorem, the Finite Basis Theorem for Cones, states that the solution set to a finite set of homogeneous linear inequalities $\{b \mid Eb \leq 0\}$ for some rational matrix E, has a finite basis, i.e. there is a matrix A such that $Eb \leq 0$ if and only if $Ax = b$ for some $x \geq 0$. Viewing the statement "$Eb \leq 0$" as the "dual" statement it turns out <u>not</u> to be the case that for all finite sets of Chvatal functions $f_1,\ldots,f_t$ there exists a rational matrix A with:

b rational and $f_i(b) \leq 0$ for $i = 1, \ldots, t$ if and only if there is an $x \geq 0$ integer with $Ax = b$. (15x)

However, this conjecture was almost correct, as one need only change "rational" to "integer". Indeed, there is an integer matrix A with:

b integer and $f^i(b) \leq 0$ for $i = 1, \ldots, t$ if and only if there is an $x \geq 0$ integer with $Ax = b$ (15)

The discovery of integer analogues is recent, and we do not know the extent and complete nature of the phenomenon. The mixed-integer program can be indirectly treated by the ideas presented here, but a direct treatment is not possible since optimal value functions of a mixed-integer program are not closed under the inductive operations which construct Chvatal functions. On the other hand, constraint sets of the form

$$\begin{aligned} Ax + By &= Cb \\ x, y &\geq 0 \\ x &\text{ integer} \end{aligned}$$

in which a general rational matrix C pre-multiplies the right-hand-side, do allow much of the treatment of integer programming to go over to the mixed case. We shall report our recent joint results in the near future.

4. Conclusions

We have shown how ideas from convexity and generalized convexity have influenced disjunctive programming, and we have indicated that even the concept of a linear function can be generalized and adapted to the discrete setting.

The generalizations that one studies depend on which aspect of linearity or convexity is retained, and which new feature of some non-convexity one choses to underscore. We can expect furthur fruitful generalizations in the future.

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The Limiting Lagrangean

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Abstract. A somewhat modified form of the lagrangean closes the duality gap in convex optimization, in many circumstances that the ordinary lagrangean and the augmented lagrangeans leave a duality gap. For example, the duality gap for a consistent program is always zero using this modified lagrangean, when the objective function and constraints are closed, convex functions; in other instances, there are constraint qualifications, but these are typically weaker than the usual Slater point requirements.

Key Words. Convex optimization, lagrangean, nonlinear programming.

1. Introduction

In Ref. 1, the first author showed that a certain kind of perturbation in the ordinary lagrangean, involving only the addition of a term for a linear functional, together with a limiting operation in which that functional is sent to zero, could close duality gaps in instances where there was a duality gap for the ordinary Lagrangean, and even for the augmented Lagrangeans in the sense of Ref. 2. The second author (Ref. 3) extended the hypotheses in which this "limiting lagrangean" closed duality gaps in R^n, and also showed that in R^n the limiting process could be taken along a line to zero, i.e. is one-dimensional, by utilizing the "ascent ray" analysis in Blair's Ref. 4. For an alternate proof of the one-dimensional limiting Lagrangean, and some additional results, see Ref. 5 and Ref. 6.

Either the multi-dimensional limiting lagrangean of Ref. 1, or the one-dimensional limiting lagrangean of Ref. 3, puts duality gaps of the ordinary lagrangean in a new perspective. Duality gaps are usually associated with the lack of a "constraint qualification," i.e. some "defect" in the constraints. Yet the limiting lagrangean shows how to close duality gaps by perturbations of the objective function. The constraints are also involved, but to a lesser degree. For example, as we shall see in this paper, if the convex functions and the convex set involved in the convex program are all closed, one need only assume that the program is consistent, for the multi-dimensional form of the limiting lagrangean to close the duality gap. For the one-dimensional limiting lagrangean in R^n, it was shown in Ref. 7 that the duality gap is closed under hypotheses substantially weaker than those sufficient conditions usually cited for a Kuhn-Tucker vector to exist.

The limiting lagrangean also allows a simultaneous treatment of an infinite set of convex constraining functions, with the same hypotheses as for a finite set of constraining functions.

The purpose of this paper is to place the multi-dimensional limiting lagrangean in a broad setting, that of set-valued convex functions in a locally convex space U, whose second continuous dual U^{**} is U under the usual injection map of U into U^{**}.

Our method of proof proceeds by an analysis of the valid implied inequalities ("cutting places") of an infinite system of linear inequalities. It is thus an outgrowth of the work on "semi-infinite systems" of Charnes, Cooper, and Kortanek (Ref. 8), Duffin and Karlovitz (Ref. 9), and Blair (Ref. 4), and also the linear analysis of Duffin (Ref. 10). Earlier versions of some of the results given here have appeared in Ref. 11.

2. Set-Valued Convex Functions

Let U and W be linear spaces, and let $\mathcal{P}(\cdot)$ denote the set of all subsets (i.e., the power set) of the set denoted within the parentheses. Throughout the paper, R denotes the real numbers.

We call a function $g: U \to \mathcal{P}(W)$ convex, if the set epi(g) is convex in U × W, where:

$$\mathrm{epi}(g) = \{(u,w) \mid u \in U \text{ and } w \in g(u)\} \tag{1}$$

We use this notion of a convex function because of its breadth. It allows convex functions to be completely identified with convex sets, for, e.g., any convex set $K \subseteq U \times W$ automatically defines the convex function $g(u) = \{w \in W | (u,w) \in K\}$. For the usual definitions of a convex function, only restricted kinds of epigraph sets epi(g) can arise, which e.g. generally contain vertical half-lines or other directions of recession (Ref. 12) that have zero as their u-coordinates. However, as our proofs below only require convexity of epi(g), our present definition of a convex function is appropriate. The concept of a set-valued convex function appears to be due to Blaschke (see Ref. 13, pp. 32-34).

The usual convex function $g^*:U \to R \cup \{\infty\} \cup \{-\infty\}$ of Ref. 12 can be cast in the form here by defining $g:U \to \mathcal{P}(R)$ in this manner:

$$g(u) = \begin{cases} R & \text{, if } g^*(x) = -\infty \text{ ;} \\ \{r \in R | r \geq g^*(u)\} & \text{, if } g^*(x) \in R \text{ ;} \\ \phi & \text{, if } g^*(x) = +\infty \text{ .} \end{cases} \tag{2}$$

Then epi(g) of (1) becomes the epigraph of g^* in the sense of Ref. 12, and hence is convex. Note that the condition "$g^*(u) \leq 0$" becomes "$0 \in g(u)$."

Our present definition of a convex function also has the convenience of permitting an entire collection of such functions $g = \underset{\alpha \in A}{X} g_\alpha$, for an arbitrary index set $A \neq \phi$, to be treated as one such function, through the definition

$$g(u) = \underset{\alpha \in A}{X} g_\alpha(u) \tag{3}$$

In fact if $g_\alpha:U \to \mathcal{P}(W_\alpha)$, then $g:U \to \mathcal{P}(\underset{\alpha \in A}{X} W_\alpha)$, and if each g_α is convex, clearly g is also. Note that $g(u) = \phi$ if $g_\alpha(u) = \phi$ for even one index $\alpha \in A$.

Finally, functions $h:U \to W$ which are convex with respect to some convex cone $C \subseteq W$ also can be interpreted as convex by our present definition. For such functions h, the convexity inequality $h(\lambda u_1 + (1-\lambda)u_2) \leq \lambda h(u_1) + (1-\lambda)h(u_2)$, where $0 \leq \lambda \leq 1$, is interpreted this way in the sense of the cone C:

$$\lambda h(u_1) + (1-\lambda)h(u_2) - h(\lambda u_1 + (1-\lambda)u_2) \in C \ . \tag{4}$$

We simply define g by:

$$g(u) = h(u) + C \tag{5}$$

and it is an easy exercise to show that epi(g) is convex.

When U and W have topologies, to each convex function $g:U \to \mathcal{P}(W)$ one associates its closure $c\ell(g)$, where

$$c\ell(g)(u) = \{w \in W | (u,w) \in c\ell(\mathrm{epi}(g))\} \tag{6}$$

By construction, $\mathrm{epi}(c\ell(g)) = c\ell(\mathrm{epi}(g))$. We say that g is <u>closed</u> if $g = c\ell(g)$. If each $g_\alpha (\alpha \in A)$ is closed, so is g in (3) when the product topology is used.

The kind of convex optimization we shall treat here is the infimization of a convex function $f:U \to \mathcal{P}(R)$, subject to a membership constraint on a convex function $g:U \to \mathcal{P}(W)$. In detail, we consider the convex program:

$$\begin{gathered}\inf f(u) \\ \text{subject to } 0 \in g(u)\end{gathered} \tag{7}$$

where, of course, $0 \in W$. We always assume that (7) is consistent, i.e. $\{u \in U | 0 \in g(u), f(u) \neq \phi\} \neq \phi$.

Because of the infimization operation in (7), f(u) may as well be taken to be convex in the sense of Ref. 12, and can be replaced by $f^*(u) = \inf f(u)$, with $f^*(u) = +\infty$ if $f(u) = \phi$. Indeed, the infimization operation of (7) abbreviates

$$\begin{aligned} &\inf z \\ &\text{subject to } z \in f(u) \\ &\text{and } 0 \in g(u) \,. \end{aligned} \tag{8}$$

However, typically g in (7) cannot be replaced by a function convex with respect to some cone.

Our formulation (7) allows a set restriction to be present. For example, the program

$$\begin{aligned} &\inf f(u) \\ &\text{subject to } 0 \in g_1(u) \\ &\text{and } u \in K \end{aligned} \tag{9}$$

where K is also convex, is cast into the form (7) by putting $g = (g_1, i(K))$, where the indicator function $i(K): U \to \mathcal{P}(W)$ is defined by:

$$i(K)(u) = \begin{cases} \{0\} \text{ if } u \in K \text{ (here } 0 \in W) \;; \\ \phi \quad \text{otherwise} \,. \end{cases} \tag{10}$$

Since K is convex, and $\text{epi}(i(K)) = K \times \{0\}$, so is i(K). We have $\text{epi}(g) = \{(u,w_1,w_2) | w_1 \in g_1(u), w_2 \in i(K)(u)\} = \{(u,w_1,0) | w_1 \in g_1(u) \text{ and } u \in K\} = (\text{epi}(g_1) \times \{0\}) \cap (K \times W \times W)$. If both g_1 and K are closed, so are both sets in the intersection just mentioned; hence, g is also closed.

The value of (7) is denoted v(P) (possibly $v(P) = -\infty$).

For further information on set-valued convex functions, see Ref. 15.

3. Ordinary Lagrangeans and Extensions of Linear Inequalities: Motivation

If the value $v(P)$ of (7) is finite, this means that the linear inequality

$$z \cdot 1 + 0 \cdot u \geq v(P) \tag{11}$$

is implied by the conditions

$$z \in f(u) \text{ and } 0 \in g(u) \tag{12}$$

Now if (11) can be extended to an inequality

$$z \cdot 1 + 0 \cdot u + \lambda_0^* w \geq v(P) \ , \tag{13}$$

where $\lambda_0^* \in W^*$, W^* denoting the continuous dual of W (Ref. 14), and is valid for the conditions

$$z \in f(u) \text{ and } w \in g(u) \ , \tag{14}$$

then we have the lagrangean statement

$$\inf_{u \in U} \{f(u) + \lambda_0^* g(u)\} \geq v(P) \tag{15}$$

and lagrange dual vector λ_0^*.

To write (15), we have used the obvious conventions:

$$C + D = \{c + d | c \in C, \ d \in D\} \tag{16}$$

for sets C, D (we set $C + D = \phi$ if $C = \phi$ or if $D = \phi$), where $C + v$ will abbreviate $C + \{v\}$, and

$$\lambda^* A = \{\lambda^* a | a \in A\} \tag{17}$$

if $\lambda^* \in Z^*$ and Z is a linear topological space with $A \subseteq Z$. Also, $\inf A$ is the greatest lower bound of all elements in the set $A \subseteq R$ (it is $+\infty$ if $A = \phi$), and $\inf_{t \in T} A(t)$ abbreviates $\inf_{t \in T} \inf A(t)$.

Of course, it is well-known that actually equality holds in (15), and that (15) implies

$$\max_{\lambda^* \in W^*} \inf_{u \in U} \{f(u) + \lambda^* g(u)\} = v(P) \tag{18}$$

Indeed, since (7) is consistent,

$$\begin{aligned} \inf_{u \in U} \{f(u) + \lambda^* g(u)\} &\leq \inf_{u \in U} \{f(u) | 0 \in g(u)\} \\ &= v(P) \ ; \end{aligned} \tag{19}$$

and from (19) for arbitrary $\lambda^* \in W^*$, (18) follows at once, using (15).

The usual "perturbational analysis" of convex programs does not emphasize Lagrangean results as extension results for linear inequalities in the space $R \times U \times W$. This is possible because the coefficient of "u" in (11) is zero—i.e. "u" does not actually appear. This fact in turn allows one to analyze only the set

$$K = \{(z,w) | z \in f(u) \text{ and } w \in g(u) \text{ for some } u \in U\} \tag{20}$$

in $R \times W$ in place of the set motivated by (14), i.e.

$$K^\wedge = \{(z,u,w) | z \in f(u) \text{ and } w \in g(u)\} \tag{21}$$

in $R \times U \times W$. Then the extension $z + \lambda_0^* w \geq v(P)$ of $z \geq v(P)$ can be obtained by suitable separation principles in real linear topological spaces (as e.g. Ref. 14, Theorem 14.2) if K has non-void interior $K^0 \neq \phi$, and in addition a suitable "constraint qualification" is met. See Borwein's paper (Ref. 15) for results using this technique. In finite-dimensional spaces R^n, the relative interior K^I can be used in place of K^0, as in (Ref. 16, Theorem 6, 10.12), but the basic idea remains the same. Essentially, what is gained by the reduction from $R \times U \times W$ to $R \times W$, is that one may have $K^0 \neq \phi$ even if $K^{\wedge}$ has no interior.

Despite the value of this reduction device, it is valuable to note that the idea of extensions of linear inequalities from $R \times U$ to $R \times U \times W$ is central to the Kuhn-Tucker results (18). For not only does this extension give (18) but, conversely, if g in (7) is such that (18) holds for <u>every</u> continuous linear function $f(u) = \{u^*(u)\}$ that is bounded below on $\{u | 0 \in g(u)\}$, then extensions exist. Indeed, denoting the lower bound again by $v(P)$, we see that (12) implies

$$z \cdot 0 + u^*(u) \geq v(P) \tag{22}$$

Now if (18), or, equivalently, (15), holds, then for some $\lambda_0^* \in W^*$ we have

$$z \cdot 0 + r + \lambda_0^* w \geq v(P) \tag{23}$$

whenever $r \in f(u)$ and $w \in g(u)$. Replacing r by its value $u^*(u)$ in (23), we see that (23) becomes the desired extension of (22).

The essential equivalence of linear inequality extension results with conjugate duality results in R^n is discussed in (Ref. 16, Sec. 5.3).

Once this equivalence is recognized, the natural issue arises, as to what reduction of constrained convex optimization (7) to unconstrained optimization (18) is possible when extensions fail to exist. Clearly, the usual interiority assumptions and constraint qualifications are simply sufficient, and not necessary, for the extensions of linear inequalities to exist, for these always exist, e.g., if g is polyhedral (epi(g) is the intersection of finitely many closed half-spaces). On the other hand, without any additional hypotheses, extensions need not exist as e.g. when $g(u) = \{w | w \geq u^2\}$ for $u \in R$ and $f(u) = \{u\}$. In this latter case, of course $v(P) = 0$, yet there is no extension of $z + 0 \cdot w \geq 0$ to $z + \lambda w \geq 0$ with $\lambda \in R$. Indeed, if such an extension existed, of necessity $\lambda \geq 0$ and $\lambda \neq 0$. Then $u + \lambda u^2 \geq 0$ is impossible for all $u \in R$, as $u + \lambda u^2 = -1/4\lambda < 0$ at the minimum point $u = -1/2\lambda$ for $\lambda > 0$.

In order to treat the case that extensions (13) (valid for (14)) fail to exist for inequalities (11) (valid for (12)), one approach is to consider the case that inequalities which are valid for (14), and "arbitrarily close" to the desired extension (13), do exist. This approach does not lead to the usual lagrangean statement (18) but, as we shall see, to this "limiting lagrangean" statement:

$$\lim_{M \to 0} \sup_{u^* \in M} \sup_{\lambda^* \in W^*} \inf_{u \in U} \{f(u) + u^*(u) + \lambda^* g(u)\} = v(P) \tag{24}$$

In the above, M is an open set in U^*, and "$M \to 0$" denotes the net (or filter) consisting of a local base of open sets M which contain $0 \in U^*$.

In the setting in which we establish (24), the spaces U, W and U^* will have locally convex topologies. The exact hypotheses will be given in

Section 4. The local convexity of U and W allows one to use simply disjointness properties, rather than interiority assumptions, in order to get separation principles (as e.g. in (Ref. 14, Theorem 14.3)), so that the closure of a convex set is always defined as the intersection of closed halfspaces; and the semi-reflexivity of U (i.e. $U^{**} = U$) allows one to derive generalizations of the Farkas Lemma (see Lemma 4.2 below) which are needed to make (13) the limit of inequalities having the desired extensions (as per the plan of our work).

Before we proceed according to our plan, some simple observations on the generality implicit in (24) (or, for that matter, (18) also) are worth remarking.

If we began with the program (9), and defined $g = (g_1, i(K))$ to obtain (7), then $\lambda^* \in (W \times W)^* = W^* \times W^*$ has the form $\lambda^* = (\lambda_1^*, \lambda_2^*)$. For $g(u) \neq \phi$ we need $i(K)(u) \neq \phi$, i.e., $u \in K$, in which case $\lambda_2^* w = 0$ for $w \in i(K)(u) = \{0\}$. Thus (24) becomes

$$\lim_{M \to 0} \sup_{u^* \in M} \sup_{\lambda_1^* \in W^*} \inf_{u \in K} \{f(u) + u^*(u) + \lambda_1^* g_1(u)\} = v(P) \qquad (25)$$

and we have relativized (24) to the set K, as one would have desired.

If, even further, g_1 was itself a product function $g_1 = \underset{\alpha \in A}{X} g_\alpha(u)$, analogous to (3) for g, the continuous dual of the range space $\underset{\alpha \in A}{X} W_\alpha$ is $(\underset{\alpha \in A}{X} W_\alpha)^* = \sum_{\alpha \in A} W_\alpha^*$, when each W_α is locally convex and Hausdorff and one has the strong topology on each W_α (Ref. 14, p. 174). This direct sum $\sum_{\alpha \in A} W_\alpha$ is the "finite sequence space" of Charnes, Cooper, and Kortanek (Ref. 8) when each $W_\alpha = R$, i.e. λ_1^* in (25) is an element of the space of all multipliers $(\lambda_\alpha | \alpha \in A)$ such that $\lambda_\alpha \neq 0$ for only finitely many $\alpha \in A$.

Also in the case $W_\alpha = R$, if each g_α arises as in (2) from a convex function g_α^* in the ordinary sense (Ref. 12), since $r \in g_\alpha(u)$ can be arbitrarily increased, we infer that all $\lambda_\alpha \geq 0$. In the same manner, whenever $g_1(u)$ contains a convex cone C of recession directions for any $u \in K$ (i.e. $g_1(u) + C = g_1(u)$), one easily shows that $\lambda_1^* c \geq 0$ for all $c \in C$. This recovers the usual "sign information," and indicates what (24) becomes in familiar settings.

4. Some Lemmas on Linear Inequalities

The purpose of this section is to show that certain basic results on linear inequalities in R^n are also true in a semi-reflexive locally convex setting. Beyond these fundamental results, only algebraic manipulations are needed to derive (24), and we defer those manipulations to the next section.

For purposes of this paper, a space X is called semi-reflexive if its second continuous dual X^{**} is X ($X^{**} = X$) under the usual injection of X into X^{**}. Thus if $x^* \in X$ and $x \in X$, and $x^*(x)$ denotes the value of x^* at the point x, then the image I(x) of x in X^{**} is that functional I(x) on X^* such that I(x) evaluated at x^* is $x^*(x)$ ($I(x)(x^*) = x^*(x)$). Our condition $X^{**} = X$ of semi-reflexivity means that each functional I(x) (for $x \in X$ arbitrary) is continuous on X^*, and that all the continuous linear functionals on X^* have the form I(x) for some $x \in X$. All Hilbert spaces (including any R^n) are semi-reflexive in this sense, as are all spaces L^p for $\infty > p > 1$.

The following result is well-known; see e.g. (Ref. 10) or use (Ref. 14, Theorem 14.3).

Lemma 4.4: Let C be a closed cone in a locally convex linear topological space X.

Then the following two statements are equivalent:

(i) $y_0 \in C$;

(ii) If $f \in X^*$, and $f(y) \geq 0$ for all $y \in C$, then $f(y_0) \geq 0$.

Lemma 4.2: Let $\{f_i | i \in I\}$ be a family of continuous linear functionals on the semi-reflexive locally convex space X which has a locally convex continuous dual X^*. Suppose that, for any $x \in X$,

$$f_i(x) \geq 0 \text{ for all } i \in I \tag{26}$$

implies

$$f(x) \geq 0 \tag{27}$$

for the continuous linear functional f.

Then for any neighborhood M of 0 in X^*, there exists a finite subset $J \subseteq I$ and non-negative numbers $\lambda_j \geq 0$, $j \in J$, and a continuous linear functional $g \in X^*$, satisfying both these conditions:

$$f = g + \sum_{j \in J} \lambda_j f_j \tag{28}$$

$$g \in M \tag{29}$$

Proof: Let $C = \mathrm{cl}(\mathrm{cone}(\{f_i | i \in I\}))$, where cone $(\{f_i | i \in I\})$ is the cone (algebraically) generated by the set $\{f_i | i \in I\}$, and cl(S) denotes the closure, here in the topology of X^*.

The conclusion of this corollary can be restated as "$f \in C$," for then $g = f - \sum_{j \in J} \lambda_j f_j$ satisfies (28) and (29).

Since C is a closed cone in the locally convex linear topological space X^* (Ref. 13, 16.1), Lemma 4.I applies. Thus if $f \notin C$, we reach a contradiction as follows, where we take $y_0 = f$ in Lemma 4.1.

There exists a continuous linear functional $\tilde{F}$ on X^* with $\tilde{F}(h) \geq 0$ for all $h \in C$ and $\tilde{F}(f) < 0$. In particular, $\tilde{F}(f_i) \geq 0$ for all $i \in I$, and $\tilde{F}(f) < 0$.

Since $\tilde{F} \in X^{**}$, by semi-reflexivity there exists $\tilde{x} \in X$ with $\tilde{F}(h) = h(\tilde{x})$ for all $h \in X^*$. In particular, $f_i(\tilde{x}) \geq 0$ for all $i \in I$ and $f(\tilde{x}) < 0$, contradicting the hypothesis. This shows that $f \in C$.

Q.E.D.

Lemma 4.3: Let $\{f_i | i \in I\}$ be a family of continuous linear functionals on the semi-reflexive locally convex space X with locally convex dual X^*. Let $\{\alpha_i | i \in I\}$ be a correspondingly-indexed family of real scalars, such that there is a solution to

$$f_i(x) \geq \alpha_i, \; i \in I . \tag{30}$$

Suppose that every solution x to (30) also satisfies

$$f(x) \geq \alpha \tag{31}$$

for the continuous linear functional f and scalar $\alpha \in R$.

Then for any real scalar $\epsilon > 0$, and neighborhood M of 0 in X^*, there exists a finite subset $J \subseteq I$, non-negative numbers λ_j, $j \in J$, a non-negative scalar $\theta \geq 0$, and a continuous linear functional g on X, and $\beta \in R$, satisfying:

$$(f,-\alpha) = \theta(0,1) + (g,-\beta) + \sum_{j\in J} \lambda_j (f_j,-\alpha_j) \ . \tag{32}$$

$$g \in M \tag{33}$$

$$|\beta| < \epsilon \tag{34}$$

In particular,

$$f = g + \sum_{j\in J} \lambda_j f_j \tag{35}$$

$$\alpha \leq \epsilon + \sum_{j\in J} \lambda_j \alpha_j \tag{36}$$

Proof: The particular conclusions (35), (36) follow from (32)-(34) by taking components in (32). We prove only (32), (33), (34).

To do so, note that, in the locally convex space X × R,

$$\begin{gathered} f_i(x) - \alpha_i r \geq 0, \ i \in I \\ r \geq 0 \end{gathered} \tag{37}$$

implies

$$f(x) - \alpha r \geq 0 \ . \tag{38}$$

Indeed, if $r > 0$, (37) implies (38) by the fact that (30) implies (31) and the linearity of the functionals $\{f_i | i \in I\}$ and f. If $r = 0$, again (39) implies (38), as we see by the following contradiction.

Let $\tilde{x}$ be such that $f_i(\tilde{x}) \geq 0$ for $i \in I$ yet $f(\tilde{x}) < 0$. By hypothesis there exists x^* with $f_i(x^*) \geq \alpha_i$ for $i \in I$. Then for any scalar $\rho \geq 0$, $f_i(x^* + \rho\tilde{x}) = f_i(x^*) + \rho f_i(\tilde{x}) \geq f_i(x^*) + 0 \geq \alpha_i$ for all $i \in I$. However for large ρ, $f(x^* + \rho\tilde{x}) = f(x^*) + \rho f(\tilde{x}) < \alpha$ as $f(\tilde{x}) < 0$. This contradicts that (30) implies (31), and proves that (37) implies (38).

One easily proves that $(X \times R)^{**} = (X^* \times R^*)^* = X^{**} \times R^{**} = X \times R$, i.e. $X \times R$ is semi-reflexive.

We apply Lemma 4.2 to the system (37), (38) with (26) taken as (37), and the functionals $\{f_i | i \in I\}$ of (26) taken as $\{(f_i, -\alpha_i) | i \in I\} \cup \{(0,1)\}$. Likewise the functional f of (27) is $(f, -\alpha)$ in (38). The lemma applies since $(X \times R)^* = X^* \times R$ is also locally convex.

Upon application of Lemma 4.2 with the neighborhood $M \times [-\epsilon, \epsilon]$ of $(0,0)$ in $X^* \times R$, we at once obtain (32)-(34) since θ is simply the multiplier of the functional (0,1), where here "0" is the identically zero linear functional on X.

Q.E.D.

5. The Main Result

To the program (7), we associate a second program

$$\begin{aligned} &\inf r \\ &\text{subject to } (r,0) \in \mathcal{cl}(f,g)(u) \end{aligned} \tag{39}$$

where (f,g) denotes the product function we denoted by f × g in (3). The value of (39) is denoted $v(P')$.

We recall that

$$\mathrm{epi}(f,g) = \{(u,r,w) \mid r \in f(u) \text{ and } w \in g(u)\} \tag{40}$$

and that by definition

$$\mathrm{epi}(c\ell(f,g)) = c\ell(\mathrm{epi}(f,g)) \tag{41}$$

If f and g are closed, so is epi(f,g) and hence, in this case

$$\begin{aligned}(r,0) \in c\ell(f,g)(u) &\leftrightarrow (u,r,0) \in \mathrm{epi}(c\ell(f,g)) = \mathrm{epi}(f,g) \\ &\leftrightarrow r \in f(u) \text{ and } 0 \in g(u)\end{aligned} \tag{42}$$

Thus $v(P') = v(P)$ when f and g are closed. In general, however, we have only the direction

$$\begin{aligned} r \in f(u) \text{ and } 0 \in g(u) \rightarrow (u,r,0) &\in \mathrm{epi}(f,g) \subseteq c\ell(\mathrm{epi}(f,g)) \\ &= \mathrm{epi}(c\ell(f,g)) \\ \rightarrow (r,0) &\in c\ell(f,g)(u)\end{aligned} \tag{43}$$

and hence always

$$v(P') \leq v(P) \ . \tag{44}$$

It is always the case that

$$\begin{aligned}(u,r,w) \in c\ell(\mathrm{epi}(f,g)) \rightarrow (u,r) &\in c\ell(\mathrm{epi}(f)) \\ \text{and } (u,w) &\in c\ell(\mathrm{epi}(g)) \ .\end{aligned} \tag{45}$$

One easily proves this from the definitions. However, the double implication

$$(u,r,w) \in c\ell(\text{epi}(f,g)) \leftrightarrow (u,r) \in c\ell(\text{epi}(f)) \text{ and } (u,w) \in c\ell(\text{epi}(g)) \tag{46}$$

may be desired.

For example, if both U and W are finite dimensional, there is always a point (u_0,r_0,w_0) in the relative interior of epi(f,g). Then (u_0,r_0) is in the relative interior of epi(f) and (u_0,w_0) is in the relative interior of epi(g), provided that f and g have the same effective domain (i.e. $\{u|f(u) \neq \phi\} = \{u|g(u) \neq \phi\}$). Then if $(u,r) \in c\ell(\text{epi}(f))$ and $(u,w) \in c\ell(\text{epi}(g))$, the Accessibility Lemma (Ref. 16) establishes that $\lambda(u_0,r_0) + (1 - \lambda)(u,r) \in \text{epi}(f)$ and $\lambda(u_0,w_0) + (1 - \lambda)(u,r) \in \text{epi}(g)$, whenever $0 < \lambda \leq 1$. This gives

$$\lambda(u_0,r_0,w_0) + (1 - \lambda)(u,r,w) \in \text{epi}(f,g) \tag{47}$$

Hence (46) holds.

When (46) holds, (39) can be replaced by the somewhat simpler program

$$\begin{gathered} \inf c\ell(f)(u) \\ \text{subject to } 0 \in c\ell(g)(u) \end{gathered} \tag{48}$$

We next state the main result of this paper.

Theorem 5.1: If U, U^*, and W are locally convex, U is semi-reflexive, f and g in (7) are convex, and (7) is consistent, then

$$\lim_{M\to 0}\ \sup_{u^*\in M}\ \sup_{\lambda^*\in W^*}\ \inf_{u\in U}\{f(u) + u^*(u) + \lambda^* g(u)\} = v(P') \qquad (49)$$

where "$M\to 0$" denotes the net consisting of a local base of open sets M which contain $0 \in M^*$.

Thus a necessary and sufficient condition for (24) is

$$v(P') = v(P) \qquad (50)$$

Corollary 5.2: With the hypotheses of Theorem 5.1, whenever f and g in (7) are closed, (24) holds.

Proof: From the remark following (42), we see that (50) holds. The result then follows from Theorem 5.1.

Q.E.D.

In R^n, it was shown (Ref. 7), that (50) (and hence (24)) held even for many nonclosed situations under an hypothesis weaker than a Slater point. In fact, the typical Slater point condition implied (50) (but the converse implication fails), so that the distinction between general convex and closed convex optimization disappears if one assumes the usual hypotheses for Lagrangean duality.

The proof of Theorem 5.1 requires two lemmas, and we give the easier one first.

Lemma 5.3: If U provides continuous linear functionals on U^* under the natural pairing $\langle u^*,u\rangle = u^*(u)$, then

$$\limsup_{M\to 0}\ \sup_{u^*\in M}\ \sup_{\lambda^*\in W}\ \inf_{u\in U}\{f(u) + u^*(u) + \lambda^* g(u)\} \le v(P') \qquad (51)$$

Proof: Let $u^* \in U^*$, $\lambda^* \in W^*$, and $\delta > 0$ be given, $\delta \leq 1$. Then there exists $u_0 \in U$ and $v \in R$ with $(v,0) \in c\ell(f,g)(u_0)$, and such that $v \leq v(P') + \delta/2$, if $v(P')$ is finite, and $v \leq -n - 1$, if $v(P') = -\infty$ (n arbitrary).

Hence, for any open neighborhood N of 0 in U, and N' of 0 in W, there are v', u_1 and w_1 satisfying:

$$w_1 \in g(u_1) \tag{52}$$

$$w_1 \in N' \tag{53}$$

$$v' \in f(u_1) \tag{54}$$

$$|v' - v| < \delta/2 \tag{55}$$

$$u_1 \in u_0 + N \tag{56}$$

From (51), $v' \leq v(P') + \delta$ if $v(P')$ is finite and $v' \leq -n$ if $v(P') = -\infty$.

We therefore have

$$\inf \{f(u_1) + u^*(u_1) + \lambda^* g(u_1)\} \leq v(P') + u^*(u_1) + \lambda^* w_1 + \delta \tag{57}$$

Since N and N' are arbitrary, and u^* and λ^* are continuous, (57) gives

$$\inf_{u \in U} \{f(u) + u^*(u) + \lambda^* g(u)\} \leq v(P') + u^*(u_0) + \delta \tag{58}$$

from which

$$\sup_{\lambda^* \in W^*} \inf_{u \in U} \{f(u) + u^*(u) + \lambda^* g(u)\} \leq v(P') + u^*(u_0) + \delta \tag{59}$$

follows by the arbitrary nature of λ^* in (58). Taking the limsup on both sides of (59) as $u^* \to 0$, we obtain (51), since $\delta > 0$ was arbitrary.

Q.E.D.

Lemma 5.4: With the hypotheses of Theorem 5.1,

$$\liminf_{M \to 0} \sup_{u^* \in M} \sup_{\lambda^* \in W} \inf_{u \in U} \{f(u) + u^*(u) + \lambda^* g(u)\} \geq v(P') \tag{60}$$

Proof: Since $R \times U \times W$ is locally convex and f and g are convex,

$$\begin{aligned} \operatorname{epi}(c\ell(f,g)) &= c\ell(\operatorname{epi}(f,g)) \\ &= \{(u,r,w) \mid u_i^*(u) + r_i^* r + w_i^*(w) \geq a_0^i,\ i \in I\} \end{aligned} \tag{61}$$

where $I \neq \phi$ is some index set (Ref. 14, Theorem 14.3). All functions u_i^*, r_i^*, and w_i^* are continuous. Thus $(r,0) \in c\ell(f,g)(u)$ is equivalent to

$$u_i^*(u) + r_i^* r \geq a_0^i,\ i \in I \tag{62}$$

We are given that (62) is consistent, and that it implies

$$r \geq v(P') \tag{63}$$

We now apply Lemma 4.3 to the implication of (62) to (63).

From (33), (34), (35) and (36) of Lemma 4.3, for any $\epsilon > 0$ and any neighborhood M of 0 in $\bar{U}^*$, there exist multipliers $y_i \geq 0$, $i \in I$, only finitely non-zero, a functional $u^* \in U^*$, and a real $r \in R$, satisfying

$$\sum_{i \in I} y_i a_0^i \geq v(P') - \epsilon \tag{64}$$

$$\sum_{i \in I} y_i (u_i^*, r_i^*) + (-u^*, r) = (0,1) \tag{65}$$

$$u^* \in M^* \tag{66}$$

$$|r| < \epsilon . \tag{67}$$

In (66), M^* is a barrel neighborhood of 0 in U^* with $M^* \subseteq (1 - \epsilon)M$ ($\epsilon < 1$). M^* exists because $(1 - \epsilon)M$ contains an open set about 0, and a locally convex space contains a local base which consists only of barrels (Ref. 14, 6.5).

Taking components in (65), we obtain

$$\sum_{i\in I} y_i u_i^* - u^* = 0 \tag{68}$$

$$\sum_{i\in I} y_i r_i^* + r = 1 . \tag{69}$$

Defining $\lambda_0 = \sum_{i\in I} y_i r_i^*$, we have

$$|\lambda_0 - 1| < \epsilon \tag{70}$$

by (67). Next, applying both sides of (68), as a functional, to an arbitrary element $u \in U$, and subtracting the result from (64), we obtain:

$$u^*(u) + \sum_{i\in I} y_i (a_0^i - u_i^*(u)) \geq v(P') - \epsilon \tag{71}$$

Now we compare with the definition (61). If $(r,w) \in c\ell(f,g)(u)$, we have

$$r_i^* r + w_i^*(w) \geq a_0^i - u_i^*(u), \ i \in I \tag{72}$$

Multiplying (72) by $y_i \geq 0$ and adding,

$$\begin{aligned} \lambda_0 r + \lambda^* w &= \sum_{i \in I} y_i (r_i^* r + w_i^*(w)) \\ &\ge \sum_{i \in I} y_i (a_0^i - u_i^*(u)) \; . \end{aligned} \tag{73}$$

where we have defined

$$\lambda^* = \sum_{i \in I} y_i w_i^* \in W^* \tag{74}$$

By combining (71) and (73) we have

$$u^*(u) + \lambda_0 r + \lambda^*(w) \ge v(\hat{P}) - \epsilon \tag{75}$$

if $(r,w) \in c\ell(\mathrm{epi}(f,g))$; thus (75) holds if $r \in f(u)$ and $w \in g(u)$. Therefore from (75),

$$\inf\{\lambda_0 f(u) + u^*(u) + \lambda^* g(u)\} \ge v(\hat{P}) - \epsilon \tag{76}$$

Since $|\lambda_0 - 1| < \epsilon$ and we may take $\epsilon < 1$, without loss of generality $\lambda_0 > 0$.

We now divide both sides of (76) by λ_0. Clearly, $\lambda^*/\lambda_0 \in W^*$. Since $|\lambda_0 - 1| < \epsilon$, $\frac{1}{1-\epsilon} M^* \subseteq M$, and M^* is a barrel, we have $u^*/\lambda_0 \in M$. In detail, if $\lambda_0 \ge 1$, then $0 < 1/\lambda_0 \le 1$, hence (as M^* is balanced), $u^*/\lambda_0 \in \frac{1}{\lambda_0} M^* \subseteq M^*$, and $M^* \subseteq (1 - \epsilon)^{-1} M^* \subseteq M$, so $u^*/\lambda_0 \in M$. Also, if $\lambda_0 < 1$, since $(1 - \epsilon)^{-1} > 1/\lambda_0 > 1$ (as $|\lambda_0 - 1| < \epsilon$), we have $u^*/\lambda_0 \in \frac{1}{\lambda_0} M^* \subseteq (1 - \epsilon)^{-1} M^* \subseteq M$, so again $u^*/\lambda_0 \in M$.

Thus in (76) we can assume that $\lambda_0 = 1$, if we replace the right-hand-side by $f(\epsilon) = v(P)/(1 + \epsilon) - \epsilon/(1 + \epsilon)$. Since $f(0) = v(P)$ and f is continuous at $\epsilon = 0$, in fact we can retain $v(P) - \epsilon$ as right-hand-side in (76). Now (76) with $\lambda_0 = 1$ gives (60), since $\epsilon > 0$ is arbitrary.

Q.E.D.

The proof of (49) is obtained by simply combining (51) and (60). In this manner, Theorem 5.1 is proven.

Results of a lagrangean type (18), but with "sup" replacing "max" in (18), can also be established by our methods, under suitable hypotheses of boundedness in (9). We need only apply the previous results to $g = (g_1, i(K))$.

Corollary 5.5: Suppose U is a reflexive normed space (with the norm topology on U^*), W is locally convex, f, g_1 and K in (9) are convex, and (9) is consistent. If K is bounded, then

$$\sup_{\lambda^* \in W^*} \inf_{u \in K} \{f(u) + \lambda^* g_1(u)\} = v(P') \ . \qquad (77)$$

Thus a necessary and sufficient condition for

$$\sup_{\lambda^* \in W^*} \inf_{u \in K} \{f(u) + \lambda^* g_1(u)\} = v(P) \qquad (78)$$

is that (50) hold. In particular, when f, g_1 and K are closed, (78) holds.

Proof: Since $0 \in M$ for any open set M of the origin in U^*, Lemma 5.3 implies that a "$\leq$" holds in (77). Hence we need only prove that "$\geq$" holds in (77), i.e. that for any $\epsilon > 0$ there exists $\lambda^* \in W^*$ with

$$\inf\{f(u) + \lambda^* g_1(u)\} \geq v(P') - \epsilon \qquad (79)$$

for all $u \in K$.

Let $L = \{\|u\| \,|\, u \in K\} < +\infty$. From (60), for any neighborhood of 0 in U^*, say $M_\rho = \{u^* \in U^* \,|\, \|u^*\| \leq \rho\}$, there exists $u^* \in M_\rho$ and $\lambda^* \in W^*$ with

$$\inf\{f(u) + u^*(u) + \lambda^* g_1(u)\} \geq v(P') - \epsilon/2 \qquad (80)$$

for all $u \in K$. Setting $\rho = \epsilon/(2L)$, we have $|u^*(u)| \leq \epsilon/2$ in (80), which at once gives (79).

Q.E.D.

6. The Case $U = R^n$

In this section we give results for the case that the domain U of both multi-valued maps f and g in (7) is finite-dimensional real-space R^n. As before, the range W can be any locally convex space. This is a continuation of results in [3]. We begin by citing a result from [3].

Theorem 6.1: [3, Theorem 3.3] Let $I \neq \phi$ be an arbitrary index set, and suppose that the system

$$a^i x \geq b_i, \quad \text{all } i \in I \qquad (81)$$

has a solution in R^n.

Suppose also that (81) implies

$$cx \geq d \qquad (82)$$

for any $x \in R^n$.

Then there is a vector $w \in R^n$ and a scalar $w_0 \in R$, with the following property:

For every $0 < \theta \leq 1$ there are nonnegative scalars $\{\lambda_i | i \in I\}$, only finitely non-zero, which satisfy

$$c + \theta w = \sum_{i \in I} \lambda_i a^i \tag{83}$$

$$d + \theta w_0 \leq \sum_{i \in I} \lambda_i b_i \tag{84}$$

In fact, if $(v,-v_0)$ is any point in the relative interior of the set

$$C'' = \text{cone}(\{(a^i,-b_i)|i \in I\} \cup \{(0,1)\}) \tag{85}$$

we may set

$$(w,-w_0) = (v,-v_0) - (c,-d) \tag{86}$$

i.e. $w = v - c$ and $w_0 = v_0 - d$.

Lemma 6.2: Suppose that the domain U of f and g in (7) is $U = R^n$, W is locally convex, f and g in (7) are convex, and (7) is consistent.

Then there exists a fixed vector $u \in R^n$ and scalars $w_0, w_1 \in R$ with the following property:

For any θ in the range $0 < \theta \leq 1$, there exists $\lambda^* \in W^*$ with

$$(1 + \theta w_1)r + \theta ux + \lambda^* w \geq v(P') + \theta w_0 \tag{87}$$

whenever $r \in f(x)$ and $w \in g(x)$, for any $x \in R^n$.

Proof: The proof entirely parallels that of Lemma 5.4 up to (75), and differs primarily in citing Theorem 6.1 in place of Lemma 4.3.

We have (61), and the fact that (62) implies (63), and now $U = R^n$. By Theorem 6.1, there exists a vector $u \in R^n$ and scalars w_0 and w_1 in R, with the following property: for any $0 < \theta \leq 1$, there are nonnegative scalars $\{\lambda_i | i \in I\}$, only finitely non-zero, which satisfy:

$$(0,1) + \theta(-u,w_1) = \sum_{i \in I} \lambda_i (u_i^*, r_i^*) \quad , \tag{88}$$

$$v(P') + \theta w_0 \leq \sum_{i \in I} \lambda_i a_0^i \tag{89}$$

Taking components in (88), we obtain

$$\sum_{i \in I} \lambda_i u_i^* - \theta u = 0 \tag{90}$$

$$1 + \theta w_1 = \sum_{i \in I} \lambda_i r_i^* \tag{91}$$

Applying both sides of (90), as a functional, to an arbitrary element $x \in R^n$, and subtracting the result from (89), we obtain

$$\theta ux + \sum_{i \in I} \lambda_i (a_0^i - u_i^* x) \geq v(P') + \theta w_0 \tag{92}$$

Again, as in (72), we have

$$r_i^* r + w_i^*(w) \geq a_0^i - u_i^*(x) \quad , \; i \in I \tag{93}$$

whenever $(r,w) \in c\ell(f,g)(x)$. Multiplying (93) by $\lambda_i \geq 0$, adding, and using (91), we have

$$\begin{aligned}(1 + \theta w_1)r + \lambda^* w &= \sum_{i \in I} \lambda_i (r_i^* r + w_i^* w) \\ &\geq \sum_{i \in I} \lambda_i (a_0^i - u_i^*(x))\end{aligned} \tag{94}$$

where we have set

$$\lambda^* = \sum_{i\in I} \lambda_i w_i^* \in W^* \tag{95}$$

Combining (92) and (95), we have

$$(1 + \theta w_1)r + \theta ux + \lambda^* w \geq v(P') + \theta w_0 \tag{96}$$

if $(r,w) \in c\ell(f,g)(x)$. In particular, (96) holds if $r \in f(x)$ and $w \in g(x)$.

Q.E.D.

Theorem 6.3: Suppose that the domain U of f and g in (7) is $U = R^n$, W is locally convex, f and g in (7) are convex, and (7) is consistent. Then there exists a fixed vector $u \in R^n$ such that

$$\lim_{\theta\to 0^+} \sup_{\lambda^*\in W^*} \inf_{x\in R^n} \{f(x) + \theta ux + \lambda^* g(x)\} = v(P') \ . \tag{97}$$

Therefore

$$\lim_{\theta\to 0^+} \sup_{\lambda^*\in W^*} \inf_{x\in R^n} \{f(x) + \theta ux + \lambda^* g(x)\} = v(P) \tag{98}$$

precisely if $v(P) = v(P')$. In particular, (98) holds if f and g are closed.

Proof: The particular fact follows from the general one (97) as in Corollary 5.2; we prove only (97). To this latter end, only the result

$$\liminf_{\theta\to 0^+} \sup_{\lambda^*\in W^*} \inf_{x\in R^n} \{f(x) + \theta ux + \lambda^* g(x)\} \geq v(P') \tag{99}$$

is necessary, by Lemma 5.3.

However, (99) is itself a direct consequence of (87). Indeed, the variable θ of (87) and $\theta' = \theta/(1 + \theta w_1)$ each go to zero if the other does, and hence for $\theta' > 0$ sufficiently small that $1 + \theta w_1 > 0$, we have from (87) that

$$r + \theta' ux + (\lambda^* w)/(1 + \theta w_1) \geq (v(P') + \theta w_0)/(1 + \theta w_1) \qquad (100)$$

if $r \in f(x)$ and $w \in g(x)$, i.e.

$$\begin{aligned} \inf_{x \in R^n} \{f(x) + \theta' ux + (\lambda^* g(x))/(1 + \theta w_1)\} \\ \geq (v(P') + \theta w_0)/(1 + \theta w_1) \end{aligned} \qquad (101)$$

Of course (101) implies

$$\sup_{\lambda \in W^*} \inf_{x \in R^n} \{f(x) + \theta' ux + \lambda^* g(x)\} \geq (v(P') + \theta w_0)/(1 + \theta w_1) \qquad (102)$$

If we take the lim inf as $\theta' \to 0^+$ on both sides of (102), we obtain (99).

Q.E.D.

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Abstract

We show that duality gaps can be closed under broad hypotheses in minimax problems, provided certain changes are made in the maximin part which increase its value. The primary device is to add a linear perturbation to the saddle function, and send it to zero in the limit. Suprema replace maxima, and infima replace minima. In addition to the usual convexity-concavity type of assumptions on the saddle function and the sets, a form of semi-reflexivity is required for one of the two spaces of the saddle function.

A sharpening of our result is possible when one of the spaces is finite-dimensional.

A variant of the proof of the previous results leads to a generalization of a result of Sion, from which the theorem of Kneser and Fan follows.

Key Words:

1) Convexity
2) Limiting Lagrangean
3) Lagrangean
4) Minimax

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A LIMITING INFISUP THEOREM

by

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A minimax theorem is one which asserts that, under suitable hypotheses,

$$\max_{y\in D}\min_{x\in C} F(x,y) = \min_{x\in C}\max_{y\in D} F(x,y) \tag{1}$$

for certain functions F and sets C, D. These have been of wide interest in the literature; see e.g. [9] and [12].

Recently, results have been obtained, motivated by the second author's paper [3], which indicate that in cases when lagrangean duality does not hold, a certain type of limiting lagrangean duality does hold (see e.g. [2], [5]). Here we extend these limiting phenomena to the general minimax setting, of which lagrangean duality is but one case, in which $F(x,y)$ is the lagrangean function with y the dual multipliers (and hence $D = R^m_+$ and $\inf_{x\in C}\sup_{y\geq 0} F(x,y)$ is the value of the primal program).

We will establish, in place of (1), this result (see Theorem 3 below):

$$\lim_{M\to 0}\ \sup_{x^*\in M}\ \sup_{y\in D}\ \inf_{x\in C}\{x^*(x) + F(x,y)\} = \inf_{x\in C}\sup_{y\in D} F(x,y) \tag{2}$$

under suitable hypotheses. In (2), the notation "M→0" indicates that M

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is an arbitrary neighborhood of zero, in a net that tends to zero, in the dual space X^* of the space X in which the set C lies $(C \subseteq X)$. We then strengthen (2) in the case that $X = R^n$.

Our method of proof in Theorem 3, is to reduce minimax problems to programming problems, by finding a suitable program (given in equation (10) below) whose value is $\inf_{x \in C} \sup_{y \in D} F(x,y)$, and then using the limiting lagrangean dual for this primal program, to obtain (2). The proof is somewhat complicated by a purely technical detail, as our reference paper [5] on this limiting lagrangean is done in the generality of set-valued convex functions, while here we only need point-valued convex functions. The reader will note below, that the devices we employ to convert a point-valued program into its set-valued equivalent are those discussed in [5] for this transformation, and we have repeated them here to have the paper more self-contained.

Following the proof of Theorem 3, we will indicate how an important special case of that result can be derived from the conjagate duality theory as developed by Rockafellar [10]. Related results can be found in [7], [8], and [13].

In a concluding section, we return to the primal program of equation (10), and by analyzing its finitely-constrained subprograms, we obtain a generalization of a result of Sion [11]. Essentially, our result shows how finite subprograms approach the value $\sup_{y \in D} \inf_{x \in C} F(x,y)$, and thus are also of a limiting nature; and the same kind of "finite approximation" is possible for the quantity $\inf_{x \in C} \sup_{y \in D} F(x,y)$.

Section I: Limiting Linear Perturbations

First, we establish the easy direction in (2).

Lemma 1: If X provides continuous linear functionals on X^*, then

$$(3)\qquad \limsup_{M\to 0}\ \sup_{x^*\in M}\ \sup_{y\in D}\ \inf_{x\in C}\{x^*(x) + F(x,y)\} \le \inf_{x\in C}\ \sup_{y\in D} F(x,y)$$

Proof: Assume $v^* = \inf_{x\in C}\sup_{y\in D} F(x,y) < +\infty$; otherwise, there is nothing to prove.

Let $x^* \in X^*$, an integer n, and $\delta > 0$ be given. Then there exists $x_0 \in C$ such that, for all $y \in D$,

$$(4)\qquad F(x_0,y) \le \begin{cases} v^* + \delta\ , & \text{if } v^* \text{ is finite;} \\ -n\ , & \text{if } v^* = -\infty. \end{cases}$$

From (4) it follows at once that, for any $y \in D$,

$$(5)\qquad \inf_{x\in C}\{x^*(x) + F(x,y)\} \le x^*(x_0) + F(x_0,y)$$

$$\le \begin{cases} v^* + \delta + x^*(x_0)\ , & \text{if } v^* \text{ is finite;} \\ -n \quad + x^*(x_0)\ , & \text{if } v^* = -\infty. \end{cases}$$

After taking the supremum over $y \in D$ on the left-hand-side in (5), and noting, that as $M \to 0$, if $x^* \in M$, then $x^*(x_0) \to 0$, we have

$$(6)\ \limsup_{M\to 0}\ \sup_{x^*\in M}\ \sup_{y\in D}\ \inf_{x\in C}\{x^*(x) + F(x,y)\} \le \begin{cases} v^* + \delta\ , & \text{if } v^* \text{ is finite;} \\ -n\ , & \text{if } v^* = -\infty. \end{cases}$$

Since $\delta > 0$, or n, is arbitrary in (6), we obtain (3).

Q.E.D.

We now recall the setting of the paper [5]. Both f and g are set-valued functions on a space U, f with subsets of the reals as values, and g with subsets of a space W as values, in this program of value v(P):

$$\text{(7)} \qquad \inf f(u) \quad \text{subject to } 0 \in g(u) \; .$$

We shall say that U is semi-reflexive, if these two conditions hold: (i) For each $u \in U$, the function $u^*(u)$ on U^* is continuous; (ii) For every continuous linear functional u^{**} on U^* there exists $u \in U$ such that $u^{**}(u^*) = u^*(u)$ for all $u^* \in U^*$.

We now summarize Corollary 5.2 of [5].

Theorem 2: [5]

If U, U^*, and W are locally convex, U is semi-reflexive, f and g in (7) are closed and convex, and $v(P) < +\infty$, then

$$\text{(8)} \qquad \lim_{M\to 0} \sup_{u^*\in M} \sup_{\lambda^*\in W} \inf_{u\in U}\{f(u) + u^*(u) + \lambda^* g(u)\} = v(P) \; .$$

We next present our main result. As regards the hypothesis β) of Theorem 3, the definition of a concavelike function is as in [11].

Theorem 3: Suppose that X is a semi-reflexive locally convex space, X* is locally convex, and $\inf_{x\in C} \sup_{y\in D} F(x,y)$ is not $+\infty$.

Suppose in addition, that C is a non-empty, closed, convex set in X, D is a non-empty set in a space Y, and F(x,y) is a function with values in $R \cup \{+\infty\}$, such that:

α) For each fixed $y\in Y$, F(x,y) is a closed convex function of $x\in X$;

β) $F(x,y)$ is concavelike in $y \in Y$ on $C \times D$.

Then (2) holds.

Proof: Using (3) of Lemma 1, it suffices for us to prove

$$\liminf_{M \to 0} \sup_{x^* \in M} \sup_{y \in D} \inf_{x \in C} \{x^*(x) + F(x,y)\} \geq v^* \tag{9}$$

where $v^* = \inf_{x \in C} \sup_{y \in D} F(x,y)$. Let $\epsilon > 0$ be given, as well as a neighborhood M of $0 \in X^*$.

Consider this program:

$$\begin{aligned} &\inf t \\ &\text{subject to } F(x,y) - t \leq 0 \text{ , for all } y \in D. \\ &\qquad\quad x \in C \end{aligned} \tag{10}$$

If (x,t) is feasible in (10), for some $x \in C$ we have $t \geq \sup_{y \in D} F(x,y)$; hence $t \geq v^*$. On the other hand, if $\delta > 0$ and n are arbitrary, there is some $x_0 \in C$ such that, for all $y \in D$,

$$F(x_0,y) \leq \begin{cases} v^* + \delta \text{ , if } v^* \text{ is finite;} \\ -n \quad\text{ , if } v^* = -\infty. \end{cases} \tag{11}$$

Putting

$$t = \begin{cases} v^* + \delta \text{ , if } v^* \text{ is finite;} \\ -n \quad\text{ , if } v^* = -\infty; \end{cases} \tag{12}$$

we have a feasible solution to (10). Since δ and n are arbitrary, the value of (10) is exactly v^*.

Now (10) can be cast as a convex program (7) in the sense of [5]. We put $f(x,t) = \{t\}$, and f is on the space $U = X \times R$, which is semireflexive, since X is. The function $g(x,t)$ maps into the product space $W = \mathop{\mathsf{X}}\limits_{y \in D} R_y$, where all $R_y = R$, of $|D|$ copies of the reals R. We set g to be a product function (as discussed in [5]) by putting, for each $(x,t) \in C \times R$

$$g(x,t) = \mathop{\mathsf{X}}\limits_{y \in D} g_y(x,t) \tag{13}$$

where we have

$$g_y(x,t) = \begin{cases} \{w | w \geq F(x,y) - t\}, & \text{if } x \in C; \\ \phi & \text{, if } x \notin C. \end{cases} \tag{14}$$

Since C is closed and $F(x,y)$ is closed in $x \in X$ for each fixed $y \in D$, g is closed. It is easy to see that $0 \in g(x,t)$ is equivalent to $0 \in g_y(x,t)$ for all $y \in D$, which is equivalent to $0 \geq F(x,y) - t$ for each $y \in D$. Therefore the program

$$\begin{gathered} \inf f(x,t) \\ \text{subject to } 0 \in g(x,t) \end{gathered} \tag{7}'$$

is entirely equivalent to the program (10), and so has value $v^* < +\infty$. Thus Theorem 2 applies.

The conjugate space W^* of W is $W^* = \bigoplus\limits_{y \in D} R_y$, i.e. all finitely-nonzero vectors $(\lambda_y | y \in D)$ indexed by $y \in D$. We conclude the following from (8): for any $\epsilon_1 > 0$, $1 > \epsilon_2 > 0$, there exists a functional $z^* \in X^*$, a real number $t^* \in R$, and a finitely non-zero vector of reals $(\lambda_y^* | y \in D)$, with:

(15a) $$t + z^*(x) + t^* t + \sum_{y \in D} \lambda_y^* w_y \geq v^* - \epsilon_1 ,$$

for all $x \in C$ and $t \in R$, and $w_y \in g_y(x,t)$;

(15b) $$|t^*| < \epsilon_2 ;$$

(15c) $$z^* \in M^* ;$$

where M^* is a convex, circled neighborhood of 0 such that $M^* \subseteq (1 - \epsilon_2)M$.

Since $w_y \in g_y(x,t)$ can be arbitrarily increased in (15a), we conclude that $\lambda_y^* \geq 0$ for all $y \in D$. Then (15a) is equivalent to (using $w_y = F(x,y) - t$)

(16) $$(1 + t^* - \sum_{y \in D} \lambda_y^*)t + z^*(x) + \sum_{y \in D} \lambda_y^* F(x,y) \geq v^* - \epsilon_1 ,$$

for all $t \in R$, $x \in C$.

Upon fixing $x \in C$ in (16), since $t \in R$ is arbitrary, we conclude that

(17a) $$1 + t^* = \sum_{y \in D} \lambda_y^* ;$$

(17b) $$z^*(x) + \sum_{y \in D} \lambda_y^* F(x,y) \geq v^* - \epsilon_1 ,$$

for all $x \in C$.

Putting $x^* = z^*/(1 + t^*)$, one can prove, using the fact that M^* is circled, (15b), and $M^* \subseteq (1 - \epsilon_2)M$, that

(18) $$x^* \in M .$$

Putting

(19) $$\lambda_y' = \lambda_y^*/(1 + t^*) , \text{ for } y \in D,$$

and dividing both sides of (17b) by $1 + t^*$, we obtain

(20) $$x^*(x) + \sum_{y \in D} \lambda'_y F(x,y) \geq (v^* - \epsilon_1)/(1 + t^*) ,$$

for all $x \in C$;

(21) $$\text{All } \lambda'_y \geq 0 \text{ , and } \sum_{y \in D} \lambda'_y = \frac{1 + t^*}{1 + t^*} = 1 .$$

By a suitable preselection of ϵ_1 and ϵ_2 so that $(v^* - \epsilon_1)/(1 + \epsilon_2) \geq v^* - \epsilon$, using (15b), (20) becomes

(22) $$x^*(x) + \sum_{y \in D} \lambda'_y F(x,y) \geq v^* - \epsilon ,$$

for all $x \in C$.

Using (21) and the concavelike property of $F(x,y)$ in $y \in D$ (i.e., hypothesis β)) from (21) there exists $\bar{y} \in D$ with $F(x,\bar{y}) \geq \sum_{y \in D} \lambda'_y F(x,y)$ for any $x \in C$. Hence by (22),

(23) $$x^*(x) + F(x,\bar{y}) \geq v^* - \epsilon \text{ , for all } x \in C.$$

From (23) it follows at once that

(24) $$\sup_{y \in D} \inf_{x \in C} \{x^*(x) + F(x,y)\} \geq v^* - \epsilon .$$

We have obtained (24) for M any neighborhood of $0 \in X^*$, yet (18) holds; hence

(25) $$\liminf_{M\to 0} \sup_{x^*\in M} \sup_{y\in D} \inf_{x\in C}\{x^*(x) + F(x,y)\} \geq v^* - \epsilon .$$

Since (25) holds for $\epsilon > 0$ arbitrary, we have (9).

Q.E.D.

An important special case of Theorem 3 can be deduced from the conjugate duality theory. Specifically, assume that (in place of hypothesis β) of Theorem 3) $F(x,y)$ is concave in y for each $x \in C$, and that the closure condition of [10, (8.27) on page 50] holds (this is stronger than the closure condition α) above). Then one can establish the conclusion of Theorem 3 using [10, (8.28) on page 51, (4.16) and (4.20) of page 21].

Corollary 4: Suppose that the hypotheses of Theorem 3 hold, and that also X and X^* are normed spaces, and the set C is bounded in X. Then

(26) $$\sup_{y\in D} \inf_{x\in C} F(x,y) = \inf_{x\in C} \sup_{y\in D} F(x,y) .$$

Proof: Since this proof entirely parallels the proof of Corollary 5.5 from Theorem 2 in [5], we omit details.

We next give a result for the case $X = R^n$, which is derived from [5, Theorem 6.3] by use of the same argument as in the proof of Theorem 3 above.

Theorem 5: Suppose that the hypotheses of Theorem 3 hold, and in addition $X = R^n$. Then there is some fixed $w \in R^n$ such that

(27) $$\lim_{\theta\to 0^+} \sup_{y\in D} \inf_{x\in C}\{\theta wx + F(x,y)\} = \inf_{x\in C} \sup_{y\in D} F(x,y) .$$

Theorem 6: Suppose that Y is a semi-reflexive locally convex space, Y^* is locally convex, and $\sup_{y \in D} \inf_{x \in C} F(x,y)$ is not $-\infty$.

Suppose, in addition, that C is a non-empty set in a space X, D is a non-empty, closed, convex set in Y, and F(x,y) is a function iwth values in $R \cup \{-\infty\}$, R the reals, such that:

i) F(x,y) is a convexlike function of $x \in X$ on $C \times D$.

ii) For each fixed $x \in X$, F(x,y) is a closed, concave function of $y \in Y$.

Then

$$(2)' \qquad \lim_{M \to 0} \inf_{y^* \in M} \inf_{x \in C} \sup_{y \in D} \{y^*(y) + F(x,y)\} = \sup_{y \in D} \inf_{x \in C} F(x,y)$$

Proof: Apply Theorem 3 to the function $G(y,x) = -F(x,y)$.

Q.E.D.

We leave the derivation of a "sup inf" theorem, analogous to Theorem 5, to the reader; as in the case of Theorem 6, it arises by applying Theorem 5 to $G(y,x) = -F(x,y)$.

Section II: A Generalization of a Theorem of Sion

By a further study of the program (10) of the preceding section, and our argument following it, we are able to derive a second result. Our general hypotheses change, in that no linear structure or topology is needed. We derive the results directly from the assumption that F is convex-concavelike (see [11], [12] for the definitions).

Theorem 7: Suppose that C and D are nonempty sets and F(x,y) is convex-concavelike on C × D.

Then

(28)
$$\sup_{y\in D}\inf_{x\in C} F(x,y) = \sup_{\substack{G\subseteq D\\ G \text{ finite}}}\ \inf_{x\in C}\max_{y\in G} F(x,y)$$

Proof: The direction (≤) in (28) is trivial, since we may use singleton sets:

(29)
$$\begin{aligned}\sup_{y\in D}\inf_{x\in C} F(x,y) &= \sup_{\{y_0\}\subseteq D}\inf_{x\in C} F(x,y_0)\\ &= \sup_{\{y_0\}\subseteq D}\inf_{x\in C}\max_{y\in\{y_0\}} F(x,y)\\ &\le \sup_{\substack{G\subseteq D\\ G \text{ finite}}}\ \inf_{x\in C}\max_{y\in G} F(x,y)\end{aligned}$$

To obtain the reverse direction (≥) in (28), we examine finite subsets of the constraints of the program (10).

Let $G \subseteq D$ be finite. Then the value $v^*(G)$ of the program

$$(10)' \qquad \inf_{x \in C} t \quad \text{subject to } F(x,y) - t \le 0, \text{ for all } y \in G$$

is $v^*(G) = \inf_{x \in C} \max_{y \in G} F(x,y)$, by reasoning similar to that following equation (10) above. Note that $v^*(G) < +\infty$. Moreover, the program (10)′ has a Slater point, for upon setting $x = x_0 \in C$ arbitrarily, and putting $t_0 = 1 + \max_{y \in G} F(x_0,y)$, we see that each functional constraint in (10)′ is satisfied as a strict inequality. Now $F(x,y) - t$ is convexlike in (x,t) on $(C \times R) \times G$, and so there exist Lagrange multipliers $\lambda_y \ge 0$, $y \in G$ with

$$(30) \qquad t + \sum_{y \in G} \lambda_y (F(x,y) - t) \ge \inf_{x \in C} \max_{y \in G} F(x,y)$$

for all $x \in C$ and $t \in R$.

Since $t \in R$ is arbitrary in (30), we conclude that

$$(31) \qquad \sum_{y \in G} \lambda_y = 1$$

and

$$(32) \qquad \sum_{y \in G} \lambda_y F(x,y) \ge \inf_{x \in C} \max_{y \in G} F(x,y)$$

for all $x \in C$. By the concavelike property of $F(x,y)$ in y, there exists $\bar{y} \in D$ with

(33) $$F(x,\bar{y}) \geq \inf_{x\in C} \max_{y\in G} F(x,y)$$

for all $x \in C$. From (33) we immediately deduce

(34) $$\sup_{y\in D} \inf_{x\in C} F(x,y) \geq \inf_{x\in C} \max_{y\in G} F(x,y)$$

and, since $G \subseteq D$ was an arbitrary finite set, we at once have the desired direction ($\geq$) in (28).

Q.E.D.

Theorem 8. With the hypotheses of Theorem 6,

(35) $$\inf_{x\in C} \sup_{y\in D} F(x,y) = \inf_{\substack{H\subseteq C \\ H \text{ finite}}} \sup_{y\in D} \min_{x\in H} F(x,y)$$

Proof: Apply Theorem 7 to the convex-concavelike function $G(y,x) = -F(x,y)$.

Q.E.D.

Corollary 9: [11, Theorem 4.1]

Let C and D be nonempty sets and let F be convex-concavelike on C × D.

If for any $\alpha < \inf_{x\in C} \sup_{y\in D} F(x,y)$ there exists a finite $G \subseteq D$ such that, for any $x \in C$ there is $y \in G$ with $F(x,y) \geq \alpha$, then

(36) $$\sup_{y\in D} \inf_{x\in C} F(x,y) = \inf_{x\in C} \sup_{y\in D} F(x,y)$$

Proof: The hypotheses state that, for any $\alpha < \inf_{x\in C} \sup_{y\in D} F(x,y)$, we have $\inf_{x\in C} \max_{y\in G} F(x,y) \geq \alpha$ for some finite $G \subseteq D$, and hence

$$(37) \qquad \inf_{x\in C}\sup_{y\in D} F(x,y) \le \sup_{\substack{G\subseteq D \\ G \text{ finite}}} \inf_{x\in C}\max_{y\in G} F(x,y)$$

$$= \sup_{y\in D}\inf_{x\in C} F(x,y),$$

the last equality by (28) of Theorem 7. Since the reverse inequality ($\le$) of (36) always holds, we obtain (36).

Q.E.D.

In a similar manner, [11, Theorem 4.1′] can be proven from Theorem 8, and Sion gives a derivation of a result of Kneser and Fan [11, Theorem 4.2] from these corollaries.

We conclude with a result that shows how the quantification over finite sets in (28) can be replaced, if one wishes, by a limit over a suitable sequence.

Corollary 10: Suppose that C and D are nonempty sets and $F(x,y)$ is convex-concavelike on $C \times D$.

Then there is a sequence $y_1, y_2, y_3, \ldots$ in D such that

$$(28)' \qquad \sup_{y\in D}\inf_{x\in C} F(x,y) = \lim_{t\to\infty}\inf_{x\in C}\max_{y\in G_t} F(x,y)$$

where $G_t = \{y_1, y_2, \ldots, y_t\}$.

Proof: Let $v_* = \sup_{\substack{G\subseteq D \\ G \text{ finite}}} \inf_{x\in C}\max_{y\in G} F(x,y).$

Inductively define the sets $H_j = \{y_{h(j)+1}, \ldots, y_{h(j+1)}\}$ by the conditions that $h(1) = 0$ and

(38) $$\inf_{x\in C}\ \max_{y\in H_j} F(x,y) \geq \begin{cases} v_* - \frac{1}{j}, & \text{if } v_* < +\infty ; \\ j, & \text{if } v_* = +\infty . \end{cases}$$

Then $G_{h(j+1)} = H_1 \cup H_2 \cup \dots \cup H_j$, and so by (38)

(39) $$\inf_{x\in C}\ \max_{y\in G_{h(j+1)}} F(x,y) \geq \begin{cases} v_* - \frac{1}{j}, & \text{if } v_* < +\infty ; \\ j, & \text{if } v_* = +\infty . \end{cases}$$

Defining $v(G) = \inf_{x\in C} \max_{y\in G} F(x,y)$ for finite subsets G of C, we observe the monotonicity property

(40) $$G \subseteq G' \text{ implies } v(G) \leq v(G').$$

Combining (39) and (40), we have

(41) $$\lim_{t\to+\infty} \inf_{x\in C}\ \max_{y\in G_t} F(x,y) = v_* = \sup_{\substack{G\subseteq D \\ G \text{ finite}}} \inf_{x\in C} \max_{y\in G} F(x,y)$$

The result (28)′ then follows from (28).

Q.E.D.

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